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Parameter Identification Applied to Aircraft

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SUMMARY

All three steps in the identification, namely characterization, parameter estimation and verification are considered and applied to the determination of aircraft parameters from flight data. The estimation procedure includes the equation error method and the output error method with the weighted least squares, maximum likelihood and Bayesian estimation technique. The problems concerning accuracy and identifiability are also discussed.

A general computing algorithm is developed covering all estimation techniques described. It is applicable to linear as well as nonlinear systems and is flexible enough for the solution of various identifiability problems. Using this algorithm the computing program has been compiled in general terms to enable the user to achieve the objectives mentioned.

As examples, the results of the identification of aircraft parameters and aerodynamic derivatives for four different aircraft are presented. They include the analysis of flight data from the Basset variable stability aircraft and from the M.S. 760 Paris aircraft, the simulated data for a nonlinear model of the F-100 aircraft, and the flight data from the longitudinal motion of the HP-115 slender delta-wing research aircraft.

The report is completed by the recommendation for future work in the area of aircraft parameter identification.

FOREWORD

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NOTATION

A	state matrix ($n \times n$)
B	input matrix ($n \times p$)
b	wing span, [m]
C	transformation matrix ($m \times n$)
$C_{l,m,n}$	rolling -, pitching -, and yawing - moment coefficients
$C_{x,y,z}$	longitudinal -, lateral -, and vertical - force coefficients
$\bar{c}$	mean aerodynamic chord, [m]
D	transformation matrix ($m \times p$)
$E \{ \cdot \}$	expected value
g	a) constant bias vector ($m \times 1$) b) acceleration due to gravity, [m/s^2]
H	gradient (sensitivity) matrix ($m \times q$)
I	identity matrix
$I_{x,y,z}$	moment of inertia of aircraft about x,y,z body axes, respectively, [$kg \ m^2$]
I_{xz}	product of inertia of aircraft referred to x a z body axes, [$kg \ m^2$]
J	cost function
L	likelihood function
L,M,N	rolling, pitching, and yawing moments, [Nm]
M	Fisher information matrix ($q \times q$) for the unknown parameters
$\bar{M}^{-1}$	error covariance matrix ($q \times q$) for the estimated parameters (Cramèr-Rao lower bound)
m	a) number of outputs b) aircraft mass, [kg]
N	number of data points
n	a) number of states b) measurement noise vector ($m \times 1$)

$n_{x,y,z}$ readings of the longitudinal, lateral, and vertical accelerometer, [g units]

$$n_{y,z}^* = \frac{g}{V_e} n_{y,z}$$

P_j percentage cumulative distribution

p number of inputs

p,q,r rolling, pitching, and yawing velocities, [rad/s]

q number of unknown parameters

R_1 measurement noise covariance matrix ($m \times m$)

R_2 covariance matrix of a priori known parameters ($q \times q$)

R_x autocorrelation function

R_{xo} autogovariance function

r correlation lag number

S wing area

$s(\cdot)$ standard error

$s^2(\cdot)$ variance estimate

T transformation matrix

t time, [s]

u,v,w longitudinal, lateral, and vertical airspeed components, [m/s]

u input vector ($p \times 1$)

u^* augmented input vector ($p \times 1$)

V true airspeed, [m/s]

W diagonal transformation matrix ($q \times q$) with elements $1/\sigma_i$

$W_{1,2}$ weighting matrices ($m \times m$) and ($q \times q$)

X,Y,Z longitudinal, lateral, and vertical forces, [N]

x state vector ($n \times 1$)

y output vector ($m \times 1$)

z measurement vector ($m \times 1$)

α	a) vector of initial conditions ($n \times 1$) b) angle of attack, [rad]
β	a) vector of unknown parameters in A, B, C and D b) sideslip angle, [rad]
γ	parameter vector ($q \times 1$)
ξ, η, ζ	rudder, elevator, and aileron deflections, [rad]
θ	pitch angle, [rad]
μ	relative density parameter
v	residual
ρ	air density, [kg/m ³]
ρ_{ij}	correlation coefficients
σ	standard deviation
σ^2	variance
τ	unit of time, [s]
ϕ	bank angle, [rad]

Matrix exponents:

T	indicates transpose matrix operation
-1	indicates inverse matrix operation

Supscripts:

E	measured quantity
o	initial value
e	steady-state value

Additional notation:

FPE	final prediction error
ML	maximum likelihood
MSE	mean square error
WLS	weighted least squares
$ \cdot $	norm of quantity enclosed
$ \cdot $	determinant of a matrix enclosed

Non-dimensional aerodynamic derivatives:

$$C_{Xu} = \frac{\partial C_X}{\partial \frac{u}{V}}$$

$$C_{X\alpha} = \frac{\partial C_X}{\partial \alpha}$$

$$C_{X\eta} = \frac{\partial C_X}{\partial \eta}$$

$$C_{Y\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{Yp} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$$

$$C_{Y\xi} = \frac{\partial C_Y}{\partial \xi}$$

$$C_{Y\zeta} = \frac{\partial C_Y}{\partial \zeta}$$

$$C_{Zu} = \frac{\partial C_Z}{\partial \frac{u}{V}}$$

$$C_{Z\alpha} = \frac{\partial C_Z}{\partial \alpha}$$

$$C_{Zq} = \frac{\partial C_Z}{\partial \frac{qc}{2V}}$$

$$C_{Z\xi^2} = \frac{\partial C_Z}{\partial \xi^2}$$

$$C_{Z\eta} = \frac{\partial C_Z}{\partial \eta}$$

$$C_{l\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{lp} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$$

$$C_{lr} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$$

$$C_{l\xi} = \frac{\partial C_l}{\partial \xi}$$

$$C_{l\zeta} = \frac{\partial C_l}{\partial \zeta}$$

$$C_{l\eta} = \frac{\partial C_l}{\partial \eta}$$

$$C_{mu} = \frac{\partial C_m}{\partial \frac{u}{V}}$$

$$C_{m\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m\dot{\alpha}} = \frac{\partial C_m}{\partial \frac{\dot{\alpha}c}{2V}}$$

$$C_{mq} = \frac{\partial C_m}{\partial \frac{qc}{2V}}$$

$$C_{m\eta} = \frac{\partial C_m}{\partial \eta}$$

$$C_{m\xi^2} = \frac{\partial C_m}{\partial \xi^2}$$

$$C_{n\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{np} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$C_{nr} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{n\xi} = \frac{\partial C_n}{\partial \xi}$$

$$C_{n\zeta} = \frac{\partial C_n}{\partial \zeta}$$

1. INTRODUCTION

Parameter identification has been developed over the last ten years as a strategy and technique for establishing the properties of any system by the measurement of its input and output time histories. During development several different approaches and methods have been proposed and tested. Their review may be found in two excellent survey papers by Eykhoff, Ref. 1, and by Åström and Eykhoff, Ref. 2.

The process of parameter identification has also been applied to the determination of stability and control derivatives of an aircraft from flight test data. There are several reasons why aircraft parameter identification forms a very important field of flight test data analysis.

These include:

1. many instances where prototype aircraft do not have the predicted characteristics, when cost and safety considerations control the ways of obtaining a better knowledge of aircraft parameters,
2. requirements for a better understanding of theoretical predictions and wind tunnel testing and the possibility of deeper understanding of aerodynamic phenomena and their relationship to aircraft stability and control,
3. requirements for simulators which need to be more accurate representations of the aircraft in all flight regimes,
4. requirements for better stability augmentation and adaptive flight control systems.

Previous approaches to the measurement of aerodynamic characteristics of an aircraft were based mainly on time consuming steady-state measurements and on the measurement of free oscillations. The analysis of transient manoeuvres was firstly proposed by Greenberg in Ref. 3 and by Shinbrot in Ref. 4. It was, however, applied to very simple manoeuvres and it resulted in only limited information about system parameters and their accuracies.

The identification of aircraft parameters using modern control theory, theory of statistical inference and new digital techniques has brought quantitatively new ways of aircraft testing and flight data analysis. This approach enables us to evaluate from one test run all or almost all stability and control derivatives involved, together with their accuracies and confidence

intervals. At the same time the accuracy of measured data is also estimated so that this data can be used in the analysis with a corresponding level of confidence.

If necessary, there is the possibility of separating the measurement noise in the output variables from the external disturbances to the system caused by gust wind effects or modeling errors.

The identification techniques give the opportunity of including in the analysis the a priori knowledge of aircraft parameters obtained from theoretical predictions, wind tunnel measurements and previous flight measurement regardless of the testing and/or analysis techniques used.

Finally the identification methods provide tools for a proper design of an experiment (e.g. optimum shape of the input signal) to obtain the most accurate results and for testing the hypothesis about the correct form of the mathematical model describing the analysed motion of an aircraft.

Substantial contributions in the field of aircraft parameter identification were made by several authors, namely by Taylor and Iliff in Ref. 5, by Denery in Ref. 6, by Mehra in Ref. 7, by Mehra and Stephner in Ref. 8 and by Rault in Ref. 9. Besides these contributions the merits of NASA as an organization should be mentioned as well. In this establishment the new technique of parameter identification has found very positive acceptance and so far it has been used on data analysis of more than fifteen different aircraft.

At the Cranfield Institute of Technology the research in aircraft parameter identification was initiated about two years ago. This report summarizes the results achieved and is concentrated on two main objectives:

1. to present a computing algorithm upon which the general computing program has been compiled and applied to linear as well as nonlinear systems using different estimation techniques,
2. to outline several methods of data analysis using various tools provided by the parameter identification technique.

For that reason no detailed and rigorous development of estimation methods is presented. It can be found in the references mentioned. However, even if this report is limited in its objectives, it is believed that it might form the basis for further research in the aircraft parameter identification at the Cranfield Institute of Technology and might contribute to the wider acceptance of these techniques in the analysis of flight test data.

2. PROBLEM STATEMENT

The problem under consideration is to identify an aircraft as a dynamic system when given the equations governing its motion and the measurements of its output and input variables. All three steps in the identification procedure, namely characterization, parameter estimation and verification will be considered.

Characterization is a qualitative operation defining the structure of a system. The system is assumed to be deterministic with time-invariant parameters. Its description is in the form of differential equations which may be either linear or nonlinear. The initial conditions are, in general, different from zero. The unknown parameters include the coefficients of equations of motion, the initial conditions and constant bias errors in measured output variables. The measured variables are used in the form of sampled time histories with a constant sampling rate.

The second step of the identification forms the parameter estimation. Several estimation techniques based on methods of estimation theory are used. For their development no external disturbances to a system (process noise) and no measurement errors in the input variables are assumed.

Parameter estimation results in the determination of the numerical values of unknown parameters, in the estimation of their accuracy and the accuracy of whole identification process.

The purpose of verification is to relate the results obtained to well-known physical points of the system under investigation. This approach can raise the question of the reliability of estimates in general and the problem of correct mathematical modelling in particular.

3. CHARACTERIZATION

It is assumed that the system under investigation is described, in general case, by the state equation

$$\dot{x} = f(x, u, \beta) \quad , \quad x(0) = \alpha \quad (3.1)$$

and by the output equation

$$y = g(x, u, \beta) \quad (3.2)$$

In these equations x is an $(n \times 1)$ state vector, u is an $(p \times 1)$ input vector, β is a $\{(q - n) \times 1\}$ vector of unknown parameters, α is an $(n \times 1)$ vector of initial conditions and y is an $(m \times 1)$ output vector.

If the system is linear then the above mentioned equations become

$$\dot{x} = Ax + Bu \quad , \quad x(0) = \alpha \quad (3.3)$$

$$y = Cx + Du \quad (3.4)$$

where the unknown parameters β are contained in all four matrices A , B , C and D .

To represent any realistic flying vehicle completely would be a task of immense difficulty. The problem is to select the simplest approximate representation that will permit the successful determination of the unknown parameters from measured data.

If a flight vehicle consists of a single mass with no substantial amount of internal liquid, and if the vehicle is relatively inflexible, then it may be possible to idealise it as a rigid body.

If, in addition, only small perturbations about some steady-state equilibrium conditions are considered, and the aircraft is of usual design and in no extreme flight condition, then the equations of motion may be linearized. The linearization results in two independent systems of equations describing the longitudinal and the lateral motion of an aircraft. The equations have the form

$$T\dot{x} = A'x + B'u \quad , \quad x(0) = 0 \quad (3.5)$$

Because T is a nonsingular square matrix then by letting $A = \bar{T}^{-1}A'$ and $B = \bar{T}^{-1}B'$ equations (3.5) can be modified to the more standard form (3.3).

The equations for both the longitudinal and the lateral motion of an aircraft are presented in Appendix A using the dimensional as well as nondimensional expressions for the aerodynamic derivatives.

The form of output equation depends upon the variables measured in flight. These are usually represented by the airspeed, linear accelerations, angular velocities and incidence angles. In some cases the angular accelerations and attitude angles are measured as well. Many of the measured quantities are the state variables considered. The linear accelerations are usually treated as output variables. However, the possibility of including them as state variables is also considered.

Whereas the equations describing the longitudinal and lateral motion of an aircraft are well developed the modelling of an aircraft during large disturbance manoeuvres or in extreme flight conditions (high angles of attack, transonic regime) is a difficult task demanding the formulation of a set of nonlinear equations. Further, for the new families of STOL and VTOL aircraft nonlinear forms of their models must normally be used.

The problem of modelling a complicated system raises the fundamental question of how complex the model should be. Although a more complex model can be justified for proper description of aircraft motion it is not clear in the case of parameter identification what should be the best relationship between model complexity and measurement information. If too many unknown parameters are sought for a limited amount of data then a reduced reliability of evaluated parameters can be expected, or attempts to identify all parameters might fail.

As an example of a nonlinear model the equations for a rapid rolling manoeuvre of an aircraft are developed in Appendix B. This type of manoeuvre is a combined longitudinal and lateral motion which follows rapid and large deflection of the ailerons. During this motion the airspeed changes can very often be neglected.

Because of the complexity of this model further simplification is proposed which constitutes splitting the combined motion into independent longitudinal and lateral movements.

4. PARAMETER ESTIMATION PROCEDURE

As was stated previously the identification of system parameters refers to the evaluation of coefficients in the known mathematical model from measured input-output data. Part of this problem is, therefore, reduced to a parameter estimation which uses the tools of estimation theory.

4.1 The estimation methods

There are several methods for the solution. Their basic differences are due to the variety of assumptions regarding the prior probability and an optimal criterion.

Four estimation methods will be described. The first two methods are the least squares technique. The optimality criterion is a scalar quantity, termed the cost function. Its minimization requires the minimization of the sum of squares of differences between data points and corresponding points derived from the solution.

The remaining two methods are based on the maximization of probability density functions.

Regression analysis:

In this case it is assumed that the input variables and all the state variables and their derivatives can be obtained from measurement, and that only the state variable derivatives are corrupted by noise.

The cost function has the form

$$\begin{aligned}
 J &= \frac{1}{2} \sum_{i=1}^N (\dot{x}_{Ei} - \dot{x}_i)^T W_1 (\dot{x}_{Ei} - \dot{x}_i) = \\
 &= \frac{1}{2} \sum_{i=1}^N ||\dot{x}_{Ei} - \dot{x}_i||_{W_1}^2
 \end{aligned} \tag{4.1}$$

where T denotes a transpose matrix, index E denotes measured values, N is the number of data points and $||\cdot||$ is the norm of quantity enclosed.

The introduction of a positive definite weighting matrix, W_1 , into (4.1) means only a generalized approach to the problem. Because of the independence of the solution for individual equations in (3.1) the values of W_1 do not enter the calculations for unknown parameters.

The least squares solution is obtained by finding the minimum of J . This requires forming the first variation of J and setting it to zero, see e.g. Ref. 5.

Then each of n equations will have the form

$$G_j^T = \left\{ \sum_{i=1}^N X_i X_i^T \right\}^{-1} \sum_{i=1}^N X_i \dot{x}_{Eji} \quad (4.2)$$

$$j = 1, 2, \dots, n$$

where matrix G_j contains the unknown parameters in the j th state equations and matrix X_i is formed by measured state and input variables at time intervals t_i . If some of the coefficients in (3.1) are known constants they are included in the elements of vector $\dot{x}_{Eji}$ after their multiplication by corresponding value of x_{Eji} or u_i .

So far it has been considered that all the state variables and their derivatives are known from measurements taken. The measurement of the derivatives of the state variables is, however, a difficult problem. Numerical differentiation of the state variables is straightforward but tends to decrease the signal to noise ratio. Several ways have been introduced to overcome these difficulties. Some of them are described in Ref. 5 and Ref. 10.

One of the modifications is based on the integration of equation (3.1). It allows the use of directly measured state variables and integrated state and input variables. The cost function then becomes

$$J = \frac{1}{2} \sum_{i=1}^N \left\| x_{Ei} - x_i \right\|_{W_1}^2 \quad (4.3)$$

where x is the solution of the equation

$$\dot{x} = f(x_E, \beta, u) \quad (4.4)$$

and where W_1 has different meaning from that in (4.1).

The function x can be further expressed as

$$x = \bar{x}_0 + H_E \Delta\beta \quad (4.5)$$

where H_E is the gradient matrix of x_0 with respect to the unknown parameters β . The vector x_0 is calculated from (4.4) for the estimated values β_0 .

The individual components of H_E can be computed by the solution of the equations

$$\frac{d}{dt} \left(\frac{\partial x}{\partial \beta_j} \right) = \frac{\partial}{\partial \beta_j} f(x_E, u, \beta) \quad , \quad \frac{\partial x}{\partial \beta_j}(0) = 0 \quad (4.6)$$

$$j = 1, 2, \dots, (q - n)$$

Setting the gradient of (4.3) equal to zero and using (4.5) and (4.6) the solution for the increments $\Delta\beta$ will have the form

$$\hat{\Delta\beta} = \left\{ \sum_{i=1}^N H_{Ei}^T W_1 H_{Ei} \right\}^{-1} \sum_{i=1}^N H_{Ei}^T W_1 (x_{Ei} - x_{oi}) \quad (4.7)$$

Because of (4.4) x is linearly related to the unknown parameters β . Their final values are, therefore, obtained directly from

$$\hat{\beta} = \beta_0 + \hat{\Delta\beta} \quad (4.8)$$

and no iterative procedure is necessary.

The standard error of the estimated parameters in each state equation is found from the relation

$$s(\beta_j) = \sigma_r \sqrt{d_{jj}} \quad , \quad j = 1, 2, \dots, q_r \quad (4.9)$$

where d_{jj} is the diagonal term of the inverse matrix in the expression for the least squares solution and q_r is the number of unknown parameters in the r th state equation. The variance, σ_r^2 , is estimated by

$$\sigma_r^2 = \frac{\sum_{i=1}^N \epsilon_{ri}^2}{N - q_r} \quad (4.10)$$

where ϵ_{ri} is the resulting error after the insertion of measured values into the r th state equation at the time interval t_i .

Weighted least squares (WLS) estimation:

The measurement equation is

$$z = y + n \quad (4.11)$$

where z is an $(m \times 1)$ vector of measurement and n is the vector of measurement errors. The cost function is formed as

$$J = \frac{1}{2} \sum_{i=1}^N ||z_i - y_i||_{w_1}^2 \quad (4.12)$$

It is minimized with respect to the unknown parameters γ , where $\gamma^T = \{\alpha^T, \beta^T\}$. The unknown parameters are contained in the output y in a nonlinear manner. For the minimization of J the method of quasilinearization is used. It begins with the linearization of the state and the output about the previous estimates α_0 and β_0 .

$$\begin{aligned} x(\alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta) &\approx x(\alpha_0 + \beta_0) + \\ &+ \left\{ \frac{\partial x}{\partial \alpha}, \frac{\partial x}{\partial \beta} \right\} \begin{Bmatrix} \Delta\alpha \\ \Delta\beta \end{Bmatrix} \end{aligned} \quad (4.13)$$

$$\begin{aligned} y &= g\{x(\alpha_0, \beta_0), u, \beta_0\} + \\ &+ \frac{\partial g}{\partial x} \left\{ \frac{\partial x}{\partial \alpha}, \frac{\partial x}{\partial \beta} \right\} \begin{Bmatrix} \Delta\alpha \\ \Delta\beta \end{Bmatrix} + \frac{\partial g}{\partial \beta} \Delta\beta \end{aligned} \quad (4.14)$$

The sensitivity matrices $\partial x / \partial \alpha$ and $\partial x / \partial \beta$ are obtained from the solution of the following set of linear time-varying differential equations

$$\frac{d}{dt} \left(\frac{\partial x}{\partial \alpha_j} \right) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \alpha_j}, \quad \frac{\partial x}{\partial \alpha_j}(0) = \frac{\partial \alpha}{\partial \alpha_j} \quad (4.15)$$

$$j = 1, 2, \dots, n$$

$$\frac{d}{dt} \left(\frac{\partial x}{\partial \beta_j} \right) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \beta_j} + \frac{\partial f}{\partial \beta_j}, \quad \frac{\partial x}{\partial \beta_j}(0) = 0 \quad (4.16)$$

$$j = 1, 2, \dots, (q - n)$$

When the gradient $\nabla_{\gamma} J$ is equated to zero a new estimate is obtained as

$$\Delta \gamma = \left\{ \sum_{i=1}^N H_i^T W_1 H_i \right\}^{-1} \sum_{i=1}^N H_i^T W_1 (z_i - y_i) \quad (4.17)$$

The j th column of H is $\partial y / \partial \gamma_j$, where γ_j is the j th parameter in the vector γ and $j = 1, 2, \dots, g$.

After one iteration the new estimate of unknown parameters is obtained as

$$\hat{\gamma} = \gamma_0 + \Delta \gamma$$

The iterative process is repeated until the differences between two successive estimates in $\Delta \gamma$ are not significant.

The lack of probabilistic statements regarding the measurement errors means that it is impossible to make any probabilistic statement about the least squares estimate. It follows from the solution that the error in estimate $\hat{\gamma}$ is proportional to the measurement error n , i.e.

$$e(\hat{\gamma}) = \gamma - \hat{\gamma} = - \left\{ \sum_{i=1}^N H_i^T W_1 H_i \right\}^{-1} \sum_{i=1}^N H_i^T W_1 n_i \quad (4.18)$$

If $E\{n\} = 0$ and the measurement errors are independent then $e(\hat{\gamma})$ has zero mean and the least squares estimate are unbiased. If $E\{nn^T\} = \sigma^2 \Theta$, where Θ is known diagonal matrix, then the covariance of $e(\hat{\gamma})$ is

$$E\{ee^T\} = \sigma^2 \left\{ \sum_{i=1}^N H_i^T W_1 H_i \right\}^{-1} \quad (4.19)$$

with $W_1 = \Theta^{-1}$. This covariance matrix provides a measure of the precision of the estimate.

The variance σ^2 is identical with the mean square error (MSE) and is estimated from

$$MSE = \frac{1}{mN} \sum_{i=1}^N (z_i - \hat{y}_i)^T W_1 (z_i - \hat{y}_i) \quad (4.20)$$

The mean square error can be used as the performance index function to give a measure of performance for the iterative estimation procedure, i.e. for its convergence.

Maximum likelihood (ML) estimation.

It is assumed for this technique that the measurements are

$$z_i = y_i + n_i, \quad i = 1, 2, \dots, N \quad (4.21)$$

where n_i is a sequence of independent gaussian random variables with $E\{n_i\} = 0$ and $E\{n_i n_j^T\} = R_1 \delta_{ij}$.

The maximum likelihood estimate of the unknown parameters γ and unknown coefficients in R_1 can be obtained by maximizing the conditional probability density of z_i given α, β, R_1 , i.e.

$$p(z_1, z_2, \dots, z_N / \alpha, \beta, R_1) = p(n_1, n_2, \dots, n_N)$$

Maximization of p means also maximization of $\ln p$. The logarithm of the conditional probability density is called the likelihood function

$$L(p) = -\frac{1}{2} \sum_{i=1}^N ||z_i - y_i||_{\frac{1}{R_2}}^2 - \frac{N}{2} \ln |R_1| - \frac{mN}{2} \ln 2\pi \quad (4.22)$$

where $|R_1|$ denotes the determinant of R_1 , see e.g. Ref.7 or Ref. 12.

To maximize linearized equation (4.22) the partial derivatives of $L(\gamma_0 + \Delta\gamma, R_1)$ with respect to each parameter change $\Delta\gamma$ are set to zero giving the new estimate

$$\Delta\gamma = \left\{ \sum_{i=1}^N H_i^T \bar{R}_1^{-1} H_i \right\}^{-1} \sum_{i=1}^N H_i^T \bar{R}_1^{-1} (z_i - y_i) \quad (4.23)$$

while the estimate of R_1 is obtained from

$$R_1 = \frac{1}{N} \sum_{i=1}^N v_i v_i^T \quad (4.23)$$

where $v_i = z_i - \hat{y}_i$ is the residual at the time interval t_i .

A lower bound on the error covariance matrix for the estimated parameters is given by the Cramér-Rao inequality, Ref. 11.

$$E \{ (\hat{\gamma} - \gamma) (\hat{\gamma} - \gamma)^T \} \geq \bar{M}^{-1} \quad (4.24)$$

where M represents the Fisher information matrix defined as

$$M = E \left\{ \left(\frac{\partial L}{\partial \gamma} \right) \left(\frac{\partial L}{\partial \gamma} \right)^T \right\} \quad (4.25)$$

The estimate of the information matrix can be, therefore, found from

$$M = \left\{ \sum_{i=1}^N H_i^T \bar{R}_1^{-1} H_i \right\} \quad (4.26)$$

Finally the performance index function $J_m(\gamma_0)$ is derived by substituting the covariance matrix of the measurement noise into the likelihood function for the nominal solution with γ_0 and R_{10} , Ref. 12. The maximization of this likelihood function is equivalent to the minimization of

$$J_m(\gamma_0) = |R_{10}(m)| = \quad (4.27)$$

Bayesian estimation:

The Bayesian estimation techniques uses for the parameter estimation both the information contained in measured data and the a priori information about the parameters involved. Treating γ as a random vector and applying Bayes rule yields the posterior distribution

$$p(\gamma/z) = \frac{p(\gamma) p(z/\gamma)}{p(z)} = \frac{p(\gamma) p(z/\gamma)}{\int_{\gamma} p(z/\gamma) p(\gamma) d\gamma} \quad (4.28)$$

see e.g. Ref. 1.

The most probable value of γ is obtained by maximizing $p(\gamma/z)$ with respect to γ . In calculating this estimate, $p(z)$ can be treated as a constant. For $p(\gamma)$ and $p(z/\gamma)$ assumed gaussian the distribution $p(\gamma/z)$ is also gaussian and is expressed as

$$p(\gamma/z) = \text{const} \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \left\| z_i - y_i \right\|_{R_1}^2 - \frac{1}{2} \left\| \gamma - \gamma_0 \right\|_{R_2}^2 \right\} \quad (4.29)$$

where $R_2 = E \{(\gamma - \gamma_0)(\gamma - \gamma_0)^T\}$ and $\gamma_0 = E \{\gamma\}$.

Expression (4.29) is maximized by maximizing the exponent. Using the linearization and setting the gradient with respect to γ equal zero yields

$$\Delta\gamma = \{\bar{R}_2^{-1} + \sum_{i=1}^N H_i^T \bar{R}_1^{-1} H_i\}^{-1} \{-\bar{R}_2^{-1} (\gamma - \gamma_0) + \sum_{i=1}^N H_i^T \bar{R}_1^{-1} (z_i - y_i)\} \quad (4.30)$$

The estimate of R_1 is obtained from (4.23) and the performance index is equivalent to that given by (4.27). The Cramér-Rao bound on the error covariance matrix of estimates is, however, found from

$$\bar{M}^{-1} = \{\bar{R}_2^{-1} + \sum_{i=1}^N H_i^T \bar{R}_1^{-1} H_i\}^{-1} \quad (4.31)$$

4.2 Comparison of various estimates

The method of regression analysis minimizes the sum of squared errors which result from the insertion of measured data into the equation (3.1) or (3.3). For that reason it is also called the equation error method. The computational scheme for this method with the modification using integrated measured data is presented in Fig. 1.

As it is pointed out in Ref. 6 the output error method gives biased estimates when noise is present in measured state variables. This method is also very sensitive to bias errors in measured data which is demonstrated on several examples in Ref. 10. The other disadvantage is the independence of each equation which is to be minimized. Despite this, the technique is very simple to apply both to linear and nonlinear systems and it can be used as a good start-up procedure for the iterative methods.

The weighted least squares estimation provides unbiased estimates of unknown parameters provided that the mathematical model is correct and there is no noise in the measured inputs, see e.g. Ref. 6. The only significant cost is due to inaccuracy in the estimates which means that the weighted sum of squared errors between the measured and computed outputs is minimized. The WLS estimation is known as the output error or the response curve fitting method.

For the maximum likelihood technique it is assumed that the distribution of measured variables is known. In this distribution only a finite number of parameters is to be determined. There is no prior knowledge regarding the values of the unknown parameters γ . If the gaussian distribution is further assumed then, according to the Gauss-Markoff theorem, the ML estimate is identical to the minimum variance estimate. This is also apparent from the comparison of expressions (4.17) and (4.23) which are identical for $W_1 = \bar{R}_1^{-1}$. Because, in general, the coefficients in the matrix $\bar{R}_1^{-1}$ are not known in advance the estimate of this matrix is updated during the iterative process. Then the final values of its coefficients are related to the variances of measurement noise.

The cost function for the ML technique is given as

$$J = \frac{1}{2} \sum_{i=1}^N ||z_i - y_i||_{\bar{R}_1^{-1}}^2 + \frac{N}{2} \ln |R_1| \quad (4.32)$$

which means that its value depends upon the second term on the right hand side of (4.32).

In the practical application of ML estimation, usually more iterations and better starting estimates are required than with the weighted least squares approach. This could be the reason for the combination of both techniques.

In the Bayesian estimation technique the unknown parameters are treated as random variables. The optimum estimate is taken as the mean of the conditional distribution $p(\gamma/z)$. This result is reminiscent of the basic concept of maximum likelihood estimation. The posterior density function $p(\gamma/z)$ is, however, more or less different from $p(z/\gamma)$ depending upon the distribution of $p(\gamma)$. The distribution $p(\gamma)$ is the prior knowledge concerning the unknown parameters.

If the gaussian distribution for $p(z/\gamma)$ and $p(\gamma)$ is adopted then the Bayesian estimate can be treated as the minimum variance estimate with expanded cost function

$$J = \frac{1}{2} \sum_{i=1}^N ||z_i - y_i||_{\bar{R}_1^{-1}}^2 + \frac{1}{2} ||\gamma - \gamma_0||_{\bar{R}_2^{-1}}^2 + \frac{N}{2} \ln |R_1| \quad (4.33)$$

It includes a penalty for departure from a priori values γ_0 . The inclusion of a priori values in (4.33) means more constraint and can improve the identification process in various sensitivity cases. On the other hand there can be limitation in the use of this technique because the statistical data of a priori values are often not available. For that reason the Bayesian estimation can be simplified to the ML estimation with the a priori weighting of some or all parameters. The expression (4.30) for the estimates $\Delta\gamma$ remains the same. Only the matrix $\bar{R}_2^{-1}$ is replaced by the weighting matrix W_2 .

Because of the similarity of the iterative approaches described, all of them can be treated as output error methods provided that the simplified assumptions for the ML and Bayesian estimations are adopted. The general computational scheme for the output error method is given in Fig. 2.

In Table 1 the cost functions and the expressions for the estimates for the output and equation error method are summarized and are completed by basic assumptions and by a priori knowledge.

4.3 Accuracy of the Estimation Procedure

The accuracy of an estimation procedure may be defined in several ways. Very often it is judged by deviations both in the parameters of the mathematical model and in the output variables.

From the previous comparison of the estimation methods it is apparent that the maximum likelihood estimation provides the most accurate estimate of unknown parameters. As shown in Ref. 13 these estimates are consistent, asymptotically normal and efficient under very general conditions.

The unknown parameters, γ , form a q dimensional random vector with mean value $\hat{\gamma}$ and with the lower bound on the covariance matrix $\bar{M}^{-1}$. The main diagonal elements of this matrix are $\sigma_{\gamma j}^2$, whereas the off-diagonal terms are formed by the expressions $\rho_{ij} \sigma_{\gamma i} \sigma_{\gamma j}$, where ρ_{ij} is the correlation coefficient for γ_i and γ_j . The accuracy of parameter γ_j is thus bounded by the estimate of $\sigma_{\gamma j}$, which is the standard error.

The error covariance matrix $\bar{M}^{-1}$ defines the hyperellipsoid

$$(\gamma - \hat{\gamma})^T M (\gamma - \hat{\gamma}) = k \quad (4.34)$$

where k is a constant. The intercepts of this hyperellipsoid with the coordinate axes are given by the reciprocal square roots of the diagonal elements of the matrix M . These quantities, however, only provide an indication of the q -dimensional k -sigma bounds and do not, in general, define the limits of this region.

From the gaussian probability density function for γ two probability statements may be formed, namely:

1. probability that the random q -vector lies within the boundaries of the hyperellipsoid contour. This probability is given, according to Ref. 14 by

$$p_q(k) = \text{erf}(k/\sqrt{2}) - \sqrt{2/\pi} \times \\ \times \exp(-k^2/2) \left\{ k + \frac{k^3}{1.3} + \dots + \frac{k^{q-2}}{1.3.5 \dots (q-2)} \right\} \quad (4.35)$$

for odd q

$$p_q(k) = \exp(-k^2/2) \left\{ 1 + \frac{k^2}{1.2} + \frac{k^4}{1.2.4} + \dots + \right. \\ \left. + \frac{k^{q-2}}{1.2.4 \dots (q-2)} \right\} \quad (4.36)$$

for even q

2. probability that any one element of the random vector lies between the intercepts of the hyperrectangular, which encloses the hyperellipsoid, with the corresponding coordinate axis without regard to where the remaining elements lie. This probability is expressed as $p_1(k)$.

The measure of overall parameter uncertainty is connected with the volume of the hyperellipsoid and is, therefore, given by determinant of the error covariance matrix.

A hyperellipsoid in q -space is not very useful since it cannot be readily visualized. Therefore its projection into two-dimensional space is often used instead. Then k -sigma ellipses can be found for two-element sets of the random vector γ without regard to the behaviour of the remaining elements.

The desired ellipse is obtained by partitioning the covariance matrix, then multiplying by the value of k^2 for selected $p_2(k)$ and finally performing a matrix inverse. The magnitudes and directions of the principal axes of the final contour follow from the solution of the related eigenvalue - eigenvector problem, see e.g. Ref. 14.

The probability for $q = 1$ and $q = 2$, i.e. $p_1(k)$ and $p_2(k)$ may be obtained from (4.35) and (4.36), respectively. Of particular interest are the values for $k = 1, 2$ and 3 .

q/k	1	2	3
1	.683	.955	.997
2	.394	.865	.989

These values are called the one-, two- and three- sigma probabilities.

The explicit values of correlation coefficients can be found in the normalized error covariance matrix $\bar{M}^{-1}$. The normalization is achieved by pre- and post- multiplication of the matrix $\bar{M}^{-1}$ by matrices W^T and W , respectively, where W is the diagonal matrix with elements $1/\sigma_{y_j}$. Then the main diagonal elements of the normalized error covariance matrix are equal to 1 and for the off-diagonal correlation coefficients the relation $-1 \leq \rho_{ij} \leq 1$ holds.

The closeness of computed and measured outputs may be defined by a criterion developed either from the performance index or the cost function. For the ML estimation technique the numerical values of both quantities are given by the determinant of the measurement noise covariance matrix as follows from (4.27) and (4.32).

If $\ln |R_1|$ is adopted as a measure for the fit error, it is possible to use its value for a comparison of results of repeated measurements under the same conditions or for studying of changes in the structure of the model assumed. Increasing the number of unknown parameters increases the number of statistical degrees of freedom which results in decreasing value of $\ln |R_1|$.

Experience has shown that at the same time the lengths of the error hyperellipsoid along each of the axes could become less uniform and the total volume of this ellipsoid could increase.

This problem has also been mentioned in Ref. 8 and as a result a new criterion for fit error has been proposed. This criterion was originally developed by Akaike in Ref. 15 as a one step ahead final predicted error (FPE) of the predictor obtained by using the least squares estimate of autoregression coefficients. For the ML estimation procedure it takes the form

$$\text{FPE} = \frac{N - q}{N + q} \ln |R_1| \quad (4.37)$$

This criterion does not improve monotonically with the increasing number of unknown parameters but has some minimum value which could determine the optimal number of parameters included in a model. It is possible that the most advantage of FPE as a fit error criterion would be during the analysis of a system where little is known about its structure.

The accuracy of estimated parameters is closely related to the sensitivity. The concept of sensitivity can be treated as the sensitivity of the function $y = g(x, u, \beta)$ with respect to the parameters. This approach means that the system is sensitive to changes in one particular parameter if a small change results in a large change in y . Then it is possible to expect that any meaningful theory of errors should lead to relatively small confidence in such parameters. On the other hand, if y changes by only a small amount when a parameter is changed, y is insensitive with respect to that parameter and the theory should result in a large certainty.

The concept of sensitivity mentioned can be mathematically expressed and connected with the nonprobabilistic definition of certain maximum allowable errors in evaluated parameters, see Ref. 16. The sensitivity of the parameters defined in this way is then given by the error covariance matrix.

It follows from the previous explanation that both the accuracy and the sensitivity of the parameters express the same feature and this depends upon the accuracy in measured data, correctness of a model and design of an experiment.



4.4 Identifiability

The concept of identifiability is usually related to various problems connected with the ability to identify the parameters in the model assumed for the system. The first group of problems mentioned is referred to the identifiability of the parameters of the system. They can be solved by finding the maximum number of parameters, which can be identified from measurements of the system input - output data, and their location in the matrices.

The remaining problems are mostly concerned with the identifiability of a description of the system which means:

1. the possibility of a numerical solution of the estimation process,
2. the reliability of the estimates in terms of physically realistic values and small error covariances for the parameters.

Identifiability of the parameters:

The determination of the maximum number of parameters, which can be identified from measurement of input - output data, and the location of these parameters in the matrices were studied by Denery in Ref. 6. for a linear system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (4.37)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (4.38)$$

From results obtained it can be concluded:

1. all parameters which are coefficients of the measured states are identifiable,
2. if $m \geq n_p$, where n_p is the number of state equations containing the unknown parameters (i.e. $n_p \leq n$), all parameters in (4.37) may be identified,
3. if $m < n_p$, the equations must be transformed into the canonical form proposed which contains a maximum n ($m + p$) parameters all of which are uniquely defined by the input - output behaviour of the system.

A

For the nonlinear system the parameter identification can be solved for $m \geq n_p$. To the author's knowledge no theory exists for the determination of the maximum number of identifiable parameters and their location in the equations for this type of system.

In the application of parameter identification to aeronautics the identifiability of the aerodynamic derivatives must be considered in addition to the identifiability of the parameters. As follows from Appendix A and Appendix B the aerodynamic derivatives are contained in parameters (coefficients of state and output equations) as single terms or in combinations. In some cases it is not possible to determine all derivatives included in the estimated parameters. One very well known example is the impossibility of evaluating C_{mq} and $C_{m\dot{\alpha}}$ separately for measured elevator deflection as the input and α , q and n_z as the output variables.

The limited possibility of extracting all derivatives was mentioned by Greenberg in Ref. 3 and later, more rigorously, in Ref. 17 and Ref. 18 together with the working rule for the decision which derivatives can be uniquely determined.

Identifiability of a description of the system:

Whereas the identification of the parameters depends upon the structure of a system, the identifiability of a description of the system is, in addition, influenced by the design of the experiment, the adequacy of the model, the number of unknown parameters and the sensitivity of output variables in these parameters. The design of the experiment itself represents the wide range of characteristics, e.g. the input form, the length of measured data and its accuracy.

It is beyond the scope of this report to discuss all factors mentioned which can cause identifiability problems. Dealing with them is the matter of experience and good a priori knowledge about the system under investigation. Nevertheless a brief remark about the appropriate input will be presented and some approaches to the identifiability problems mentioned.

The basic demand on the input is the proper excitation of all modes involved (i.e. large signal to noise ratio and a minimal correlation between response variables) within the frequency range of interest. In addition it must not be possible to express the input as a linear combination of system response variables.

The previous approaches to the selection of suitable forms of an input were purely qualitative ones. Only recently in Ref. 9. and mainly in Ref. 8 the quantitative approach has been used for the development of the optimum input form for a given linear system. The design of an input is formulated as an optimal control problem with the sensitivity of the system response to the unknown parameters as the performance criterion. Pilot acceptability and other criteria are taken into account indirectly.

An efficient tool for improving the identifiability is the use of a priori known values of the parameters. This information is obtained from theoretical predictions, wind tunnel or previous flight measurement. Then the a priori values can be included in the estimation procedure as a set of fixed values, a set of values with weights expressing their uncertainty (the a priori weighting) and finally in the form of probability distribution (the Bayesian estimation).

The introduction of fixed parameters generally improves numerical convergence. It can be useful in such cases where the numerical solution with all unknown parameters failed. Because the true values of fixed parameters are not known, the estimated parameters will depend upon these fixed values. To remove this dependence the number of fixed parameters should be successively reduced to zero during repeated computing.

The other possibility of using fixed parameters is in those cases where the estimated parameters are strongly correlated. Then fixing dependent parameters can decrease the error covariances of remaining parameters. So far it is not known which values should be given to the dependent parameters when their a priori values are not available.

The significant improvement in identifiability may be achieved by a priori weighting. This approach improves numerical convergence, could ensure physically realistic values for estimated parameters and small error covariances. Very often the standard errors of the a priori values are not known. In these cases the rough estimates of corresponding weights may be obtained from the assumed maximum errors in the a priori values, $\delta_{\max}$, as $w_{2j} = 3/\delta_{j\max}$. In all cases the obtained fit error should be compared with that obtained without a priori values included. Their difference should not be too large.

The identifiability of a description of the system is also discussed in Ref. 8 and some ways for its improvement are demonstrated in examples. In addition two other approaches are introduced. The first one is the constrained optimization technique. It uses a range of allowable values or

relationships between aerodynamic derivatives as the constraints on the parameter to avoid non-physical estimates.

The second technique is the rank-deficient solution of matrix inverse. Using it the inverse of the information matrix is first computed leaving out one or more of its smallest eigenvalues. Then the inversion is repeated until all previously omitted eigenvalues are added in again. The last inverse is the full rank inverse and the procedure is repeated at every iteration.

Experience with various identification techniques shows that the identifiability problems are probably the most difficult ones in the whole process of parameter identification. Further research is, therefore, needed to enable parameter identification to be used with ever increasing confidence and thus make this technique more attractive.

5. VERIFICATION

The problems concerning the model structure have been discussed in connection with characterization of the system and identifiability. It follows from these discussions some doubts about the correctness of the model used could still remain even if the parameter estimation has been successfully completed. One of the possible ways of checking the validity of the structure is based on the analysis of residuals. For the ML estimation these residuals should form a sequence of uncorrelated random variables with gaussian distribution and zero mean, as pointed in Chapter 4.1.

The simplest way for checking some of the assumptions mentioned is the time sequence plot of residuals. From this time history trends can be apparent and eventual deterministic components discovered.

The check of normality uses the calculated percentage cumulative distribution, P_j , for each of N samples of the residuals $v_k(t_i)$, $k = 1, 2, \dots, m$. Because of large number of data points the grouped residuals are usually used in the calculation. The probability of the residuals v_i less than or equal to the j th class limit is

$$P_j = P \left\{ v_i \leq v_j + \frac{\Delta v}{2} \right\} \quad (5.1)$$

where v_j are the midpoints of the class intervals and Δv is the length of the class intervals.

It is advantageous to plot v_j as abscissas and the corresponding P_j values as ordinates on the probability paper, on which the ordinate scale is graduated according to the area under a normal distribution function. Then the fitted line of all values of P_j must, in the case of a normal distribution, be a straight line. The mean value can also be checked and is zero if the condition

$$P \left\{ v_i < v_j = 0 \right\} = 50\% \quad (5.2)$$

is satisfied.

The assumption of uncorrelated residuals is checked by the autocorrelation function of the residuals. Its estimate is found from the expression

$$\hat{R}_v(r) = \frac{1}{N-r} \sum_{i=1}^{N-r} v_i v_{i+r}, \quad r = 0, 1, \dots, r_{\max} \quad (5.3)$$

and the relative standard error of this estimate from the expression

$$\bar{s}(\hat{R}_v) = \sqrt{\frac{r_{\max}}{2N}} \quad (5.4)$$

as presented in Ref. 19.

If the residuals are to be uncorrelated, and in the normal case also independent, then the condition $R_v(r \neq 0) = 0$ should be satisfied.

6. COMPUTING ALGORITHM AND PROGRAM

A lot of effort has been devoted to the development of a computing algorithm and program which would:

1. cover all four estimation techniques described,
2. be applicable to linear as well as nonlinear systems,
3. be flexible enough for solving various identifiability problems.

If the modified equation method is used then its similarity to the output error method is apparent from expressions (4.3) and (4.12), and (4.7) and (4.17). Therefore the problem of covering both linear and nonlinear models with one algorithm remained. It consisted in finding general expressions for the constraint and sensitivity equations which could be used for both systems. The solution has been, in the main, found in the introduction of the augmented input vector which includes nonlinear terms in x , ux and u , as well as the input variables u .

The constraint equations (3.1) and (3.2) can be now expressed as

$$\dot{x} = Ax + Bu^* \quad , \quad x(0) = \alpha \quad (6.1)$$

$$y = Cx + Du^* + g$$

where $u^* = u^*(x, u)$ is the augmented input vector mentioned. In addition an $(m \times 1)$ vector of measurement biases, g , was added in the output equation.

An $(q \times 1)$ vector of unknown parameters is

$$\gamma^T = \{\alpha, \beta, g\}$$

where

$$\beta^T = \{a, b, c, d\}$$

and where

a is an $(l_a \times 1)$ vector of unknown parameters in A ,

b is an $(l_b \times 1)$ vector of unknown parameters in B ,

c is an $(l_c \times 1)$ vector of unknown parameters in C

which are different from those in A ,

d is an $(l_d \times 1)$ vector of unknown parameters in D

which are different from those in B .

For the number of unknown parameters the following relations hold:

$$l_a + l_b + l_c + l_d = l$$

$$l + n + m = q$$

NOTE:

1. for the regression analysis

$$\dot{x} = A x_E + Bu^*, \quad x(0) = \alpha \quad (6.3)$$

$$y = x_E$$

$$u^* = u^*(x_E, u)$$

where α is a vector with known elements.

2. for the linear system the augmented input vector u^* is simplified to the input vector u .

The sensitivity equations developed from (6.1) and (6.2) have the form

$$\begin{aligned} \dot{x}_{aj} &= A^* x_{aj} + \frac{\partial A}{\partial a_j} x, \quad x_{aj}(0) = 0, \quad j = 1, 2, \dots, l_a \\ \dot{x}_{bj} &= A^* x_{bj} + \frac{\partial B}{\partial b_j} u^*, \quad x_{bj}(0) = 0, \quad j = 1, 2, \dots, l_b \end{aligned} \quad (6.4)$$

$$\dot{x}_{\alpha j} = A^* x_{\alpha j}, \quad x_{\alpha j}(0) = \frac{\partial \alpha}{\partial a_j}, \quad j = 1, 2, \dots, n$$

and

$$y_{aj} = C^* x_{aj} + \frac{\partial C}{\partial a_j} x$$

$$y_{bj} = C^* x_{bj} + \frac{\partial C}{\partial b_j} u^*$$

$$y_{cj} = \frac{\partial C}{\partial c_j} x, \quad j = 1, 2, \dots, l_c \quad (6.5)$$

$$y_{dj} = \frac{\partial D}{\partial d_j} u^*, \quad j = 1, 2, \dots, l_d$$

$$y_{\alpha j} = C x_{\alpha j}$$

$$y_{gj} = \frac{\partial y}{\partial g_j}, \quad j = 1, 2, \dots, m$$

where

$$A^* = A + B \frac{\partial u^*}{\partial x}, \quad C^* = C + D \frac{\partial u^*}{\partial x}$$

$$x_{aj} = \frac{\partial x}{\partial a_j}, \quad \dots, \quad y_{aj} = \frac{\partial y}{\partial a_j}$$

The expression $\partial u^*/\partial x$ is a $(p \times n)$ matrix the columns of which are $\partial u^*/\partial x_j$, $j = 1, 2, \dots, n$.

