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A Class of Upwind Methods for Conservation Laws

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Abstract

Various new methods for the solution of hyperbolic systems of conservation laws in one, two and three space dimensions are developed. All are explicit, conservative time-marching methods that are second order accurate in space and time in regions of smooth flow and make use of local Riemann problems at intercell boundaries.

In one space dimension, the Weighted Average Flux (waf) approach of Toro is extended to generate a scheme that is stable with timesteps twice as large as those allowed by the stability conditions of the original scheme. A Riemann problem based extension of the Warming-Beam scheme is considered. Total Variation Diminishing (tvd) conditions are enforced for both schemes. Numerical results for the Euler Equations of Gas Dynamics are presented.

In two and three space dimensions, finite volume versions of the waf scheme on Cartesian grids are derived for the linear advection equation. Two two dimensional schemes are found that are second order accurate in space and time. One of these is extended for the solution of nonlinear systems of hyperbolic conservation laws in two separate ways. The resulting schemes are tested on the Shallow Water equations. The equivalent three dimensional schemes are also discussed. The two dimensional schemes are then extended for use on structured, body-fitted grids of quadrilaterals and one of these extensions is used to demonstrate the phenomena of Mach reflection of shallow water bores.

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Notation

A	Jacobian of a flux function
A^n	Jacobian of the projection of a flux function onto the vector $\mathbf{n}$
A	coefficient in a numerical scheme
a	wave speed in the x direction for the linear advection equation or sound speed in Gas dynamics
B	flux limiter
b	wave speed in the y direction for the linear advection equation
$C_{i+\frac{1}{2}}, C_i$	coefficient in TVD theorems
c	wave speed in the z direction for the linear advection equation or celerity for the Shallow Water equations
$D_{i+\frac{1}{2}}, D_i$	coefficient in TVD theorems
E	total energy
$E_{i+\frac{1}{2},j}, E_{i,j+\frac{1}{2}}$	intercell fluxes
e	internal energy
F	vector flux function in the x direction
$F_{i+\frac{1}{2}}, F_{i+\frac{1}{2},j}, F_{i+\frac{1}{2},j,k}$	intercell flux in the x direction for systems of equations
F	Froude number
F_S	Froude number of a bore with speed S
f	scalar flux function in the x direction
$f_{i+\frac{1}{2}}, f_{i+\frac{1}{2},j}, f_{i+\frac{1}{2},j,k}$	intercell flux in the x direction for scalar equations
G	vector flux function in the y direction
$G_{i,j+\frac{1}{2}}, G_{i,j+\frac{1}{2},k}$	intercell flux in the y direction for systems of equations

g	scalar flux function in the y direction
$g_{i,j+\frac{1}{2}}, g_{i,j+\frac{1}{2},k}$	intercell flux in the y direction for scalar equations
$\mathbf{H}$	vector flux function in the z direction
$\mathbf{H}_{i,j,k+\frac{1}{2}}$	intercell flux in the z direction for systems of equations
h	scalar flux function in the z direction
$h_{i,j,k+\frac{1}{2}}$	intercell flux in the z direction for scalar equations
I	space integration range or $\sqrt{-1}$ in von Neumann stability analysis
i	cell index (x -direction for Cartesian grids)
j	cell index (y -direction for Cartesian grids)
k	cell index (z -direction for Cartesian grids) or index for waves in the solution a Riemann problem
L	lower limit on a limiter function or operator for a one dimensional scheme
$l_{i+\frac{1}{2},j}, l_{i,j+\frac{1}{2}}$	cell side lengths
$l_{i+\frac{1}{2},j}^L, l_{i+\frac{1}{2},j}^R, l_{i,j+\frac{1}{2}}^L, l_{i,j+\frac{1}{2}}^R$	integration lengths
M	Mach number or number of space dimensions
M_S	shock Mach number
M_i, M_j, M_k	Number of cells in i, j, k directions respectively
N	number of equations in a system/waves in the solution of a Riemann problem
$\mathbf{n}$	unit vector normal to an intercell boundary
n	time level

P	stretching parameter
$P(x_1, \dots, x_M)$	pointwise error in M dimensions
p	pressure
$RP(\cdot, \cdot)$	Riemann problem
r	flow parameter
S	amplification factor in von Neumann stability analysis or speed of a shock wave
t	time
TV	total variation
t_1, t_2	limits on time integral
$\mathbf{U}$	vector of conserved variables
$\mathbf{U}_0$	initial conditions for a system of equations
$\bar{\mathbf{U}}(\cdot; \cdot, \cdot)$	solution to a Riemann problem
U	upper limit on a limiter function
u	velocity in the x direction or conserved variable in the linear advection equation
u_0	initial conditions for a scalar equation
u^{ex}	exact solution to the linear advection equation
$V(\cdot)$	area in 2D, volume in 3D
v	velocity in the y direction
W	weight in WAF intercell flux
$w^\perp, w^\parallel$	components of velocity perpendicular and parallel to the normal to a cell boundary
$\mathbf{x}$	coordinate vector

x	coordinate direction
y	coordinate direction
z	coordinate direction

Greek alphabet

α	compression parameter
$\Delta x, \Delta y, \Delta z$	mesh spacings for Cartesian grids
Δt	timestep
$\Delta_i, \Delta_{i,j}, \Delta_{i,j,k}$	computing cell
ϵ	test error in von Neumann stability analysis
γ	ratio of specific heats
λ	wavespeed <i>or</i> characteristic speed
ν	Courant number
ν_{max}	Courant number coefficient
ϕ	limiter function <i>or</i> geopotential in the Shallow Water equations <i>or</i> a phase angle in 2D von Neumann stability analysis
ψ	limiter function
ρ	density
θ	a phase angle in von Neumann stability analysis
$\theta_{i+\frac{1}{2},j}, \theta_{i,j+\frac{1}{2}}$	angles between cell sides and x direction

ζ	coordinate direction or a phase angle in 3D von Neumann stability analysis
---------	---

Subscripts

i, j, k	in gridcell $\Delta_{i,j,k}$
$i + \frac{1}{2}$	at boundary $i + \frac{1}{2}$
$j + \frac{1}{2}$	at boundary $j + \frac{1}{2}$
$k + \frac{1}{2}$	at boundary $k + \frac{1}{2}$
L	left
loc	at the current local boundary
M	middle
R	right
upw	at the nearest upwind boundary
x	in the x direction or x derivative
y	in the y direction or y derivative
z	in the z direction or z derivative

Superscripts

(β)	from the β -scheme
-----------	--------------------------

(LW)	from the Lax-Wendroff scheme
n	time level
(k)	k^{th} characteristic field
(WB)	from the Warming-Beam scheme

Chapter 1

Introduction

1.1 Aims

Many physical problems can be accurately described by *hyperbolic systems of conservation laws*, i.e coupled systems of partial differential equations that can be written in differential form

$$\mathbf{U}_y + \sum_{k=1}^M \mathbf{F}^{(k)}(\mathbf{U})_{x^{(k)}} = 0 \quad (1.1)$$

where y and $\mathbf{x} = (x^{(1)}, \dots, x^{(M)})^T$ are independent variables, $\mathbf{U}(\mathbf{x}, y)$ is a vector of N *conserved variables* and for each k , $\mathbf{F}^{(k)}(\mathbf{U})$ is a vector valued function of $\mathbf{U}$ called the *flux function*. The system (1.1) is called *strictly hyperbolic* if, for each k , the Jacobian matrix

$$A^{(k)} = \frac{\partial \mathbf{F}^{(k)}}{\partial \mathbf{U}} \quad (1.2)$$

has N real and distinct eigenvalues. If the eigenvalues are not distinct, the system is *hyperbolic*. The differential form (1.1) of the conservation law is only valid for smooth solutions. However, it is well known that solutions to systems of hyperbolic conservation laws with initial conditions $\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x})$ can develop discontinuities in a finite time, even if $\mathbf{U}_0(\mathbf{x})$ is smooth; solutions to an integral form of (1.1) are therefore usually sought. The theory behind hyperbolic systems of conservation laws is discussed in many texts; see for example the paper by Lax [30] or the book by LeVeque [33].

Many systems of the form (1.1) exist in mathematical physics. Consider for example, the time dependent flow of an inviscid, compressible gas. The equations describing this type of flow are called the *Euler equations of Gas Dynamics*, and form a hyperbolic system of conservation laws. When written in differential form (1.1), $y \equiv t$ is the time variable, $\{x^{(k)} : k = 1, \dots, M\}$ are the space variables and $M = 1, 2$ or 3 is the number of space dimensions. The derivation of the Euler equations can be found in most texts on compressible fluid dynamics; see, for example, the book by Anderson [1]. One application of the Euler equations is the flow of air over a solid body, such as an aerofoil or complete aircraft. The effects of heat conduction and viscosity are only significant in a thin region

adjacent to the solid boundaries called the *boundary layer*. Outside this region, the flow is effectively inviscid and the Euler equations can be used. If the flow is steady, i.e independent of time, and everywhere supersonic in one particular direction, the flow can be modelled by a hyperbolic system called the *steady supersonic Euler equations*. When this system is written in differential form (1.1), y represents the direction in which the flow is supersonic, and x represents the remaining space coordinates. The steady supersonic Euler equations and some methods for their solution are discussed by Glaister [14] and Toro and Chou [71]. Another example of a system of hyperbolic conservation laws is provided by the Shallow Water equations. These describe the flow of a fluid with a free surface under the influence of gravity. A key assumption in their derivation is that the pressure variation is hydrostatic, or, equivalently, the vertical component of acceleration has a negligible effect on the pressure. If the effects of surface tension and viscosity are neglected, and the bed of the fluid is assumed horizontal then the system is hyperbolic, with $y = t$ as the time variable, $x^{(1)}, \dots, x^{(M)}$ as space variables and $M = 1$ or 2 the number of space dimensions. For further details, see the book by Stoker [52].

Many systems of hyperbolic conservation laws, such as the Euler equations or the Shallow Water equations, are nonlinear and cannot be solved analytically in more than a few idealized situations. Therefore, in general, numerical solutions must be sought. The development of numerical methods for these systems goes back as far as the classic paper by von Neumann and Richtmyer [79] in 1950, and even earlier. A worldwide research effort into these methods over the last forty or fifty years has led to a vast number of schemes being available today, as can be seen in the review papers by Rizzi and Engquist [42] and Moretti [36]. All of these schemes are based on the idea of evolving the initial conditions $U_0(x) = U(x, 0)$ in the direction of increasing y in a series of discrete jumps, a process known as *time marching* or *space marching*, depending on the role of y as a time or space coordinate. For notational convenience, it will be assumed that y in (1.1) represents time and will be written as t in the remainder of this thesis. At each timelevel, a finite dataset is stored. The methods can be divided into two main types: *implicit methods* and *explicit methods*. An *implicit method* is a method for which the data at a given timelevel depends on other data at that timelevel as well as data at previous timelevels. The update from one timelevel to the next thus involves the solution of a system of linear equations. This can be costly, but many implicit methods can be used without restriction on the timestep. However, when applied to nonlinear systems of equations, an upper limit may have to be set on the timestep if time accuracy is to be maintained. An *explicit method* is one for which the data at a given timelevel depends only on data at previous timelevels. Explicit methods are thus special case of implicit methods in which the matrix defining the linear system is diagonal, so solution of the system is trivial. However, the requirement of stability (discussed later) sets an upper bound on the timestep for explicit methods which can be orders of magnitude smaller than the timesteps used by implicit methods. The book by Hirsch [25] gives a good account of both implicit and explicit methods. The aim of this thesis is to introduce some *explicit* timemarching methods for the solution of hyperbolic systems of conservation laws in one, two and three space dimensions.

1.2 Background

The vast number of explicit timemarching schemes for the solution of hyperbolic systems of conservation laws in existence today can be classified in various ways. Perhaps the most fundamental classification is between *shock tracking* or *shock fitting* methods and *shock capturing* methods. *Shock tracking* methods are those for which discontinuities in the solution are treated as internal boundaries. Boundary conditions across the discontinuities are provided by the *Rankine-Hugoniot conditions*, which are algebraic conditions all discontinuities in solutions to hyperbolic conservation laws must satisfy. Smooth regions of flow are computed by the use of methods designed to solve the differential form (1.1) of the conservation law. A good discussion of shock tracking methods and references to specific works can be found in the review by Moretti [36]. The main problem with these methods appears to be that the number of discontinuities in the solution and the way they interact must be known in advance, and, for this reason, it is hard (if at all possible) to write a program that is general enough to solve all problems, particularly in two or three space dimensions. An alternative to shock tracking methods is provided by *shock capturing* methods, which appear to be far more popular today. These methods can be defined as those methods that treat all points in the flow and all flow features with the same algorithm, i.e. do not resort to special procedures at discontinuities. The only requirement, due to a theorem by Lax and Wendroff [29], is that shock capturing schemes must be constructed in *conservation form*, as discussed in Chapter 2. This is a special form of explicit scheme derived from the integral form of the conservation laws, and ensures that discontinuities in the solution move at the correct speed. Shock capturing methods offer the advantage that no information about the solution is needed in advance, except of course boundary conditions around the edges of the computational domain. However, due to a theorem of Godunov [17] (see Roe [44]), all linear shock capturing methods that are more than first order accurate will suffer from spurious oscillations in the region of discontinuities. These oscillations are unphysical, and therefore undesirable. In order to obtain oscillation-free solutions, special procedures must be incorporated into the schemes to eliminate the oscillations.

The earliest successful explicit methods of the shock capturing type in one space dimension appear to be the so-called *artificial viscosity* methods. These were pioneered by von Neumann and Richtmyer [79] and update the whole flowfield via simple finite difference formulae, such as the Lax-Wendroff schemes [29]. These finite difference schemes are usually second order accurate, and therefore suffer from spurious oscillations near discontinuities. Artificial viscosity methods are characterized by the fact that they overcome this problem by adding extra terms into the update of the solution that act in a similar manner to viscosity — hence their name. The viscous terms broaden any discontinuous profiles and damp the unwanted oscillations. Many methods of the artificial viscosity type exist, the most popular of which are probably those of Jameson [26]. Other methods include the two-step Lax-Wendroff schemes of Richtmyer, described in [25], and MacCormack's scheme [35]. See Rizzi and Engquist [42] for further references. The main advantage of artificial viscosity methods is that they are simple to program and are relatively cheap in

terms of CPU time. However the size of the artificial viscosity terms needed is problem dependent, so the program must be 'tuned' to each particular problem and a number of calculations may be required before an acceptable solution is obtained.