NOTE:

1. for the regression analysis

$$x = x_E, \quad C = I \text{ and } D = 0$$

The sensitivity equations are simplified as

$$\dot{x}_{aj} = \frac{\partial A}{\partial a_j} x_E, \quad x_{aj}(0) = 0, \quad j = 1, 2, \dots, l_a$$

$$\dot{x}_{bj} = \frac{\partial B}{\partial b_j} u^*, \quad x_{bj}(0) = 0, \quad j = 1, 2, \dots, l_b \quad (6.6)$$

where $l_a + l_b = l = q$.

2. for the linear model

$$u^* = u, \quad A^* = A \quad \text{and} \quad C^* = C.$$

The estimate for $\Delta\gamma$ can be computed from the expression

$$\begin{aligned} \Delta\gamma &= -M^{-1} \left(\frac{\partial J}{\partial \gamma} \right)^T = \\ &= \{W_2 + \sum_{i=1}^N H_i^T W_1 H_i\}^{-1} \{-W_2 (\gamma - \gamma_0) + \sum_{i=1}^N H_i^T W_1 (z_i - \hat{y}_i)\} \end{aligned} \quad (6.7)$$

The $(m \times q)$ sensitivity matrix, H , is composed as

$$H = \{Y_\beta, \quad Y_\alpha, \quad I\}$$

where the i th column of Y_β is $\partial y / \partial \beta_j$, $j = 1, 2, \dots, l$, the i th column of Y_α is $\partial y / \partial d_j$, $j = 1, 2, \dots, n$ and I is the $(m \times m)$ identity matrix.

NOTE:

1. for the estimation without a priori weighting $W_2 = 0$,
2. for the WLS estimation W_1 is the weighting matrix with constant elements,
3. for the ML estimation $W_1 = \bar{R}_1^{-1}$ and the matrix is either fixed or updated after each iteration using the relation

$$R_1 = \frac{1}{N} \sum_{i=1}^N (z_i - \hat{y}_i) (z_i - \hat{y}_i)^T \quad (6.8)$$

4. for the Bayesian estimation $W_1 = \bar{R}_1^{-1}$ and $W_2 = \bar{R}_2^{-1}$.

The iterative process is completed if

$$\frac{J_k - J_{k-1}}{J_k} < 0.01 \quad (6.9)$$

where J_k is the cost function corresponding to the estimation technique used after the k th iteration.

For the regression analysis and the WLS estimation the estimates of standard error of evaluated parameters are obtained from the main diagonal elements of the matrix $\bar{M}^{-1}$ after their multiplication by $\sqrt{2J/mN}$.

The normalized matrix $\bar{M}_{\text{NORM}}^{-1}$ is obtained from

$$\bar{M}_{\text{NORM}}^{-1} = W^T \bar{M}^{-1} W \quad (6.10)$$

The computing program has been written in general terms to enable the user to achieve the objectives mentioned previously. The generality of the program was made possible by a storage compression algorithm which allowed the sensitivity equations (6.4) to be written as

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial x}{\partial \beta_A} \right) &= A^* \frac{\partial x}{\partial \beta_A} + F_A(x), \quad \frac{\partial x}{\partial \beta_A}(0) = 0 \\ \frac{d}{dt} \left(\frac{\partial x}{\partial \beta_B} \right) &= A^* \frac{\partial x}{\partial \beta_B} + F_B(u^*), \quad \frac{\partial x}{\partial \beta_B}(0) = 0 \\ \frac{d}{dt} \left(\frac{\partial x}{\partial \alpha} \right) &= A^* \frac{\partial x}{\partial \alpha}, \quad \frac{\partial x}{\partial \alpha}(0) = I \end{aligned} \quad (6.11)$$

where $\partial x / \partial \beta_A$ is an $(n \times \ell_a)$ matrix for the ℓ_a unknown parameters in A,

$\partial x / \partial \beta_B$ is an $(n \times \ell_b)$ matrix for the ℓ_b unknown parameters in B,

$F_A(x)$ is a matrix whose element $f_{j,k}$ contains the element x_{ℓ} of x as defined by an $(n \times n)$ diagram whose element (j,ℓ) contains the value k ,

$F_B(u^*)$ is a matrix whose elements are defined by a diagram in a similar way to $F_A(x)$.

The scheme for the set up of the diagram and for the design of the F_A matrix are presented in Ref. 20.

The submatrices of H are then given by

$$\begin{aligned} \frac{\partial y}{\partial \beta_A} &= C^* \frac{\partial x}{\partial \beta_A} + F_{CA}(x) \\ \frac{\partial y}{\partial \beta_B} &= C^* \frac{\partial x}{\partial \beta_B} + F_{DB}(u^*) \\ \frac{\partial y}{\partial \beta_C} &= F_{CC}(x), \quad \frac{\partial y}{\partial \beta_D} = F_{DD}(u^*) \\ \frac{\partial y}{\partial \alpha} &= C \frac{\partial x}{\partial \alpha}, \quad \frac{\partial y}{\partial g} = I \end{aligned} \quad (6.12)$$

where $F_{CA}(x)$, $F_{DB}(u^*)$, $F_{CC}(x)$ and $F_{DD}(u^*)$ are defined by diagrams in a similar way to $F_A(x)$.

At each time interval t_i equations (6.11) are solved using the fourth-order Runge-Kutta algorithm together with the estimated state vector from (6.1). The matrix H_i is compiled from submatrices obtained by using equations (6.12), and the quantities

$$H_i^T W_1 H_i \quad \text{and} \quad H_i^T W_1 (z_i - \hat{y}_i)$$

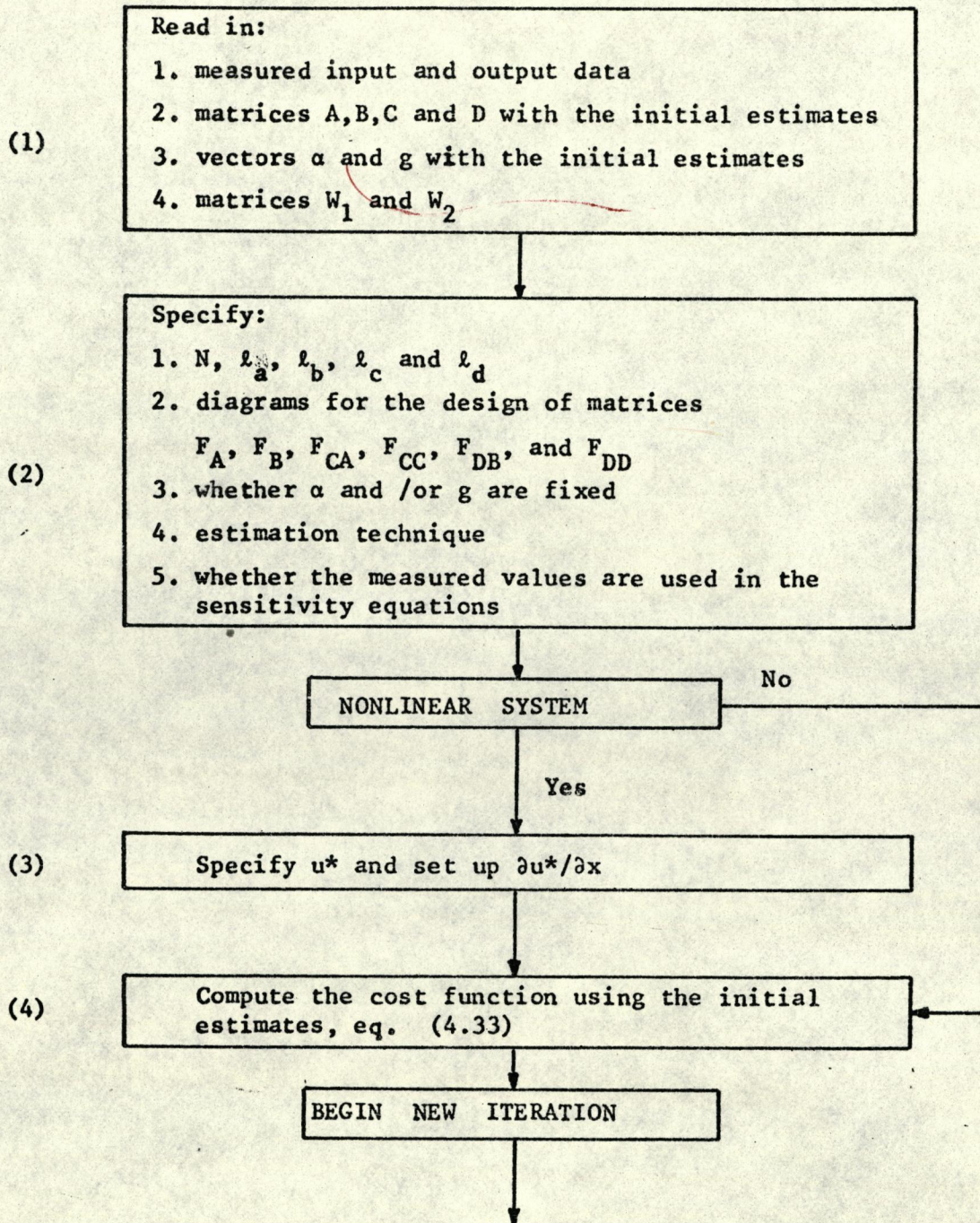
are evaluated and accumulated to give M and $\partial J / \partial \gamma$.

For the inverse of the matrix M the iterative process described in Ref. 20 has been included in the program. Experience has shown that the use of this approach can lead to a single pass improvement of about four significant figures in the accuracy of an inversion of a poorly conditioned matrix.

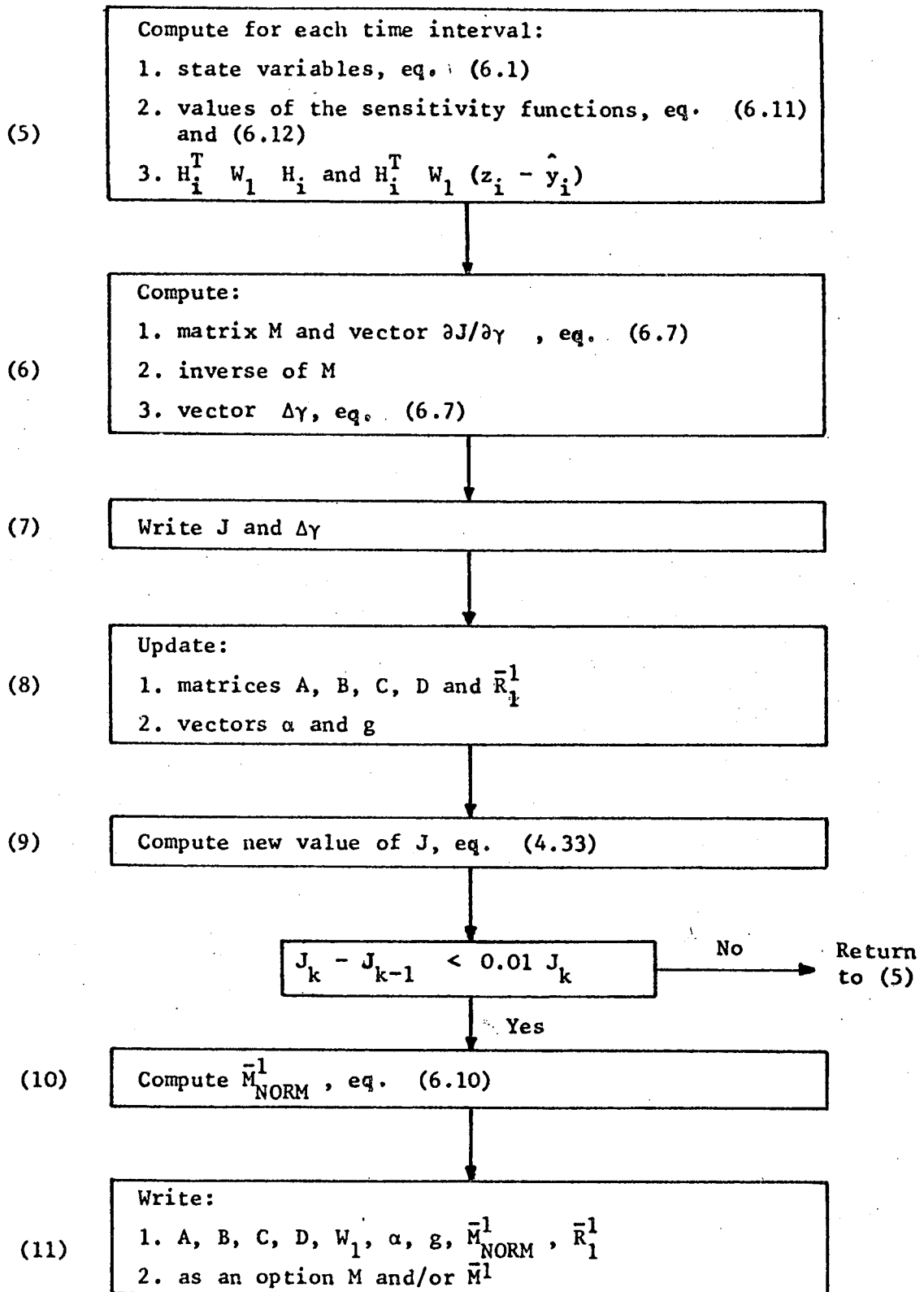
The flowchart for the computing program is presented on page 31 and 32. The scheme has been implemented at Cranfield Institute of Technology on a small 16 bit word length computer using an intermediate level interpretation language which was designed specifically for experimental data analysis applications. The language has been developed by Williams in Ref. 21 who has also compiled the program described.

The interpretation language offers considerable operational flexibility. In addition to enabling acquisition, scaling, transformation and processing to take place without recourse to intermediate storage, it allows "hands on" control even, under some circumstances, during program execution. The consequences of this last feature are that wayword runs can often be "rescued" without the need for restarting the entire process, and that the general program described here can be incorporated, without modification, into a special purpose program suite to enable data to be input and output in a user oriented format. The only modification to this general program is that connected with a specific form of the nonlinear system. This modification is prepared for each nonlinear system under investigation as a short sub-routine.

The disadvantage of the language is the relative slowness of execution due to its interpretative nature. It is assumed that computing is roughly ten times slower than that based on the compiling program.



Parameter Identification Program Flowchart.



Parameter Identification Program Flowchart (Concluded).

7. EXAMPLES

As examples, the results of the identification of aircraft parameters and aerodynamic derivatives for four different aircraft are presented. They include the analysis of flight data from the BASSET variable stability aircraft and from the M.S. 760 PARIS aircraft, the simulated data for a nonlinear model of the F-100 aircraft, and the flight data of the HP-115 aircraft.

The flight test programme on the BASSET aircraft was carried out at C.I.T. after installation of a variable stability and control (V.S.C.) system. The variable characteristics of the aircraft are selected by setting of the potentiometers on the special purpose analogue computer. A substantial part of the flight test programme was formed by the measurement of basic aircraft parameters and by the measurement of parameters for preselected values of stability derivatives or aircraft response characteristics. As a final result of these measurements the predicted calibration curves for setting of the potentiometers were checked and eventually modified accordingly. Detailed description of the aircraft, its systems, flight test programme and results is given in Ref. 22. In the report presented the results from the measurement on the basic aircraft and some interesting cases from the variable stability simulation are mentioned.

The measurement on the M.S. 760 PARIS aircraft was carried out at C.I.T. as well. During this experiment the input-output data for the short period longitudinal and lateral motion was obtained. The main purpose of the parameter identification was a comparison of the ML estimates with those from theoretical predictions, wind tunnel measurement and previous flight tests. A description of the M.S. 760 PARIS aircraft can be found in various C.I.T. reports. The basic characteristics of the instrumentation system installed on the aircraft were evaluated and summarized in Ref. 23.

In the third example the responses of the F-100 aircraft in two rapid rolling manoeuvres were computed using the aircraft characteristics from Ref. 24. The flight conditions and input forms were chosen to represent typical pilot outputs and aircraft performance.

In the fourth example the data is taken from the measurement of the HP-115 slender delta-wing research aircraft in its longitudinal motion. The responses were excited from the horizontal steady-state flights at different airspeeds by the elevator deflection. Because of the input form used the airspeed changes during the transient motion were negligible. For the identification the time histories of η , q and n_z were available. The detailed description of the identification process applied is in Ref. 27. In this report only a brief idea about the analysis is given.

7.1 BASSET Variable Stability Aircraft

The BASSET data for the short period longitudinal motion of the basic aircraft was obtained from six runs. During these measurements the same flight conditions were maintained within the best possible limits. For the parameter identification the ML estimation was used and the model of the aircraft was assumed in the form

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_{\alpha} & 1 \\ M_{\alpha} & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\eta} & Z_o \\ M_{\eta} & M_o \end{bmatrix} \begin{bmatrix} \eta \\ 1 \end{bmatrix} \quad (7.1)$$

and

$$\begin{bmatrix} \alpha \\ q \\ \eta_z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ Z_{\alpha} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z_{\eta} & Z'_o \end{bmatrix} \begin{bmatrix} \eta \\ 1 \end{bmatrix} \quad (7.2)$$

The results of the identification are summarized in Table 2 which includes the values of parameters, lower bounds on their standard errors, mean square errors of output variables and fit errors. In the last column of this table the sample mean values and their standard errors are given. The estimated parameters indicate slightly higher scatter than would be expected from their standard errors. This could be caused by the input form used which was larger in amplitude and in duration than was necessary for the proper excitation of the short period motion only. Both the airspeed changes and possible nonlinearities were not included in the model representing the aircraft. Also not included were the dynamic characteristics of the V.S.C. system.

One typical set of measured input and output data is shown in Fig 3. together with the computed time histories of the output variables. The corresponding residuals are plotted in Fig. 4. The oscillatory components in all three time histories are presented. However no further attempt was made to find the reason for that.

The accuracy of the mean values of the estimated parameters was checked in Fig. 5 by comparison of predicted outputs, based on the mean values, with those actually measured. The disagreement in q and n_z is mainly due to constant bias errors in measured variables. These errors were not included in the predicted time histories.

The next step in the BASSET experiment was measurement on the aircraft with the V.S.C. system operating. The cases were selected to demonstrate the ability of the ML estimation technique to identify the parameters of the aircraft with completely different mode characteristics.

The results of the identification of the variable stability aircraft performing the short period longitudinal motion are presented in Table 3. The changes in aircraft characteristics were achieved by altering the values of derivatives $C_{m\dot{\alpha}}$, $C_{m\ddot{\alpha}}$ and C_{mq} in both directions with respect to values for the basic aircraft. The predicted values included in the table are based on evaluated mean values, on theoretical calibration curves for the potentiometer setting and on the secondary effects of the V.S.C. system. In most cases the disagreement between the predicted and evaluated values occurs for those parameters which are strongly correlated (for strong correlation the values of the correlation coefficient equal to or greater than 0.85 were chosen). It would be possible to expect some improvement in results if fixing parameters or a priori weighting is used.

The measured data for three cases in Table 3 is plotted in Figure 6 to 8 inclusive, and compared with computed outputs. The first case demonstrates the effect of positive e.g. margin, the other two changes in the damping coefficient. In Figure 7 the damping ratio is equal to 0.06. This low value was achieved by setting the positive value of $C_{m\dot{\alpha}}$. In Figure 8 the damping ratio is equal to 0.95 and was achieved by a large value of C_{mq} . In all cases the fit between measured and computed variables is satisfactory.

The lateral motion of the aircraft was excited simultaneously by the rudder and aileron. Five repeated runs were chosen for the parameter identification using the ML estimation. Equations (A.6) with bias terms added described the aircraft. In these equations the parameters Y_p , Y_ϕ and Y_ξ were treated as known constants defined as

$$Y_p = \alpha_e, \quad Y_\phi = \frac{g}{u_e} \cos \theta_e \quad \text{and} \quad Y_\xi = 0$$

The output vector included the variables p , r and β .

The results of the identification are given in Table 4. The degree of repeatability for individual parameter varies. Under ideal conditions the lower bounds on standard errors should be close to the estimates of standard errors for the mean values. The differences are mainly due to the fact that the measurements were not repeated often enough, but can also be caused by decreased identifiability in some parameters (e.g. L_r , L_ζ and N_p) or by other unknown reasons (see e.g. large scatter in L_ξ).

The measured and computed variables for one of the five runs are presented in Figure 9 and the resulting residuals in Figure 10. The residuals in r and β contain deterministic oscillatory components the frequency of which agree with the Dutch-roll frequency. This is very likely caused by the non-steady flight regime before the introduction of inputs and in more detailed analysis, therefore, the non-zero initial conditions should be included.

The non-steady conditions are probably the reason for greater differences in measured and predicted time histories in r . The comparison of both sets of data is given in Figure 11 where the predicted outputs are based on sample mean values from Table 4. The remaining pairs of output variables in p and β show no substantial differences.

In addition to the previous measurement on the basic aircraft one run with the input variable in ζ only was analysed. Because the complete lateral motion was not properly excited the parameter identification created some identifiability problems. The results in Table 5 indicate high inconsistency in the estimated parameters from the regression analysis, the WLS and ML estimation. The evaluated parameters are far apart from the a priori values obtained as the mean values from three previous measurements. Some of the estimated parameters have also non-physical values.

Substantial improvement in the ML estimates has been achieved by fixing parameters L_r and N_p on their a priori values and by the Bayesian estimation. These results are given in the last two columns of Table 5 and are in very good agreement with the a priori values. Despite large differences in the parameters from various estimations and under various conditions the mean square errors and fit errors remain almost the same. This emphasizes the known fact that those two errors do not fully characterize the accuracy of the parameter identification. The measured and computed output variables are compared in Figure 12. The computed curves are based on results of the Bayesian estimation.

The last set of results was taken from the identification of the lateral motion of the aircraft with the V.S.C. system operating. The data included six runs in which all stability derivatives involved were changed individually. For the determination of the parameters the WLS estimation was used because convergence is generally more rapid than for the ML estimation technique. The weighting matrix was based on the data from previous results. The evaluated cost functions indicated the departure of the WLS estimates from the minimum variance ones for which $J = 0.5$.

The results obtained are presented in Table 6 where the predicted values of parameters are also included. The changes in mode characteristics of the basic aircraft and no appropriate changes in the forms of the input variables resulted in strong single or even multiple correlation between parameters. This decreased the identifiability for some parameters and, consequently, could have caused the differences between their estimated and predicted values. All strongly correlated parameters are summarized in Table 7.

Measurement and computed data for three runs are plotted in Figure 13 to 15 inclusive. The first example demonstrates small dihedral effect on the resulting response. In Figure 14 the aircraft has positive damping in yaw and in Figure 15 small directional stability. The fit between measured and computed outputs is good in almost all cases.

7.2 M.S. 760 PARIS aircraft

The longitudinal responses of the aircraft were measured in horizontal flights at different airspeeds. Only the short period motion was excited by pulse input in the elevator deflection. The aircraft parameters were evaluated by the ML estimation and equations (7.1) and (7.2) represented the model. The nondimensional aerodynamic derivatives were obtained from expressions given in Appendix A. These derivatives are plotted in Figure 16 as functions of the lift coefficient C_L , and fitted by the first or second order polynomials. It was assumed that $C_{L\alpha} \approx -C_{Z\alpha}$ and $C_{L\eta} \approx -C_{Z\eta}$. The plotted derivatives $C_{m\alpha}$ and $C_{m\eta}$ contain additional terms with the derivative $C_{m\dot{\alpha}}$ having the forms

$$\frac{C_{L\alpha} C_{m\dot{\alpha}}}{2\mu} \quad \text{and} \quad \frac{C_{L\eta} C_{m\dot{\alpha}}}{2\mu}$$

respectively. The differences between $C_{m\eta}$ and those values plotted are less than one percent. However, for the derivative $C_{m\alpha}$ the differences could be about eight percent (the plotted points have larger negative values) and should be, therefore, considered when comparison with $C_{m\alpha}$ values from other data is made.

In Figure 16 the ML estimates are compared with those determined from wind tunnel and steady-state flight measurements. The differences ranging from ten to fourteen percent are apparent in the lift curve slope values from the unsteady and steady-state measurements. The ML estimates of $C_{L\eta}$ are also higher than values from steady-state flights. The remaining derivatives are in good agreement.

One of the sets of measured and computed time histories is presented in Figure 17. The corresponding residuals and their autocovariance functions are plotted in Figure 18. The end parts of the measured data in Figure 17 are distorted, probably by a gust wind. This distortion is also seen in the residuals and it is, very likely, the reason for the shape of the autocovariance functions which are different from the expected ones for white noise.

An attempt was made to investigate the reason for the discrepancies in values of $C_{L\alpha}$ obtained from the unsteady and steady-state measurements. The results based on different model forms are presented in Table 8. The omission of biased terms decreased the number of unknown parameters but effected only the parameter M_q and the fit in n_z . Then the additional term Z_q was included in the model. As a consequence its previous fixed value was only slightly changed but the values of other parameters remained almost the same.

For the a priori weighting the value $Z_\alpha = -1.32 \pm 0.01$, which corresponds to $C_{L\alpha} = 4.8$, was used. The resulting ML estimate of Z_α was very close to the a priori value. At the same time the value of Z_η was also decreased. This could mean that the data obtained from the unsteady measurements was not very sensitive to changes in Z_α .

The last three columns in Table 8 include the results where the combinations of two measured outputs were used for the parameter identification. This approach could eventually eliminate one of the output variables if this were distorted by bias errors. In all cases, however, the value of $C_{L\alpha}$ changed almost insignificantly. The different value of M_q and M_η for α_E and n_{zE} analysed are due to decreased information on the pitching motion of the aircraft contained in these variables.

The aircraft parameters were also identified from each output variable separately. In this case all coefficients in (7.1) could not be identified. The equations of motion, therefore, had to be transformed into the observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -K_0 \\ 1 & -K_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_{y0} \\ K_{y1} \end{bmatrix} n \quad (7.3)$$

$$y = [0, 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K_{y2} n \quad (7.4)$$

The coefficients in these equations are identical to those in the transfer function

$$\frac{y}{n}(s) = K_{y2} + \frac{K_{y1}s + K_{y0}}{s^2 + K_1s + K_0} \quad (7.5)$$

where

$$K_1 = -Z_\alpha - M_q \quad (7.6)$$

$$K_0 = -M_\alpha + Z_\alpha M_q$$

for $y = \alpha$

$$K_{\alpha 2} = 0, \quad K_{\alpha 1} = Z_\eta, \quad K_{\alpha 0} = M_\eta - Z_\eta M_q$$

for $y = q$

$$K_{q2} = 0, \quad K_{q1} = M_\eta, \quad K_{q0} = -Z_\alpha M_\eta + Z_\eta M_\alpha \quad (7.7)$$

and for $y = n_z^*$

$$K_{n2} = Z_\eta, \quad K_{n1} = Z_\eta Z_\alpha, \quad K_{n0} = Z_\alpha M_\eta - Z_\eta Z_\alpha M_\alpha$$

The results of the ML estimation are given in Table 9. They are compared with the predicted values based on parameters in Table 8 for the basic model and the model with a priori weighting of Z_α . The parameters $K_{\alpha 1}$, K_{n1} and K_{n2} are excluded from the table. Their identification is unrealistic because of their small values and great sensitivity.

As follows from expression (7.6) and (7.7) the changes in the parameter Z_α can significantly influence the coefficients K_1 , K_{q0} and $K_{n0} = -K_{q0}$, and a reasonably good estimate of Z_α can be found from the ratio of K_{q0} and K_{q1} . The results from q_E and n_{zE}^* are close to the predicted values. If K_{q1} is assumed equal to -12 then $Z_\alpha \approx -1.32$ for n_{zE}^* and $Z_\alpha \approx -1.47$ for q_E analysed.

All parameters obtained from q_E differ from those predicted. This could be due to some bias error in q_E or due to identifiability problems because all parameters considered are strongly correlated.

Because the investigation of various model forms did not give the answer to the problem further measurement was carried out with different input forms. In addition to symmetric pulse, asymmetric pulse with sharp increase of deflection and double pulses with either symmetric or asymmetric forms were applied. The forms of the inputs used in four runs analysed can be seen in Figure 16 and Figure 19. The ML estimates from those runs are presented in Figure 10. In addition to results usually given the values of the trace of the matrix $\bar{M}^{-1}$ are also included. As pointed out in Ref. 8 they provide information on the overall performance of the input. In this case, a lower value of the trace indicates the better performance.

For the single and double symmetric pulses the values of Z_α and M_α are in agreement. The differences in M_q and M_η could be caused by strong correlation in these parameters. The values of Z_α are now not very far from those obtained from steady-state and wind tunnel measurements.

The estimates from the remaining two runs (run 2.8 and run 2.11) are very consistent. The value of Z_α and M_α are different, however, from values in two previous runs. On the other hand the remaining parameters and fit errors are almost the same in all four cases. This demonstrates the influence of the design of an experiment on the accuracy of the estimated parameters and thus creates a problem for further research.

The lateral data on the aircraft was obtained from measurement in horizontal flight at different airspeeds using the aileron and rudder deflection simultaneously. The ML estimation was applied and the model was based on equation (A.6) with known parameters Y_p , Y_ϕ and $Y_\xi = 0$. The output vector was $y^T = \{p, r, \beta, n_y^*\}$. From the estimated parameters the nondimensional aerodynamic derivatives were evaluated. They are plotted in Figure 20 in relation to C_L and are fitted by first or second order polynomials. The two-sigma confidence limits for each set of points were also established.

In Figure 21 the ML estimates are compared with predicted values and with the results of wind tunnel and flight measurements. The flight measurements in this case are obtained from steady-state data or from free oscillations. It follows from the comparison that the only significant disagreement is in the derivative $C_{\dot{\alpha}\beta}$. The large negative value in this derivative obtained from free oscillations could be caused by the large value of $C_{\dot{\alpha}r}$ used in data analysis by the time vector method, see Ref. 25.

The measured input and output variables in one run are presented in Figure 22 and compared with the computed outputs. There is very good fit in all four variables. The residuals for this data are plotted in Figure 23. Their time histories for r and β include oscillatory components. Some changes in the model structure were made to enable these components to be removed. The best improvement was due to the introduction of $Y_{\dot{\alpha}}$ and $Y_{\dot{\beta}}$ as additional unknown parameters. The results for this extended model are given in Table 11 together with the estimates for the basic model. The significant improvement in the mean square error and in the residuals for r can be seen. The oscillatory character of residuals in β , however, remains. It could be caused by the error in β vane reading corrections, by external disturbances or by both.

The autocovariance functions for the residuals in the extended model are given in Figure 25. These functions for the variables p and n_y^* are very close to those for white noise. In order to check the normality of the residuals their cumulative frequencies are plotted in Figure 26.

Also included in Table 11 are two cases with different constraints. First of all the bank angle, ϕ , was added to the output variables. In this particular case more constraint on the roll rate data resulted in the departure of corresponding parameters from their previously estimated values. This was accompanied by the decrease of fitness in the p and n_y data.

In the second case the lateral acceleration was excluded from the output vector. As expected the main differences appeared in the parameters of the side force equation and in the mean square error for β . There were also some changes in parameters of the yawing moment equation but the fit between measured and computed yaw rate is the same as that for the basic model.

The effect of various input forms was demonstrated in three runs in which other flight conditions were maintained approximately the same. The various forms of inputs are given in Figure 22 and Figure 27. The ML estimates are presented in Table 12 and are again completed by the trace of the matrix $\bar{M}^{-1}$.

Even if the input forms in run 2.4 and 3.4B are similar their overall performances are quite different. The third input form, which consists of a series of pulses in aileron deflection followed by pulses in rudder deflection, has the best performance index.

Some changes in the parameters can be also found. It is not clear, however, whether the differences in the corresponding parameters are caused by different input forms or by the decreased identifiability in parameters, some of which are strongly correlated. As with the longitudinal motion the demand on further research into the optimal input form seems to be obvious.

7.3 F-100 Aircraft in Simulated Rapid Rolling Manoeuvres.

Equations (B.4) and (B.5) for a rapid rolling manoeuvre of an aircraft developed in Appendix B were used in the calculation of the simulated data. The aircraft responses were calculated as rolling pull-outs initiated from a 3g pull-up at Mach number 0.9 and height approximately 6000 m. In the first run the rudder was locked, in the second run the rudder deflection was used to regain coordination.

The time interval during the calculation was $\Delta t = 0.05s$ and the number of samples for each variables was $N = 160$. To the computed outputs the gaussian random noise with zero mean was added. The variance of this simulated measurement noise had the following values

y	s^2
β, α	1×10^{-6}
p, q, r	$2.5 \times 10^{-5} \{1/s^2\}$
n_y^*	$2.5 \times 10^{-7} \{1/s^2\}$
n_z^*	$4 \times 10^{-6} \{1/s^2\}$

The inputs chosen and the resulting outputs are presented in Figure 28 and Figure 29.

The prevailing rolling motion of the aircraft and the complexity of the model make the evaluation of physically realistic and sufficiently accurate parameters almost impossible, even if simulated data is used. Therefore for the parameter identification only simplified equations (B.6) were used. This model describes the lateral part of the complete motion with the output variables β, p, r and n_y^* and with the input variables ξ, η, α and q .

In run 2 the input variables also include ζ . The previous number of 24 unknown parameters in run 2 was reduced to 18 by this simplification.

The true values of parameters and the ML estimates in run 1 are given in Table 13. Two cases are included. In the first one all parameters in the simplified model were identified. The main difference between true and estimated parameters is only in values of the parameter L_β . This is caused by the small excitation of the sideslip angle during the aircraft motion and by very large term L_p . Even if the values of the other parameters are satisfactory, the simple and multiple correlation between them exists. The correlated parameters are reviewed in Table 15 together with the cases analysed.

In order to remove this correlation four of the correlated parameters were fixed at their estimated values. The estimates of remaining parameters are given in the last column of Table 15. There are almost no changes in their values comparing to the previous case. The number of strongly correlated parameters, however, was substantially reduced.

Because of the rudder deflection used in run 2 the sideslip angle was decreased. This effected the accuracy of the estimated parameter L_β even more, as can be seen from the results in Table 14. The fixing of parameters in case B and C removed the strong correlation, first of all between some and then between all remaining parameters. The accuracy of the estimates, however, was not improved.

Finally the a priori weighting was used in case D with a priori values

$$L_\beta = -9.5 \pm 0.5$$

$$L_r = 0.135 \pm 0.02 \quad , \quad N_p = 0.0021 \pm 0.0002$$

This approach resulted in very accurate estimates of all parameters and small number of strongly correlated parameters.

The limited experience with the identification of a rapid rolling manoeuvre of the aircraft indicates that difficult identifiability problems leading to inaccurate values of parameters and their strong correlation can be met. The results will be substantially influenced by the design of an experiment, by the complexity of a model and by the amount and accuracy of the a priori information on parameters. The further identification of real flight test data from carefully planned experiments could significantly contribute to the solution of this problem.

7.4 HP-115 Slender Delta-Wing Research Aircraft

Taking into account the wind tunnel test and preliminary flight test results, the linearized equations of motion are considered in the form

$$\begin{aligned}\ddot{\alpha} &= Z_{\alpha}\alpha + Z_q q + Z_{\alpha^2}\alpha^2 + Z_{q\alpha}q\alpha + Z_{\eta\alpha}\eta\alpha + Z_{\dot{\eta}}\dot{\eta} + Z_{\ddot{\eta}}\ddot{\eta} + Z_o \\ \ddot{q} &= M_{\alpha}\alpha + M_q q + M_{\alpha^2}\alpha^2 + M_{q\alpha}q\alpha + M_{\eta\alpha}\eta\alpha + M_{\dot{\eta}}\dot{\eta} + M_{\ddot{\eta}}\ddot{\eta} + M_o \\ \ddot{\theta} &= q\end{aligned}\quad (7.8)$$

The vertical acceleration can then be expressed as

$$n_z^* = Z_{\alpha}\alpha + Z_q^1 q + Z_{\theta}\theta + Z_{\alpha^2}\alpha^2 + Z_{q\alpha}q\alpha + Z_{\eta\alpha}\eta\alpha + Z_{\dot{\eta}}\dot{\eta} + Z_{\ddot{\eta}}\ddot{\eta} + Z_o^1 \quad (7.9)$$

The parameters in (7.8) and (7.9) are defined in Ref. 25.

For the parameter estimation, the linear model with $Z_{\dot{\eta}} = M_{\dot{\eta}} = 0$ was first used. The further simplification was achieved by fixing Z_q and Z_q^1 on their wind tunnel values and Z_{η} on values from the steady-state flights. Comparison of measured and computed time histories at a low angle of attack is given in Fig. 30. The fit between the two output variables is very good. The resulting parameters and the lower bounds on their standard errors are presented in the third column of Table 16. The estimated parameters accord well with predictions.

The use of the linear model in the case of low angle of attack ($\alpha_e = 7.2$ deg) was also substantiated by comparing previous results with those from the ML estimation with a priori weighting and the nonlinear model having $Z_{\eta\alpha} = M_{\eta\alpha} = 0$. The a priori values were taken from the linear case with weights proportional to their standard errors.

The results of the two approaches mentioned did not produce significantly different values of parameters. The estimates of variance $s^2(n_z^*)$, however, can be considered as significantly different. The measure for significant difference in parameters is based on 2σ confidence intervals, and that for those in variances of measurement noise on the critical value for their ratio.

In the second test case of a high angle of attack ($\alpha_e = 20.3$ deg) the linear model was proved to be inadequate. This follows from the comparison of the measured and computed output time histories presented in Fig. 31 and Fig. 32 for the linear model and the nonlinear model with a priori weighting. The substantial improvement of the fit in the vertical acceleration is apparent and is confirmed by the numerical values of corresponding variance estimates in Table 16.

The use of a nonlinear model also had the significant effect on the damping parameter M_q , whereas the remaining parameters from the linear model and their counterparts in the nonlinear model do not differ significantly. These results are given in the second part of Table 16. The strong correlation between M_α and M_q in the linear case was removed by the addition of nonlinear terms.

The time histories of residuals in both output variables and the corresponding autocovariance functions were computed and plotted as in Fig. 33 to Fig. 35. The effect of nonlinear terms on $\hat{R}_{nzo}$ is apparent whereas the function $\hat{R}_{qo}$ did not change. In all cases the autocovariance function $\hat{R}_{qo}$ is different from that assumed for the random white noise. This could be caused by bias errors in the measured time histories of q .

The resulting non-dimensional derivatives of the aircraft are plotted in Fig. 36 as functions of α_e and compared with predictions. Both the simplified linear model ($7 \text{ deg.} < \alpha_e < 16 \text{ deg.}$) and the nonlinear model with $Z_{\eta\alpha} = M_{\eta\alpha} = 0$ ($\alpha_e > 16 \text{ deg.}$) were used for the evaluation of $C_{L\alpha}$, $C_{m\alpha}$, $C_{m\eta}$ and $C_{mq} + C_{m\dot{\alpha}}$. The remaining derivatives were estimated by applying the nonlinear model only.

8. RECOMMENDATION FOR FUTURE WORK

Some areas for future work in the aircraft parameter identification have been proposed in Ref. 8. These proposals are in agreement with trends based upon the experience and achievements at C.I.T. and other establishments. The topics for the further research mentioned can, therefore, be summarized as:

1. extensive application of the generalized maximum likelihood technique, which considers the measurement noise in the input variables and the external disturbances, for the estimation of the aircraft states and parameters. This very general approach would cover the determination and analysis of aircraft performance, stability and control characteristics, and handling and riding qualities,
2. identification of aerodynamic derivatives from flight data for specific important flight regimes (e.g. high angle of attack and transonic flights, large disturbance combined manoeuvres) and unorthodox aircraft (STOL, VTOL) design. This will require the continuing development in methods for model structure determination and verification since the models in such regimes and for such aircraft are not well known,
3. correlation of the flight test results with wind tunnel results for high performance aircraft and aircraft of special design,
4. development of methods for solving identifiability problems and the determination of the effect of a stability augmentation system and a human pilot on the identifiability of the parameters,
5. development of methods for the design of optimal inputs for linear as well as nonlinear systems. Modification of the optimal input for pilot acceptability and ease of implementation, and flight test validation of optimal inputs.

9. CONCLUSION

Aircraft parameter identification is a qualitatively new strategy and technique for the design of an experiment and for flight test data analysis. The identification procedure includes three steps, namely characterization, parameter estimation and verification.

Characterization is a qualitative operation defining the structure of a system. If the aircraft is of usual design and in no extreme flight conditions, then the equations of motion are well defined. On the other hand the modelling of an aircraft during large disturbance manoeuvres, in extreme flight conditions or of unorthodox design could create problems. Very often it is not clear what should be the best relationship between model complexity and measurement information.

For parameter estimation two methods were introduced and compared. The first one is the equation error method and is usually used as the start up procedure for the output error method. This second method can include different estimation techniques from which the weighted least squares, maximum likelihood and Bayesian estimation were used. Further generalization of the maximum likelihood estimation, which takes into account the measurement noise in the input variables and the effect of external disturbances, was not considered in this report.

The problems of accuracy and identifiability were broadly discussed. Experience with various identification techniques shows that the possibility of a numerical solution of the estimation process and the reliability of the parameters, in terms of physically realistic values and small error covariances for the parameters, are the most important problems in the whole process of parameter identification.

Even if the parameter estimation has been successfully completed some doubts about the correctness of the model may remain. Therefore one of the possible ways of checking the validity of the structure based on the analysis of residuals was introduced.

A general computing algorithm was developed which covers all estimation techniques described in the report and which is applicable to linear as well as nonlinear systems, and which is also flexible enough for the solution of various identifiability problems. Using this algorithm the computing program has been compiled in general terms to enable the user to achieve the objectives mentioned.

The estimation methods were applied successfully in three different examples:

1. BASSET variable stability aircraft data for longitudinal and lateral motion with different mode characteristics were analysed. The repeatability of the estimates and the predictions based on sample mean values obtained were checked. Some identifiability problems were met during the analysis. For their solution fixing parameters and a priori weighting was used.
2. PARIS aircraft data was used for the estimation of nondimensional aerodynamic derivatives. These derivatives were compared with theoretical predictions and results from wind tunnel and steady-state flight measurements. The effects of different model structures and various input forms were investigated.
3. F-100 aircraft simulated data on two rapid rolling manoeuvres was identified using a simplified nonlinear model of the system. Difficult identifiability problems resulted in large covariances for some estimated parameters. The results of the identification of rapid rolling manoeuvres will be substantially influenced by the design of an experiment, by the complexity of a model and by the amount and accuracy of the a priori information on parameters. The further identification of real flight test data from carefully planned experiments could significantly contribute to the solution of this problem.
4. HP-115 aircraft data demonstrates the ability of the ML technique with and without a priori weighting to evaluate the significant parameters even if the measurement information is limited.

It is believed that the results achieved might form the basis for further research in the aircraft parameter identification at the Cranfield Institute of Technology. The areas for this research have been proposed. The results of the report presented might also contribute to the wider acceptance of identification techniques in the analysis of flight test data.

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APPENDIX A

Linear models of an aircraft.

In Ref.26 the complete set of linearized equations of aircraft motion are developed. The datum conditions correspond to steady rectilinear flight. These equations are referred to the system of body axis and have the form:

the longitudinal equations

$$\begin{aligned} \dot{u} + x_u u + x_w w + w_e q + g_1 \dot{\theta} + x_\eta \eta &= 0 \\ z_u u + (1 + z_w) \dot{w} + z_w w + z_q q - u_e q + g_2 \dot{\theta} + z_\eta \eta &= 0 \\ m_u u + m_w \dot{w} + m_w w + \dot{q} + m_q q + m_\eta \eta &= 0 \\ - q + \dot{\theta} &= 0 \end{aligned} \quad (A.1)$$

the lateral equations

$$\begin{aligned} \dot{v} + y_v v + y_p p - w_e p + u_e r - g_1 \dot{\phi} + y_\xi \xi + y_\zeta \zeta &= 0 \\ l_v v + \dot{p} + l_p p + e_x \dot{r} + l_r r + l_\xi \xi + l_\zeta \zeta &= 0 \\ n_v v + e_z \dot{p} + n_p p + \dot{r} + n_r r + n_\xi \xi + n_\zeta \zeta &= 0 \\ - p + \dot{\phi} &= 0 \end{aligned} \quad (A.2)$$

For the analysis of experimental data state variables w and v are replaced by variables α and β using the relations $w \approx u_e \alpha$ and $v \approx u_e \beta$. Then for the new sets of state variables equations (A.1) and (A.2) may be rewritten as

$$T\dot{x} = A'x + B'u \quad (A.3)$$

where for the longitudinal motion

$$x^T = \{u, \alpha, q, \theta\}$$

$$u = \eta$$

$$A' = \begin{bmatrix} -x_u & -u_e x_w & -w_e & -g_1 \\ -\frac{z_u}{u_e} & -z_w & -\frac{z_q}{u_e} + 1 & -\frac{g_2}{u_e} \\ -m_u & -u_e m_w & -m_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} -x_\eta \\ -\frac{z_\eta}{u_e} \\ -m_\eta \\ 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + z_{\dot{w}} & 0 & 0 \\ 0 & u_e m_{\dot{w}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and for the lateral motion

$$x^T = \{ \beta, p, q, \theta \}$$

$$u^T = \{ \xi, \zeta \}$$

$$A' = \begin{bmatrix} -y_v & a_e - \frac{y_p}{u_e} & -1 & \frac{g_1}{u_e} \\ -u_e l_v & -l_p & -l_r & 0 \\ -u_e n_v & -n_p & -n_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} -\frac{y_\xi}{u_e} & -\frac{y_\zeta}{u_e} \\ -l_\xi & -l_\zeta \\ -n_\xi & -n_\zeta \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e_x & 0 \\ 0 & e_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the aircraft parameter identification it is more convenient to rearrange equation (A.3) into the more common form

$$\dot{x} = Ax + Bu \quad (A.4)$$

and to use the concise form for the coefficients in matrices A and B.

The longitudinal equations become

$$\begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & X_q & X_\theta \\ Z_u & Z_\alpha & Z_q & Z_\theta \\ M_u & M_\alpha & M_q & M_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} X_\eta \\ Z_\eta \\ M_\eta \\ 0 \end{bmatrix} \quad (\text{A.5})$$

where

$$\begin{aligned} X_u &= -x_u & Z_u &= -\frac{z_u}{1+z_w} \frac{1}{u_e} \\ X_\alpha &= -u_e x_w & Z_\alpha &= -\frac{z_w}{1+z_w} \\ X_q &= -w_e & Z_q &= -\frac{z_q - u_e}{1+z_w} \frac{1}{u_e} \\ X_\theta &= -g \cos \theta_e & Z_\theta &= -\frac{g \sin \theta_e}{1+z_w} \frac{1}{u_e} \\ X_\eta &= -x_\eta & Z_\eta &= -\frac{z_\eta}{1+z_w} \frac{1}{u_e} \\ M_u &= -m_u + \frac{z_u m_w}{1+z_w} & M_\theta &= \frac{m_w g \sin \theta_e}{1+z_w} \\ M_\alpha &= -u_e m_w + u_e \frac{z_w m_w}{1+z_w} & M_\eta &= -m_\eta + \frac{z_\eta m_w}{1+z_w} \\ M_q &= -m_q + \frac{m_w (z_q - u_e)}{1+z_w} \end{aligned}$$

The relationships between the coefficients in the longitudinal equations and the non-dimensional aerodynamic derivatives are given by

$$\begin{aligned} X_u &= \frac{1}{\tau} C_{Xu}, & X_\alpha &= \frac{v_e}{\tau} C_{X\alpha}, & X_\eta &= \frac{v_e}{\tau} C_{X\eta} \\ Z_u &= \frac{1}{\mu} C_{Zu}, & Z_\alpha &= \frac{1}{\tau} C_{Z\alpha}, & Z_q &= 1 + \frac{1}{2\mu} C_{Zq}, & Z_\eta &= \frac{1}{\tau} C_{Z\eta} \end{aligned}$$

$$M_u = \frac{C_{mu}}{i_y \tau \bar{c}} + \frac{C_{zu} C_{m\dot{\alpha}}}{2i_y \tau^2 V_e}$$

$$M_q = \frac{1}{2i_y \tau} (C_{mq} + C_{m\dot{\alpha}})$$

where

$$\tau = \frac{2m}{\rho S V_e},$$

$$\mu = \frac{2m}{\rho S \bar{c}},$$

$$i_y = \frac{I_y}{m \bar{c}^2},$$

and where $u = V_e$.