More advanced shock-capturing schemes, known as *linear-hybridized* schemes, were pioneered by Boris and Book [4] with their Flux Corrected Transport (FCT) method. In this approach, a second order accurate scheme is used in regions of smooth flow, and in regions of high gradient a first order, monotone scheme is blended with the second order scheme in such a way that oscillations in the solution are minimised. Examples of linear-hybridized schemes include the monotonic version of Fromm's scheme [13] introduced by van Leer [73] and the FCT scheme of Zalesak [84]. The linear-hybridized approach was generalized by Harten [21] who introduced the idea of *Total Variation Diminishing* (TVD) schemes, and provided a theorem that enables their construction. In [54], Sweby discussed the concept of the *flux limiter* as a way of formally viewing TVD methods. The flux limiter is essentially the function that provides the interpolation between first and second order schemes. Sweby showed that various methods, including van Leer's method [73], can be rewritten in flux limiter form. The main fault of TVD schemes is that they automatically add dissipation at any extrema in the solution, leading to first order accuracy at those points. To overcome this problem, Harten and Osher [24] developed their *Essentially Non-Oscillatory* (ENO) schemes, in which the Total Variation is allowed to increase sometimes. In general, all linear-hybridized methods have a great advantage over artificial viscosity methods in that the addition of the minimum amount of dissipation into the solution necessary to guarantee oscillation free results is a function only of the data at the current timelevel, and does not depend on any parameters that have to be 'tuned' by the user.

Perhaps the most significant advance so far in the development of explicit methods for hyperbolic systems of conservation laws (1.1) has been the introduction of *upwind methods*, in which the update of a solution from one timelevel to the next depends locally on data biased towards the direction from which information is propagated. The first upwind method was that due to Courant, Isaacson and Rees [10], now called the CIR scheme. Modern upwind methods fall into two main classes: *flux vector splitting* methods and *flux difference splitting* methods. The first *flux vector splitting* method was introduced by Steger and Warming [51]. Other examples include those by van Leer [76] or, more recently, Zha and Bilgen [85]. Flux vector splitting methods involve splitting the flux $\mathbf{F}(\mathbf{U})$ in each computing cell into two parts as $\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-$ where $\mathbf{F}^+$ and $\mathbf{F}^-$ correspond to right and left running characteristics respectively. Forward and backward differences of $\mathbf{F}^+$ and $\mathbf{F}^-$ are then used to update the solution from one timelevel to the next. On the other hand, *flux difference splitting* methods involve splitting the difference in flux between two states either side of an intercell boundary into a series of jumps across waves in the solution to the initial value problem defined by those states. This initial value problem is called the *Riemann problem*. The idea of using solutions to local Riemann problems in numerical methods for hyperbolic systems of conservation laws was first introduced by Godunov [17]. His method was the first conservative extension of the CIR scheme to nonlinear systems. Godunov's method involves taking the data value in each cell as the integral average of the conserved variables over that cell and assumed constant throughout that cell. He

then solved the Riemann problem at each intercell boundary exactly and allowed waves to propagate into neighbouring cells. The piecewise constant data was thus evolved exactly in time. Before waves from different Riemann problems interacted, Godunov applied the averaging procedure again to obtain the data at the next timelevel. This process leads to a first order, monotone scheme. A modification of Godunov's method, known as Glimm's method or the Random Choice Method (rcm), has been provided by Chorin [5] and is based on a proof of existence of solutions of hyperbolic conservation laws by Glimm [16]. Glimm's method is the same as Godunov's method up to the solution of the Riemann problems, but rather than average the solutions in each cell to obtain data at the new time level, the new solution in each cell is taken from a randomly chosen point in the combined solution of the Riemann problems in that cell at the new timelevel. The advantage of Glimm's method over Godunov's method is that discontinuous waves are represented by sharp jumps in the relevant flow variables, rather than being spread over several cells as in Godunov's method, although the location of the discontinuity may be wrong by a few cells. This sharp representation of discontinuities is important for some applications. Colella [6] proposed a sampling procedure for Glimm's method based on Van der Corput's sampling sequence and presented a hybrid of rcm and Godunov's method that has advantages over Glimm's method in multidimensional flows. Another method based on random sampling is the Random Flux approach of Toro [60] which involves randomly sampling the solution of a Riemann problem at the half timestep level to obtain a numerical flux for the conservation form. The Random Flux approach was proved second order accurate in a statistical sense by Toro and Roe [72]. In his original work, Godunov proposed two iterative algorithms for the exact solution of the Riemann problem, as well as some approximate procedures that are computationally cheaper (see Gottlieb and Groth [20]). Since that time, many authors have introduced more efficient exact iterative solvers, for example Gottlieb and Groth [20], Toro [60] and, more recently, Pike [40] and Schleicher [47]. Alternative approximate Riemann solvers have also been proposed, such as those of Roe [43], Osher and Solomon [39], Harten, Lax and van Leer [23] and Toro [64]. As a result of these developments, the introduction of many high order extensions of Godunov's method and the enormous expansion in available computing power, flux difference splitting methods have become very popular over the last decade. They are often referred to as *Riemann problem based methods* or *Godunov-type methods* and tend to be preferred over flux vector splitting methods because of their greater robustness and superior resolution of features such as contact waves.

The first higher order extensions of Godunov's method, called *Monotone Upwind-centred Schemes for Conservation Laws* (muscl) schemes, were introduced by van Leer [74], [75]. Second order accuracy is achieved in van Leer's scheme by replacing the constant data in cells used by Godunov with piecewise linear data. This linear reconstruction of data gives two data values on each intercell boundary which are then used as input for a Riemann solver. The solution on the time-axis to the Riemann problem at each intercell boundary is then evolved through half a timestep before being used to calculate intercell fluxes for the conservative form. Despite the extra work involved in evolving slopes as well as cell-averaged data in time, the improvement in accuracy provided by van Leer's schemes

over Godunov's scheme made the method attractive. Since van Leer's schemes, other MUSCL-type schemes have been presented, such as the *Generalized Riemann Problem* (GRP) method of Ben-Artzi and Falcovitz [2], the *Piecewise Linear Method* (PLM) of Colella [7] and the MUSCL-Hancock scheme [77]. To avoid spurious oscillations inherent in all second order schemes, MUSCL-type schemes place bounds on the slopes on the reconstructed data within each cell. Hence these methods are often called *slope limiter* methods. A MUSCL-type scheme that achieves higher than second order accuracy in space in smooth flow is the *Piecewise Parabolic Method* (PPM) of Colella and Woodward [9] in which the constant slabs in Godunov's method are replaced by parabolic elements rather than the linear elements in the original MUSCL scheme. This enables the PPM method to achieve third order accuracy in space in smooth regions of flow.

The other main class of high order extensions of Godunov's method consists of those that fall into the category of *flux limiter* methods discussed earlier. These methods are those that are constructed as a first order scheme plus a limited amount of a higher order correction and are distinguished from MUSCL schemes in that they do not depend on a reconstruction of the data before the solution of Riemann problems. This class includes the high order accurate schemes of Sweby [54], Harten [22] and Roe [45]. Another second order extension of Godunov's method that can be thought of as a flux limiter method is the *Weighted Average Flux* (WAF) method of Toro [62], [67]. In the WAF method, the initial conditions for the Riemann problems are taken as the integral average of conserved variables within the cells either side of the intercell boundary, as is done in Godunov's method. Second order accuracy is achieved by taking each intercell flux as an integral average of the flux function over the solution to the local Riemann problem at the half timestep level. The origins of the WAF approach lie in the *Random Flux* approach of Toro [60] and Toro and Roe [72] discussed earlier. The WAF method has been made TVD by computing a limiter function for each wave in the solution to each Riemann problem. The method is discussed in detail in Chapter 2 since it forms the basis of much of the work presented later in this thesis.

The rate of advance of all explicit time marching methods is restricted by the requirement that they remain *stable*, i.e. that errors introduced by the method, or some other mechanism, should not grow unboundedly as time progresses. The timestep Δt for explicit methods is usually restricted by the Courant, Friedrichs and Lewy (CFL) condition

$$\lambda_{max} \frac{\Delta t}{\Delta l} \leq 1 \quad (1.3)$$

where λ_{max} is either a maximum absolute characteristic speed on the data over the whole flow domain, or the maximum absolute wavespeed in the solution to the Riemann problems at all intercell boundaries, and Δl is some length characteristic of the shortest distance across a cell, such as the minimum sidelength. The term on the left hand side of (1.3) is called the *Courant number*. Various methods have been designed to avoid the restriction on Δt posed by (1.3). These are commonly referred to as *large timestep schemes*. In [80], Warming and Beam proposed a finite difference scheme that is an extension of MacCormack's two-step scheme [35] and is stable up to a Courant number of two. Greater stability is achieved by using more data in the update of each cell value. LeVeque [31] has

proposed a **MUSCL**-type scheme that is stable for timesteps of arbitrary size. The scheme achieves stability by allowing waves from local Riemann problems to pass through more than one cell during each timestep. Second order accuracy is achieved by a **MUSCL**-type reconstruction of data. Interactions of waves from different Riemann problems are treated linearly, which effectively places an upper limit on the timestep when the scheme is applied to non-linear problems if acceptable accuracy is to be maintained.

In most practical problems in Computational Fluid Dynamics (**CFD**), details of fluid flow phenomena are required in two or three space dimensions. The first step towards obtaining numerical solutions to conservation laws in more than one space dimension is to decide on the type of grid used. The types of grid possible can be classified into two main types: *structured* grids and *unstructured* grids. In two space dimensions, a structured grid is one in which the cells, and the way they are connected, can be described by labeling the cells with two integer indices (i, j) . Three integer indices are needed in three dimensions. An unstructured grid is one that is not structured. Many numerical methods for the generation of grids can be found in the literature. For structured grids, see, for example, Eiseman [11] or Thompson *et. al.* [57], or the review of **CFD** by Kutler [28] and the papers cited there. A good discussion of numerical methods for the generation of unstructured grids is given in the lecture series by Weatherill [81].

The grids of interest in this work will be *structured body-fitted grids of quadrilaterals*. Such grids are called *curvilinear* grids here. The term *body-fitted* means that the grid cells completely cover the flow domain, but no cell covers, even partially, any region outside the domain. A curvilinear grid for which all grid cells are rectangular is called *Cartesian*. A popular technique for the generation of numerical methods for hyperbolic systems of conservation laws on curvilinear grids in two or three dimensions is to extend a one dimensional method by the use of *space operator splitting*, as described by Yanenko [83] and Strang [53]. This involves updating the solution from one timelevel to the next is a series of *sweeps* of the one dimensional method along one dimensional strips of cells. Once a one dimensional scheme is available, operator splitting offers a very simple way of generating efficient schemes for two and three space dimensions. The power of the operator splitting technique combined with modern shock capturing methods has been shown in the review paper by Woodard and Colella [82], where the flow of an inviscid, compressible gas containing strong shocks waves was computed. However, operator split schemes suffer from the fact that they are strongly influenced by one dimensional physics. It has been reported in the review paper by van Leer [78] that this can lead to bad resolution of flow features such as shock waves that are not aligned to the grid directions.

An alternative to operator split schemes are the so-called *finite volume* schemes, in which the update from one timelevel to the next is performed in a single step using the multidimensional conservative formula. The finite volume methods that are of interest here are those that are Riemann-problem based, although other types of finite volume schemes do exist, such as the cell vertex schemes of Morton [37] and Morton and Suli [38]. The most up to date Riemann problem based finite volume schemes fall into two main classes. The first are those that depend on solutions to one-dimensional Riemann problems aligned to cell boundaries, such as the schemes of Colella [8] and LeVeque [32]. These have

been labelled 'one-and-a-half' dimensional schemes by Roe [46], since, as in the case of operator split schemes they depend heavily on one-dimensional physics and are strongly influenced by the directions of the grid lines. One way to remove the dependency on one dimensional physics may be to construct schemes based on the solution of two dimensional Riemann problems, such as those discussed by Zhang and Zheng [86]. However, although some analytical and numerical results are available on the solution to these problems, as discussed in the work of Zhang and Zheng [86] and Shultz-Rinne *et. al.* [48], there does not appear to be a sufficiently general solution procedure to make the construction of such a scheme possible. Even if it were possible, the scheme would still be strongly tied to the grid directions, and it is claimed (for example, by Roe [46] or van Leer [78]) that this would limit the attainable accuracy. An alternative to such schemes is to construct *multidimensional upwind* (MDU) schemes, as discussed in the review paper by van Leer [78]. An important type of MDU methods are the *multidirectional methods* in which Riemann problems are solved in a direction aligned with the flow velocity or steepest pressure gradient, rather than normal to grid interfaces. The main problem with these methods is that a second Riemann problem has to be solved in a direction normal to the direction of the first Riemann problem to complete the calculation of an intercell flux, which reduces the cost-effectiveness of the methods. A second type of MDU scheme comes from the *multidimensional fluctuation approach*. In this approach, the finite volume formula is abandoned altogether, and data values are stored at particular points in the flow, or *grid nodes*, and are not interpreted as integral averages over grid cells. At each timestep, a residual is computed in each cell and distributed among nearby cell vertices in a way determined by the local flow direction. The construction of multidimensional wavemodells, ways to ensure conservation and convection schemes for the propagation of waves are still very much topics of research. See van Leer [78] for a detailed discussion and a list of references.

1.3 Outline

The primary objective of this thesis is the development of new schemes for the solution of hyperbolic conservation laws in one, two and three space dimensions. The numerical methods presented here will be of the explicit, shock capturing type and can be considered as second order accurate extensions of Godunov's method. The majority of schemes presented here will be based on the Weighted Average Flux (waf) approach of Toro [62], [67], and none will achieve second order accuracy by the use of MUSCL-type reconstruction steps.

The first development, presented in Chapter 3, will be to consider an extension of the waf approach in one space dimension to develop a large timestep scheme, by taking into account not only the Riemann problem at the local boundary when computing an intercell flux, but also its nearest neighbours. The resulting scheme will be second order accurate in space and time in regions of smooth flow, and will have twice the stability limit of the original waf scheme. It will be effectively a combined extension of the Warming-Beam

[80] and Lax-Wendroff [29] schemes on the linear advection equation. The scheme is made **tvD**, at least on scalar problems, to prevent spurious oscillations developing in regions of high gradient.

An alternative Riemann-problem based extension of the Warming Beam scheme that is not based on the **WAF** approach is presented in Chapter 4. It differs from the scheme of Chapter 3 in that it does not reduce to the Lax-Wendroff scheme at low Courant numbers. The scheme is linearly stable and **tvD** up to a Courant number of two, and is second order accurate in space and time in smooth regions of flow. A class of **tvD**, Riemann problem based schemes is constructed by averaging this new scheme with the original **WAF** scheme. In smooth regions of flow, all schemes in this class are second order accurate in space and time and one is third order accurate in space and time.