The lateral equations are

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & -1 & Y_\phi \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} Y_\xi & Y_\zeta \\ L_\xi & L_\zeta \\ N_\xi & N_\zeta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \quad (\text{A.6})$$

where

$$Y_\beta = -Y_v$$

$$Y_\phi = \frac{g}{V_e} \cos \theta_e,$$

$$L_\beta = -\frac{l_v^{-e} n_v}{1 - e_x e_z}$$

$$L_p = -\frac{l_p^{-e} n_p}{1 - e_x e_z}$$

$$L_r = -\frac{l_r^{-e} n_r}{1 - e_x e_z}$$

$$L_\xi = -\frac{l_\xi^{-e} n_\xi}{1 - e_x e_z}$$

$$L_\zeta = -\frac{l_\zeta^{-e} n_\zeta}{1 - e_x e_z}$$

$$Y_p = \alpha_e - \frac{y p}{u_e}$$

$$Y_\xi = -\frac{y \xi}{u_e}, \quad Y_\zeta = -\frac{y \zeta}{u_e}$$

$$N_\beta = -\frac{n_v^{-e} l_v}{1 - e_x e_z}$$

$$N_p = -\frac{n_p^{-e} l_p}{1 - e_x e_z}$$

$$N_r = -\frac{n_r^{-e} l_r}{1 - e_x e_z}$$

$$N_\xi = -\frac{n_\xi^{-e} l_\xi}{1 - e_x e_z}$$

$$N_\zeta = -\frac{n_\zeta^{-e} l_\zeta}{1 - e_x e_z}$$

When non-dimensional aerodynamic derivatives are used the coefficients in (A.6) may be expressed as

$$\begin{aligned}
 Y_\beta &= \frac{1}{\tau} C_{Y\beta} & Y_p &= \alpha_e + \frac{1}{2\mu} C_{Yp} & Y_\xi &= \frac{1}{\tau} C_{Y\xi} & Y_\zeta &= \frac{1}{\tau} C_{Y\zeta} \\
 L_\beta &= \frac{\mu}{i_x \tau^2} \frac{C_{L\beta} - e_z C_{n\beta}}{1 - e_x e_z} & N_\beta &= \frac{\mu}{i_z \tau^2} \frac{C_{n\beta} - e_x C_{L\beta}}{1 - e_x e_z} \\
 L_p &= \frac{1}{2i_x \tau} \frac{C_{Lp} - e_z C_{np}}{1 - e_x e_z} & N_p &= \frac{1}{2i_z \tau} \frac{C_{np} - e_x C_{Lp}}{1 - e_x e_z} \\
 L_r &= \frac{1}{2i_x \tau} \frac{C_{Lr} - e_z C_{nr}}{1 - e_x e_z} & N_r &= \frac{1}{2i_z \tau} \frac{C_{nr} - e_x C_{Lr}}{1 - e_x e_z} \\
 L_\xi &= \frac{\mu}{i_x \tau^2} \frac{C_{L\xi} - e_z C_{n\xi}}{1 - e_x e_z} & N_\xi &= \frac{\mu}{i_z \tau^2} \frac{C_{n\xi} - e_x C_{L\xi}}{1 - e_x e_z} \\
 L_\zeta &= \frac{\mu}{i_x \tau^2} \frac{C_{L\zeta} - e_z C_{n\zeta}}{1 - e_x e_z} & N_\zeta &= \frac{\mu}{i_z \tau^2} \frac{C_{n\zeta} - e_x C_{L\zeta}}{1 - e_x e_z}
 \end{aligned}$$

where

$$\mu = \frac{2m}{\rho S b}, \quad i_x = \frac{I_x}{mb^2}, \quad i_z = \frac{I_z}{mb^2}$$

For the output equations the acceleration equations are transformed to the instrument location at the centre of gravity. These equations are

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \frac{1}{g} \begin{bmatrix} \dot{u} + qw - rv + g \sin \theta \\ \dot{v} + ru - pw - g \cos \theta \sin \phi \\ \dot{w} + pv - qu - g \cos \theta \cos \phi \end{bmatrix} \quad (A.7)$$

where n_x , n_y and n_z are readings of the longitudinal, lateral and vertical accelerometer in g units, respectively.

The perturbed acceleration equations where w and v are replaced by α and β have the form

$$\begin{aligned}
 n_x^* &= g n_x = \dot{u} + w_e q + g \cos \theta_e \theta \\
 n_y^* &= \frac{g}{V_e} n_y = \dot{\beta} + r - \alpha_e p - \frac{g}{V_e} \cos \theta_e \phi \\
 n_z^* &= \frac{g}{V_e} n_z = \dot{\alpha} - q + \frac{g}{V_e} \sin \theta_e \theta
 \end{aligned} \quad (A.8)$$

In steady - state flight there exists

$$\begin{aligned} n_{xe} &= \sin \theta_e \\ n_{ye} &= 0 \\ n_{ze} &= -\cos \theta_e \end{aligned} \quad (A.9)$$

Using (A.5), (A.6) and (A.8) and including the state variables as measured quantities the matrices in the output equation $y = Cx + Du$ may be formulated as follows:

for the longitudinal motion

$$y^T = \{u, \alpha, q, \theta, n_x^*, n_z^*, \dot{q}\} \quad (A.10)$$

$$C = \begin{bmatrix} I \\ C' \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ D' \end{bmatrix} \quad (A.11)$$

$$C' = \begin{bmatrix} X_u & X_\alpha & 0 & 0 \\ Z_u & Z_\alpha & 0 & 0 \\ M_u & M_\alpha & M_q & 0 \end{bmatrix}, \quad D' = \begin{bmatrix} X_\eta \\ Z_\eta \\ M_\eta \end{bmatrix} \quad (A.12)$$

for the lateral motion

$$y^T = \{\beta, p, r, \phi, n_y^*, \dot{p}, \dot{r}\} \quad (A.13)$$

$$C' = \begin{bmatrix} Y_\beta & Y'_p & 0 & 0 \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \end{bmatrix}, \quad D' = \begin{bmatrix} Y_\xi & Y_\zeta \\ L_\xi & L_\zeta \\ N_\xi & N_\zeta \end{bmatrix} \quad (A.14)$$

It is also possible to include the linear acceleration as the state variable provided that this variable is replaced by the filtered one. Therefore if the variables n_x , n_y and n_z are passed through a first order low-pass filter with unit gain and time constant τ_F then the filtered accelerations are

$$n_{xF} = \frac{n_x}{1 + \tau_F s} \quad (A.15)$$

and similarly for the others.

The state equations can then be formulated as

$$\begin{aligned}
 \dot{n}_{xF}^* &= X_u + X_\alpha + \frac{1}{\tau_F} n_{xF}^* + X_\eta \\
 \dot{n}_{yF}^* &= Y_\beta + Y'_p + \frac{1}{\tau_F} n_{yF}^* + Y_\xi + Y_\zeta \\
 \dot{n}_{zF}^* &= Z_u + Z_\alpha + \frac{1}{\tau_F} n_{zF}^* + Z_\eta
 \end{aligned}
 \tag{A.16}$$

where

$$n_{xF}^* = \tau_F g n_{xF}, \quad n_{yF}^* = \tau_F \frac{g}{V_e} n_{yF}, \quad \text{and} \quad n_{zF}^* = \tau_F \frac{g}{V_e} n_{zF}$$

APPENDIX B

Nonlinear models for rapid rolling manoeuvres of an aircraft.

In a rapid rolling manoeuvre little change in the forward speed will be expected. The mathematical model for an aircraft can, therefore, be developed from equations of the concise form. With respect to body axes there are:

$$\begin{aligned}
 \dot{v} + u_e r - p w - g m_3 &= \frac{Y}{m} \\
 \dot{w} + p v - u_e q - g n_3 &= \frac{Z}{m} \\
 \dot{p} + e_x (\dot{r} + p q) + b_x q r &= \frac{L}{I_x} \\
 \dot{q} + e_y (r^2 - p^2) + b_y r p + c_{xy} r &= \frac{M}{I_y} \\
 \dot{r} + e_z (\dot{p} - q r) + b_z p q - c_{xz} q &= \frac{N}{I_z} \\
 \dot{l}_3 &= r m_3 - q n_3 \\
 \dot{m}_3 &= p n_3 - r l_3 \\
 \dot{n}_3 &= q l_3 - p m_3
 \end{aligned} \tag{B.1}$$

where

$$l_3 = -\sin \theta$$

$$m_3 = \sin \phi \cos \theta$$

$$n_3 = \cos \phi \cos \theta$$

The equations may be expressed in terms of perturbations from a symmetrical equilibrium flight conditions defined by the steady-state values of u_e , w_e , p_e , q_e , r_e , θ_e and ϕ_e .

The following simplifying assumptions are introduced:-

1. the perturbations in all variables except in rate of roll can be treated as small quantities,
2. the gravity terms have only small effect. They can be either neglected or expressed in a simple form,
3. the set of variables in (B.1) is replaced by a set V_e , β , α , p , r , ϕ using the relations $V \approx u$, $\beta \approx v/V$ and $\alpha \approx w/V$,

4. the coefficients of aerodynamic forces and moments have the form

$$\begin{aligned}
 C_Y &= C_{Y\beta} + C_{Yp} \frac{pb}{2V_e} + C_{Y\xi} \xi + C_{Y\zeta} \zeta \\
 C_Z &= C_{Z0} + C_{Z\alpha} \alpha + C_{Zq} \frac{\bar{qc}}{2V_e} + C_{Z\eta} \eta + C_{Z\xi}^2 \xi^2 \\
 C_l &= C_{l\beta}(\alpha)\beta + C_{lp}(\alpha) \frac{pb}{2V_e} + C_{lr}(\alpha) \frac{rb}{2V_e} + C_{l\xi}(\alpha)\xi + C_{l\zeta} \zeta + C_{l\eta} \eta \\
 C_m &= C_{m\alpha} \alpha + C_{mq} \frac{\bar{qc}}{2V_e} + C_{mq} \frac{\bar{qc}}{2V_e} + C_{m\xi}^2(\alpha)\xi^2 + C_{m\eta} \eta \\
 C_n &= C_{n\beta}(\alpha)\beta + C_{np}(\alpha) \frac{pb}{2V_e} + C_{nr}(\alpha) \frac{rb}{2V_e} + C_{n\xi}(\alpha)\xi + C_{n\zeta} \zeta
 \end{aligned} \tag{B.2}$$

where $C_{l\beta}(\alpha)$, $C_{lr}(\alpha)$, ..., $C_{n\xi}(\alpha)$ are considered as linear functions of α , i.e.

$$C_{l\beta}(\alpha) = C_{l\beta} + C_{l\beta\alpha} \alpha$$

$$C_{n\xi}(\alpha) = C_{n\xi} + C_{n\xi\alpha} \alpha$$

Using the above mentioned assumptions equations (B.1) can be rewritten as

$$\begin{aligned}
 \dot{\beta} &= -y_v \beta + (\alpha_e - \frac{1}{V_e} y_p) p - r + \frac{g}{V_e} \cos \theta_e \cos \phi_e \phi + \\
 &+ \alpha p - \frac{1}{V_e} y_{\xi} \xi - \frac{1}{V_e} y_{\zeta} \zeta
 \end{aligned}$$

$$\dot{\alpha} = -z_w \alpha + q - \frac{g}{V_e} \cos \theta_e \sin \phi_e \phi - \beta p - \frac{1}{V_e} z_{\xi}^2 \xi^2 - \frac{1}{V_e} z_{\eta} \eta$$

$$\begin{aligned}
 \dot{p} + e_x \dot{r} &= -V_e l_v \beta - l_p p - (l_r + b_x q_e) r - V_e l_{v\alpha} \beta \alpha + \\
 &- l_{r\alpha} \alpha r - e_x p q - b_x q r - l_{\xi\alpha} \alpha \xi - l_{\xi} \xi + \\
 &- l_{\zeta} \zeta - l_{\eta} \eta
 \end{aligned} \tag{B.3}$$

$$\begin{aligned}
 \dot{q} + V_e m_w \dot{\alpha} &= -V_e m_w \alpha - m_q q + c_{xy} r + e_y p^2 - b_y p r + \\
 &- m_{\xi}^2 \alpha \xi^2 - m_{\xi}^2 \xi^2 - m_{\eta} \eta
 \end{aligned}$$

$$\begin{aligned}
\dot{\beta} + e_z \dot{p} = & -V_e n_v \beta - (n_p - b_z q_e) p + c_{xz} q - n_r r - V_e n_{va} \beta \alpha + \\
& - n_{pa} ap - n_{ra} ar - b_z pq - n_{\xi\alpha} \alpha \xi + \\
& - n_{\xi} \xi - n_{\zeta} \zeta - \\
\dot{\phi} = & p
\end{aligned}$$

For parameter identification purposes it is convenient to express the state and output equations of an aircraft as

$$\begin{aligned}
T\dot{x} &= A'x + B'u^* , \quad x(0) = \alpha \\
y &= Cx + Du^*
\end{aligned} \tag{B.4}$$

where the state vector and the augmented input vector are defined as

$$\begin{aligned}
x^T &= \{\beta, \alpha, p, q, r, \phi\} \\
u^{*T} &= \{\beta\alpha, \beta p, ap, ar, p^2, pq, pr, qr, \alpha\xi^2, \\
&\quad \alpha\xi, \xi^2, \xi, \zeta, \eta, 1\} \\
y^T &= \{x^T, n_y^*, n_z^*\}
\end{aligned} \tag{B.5}$$

and the matrix T as

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & e_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & e_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} -\frac{1}{C'} \end{bmatrix} , \quad D = \begin{bmatrix} -\frac{0}{D'} \end{bmatrix}$$

The matrices A' , B' , C' and D' are presented on page 61. They also include the unknown bias terms Y_o , Z_o , ..., N_o . The expressions for the coefficients in the above mentioned matrices are as follows:

$$A' = \begin{bmatrix} Y_{\beta} & 0 & Y_p & 0 & -1 & Y_{\phi} \\ 0 & Z_{\alpha} & 0 & 1 & 0 & Z_{\phi} \\ L_{\beta} & 0 & L_p & 0 & L_r & 0 \\ 0 & M_{\alpha} & 0 & M_q & M_r & M_{\phi} \\ N_{\beta} & 0 & N_p & 0 & N_r & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_{\xi} & Y_{\zeta} & 0 & Y_o \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_{\xi^2} & 0 & 0 & Z_{\eta} & Z_o \\ L_{\beta\alpha} & 0 & L_{p\alpha} & L_{r\alpha} & 0 & E_x & 0 & B_x & 0 & L_{\xi\alpha} & 0 & L_{\xi} & L_{\zeta} & L_{\eta} & L_o \\ 0 & M_{\alpha} & 0 & 0 & E_y & 0 & B_y & 0 & M_{\xi^2\alpha} & 0 & M_{\xi^2} & 0 & 0 & M_{\eta} & M_o \\ N_{\beta\alpha} & 0 & N_{p\alpha} & N_{r\alpha} & 0 & B_z & 0 & E_z & 0 & N_{\xi\alpha} & 0 & N_{\xi} & N_{\zeta} & 0 & N_o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C' = \begin{bmatrix} Y_{\beta} & 0 & Y'_p & 0 & 0 & 0 \\ 0 & Z_{\alpha} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_{\xi} & Y_{\zeta} & 0 & Y_o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_{\xi^2} & 0 & 0 & Z_{\eta} & Z_o \end{bmatrix}$$

Matrices in Equations (B.4) Describing a Rapid Rolling Manoeuvre of an Aircraft

a) known constants

$$B_x = \frac{I_y - I_z}{I_x} = -b_x$$

$$E_x = \frac{I_{zx}}{I_x} = -e_x$$

$$Y_\phi = \frac{g}{V_e} \cos \theta_e \cos \phi_e$$

$$M_r = c_{xz}$$

$$B_y = \frac{I_z - I_x}{I_y} = -b_y$$

$$E_y = -\frac{I_{zx}}{I_x} = e_y$$

$$Z_\phi = -\frac{g}{V_e} \cos \theta_e \sin \phi_e$$

$$N_q = c_{xy}$$

$$B_z = \frac{I_x - I_y}{I_z} = -b_z$$

b) unknown parameters

$$Y_\beta = \frac{1}{\tau} C_{Y\beta}$$

$$Y_\xi = \frac{1}{\tau} C_{Y\xi}$$

$$Z_\alpha = \frac{1}{\tau} C_{z\alpha}$$

$$Z_\eta = \frac{1}{\tau} C_{z\eta}$$

$$L_\beta = \frac{\mu}{i_x \tau^2} C_{l\beta}$$

$$L_r = \frac{C_{lr}}{2i_x \tau} - b_x q_e$$

$$L_{pa} = \frac{C_{lpa}}{2i_x \tau}$$

$$L_{\xi\alpha} = \frac{\mu}{i_x \tau^2} C_{l\xi\alpha}$$

$$L_\zeta = \frac{\mu}{i_x \tau^2} C_{l\zeta}$$

$$M_\alpha = \frac{\mu}{i_y \tau^2} C_{m\alpha} + \frac{C_{z\alpha} C_{m\dot{\alpha}}}{2i_y \tau^2}$$

$$Y_p = \alpha_e + \frac{1}{2\mu} C_{Yp}$$

$$Y_\zeta = \frac{1}{\tau} C_{Y\zeta}$$

$$Z_\xi^2 = \frac{1}{\tau} C_{z\xi}^2$$

$$L_p = \frac{C_{lp}}{2i_x \tau}$$

$$L_{\beta\alpha} = \frac{\mu}{i_x \tau^2} C_{l\beta\alpha}$$

$$L_{ra} = \frac{C_{lra}}{2i_x \tau}$$

$$L_\xi = \frac{\mu}{i_x \tau^2} C_{l\xi}$$

$$L_\eta = \frac{\mu}{i_x \tau^2} C_{l\eta}$$

$$M_q = \frac{C_{mq} + C_{m\dot{q}}}{2i_y \tau}$$

$$M_{\phi} = \frac{C_{m\dot{\alpha}}}{2i_y \tau} \frac{g}{V_e} \cos \theta_e \sin \phi_e$$

$$M_{\dot{\alpha}} = - \frac{C_{m\dot{\alpha}}}{2i_y \tau}$$

$$M_{\xi 2\alpha} = \frac{\mu}{i_y \tau^2} C_{m\xi 2\alpha}$$

$$M_{\xi 2} = \frac{\mu}{i_y \tau^2} C_{m\xi \alpha} + \frac{C_{z\xi}^2 C_{m\dot{\alpha}}}{2i_y \tau}$$

$$M_{\eta} = \frac{\mu}{i_y \tau^2} C_{m\eta} + \frac{C_{z\eta} C_{m\dot{\alpha}}}{2i_y \tau}$$

$$N_{\beta} = \frac{\mu}{i_z \tau^2} C_{n\beta}$$

$$N_p = \frac{C_{np}}{2i_z \tau} + b_z q_e$$

$$N_r = \frac{C_{nr}}{2i_z \tau}$$

$$N_{\beta\alpha} = \frac{\mu}{i_z \tau^2} C_{n\beta\alpha}$$

$$N_{p\alpha} = \frac{C_{np\alpha}}{2i_z \tau}$$

$$N_{r\alpha} = \frac{C_{nr\alpha}}{2i_z \tau}$$

$$N_{\xi\alpha} = \frac{\mu}{i_z \tau^2} C_{n\xi\alpha}$$

$$N_{\xi} = \frac{\mu}{i_z \tau^2} C_{n\xi}$$

$$N_{\zeta} = \frac{\mu}{i_z \tau^2} C_{n\zeta}$$

$$\Psi'_p = \frac{C_{Yp}}{2\mu}$$

Equations (B.3), which describe a rapid rolling manoeuvre of an aircraft, can still be too complicated for the estimation of the unknown parameters, mainly because of their large number. To decrease this complexity a simplification of equations (B.3) is proposed based on splitting of them into two independent sets. The first set of equations represents a simplified model which describes the lateral motion only. The variables α and q are considered as additional inputs. Therefore, the state, augmented input and output vector are defined as

$$\mathbf{x}^T = \{\beta, p, r, \phi\}$$

$$\mathbf{u}^{*T} = \{\beta\alpha_E, p\alpha_E, r\alpha_E, pq_E, rq_E, \alpha_E \xi, \xi, \zeta, \eta, 1\}$$

(B.6)

$$\mathbf{y}^T = \{\mathbf{x}^T, \mathbf{n}^*\}$$

where the index E denotes measured quantities. For the simplified model the matrices A', B', C' and D' in (B.4) are expressed as

$$A' = \begin{bmatrix} Y_\beta & Y_p & -1 & Y_\phi \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad B' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & Y_\xi & Y_\zeta & 0 & Y_o \\ L_{\beta\alpha} & L_{p\alpha} & L_{r\alpha} & E_x & B_x & L_{\xi\alpha} & L_\xi & L_\zeta & L_\eta & L_o \\ N_{\beta\alpha} & N_{p\alpha} & N_{r\alpha} & B_z & E_z & N_{\xi\alpha} & N_\xi & N_\zeta & 0 & N_o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C' = \{Y_\beta, Y_p, 0, 0\} \quad D' = \{0, 0, 0, 0, 0, 0, Y_\xi, Y_\zeta, 0, Y_o\}$$

The matrix T is now equal to

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e_x & 0 \\ 0 & e_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the second simplified model, describing the longitudinal motion only, the input variable β_E , p_E and r_E are added.

$$x^T = \{\alpha, q\}$$

$$u^{*T} = \{\beta_E p_E, p_E^2, p_E r_E, \alpha \xi^2, \xi^2, \eta, 1\}$$

$$y^T = \{x^T, n_z^*\}$$

and the matrices in (B.4) takes the form

$$A' = \begin{bmatrix} Z_\alpha & 1 \\ M_\alpha & M_q \end{bmatrix} \quad B' = \begin{bmatrix} -1 & 0 & 0 & 0 & Z_{\xi^2} & Z_\eta & Z_o \\ M_d & E_y & B_y & M_{\xi^2\alpha} & M_{\xi^2} & M_\eta & M_o \end{bmatrix}$$

$$C' = \{Z_\alpha, 0\} \quad D' = \{0, 0, 0, 0, Z_{\xi^2}, Z_\eta, Z_o\}$$

The matrix T is now equal to the unit matrix.

METHOD	ESTIMATION	BASIC ASSUMPTIONS	A PRIORI KNOWLEDGE	COST FUNCTION	ESTIMATE
Equation Error Method	Regression Analysis	$z = x_E = x$	-	$\frac{1}{2} \sum_{i=1}^N \ \dot{x}_{Ei} - \dot{x}_i \ ^2_{W_1}$	$\gamma = \left\{ \sum_{i=1}^N x_i x_i^T \right\} \sum_{i=1}^N x_i \dot{x}_{Ei}$
Output Error Method	Weighted Least Squares	-	-	$\frac{1}{2} \sum_{i=1}^N \ z_i - \gamma_i \ ^2_{W_1}$	$\Delta \gamma = \left\{ \sum_{i=1}^N H_i^T W_1 H_i \right\} \sum_{i=1}^N H_i^T W_1 (z_i - \gamma_i)$
	Maximum Likelihood	$p(z \gamma)$ gaussian	-	$\frac{1}{2} \sum_{i=1}^N \ z_i - \gamma_i \ ^2_{R_1} + \frac{N \ell n}{2} R_1 $	$\Delta \gamma = \left\{ \sum_{i=1}^N H_i^T R_1^{-1} H_i \right\} \sum_{i=1}^N H_i^T R_1^{-1} (z_i - \gamma_i)$
	Bayesian	$p(z \gamma)$ and $p(\gamma)$ gaussian	$\gamma_0 = E\{\gamma\}$ $R_2 = E\{(\gamma - \gamma_0)(\gamma - \gamma_0)^T\}$	$\frac{1}{2} \sum_{i=1}^N \ z_i - \gamma_i \ ^2_{R_1} + \frac{1}{2} \ \gamma - \gamma_0 \ ^2_{R_2} + \frac{N \ell n}{2} R_1 $	$\Delta \gamma = \left\{ \sum_{i=1}^N H_i^T R_1^{-1} H_i + R_2^{-1} \right\}^{-1} \sum_{i=1}^N H_i^T R_1^{-1} (z_i - \gamma_i) - R_2^{-1} (\gamma - \gamma_0)$

TABLE I Basic Assumptions, Cost Functions, and Parameter Estimates of various Estimation Techniques

ITEM	FLIGHT 6		FLIGHT 9	FLIGHT 11			MEAN VALUE
	RUN 3	RUN 4	RUN 5	RUN 1	RUN 3	RUN 4	
Z_{α}	-1.32 ± 0.02	-1.38 ± 0.02	-1.38 ± 0.02	-1.46 ± 0.04	-1.32 ± 0.03	-1.21 ± 0.02	-1.39 ± 0.04
Z_{η}	-0.15 ± 0.02	-0.17 ± 0.02	-0.07 ± 0.02	-0.19 ± 0.03	-0.07 ± 0.03	0.00 ± 0.02	-0.11 ± 0.04
M_{α}	-9.4 ± 0.2	-9.7 ± 0.1	-9.2 ± 0.1	-10.1 ± 0.3	-9.2 ± 0.2	-8.6 ± 0.2	-9.4 ± 0.2
M_q	-3.1 ± 0.1	-3.7 ± 0.1	-3.3 ± 0.1	-3.3 ± 0.2	-3.5 ± 0.2	-3.7 ± 0.2	-3.4 ± 0.1
M_{η}	-9.9 ± 0.3	-11.4 ± 0.3	-13.9 ± 0.3	-11.5 ± 0.6	-10.8 ± 0.5	-10.4 ± 0.4	-11.3 ± 0.6
$S^2(\alpha)$	8.9×10^{-6}	1.6×10^{-6}	7.3×10^{-6}	4.9×10^{-5}	6.5×10^{-6}	6.4×10^{-6}	-
$S^2(q)$	1.6×10^{-4}	1.3×10^{-4}	2.0×10^{-5}	2.5×10^{-4}	6.7×10^{-5}	4.7×10^{-5}	-
$S^2(n_z^x)$	6.8×10^{-5}	3.2×10^{-4}	6.8×10^{-5}	4.1×10^{-4}	8.9×10^{-4}	3.8×10^{-4}	-
$\ln R_1 $	-29.9	-28.0	-34.5	-27.2	-28.6	-29.7	-

TABLE 2 Comparison of Results from Repeated Flight Measurements by ML Estimation. BASSET Aircraft

ITEM	RUN 11.12 ($C_{m\alpha}$)		RUN 11.05 ($C_{m\alpha}$)		RUN 9.30 ($C_{m\alpha}$)	
	PRED.	ESTIMATE	PRED.	ESTIMATE	PRED.	ESTIMATE
Z_α	- 1.39	- 1.27 $\pm$ 0.02	- 1.39	- 1.36 $\pm$ 0.04	- 1.39	- 1.38 $\pm$ 0.03
Z_η	- 0.11	- 0.02 $\pm$ 0.02	- 0.11	- 0.04 $\pm$ 0.02	- 0.11	- 0.13 $\pm$ 0.03
Z_o	0	- 0.0029	0	0.0064	0	0.0018
Z_o^1	0	0.0039	0	0.0047	0	0.0020
M_α	3.35	- 1.0*** $\pm$ 0.1	-15.8	-15.2 $\pm$ 0.2	- 9.4	- 9.9 $\pm$ 0.3
M_q	- 3.4	- 2.9*** $\pm$ 0.1	- 3.4	- 2.3* $\pm$ 0.1	- 4.2	- 4.0* $\pm$ 0.2
M_η	-11.3	- 8.6* $\pm$ 0.2	-11.3	- 9.0* $\pm$ 0.2	-11.3	-13.9* $\pm$ 0.6
M_o	0	0.015	0	- 0.019	0	- 0.0016
$S^2(\alpha)$	-	2.3 $\times 10^{-6}$	-	5.7 $\times 10^{-6}$	-	1.3 $\times 10^{-6}$
$S^2(q)$	-	5.4 $\times 10^{-5}$	-	8.4 $\times 10^{-5}$	-	2.0 $\times 10^{-5}$
$S^2(n_z^x)$	-	1.2 $\times 10^{-5}$	-	1.7 $\times 10^{-5}$	-	3.3 $\times 10^{-6}$
$\ln R_1 $	-	-34.1	-	-32.4	-	-37.0

*** strong correlation

TABLE 3 Comparison of Predicted and Estimated Parameters from Flight Data by ML Estimation. BASSET Variable Stability Aircraft

ITEM	RUN 9.37 (C_{mq})		RUN 9.09 (C_{mq})		RUN 9.08 (C_{mq})	
	PRED.	ESTIMATE	PRED.	ESTIMATE	PRED.	ESTIMATE
Z_α	- 1.39	- 1.34* $\pm$ 0.02	- 1.39	- 1.21 $\pm$ 0.03	- 1.39	- 1.25 $\pm$ 0.02
Z_η	- 0.11	- 0.01 $\pm$ 0.03	- 0.11	0.00 $\pm$ 0.02	- 0.11	- 0.04 $\pm$ 0.05
Z_o	0	0.0040	0	0.0020	0	0.0057
Z_o^1	0	0.0025	0	0.0040	0	0.0034
M_α	- 9.4	- 9.96 $\pm$ 0.06	- 9.9	- 9.9* $\pm$ 0.3	- 8.9	- 8.42 $\pm$ 0.05
M_q	0.45	0.98* $\pm$ 0.02	- 6.4	- 6.7* $\pm$ 0.4	0.42	0.13 $\pm$ 0.03
M_η	-11.3	- 9.8 $\pm$ 0.1	-11.3	-13.4* $\pm$ 0.7	-11.3	-14.3 $\pm$ 0.2
M_o	0	0.012	0	- 0.023	0	- 0.035
$S^2(\alpha)$	-	1.7 $\times 10^{-6}$	-	8.3 $\times 10^{-7}$	-	7.6 $\times 10^{-6}$
$S^2(q)$	-	1.7 $\times 10^{-5}$	-	1.7 $\times 10^{-5}$	-	8.4 $\times 10^{-5}$
$S^2(n_z^x)$	-	3.7 $\times 10^{-6}$	-	5.6 $\times 10^{-6}$	-	1.1 $\times 10^{-5}$
$\ln R_1 $	-	-36.8	-	-37.1	-	-32.6

TABLE 3 Comparison of Predicted and Estimated Parameters from Flight Data by ML Estimation. BASSET Variable Stability Aircraft — Concluded

ITEM	FLIGHT 6	FLIGHT 10			FLIGHT 12			MEAN VALUE
	RUN 2	RUN 2			RUN 2	RUN 4	RUN 5	
L_p	-3.40 ± 0.04	-3.41 ± 0.06			-3.28 ± 0.05	-3.2 ± 0.1	-3.39 ± 0.07	-3.34 ± 0.04
L_r	0.67 ± 0.05	0.53 ± 0.05			0.25 ± 0.06	0.9 ± 0.1	0.50 ± 0.06	0.6 ± 0.1
L_β	-6.58 ± 0.09	-6.54 ± 0.09			-5.8 ± 0.1	-7.2 ± 0.2	-6.1 ± 0.1	-6.4 ± 0.3
L_ξ	-9.34 ± 0.09	-9.3 ± 0.1			-9.8 ± 0.1	-7.9 ± 0.3	-10.0 ± 0.2	-9.3 ± 0.4
L_ζ	0.78 ± 0.07	0.4 ± 0.1			0.50 ± 0.08	1.4 ± 0.1	0.50 ± 0.07	0.7 ± 0.2
N_p	-0.52 ± 0.01	-0.34 ± 0.02			-0.52 ± 0.02	-0.49 ± 0.04	-0.58 ± 0.01	-0.49 ± 0.06
N_r	-0.63 ± 0.02	-0.51 ± 0.02			-0.25 ± 0.03	-0.36 ± 0.05	-0.39 ± 0.02	-0.43 ± 0.06
N_β	3.45 ± 0.02	3.45 ± 0.02			3.28 ± 0.04	3.37 ± 0.07	3.08 ± 0.02	3.33 ± 0.07
N_ξ	-0.42 ± 0.02	-0.14 ± 0.05			-0.48 ± 0.05	-0.57 ± 0.06	-0.74 ± 0.04	-0.47 ± 0.01
N_ζ	-1.66 ± 0.02	-1.67 ± 0.03			-1.61 ± 0.03	-1.81 ± 0.02	-1.56 ± 0.01	-1.66 ± 0.01
Y_β	-0.17 ± 0.01	-0.23 ± 0.02			-0.38 ± 0.02	-0.25 ± 0.05	-0.32 ± 0.02	-0.27 ± 0.04
Y_ξ	0.008 ± 0.007	0.04 ± 0.01			0.06 ± 0.01	0.08 ± 0.02	-0.06 ± 0.01	0.03 ± 0.08
$S^2(p)$	2.0×10^{-5}	1.8×10^{-5}			1.7×10^{-5}	2.9×10^{-5}	1.1×10^{-5}	---
$S^2(r)$	9.3×10^{-6}	8.0×10^{-6}			1.5×10^{-6}	2.2×10^{-6}	4.3×10^{-7}	---
$S^2(\beta)$	2.8×10^{-6}	3.9×10^{-6}			1.1×10^{-5}	1.7×10^{-5}	1.3×10^{-5}	---
$\ln R_1 $	-35.2	-35.1			-35.9	-34.4	-37.3	---

TABLE 4 Comparison of Results from Repeated Flight Measurements by ML Estimation. BASSET Aircraft

ITEM	A PRIORI	REGRESSION ANALYSIS	WLS ESTIMATE	ML ESTIMATE		BAYESIAN ESTIMATE
				CASE A	CASE B	
L_p	-3.30 ± 0.08	-11.3 ± 0.2	-7.4 ± 1.5	-5.6 ± 1.1	-3.5 ± 0.1	-3.31 ± 0.08
L_r	0.6 ± 0.3	1.25 ± 0.05	-0.3 ± 0.4	0.1 ± 0.3	0.6	0.64 ± 0.04
L_β	-6.4 ± 0.7	-2.4 ± 0.3	-11 ± 2.1	-9 ± 1.6	-5.7 ± 0.2	-5.4 ± 0.1
L_ζ	0.8 ± 0.5	0.38 ± 0.03	1.0 ± 0.1	0.7 ± 0.1	0.56 ± 0.06	0.51 ± 0.05
N_p	-0.53 ± 0.05	0.3 ± 0.1	4.4 ± 0.9	3.2 ± 0.6	-0.53	-0.44 ± 0.04
N_r	-0.33 ± 0.07	-0.17 ± 0.04	0.9 ± 0.2	0.6 ± 0.2	-0.40 ± 0.02	-0.35 ± 0.02
N_β	3.3 ± 0.2	4.2 ± 0.2	10 ± 1.3	8.3 ± 0.9	2.88 ± 0.01	3.01 ± 0.06
N_ζ	-1.7 ± 0.1	-1.78 ± 0.02	-1.92 ± 0.07	-1.81 ± 0.04	-1.76 ± 0.01	-1.58 ± 0.01
Y_β	-0.32 ± 0.06	-0.24 ± 0.02	-0.23 ± 0.01	-0.22 ± 0.02	-0.24 ± 0.02	-0.26 ± 0.02
Y_ζ	0.07 ± 0.02	-0.04 ± 0.01	0.055 ± 0.008	0.054 ± 0.008	0.040 ± 0.001	0.046 ± 0.008
$S^2(p)$	-	-	2.3×10^{-5}	2.2×10^{-5}	2.3×10^{-5}	2.2×10^{-5}
$S^2(r)$	-	-	8.4×10^{-6}	6.6×10^{-6}	9.5×10^{-6}	9.8×10^{-6}
$S^2(\beta)$	-	-	7.4×10^{-6}	8.8×10^{-6}	1.1×10^{-5}	1.1×10^{-5}
$\ln R_1 $	-	-	-34.2	-34.3	-33.7	-33.7

TABLE 5 Comparison of Results from Flight Data using various Estimates.
BASSET Aircraft. (Rudder Input Only)

ITEM	RUN 12.06 (C_{1p})		RUN 12.18 (C_{1r})		RUN 10.07 ($C_{1\beta}$)	
	PRED.	ESTIMATE	PRED.	ESTIMATE	PRED.	ESTIMATE
L_p	-0.82	- 0.9 $\pm$ 0.1	-3.34	- 4.1 $\pm$ 0.01	-3.34	- 3.7 $\pm$ 0.1
L_r	0.6	- 0.2 $\pm$ 0.3	-1.2	- 1.2 $\pm$ 0.1	0.6	1.09 $\pm$ 0.08
L_β	-6.4	- 6.3 $\pm$ 0.3	-6.6	- 6.4 $\pm$ 0.3	-1.4	- 1.4 $\pm$ 0.2
L_ξ	-9.3	-10.5 $\pm$ 0.1	-9.3	-11.2 $\pm$ 0.4	-9.3	-14.9 $\pm$ 0.3
L_ζ	0.7	0.2 $\pm$ 0.2	0.7	0.7 $\pm$ 0.2	0.7	1.1 $\pm$ 0.2
L_o	0	- 0.019	0	- 0.032	0	- 0.016
N_p	-0.42	- 0.33 $\pm$ 0.04	-0.49	0.07 $\pm$ 0.02	-0.49	- 0.72 $\pm$ 0.02
N_r	-0.43	- 0.61 $\pm$ 0.1	-0.26	- 0.76 $\pm$ 0.04	-0.43	- 0.39 $\pm$ 0.03
N_β	3.33	3.1 $\pm$ 0.1	3.33	4.61 $\pm$ 0.05	4.35	4.48 $\pm$ 0.05
N_ξ	-0.47	0.09 $\pm$ 0.06	-0.47	1.04 $\pm$ 0.04	-0.47	- 0.89 $\pm$ 0.08
N_ζ	-1.66	- 1.62 $\pm$ 0.08	-1.66	- 2.08 $\pm$ 0.03	-1.66	- 2.29 $\pm$ 0.04
N_o	0	- 0.0025	0	0.011	0	- 0.0077
Y_β	-0.27	- 0.09 $\pm$ 0.04	-0.27	0.04 $\pm$ 0.03	-0.27	- 0.38 $\pm$ 0.01
Y_ζ	0.03	- 0.06 $\pm$ 0.01	0.03	- 0.03 $\pm$ 0.01	0.03	0.04 $\pm$ 0.01
Y_o	0	0.0023	0	- 0.0067	0	0.0034
$S^2(p)$	-	1.6 $\times 10^{-5}$	-	1.9 $\times 10^{-5}$	-	1.7 $\times 10^{-3}$
$S^2(r)$	-	8.7 $\times 10^{-7}$	-	6.8 $\times 10^{-6}$	-	7.2 $\times 10^{-4}$
$S^2(\beta)$	-	1.1 $\times 10^{-5}$	-	2.8 $\times 10^{-5}$	-	4.6 $\times 10^{-4}$
J	-	0.35	-	0.90	-	0.32

TABLE 6 Comparison of Predicted and Estimated Parameters from Flight Data by WLS Estimation. BASSET Variable Stability Aircraft

ITEM	RUN 10.34 (C_{np})		RUN 10.25 (C_{nr})		RUN 10.19 ($C_{n\beta}$)	
	PRED.	ESTIMATE	PRED.	ESTIMATE	PRED.	ESTIMATE
L_p	-3.26	- 3.12 $\pm$ 0.07	-3.34	- 3.67 $\pm$ 0.07	-3.34	- 3.27 $\pm$ 0.05
L_r	0.6	- 0.8 $\pm$ 0.2	0.4	0.19 $\pm$ 0.04	0.6	0.41 $\pm$ 0.06
L_β	-6.4	- 6.6 $\pm$ 0.4	-6.4	- 6.9 $\pm$ 0.1	-6.0	- 5.75 $\pm$ 0.05
L_ξ	-9.3	-11.6 $\pm$ 0.3	-9.3	-17.9 $\pm$ 0.3	-9.3	-13.0 $\pm$ 0.2
L_ζ	0.7	0.3 $\pm$ 0.3	0.7	0.4 $\pm$ 0.03	0.7	0.6 $\pm$ 0.1
L_o	0	0.040	0	0.0095	0	- 0.024
N_p	0.22	0.01 $\pm$ 0.02	-0.49	- 0.44 $\pm$ 0.02	-0.49	- 0.31 $\pm$ 0.01
N_r	-0.43	0.31 $\pm$ 0.04	0.38	0.53 $\pm$ 0.02	-0.43	- 0.59 $\pm$ 0.02
N_β	3.33	2.9 $\pm$ 0.1	3.43	3.38 $\pm$ 0.03	1.50	1.544 $\pm$ 0.009
N_ξ	-0.47	- 1.18 $\pm$ 0.07	-0.47	- 0.45 $\pm$ 0.08	-0.47	- 0.031 $\pm$ 0.04
N_ζ	-1.66	- 1.45 $\pm$ 0.07	-1.66	- 2.12 $\pm$ 0.04	-1.66	- 1.95 $\pm$ 0.03
N_o	0	- 0.020	0	0.023	0	0.012
Y_β	-0.27	- 0.07 $\pm$ 0.03	-0.27	- 0.33 $\pm$ 0.02	-0.27	- 0.213 $\pm$ 0.007
Y_ζ	0.03	- 0.23 $\pm$ 0.03	0.03	0.06 $\pm$ 0.02	0.03	0.16 $\pm$ 0.02
Y_o	0	0.012	0	0.00028	0	0.0025
$S^2(p)$	-	2.9 $\times 10^{-5}$	-	3.4 $\times 10^{-5}$	-	2.0 $\times 10^{-3}$
$S^2(r)$	-	1.3 $\times 10^{-5}$	-	4.5 $\times 10^{-6}$	-	3.3 $\times 10^{-4}$
$S^2(\beta)$	-	2.7 $\times 10^{-5}$	-	5.1 $\times 10^{-6}$	-	1.4 $\times 10^{-3}$
J	-	1.23	-	0.56	-	0.33

TABLE 6 Comparison of Predicted and Estimated Parameters from Flight Data by WLS Estimation. BASSET Variable Stability Aircraft - Concluded

RUN	12.06	12.18	10.07	10.34	10.25	10.19
CHANGE IN	C_{1p}	C_{1r}	$C_{1\beta}$	C_{np}	C_{nr}	$C_{n\beta}$
STRONG CORRELATION BETWEEN	L_r, L_p $L_\beta(L_r, L_p)$ N_r, N_p $N_\beta(N_r, N_p)$ $L_\zeta(L_p, L_r)$ $N_\zeta(N_p, N_r)$ Y_ζ, Y_β	L_ξ, L_p Y_β, N_r	L_ξ, L_r N_ξ, N_p	Y_β, N_p N_ζ, N_β	L_β, L_p N_β, N_p Y_β, N_p $L_\xi(L_p, L_\beta)$ L_o, L_ζ N_ξ, N_p	L_ξ, L_p N_ξ, N_p Y_ζ, Y_o

TABLE 7 Parameters with Strong Correlation. BASSET Variable Stability Aircraft

ITEM	CASE						
	A BASIC MODEL	B NO BIAS TERMS	C Z_q AS UNKNOWN PARAMETER	D A PRIORI WEIGHTING OF Z_α	E α AND q MEASURED	F q AND n_z MEASURED	G α AND n_z MEASURED
Z_α	- 1.43 $\pm$ 0.02	- 1.41 $\pm$ 0.02	- 1.44 $\pm$ 0.02	- 1.35 $\pm$ 0.009	- 1.45 $\pm$ 0.03	- 1.41 $\pm$ 0.02	- 1.44 $\pm$ 0.02
Z_q	1.0	1.0	0.98 $\pm$ 0.01	1.0	1.0	1.0	1.0
Z_η	- 0.14 $\pm$ 0.02	- 0.14 $\pm$ 0.02	- 0.15 $\pm$ 0.02	- 0.11 $\pm$ 0.02	- 0.07 $\pm$ 0.04	- 0.16 $\pm$ 0.02	- 0.18 $\pm$ 0.02
Z_o	- 0.00021	0	0.00010	- 0.00099	0.00065	- 0.0022**	0.0061
Z_o^1	0.0037	0	0.0037	0.0030	-	0.0020**	0.0038
M_α	- 6.12 $\pm$ 0.05	- 6.02 $\pm$ 0.06	- 6.21 $\pm$ 0.08	- 6.03 $\pm$ 0.05	- 6.21 $\pm$ 0.07	- 6.02 $\pm$ 0.07	- 6.07 $\pm$ 0.06
M_q	- 1.78* $\pm$ 0.04	- 1.86 $\pm$ 0.04	- 1.78 $\pm$ 0.04	- 1.86* $\pm$ 0.04	- 1.82 $\pm$ 0.05	- 1.88* $\pm$ 0.04	- 1.30* $\pm$ 0.06
M_η	-11.9* $\pm$ 0.1	-12.2 $\pm$ 0.1	-11.9 $\pm$ 0.1	-12.0* $\pm$ 0.1	-12.0 $\pm$ 0.1	-12.3* $\pm$ 0.1	-10.1* $\pm$ 0.2
M_o	0.0084	0	0.0088	0.0075	0.0092	- 0.000020**	0.0047
$S^2(\alpha)$	2.9 $\times 10^{-6}$	3.6 $\times 10^{-6}$	2.9 $\times 10^{-6}$	3.2 $\times 10^{-6}$	3.0 $\times 10^{-6}$	-	2.1 $\times 10^{-6}$
$S^2(q)$	5.7 $\times 10^{-6}$	7.7 $\times 10^{-6}$	5.4 $\times 10^{-6}$	5.9 $\times 10^{-6}$	4.9 $\times 10^{-6}$	5.0 $\times 10^{-6}$	-
$S^2(n_z^x)$	5.9 $\times 10^{-6}$	1.2 $\times 10^{-5}$	5.9 $\times 10^{-6}$	6.4 $\times 10^{-6}$	-	5.9 $\times 10^{-6}$	4.6 $\times 10^{-6}$
$\rho_n R_1 $	-36.9	-35.6	-36.9	-36.6	-	-	-

*** strong correlation

TABLE 8 Effect of various Model Forms on Parameters Determined from Flight Data using ML Estimation. PARIS Aircraft

ITEM	PREDICTED FROM		EVALUATED FROM		
	CASE A	CASE D	α_E	q_E	n_{zE}^x
K_0	8.66	8.54	$7.4^* \pm 0.2$	$9.0^* \pm 0.2$	$8.4^* \pm 0.3$
K_1	3.21	3.21	$2.53^* \pm 0.08$	3.33 ± 0.04	$2.9^* \pm 0.1$
K_{q0}	-16.1	-15.5	-	$-17.7^* \pm 0.6$	$-15.8^* \pm 0.6$
K_{q1}	-11.9	-12.0	-	-12.2 ± 0.2	-
$K_{\alpha 0}$	-11.6	-11.8	$-9.5^* \pm 0.3$	-	-
$S^2()$	-	-	1.8×10^{-6}	4.7×10^{-6}	6.2×10^{-6}

* strong correlation

TABLE 9 Comparison of Predicted and Estimated Parameters from Flight Data by ML Estimation. PARIS Aircraft

ITEM	RUN 1.4	RUN 2.8	RUN 2.9	RUN 2.11
Z_α	- 1.370 $\pm$ 0.005	- 1.43 $\pm$ 0.02	- 1.34 $\pm$ 0.01	- 1.435 $\pm$ 0.006
Z_η	- 0.169 $\pm$ 0.005	- 0.14 $\pm$ 0.02	- 0.16 $\pm$ 0.01	- 0.158 $\pm$ 0.007
Z_o	- 0.0019	- 0.00021	0.0049	- 0.0071
Z_o^1	- 0.00054	0.0037	- 0.00012	0.0010
M_α	- 5.94 $\pm$ 0.02	- 6.12 $\pm$ 0.05	- 5.65 $\pm$ 0.06	- 6.53 $\pm$ 0.02
M_q	- 1.75 $\pm$ 0.01	- 1.78* $\pm$ 0.04	- 1.95* $\pm$ 0.03	- 1.80* $\pm$ 0.02
M_η	-11.59 $\pm$ 0.04	-11.9* $\pm$ 0.1	-12.44* $\pm$ 0.09	-12.07* $\pm$ 0.06
M_o	0.0027	0.0084	0.0085	0.0019
$S^2(\alpha)$	5.7 $\times 10^{-7}$	2.9 $\times 10^{-6}$	9.0 $\times 10^{-6}$	1.7 $\times 10^{-6}$
$S^2(q)$	8.2 $\times 10^{-6}$	5.7 $\times 10^{-6}$	8.3 $\times 10^{-6}$	1.5 $\times 10^{-5}$
$S^2(n_z^x)$	2.1 $\times 10^{-6}$	5.9 $\times 10^{-6}$	6.8 $\times 10^{-6}$	8.9 $\times 10^{-6}$
$\ln R_1 $	-39.2	-36.9	-35.2	-36.0
$\text{Tr}(\bar{M}^{-1})$	2.20 $\times 10^{-3}$	2.14 $\times 10^{-2}$	1.30 $\times 10^{-2}$	4.60 $\times 10^{-3}$