Since the original **WAF** scheme can only be used in more than one space dimension by the use of space operator splitting, the other chapters in the thesis will be devoted to the construction of finite volume **WAF**-type schemes in two and three space dimensions. In Chapter 5, such schemes are constructed for the linear advection equation. Intercell fluxes for the schemes are generated from integrals over the exact solution to the linear advection equation with initial conditions as data at the previous timelevel. The schemes therefore have multidimensional upwind aspects. They differ from each other since different methods of integration are used to find the fluxes. In two space dimensions, two schemes are found that are second order accurate in space and time, and have good stability properties when compared to the two dimensional Lax-Wendroff schemes [29]. The natural way to extend these two schemes to non-linear systems would be to integrate over the solutions of two and three dimensional Riemann problems. This approach is, however, not possible due to the lack of suitably general two and three dimensional Riemann solvers. Two other approaches to the extension of one of the schemes on Cartesian grids are considered in Chapter 6. The resulting schemes are extended to curvilinear grids in Chapter 7.

The layout of the thesis can be summarized as follows. In Chapter 2, background material is discussed and a detailed description of the original **WAF** method is given. In Chapter 3, the large timestep version of the **WAF** method is developed. In Chapter 4, a **tvD** Riemann problem based extension of the Warming-Beam scheme is developed and averaged with the original **WAF** scheme to produce a class of second order accurate Riemann problem based **tvD** schemes. In Chapter 5, various multidimensional **WAF**-type schemes for the linear advection equation on Cartesian grids are derived. One of these schemes is extended to non-linear systems on Cartesian grids via two distinct approaches in Chapter 6. In Chapter 7 the two schemes of Chapter 6 are extended for use on curvilinear grids. In Chapter 8 the thesis is summarized and future work is discussed.

Chapter 2

Numerical Methods for Conservation Laws

2.1 Introduction

The aim of this chapter is to introduce hyperbolic systems of conservation laws and their properties, and discuss some numerical methods for their solution that form the basis for developments presented in later chapters. The theory of hyperbolic conservation laws will not be considered here, but a detailed discussion of the subject can be found in the report by Lax [30] or the book by LeVeque [34].

This Chapter is laid out as follows. In §2.2 some one dimensional systems of conservation laws will be introduced. In §2.3 properties of numerical schemes for scalar (single) conservation laws in one dimension will be considered. In §2.4, two methods for systems of conservation laws in one dimension, namely Godunov's method [18] and the Weighted Average Flux (waf) method of Toro [62], [63], [67], [66], will be considered in detail as these form the basis of developments presented in later chapters. In §2.5, two dimensional systems of conservation laws are introduced and methods for their solution are considered in §2.6. Three dimensional systems of conservation laws will not be considered here, since basic features of numerical methods for their solution follow easily from the two dimensional case.

2.2 One Dimensional Systems of Conservation Laws

The aim of this section is to introduce hyperbolic systems of conservation laws in one space dimension. The problems of interest are therefore initial value problems of the form:

$$U_t + F(U)_x = 0 \tag{2.1}$$

$$U(x, 0) = U_0(x) \tag{2.2}$$

where $x \in (-\infty, +\infty)$ and $t \geq 0$ represent space and time, $\mathbf{U}(x, t)$ is a vector of N conserved variables, $\mathbf{F}(\mathbf{U})$ is a vector valued function of $\mathbf{U}$ called the *flux function*, and $\mathbf{U}_0(x)$ is some initial distribution of $\mathbf{U}$ called the *initial conditions*. Equation (2.1) is called the *differential form* of the conservation law. The equivalent *quasi-linear form* is

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = 0 \quad (2.3)$$

where $\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ is the Jacobian of $\mathbf{F}$. The system (2.1) is said to be *hyperbolic* if for each $x, t, \mathbf{U}$ the matrix $\mathbf{A}$ has N real eigenvalues $\lambda^{(k)}$ and a complete set of eigenvectors, or, equivalently, $\mathbf{A}$ is diagonalizable. The eigenvalue $\lambda^{(k)}$ is called the k^{th} *characteristic speed* of the system. If the eigenvalues are distinct, the system is called *strictly hyperbolic*.

A well known system of the form (2.1) is the *Euler equations of Gas Dynamics*:

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix}_x = 0 \quad (2.4)$$

where ρ is the density, u is the velocity, p is the pressure,

$$E = \frac{1}{2}\rho u^2 + \rho e \quad (2.5)$$

is the total energy and e is the specific internal energy. Equations (2.4) describe the flow of an inviscid, compressible fluid. Since there are four variables and three equations, an extra equation is needed to close the system. This is done here by adding an *Equation of State*; the Equation of State for the Euler equations throughout this work, even in more than one space dimension, will be taken as:

$$e(\rho, p) = \frac{p}{(\gamma - 1)\rho} \quad (2.6)$$

where γ is the ratio of specific heats, assumed constant at $\gamma = 1.4$ here. The system (2.4) is strictly hyperbolic with characteristic speeds $u - a$, u and $u + a$ where $a = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed. A good discussion of the derivation and properties of the Euler equations can be found in the book by Anderson [1].

Another example of a system of the form (2.1) is given by the *Shallow Water equations*:

$$\begin{pmatrix} \phi \\ \phi u \end{pmatrix}_t + \begin{pmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \end{pmatrix}_x = 0 \quad (2.7)$$

These model the flow of a fluid with a free surface under the influence of gravity. Here $\phi = gH$ is the geopotential, $g \approx 9.81$ is the acceleration due to gravity, H is the total depth of fluid and u is the velocity. The characteristic speeds of this system are $u - c$ and $u + c$, where $c = \sqrt{\phi}$ is the *celerity*, analogous to the sound speed in gas dynamics. The derivation of (2.7) assumes that the pressure varies hydrostatically, or, equivalently, that the vertical acceleration of the fluid has a negligible effect on the pressure. The effects

of viscosity and surface tension are neglected and the bed of the fluid is assumed to be horizontal. A further, more detailed, discussion of these equations can be found in Stoker [52].

It is well known that systems of conservation laws can develop discontinuous solutions in a finite time, even if the initial data $U_0(x)$ is smooth. Such solutions are not differentiable and are therefore not admissible as solutions to (2.1), (2.2). However, solutions to the integral form

$$\int_{x_1}^{x_2} U(x, t_2) dx - \int_{x_1}^{x_2} U(x, t_1) dx = \int_{t_1}^{t_2} F(U(x_1, t)) dt - \int_{t_1}^{t_2} F(U(x_2, t)) dt \quad (2.8)$$

for an arbitrary domain $[x_1, x_2] \times [t_1, t_2]$ can admit discontinuities and are therefore normally sought. Solutions to (2.1), (2.2) are called *classical* or *strong* solutions to the conservation law; solutions to (2.8), (2.2) are called *generalized* or *weak* solutions to the conservation law.

Consider a weak solution $U(x, t)$ to the conservation law, with an isolated discontinuity of speed S . By considering the integral form (2.8) over arbitrary domains $[x_1, x_2] \times [t_1, t_2]$ containing the discontinuity, it can be shown that, to be a weak solution, $U(x, t)$ must satisfy

$$F(U_R) - F(U_L) = S (U_R - U_L) \quad (2.9)$$

where subscripts $\cdot_L$ and $\cdot_R$ denote values immediately to the left and right of the discontinuity. Condition (2.9) is called the *Rankine Hugoniot* condition, and must hold for any discontinuity in the solution. Two types of discontinuity can occur in solutions to the one dimensional Euler equations of Gas dynamics. *Shock waves* are discontinuous waves across which all variables increase. As a consequence of the Second Law of Thermodynamics, entropy must increase across a shock wave, or, equivalently, a shock of speed S in characteristic field k must satisfy

$$\lambda_2^{(k)} \geq S \geq \lambda_1^{(k)} \quad (2.10)$$

where $\lambda_2^{(k)}$ is the k^{th} characteristic speed on the state *behind* the shock and $\lambda_1^{(k)}$ is the k^{th} characteristic speed on the state *ahead* of the shock. *Contacts* are discontinuous waves admitted by the Euler equations across which the density changes and pressure and velocity remain constant. The speed of a contact and the characteristic speeds of the same field either side of the contact are always equal. The equivalent of a shock wave in the Shallow Water equations is called a *bore wave*; there is no equivalent of a contact wave in the Shallow Water equations. Waves that are not discontinuous can also exist in the solution of systems (2.1), i.e there exist waves which are of finite width and across which all variables vary continuously. If any point P within such a wave in characteristic field k satisfies

$$\lambda_2^{(k)} \leq \lambda_P^{(k)} \leq \lambda_1^{(k)} \quad (2.11)$$

where $\lambda_P^{(k)}$ is the k^{th} characteristic speed at point P then the wave is called a *rarefaction wave*; if any point P within the wave satisfies

$$\lambda_2^{(k)} \geq \lambda_P^{(k)} \geq \lambda_1^{(k)} \quad (2.12)$$

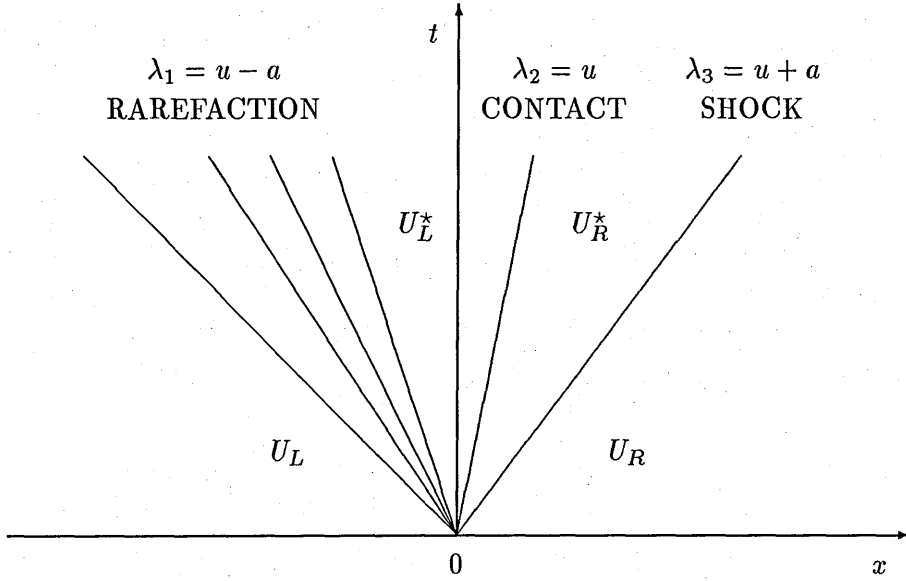


Figure 2.1: An example of the structure of the solution of the Riemann problem for the Euler equations of Gas Dynamics.

then the wave is called a *compression wave*.

An initial value problem of the form (2.1), (2.2) that is of particular interest in this thesis is the *Riemann problem*. This is defined for any system (2.1) as the problem with initial conditions (2.2) of the form

$$\mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L & \text{if } x < 0 \\ \mathbf{U}_R & \text{if } x > 0 \end{cases} \quad (2.13)$$

for constant states $\mathbf{U}_L, \mathbf{U}_R$. This problem will be denoted $RP(\mathbf{U}_L, \mathbf{U}_R)$.

In general, the solution of a Riemann problem $RP(\mathbf{U}_L, \mathbf{U}_R)$ will consist of N waves of speeds $a^{(k)}, k = 1, \dots, N$ separating $N + 1$ constant states, called *star states*, and is a function only of the similarity variable $\frac{x}{t}$. The solution will be denoted $\bar{\mathbf{U}}(\frac{x}{t}; \mathbf{U}_L, \mathbf{U}_R)$ here. Fig. 2.1 shows an example of the structure of the solution of a Riemann problem for the Euler equations. In this situation, the left wave is a rarefaction wave, the right wave is a shock wave, and the contact has positive speed. For many important systems of equations, exact iterative Riemann solvers exist, such as those of Gottlieb and Groth [20], Toro [61] and more recently Pike [40], as well as many non-iterative approximate Riemann solvers, such as those by Roe [43], Harten, Lax and van Leer [23] and Toro [64]. Reviews of Riemann solvers for the Euler equations and the Shallow Water equations have been given by Toro in [67] and [66] respectively. The idea of using solutions to local Riemann problems in the update of numerical solutions to (2.1), (2.2) was first introduced by Godunov [18]. His method is discussed later in this chapter.

Before introducing details of particular methods for the solution of initial value prob-

lems (2.1), (2.2), some general properties of such methods will be introduced. Throughout this thesis, whenever one dimensional initial value problems (2.1), (2.2) are considered it will be assumed that a weak solution $U(x, t)$ to the conservation law is sought in the time interval $t \in [0, T]$ for some $T > 0$ on some fixed, finite interval $D = [x_1, x_2]$ and that boundary conditions at x_1, x_2 are defined for all $t \in [0, T]$. The interval D is divided into M computing cells with constant width $\Delta x = (x_2 - x_1)/M$. The class of methods of interest here will be explicit, timemarching methods, where the initial conditions will be evolved to time T in a series of finite, variable time steps of size Δt^n . At any time t , the integral average

$$U_i^n = \frac{1}{\Delta x} \int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} U(x, t^n) dx \quad (2.14)$$

of U over cell i will be stored for each cell. A method is called *conservative* if it can be written in the form

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} [F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}] \quad (2.15)$$

where $F_{i+\frac{1}{2}}(U_{i-m_1}^n, \dots, U_{i+m_2}^n)$ for nonnegative integers m_1, m_2 is some function of the data at timelevel n , and is called the *intercell flux*. The conservative form (2.15) is derived from the integral form (2.8) of the conservation law by taking the domain $(x, t) \in [(i - \frac{1}{2})\Delta x, (i + \frac{1}{2})\Delta x] \times [0, \Delta t]$, making the approximation

$$F_{i+\frac{1}{2}} \approx \frac{1}{\Delta t} \int_0^{\Delta t} F(W(x, t)) dt \quad (2.16)$$

where $W(x, t)$ is the solution of (2.1) with initial conditions $W(x, 0) = U(x, t^n)$ and using the definition (2.14). The scheme (2.15) is said to be *consistent with the conservation law* (2.1) if $F_{i+\frac{1}{2}}$ is a Lipschitz continuous function that satisfies

$$F_{i+\frac{1}{2}}(U, \dots, U) = F(U) \quad (2.17)$$

for all i , where $F(U)$ is the flux in (2.1). Consider a sequence of solutions $U^{(\Delta x)}(x, t)$ of the conservation law as $\Delta x \searrow 0$ on the strip $0 \leq t \leq T$, and assume $\Delta t^n = O(\Delta x)$ for all n . If there exists a function $U(x, t)$ such that

$$\lim_{\Delta x \searrow 0} U^{(\Delta x)}(x, t) = U(x, t) \quad (2.18)$$

then the scheme is said to be *convergent*. An important convergence theorem has been given by Lax and Wendroff [29], and can be stated as follows:

Theorem 1 Suppose a sequence $U^{(\Delta x)}$ is computed via some consistent, conservative method and converges to a function $U(x, t)$ as $\Delta x \searrow 0$. Then $U(x, t)$ is a weak solution to the conservation law (2.1).

To ensure convergence to weak solutions of conservation laws, the methods developed in later chapters will therefore be developed as consistent, conservative methods.