* strong correlation

TABLE 10 Effect of various Input Forms on Parameters Determined from Flight Data using ML Estimation. PARIS Aircraft

ITEM	RUN 2.4			
	BASIC MODEL	EXTENDED MODEL	ϕ IN MEASURED VARIABLES	n_y EXCLUDED
L_p	- 4.03* $\pm$ 0.06	- 4.07** $\pm$ 0.06	- 4.52* $\pm$ 0.06	- 4.15 $\pm$ 0.08
L_r	0.79 $\pm$ 0.08	1.01 $\pm$ 0.07	0.58 $\pm$ 0.07	0.65 $\pm$ 0.09
L_β	-13.3 $\pm$ 0.1	-13.5 $\pm$ 0.1	-14.2 $\pm$ 0.1	-13.6 $\pm$ 0.2
L_ξ	-32.8* $\pm$ 0.4	-33.6** $\pm$ 0.4	-36.2* $\pm$ 0.4	-34.6 $\pm$ 0.5
L_ζ	1.5 $\pm$ 0.2	1.8 $\pm$ 0.2	1.1 $\pm$ 0.2	1.7 $\pm$ 0.2
L_o	0.0077	0.0076	0.012	0.032
N_p	- 0.39** $\pm$ 0.01	- 0.28*** $\pm$ 0.01	- 0.46** $\pm$ 0.01	- 0.29 $\pm$ 0.01
N_r	- 0.40 $\pm$ 0.01	- 0.41 $\pm$ 0.01	- 0.42 $\pm$ 0.01	- 0.31 $\pm$ 0.01
N_β	3.38 $\pm$ 0.02	3.70* $\pm$ 0.03	3.27 $\pm$ 0.02	3.73 $\pm$ 0.02
N_ξ	- 1.45** $\pm$ 0.08	- 0.56*** $\pm$ 0.08	- 1.95** $\pm$ 0.07	- 0.49 $\pm$ 0.09
N_ζ	- 3.11 $\pm$ 0.03	- 3.03 $\pm$ 0.02	- 3.16 $\pm$ 0.03	- 2.95 $\pm$ 0.03
N_o	0.00097	0.0030	0.00099	0.0030
Y_p	0.04276	0.013 $\pm$ 0.001	0.04276	0.04276
Y_β	- 0.120 $\pm$ 0.002	- 0.124 $\pm$ 0.002	- 0.120 $\pm$ 0.002	- 0.26 $\pm$ 0.02
Y_ξ	0	- 0.017 $\pm$ 0.003	0	0
Y_ζ	0.037 $\pm$ 0.003	0.044 $\pm$ 0.003	0.039 $\pm$ 0.003	0.186 $\pm$ 0.009
Y_o	- 0.0062	- 0.0050	- 0.0068	0.00076
Y_o^1	- 0.00074	- 0.00055	- 0.00069	-
$S^2(p)$	2.1 $\times 10^{-5}$	2.1 $\times 10^{-5}$	2.8 $\times 10^{-5}$	2.7 $\times 10^{-5}$
$S^2(r)$	1.8 $\times 10^{-5}$	4.0 $\times 10^{-6}$	2.0 $\times 10^{-5}$	1.6 $\times 10^{-5}$
$S^2(\beta)$	1.2 $\times 10^{-6}$	1.3 $\times 10^{-6}$	1.2 $\times 10^{-6}$	4.3 $\times 10^{-7}$
$S^2(n_y^x)$	3.5 $\times 10^{-7}$	3.2 $\times 10^{-7}$	6.2 $\times 10^{-7}$	-
$\ln R_1 $	-50.0	-51.7	-	-

***** strong correlation

TABLE II Effect of various Model Forms on Parameters Determined from Flight Data using ML Estimation PARIS Aircraft

ITEM	RUN 2.4	RUN 3.4B	RUN 3.6G
L_p	- 4.03* $\pm$ 0.06	- 4.20* $\pm$ 0.08	- 4.09* $\pm$ 0.04
L_r	0.79 $\pm$ 0.08	0.9 $\pm$ 0.1	0.36 $\pm$ 0.05
L_β	-13.3 $\pm$ 0.1	-12.4 $\pm$ 0.2	-13.3 $\pm$ 0.1
L_ξ	-32.8* $\pm$ 0.4	-37.6* $\pm$ 0.6	-34.0* $\pm$ 0.3
L_ζ	1.5 $\pm$ 0.2	0.5 $\pm$ 0.2	1.2 $\pm$ 0.2
L_o	0.0077	0.29	- 0.072
N_p	- 0.39** $\pm$ 0.01	- 0.55** $\pm$ 0.02	- 0.427** $\pm$ 0.007
N_r	- 0.40 $\pm$ 0.01	- 0.56 $\pm$ 0.02	- 0.415 $\pm$ 0.007
N_β	3.38 $\pm$ 0.02	3.29 $\pm$ 0.04	3.27 $\pm$ 0.01
N_ξ	- 1.45** $\pm$ 0.08	- 2.9** $\pm$ 0.2	- 1.79** $\pm$ 0.05
N_ζ	- 3.11 $\pm$ 0.03	- 4.37 $\pm$ 0.08	- 3.27 $\pm$ 0.03
N_o	0.00097	- 0.018	- 0.0019
Y_β	- 0.120 $\pm$ 0.002	- 0.118 $\pm$ 0.002	- 0.118 $\pm$ 0.001
Y_ζ	0.037 $\pm$ 0.003	0.046 $\pm$ 0.003	0.036 $\pm$ 0.002
Y_o	- 0.0062	0.0025	- 0.0011
Y_o^1	- 0.00074	- 0.00041	- 0.00044
$S^2(p)$	2.1 $\times 10^{-5}$	3.9 $\times 10^{-5}$	7.5 $\times 10^{-5}$
$S^2(r)$	1.8 $\times 10^{-5}$	9.0 $\times 10^{-5}$	8.1 $\times 10^{-5}$
$S^2(\beta)$	1.2 $\times 10^{-6}$	1.7 $\times 10^{-5}$	1.1 $\times 10^{-5}$
$S^2(n_y^x)$	3.5 $\times 10^{-7}$	4.7 $\times 10^{-7}$	5.6 $\times 10^{-7}$
$\ln R_1 $	-50.0	-45.0	-44.7
$Tr(M^{-1})$	0.214	0.498	0.115

*** strong correlation

TABLE 12 Effect of various Input Forms on Parameters Determined from Flight Data using ML Estimation. PARIS Aircraft

ITEM	TRUE VALUE	ESTIMATED VALUE	
		CASE A	CASE B
Y_{β}	- 0.20	- 0.201 $\pm$ 0.003	- 0.201 $\pm$ 0.003
L_{β}	-10	-15 $\pm$ 2	-15.3 $\pm$ 0.4
L_p	- 3.9	- 3.89 $\pm$ 0.02	- 3.89 $\pm$ 0.01
L_r	0.13	0.3 $\pm$ 0.2	0.28 $\pm$ 0.03
N_{β}	5.7	5.67 $\pm$ 0.02	5.68 $\pm$ 0.02
N_p	0.002	- 0.0006 $\pm$ 0.009	- 0.0006 $\pm$ 0.001
N_r	- 0.24	- 0.243 $\pm$ 0.004	- 0.243 $\pm$ 0.003
Y_{ξ}	0.0071	0.0068 $\pm$ 0.0004	0.0068 $\pm$ 0.0004
$L_{\beta\alpha}$	-680	-580 $\pm$ 44	-577 $\pm$ 8
$L_{r\alpha}$	8.4	6 $\pm$ 3.4	-
$L_{\xi\alpha}$	-32	-26 $\pm$ 7	-
L_{ξ}	-23	-23.2 $\pm$ 0.5	-23.20 $\pm$ 0.06
$N_{p\alpha}$	- 1.6	- 1.6 $\pm$ 0.2	-
$N_{\xi\alpha}$	1.1	1.1 $\pm$ 0.8	-
N_{ξ}	- 0.92	- 0.93 $\pm$ 0.05	-0.934 $\pm$ 0.007

TABLE 13 Comparison of True and Determined Values of Parameters
from Simulated Data using ML Estimation. F - 100
Aircraft - Run I.

ITEM	TRUE VALUE	ESTIMATED VALUE			
		CASE A	CASE B	CASE C	CASE D
Y_{β}	- 0.20	- 0.202 $\pm$ 0.003	- 0.202 $\pm$ 0.03	- 0.202 $\pm$ 0.003	- 0.201 $\pm$ 0.003
L_{β}	- 10.0	-19.0 $\pm$ 3	-19.0 $\pm$ 3	-19.0 $\pm$ 0.2	-10.2 $\pm$ 0.5
L_p	- 3.9	- 3.89 $\pm$ 0.03	- 3.89 $\pm$ 0.02	- 3.89 $\pm$ 0.01	- 3.87 $\pm$ 0.01
L_r	0.13	0.3 $\pm$ 0.2	0.34 $\pm$ 0.03	0.34 $\pm$ 0.03	0.13 $\pm$ 0.02
N_{β}	5.7	5.65 $\pm$ 0.05	5.65 $\pm$ 0.04	5.65 $\pm$ 0.02	5.71 $\pm$ 0.03
N_p	0.002	- 0.01 $\pm$ 0.01	- 0.014 $\pm$ 0.005	- 0.014 $\pm$ 0.001	0.0002 $\pm$ 0.0002
N_r	- 0.24	- 0.245 $\pm$ 0.004	- 0.245 $\pm$ 0.003	- 0.245 $\pm$ 0.003	- 0.240 $\pm$ 0.004
Y_{ξ}	0.0071	0.0080 $\pm$ 0.0008	0.0080 $\pm$ 0.0003	-	0.0079 $\pm$ 0.0008
Y_{ζ}	0.03	0.017 $\pm$ 0.008	-	0.017 $\pm$ 0.003	0.019 $\pm$ 0.008
$L_{\beta\alpha}$	-680	-504 $\pm$ 6	-504 $\pm$ 6	-	-667 $\pm$ 10
$L_{r\alpha}$	8.4	5 $\pm$ 4	-	-	7.5 $\pm$ 0.6
$L_{\xi\alpha}$	- 32	-27 $\pm$ 6	-27 $\pm$ 4	-	-37 $\pm$ 2
L_{ξ}	- 23	-23.2 $\pm$ 0.4	-23.2 $\pm$ 0.3	-23.22 $\pm$ 0.06	-22.6 $\pm$ 0.1
L_{ζ}	2.5	3.2 $\pm$ 0.8	3.2 $\pm$ 0.7	3.2 $\pm$ 0.6	3.1 $\pm$ 0.7
$N_{p\alpha}$	- 1.6	- 1.4 $\pm$ 0.2	- 1.37 $\pm$ 0.04	-	- 1.62 $\pm$ 0.03
$N_{\xi\alpha}$	1.1	2.4 $\pm$	-	-	1.4 $\pm$ 0.4
N_{ξ}	- 0.92	- 1.01 $\pm$ 0.06	- 1.01 $\pm$ 0.01	- 1.01 $\pm$ 0.01	- 0.94 $\pm$ 0.01
N_{ζ}	- 5.3	- 5.4 $\pm$ 0.1	- 5.41 $\pm$ 0.07	- 5.41 $\pm$ 0.06	- 5.32 $\pm$ 0.06

TABLE 14 Comparison of True and Determined Values of Parameters
from Simulated Data using ML Estimation. F-100
Aircraft. Run 2

TABLE 15 Review of Cases analysed together with Parameters with
Strong Correlation. F-100 Aircraft

RUN	CASE		STRONG CORRELATION BETWEEN
1	A	All unknown parameters estimated	$L_{\beta}, L_{\beta\alpha}$ $L_r, L_{r\alpha}$ $L_{\beta}, L_{\xi\alpha}$ $N_p, N_{p\alpha}$ $N_{\xi\alpha} (N_p, N_{p\alpha})$ $L_{\xi} (L_{\beta}, L_{\beta\alpha}, L_{\xi\alpha}, L_p)$ $N_{\xi} (N_p, N_{p\alpha}, N_{\xi\alpha})$
	B	$L_{r\alpha}, L_{\xi\alpha}, N_{p\alpha}, N_{\xi\alpha}$ fixed	$L_{\beta}, L_{\beta\alpha}$ L_{ξ}, L_p N_{ξ}, N_p
2	A	All unknown parameters estimated	Y_{ξ}, Y_{ζ} $L_{\beta}, L_{\beta\alpha}$ $L_r, L_{r\alpha}$ $L_{\xi}, L_{\xi\alpha}$ $N_p, N_{p\alpha}$ $N_{\xi} (N_p, N_{p\alpha})$ $N_{\xi} (N_p, N_{p\alpha}, N_{\xi\alpha})$
	B	$Y_{\zeta}, L_{r\alpha}, N_{\xi\alpha}$ fixed	$L_{\beta}, L_{\beta\alpha}$ L_{ξ}, L_{β} $N_p, N_{p\alpha}$ N_p, N_{β} $L_{\xi} (L_{\beta\alpha}, L_{\xi\alpha})$ $L_{\xi\alpha} (L_{\beta}, L_{\beta\alpha})$
	C	$Y_{\xi}, L_{r\alpha}, L_{\xi\alpha}, N_{p\alpha}, N_{\xi\alpha}$ fixed	-
	D	A priori weighting of L_{β}, L_r, N_p	Y_{ξ}, Y_{ζ} $N_{\xi\alpha}, N_{p\alpha}$

ITEM	RUN 849-6 , $\alpha_e = 7.2$ deg			RUN 850-5 , $\alpha_e = 20.3$ deg		
	PRED.	LINEAR MODEL	NON-LINEAR MODEL	PRED.	LINEAR MODEL	NON-LINEAR MODEL
Z_α	- 1.74	- 1.52 $\pm$ 0.03	- 1.54 $\pm$ 0.03	-1.02	- 1.43 $\pm$ 0.06	- 1.48 $\pm$ 0.04
Z_{α^2}	- 5	-	- 2 $\pm$ 1	-	-	- 7 $\pm$ 4
$Z_{q\alpha}$	-	-	- 0.5 $\pm$ 0.4	-	-	- 0.9 $\pm$ 2
$Z_{\eta\alpha}$	-	-	-	-	-	- 0.6 $\pm$ 2
Z_η	- 0.38	- 0.38	- 0.35 $\pm$ 0.01	-0.26	- 0.26	- 0.23 $\pm$ 0.01
Z_η^*	-	-	- 0.009 $\pm$ 0.001	-	-	- 0.006 $\pm$ 0.002
Z_0	0	0.006 $\pm$ 0.001	0.0020 $\pm$ 0.0009	0	0.021 $\pm$ 0.002	0.020 $\pm$ 0.001
Z_0^1	0	- 0.0003 $\pm$ 0.0009	0.0009 $\pm$ 0.0007	0	0.008 $\pm$ 0.002	0.008 $\pm$ 0.001
M_α	- 6.85	- 5.37 $\pm$ 0.09	- 5.42 $\pm$ 0.09	-0.65	- 1.0 *) $\pm$ 0.1	- 0.97 $\pm$ 0.09
M_q	- 1.63	- 1.99 $\pm$ 0.06	- 2.21 $\pm$ 0.06	-1.1	- 1.03*) $\pm$ 0.07	- 0.81 $\pm$ 0.06
M_{α^2}	-	-	3 $\pm$ 5	-	-	- 73 $\pm$ 11
$M_{q\alpha}$	-	-	-17 $\pm$ 3	-	-	50 $\pm$ 8
$M_{\eta\alpha}$	-	-	-	-	-	178 $\pm$ 37
M_η	-12.6	-14.1 $\pm$ 0.1	-14.07 $\pm$ 0.02	-3.4	- 3.22 $\pm$ 0.07	- 3.23 $\pm$ 0.06
M_η^*	-	-	- 0.08 $\pm$ 0.03	-	-	- 0.17 $\pm$ 0.02
M_0	0	0.013 $\pm$ 0.003	0.018 $\pm$ 0.003	0	0.030 $\pm$ 0.001	0.034 $\pm$ 0.001
$S^2(q)$	-	2.3 $\times 10^{-5}$	2.1 $\times 10^{-5}$	-	2.3 $\times 10^{-5}$	2.4 $\times 10^{-5}$
$S^2(\eta_x^*)$	-	4.3 $\times 10^{-6}$	2.7 $\times 10^{-6}$	-	19.5 $\times 10^{-6}$	7.0 $\times 10^{-6}$
$\ln R_1 $	-	-22.9	-23.6	-	-21.5	- 22.5

*) strong correlation

TABLE 16 Comparison of Predicted and Estimated Parameters from Flight Data by ML Estimation. HP-115 Aircraft

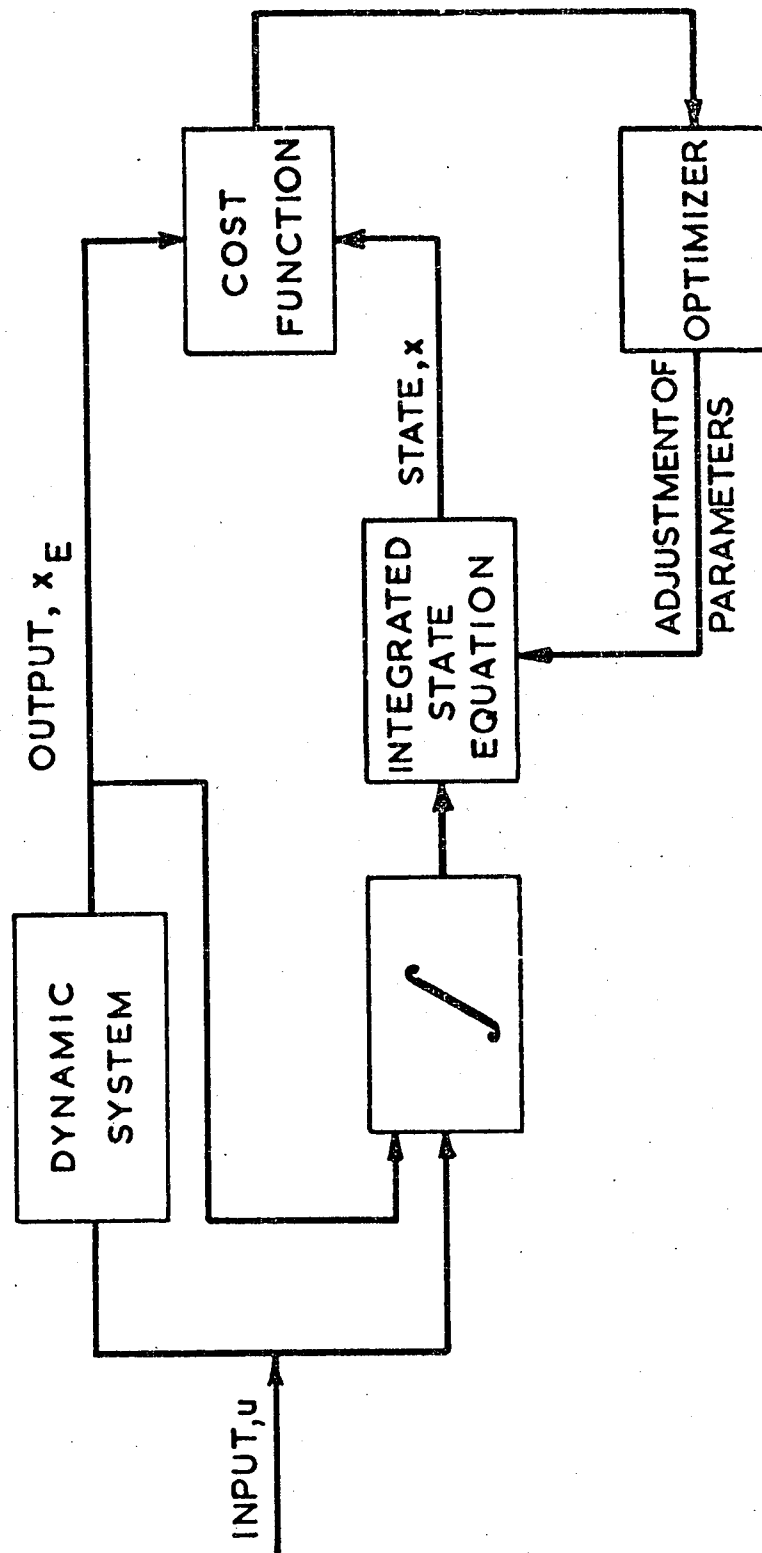


Fig.1 Parameter Identification using Modified Equation Error Method

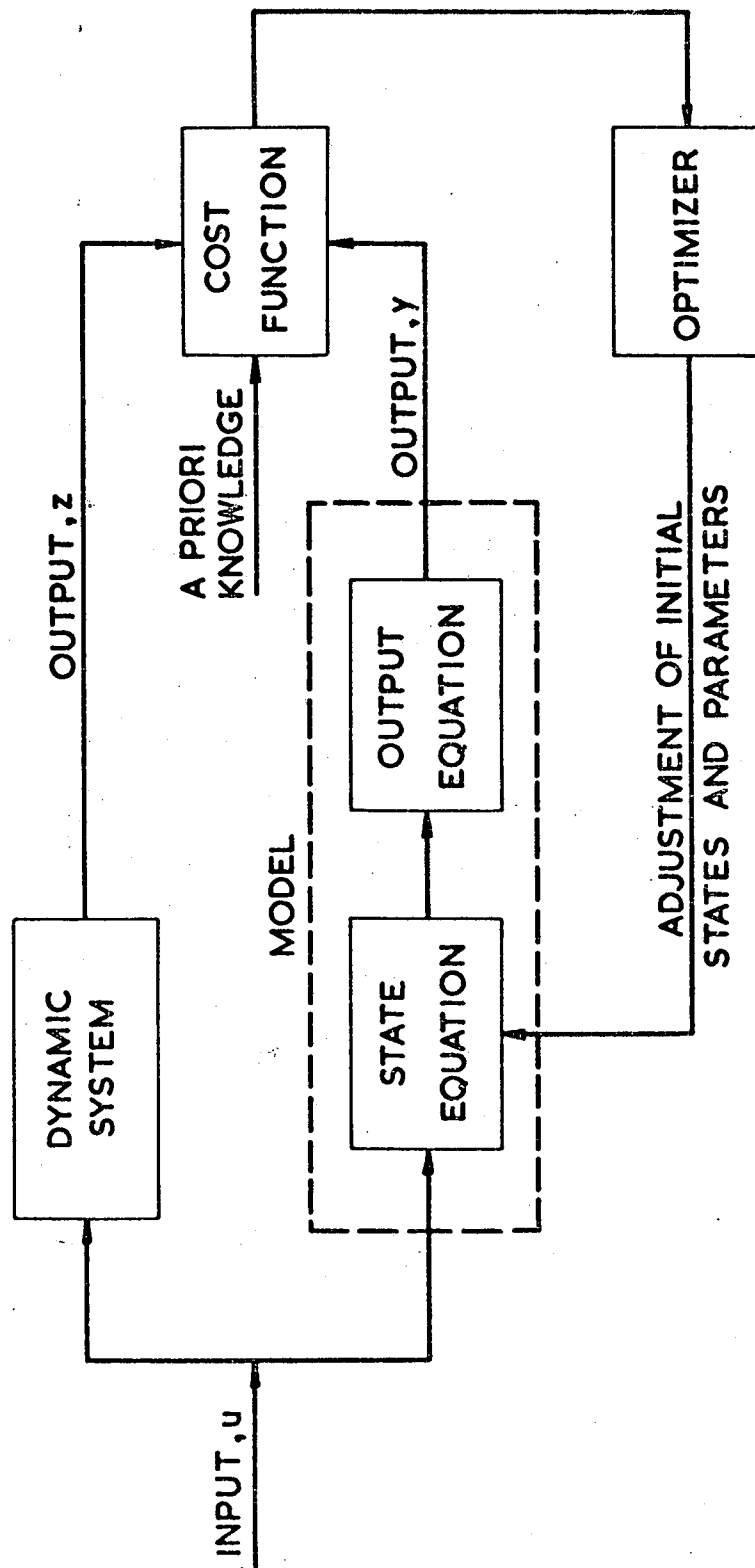


Fig. 2 Parameter Identification using Output Error Method.

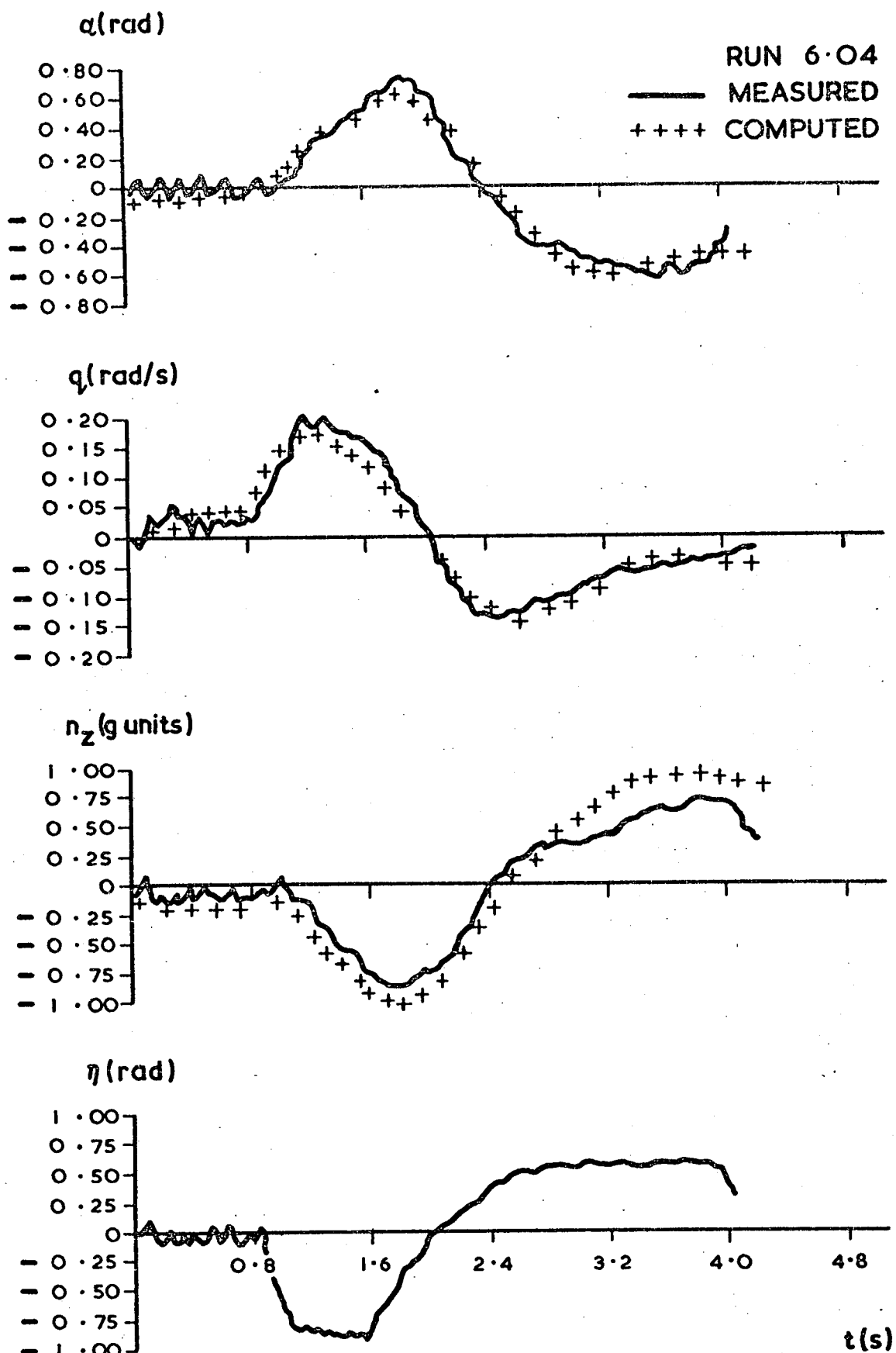


Fig. 3 Comparison of time Histories Measured in Flight with those Computed using ML Estimation. BASSET Aircraft

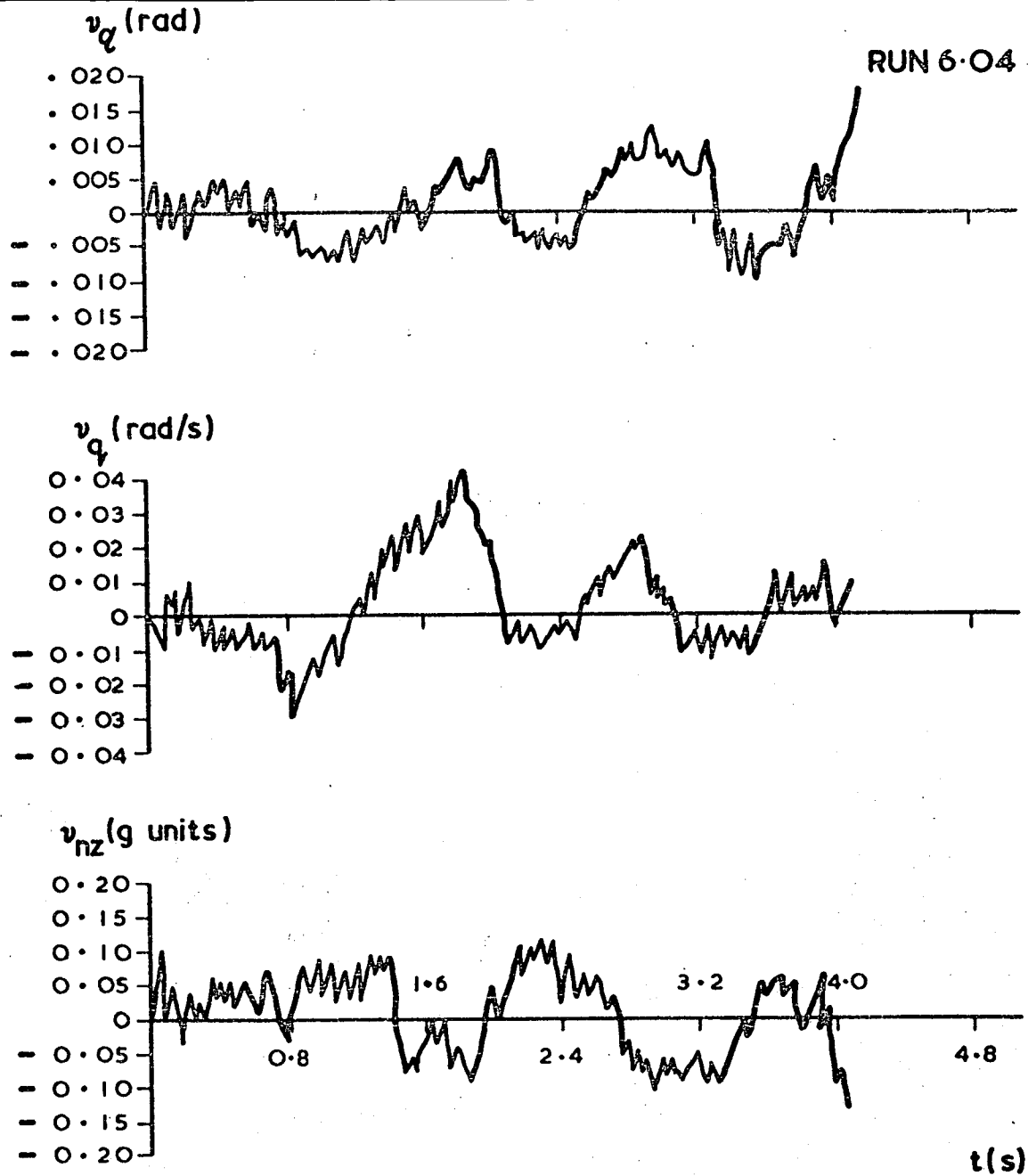


Fig. 4 Time Histories of Residuals. BASSET Aircraft

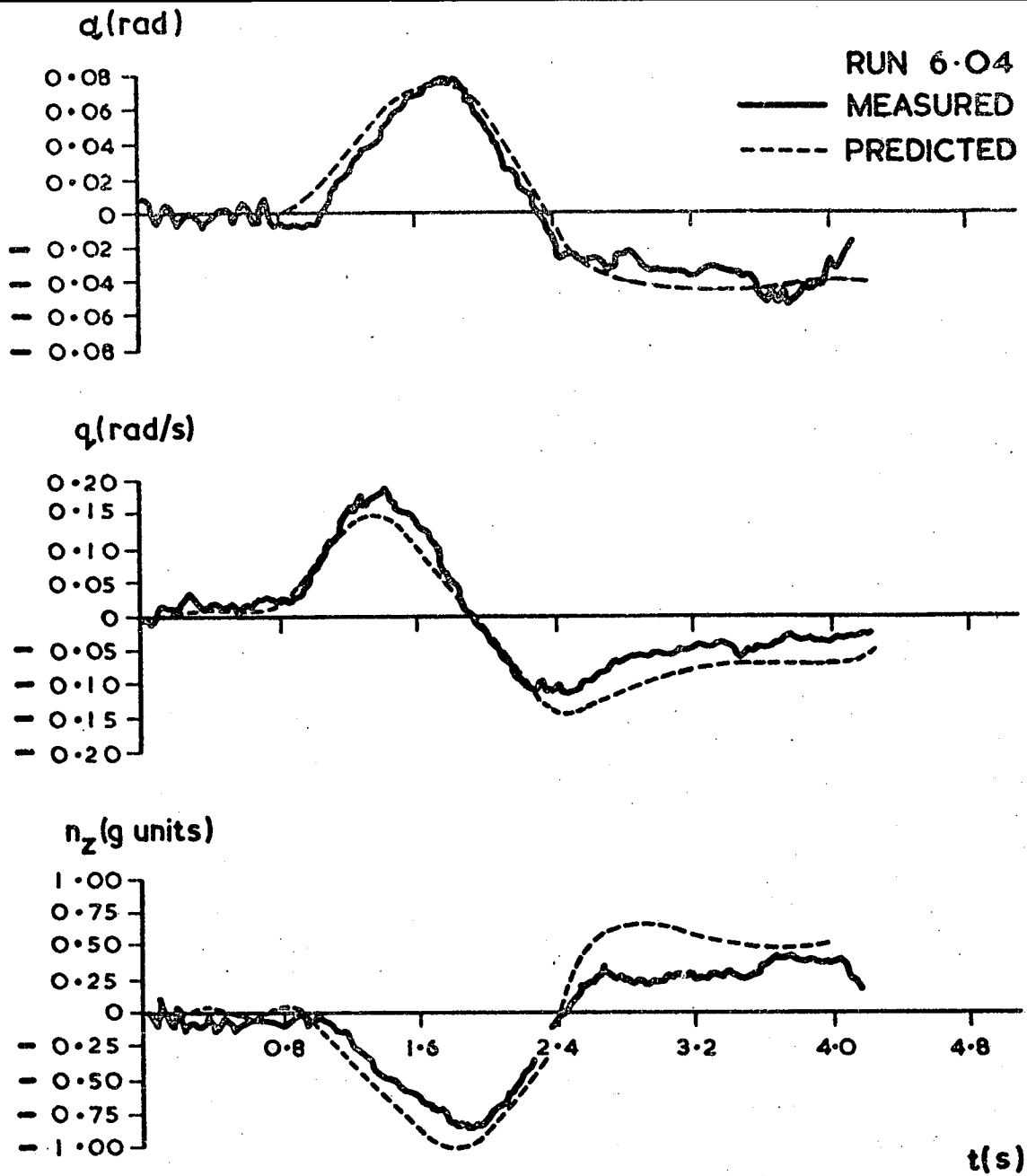


Fig.5 Comparison of Time Histories Measured in Flight with those Predicted.
BASSET Aircraft

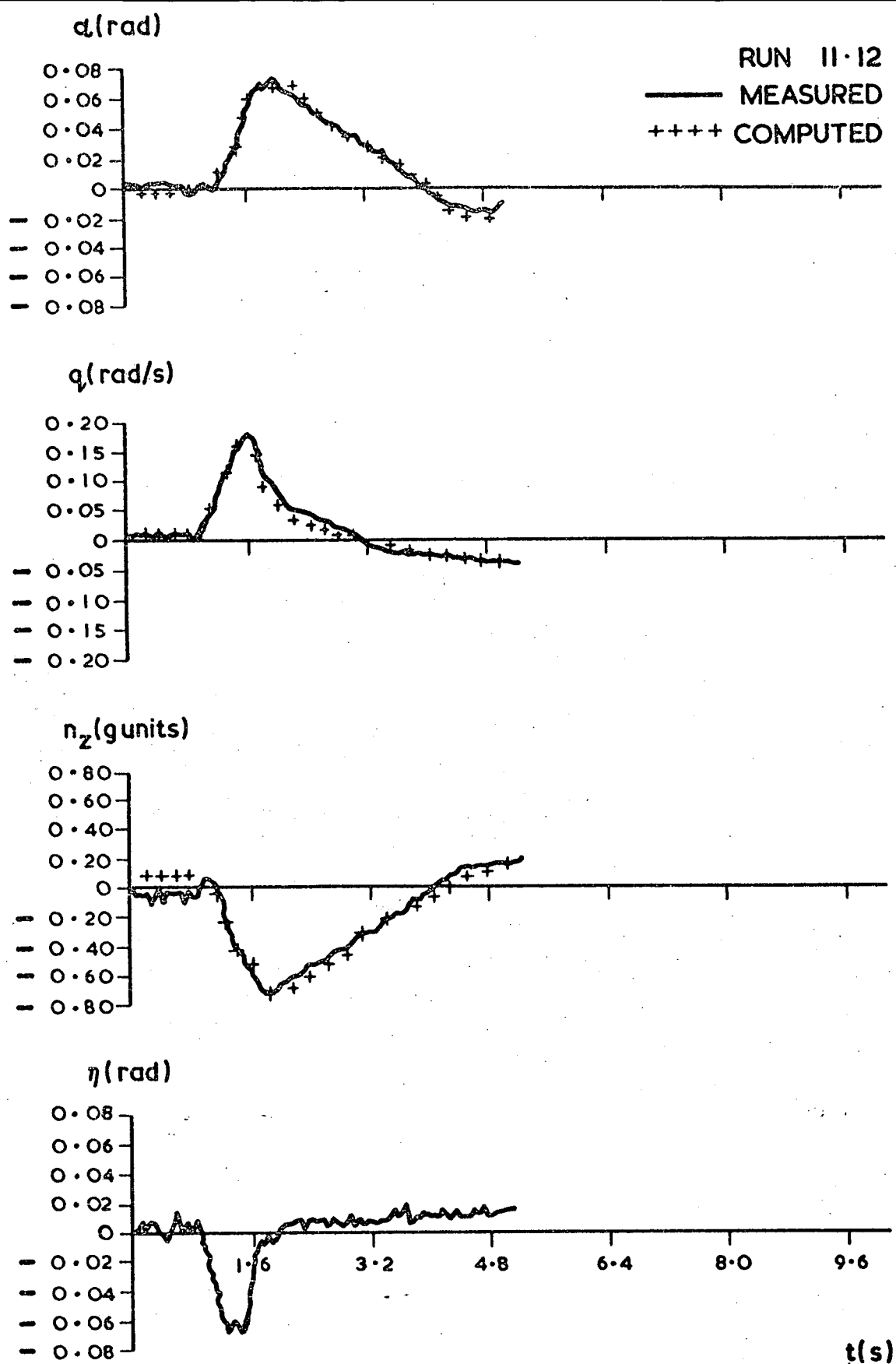


Fig. 6 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. BASSET Variable Stability Aircraft (Positive C.G. margin)

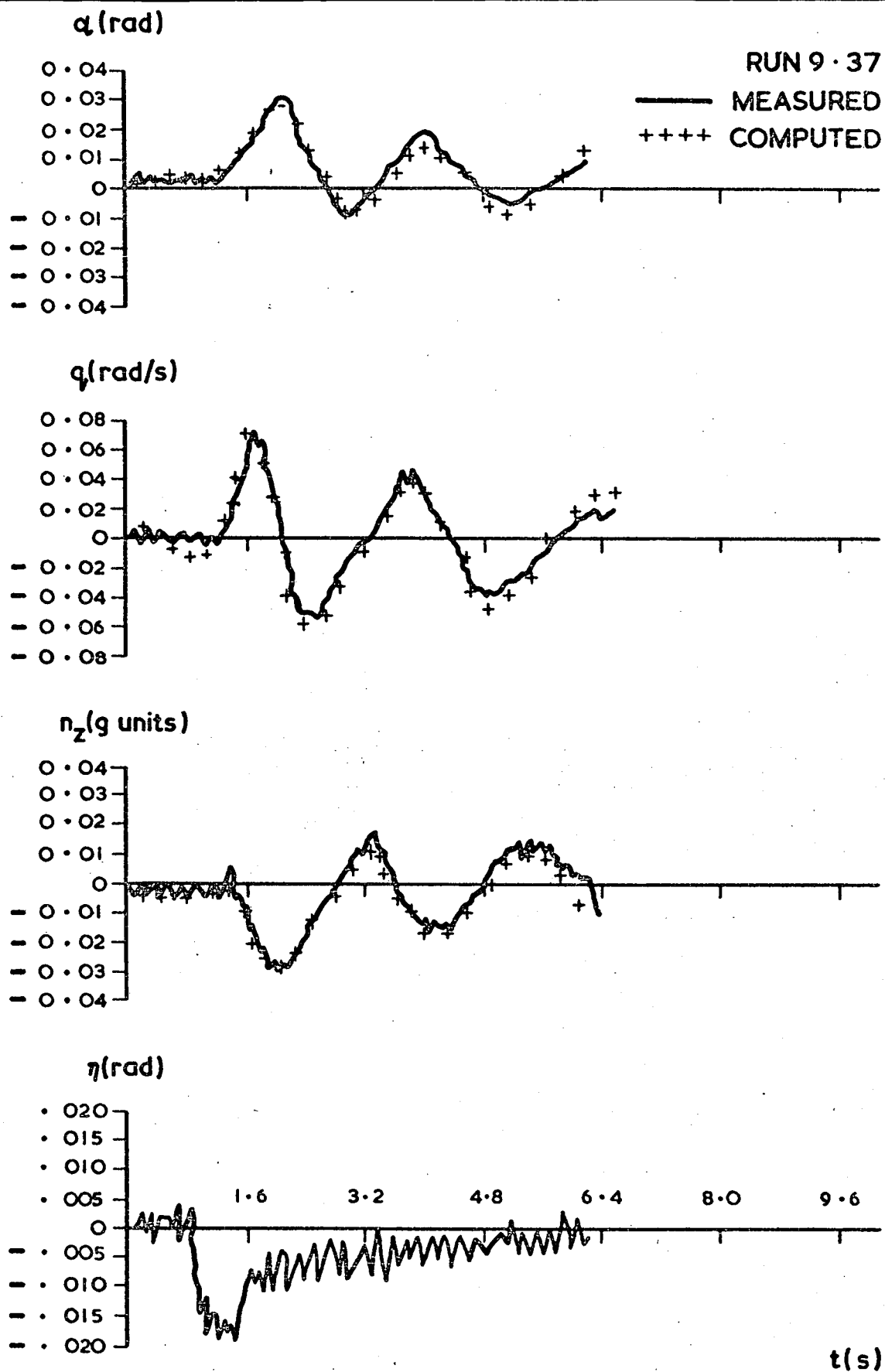


Fig.7 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. BASSET Variable Stability Aircraft (Positive damping derivative)

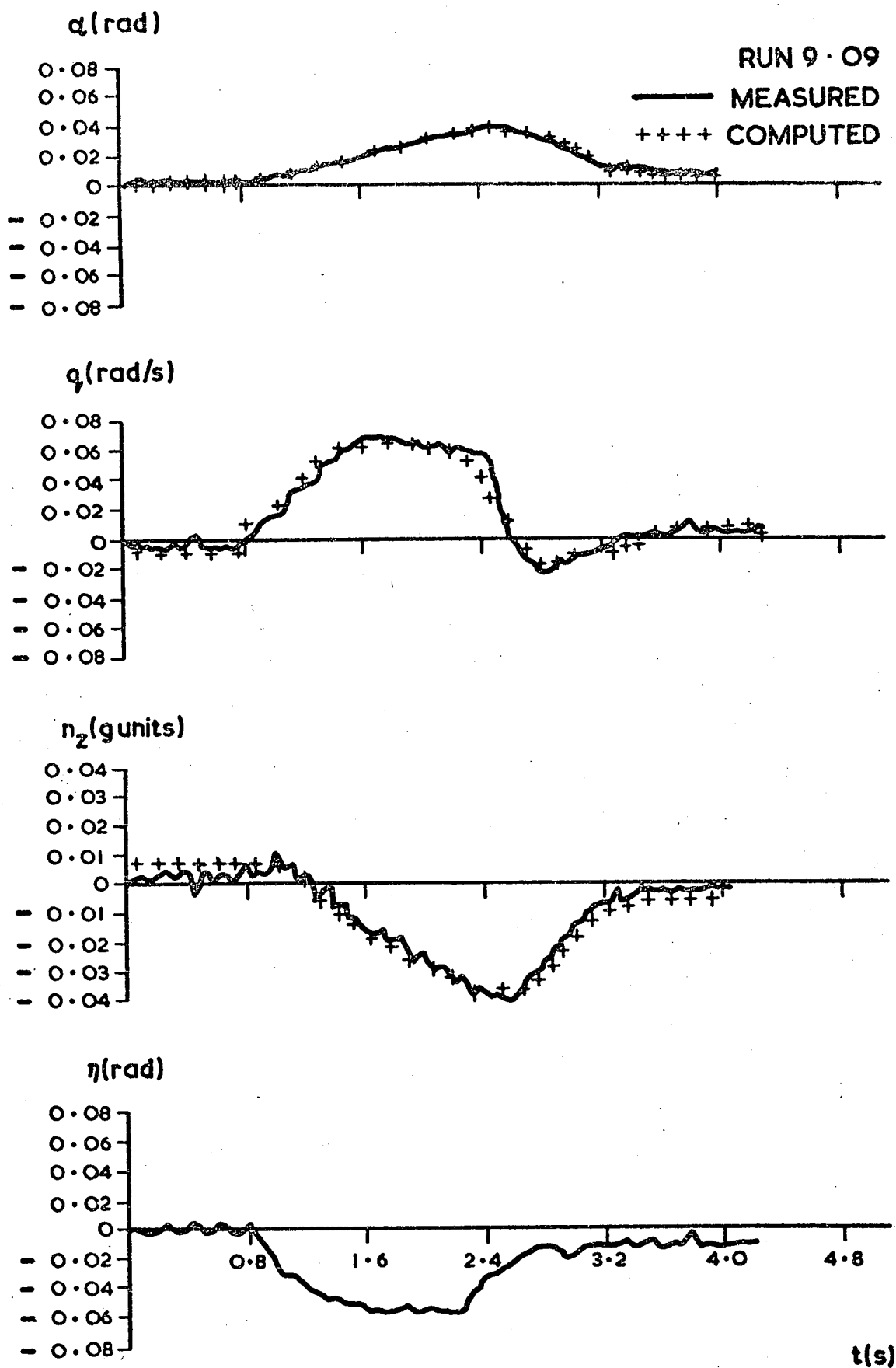


Fig.8 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. BASSET Variable Stability Aircraft (Large damping)