A more sophisticated convergence theorem has been provided by Harten [22], which guarantees that *if* a scheme converges, then the limit is a weak solution of the conservation law *and* is physically relevant in the sense that it satisfies certain entropy conditions. Harten's theorem will not be used when discussing the entropy properties of schemes constructed later in this thesis, so it is not presented here.

2.3 Properties of Schemes for Model Equations

Consider the initial value problem

$$u_t + f(u)_x = 0 \quad (2.19)$$

$$u(x, 0) = u_0(x) \quad (2.20)$$

where $u(x, t)$ is a single conserved variable, and $f(u)$ is some flux function. Equations of the form (2.19) provide very simple models of convection, and for this reason schemes are often constructed first for equation (2.19), and then extended to non-linear systems of conservation laws such as (2.1). The aim of this section is to discuss properties of schemes for the solution of (2.19), (2.20). The methods of interest here will be consistent, timemarching methods of the form

$$u_i^{n+1} = \sum_{\alpha: A_\alpha \neq 0} A_\alpha u_{i+\alpha}^n \quad (2.21)$$

where $\{A_\alpha\}$ is a set of real coefficients, only a finite number of which are non-zero. The set of points $\{\Delta_{i+\alpha} : A_{i+\alpha} \neq 0\}$ is called the *stencil* of the scheme. An alternative to (2.21) will be to express the schemes in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] \quad (2.22)$$

where $f_{i+\frac{1}{2}}$ are scalar intercell fluxes.

Consider first the linear advection equation, i.e equation (2.19) with flux

$$f(u) = au \quad (2.23)$$

for some real, constant speed a . In [44], Roe considered properties of the scheme for the initial value problem (2.19), (2.20), (2.23). Many of these properties are fundamental to the development of numerical methods for conservation laws, and will now be summarized.

The aim of any numerical method is to find an approximate solution to an equation that is as close to a true solution as possible. To gauge the success of a method, we can define the *pointwise error* $P(i\Delta x)$ at the point $i\Delta x$ as the difference between the exact solution and the numerical solution at that point after one timestep divided by the timestep, i.e

$$P(i\Delta x) = \frac{1}{\Delta t} (u^{ex}(i\Delta x, t^{n+1}) - u_i^{n+1}) \quad (2.24)$$

where $u^{ex}(x, t^{n+1})$ is the exact solution of (2.19) with 'initial condition' $u(i\Delta x, t^n) = u_i^n$ at position x and time $t + \Delta t$. It is desirable that as the number of cells increases, i.e. $\Delta x \searrow 0$, the approximate solution u_i^n should become closer to the exact solution at that point. The scheme (2.21) is said to be *convergent* if

$$\| \Delta t P(i\Delta x) \| \rightarrow 0 \quad \text{as} \quad \Delta x \rightarrow 0 \quad (2.25)$$

Here $\| \cdot \|$ is some suitable norm, such as the time integrated l_1 -norm, which is defined on a mesh function $v(x, t)$ as:

$$\| v \|_{1,T} = \int_0^T \int_{-\infty}^{\infty} |v(x, t)| dx dt \quad (2.26)$$

The term u_i^{n+1} in (2.24) can be expanded via a Taylor series as

$$u^{ex}(i\Delta x, t^{n+1}) = u + \Delta t u_t + \frac{1}{2} \Delta t^2 u_{tt} + \dots \quad (2.27)$$

where $u = u^{ex}(i\Delta x, t^n) = u_i^n$. We can also substitute the scheme (2.21) into (2.24) and use the Taylor expansions

$$u_{i+\alpha}^n = u + \alpha \Delta x u_x + \frac{1}{2} \alpha^2 \Delta x^2 u_{xx} + \dots \quad (2.28)$$

Following directly from the linear advection equation (2.19), (2.23) are the relations

$$u_t = -a u_x \quad (2.29)$$

$$u_{tt} = (-a u_x)_t = -a (u_t)_x = a^2 u_{xx} \quad (2.30)$$

and in general

$$u_{(nt)} = (-a)^n u_{(nx)} \quad (2.31)$$

for all $n > 1$, where $u_{(nt)} = \frac{\partial^n}{\partial t^n} u$ and $u_{(nx)} = \frac{\partial^n}{\partial x^n} u$. Substituting Taylor expansions (2.27) and (2.28) into the scheme (2.21), using (2.31) to convert time derivatives into space derivatives, assuming consistency and rearranging gives

$$P(i\Delta x) = \beta_2 + \beta_1 u_x + \beta_2 u_{xx} + \beta_3 u_{xxx} + \dots \quad (2.32)$$

where $\{\beta_k\}$ are coefficients that are, in general, functions of Δt , Δx , the coefficients A_α of the scheme and the *Courant number* ν defined as

$$\nu = a \frac{\Delta t}{\Delta x} \quad (2.33)$$

The Courant number can be thought of as the ratio of flow speed a to 'grid speed' $\frac{\Delta x}{\Delta t}$. The first term on the right hand side of (2.32) with $\beta \neq 0$ is called the *leading term* of the pointwise error (2.32). The pointwise error is used, along with the following theorems, to define the accuracy of the scheme (2.21).

Theorem 2 If u_i^n is a polynomial of degree p in i , scheme (2.21) will give the exact solution to (2.19), (2.23) if and only if

$$\sum_{\alpha} \alpha^q A_{\alpha} = (-\nu)^q \quad (2.34)$$

for all integers q such that $0 \leq q \leq p$.

Theorem 3 If any scheme meets the conditions of Theorem 2, the leading term in its pointwise error is

$$\frac{a(\Delta x)^p}{\nu(p+1)!} \left[(-\nu)^{p+1} - \sum_{\alpha} \alpha^{p+1} A_{\alpha} \right] \frac{\partial^p}{\partial x^p} u \quad (2.35)$$

These theorems enable the following definition. Any scheme of the form (2.21) for the linear advection equation (2.19), (2.23) that satisfies the conditions of Theorem 2 is called p^{th} order accurate. Theorems 2 and 3 and the definition of accuracy are extended to two and three dimensions in Appendix A. The accuracy of a scheme is an important property — obviously the higher the accuracy, the better the scheme (provided the scheme is stable). The scheme (2.21) is said to be *consistent* (with the linear advection equation) if it is at least first order accurate.

Another important property of the scheme (2.21) is that it must, at least under certain conditions, be *stable*, i.e any errors produced by the scheme or generated from other sources should not be amplified by the scheme at later times. If this is not the case, any errors will grow in time and scheme will be useless. Schemes are usually stable under restrictions on the Courant number ν , which amount to restrictions on the timestep Δt . The stability restrictions on the scheme (2.21) can be established by a technique known as *von Neumann stability analysis*. This is done by performing a Fourier analysis on the scheme and looking for restrictions on the coefficients A_{α} that ensure all Fourier modes are stable. Consider an error of the form

$$\epsilon_i^n = S^n e^{I\theta i} \quad (2.36)$$

where S is the amplification coefficient, θ is a phase angle and $I = \sqrt{-1}$. Due to the linearity of the problem, errors will be evolved in time via (2.21), i.e

$$\epsilon_i^{n+1} = \sum_{\alpha} A_{\alpha} \epsilon_{i+\alpha}^n \quad (2.37)$$

Substituting (2.36) into (2.37) and rearranging gives:

$$S^{n+1} e^{I\theta i} = S^n e^{I\theta i} \sum_{\alpha} A_{\alpha} e^{I\alpha\theta} \quad (2.38)$$

Here S^n means S to the power n , not S at timelevel n . Dividing through by $S^n e^{I\theta i}$ gives

$$S = \sum_{\alpha} A_{\alpha} e^{I\alpha\theta} \quad (2.39)$$

Using the standard relation

$$e^{I\eta} = \cos \eta + I \sin \eta \quad (2.40)$$

gives

$$S = \sum_{\alpha} A_{\alpha} \cos(\alpha\theta) + I \cdot \sum_{\alpha} A_{\alpha} \sin(\alpha\theta) \quad (2.41)$$

Note that, in general, S is complex. For stability, the expression for S must satisfy

$$|S| \leq 1 \quad (2.42)$$

i.e the Fourier mode θ must not be amplified. An equivalent condition

$$|S|^2 \leq 1 \quad (2.43)$$

usually proves easier to work with.

An important theorem that links the properties of consistency, stability and convergence of schemes for the linear advection equation is the *Lax Equivalence Theorem*, which can be stated as:

Theorem 4 *For a consistent, linear method applied to a well-posed initial value problem, stability is necessary and sufficient for convergence.*

For details of the proof of this theorem, see the book by LeVeque [33] and the references cited there. This theorem means that it is sufficient only to check that a scheme is consistent and stable to ensure it is convergent.

The idea of the *monotonicity* of an algorithm was introduced by Godunov in [17], and later reported by Roe [44]. An algorithm is called *monotonic* if, whenever the data $\{u_i^n\}$ is a monotonic increasing or decreasing function of i , the solution $\{u_i^{n+1}\}$ at the next timelevel is monotonic in the same direction. Roe also reported (with proofs) the following three theorems of Godunov:

Theorem 5 *An algorithm of the form (2.21) is monotonic if and only if none of the coefficients A_{α} are negative.*

Theorem 6 *All monotonic algorithms are stable.*

Theorem 7 *There are no linear monotonic algorithms of second or higher order accuracy, unless ν is an integer.*

The idea of an algorithm being *compatible* with a set of data is also introduced in [44]. An algorithm is *compatible* with a given set of data values $\{u_i^n\}$ if the solution at each point in that set is bounded by the data values from the interval containing the characteristic through the solution point. More formally, if $\{u_i^n\}$ is a set of data values, and $\{u_i^{n+1}\}$ is

the corresponding set of solution values output by the algorithm, then $\{u_i^n\}$ is *compatible* with the algorithm if, for all i ,

$$\min\{u_{i-K-1}^n, u_{i-K}^n\} \leq u_i^{n+1} \leq \max\{u_{i-K-1}^n, u_{i-K}^n\} \quad (2.44)$$

where $K \leq \nu \leq K+1$. For $0 \leq \nu \leq 1$, the above definition is equivalent to

$$0 \leq \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} \leq 1 \quad (2.45)$$

and for $-1 \leq \nu \leq 0$, it is equivalent to

$$0 \leq \frac{u_i^{n+1} - u_i^n}{u_{i+1}^n - u_i^n} \leq 1 \quad (2.46)$$

It can be seen from (2.45) that an algorithm that is compatible with the data $\{u_i^n\}$ will not introduce oscillations into the solution $\{u_i^{n+1}\}$ if they were absent from $\{u_i^n\}$. In [44], Roe successfully used the idea of compatibility to 'blend' various methods to obtain a scheme with variable A_α which achieved high order accuracy without introducing oscillations into the solution. Godunov's theorem (Theorem 7) was thus successfully circumvented. Sweby and Baines [55] proved the convergence of an extension of this scheme to general scalar conservation laws.

Example — The First Order Upwind scheme.

It is shown in [44] that the algorithm with the three point stencil $\{u_{i-1}^n, u_i^n, u_{i+1}^n\}$ that has the smallest pointwise error is and remains monotonic is

$$u_i^{n+1} = \begin{cases} \nu u_{i-1}^n + (1-\nu)u_i^n & \text{if } a \geq 0 \\ (1+\nu)u_i^n - \nu u_{i+1}^n & \text{if } a < 0 \end{cases} \quad (2.47)$$

By Theorem 5, this scheme is monotonic provided

$$|\nu| \leq 1 \quad (2.48)$$

This is the scheme of Courant, Isaacson and Rees [10], and is a first order method commonly known as the First Order Upwind (FOU) method, since u_i^{n+1} depends on the data 'upwind' of u_i^n . A simple application of von Neumann stability analysis shows the FOU scheme is stable provided (2.48) holds, and the Lax Equivalence theorem then ensures the scheme is convergent. In conservation form (2.22), the FOU scheme has intercell flux

$$f_{i+\frac{1}{2}} = \begin{cases} f_i & \text{if } a \geq 0 \\ f_{i+1} & \text{if } a < 0 \end{cases} \quad (2.49)$$

This conservative form of the `rou` scheme was extended by Godunov [18] to an upwind scheme for nonlinear systems of equation which is discussed more fully in the next section.

Example — The Lax-Wendroff scheme

Consider the scheme

$$u_i^{n+1} = \frac{1}{2}\nu(1+\nu)u_{i-1}^n + (1-\nu^2)u_i^n - \frac{1}{2}\nu(1-\nu)u_{i+1}^n \quad (2.50)$$

This scheme is due to Lax and Wendroff [29]. It has a three point, centred stencil. By studying the conditions (2.34) it can be shown to be second order accurate in space and time. Equation (2.34) shows that there are three conditions required for this accuracy, and since the Lax-Wendroff scheme has a three point stencil $\{u_{i-1}^n, u_i^n, u_{i+1}^n\}$, and thus three coefficients, it is the unique scheme on this stencil with this accuracy. By Theorem 7, this scheme is *not* monotonic, and thus spurious oscillations will appear in regions of high gradient. A simple application of the von Neumann stability analysis described above shows the scheme to be stable provided (2.48) holds. By the Lax-Equivalence theorem, (2.48) is also the condition for the scheme (2.50) to be convergent. In conservation form (2.22), the Lax-Wendroff scheme has intercell flux

$$f_{i+\frac{1}{2}} = \frac{1}{2}(1+\nu)f_i + \frac{1}{2}(1-\nu)f_{i+1} \quad (2.51)$$

Return, now, to the general scalar conservation law (2.19) and consider schemes of the form (2.21) with variable coefficients A_α . Define the Total Variation of data u^n at timelevel n as

$$TV(u^n) = \sum_i |u_{i+1}^n - u_i^n| \quad (2.52)$$

In [22] Harten introduced the idea of *Total Variation Diminishing* (TVD) schemes. A numerical method is called TVD if

$$TV(u^{n+1}) \leq TV(u^n) \quad (2.53)$$

Harten also gave sufficient conditions for a scheme to be TVD:

Theorem 8 *Suppose u is a mesh function that is either periodic or of compact support. Any scheme on u that can be written in the form*

$$u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}}(u_i^n - u_{i-1}^n) + D_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n) \quad (2.54)$$

is TVD provided

$$C_{i+\frac{1}{2}} \geq 0 \quad D_{i+\frac{1}{2}} \geq 0 \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1 \quad \forall i \quad (2.55)$$

Proof. See Harten [22].

Following directly from Theorem 8 are two Corollaries:

Corollary 1 *Any scheme written in the form*

$$u_i^{n+1} = u_i^n - C_i (u_i^n - u_{i-1}^n) \quad (2.56)$$

is TVD provided

$$0 \leq C_i = \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} \leq 1 \quad (2.57)$$

Proof. Set $D_{i+\frac{1}{2}} = 0$ in Theorem 8, write $C_i = C_{i+\frac{1}{2}}$ and rearrange.