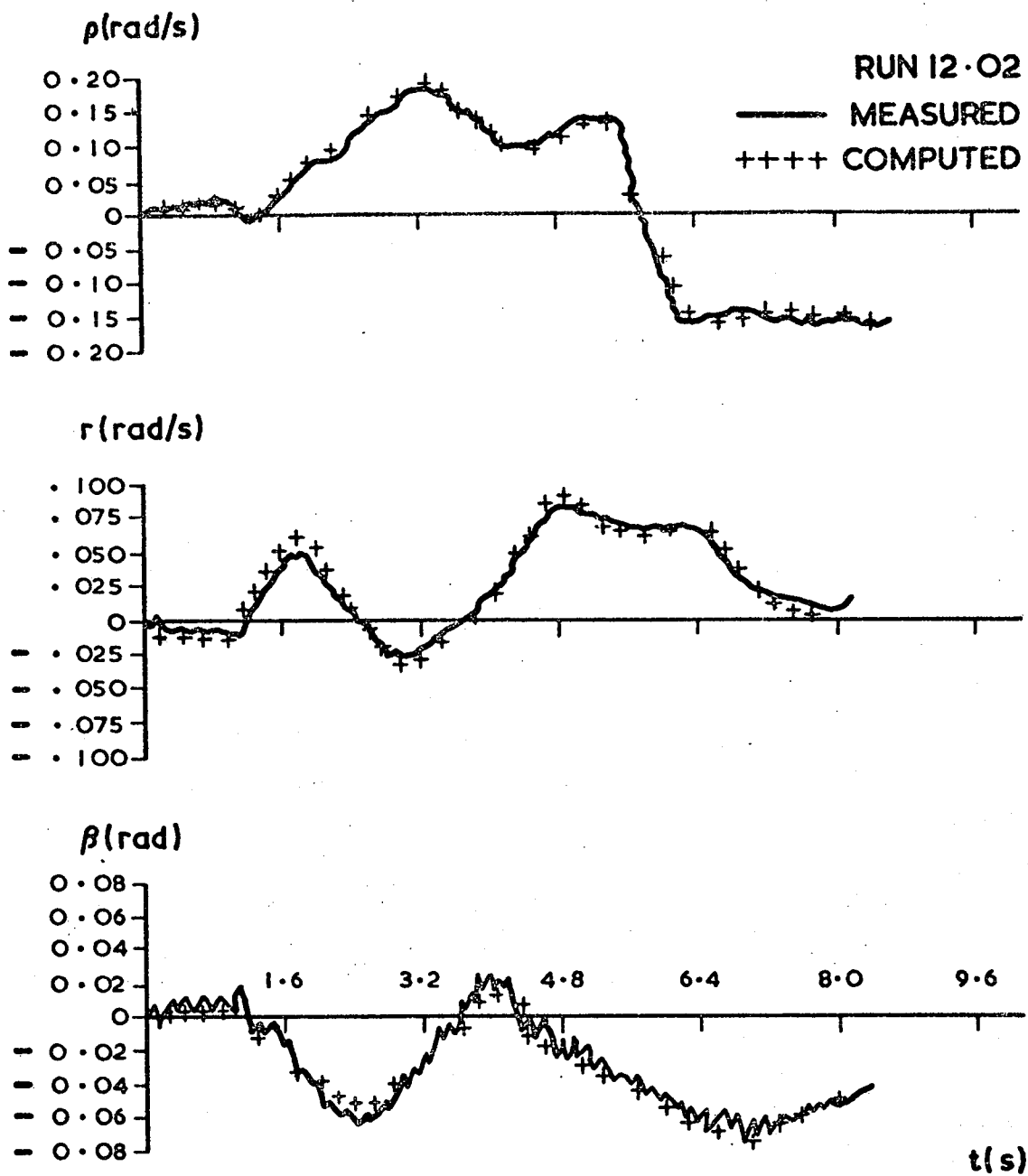


Fig. 9 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. BASSET Aircraft

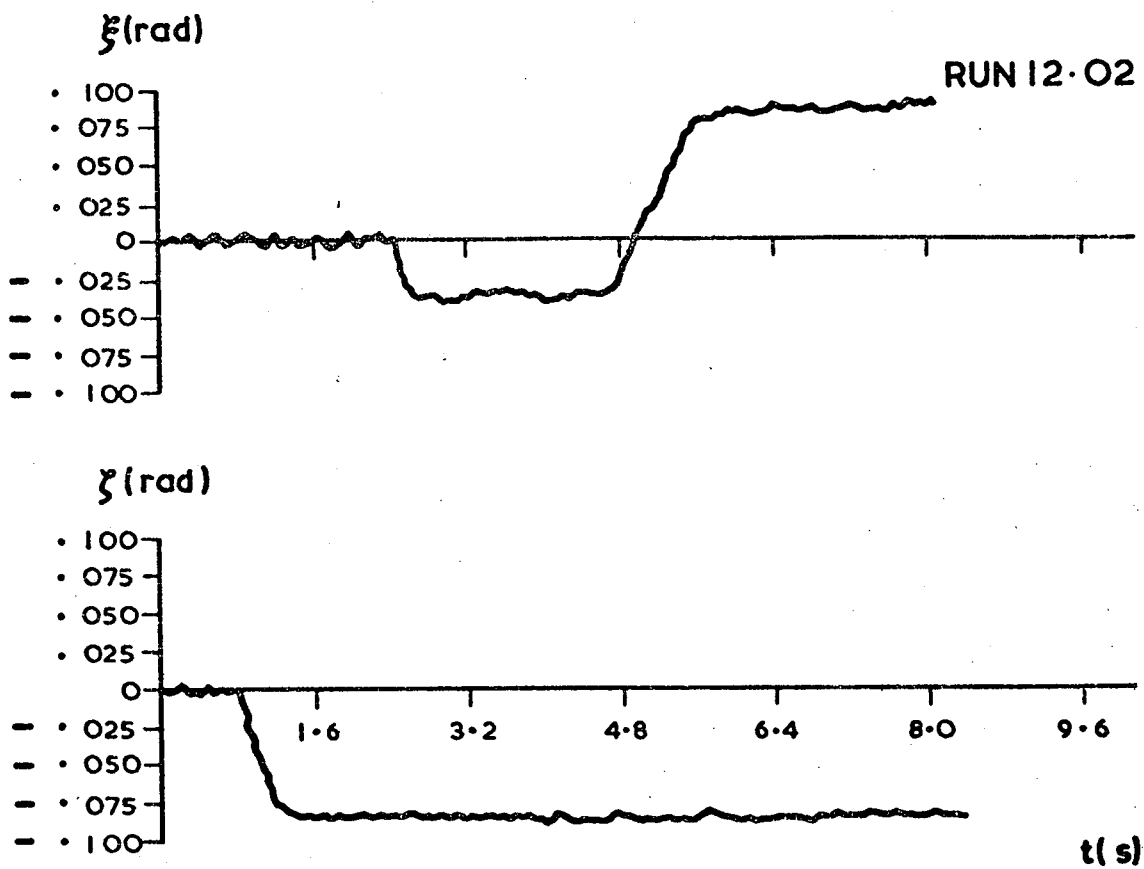


Fig. 9 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. BASSET Aircraft — Concluded

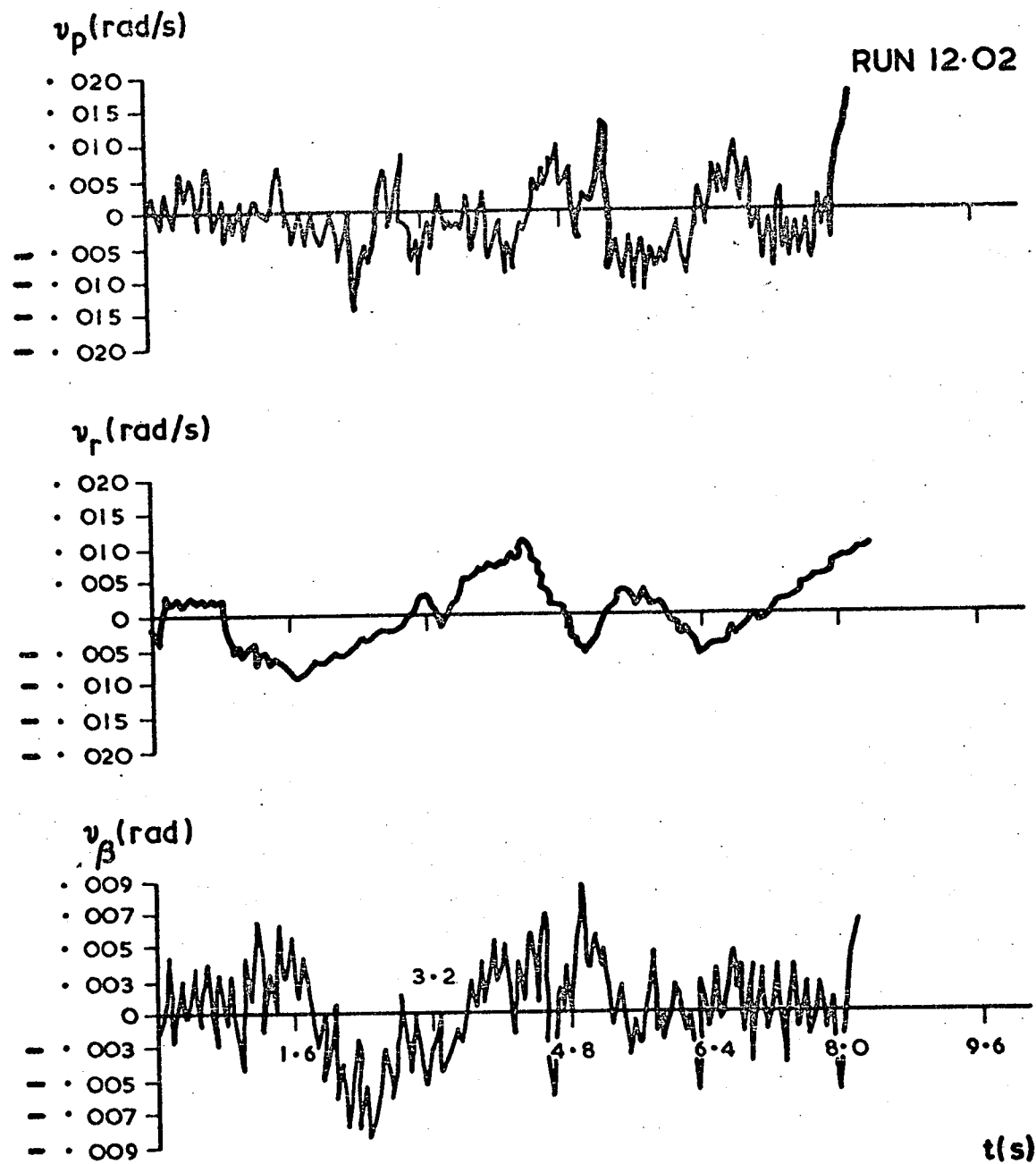


Fig.10 Time Histories of Residuals. BASSET Aircraft

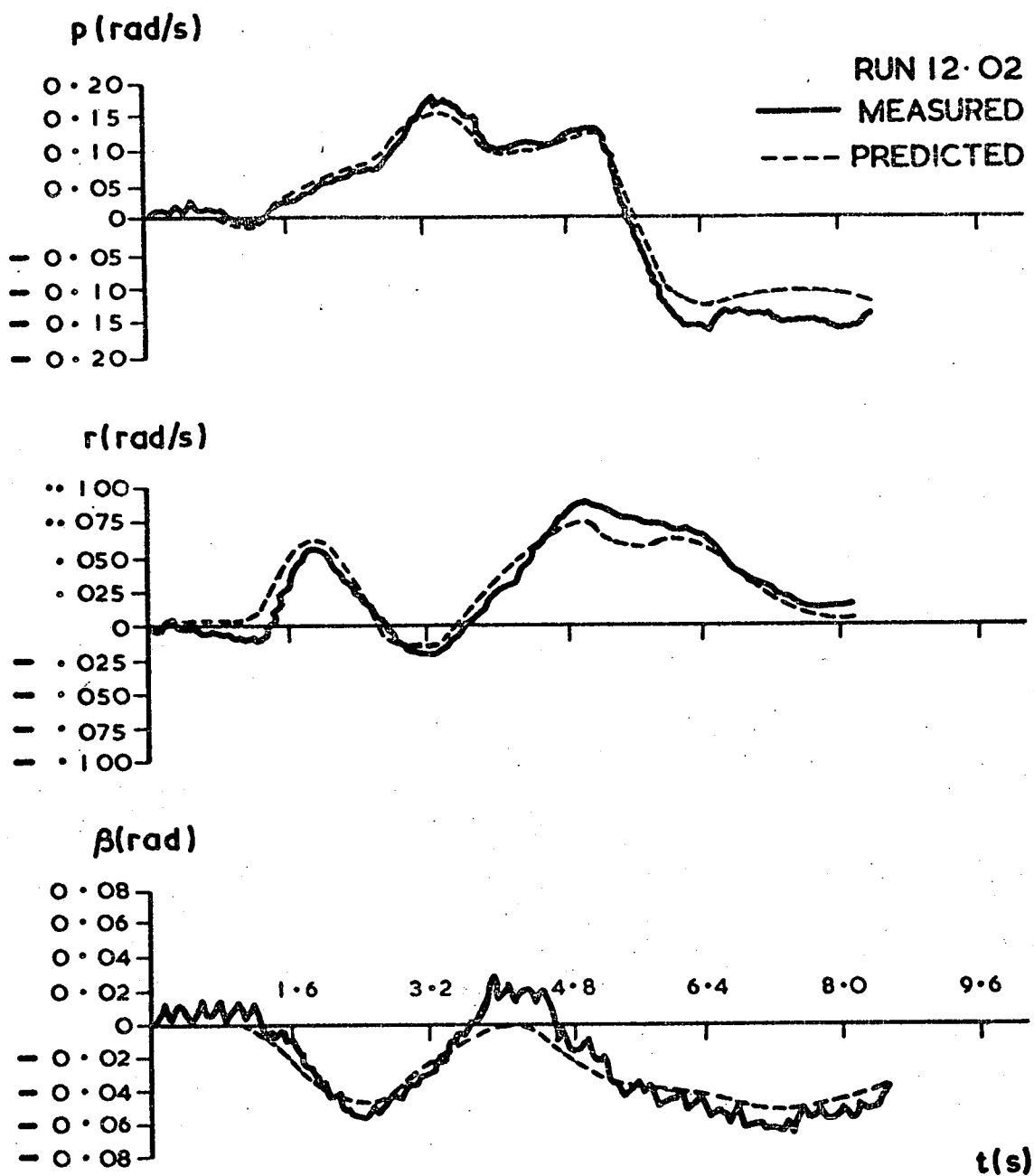


Fig. II Comparison of Time Histories Measured in Flight and those Predicted.
BASSET Aircraft

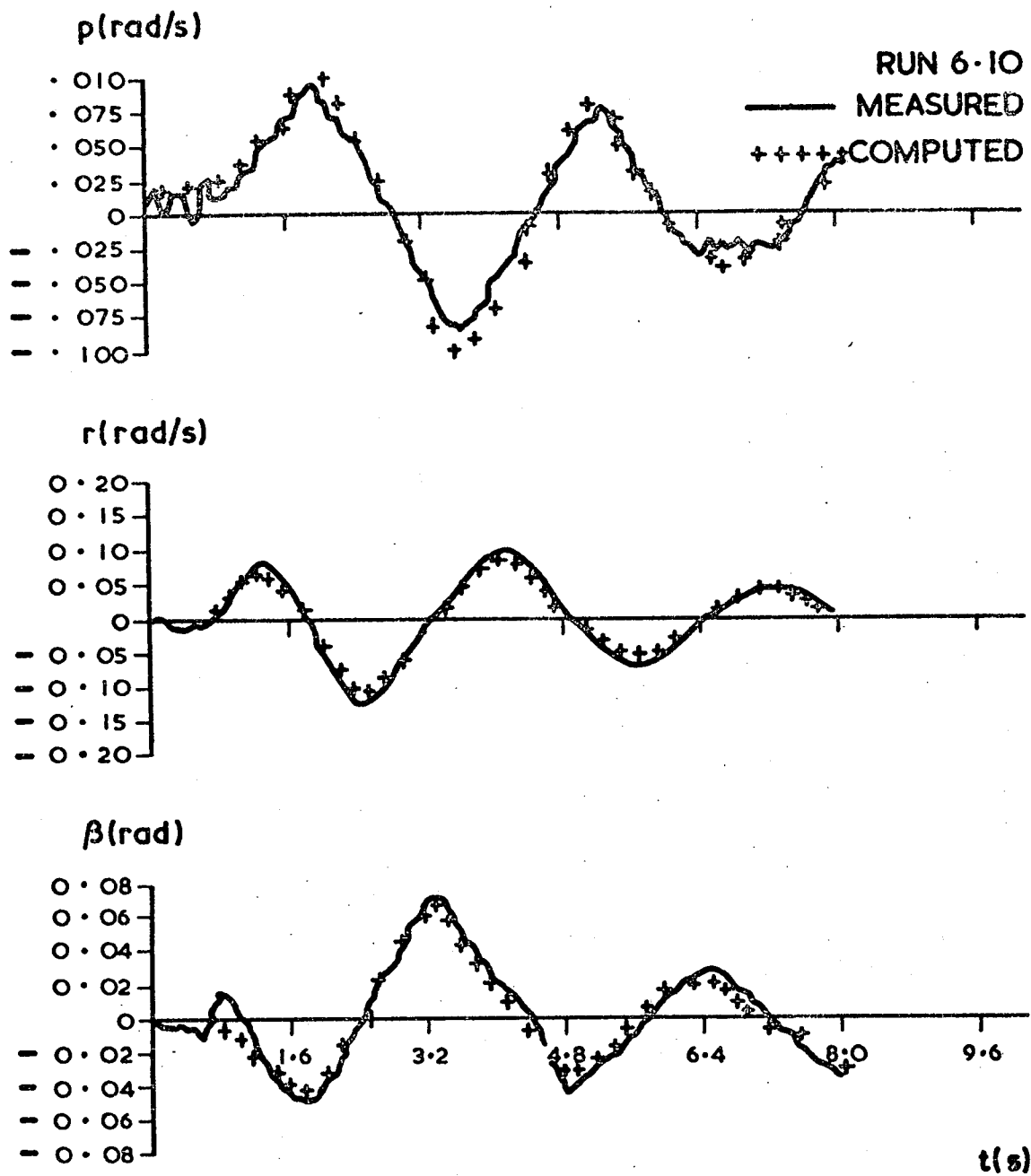


Fig.12 Comparison of Time Histories Measured in Flight with those Computed using Bayesian Estimation. BASSET Aircraft

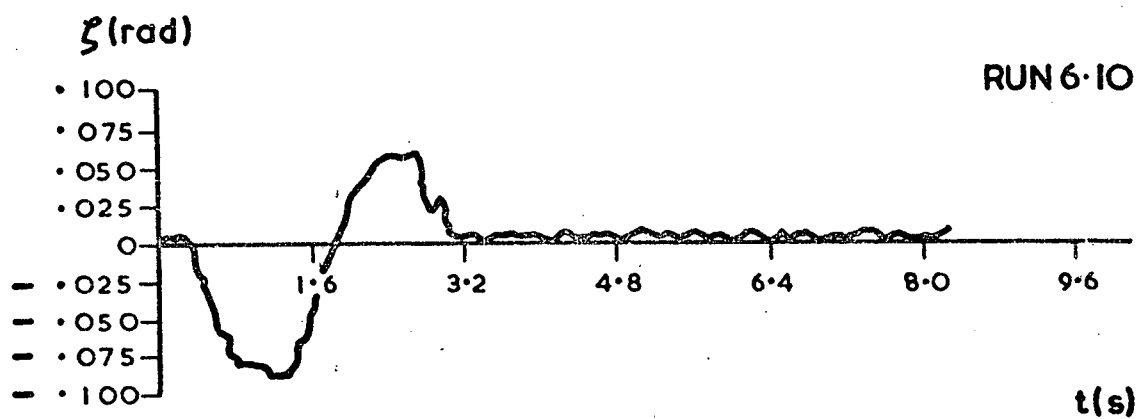


Fig.12 Comparison of Time Histories Measured in Flight with those Computed using Bayesian Estimation. BASSET Aircraft — Concluded

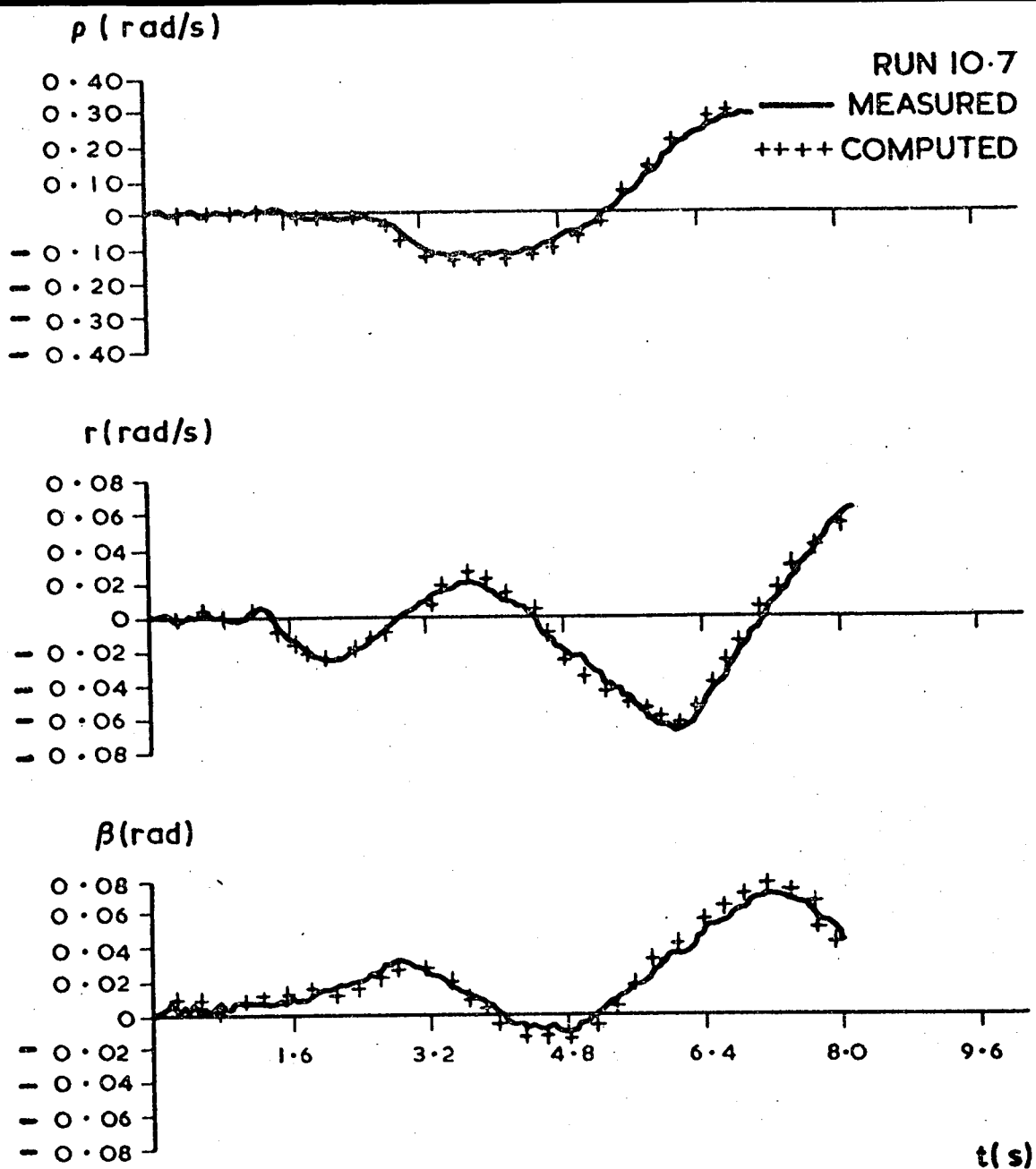


Fig.13 Comparison of Time Histories Measured in Flight with those Computed using WLS Estimation. BASSET Variable Stability Aircraft (Small dihedral effect)

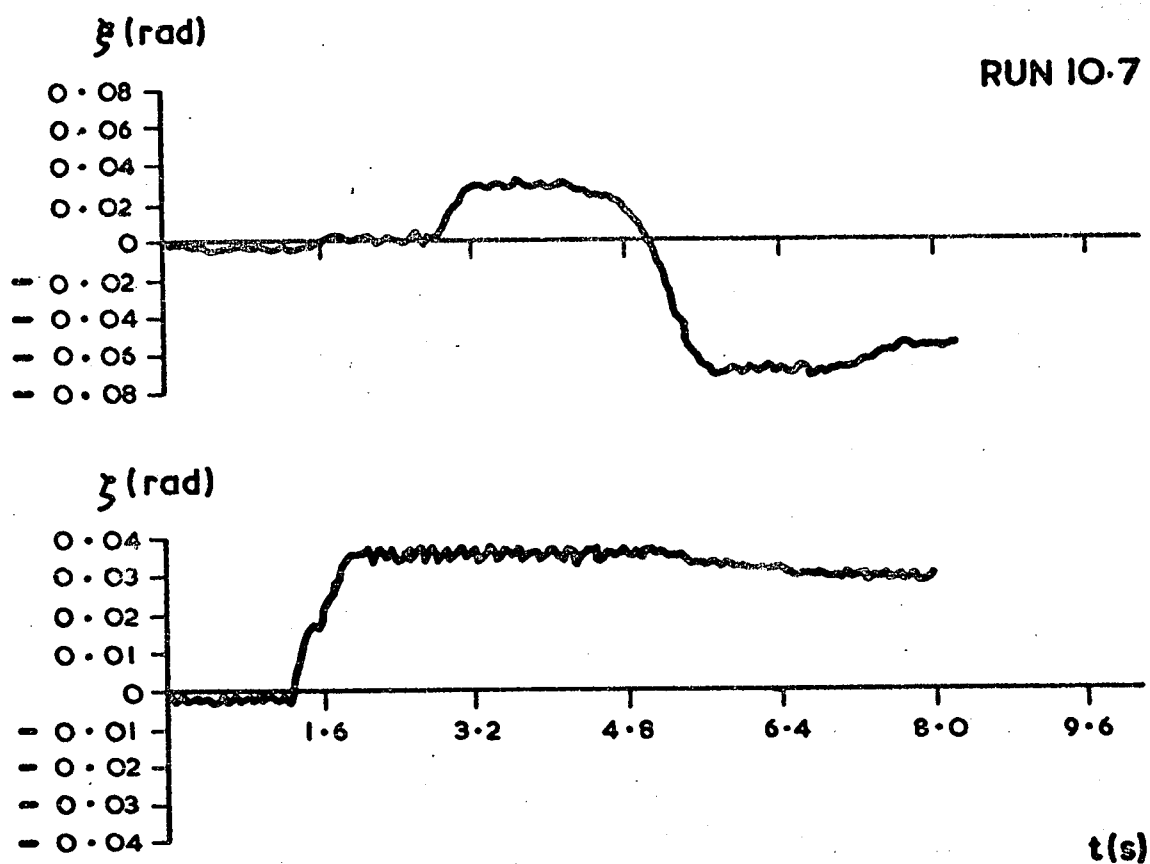


Fig.13 Comparison of Time Histories Measured in Flight with those Computed using WLS Estimation. BASSET Variable Stability Aircraft (Small dihedral effect) — Concluded

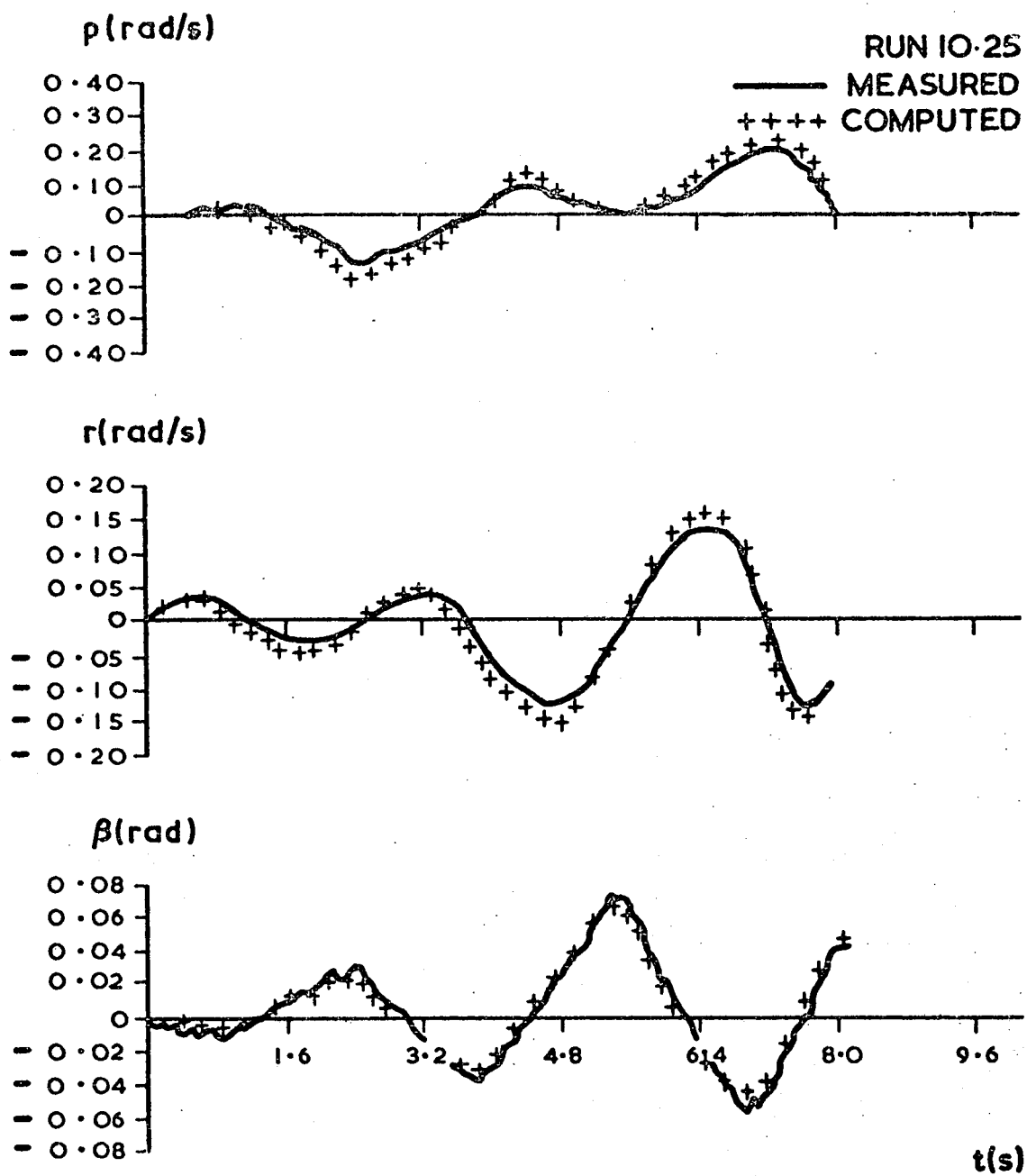


Fig. 14 Comparison of Time Histories Measured in Flight with those Computed using WLS Estimation. BASSET Variable Stability Aircraft (Positive damping in yaw)

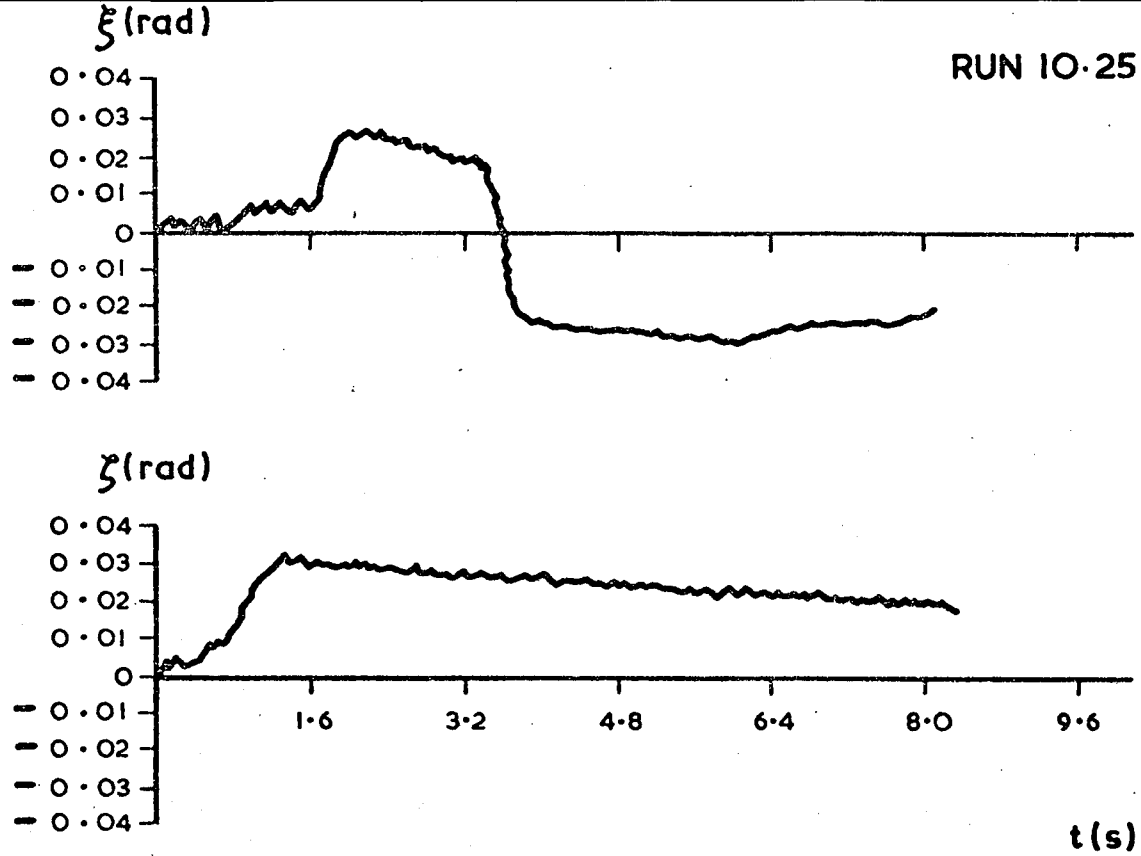


Fig. 14 Comparison of Time Histories Measured in Flight with those Computed using WLS Estimation. BASSET Variable Stability Aircraft (Positive damping in yaw) — Concluded

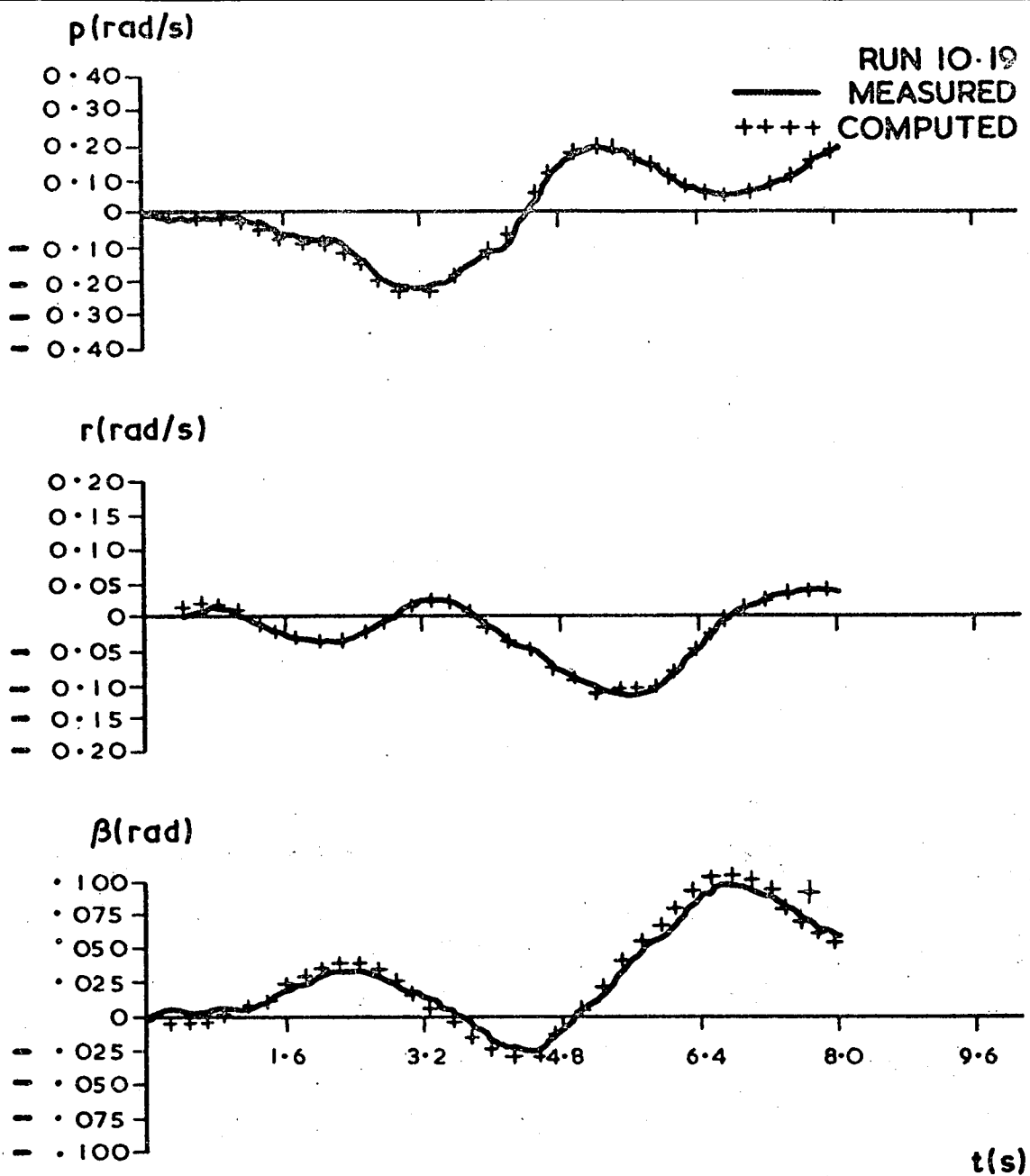


Fig. 15 Comparison of Time Histories Measured in Flight with those Computed
 using WLS Estimation. BASSET Variable Stability Aircraft
 (Small directional stability)

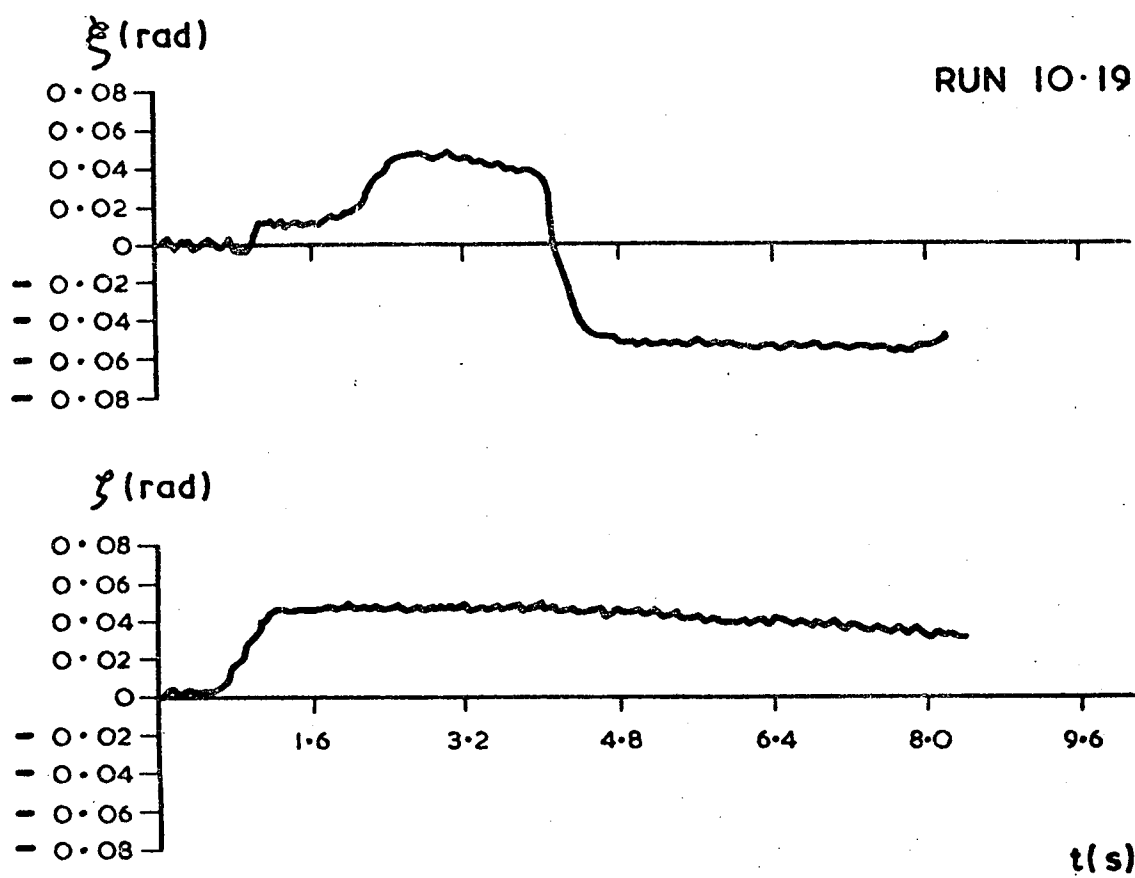


Fig. 15 Comparison of Time Histories Measured in Flight with those Computed using WLS Estimation. BASSET Variable Stability Aircraft (Small directional stability) — Concluded

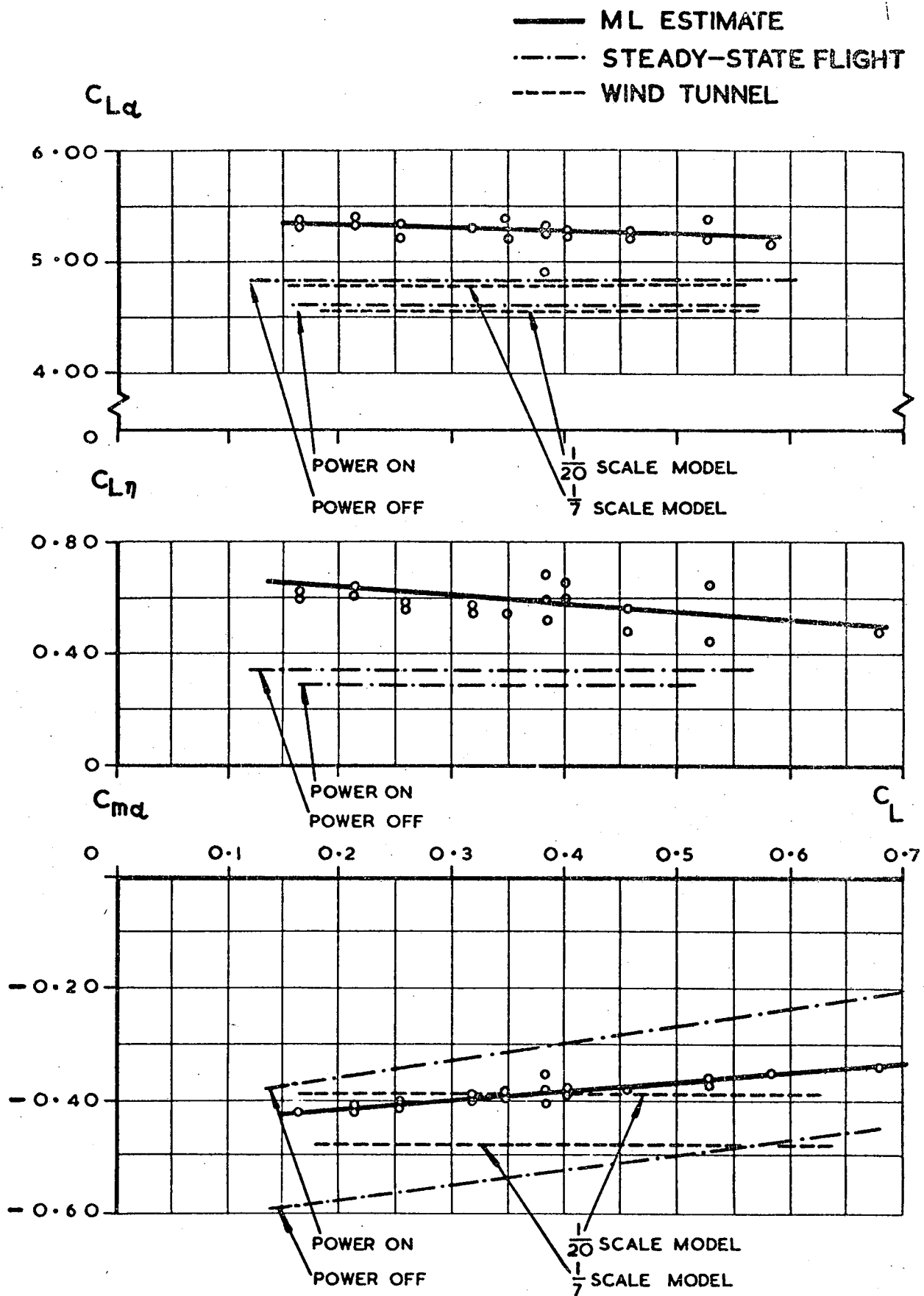


Fig. 16 Variation of Longitudinal Aerodynamic Derivatives, Determined from Flight Data and Other Sources, with Lift Coefficient. PARIS Aircraft

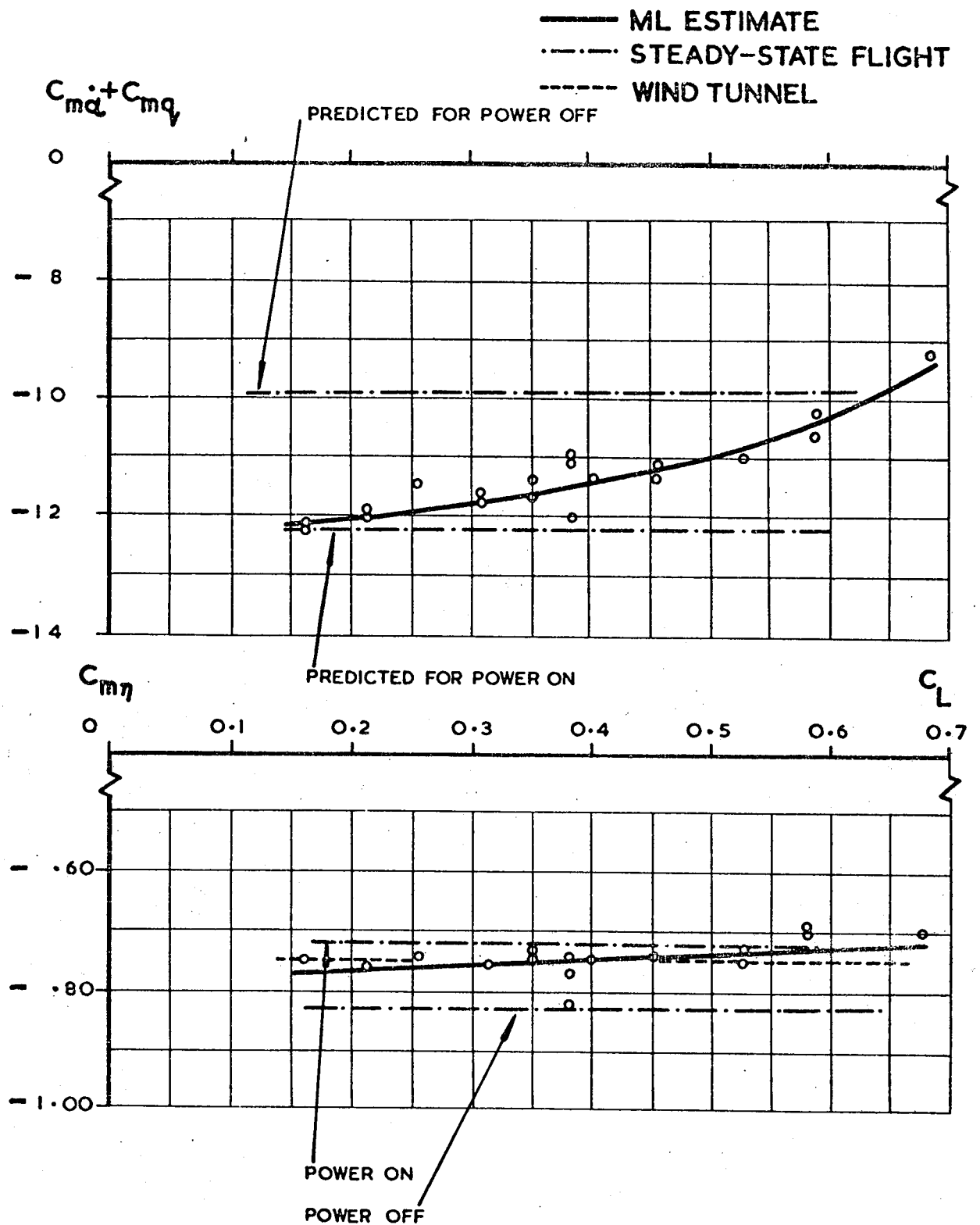


Fig.16 Variation of Longitudinal Aerodynamic Derivatives, Determined from Flight Data and other Sources, with Lift Coefficient. PARIS Aircraft- Concluded

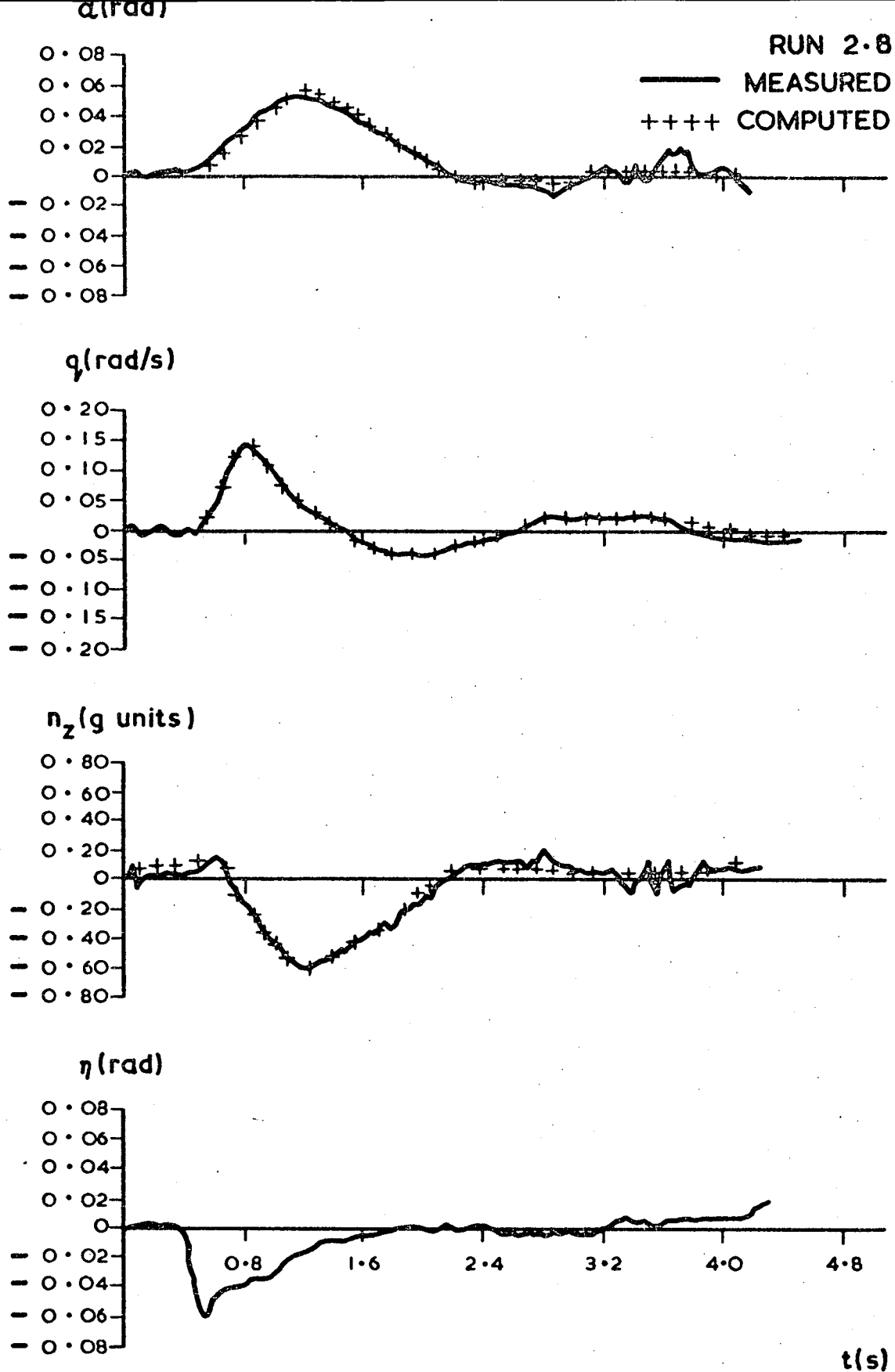


Fig.17 Comparison of Time Histories Measured in Flight with those Computed using ML Estimation. PARIS Aircraft

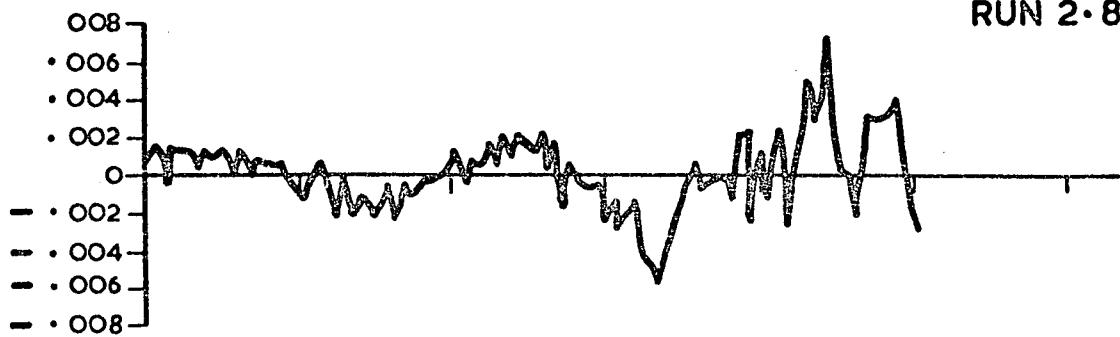
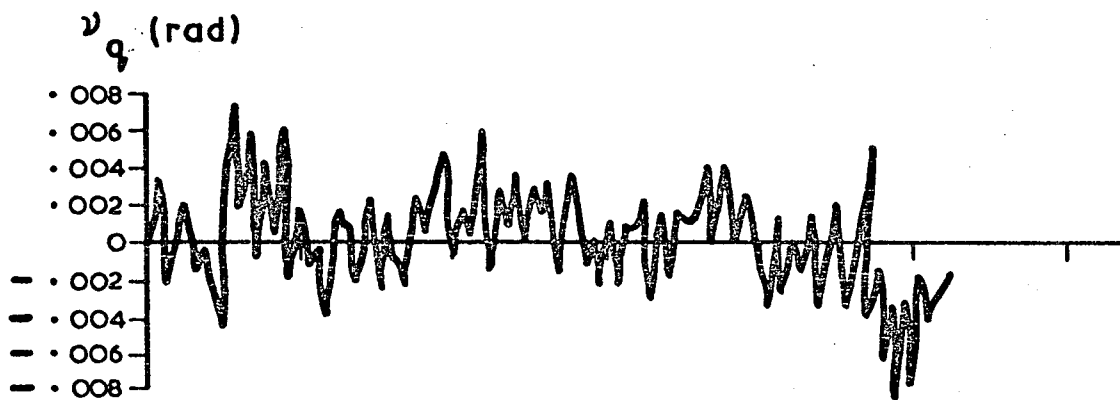
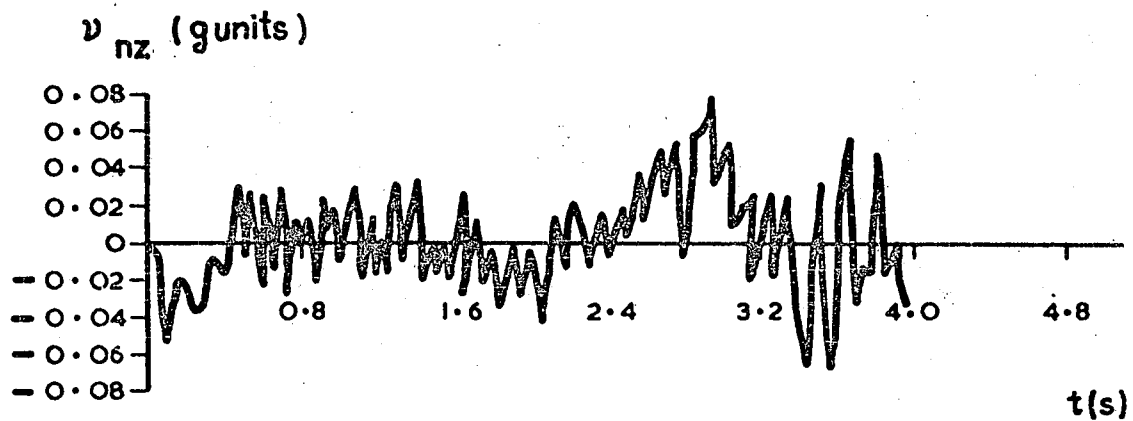


Fig.18 Time Histories and Sample Autocovariance Functions of the Residuals. PARIS Aircraft

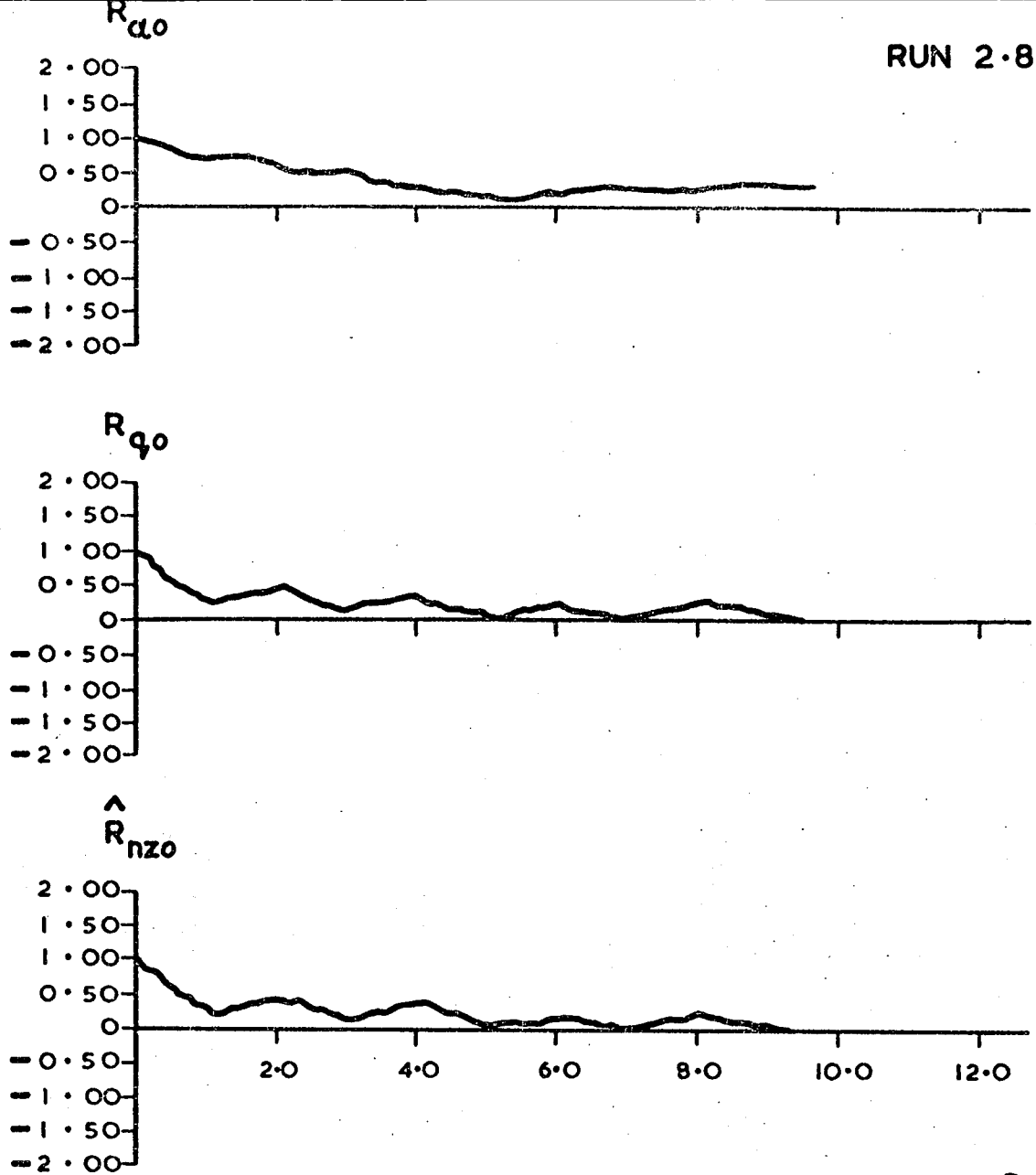


Fig.18 Time Histories and Sample Autocovariance Functions of the Residuals. PARIS Aircraft—Concluded

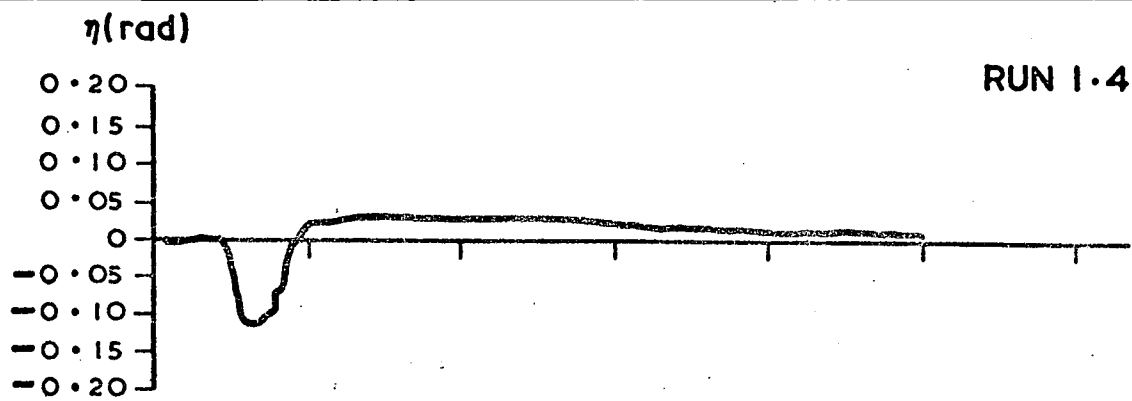
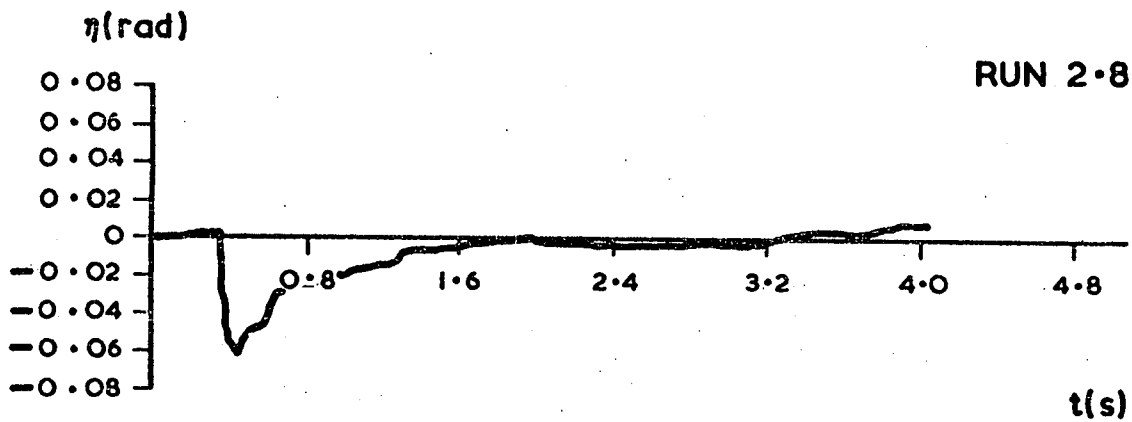


Fig.19 Different Input Forms' Used in the Excitation of the Longitudinal Motion. PARIS Aircraft

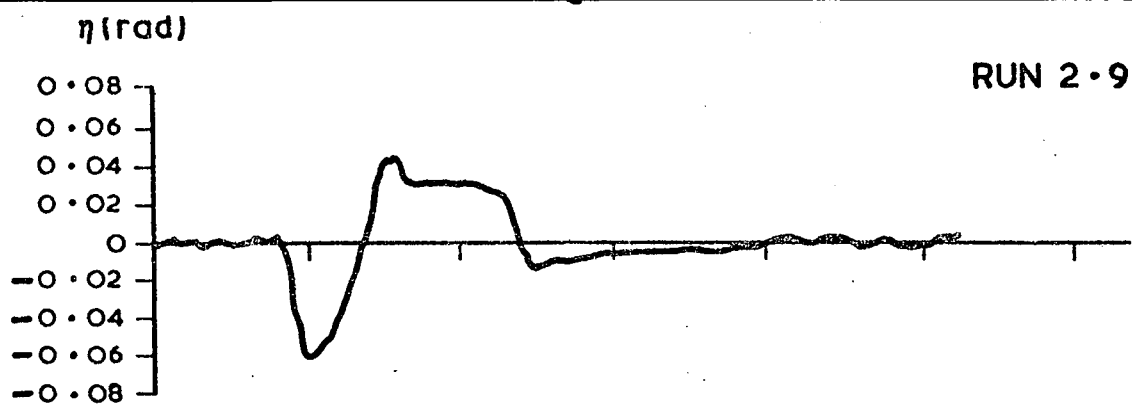
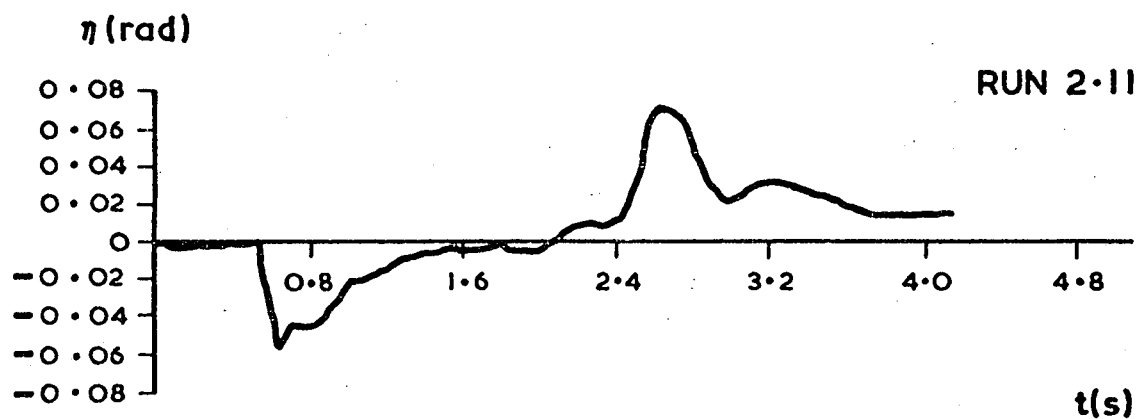


Fig.19 Different Input Forms' Used in the Excitation of the Longitudinal Motion. PARIS Aircraft — Concluded

○ MEASURED DATA
— ML ESTIMATE

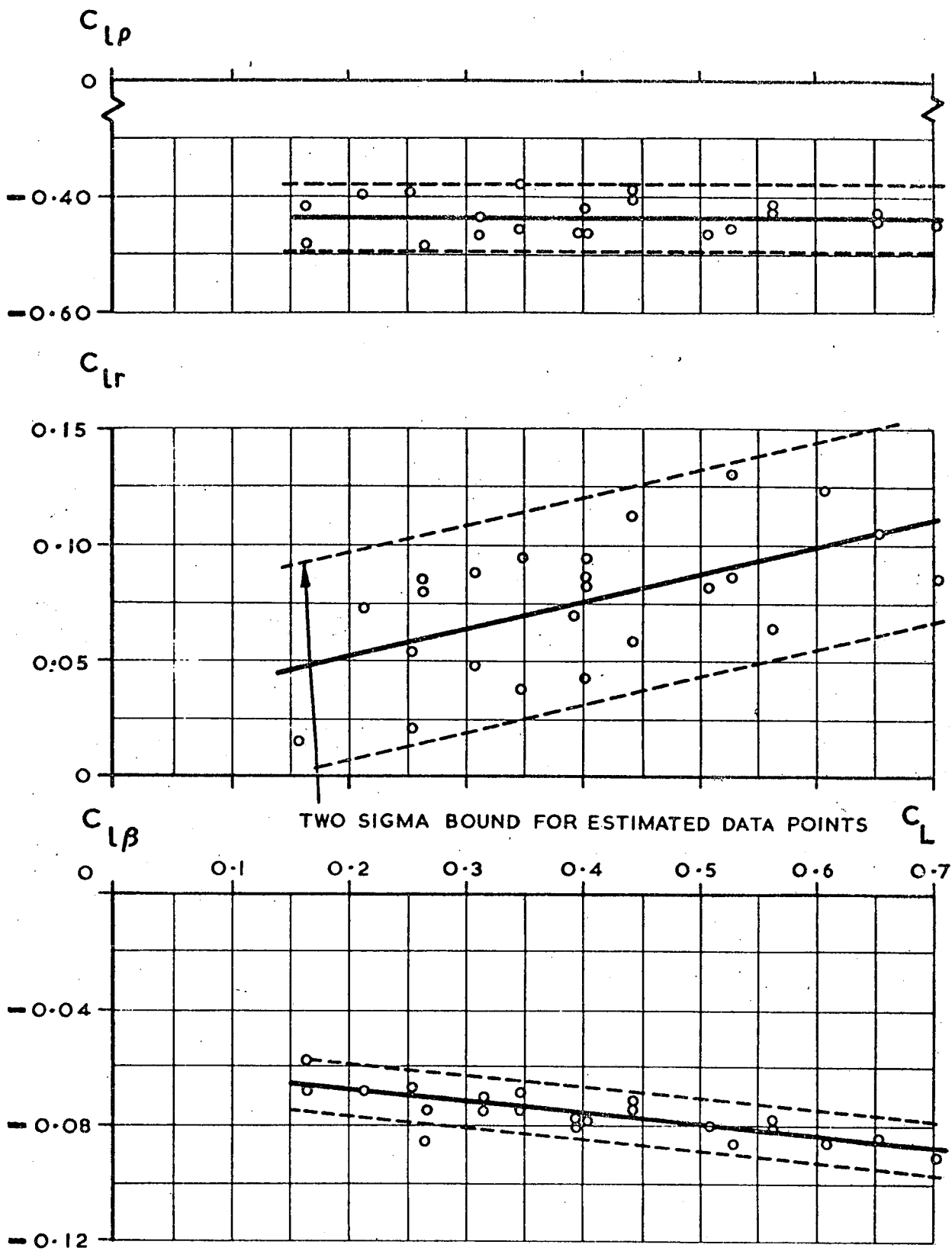


Fig. 20 Variation of Lateral Aerodynamic Derivatives, from Flight Data using ML Estimation, with Lift Coefficient. PARIS Aircraft

○ MEASURED DATA
— ML ESTIMATE

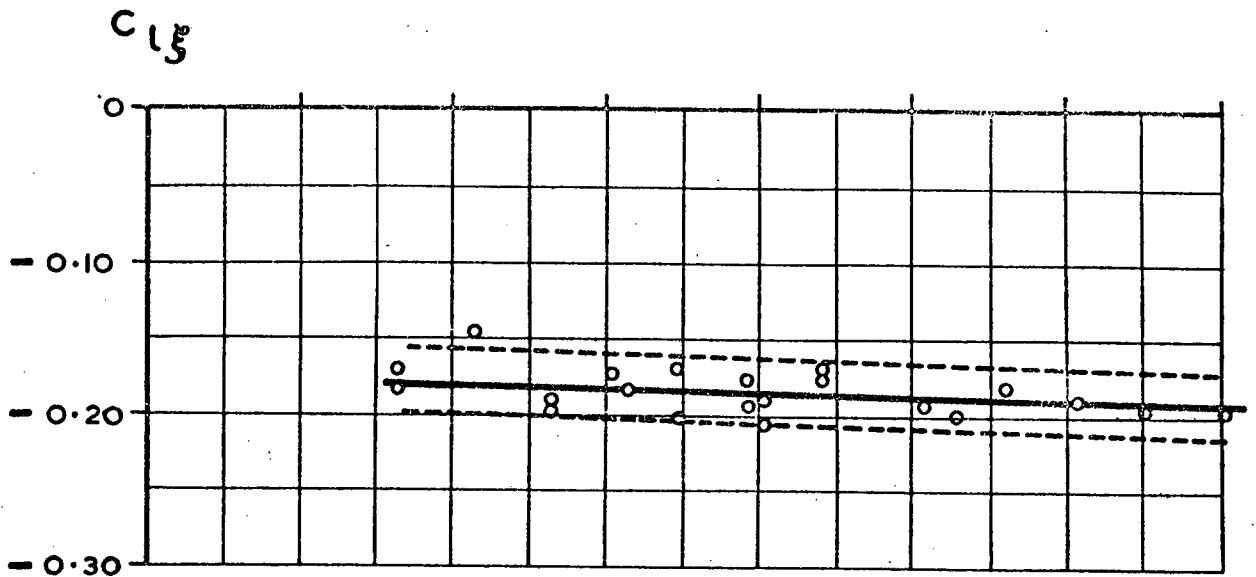
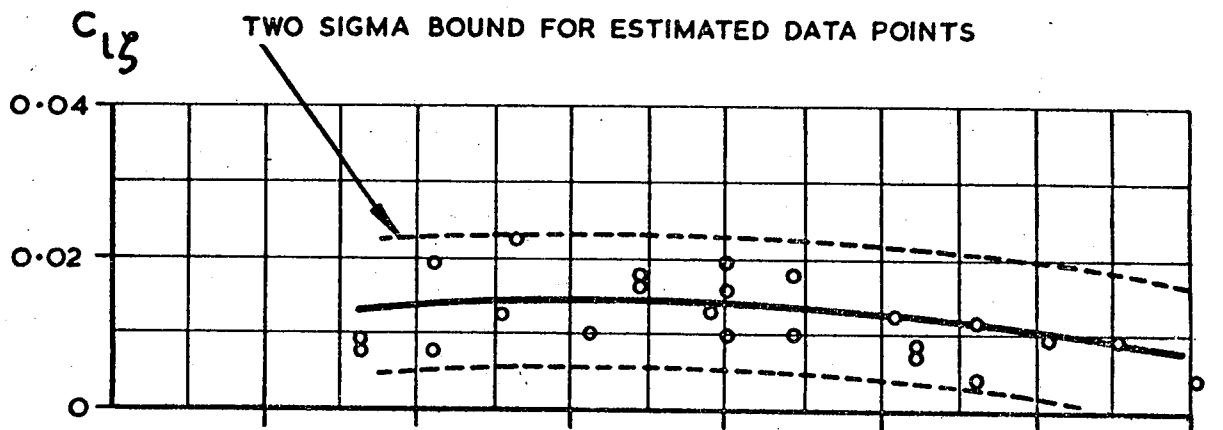
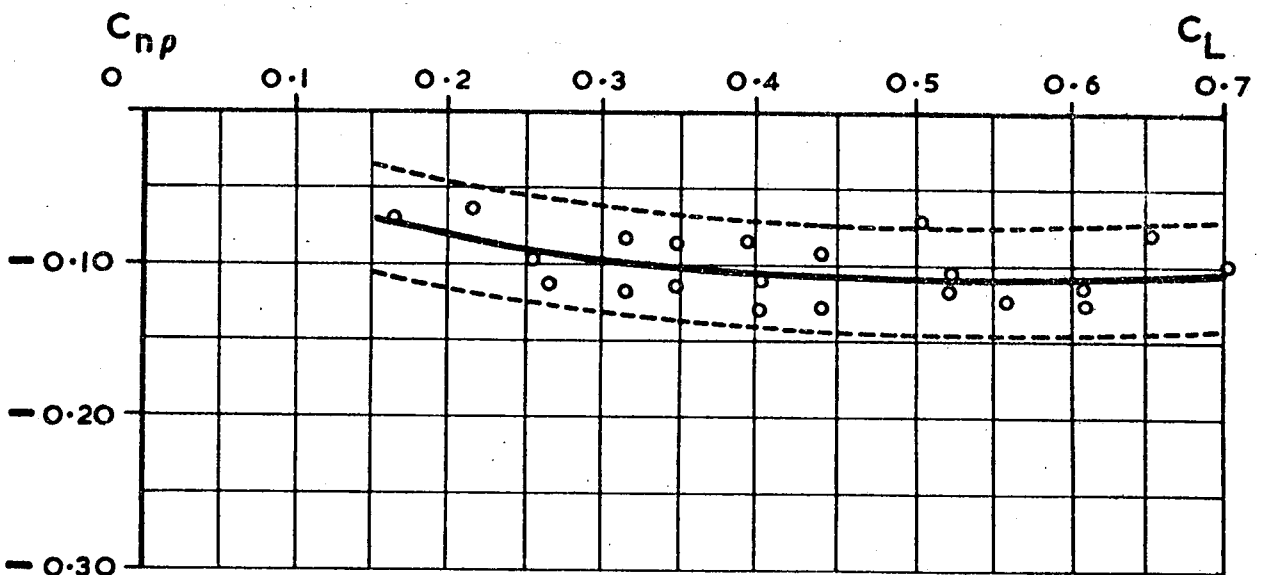


Fig. 20 Variation of Lateral Aerodynamic Derivatives, from Flight Data using Estimation, with Lift Coefficient. PARIS Aircraft—Continued

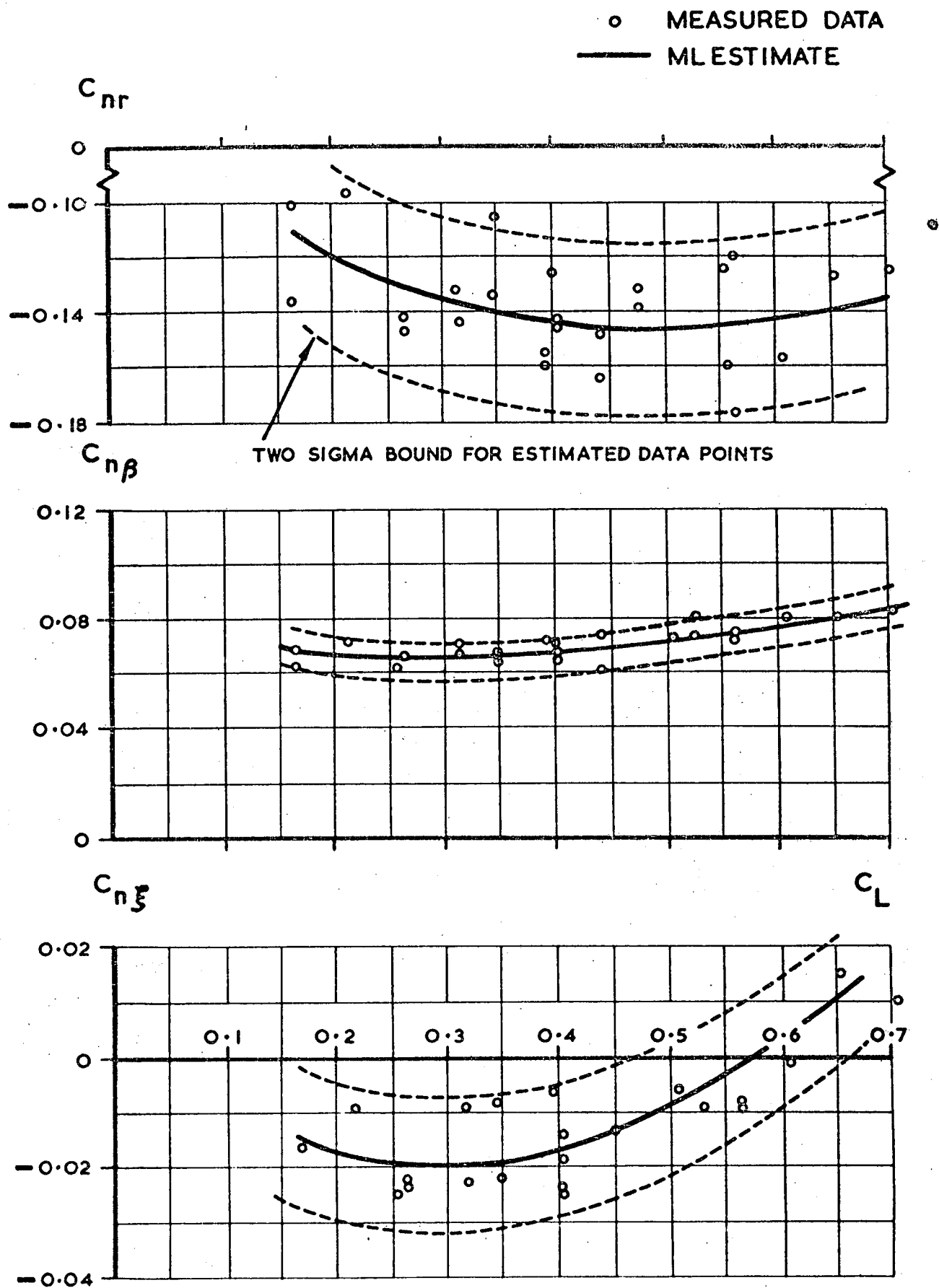


Fig.20 Variation of Lateral Aerodynamic Derivatives, from Flight Data using Estimation, with Lift Coefficient. PARIS Aircraft—Continued

○ MEASURED DATA
— ML ESTIMATE

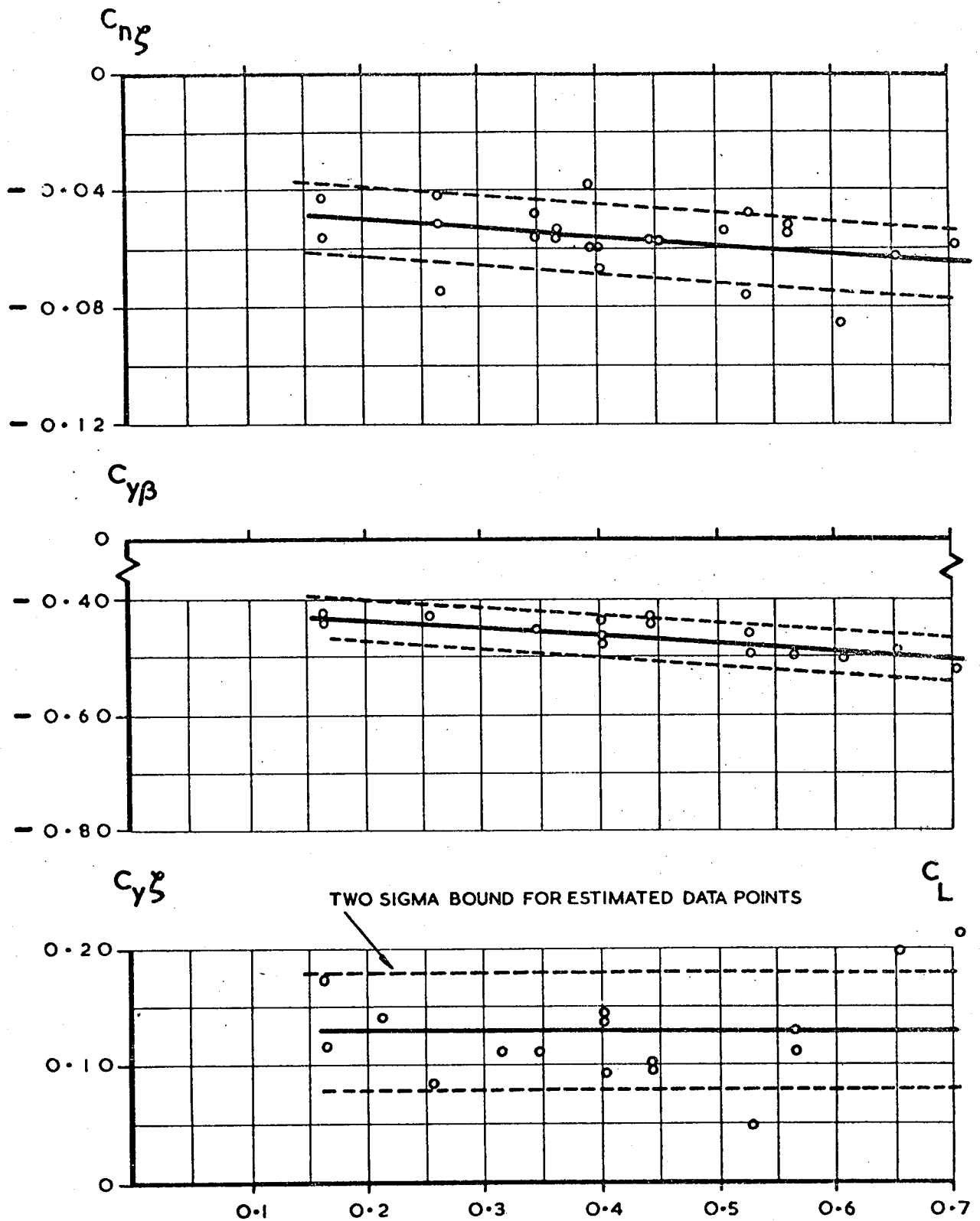


Fig.20 Variation of Lateral Aerodynamic Derivatives, from Flight Data using Estimation, with Lift Coefficient. PARIS Aircraft—
Concluded

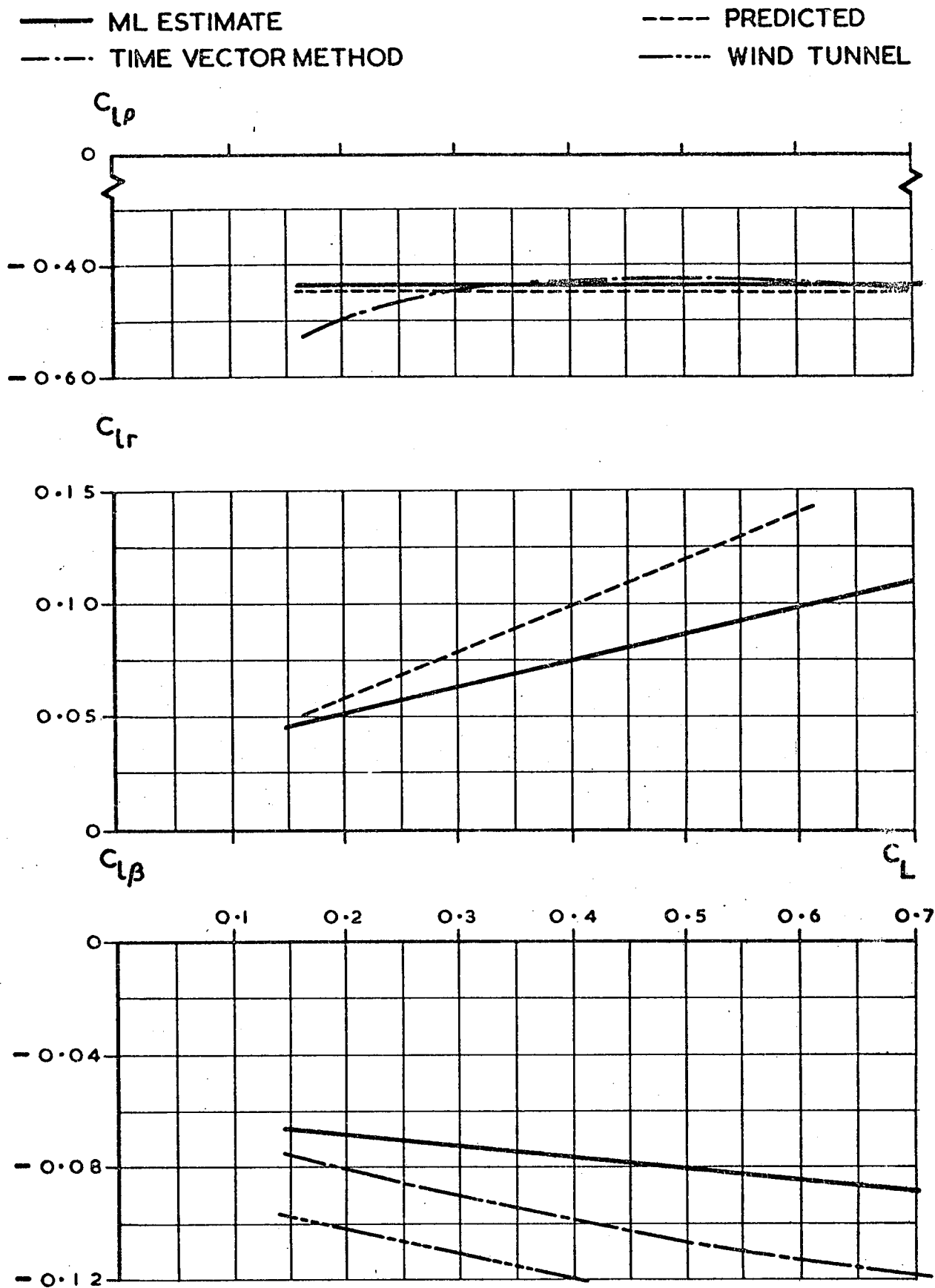


Fig.21 Comparison of Lateral Aerodynamic Derivatives from Unsteady Measurements and Other Sources. PARIS Aircraft

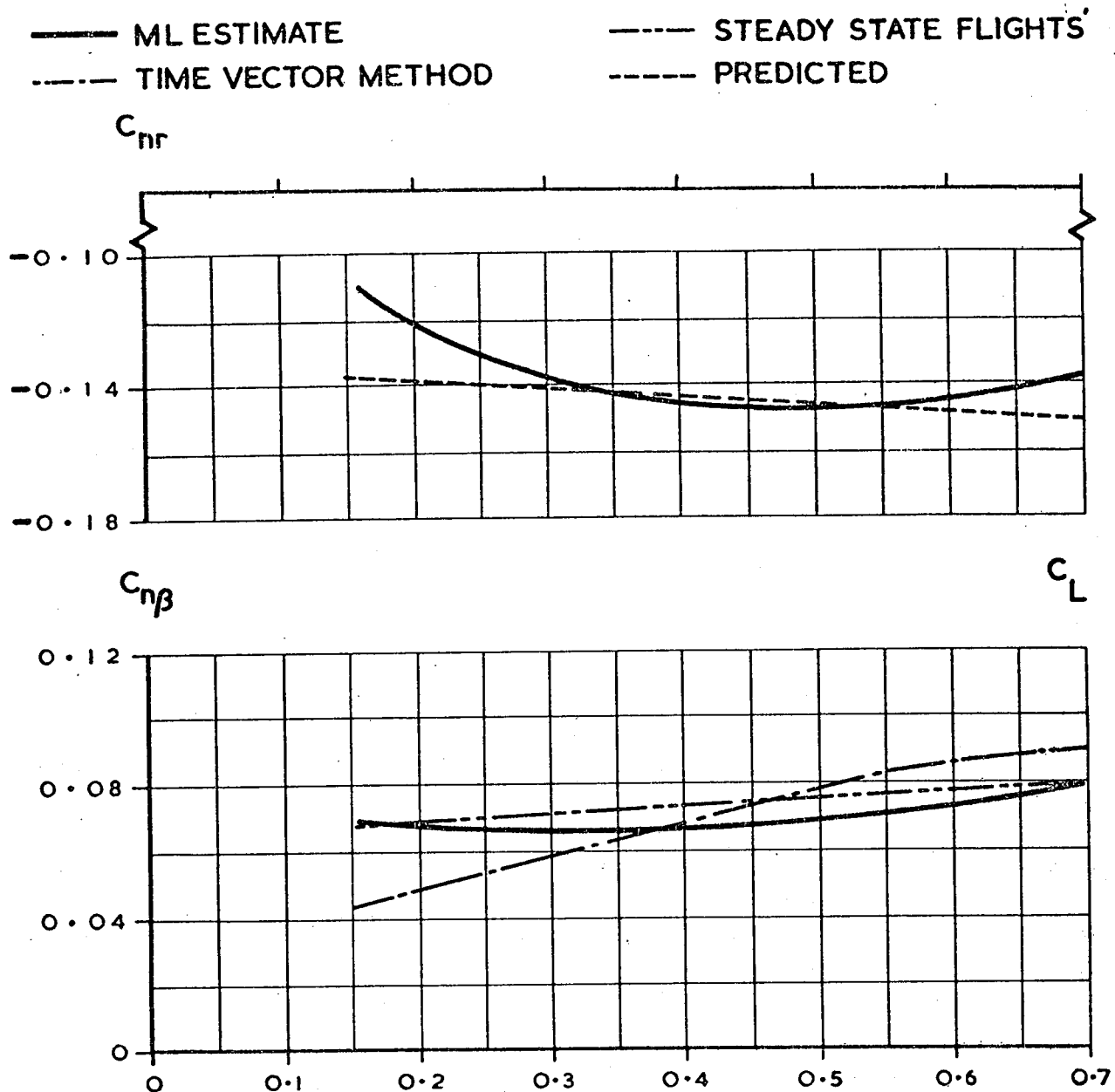


Fig.21 Comparison of Lateral Aerodynamic Derivatives from Unsteady Measurements and Other Sources. PARIS Aircraft— Continued

— ML ESTIMATE - - - - TIME VECTOR METHOD
 - - - - STEADY-STATE FLIGHTS - - - - PREDICTED

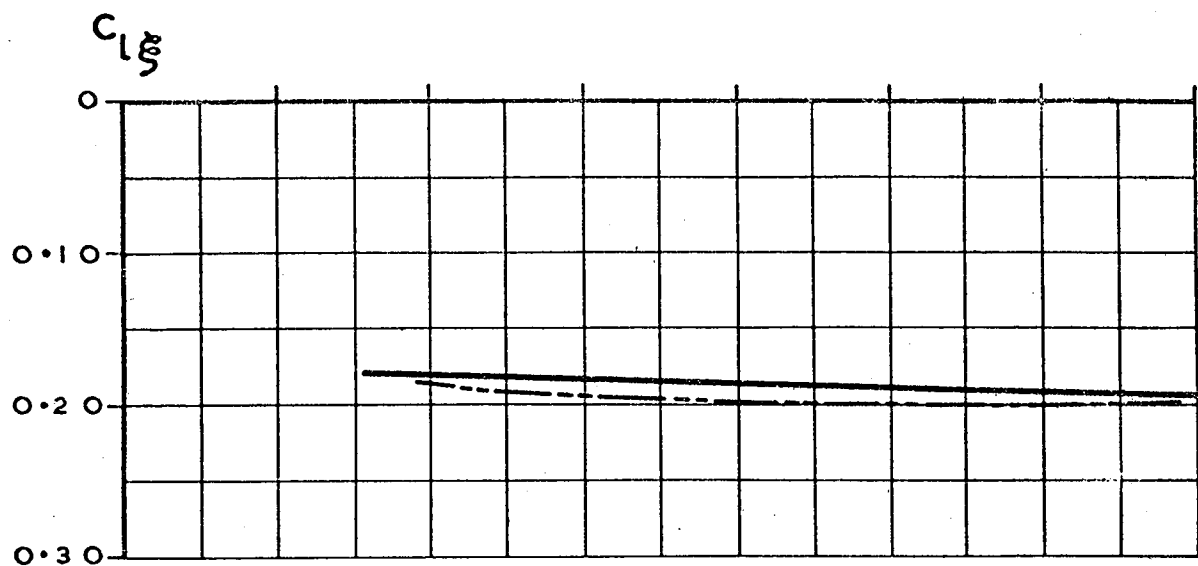
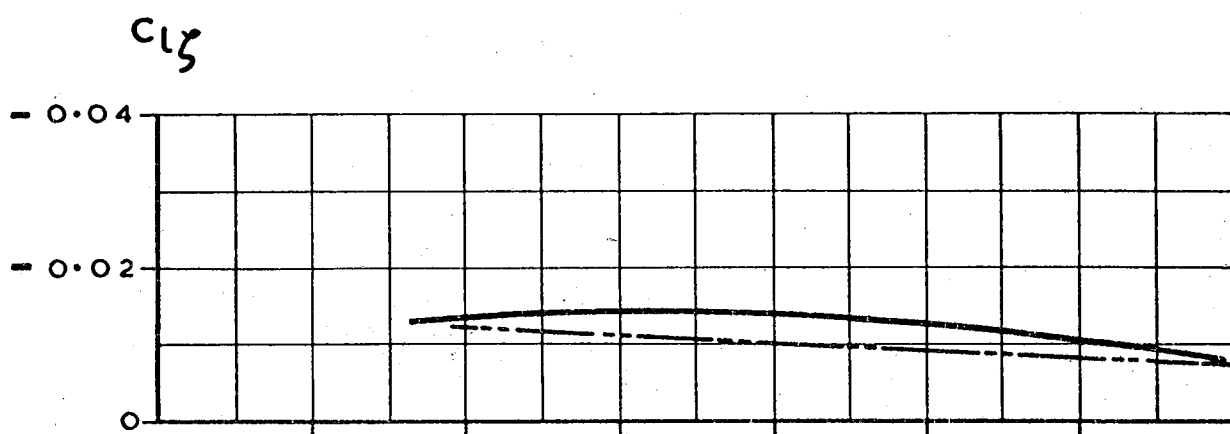
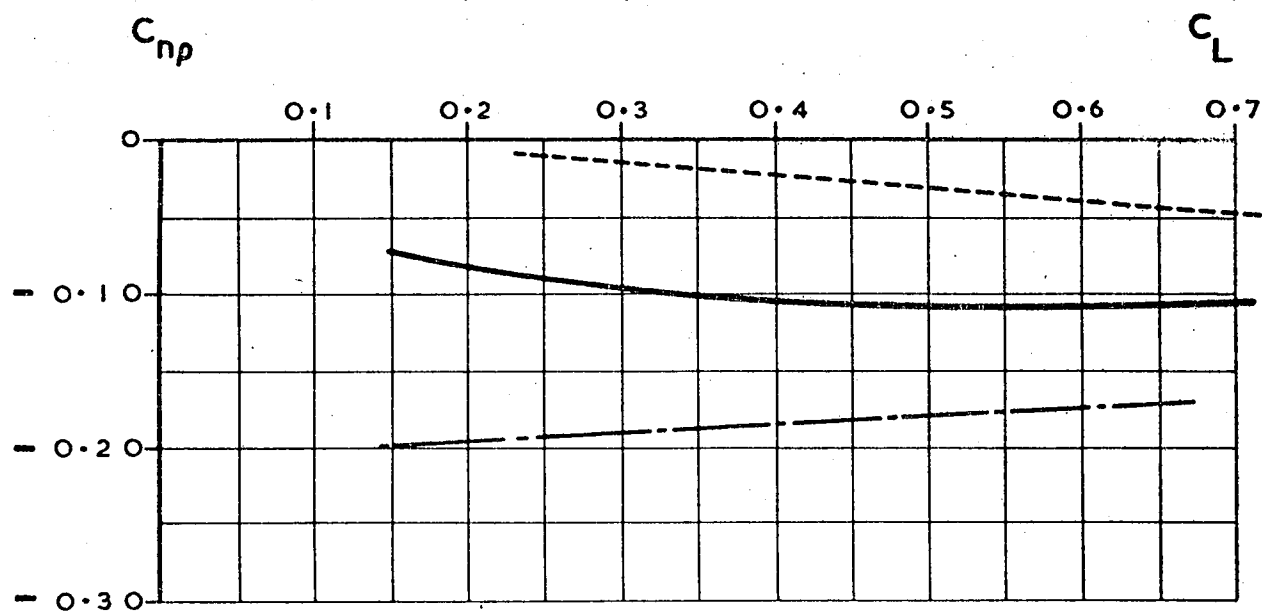