Corollary 2 *Any scheme written in the form*

$$u_i^{n+1} = u_i^n + D_i (u_{i+1}^n - u_i^n) \quad (2.58)$$

is TVD provided

$$0 \leq D_i = \frac{u_i^{n+1} - u_i^n}{u_{i+1}^n - u_i^n} \leq 1 \quad (2.59)$$

Proof. Set $C_{i+\frac{1}{2}} = 0$ in Theorem 8, write $D_i = D_{i+\frac{1}{2}}$ and rearrange.

Note that the conditions (2.57) and (2.59) are identical to the data compatibility conditions (2.45) and (2.46).

2.4 Numerical Methods for One Space Dimension

2.4.1 Introduction

Two specific numerical methods for the solution of the initial value problem (2.1), (2.2) will now be considered in detail, since they form the foundation for the developments presented later in this thesis. In §2.4.1, Godunov's method [18] will be described. This was the first method for the solution of (2.1), (2.2) to use the solution to Riemann problems at intercell boundaries to evolve the solution of the global problem. In §2.4.2, the Weighted Average Flux (waf) method of Toro [62], [63], a second order extension of Godunov's method, is described. It is a TVD scheme on the linear advection equation.

2.4.2 Godunov's Method

In [18], Godunov introduced the idea of using the solution to local Riemann problems in numerical methods for hyperbolic conservation laws of the form (2.1). The scheme he proposed is a conservative extension of the FOU scheme (2.47) and runs roughly as follows. Assume the value U_i^n stored for cell Δ_i approximates the integral average of U in cell i at timelevel n , as in (2.14), and assume this value applies throughout the cell, i.e U^n is a piecewise constant function. At each boundary $i + \frac{1}{2}$ solve the Riemann problem $RP(U_i^n, U_{i+1}^n)$, and let waves propagate away from the boundary. Define the maximum Courant number as $\nu_{max} = \lambda_{max} \frac{\Delta t}{\Delta x}$ where $\lambda_{max} = \max_{i,k} \left\{ |\lambda_{i+\frac{1}{2}}^{(k)}| \right\}$ is the maximum absolute value of the wavespeeds $\lambda_{i+\frac{1}{2}}^k$ from all the local Riemann problems solved. Provided

$$\nu_{max} \leq \frac{1}{2} \quad (2.60)$$

none of the waves will interact, and the Riemann problems describe the time evolution of the data U^n up to the accuracy of the Riemann solver used. At time $t^{n+1} = t^n + \Delta t$, the cell averages are computed from the evolved solution to obtain the data U^{n+1} at the new time level, i.e take

$$U_i^{n+1} = \frac{1}{\frac{1}{2}\Delta x} \int_{-\frac{1}{2}\Delta x}^0 \bar{U}\left(\frac{x}{t}; U_{i-1}^n, U_i^n\right) dx + \frac{1}{\frac{1}{2}\Delta x} \int_0^{\frac{1}{2}\Delta x} \bar{U}\left(\frac{x}{t}; U_i^n, U_{i+1}^n\right) dx \quad (2.61)$$

In [23], Harten, Lax and van Leer showed that Godunov's method will not produce entropy violating solutions. By integrating around the domain $[(i - \frac{1}{2})\Delta x, (i + \frac{1}{2})\Delta x] \times [0, \Delta t]$ Godunov's method can be written in conservation form (2.15) with intercell flux

$$F_{i+\frac{1}{2}} = F(\bar{U}(0; U_i, U_{i+1})) \quad (2.62)$$

The intercell flux is thus taken as the flux of the solution to the Riemann problem $RP(U_i^n, U_{i+1}^n)$ on the time axis, i.e at the point which is upwind of all waves in that solution. Note that this intercell flux is an extension to nonlinear systems of the flux (2.49) for the FOU scheme on the linear advection equation. Provided

$$\nu_{max} \leq 1 \quad (2.63)$$

holds, waves from any boundary will not reach other intercell boundaries, and thus will not affect the flux (2.62). Hence the conservative formulation of Godunov's scheme is valid for higher timesteps than the original formulation, which is restricted by condition (2.60).

On the linear advection equation (2.19), (2.23) Godunov's scheme reduces to the First Order Upwind scheme (2.47). It is thus first order accurate, and provided (2.63) holds, it will be monotonic and stable.

2.4.3 The Weighted Average Flux (WAF) Method

Much of the work presented later in this thesis will be based on the Weighted Average Flux (waf) approach of Toro [62], [63], [67]. For this reason, the waf scheme will now

be discussed in detail. It is a second order extension of Godunov's method and is a generalization of the Lax-Wendroff method (2.50) to non-linear systems. The intercell flux at boundary $i + \frac{1}{2}$ for the conservative form (2.15) is defined by integrating across the solution of the Riemann problem from $x = -\frac{1}{2}\Delta x$ to $x = +\frac{1}{2}\Delta x$, at time $t = \frac{1}{2}\Delta t$, i.e. along the line AB in Fig. 2.2. Thus:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{\Delta x} \int_{-\frac{1}{2}\Delta x}^{+\frac{1}{2}\Delta x} \mathbf{F}(\bar{\mathbf{U}}(\frac{x}{\frac{1}{2}\Delta t}; \mathbf{U}_i, \mathbf{U}_{i+1})) dx \quad (2.64)$$

Define $\nu_{i+\frac{1}{2}}^{(k)} = \lambda_{i+\frac{1}{2}}^{(k)} \frac{\Delta t}{\Delta x}$ to be the Courant number of the k -th wave. It is assumed that the timestep Δt is chosen such that (2.63) holds. In the case of the linear advection equation (2.19), (2.23) there is only one wave of speed a in the solution of each Riemann problem, as shown in Fig. 2.2, and (2.64) becomes the Lax-Wendroff flux (2.51). The flux (2.64) therefore gives a scheme that is second order accurate in space and time and is stable provided (2.63) holds, at least in the linear case.

For non-linear systems, it is sufficient to represent waves as *single ray* waves in most situations. This is obviously exact for discontinuous waves such as shock waves or contact discontinuities, and the wavespeed is obviously taken as the speed given the wave by the Riemann solver. For rarefaction waves the single ray assumption is an approximation and there is some ambiguity in the choice of wavespeed. Experience has shown that taking the speed at the head of the rarefaction, i.e. the largest speed in magnitude throughout the fan, is sufficient to obtain good resolution of rarefaction waves. Under the single ray assumption, the flux (2.64) reduces to a weighted average of the fluxes in the solution of the Riemann problem, and can be written as:

$$\mathbf{F}_{i+\frac{1}{2}} = \sum_{k=1}^{N+1} W^{(k)} \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (2.65)$$

where $\mathbf{F}_{i+\frac{1}{2}}^{(k)}$ is the k -th flux in the solution of the Riemann problem, with $\mathbf{F}_{i+\frac{1}{2}}^{(1)} = \mathbf{F}_i$ and $\mathbf{F}_{i+\frac{1}{2}}^{(N+1)} = \mathbf{F}_{i+1}$. The weights $W^{(k)}$ are defined as:

$$W^{(k)} = \frac{1}{2}(\nu_{i+\frac{1}{2}}^{(k)} - \nu_{i+\frac{1}{2}}^{(k-1)}) \quad (2.66)$$

with

$$\nu_{i+\frac{1}{2}}^{(0)} = -1 \quad \nu_{i+\frac{1}{2}}^{(N+1)} = +1 \quad (2.67)$$

These weights can be seen geometrically in Fig. 2.2. Combining (2.65) and (2.66), the intercell flux for the WAF method can be written as

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2} \sum_{k=1}^N \nu_{i+\frac{1}{2}}^{(k)} \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (2.68)$$

where $\Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} = \mathbf{F}_{i+\frac{1}{2}}^{(k+1)} - \mathbf{F}_{i+\frac{1}{2}}^{(k)}$ is the flux jump across the k -th wave.

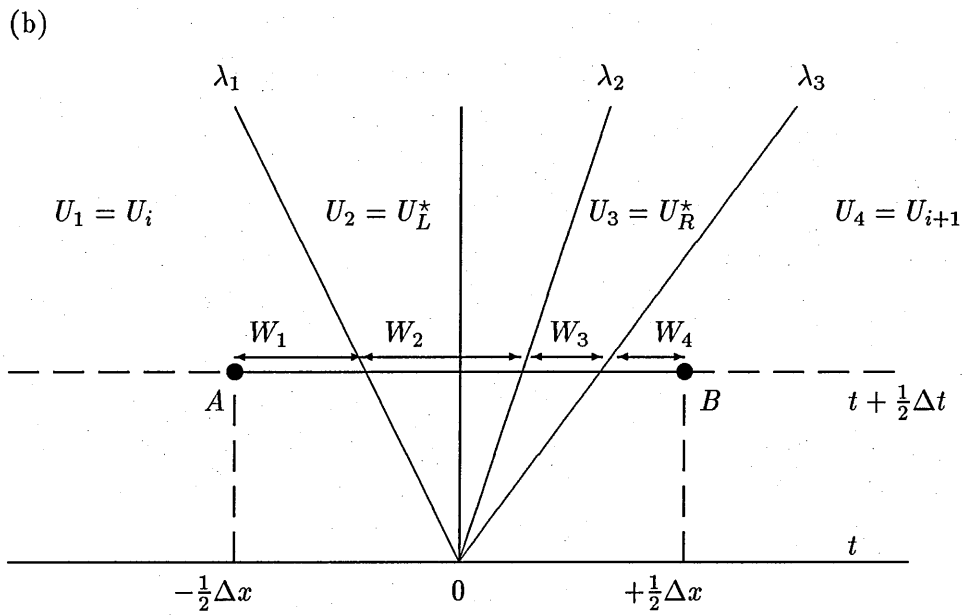
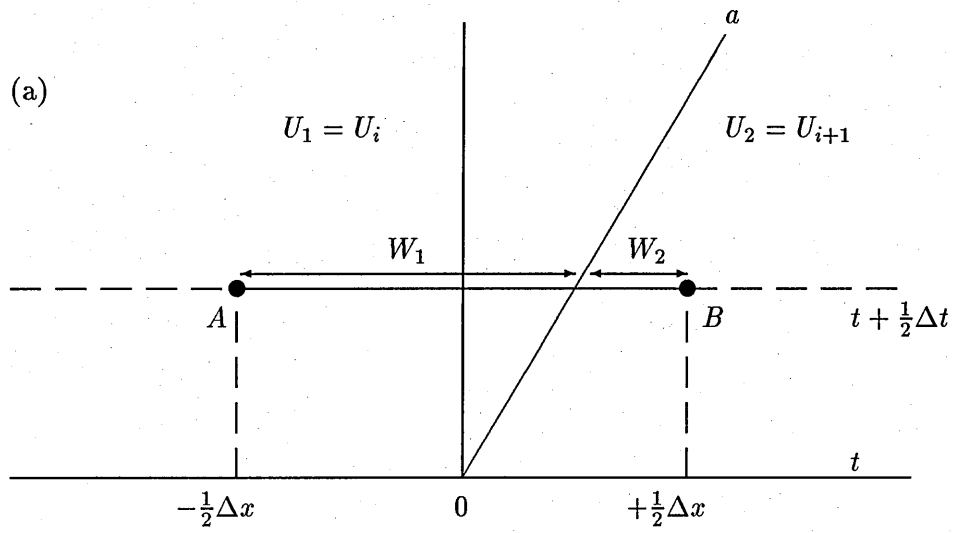


Figure 2.2: Weights W_k for the waf scheme on (a) the linear advection equation and (b) a system of three equations.

The only situation in which the assumption of single ray waves is not entirely safe is when a 'sonic rarefaction' occurs, i.e when a rarefaction wave in the solution of a local Riemann problem has head and tail speeds of opposite sign. Failure to take adequate care in this situation can lead to the growth of shocks which do not satisfy the entropy conditions and are therefore physically unrealistic. This could be resolved by discarding the single ray wave representation of the sonic rarefaction and computing the integral (2.64) through the wave exactly. In practice, this would be expensive, and experience has shown that it is sufficient to represent the rarefaction wave as a single ray wave with speed equal to the true speed of the head of the wave, and use (2.65) with a weight corresponding to the whole region between the head of the rarefaction and the wave immediately behind the rarefaction. The state corresponding to this weight is taken as $\bar{U}(0; U_i, U_{i+1})$, i.e the state on the time axis in the solution to the Riemann problem. This process ensures that the most upwind data is used in the flux calculation.

TVD Version of the WAF Method

An important property of numerical methods is that they should remain oscillation-free in regions of high gradient, at least when applied to the linear advection equation. In [54], Sweby introduced flux limiters $B_{i+\frac{1}{2}}$ into the Lax-Wendroff scheme on the linear advection equation (2.19), (2.23) when $a > 0$ by using the intercell flux

$$f_{i+\frac{1}{2}} = f_i + \frac{1}{2}(1 - \nu)B_{i+\frac{1}{2}}(f_{i+1} - f_i) \quad (2.69)$$

When $B_{i+\frac{1}{2}} \equiv 1$ this reduces to the Lax-Wendroff intercell flux (2.51); when $B_{i+\frac{1}{2}} \equiv 0$ this reduces to the **FOU** intercell flux (2.49). By using Corollaries 1 and 2 following Harten's theorem (Theorem 8), Sweby found limits of $B_{i+\frac{1}{2}}$ that ensured the resulting scheme was **TVD**. Since **WAF** reduces to the Lax-Wendroff method on the linear advection equation, it is second order accurate in space and time and will need some form of flux limiting to control spurious oscillations. In [63], Toro introduced limiter functions $\phi_{i+\frac{1}{2}}$ for the **WAF** method by studying the flux

$$f_{i+\frac{1}{2}} = \frac{1}{2}(1 + \phi_{i+\frac{1}{2}})f_i + \frac{1}{2}(1 - \phi_{i+\frac{1}{2}})f_{i+1} \quad (2.70)$$

for the linear advection equation. When $\phi_{i+\frac{1}{2}} \equiv |\nu|$, this gives the Lax-Wendroff intercell flux; when $\phi_{i+\frac{1}{2}} = \text{sgn}(\nu)$ this gives the **FOU** intercell flux. Toro performed a similar task to Sweby by finding regions for which the functions $\phi_{i+\frac{1}{2}}$ give a **TVD** scheme by making use of Corollaries 1 and 2 above. The analysis was originally published in [63], and can be found in [67] or [66]. The results are now summarized. For convenience, write $\phi = \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu)$. Provided:

$$L \leq \phi \leq 1 \quad (2.71)$$

for some lower limit $L \in [-1, |\nu|]$ the **TVD** regions are:

$$\mathbf{Z}_L = \left\{ (r_{i+\frac{1}{2}}, \phi) \text{ such that } r_{i+\frac{1}{2}} \leq 0, \phi \geq \phi_L, L \leq \phi \leq 1 \right\} \quad (2.72)$$

$$\mathbf{Z}_R = \left\{ (r_{i+\frac{1}{2}}, \phi) \text{ such that } r_{i+\frac{1}{2}} \geq 0, \phi \geq \phi_R, L \leq \phi \leq 1 \right\} \quad (2.73)$$

where

$$\phi_L = 1 + (L + 1)r_{i+\frac{1}{2}} \quad (2.74)$$

and

$$\phi_R = 1 - \frac{2}{|\nu|}(1 - |\nu|)r_{i+\frac{1}{2}} \quad (2.75)$$

Here $r_{i+\frac{1}{2}}$ is the local *flow parameter*:

$$r_{i+\frac{1}{2}} = \frac{\Delta_{upw}u}{\Delta_{loc}u} \quad (2.76)$$

$\Delta_{loc}u$ is the *local* jump in u :

$$\Delta_{loc}u = u_{i+1} - u_i \quad (2.77)$$

and $\Delta_{upw}u$ is the *upwind* jump

$$\Delta_{upw}u = \begin{cases} u_i - u_{i-1} & \text{if } \nu > 0 \\ u_{i+2} - u_{i+1} & \text{if } \nu < 0 \end{cases} \quad (2.78)$$

Note that $r_{i+\frac{1}{2}} \approx 1$ corresponds to smooth flow, and $r_{i+\frac{1}{2}} \approx 0$ corresponds to steep gradients.