Fig. 21 Comparison of Lateral Aerodynamic Derivatives from Unsteady Measurements and Other Sources. PARIS Aircraft—Continued

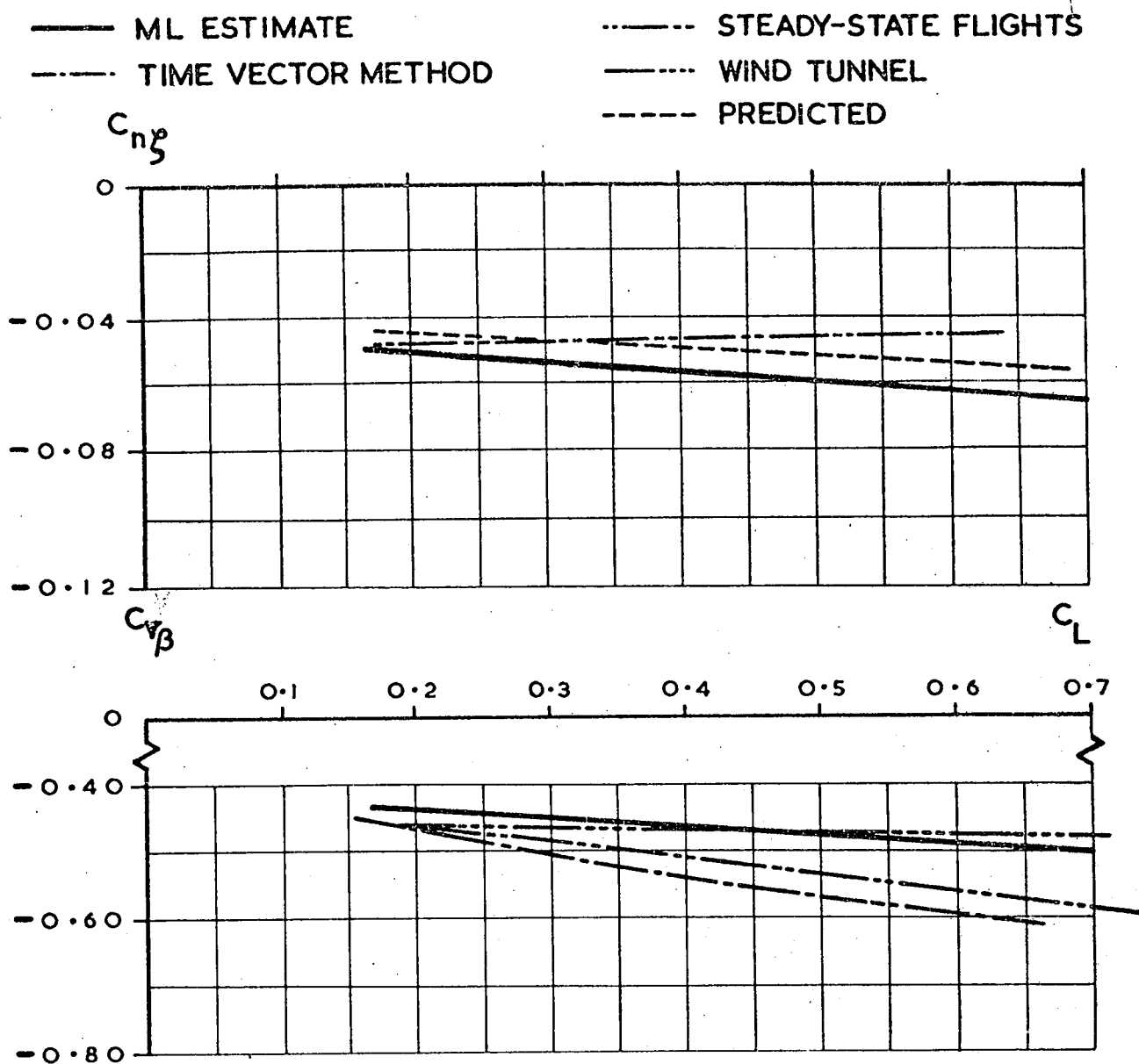


Fig.21 Comparison of Lateral Aerodynamic Derivatives from Unsteady Measurements and Other Sources. PARIS Aircraft— Concluded

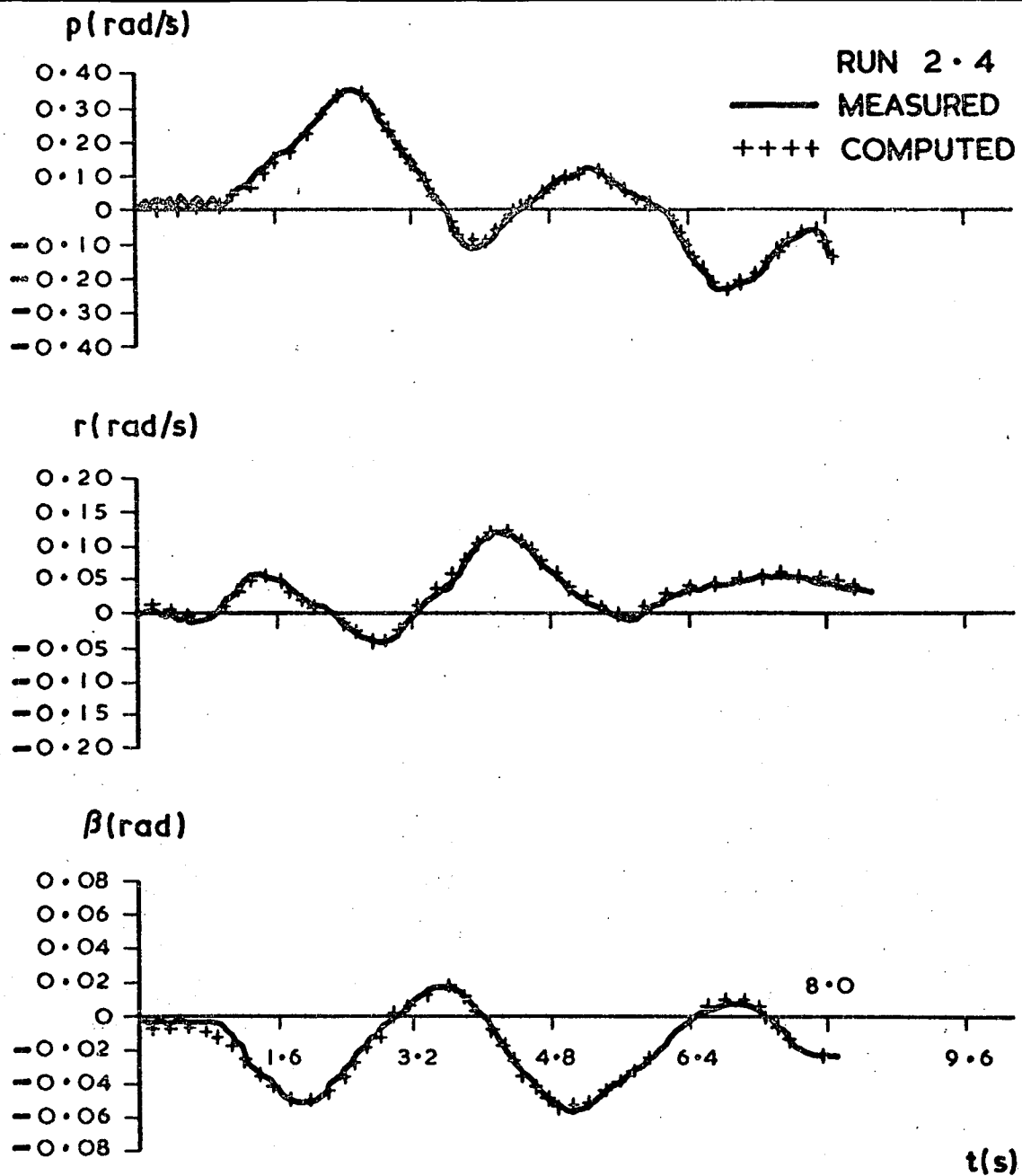


Fig.22 Comparison of Time Histories Measured in Flight and Computed Using ML Estimation. PARIS Aircraft

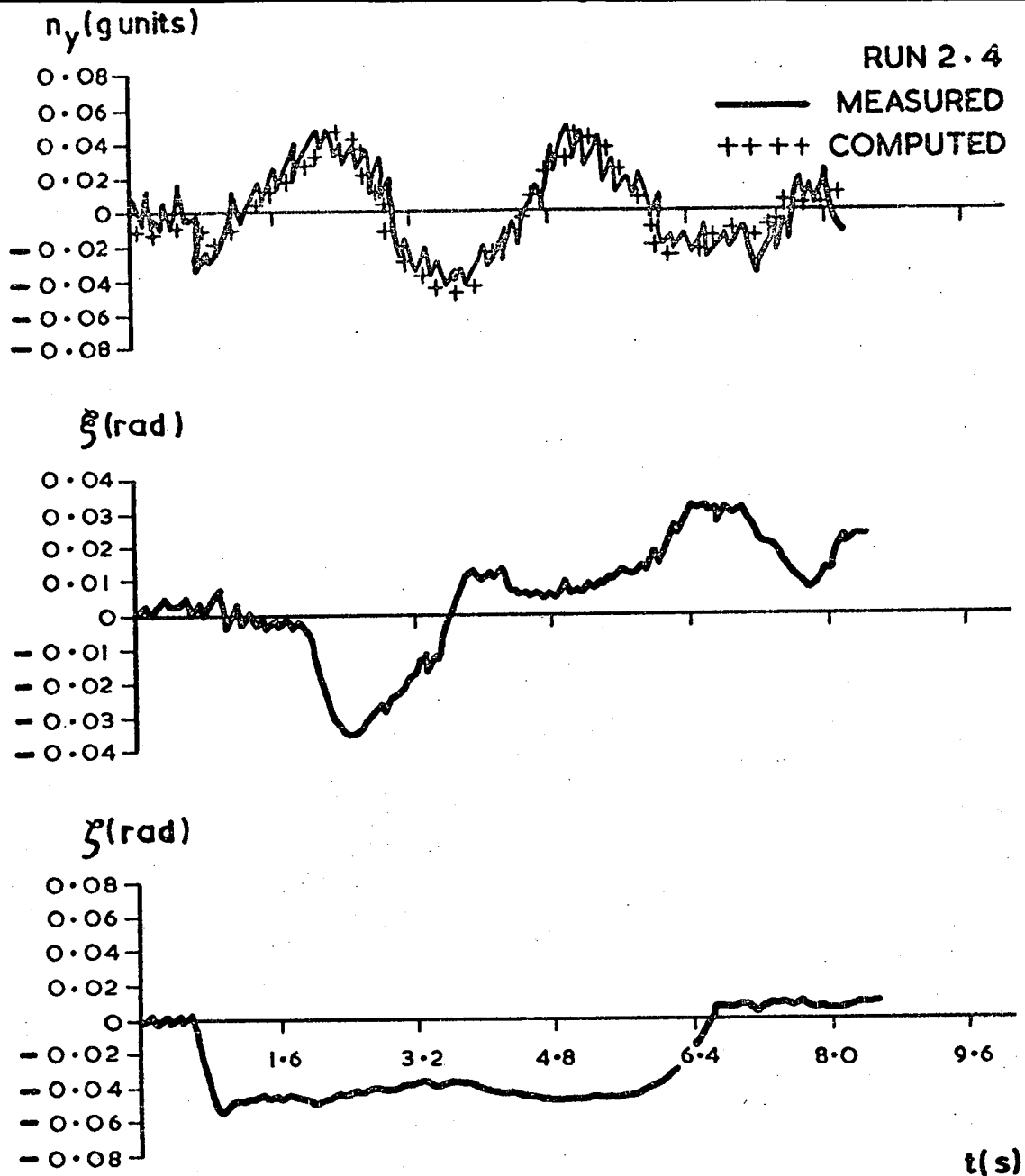


Fig.22 Comparison of Time Histories Measured in Flight and Computed Using ML Estimation. PARIS Aircraft — Concluded

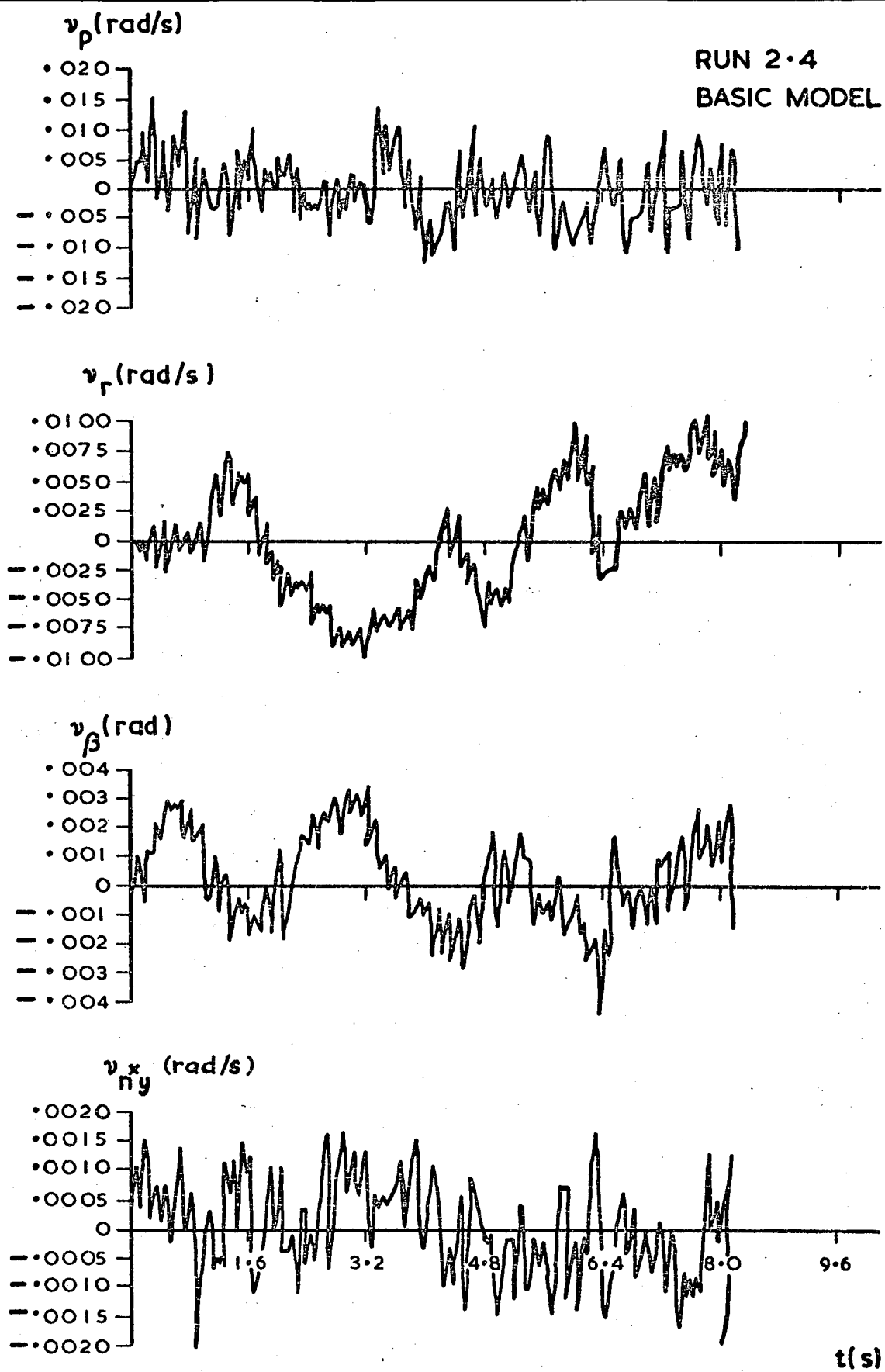


Fig.23 Time Histories of Residuals. PARIS Aircraft

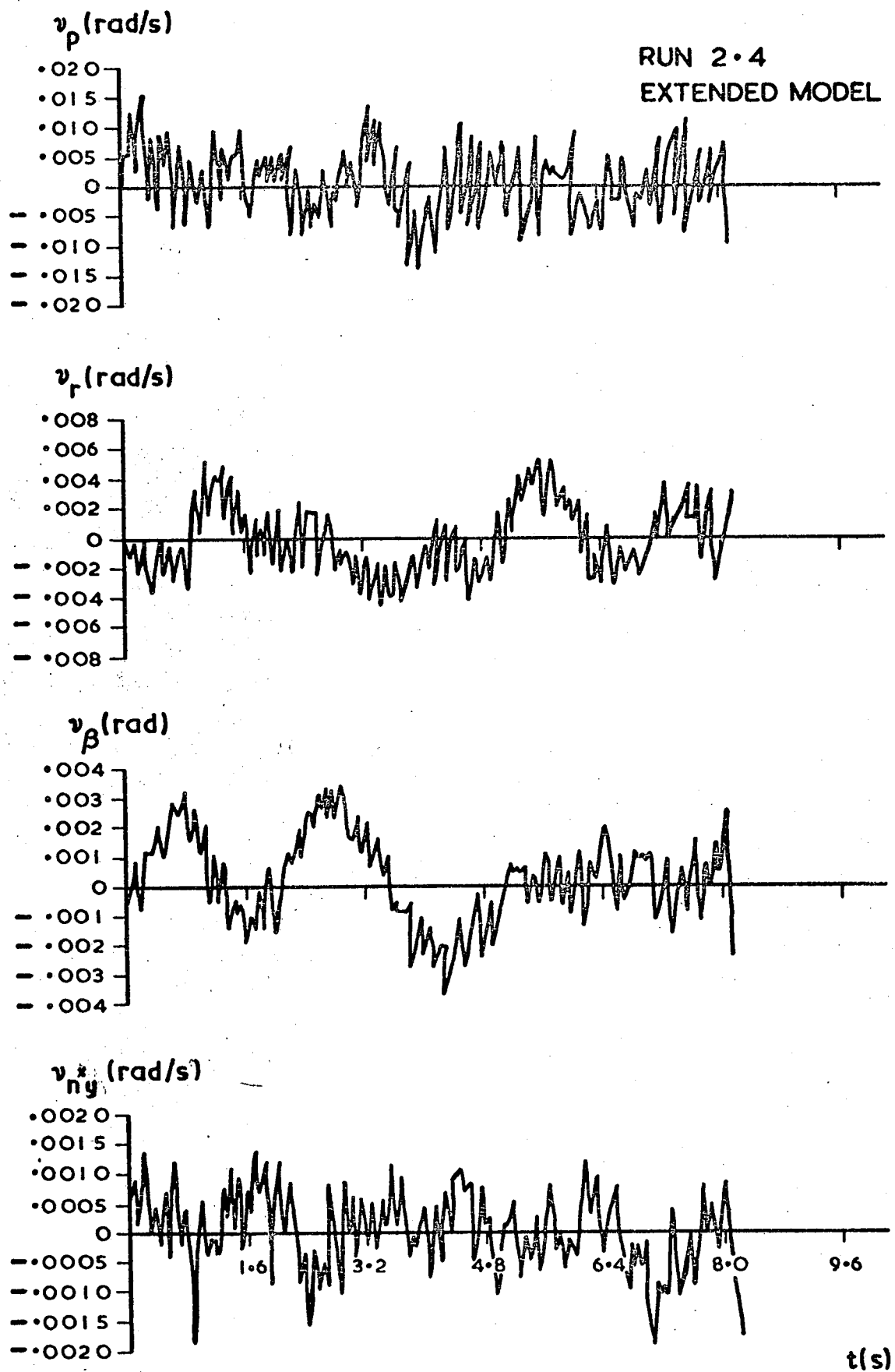


Fig.24 Time Histories of Residuals. PARIS Aircraft

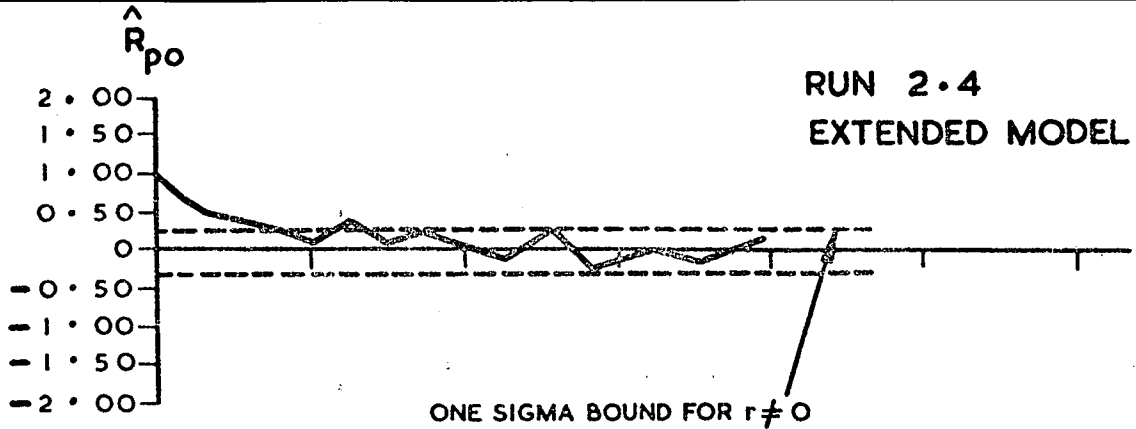
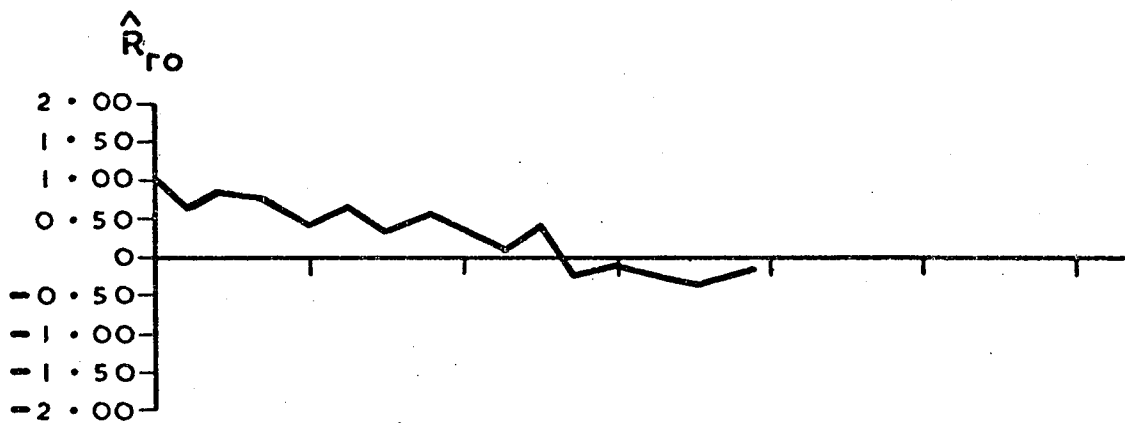
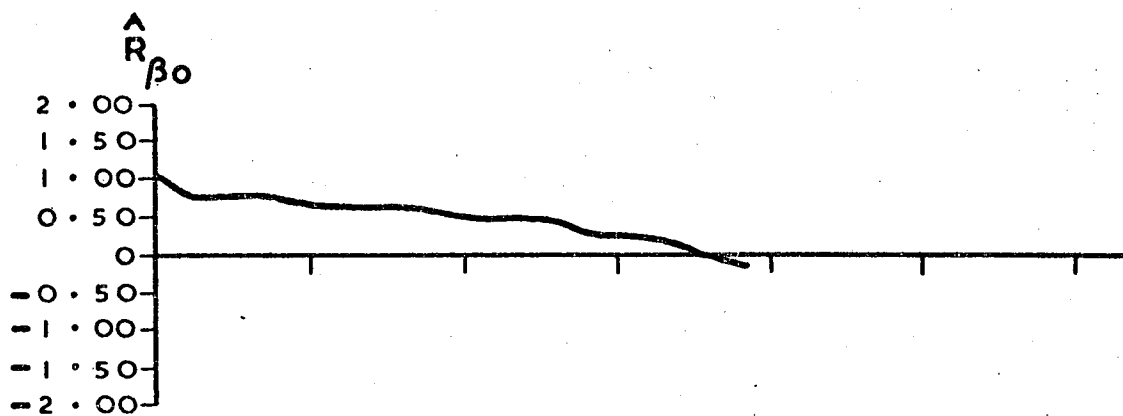
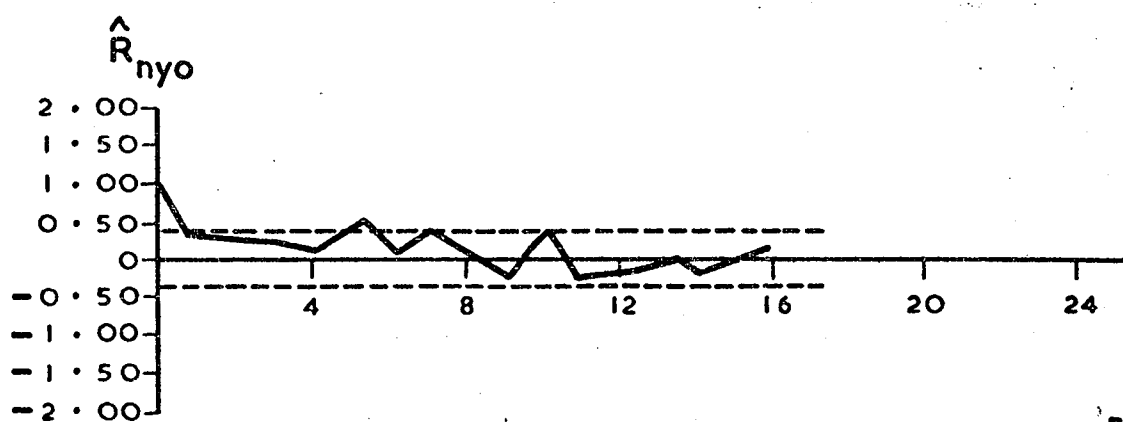


Fig.25 Sample Autocovariance Functions of Residuals. PARIS Aircraft

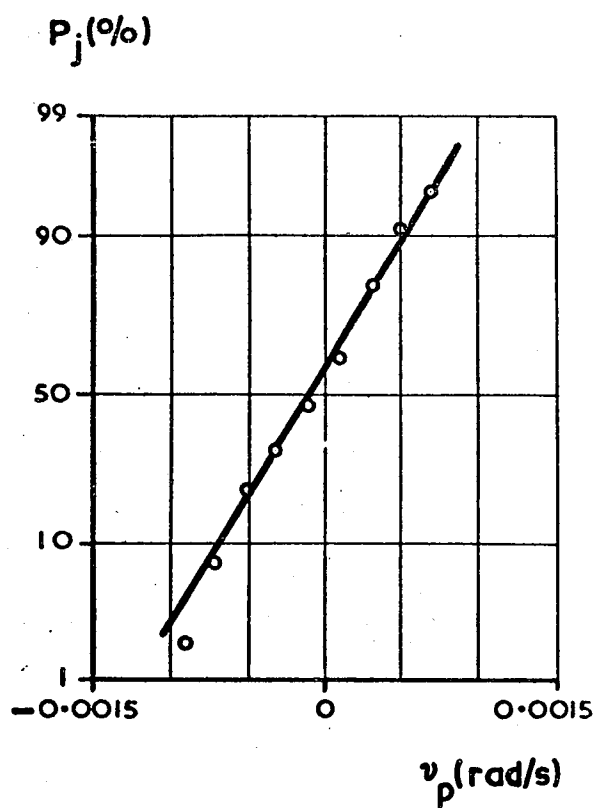
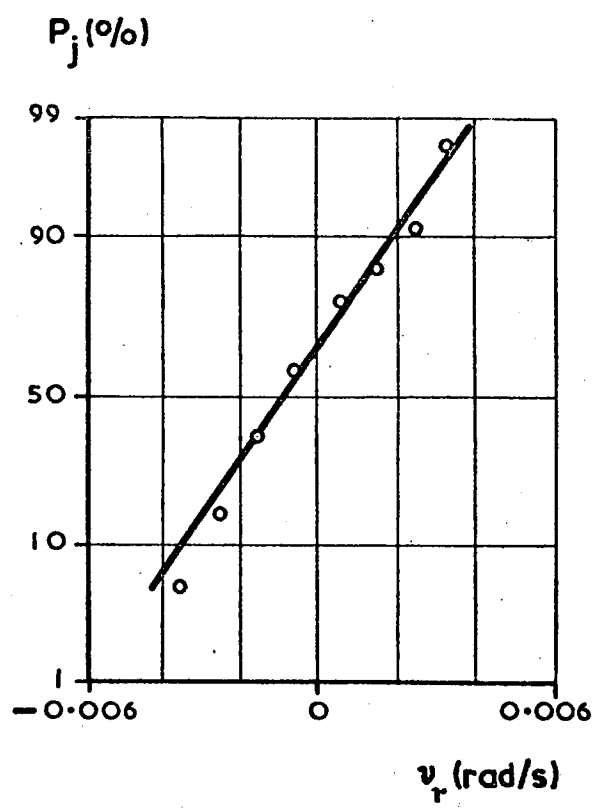
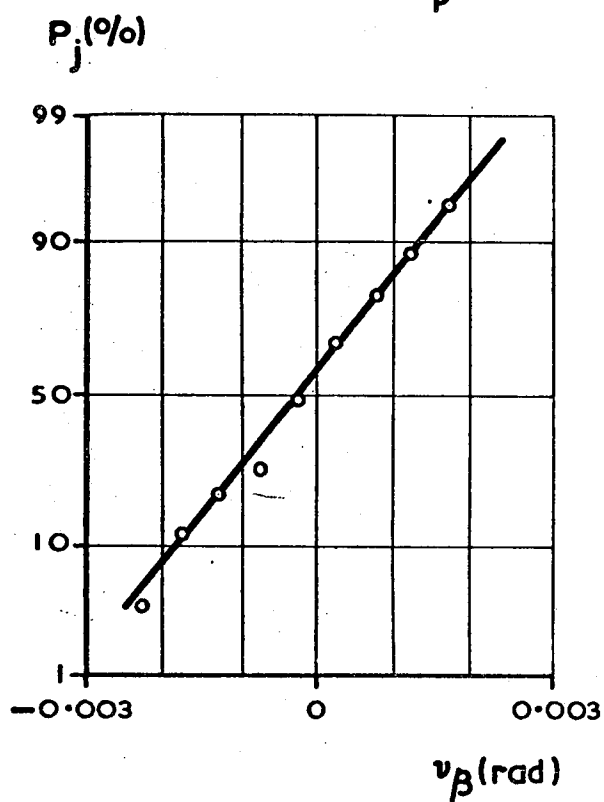
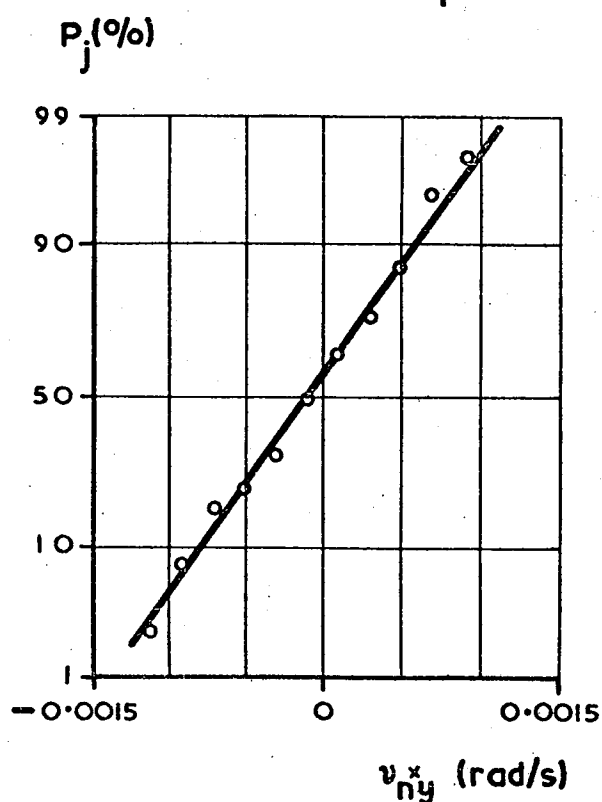


Fig.26 Cumulative Frequency of Residuals. PARIS Aircraft
(Run 2.4-Extended Model)

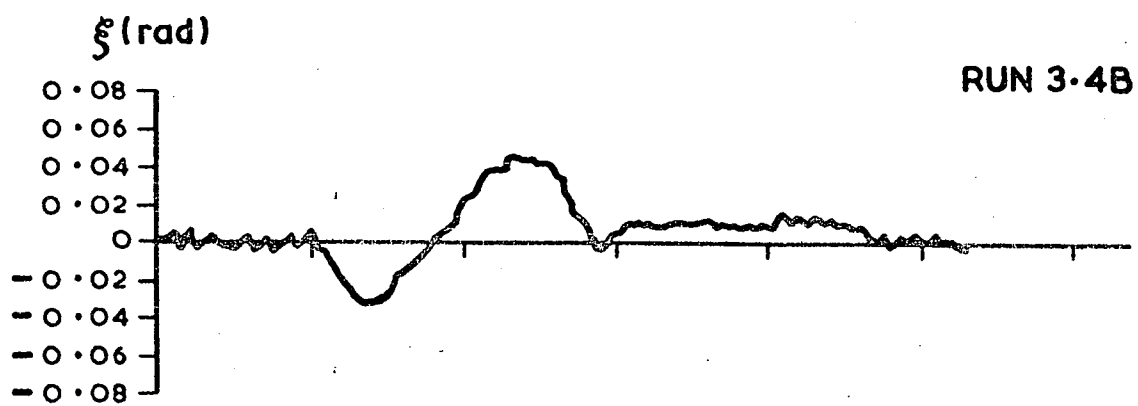
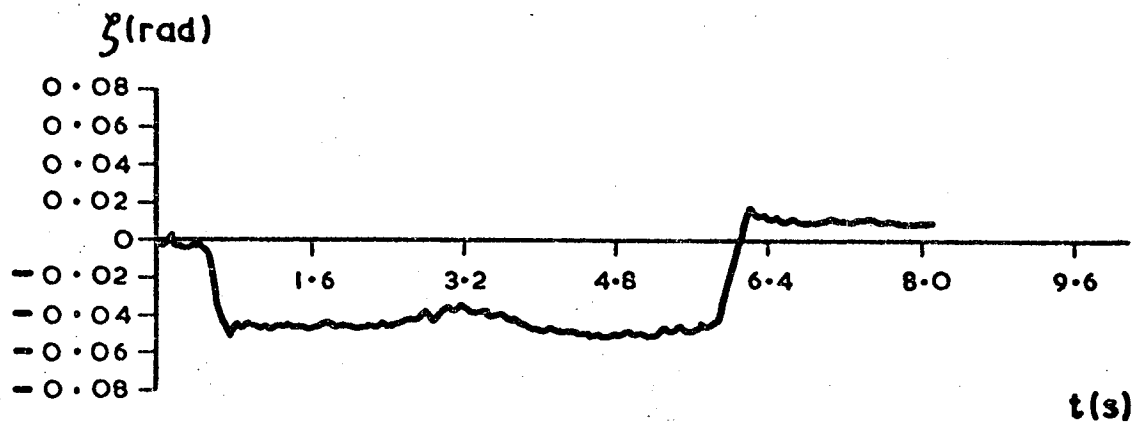
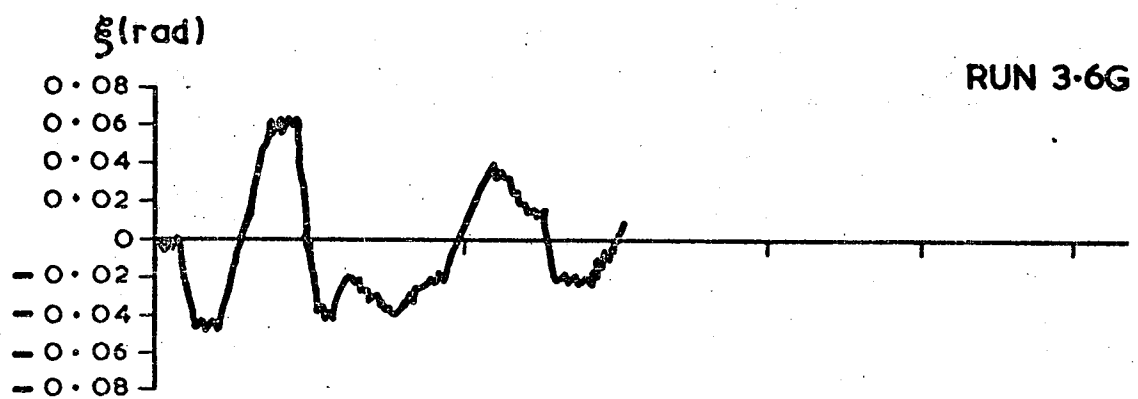
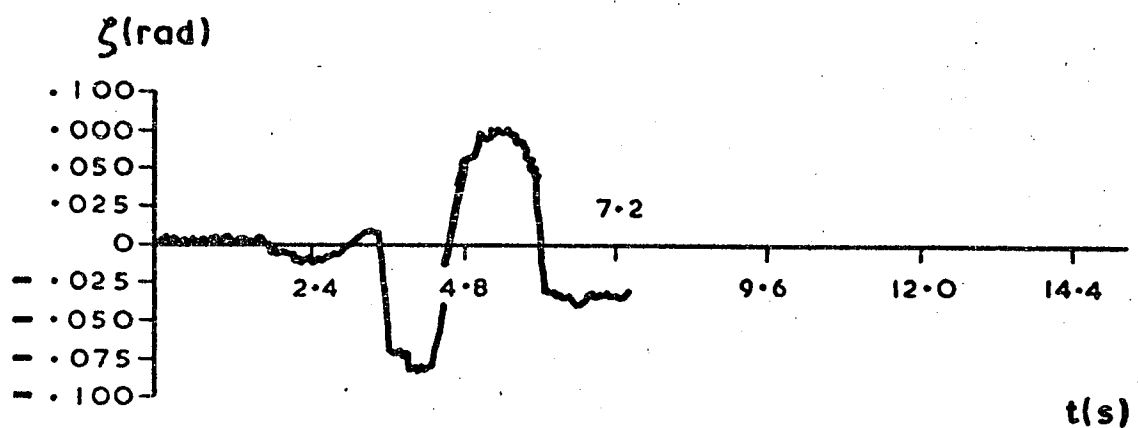


Fig.27 Different Input Forms used in the Excitation of the Lateral Motion. PARIS Aircraft

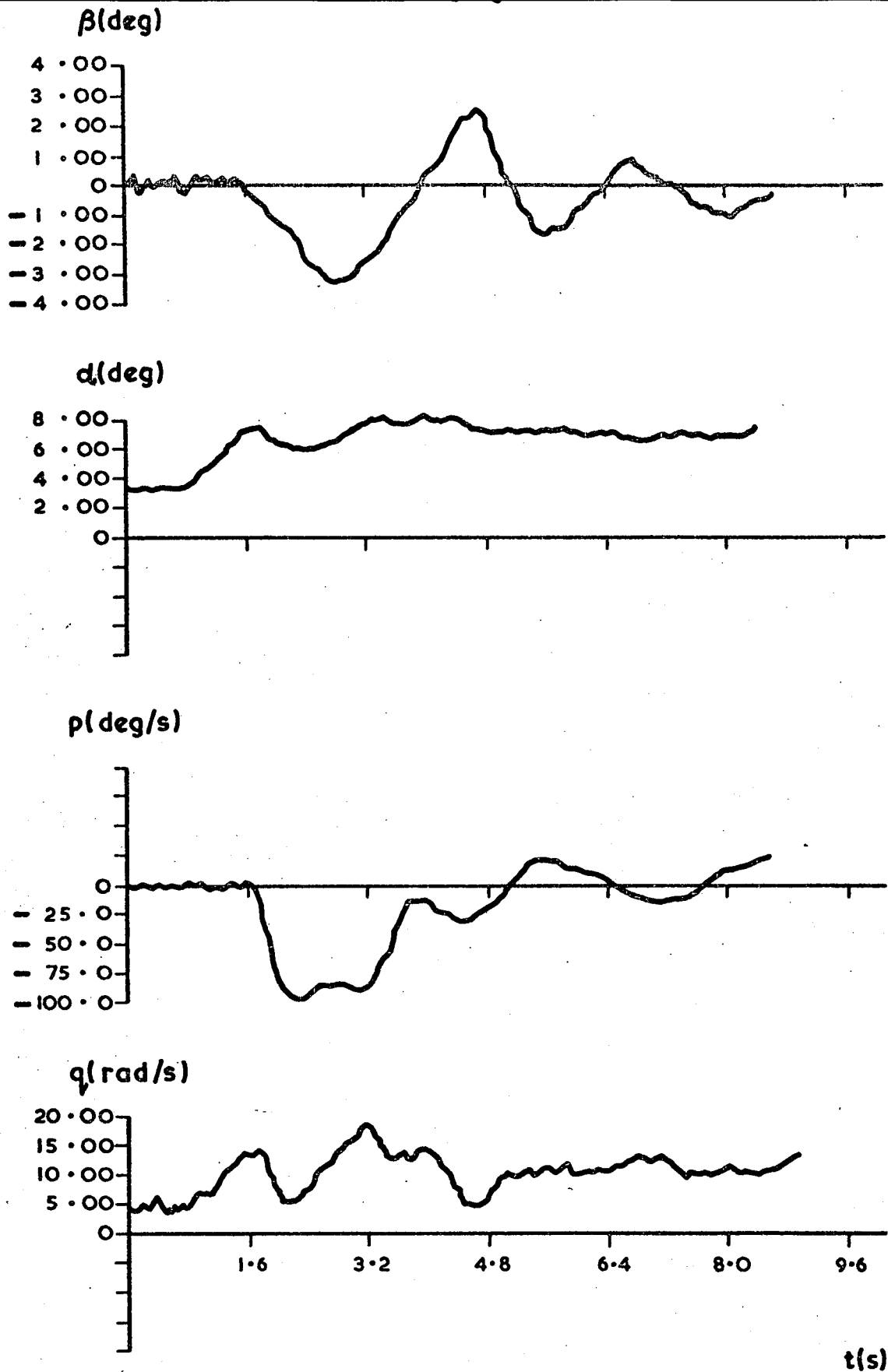


Fig.28 Simulated Rapid Rolling Manoeuvre .F-100 Aircraft (Run I)