The **TVB** regions are illustrated in Fig. 2.3. Three popular limiter functions are

$$\phi(r, \nu) = \begin{cases} 1 & r \leq 0 \\ 1 - (1 - |\nu|)r & 0 < r \leq 1 \\ |\nu| & r > 1 \end{cases} \quad (2.79)$$

$$\phi(r, \nu) = \begin{cases} 1 & r \leq 0 \\ 2r/(1 + r) & r > 0 \end{cases} \quad (2.80)$$

$$\phi(r, \nu) = \begin{cases} 1 & r \leq 0 \\ 1 - 2(1 - |\nu|)r & 0 < r \leq \frac{1}{2} \\ |\nu| & \frac{1}{2} < r \leq 1 \\ 1 - (1 - |\nu|)r & 1 < r \leq 2 \\ 1 - 2(1 - |\nu|) & r > 2 \end{cases} \quad (2.81)$$

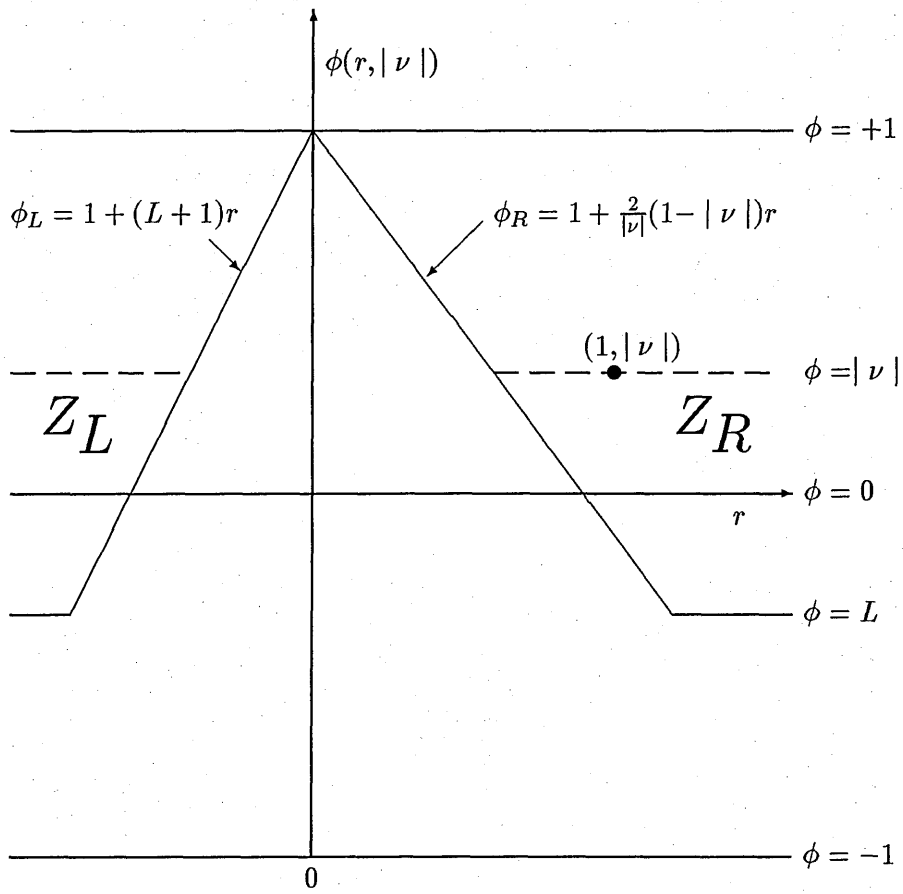


Figure 2.3: TVD regions for the waf scheme.

Many other choices are, of course, possible. All are related to Flux Limiters B via

$$\phi = 1 - (1 - |\nu|)B \quad (2.82)$$

Limiter functions (2.79), (2.80) and (2.81) are related to the **MINMOD**, **VAN LEER** and **SUPERBEE** flux limiters respectively. The **TVD WAF** flux (2.70) is extended to non-linear systems by replacing (2.68) with the more general intercell flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2} \sum_{k=1}^N \phi_{i+\frac{1}{2}}^{(k)} \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (2.83)$$

where $\phi_{i+\frac{1}{2}}^{(k)}$ is a limiter function. Note that if $\phi_{i+\frac{1}{2}}^{(k)} \equiv \nu_{i+\frac{1}{2}}^{(k)}$ this reduces to the original flux (2.68), and if $\phi_{i+\frac{1}{2}}^{(k)} \equiv \text{sgn}(\nu_{i+\frac{1}{2}}^{(k)})$, equation (2.83) reduces to the Godunov flux (2.62). The limiter functions ϕ can be used in the flux (2.83), by taking

$$\phi_{i+\frac{1}{2}}^{(k)} = \phi(r_{i+\frac{1}{2}}^{(k)}, \nu_{i+\frac{1}{2}}^{(k)}) \cdot \text{sgn}(\nu_{i+\frac{1}{2}}^{(k)}) \quad (2.84)$$

for each integer k such that $1 \leq k \leq N$. Here $r_{i+\frac{1}{2}}^{(k)}$ is the flow parameter for the k^{th} wave from the Riemann problem at boundary $i + \frac{1}{2}$, and is defined as

$$r_{i+\frac{1}{2}}^{(k)} = \frac{\Delta_{upw}^{(k)} q}{\Delta_{loc}^{(k)} q} \quad (2.85)$$

where $\Delta_{upw}^{(k)} q$ is the *upwind* jump in some variable q :

$$\Delta_{upw}^{(k)} q = \begin{cases} q_{i-\frac{1}{2}}^{(k)} - q_{i-\frac{1}{2}}^{(k-1)} & \text{if } \nu_{i+\frac{1}{2}}^{(k)} > 0 \\ q_{i+\frac{3}{2}}^{(k)} - q_{i+\frac{3}{2}}^{(k-1)} & \text{otherwise} \end{cases} \quad (2.86)$$

and $\Delta_{loc}^{(k)} q$ is the *local* jump in q :

$$\Delta_{loc}^{(k)} q = q_{i+\frac{1}{2}}^{(k)} - q_{i+\frac{1}{2}}^{(k-1)} \quad (2.87)$$

The choice of the variable q is debatable, although it is typically taken as density for the Euler equations or geopotential for the Shallow Water equations, since experience has shown that these are suitable choices.

Boundary conditions for the **WAF** method are set as follows. Two artificial cells, called *fictitious cells* or *ghost cells*, are added to either end of the grid before the computation starts. At the beginning of each timestep, values are set in these cells. If a boundary is transmissive, values in each of the two cells are taken from the matching cell inside the domain, i.e the cell that is an equal distance (in terms of numbers of cells) from the boundary. The same procedure is applied for reflective boundaries, except the sign of the velocity is changed. The intercell fluxes for the cell boundaries on the boundary of the flow domain are then computed in the same way as for intercell boundaries within the interior of the flow domain.

2.5 Two Dimensional Systems of Conservation Laws

The aim of this section is to discuss hyperbolic systems of conservation laws in two space dimensions. These can be written in the form

$$\mathbf{U}_t + \nabla \cdot \mathbf{E} = 0$$

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{U} : \mathbf{R}^2 \times [0, T] \rightarrow \mathbf{R}^N \quad (2.88)$$

$$\mathbf{E} = (\mathbf{F}, \mathbf{G}) \in \mathbf{R}^N \times \mathbf{R}^N$$

where $\mathbf{x} = (x, y)$ is the coordinate vector, $\mathbf{U}$ is the vector of N conserved variables, and $\mathbf{F}, \mathbf{G}$ are vector valued flux functions in the x and y directions respectively. For each unit vector $\mathbf{n}$ in $\mathbf{R}^2$ the *projected equations* along $\mathbf{n}$ are defined as

$$\mathbf{U}_t + \frac{\partial}{\partial \zeta} \mathbf{E}^{(\mathbf{n})} = 0 \quad \mathbf{E}^{(\mathbf{n})}(\mathbf{U}) = \mathbf{n} \cdot \mathbf{E}(\mathbf{U}) \quad (2.89)$$

This can be written as

$$\mathbf{U}_t + \mathbf{A}^{(\mathbf{n})}(\mathbf{U}) \mathbf{U}_\zeta = 0 \quad \mathbf{A}^{(\mathbf{n})} = \frac{\partial \mathbf{E}^{(\mathbf{n})}}{\partial \mathbf{U}} \quad (2.90)$$

The system (2.88) is said to be *hyperbolic* if for each $\mathbf{n}$ the projected equation (2.90) is hyperbolic in the sense of a one-dimensional system (2.1), i.e if $\mathbf{A}^{(\mathbf{n})}$ is diagonalizable for each $\mathbf{n}$. If the eigenvalues of $\mathbf{A}^{(\mathbf{n})}$ are distinct for each $\mathbf{n}$, then (2.90) is *strictly hyperbolic*.

The aim of this section is to discuss numerical methods for the solution of (2.88) with initial conditions

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}) \quad (2.91)$$

for some given function $\mathbf{U}_0(\mathbf{x})$. Before this discussion, it is appropriate to introduce two hyperbolic systems of the form (2.88) that will be of use later in this thesis. The two dimensional Euler equations of Gas Dynamics can be written in form (2.88) as

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{pmatrix}_y = 0 \quad (2.92)$$

where u and v are the x and y components of velocity, ρ and p are the density and pressure, and the total energy E is defined as

$$E = \frac{1}{2} \rho (u^2 + v^2) + \rho e \quad (2.93)$$

As in the one dimensional case, the system is closed via the equation of state (2.6). The ratio of specific heats, γ , is again assumed constant here. The eigenvalues of $\mathbf{A}^{(\mathbf{n})}$ for

the projected equations (2.89) are $\lambda^{(1)} = w^{\parallel} - a$, $\lambda^{(2)} = \lambda^{(3)} = w^{\parallel}$ and $\lambda^{(4)} = w^{\parallel} + a$, where $w^{\parallel}$ is the component of velocity in the $\mathbf{n}$ direction and $a = \sqrt{\frac{\gamma p}{\rho}}$ is the local sound speed. The system is thus not strictly hyperbolic, since the eigenvalues are not distinct. However Riemann problems for the projected equations of this system can be solved by making use of Riemann solvers for the one-dimensional Euler equations (2.4),(2.5),(2.6). This is because the eigenvalue λ_3 represents a *slip line*, across which only the component of velocity perpendicular to $\mathbf{n}$, denoted $w^{\perp}$, changes, and this quantity does not change across any other wave. Thus $w^{\perp}$ can be ignored, and any Riemann solver for the one dimensional Euler equations can be used to solve the Riemann problem to give star-state values of ρ , $w^{\parallel}$ and p . The other component of velocity, $w^{\perp}$, is determined from the position of the slip line in the $x - t$ plane.

A second example of a system of the form (2.88) is formed by the Shallow Water equations:

$$\begin{pmatrix} \phi \\ \phi u \\ \phi v \end{pmatrix}_t + \begin{pmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \\ \phi uv \end{pmatrix}_x + \begin{pmatrix} \phi v \\ \phi uv \\ \phi v^2 + \frac{1}{2}\phi^2 \end{pmatrix}_y = 0 \quad (2.94)$$

where u and v are again the x and y components of velocity, $\phi = \sqrt{gH}$ is the geopotential, H is the total depth of water and $g \approx 9.81$ is the acceleration due to gravity. The eigenvalues for the projected equations (2.89) of (2.94) are $\lambda_1 = w^{\parallel} - c$, $\lambda_2 = w^{\parallel}$ and $\lambda_3 = w^{\parallel} + c$, where $w^{\parallel}$ is the component of velocity in the $\mathbf{n}$ direction and $c = \sqrt{\phi}$ is the *celerity*, analogous to sound speed in Gas Dynamics. Since all three eigenvalues are distinct, this system is strictly hyperbolic. The solution of the Riemann problem for the projected equations of system (2.94) consists of three waves separating four constant states. The centre wave is always a slip line which, as for the Euler equations, is a discontinuity across which only the velocity component perpendicular to $\mathbf{n}$, $w^{\perp}$, changes. As in the case of the Euler equations, $w^{\perp}$ does not change across the other waves. Any Riemann solver for the one dimensional Shallow Water equations (2.7) can thus be used to find ϕ and $w^{\parallel}$ in the star-state, and $w^{\perp}$ will then be determined by the position of the slip line in the $x - t$ plane.

2.6 Numerical Methods for Two Space Dimensions

2.6.1 Introduction

Methods for the solution of (2.88), (2.91) on *curvilinear* grids, i.e *structured*, *body-fitted grids of quadrilaterals*, will be considered here. A *Cartesian* grid is a curvilinear grid in which all cells are identical rectangles. Since the grids considered here are structured, each cell can be indexed by two integers and will be denoted $\Delta_{i,j}$ here. A value $U_{i,j}^n$ will be stored for cell $\Delta_{i,j}$ at each timelevel n , and defined as the integral average of U^n over cell $\Delta_{i,j}$, i.e:

$$U_{i,j}^n = \frac{1}{V(\Delta_{i,j})} \int_{\Delta_{i,j}} U(x, y, t^n) dx dy \quad (2.95)$$

where $V(\Delta_{i,j})$ is the area of $\Delta_{i,j}$.

2.6.2 The Space Operator Splitting Technique

A popular technique for extending numerical methods for the solution of one-dimensional conservation laws (2.1) to the solution of two-dimensional conservation laws (2.88) on curvilinear grids is the Space Operator Splitting technique. A detailed discussion of the technique can be found in Strang [53] and the references cited there, and is briefly described here for two space dimensions; the extension to three space dimensions follows easily, and will not be discussed.

Consider any method for the solution of the one dimensional conservation law (2.1), written in operator form as:

$$\mathbf{U}_i^{n+1} = \left(\mathbf{L}^{(\Delta t)} \right) \mathbf{U}_i^n \quad (2.96)$$

with the superscript denoting the timestep used. For a two dimensional grid, two operators $\mathbf{L}_i^{(\Delta t)}$ and $\mathbf{L}_j^{(\Delta t)}$ that act on $\mathbf{U}^n$ can be defined as those which perform one update of scheme (2.96) along all one-dimensional strips of cells indexed by constant i and constant j respectively. The operators are effectively solving the one-dimensional projected equation (2.89) with $\mathbf{n}$ tangent to a local coordinate lines, and varying along the strip of cells. Given these definitions, a simple scheme for the solution of (2.88), (2.91) can be constructed as:

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{L}_j^{(\Delta t)} \left(\mathbf{L}_i^{(\Delta t)} \left(\mathbf{U}_{i,j}^n \right) \right) \quad (2.97)$$

or more explicitly:

$$\tilde{\mathbf{U}}_{i,j} = \mathbf{L}_i^{(\Delta t)} \left(\mathbf{U}_{i,j}^n \right) \quad \text{for all } i, j \quad (2.98)$$

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{L}_j^{(\Delta t)} \left(\tilde{\mathbf{U}}_{i,j} \right) \quad \text{for all } i, j \quad (2.99)$$

Step (2.98) is called an ' i -sweep'; step (2.99) is called a ' j -sweep'. The timestep Δt is kept the same for each sweep, and determined from the initial data $\{\mathbf{U}_i^n\}$ for the timestep, using the maximum characteristic speed over all cells, and the stability condition of the one dimensional scheme (2.96). The accuracy of operator splitting can be assessed by studying the linear system

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B}\mathbf{U}_y = 0 \quad (2.100)$$

where $\mathbf{U}$ is a vector of dependant variables and $\mathbf{A}$ and $\mathbf{B}$ are constant matrices. In general, i.e when $\mathbf{A}$ and $\mathbf{B}$ do not commute, it can be shown that the operator splitting (2.97) will introduce first order errors into the solution, whatever the accuracy of the operators (2.96). In [53], Strang suggested the splitting

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{L}_i^{(\frac{1}{2}\Delta t)} \left(\mathbf{L}_j^{(\Delta t)} \left(\mathbf{L}_i^{(\frac{1}{2}\Delta t)} \left(\mathbf{U}_{i,j}^n \right) \right) \right) \quad (2.101)$$

The roles of $\mathbf{L}_i$ and $\mathbf{L}_j$ can be interchanged. Since three sweeps are needed for (2.101) rather than two for (2.97), scheme (2.101) is more expensive. Strang [53] showed that

provided (2.8) is second order accurate in space and time, the splitting (2.101) is the same order of accuracy for (2.100), even if $\mathbf{A}$ and $\mathbf{B}$ do not commute.

Operator split results for the solution of two dimensional systems using the one dimensional **waf** method described earlier will be presented later, so the application of i -sweeps using this method will now be considered in detail. The application of j -sweeps follows similarly. The first step is the solution of the Riemann problems $RP(\mathbf{U}_{i,j}^{(1)}, \mathbf{U}_{i+1,j}^{(1)})$ at boundary $(i + \frac{1}{2}, j)$ for some initial data set $\mathbf{U}^{(1)}$. This is done as follows. Firstly, compute components of velocity from the states $\mathbf{U}_{i,j}^{(1)}, \mathbf{U}_{i+1,j}^{(1)}$ that are normal and tangential to the boundary. The normal velocities and other variables (densities and pressures for the Euler equations, geopotential for the Shallow Water equations) define a one-dimensional Riemann problem for the projected equation (2.89) normal to the boundary and any one-dimensional Riemann solver can be used to find wavespeeds and star states in the solution. Flow parameters can then be computed for each of the waves in the solution, as in the one dimensional case, using the solution to Riemann problems at boundaries $(i - \frac{1}{2}, j)$ and $(i + \frac{3}{2}, j)$. Slip lines are then added to the solution, with speed equal to the star state velocity, and magnitude equal to the jump in velocity tangential to the boundary in the initial data. Flow parameters for slip lines are computed from these velocity jumps, since no other variables change across them. Courant numbers for all waves can be computed from the wavespeeds $\lambda_{i+\frac{1}{2},j}^{(k)}$ as

$$\nu_{i+\frac{1}{2},j}^{(k)} = \lambda_{i+\frac{1}{2},j}^{(k)} \frac{\Delta t}{l_{i+\frac{1}{2},j}^K} \quad (2.102)$$

where $K = R$ (right) if $\lambda_{i+\frac{1}{2},j}^{(k)} \geq 0$ or $K = L$ (left) if $\lambda_{i+\frac{1}{2},j}^{(k)} \leq 0$. $l_{i+\frac{1}{2},j}^R$ and $l_{i+\frac{1}{2},j}^L$ are right and left *integration lengths*, and should be approximations to the distances across the cells to the right and left of the boundary $(i + \frac{1}{2}, j)$ respectively. Thus provided Courant numbers defined by (2.102) remain less than unity, waves from the Riemann problem at $(i + \frac{1}{2}, j)$ will not cross the neighbouring cells completely, and the one dimensional sweeps should remain stable. The right integration length $l_{i+\frac{1}{2},j}^R$ should be representative of the width of the cell $\Delta_{i+1,j}$, and can be defined as

$$l_{i+\frac{1}{2},j}^R = \sqrt{d_x^2 + d_y^2} \quad (2.103)$$

where

$$d_x = \frac{1}{2}(x_{i+\frac{3}{2},j+\frac{1}{2}} - x_{i+\frac{1}{2},j+\frac{1}{2}}) + \frac{1}{2}(x_{i+\frac{3}{2},j-\frac{1}{2}} - x_{i+\frac{1}{2},j-\frac{1}{2}}) \quad (2.104)$$

$$d_y = \frac{1}{2}(y_{i+\frac{3}{2},j+\frac{1}{2}} - y_{i+\frac{1}{2},j+\frac{1}{2}}) + \frac{1}{2}(y_{i+\frac{3}{2},j-\frac{1}{2}} - y_{i+\frac{1}{2},j-\frac{1}{2}}) \quad (2.105)$$

and $(x_{i+\frac{3}{2},j+\frac{1}{2}}, x_{i+\frac{3}{2},j-\frac{1}{2}})$ is the intersection between gridlines $i + \frac{1}{2}$ and $j + \frac{1}{2}$. The left integration length $l_{i+\frac{1}{2},j}^L$ is defined similarly. The Courant number and flow parameter for each wave can be used to compute a limiter function for each wave, using, for example, (2.79), (2.80) or (2.81). Components of velocity parallel to the boundary are then

determined from the magnitude of the limiter $\phi_{i+\frac{1}{2}}^{slip}$ for the slip line. If $\frac{x}{t} \leq \phi_{i+\frac{1}{2}}^{slip}$, then the component is taken from $U_{i,j}$; otherwise it is taken from $U_{i+1,j}$. All states are now determined and a WAF intercell flux $E_{i+\frac{1}{2},j}$ normal to the boundary can be computed as in the one-dimensional case. Note that the limiter for the slip line does not, in general, equal the limiter for the contact, even though these waves have the same speed, and five fluxes need to be included in the average. The intercell flux $E_{i+\frac{1}{2},j}$ can be transformed into $x - y$ coordinates by computing x and y components of momentum fluxes using the angle of the boundary to the x -axis and leaving the other components unchanged. The state $U^{(1)}$ is then updated via the conservative formula for the strip:

$$U_{i,j}^{(2)} = U_{i,j}^{(1)} - \frac{\Delta t}{V(\Delta_{i,j})} [E_{i+\frac{1}{2},j} \cdot l_{i+\frac{1}{2},j} - E_{i-\frac{1}{2},j} \cdot l_{i-\frac{1}{2},j}] \quad (2.106)$$

to obtain the data $U^{(2)}$ resulting from this i -sweep. Sweeps in the j -direction follow similarly. See Speares [50] for further details.

The power of combining operator splitting with modern shock capturing methods for solving realistic flow problems is shown in the review paper by Woodard and Colella [82]. However, the approach does suffer a serious drawback when the conservative formula (2.106) is used in sweeps on non-Cartesian grids. If the fluxes $E_{i+\frac{1}{2},j}$ are consistent, (2.106) will always introduce an error into the data and will thus not preserve constant flow, as follows. Suppose $U_{i,j}^{(1)} \equiv U \forall i, j$. The intercell fluxes must then be given by

$$E_{i+\frac{1}{2},j} = E(U) \cdot n_{i+\frac{1}{2},j} \quad (2.107)$$

where $E(U) = (F(U), G(U))$ is the general flux function and $n_{i+\frac{1}{2},j}$ is the normal to boundary $(i + \frac{1}{2}, j)$ in the direction of increasing ζ along the strip of cells. Substituting (2.107) into (2.106) gives

$$U^{(2)} = U^{(1)} - \frac{\Delta t}{V(\Delta_{i,j})} \cdot E(U) \cdot [n_{i+\frac{1}{2},j} \cdot l_{i+\frac{1}{2},j} - n_{i-\frac{1}{2},j} \cdot l_{i-\frac{1}{2},j}] \quad (2.108)$$

The terms in brackets on the right hand side of (2.108) will not cancel unless $l_{i+\frac{1}{2},j} \equiv l_{i-\frac{1}{2},j}$ and $n_{i+\frac{1}{2},j} \equiv n_{i-\frac{1}{2},j}$, i.e the cell sides are the same length and face in the same direction. Thus the grid must be uniform, i.e all cells must be of the same shape and size. It can be easily shown, by simple application on the linear advection equation, that errors introduced by one sweep are not necessarily rectified by following sweeps. An operator split scheme on a curvilinear grid will thus not preserve constant data. The error introduced by the operator splitting technique in the way described above will be called the *operator splitting error*.

2.6.3 Finite Volume Schemes

An alternative to operator split methods for multi-dimensional systems of conservation laws such as (2.88) are truly finite volume schemes, i.e schemes that can be written directly

in conservation form

$$\begin{aligned} \mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^n - \frac{\Delta t}{V(\Delta_{i,j})} & \left[\mathbf{E}_{i+\frac{1}{2},j} \cdot l_{i+\frac{1}{2},j} - \mathbf{E}_{i-\frac{1}{2},j} \cdot l_{i-\frac{1}{2},j} \right] \\ & - \frac{\Delta t}{V(\Delta_{i,j})} \left[\mathbf{E}_{i,j+\frac{1}{2}} \cdot l_{i,j+\frac{1}{2}} - \mathbf{E}_{i,j-\frac{1}{2}} \cdot l_{i,j-\frac{1}{2}} \right] \end{aligned} \quad (2.109)$$

where $V(\Delta_{i,j})$ is the area of cell $\Delta_{i,j}$, $\mathbf{E}_{i\pm\frac{1}{2},j}$, $\mathbf{E}_{i,j\pm\frac{1}{2}}$ are intercell fluxes normal to boundaries $(i \pm \frac{1}{2}, j)$ and $(i, j \pm \frac{1}{2})$ and $l_{i\pm\frac{1}{2},j}$, $l_{i,j\pm\frac{1}{2}}$ are the lengths of those boundaries. This formula is derived from the integral form of the two-dimensional conservation law. Schemes based on (2.109) will be called *finite volume* schemes here, as opposed to operator split schemes, although the one dimensional scheme used in the operator splitting may be finite volume in one space dimension. Note that provided the fluxes $\mathbf{E}_{i\pm\frac{1}{2},j}$, $\mathbf{E}_{i,j\pm\frac{1}{2}}$ are consistent, finite volume schemes (2.109) will not suffer from the operator split error. Consider constant data $\mathbf{U}_{i,j}^{(1)} \equiv \mathbf{U} \forall i, j$. Then

$$\mathbf{E}_{i\pm\frac{1}{2},j} = \mathbf{E}(\mathbf{U}) \cdot \mathbf{n}_{i\pm\frac{1}{2},j} \quad \mathbf{E}_{i,j\pm\frac{1}{2}} = \mathbf{E}(\mathbf{U}) \cdot \mathbf{n}_{i,j\pm\frac{1}{2}} \quad (2.110)$$

Substituting these into (2.109) gives:

$$\begin{aligned} \mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^n - \frac{\Delta t}{V(\Delta_{i,j})} \cdot \mathbf{E}(\mathbf{U}) & \left[\mathbf{n}_{i+\frac{1}{2},j} \cdot l_{i+\frac{1}{2},j} - \mathbf{n}_{i-\frac{1}{2},j} \cdot l_{i-\frac{1}{2},j} \right. \\ & \left. + \mathbf{n}_{i,j+\frac{1}{2}} \cdot l_{i,j+\frac{1}{2}} - \mathbf{n}_{i,j-\frac{1}{2}} \cdot l_{i,j-\frac{1}{2}} \right] \end{aligned} \quad (2.111)$$

The term in the brackets on the right hand side of (2.111) is effectively the integral around the boundary of the cell of the outward facing normal vector. Since the boundary forms a closed loop, this is necessarily zero, whatever the shape of the cell, and (2.111) gives $\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^n$. Thus constant data is preserved.

It is clear from (2.109) that the update from one step to the next for finite volume schemes involves just one step, not multiple steps as in the case of operator split methods. This single step allows better modelling of multidimensional convection inherent in systems such as (2.1), as will be seen in Chapter 5 for the simple case of linear advection. Several finite volume schemes for the solution of (2.1) exist. These include the schemes of Colella [8] and LeVeque [32], and the finite volume version of the **MUSCL**-Hancock scheme described by van Leer in [77] and by Quirk in [41], all of which depend on **MUSCL**-type reconstruction of data to achieve higher than first order accuracy. A finite volume version of **WAF**-type scheme has been suggested by Speares [50] although this scheme appears to be linearly unstable, as is discussed in Chapter 5. New finite volume **WAF**-type schemes are developed in Chapters 5, 6 and 7.

Chapter 3

A Unified, Riemann-Problem Based Extension of the Warming-Beam and Lax-Wendroff Schemes

3.1 Introduction

The aim of this chapter is to construct, via the **waf** approach, a scheme for the solution to non-linear systems of hyperbolic conservation laws in one space dimension that has stability restriction of $\nu_{max} \leq 2$. On the linear advection equation, the scheme reduces to the Lax-Wendroff scheme for $|\nu| \leq 1$ and the Warming-Beam scheme when $1 \leq |\nu| \leq 2$. **tvD** criteria are applied to the scheme to prevent spurious oscillations in regions of high gradient. The scheme is extended to non-linear systems of equations by the use of solutions to local Riemann problems at intercell boundaries. The resulting one-dimensional scheme is successfully extended to two space dimensions by the use of space operator splitting. The work discussed in this Chapter has already been presented in [69] and [68].