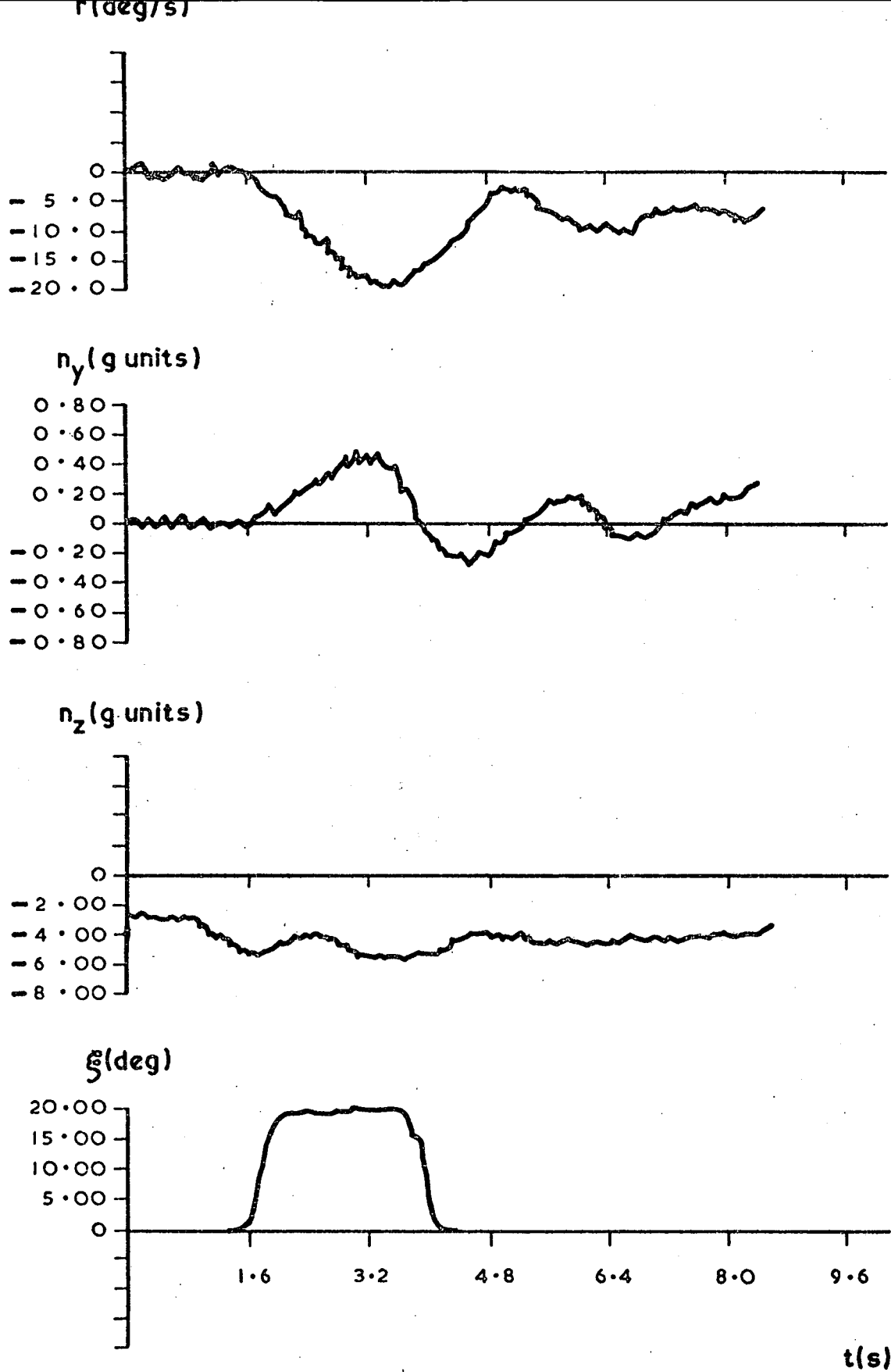


Fig.28 Simulated Rapid Rolling Manoeuvre. F-100 Aircraft (Run I)
- Continued

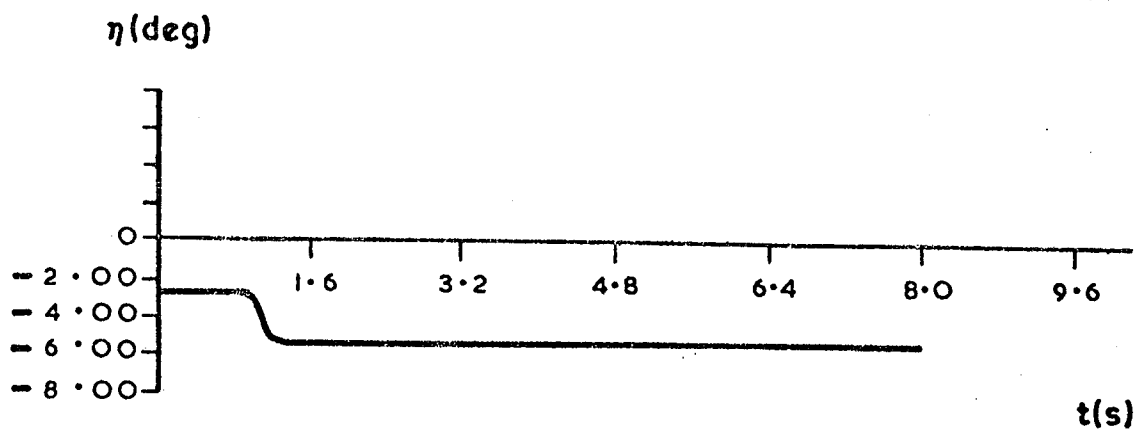


Fig.28 Simulated Rapid Rolling Manoeuvre. F-100 Aircraft (Run 1)
— Concluded

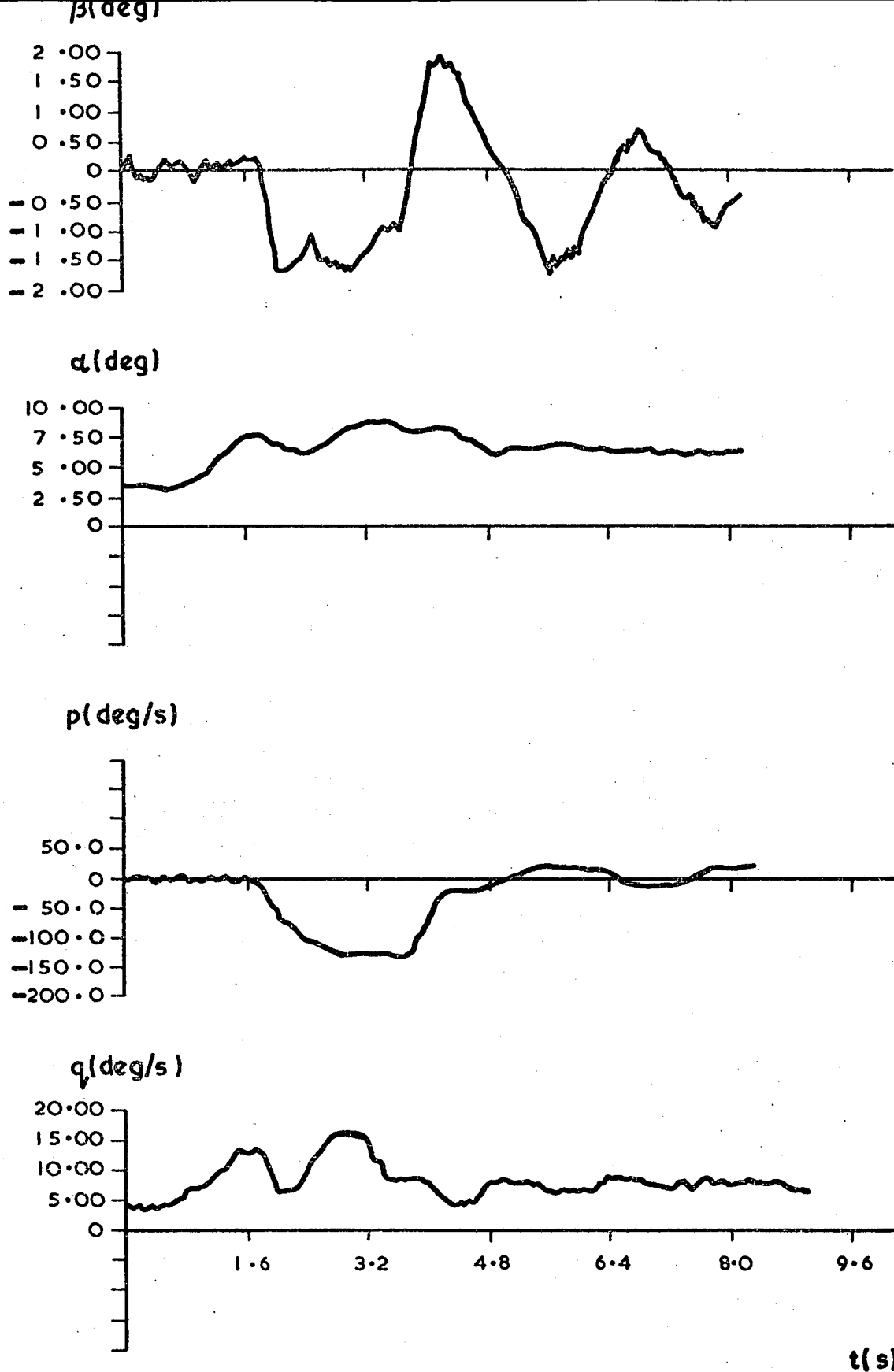


Fig.29 Simulated Rapid Rolling Manoeuvre. F-100 Aircraft (Run 2)

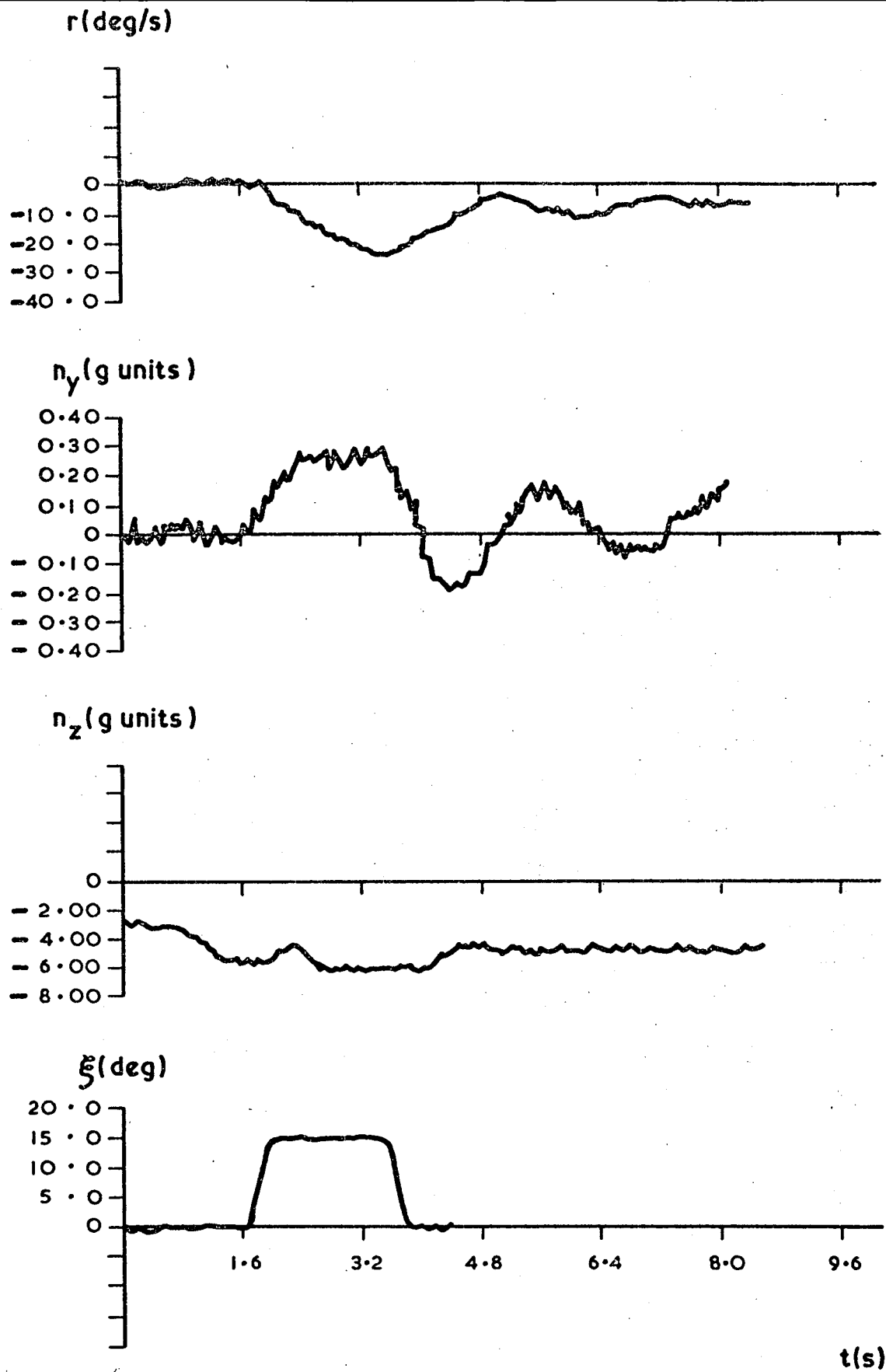


Fig.29 Simulated Rapid Rolling Manoeuvre. F-100 Aircraft (Run 2)

—Continued

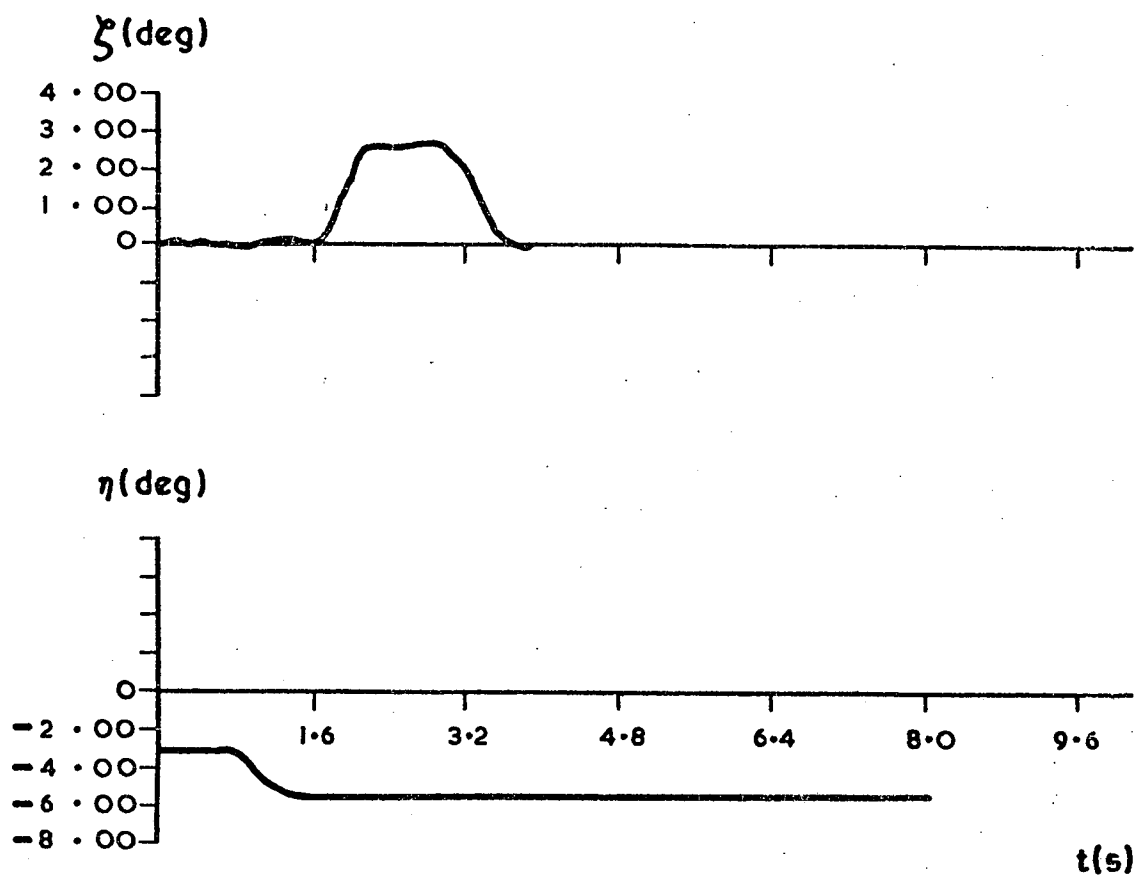


Fig.29 Simulated Rapid Rolling Manoeuvre . F-100 Aircraft (Run 2)
— Concluded

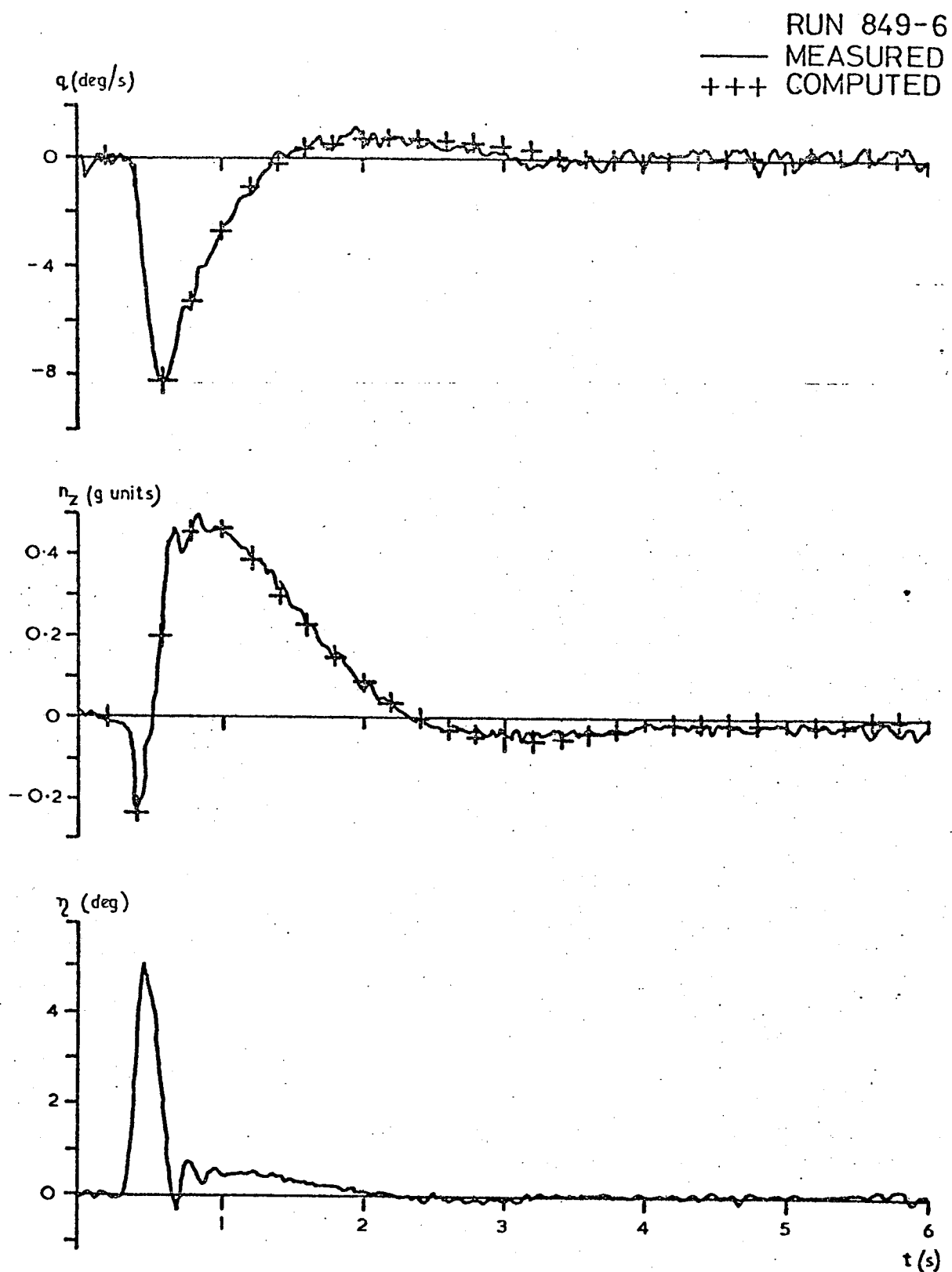


Fig.30 Comparison of Time Histories Measured with those Computed using ML Estimation. HP-115 Aircraft, Linear Model, $\alpha_e = 7.2$ deg.

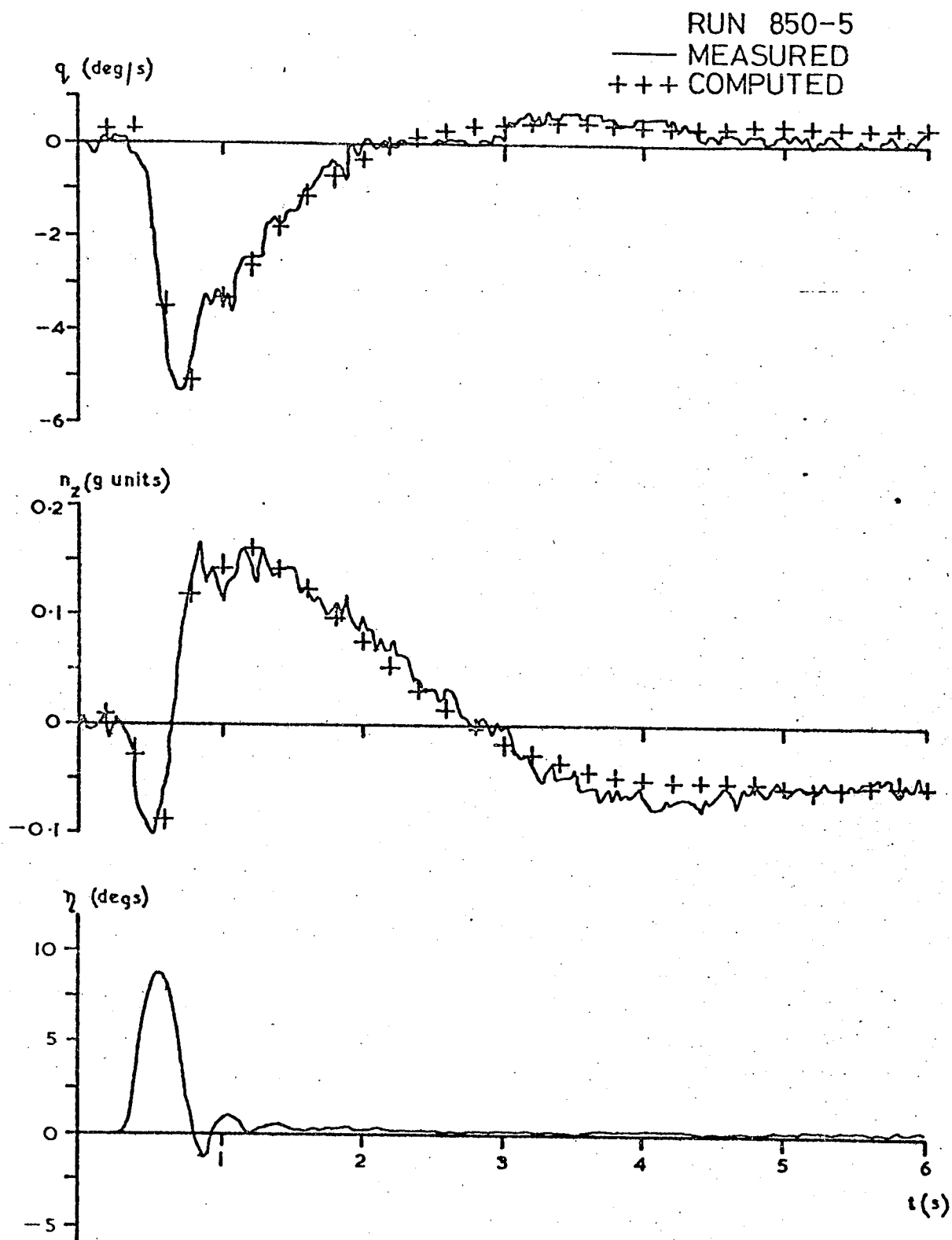


Fig.31 Comparison of Time Histories Measured with those Computed using ML Estimation. HP-115 Aircraft, Linear Model, $\alpha_e = 20.3$ deg.

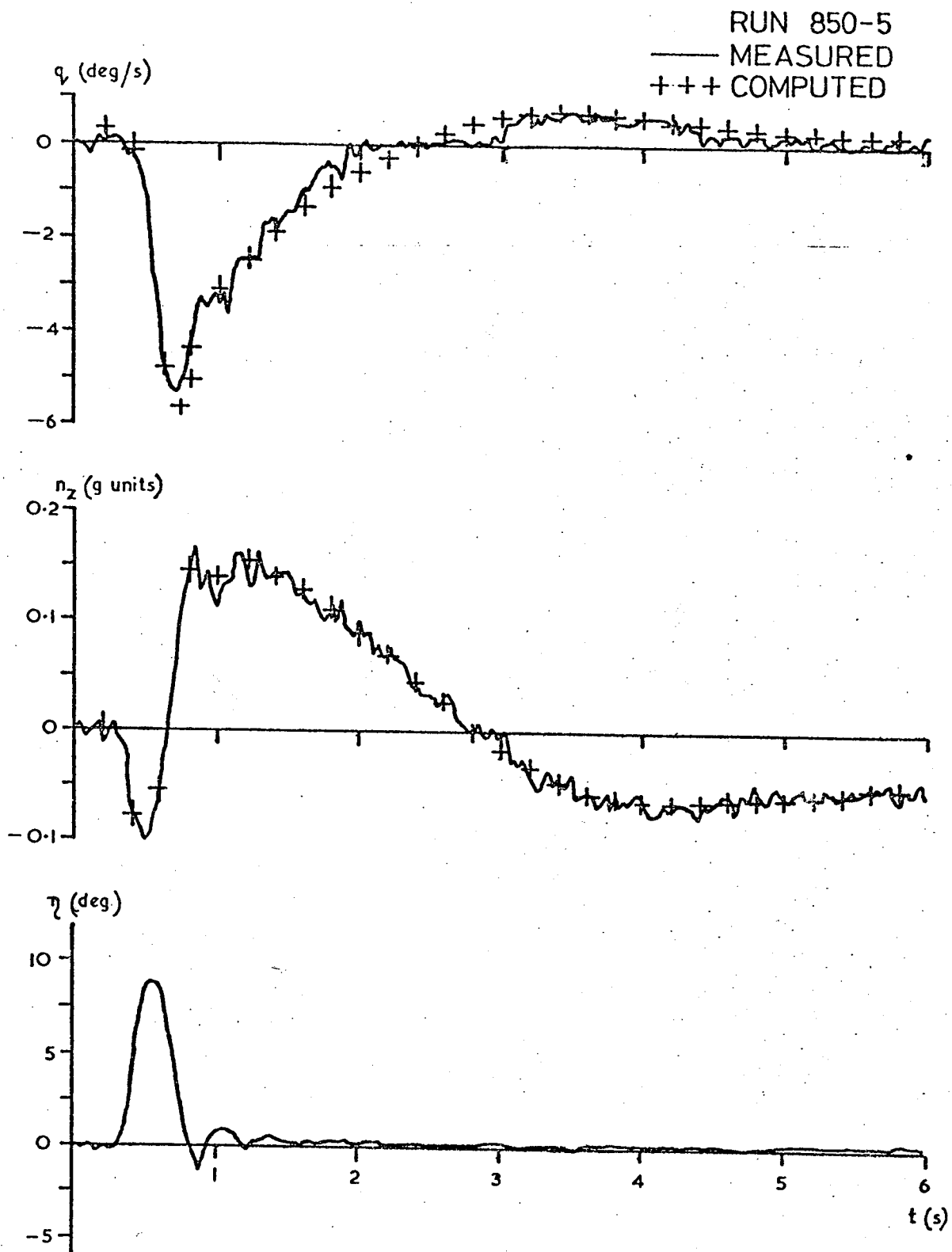


Fig.32 Comparison of Time Histories Measured with those Computed using ML Estimation. HP-115 Aircraft, Non-Linear Model, $\alpha_e = 20.3$ deg.

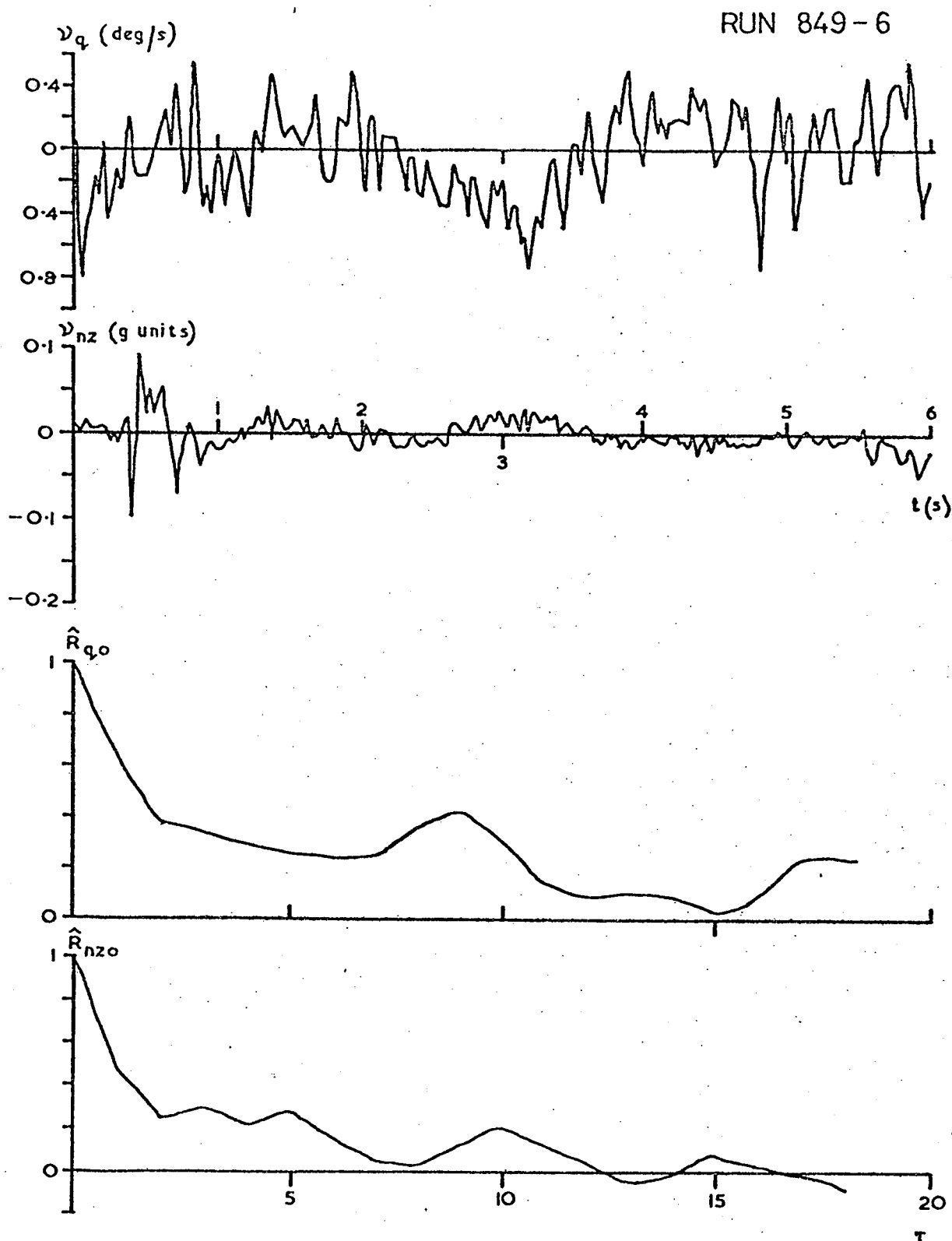


Fig. 33 Time Histories and Sample Autocovariance Functions of the Residuals. HP-115 Aircraft, Linear Model, $\alpha_e = 7.2$ deg.

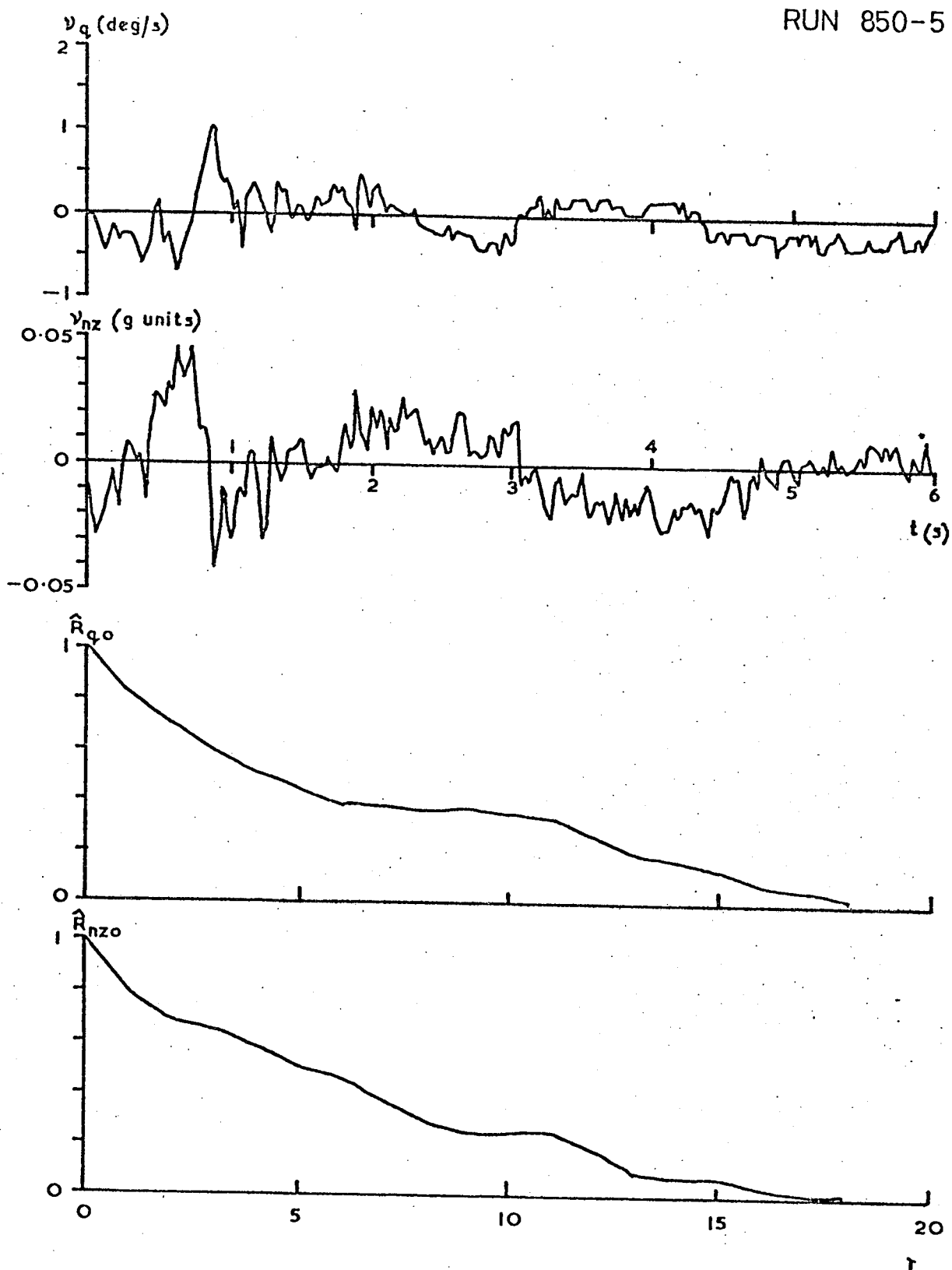


Fig. 34 Time Histories and Sample Autocovariance Functions of the Residuals. HP-115 Aircraft, Linear Model, $\alpha_e = 20.3$ deg.

RUN 850-5

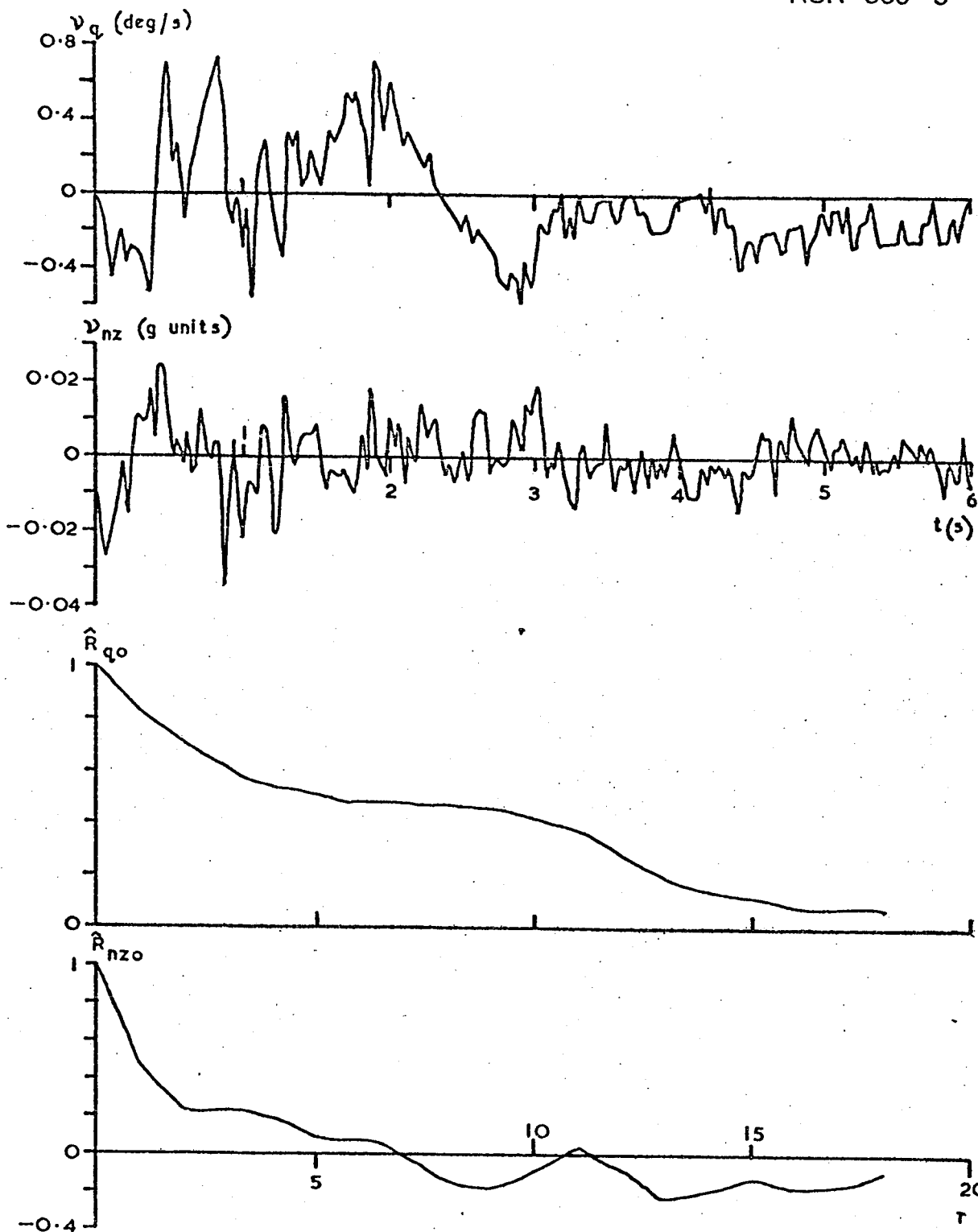


Fig. 35 Time Histories and Sample Autocovariance Functions of the Residuals. HP-115 Aircraft, Non-Linear Model, $\alpha_e = 20.3$ deg.

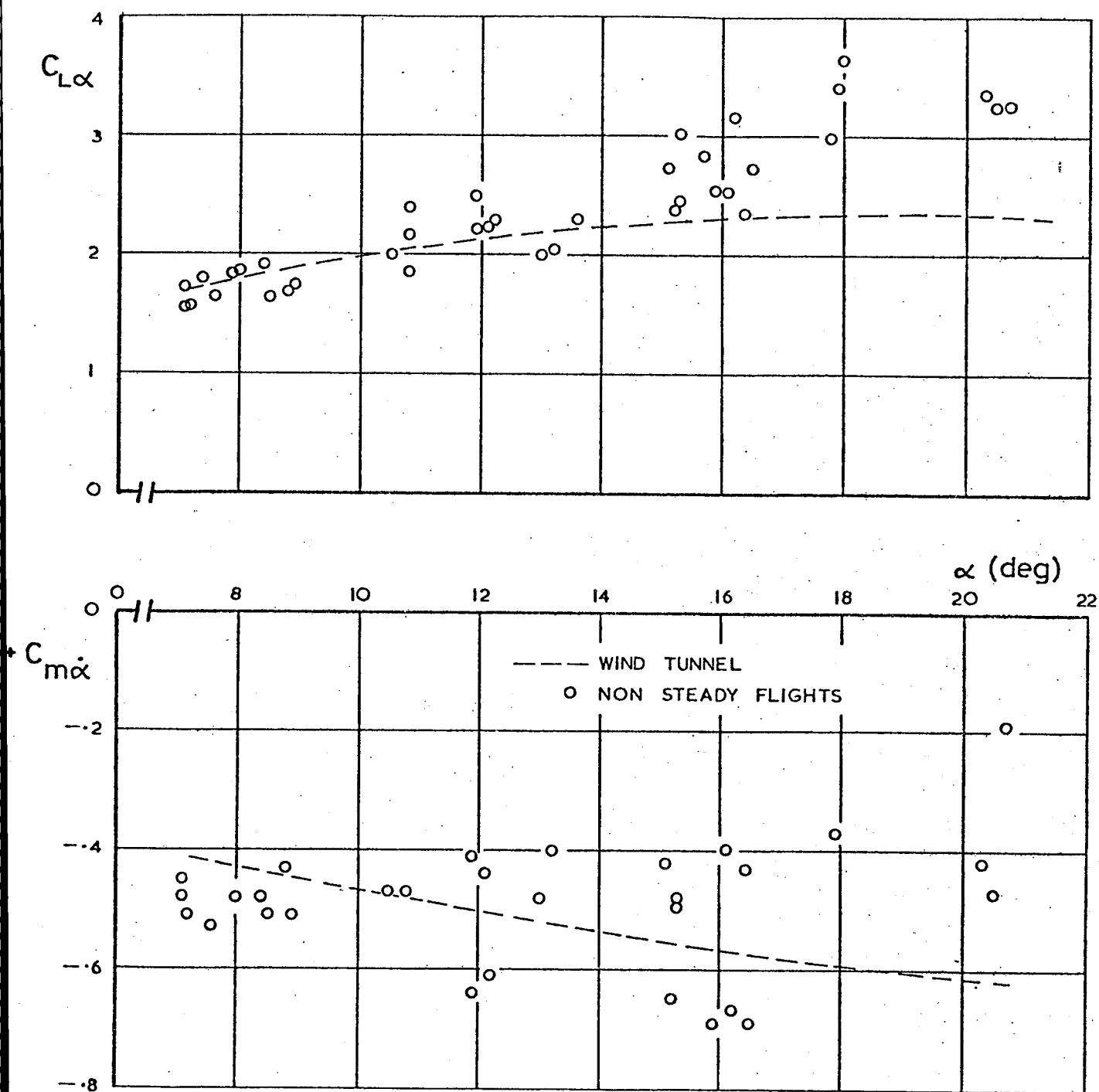


Fig.36 Comparison of Aerodynamic Derivatives from Unsteady Measurements and Other Sources.
HP-115 Aircraft.

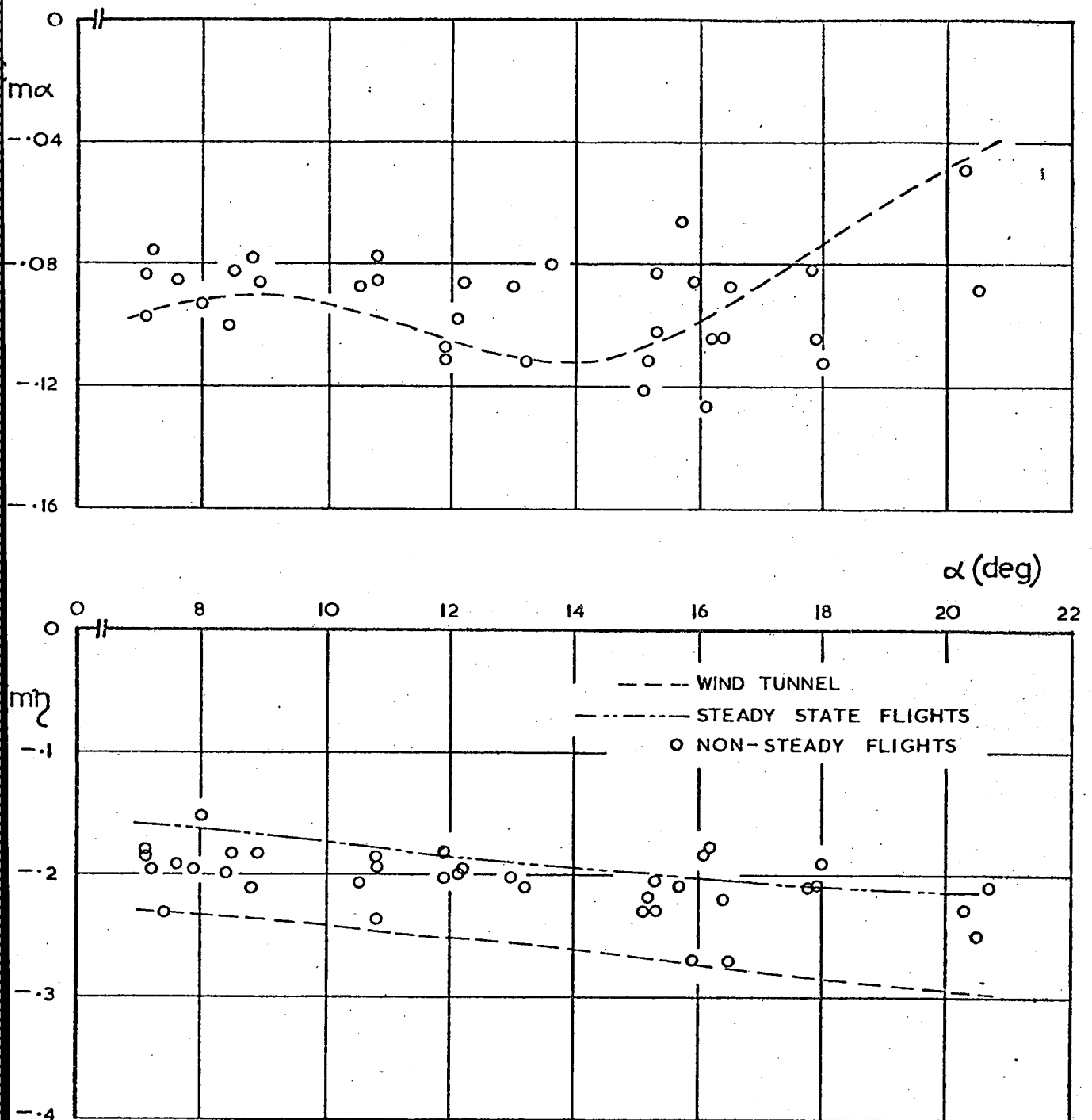


Fig.36 Comparison of Aerodynamic Derivatives from Unsteady Measurements and Other Sources.
HP-115 Aircraft. — Continued.

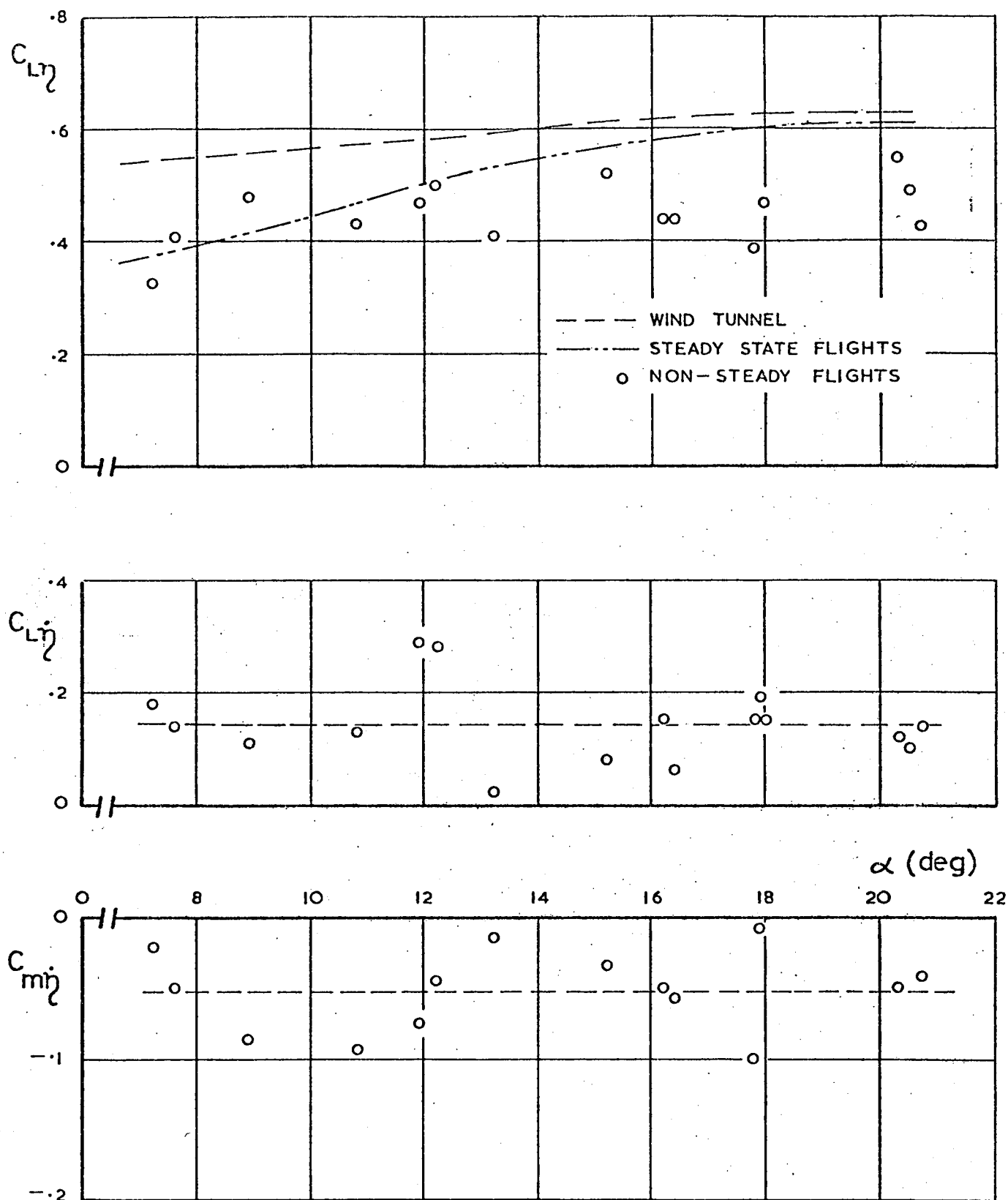


Fig.36 Comparison of Aerodynamic Derivatives from Unsteady Measurements and Other Sources.
HP-115 Aircraft. — Concluded.