The chapter is set out as follows. In §3.2, the Warming Beam scheme is discussed. In §3.3 the unified extension of the Warming-Beam and Lax-Wendroff schemes is derived on a general scalar conservation law and a **tvD** version of the scheme is constructed in §3.4. In §3.5, the scheme is extended to non-linear systems of equations. In §3.6, numerical results are presented in both one and two space dimensions and conclusions are drawn in §3.7.

3.2 The Warming-Beam Scheme

Consider the system of hyperbolic conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \quad (3.1)$$

where $\mathbf{U}$ is a vector of N conserved variables, and $\mathbf{F}(\mathbf{U})$ is some flux function of $\mathbf{U}$. A popular finite difference method for the solution of (3.1) is the two step MacCormack scheme [35]:

$$\overline{\mathbf{U}}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} \nabla \mathbf{F}_i^n \quad (3.2)$$

$$\mathbf{U}_i^{n+1} = \frac{1}{2}(\mathbf{U}_i^n + \overline{\mathbf{U}}_i^{n+1}) - \frac{1}{2} \frac{\Delta t}{\Delta x} \Delta \mathbf{F}_i^{n+1} \quad (3.3)$$

where $\mathbf{U}_i^n$ is the approximation to the solution $\mathbf{U}$ at the point $(i + \frac{1}{2})\Delta x$ at time level n and the forward and backward operators Δ and ∇ are defined as:

$$\Delta \mathbf{F}_i = \mathbf{F}_{i+1} - \mathbf{F}_i \quad \nabla \mathbf{F}_i = \mathbf{F}_i - \mathbf{F}_{i-1} \quad (3.4)$$

This scheme is a three-point, centred finite difference scheme. On the linear advection equation

$$u_t + (au)_x = 0 \quad (3.5)$$

MacCormack's scheme reduces to the Lax-Wendroff method (2.50). It is therefore second order accurate in space and time.

Warming and Beam [80] proposed a modification to MacCormack's method to give a one-sided, second-order accurate finite difference scheme. For positive wavespeeds, i.e flow from left to right, they replace the step (3.3) by:

$$\mathbf{U}_i^{n+1} = \frac{1}{2}(\mathbf{U}_i^n + \overline{\mathbf{U}}_i^{n+1}) - \frac{1}{2} \frac{\Delta t}{\Delta x} \nabla^2 \mathbf{F}_i^n - \frac{1}{2} \frac{\Delta t}{\Delta x} \nabla \mathbf{F}_i^{n+1} \quad (3.6)$$

For the linear advection equation (3.5), the scheme (3.2), (3.6) can be written as a one step scheme:

$$u_i^{n+1} = \frac{1}{2}(1 - \nu)(2 - \nu)u_i^n + \nu(2 - \nu)u_{i-1}^n + \frac{1}{2}\nu(\nu - 1)u_{i-2}^n \quad (3.7)$$

where $\nu = a \frac{\Delta t}{\Delta x}$ as in Chapter 2. Scheme (3.7) has a one sided stencil and by Theorem 1, Chapter 2 it can be shown to be second order accurate in space and time. Von Neumann stability analysis shows that (3.7) is stable provided $0 \leq \nu \leq 2$. Note that (3.7) can be written in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] \quad (3.8)$$

with intercell flux

$$f_{i+\frac{1}{2}} = f_i - \frac{1}{2}(\nu - 1)(f_i - f_{i-1}) \quad (3.9)$$

A modified version of MacCormack's scheme can be defined as

$$\overline{U_i^{n+1}} = U_i^n - \frac{\Delta t}{\Delta x} \Delta F_i^n \quad (3.10)$$

$$U_i^{n+1} = \frac{1}{2}(U_i^n + \overline{U_i^{n+1}}) - \frac{1}{2} \frac{\Delta t}{\Delta x} \nabla F_i^{n+1} \quad (3.11)$$

For negative wavespeeds, Warming and Beam use (3.10) and replace (3.11) with

$$U_i^{n+1} = \frac{1}{2}(U_i^n + \overline{U_i^{n+1}}) + \frac{1}{2} \frac{\Delta t}{\Delta x} \Delta^2 F_i^n - \frac{1}{2} \frac{\Delta t}{\Delta x} \Delta F_i^{n+1} \quad (3.12)$$

For the linear advection equation (3.5), the scheme (3.10), (3.12) can be written in one-step form as:

$$u_i^{n+1} = \frac{1}{2}(2 + \nu)(1 + \nu)u_i^n - \nu(2 + \nu)u_{i+1}^n + \frac{1}{2}\nu(1 + \nu)u_{i+2}^n \quad (3.13)$$

This scheme is second order accurate in space and time, and is stable provided $-2 \leq \nu \leq 0$. Scheme (3.13) can be written in conservation form (3.8) with intercell flux

$$f_{i+\frac{1}{2}} = f_{i+1} - \frac{1}{2}(1 + \nu)(f_{i+2} - f_{i+1}) \quad (3.14)$$

Due to the stability restrictions, schemes (3.2), (3.6) and (3.10), (3.12) can only be used when all characteristic speeds of (3.1) are of the same sign, i.e within regions of supersonic flow. To avoid this restriction, Warming and Beam combined the scheme (3.2), (3.6) with (3.2), (3.3) and (3.10), (3.12) with (3.10), (3.11) to produce a scheme for use in a general flow, with embedded supersonic regions. The resulting scheme was shown to work well when computing transonic flow around aerofoils. However, to ensure conservation, and therefore the correct speed of shock waves, they had to derive special switching operators to switch between (3.6) and (3.3), and between (3.12) and (3.11).

The scheme proposed by Warming and Beam had a larger stability region than the original MacCormack scheme, and therefore made timesavings possible. Warming and Beam also showed that the schemes (3.7) and (3.12) with one sided stencils also give other advantages over the Lax Wendroff scheme, such as better dispersive properties.

The aim of this chapter is to use the Weighted Average Flux approach discussed in Chapter 2 to combine the Lax-Wendroff intercell flux (2.51) with the fluxes (3.7) and (3.12) in a single conservative flux. Conservative switching between the two components comes naturally from the derivation of the scheme. A tvd version of the scheme is derived for scalar conservation laws and extended to non-linear systems of the form (3.1).

3.3 Development of the WAF-D2 Scheme on Model Problems

In this section, a generalized waf intercell flux is discussed and used to derive a unified extension of the Warming-Beam and Lax-Wendroff schemes on model problems. The

scheme will be referred to as the **WAF-D2** scheme, since the linear stability limit of the scheme will be 2. In the following sections a **TVD** version of the scheme will be constructed and extended to non-linear systems.

The scheme will be developed on the scalar model conservation law discussed in Chapter 2:

$$u_t + f(u)_x = 0 \quad (3.15)$$

where u is the conserved variable, and f is any linear or nonlinear scalar flux function. The solution to the Riemann problem $\mathbf{RP}(u_i, u_{i+1})$ for (3.15) has one wave of speed $\lambda_{i+\frac{1}{2}}$ separating u_i and u_{i+1} . The waves from all Riemann problems will be represented as single ray waves, as in the case of the classical **WAF** scheme described in Chapter 2. For discontinuous waves, this representation is exact as the wavespeed is taken from the Rankine-Hugoniot condition; for rarefaction waves, the speed is taken as the speed of the *head* of the rarefaction. In the case of sonic rarefaction waves, the state behind the rarefaction is replaced by the state on the time axis in the solution of the Riemann problem. The aim is to develop a scheme in conservative form (3.8) with stability limit

$$\nu_{max} \leq 2 \quad (3.16)$$

where $\nu_{max} = \lambda_{max} \frac{\Delta t}{\Delta x}$ is the Courant number coefficient and $\lambda_{max} = \max\{|\lambda_{i+\frac{1}{2}}|\}$ is the largest absolute wavespeed in the domain. This is twice the stability limit of the classical **WAF** scheme and the new scheme will thus offer potential computational savings of up to 50% over the classical **WAF** scheme. The new scheme will be based on a generalized **WAF** intercell flux:

$$f_{i+\frac{1}{2}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(\tilde{u}(x, t)) dx dt \quad (3.17)$$

where $\tilde{u}$ is the solution of some initial value problems. These initial value problems are taken as conventional Riemann problems at intercell boundaries based on constant data in cells. However, they could be taken as generalized Riemann problems based on reconstructed (sloped) data in the cells or any other initial valued problems. A detailed discussion of the schemes that can be derived from the general form (3.8), (3.17) is given in [70]. Extension of the scheme (3.8), (3.17) to two and three space dimensions will be considered in Chapter 5.

Here a special case of (3.8), (3.17) will be considered in detail. The scheme will be considered under the assumption that the cell width Δx is constant throughout the domain. It can easily be extended to the case when the cell width $\Delta_i x$ for each cell is different, when a 'grid distortion' parameter $g = \Delta_{i+1} x / \Delta_i x$ must be introduced, but this extension is not considered here. The intercell flux for the scheme presented here is derived from (3.17) by taking

$$x_1 = -\frac{1}{2}\Delta x \quad x_2 = \frac{1}{2}\Delta x \quad t_1 = 0 \quad t_2 = \Delta t \quad (3.18)$$

where, in local coordinates (x, t) , $x = 0$ at the boundary $i + \frac{1}{2}$ and $t = 0$ at time level n . These are the same as for the original **WAF** scheme. The new scheme will differ from

the original **waf** scheme in that the timestep Δt is chosen to satisfy (3.16). The **waf** scheme was derived via exact integration in space and the midpoint rule in time. The same strategy is taken here and the intercell flux becomes

$$f_{i+\frac{1}{2}} = \frac{1}{\Delta x} \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} f(\tilde{u}(x, \frac{1}{2}\Delta t)) dx \quad (3.19)$$

The function $\tilde{u}$ is taken to be the combination of solutions to conventional Riemann problems at intercell boundaries. The condition (3.16) on Δt restricts the number of Riemann problems involved in the calculation $f_{i+\frac{1}{2}}$ to those at boundaries $i - \frac{1}{2}$, $i + \frac{1}{2}$ and $i + \frac{3}{2}$. As in Chapter 2, define a Courant number at boundary $i + \frac{1}{2}$ as $\nu_{i+\frac{1}{2}} = \lambda_{i+\frac{1}{2}} \frac{\Delta t}{\Delta x}$. Since $|\nu_{i+\frac{1}{2}}| > 1$ are allowed by (3.16) waves from these Riemann problems may interact. In deriving the scheme, it will be assumed that *all wave interactions are linear*, i.e any two waves pass through each other without changes in speed, direction or strength. This will be exact if the flux function $f(u)$ in (3.15) is linear, and an approximation otherwise.

The intercell flux of the scheme will depend on which waves from the three Riemann problems pass through the space integration range $AB = [-\frac{1}{2}\Delta x, \frac{1}{2}\Delta x]$ and which do not. There are many possible such combinations of waves. Just three are considered here:

- (i) $1 \leq \nu_{i-\frac{1}{2}} \leq 2, 1 \leq \nu_{i+\frac{1}{2}} \leq 2, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$
- (ii) $-2 \leq \nu_{i-\frac{1}{2}} \leq 1, -1 \leq \nu_{i+\frac{1}{2}} \leq 1, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$
- (iii) $1 \leq \nu_{i-\frac{1}{2}} \leq 2, -1 \leq \nu_{i+\frac{1}{2}} \leq 1, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$

These cases are illustrated in Fig. 3.1.

Case (i): $1 \leq \nu_{i-\frac{1}{2}} \leq 2, 1 \leq \nu_{i+\frac{1}{2}} \leq 2, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Consider Fig. 3.1(i). In this case, the wave from boundary $i - \frac{1}{2}$ intersects the integration range AB , and the waves from boundaries $i + \frac{1}{2}$ and $i + \frac{3}{2}$ do not. Evaluation of the integral in (3.19) gives:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(\nu_{i-\frac{1}{2}} - 1)f_{i-1} + \frac{1}{2}(3 - \nu_{i-\frac{1}{2}})f_i \quad (3.20)$$

This can be written as

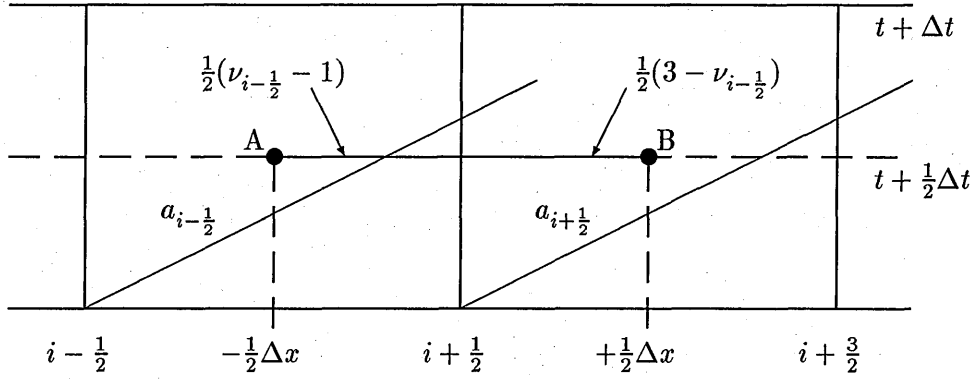
$$f_{i+\frac{1}{2}} = f_i - \frac{1}{2}\psi_{i+\frac{1}{2},L}(f_i - f_{i-1}) \quad (3.21)$$

with

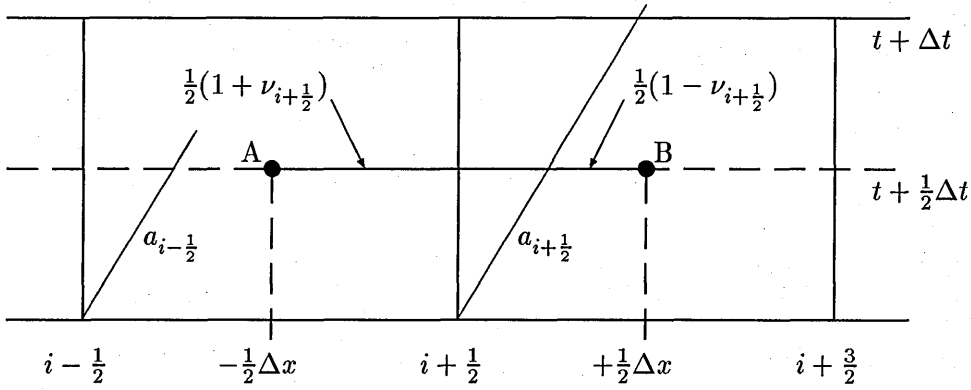
$$\psi_{i+\frac{1}{2},L} = \nu_{i-\frac{1}{2}} - 1 \quad (3.22)$$

On the linear advection equation, this is the intercell flux (3.9) for the Warming-Beam scheme. Note that the first term f_i is the integral between A and B across the solution of

Case (i)



Case (ii)



Case (iii)

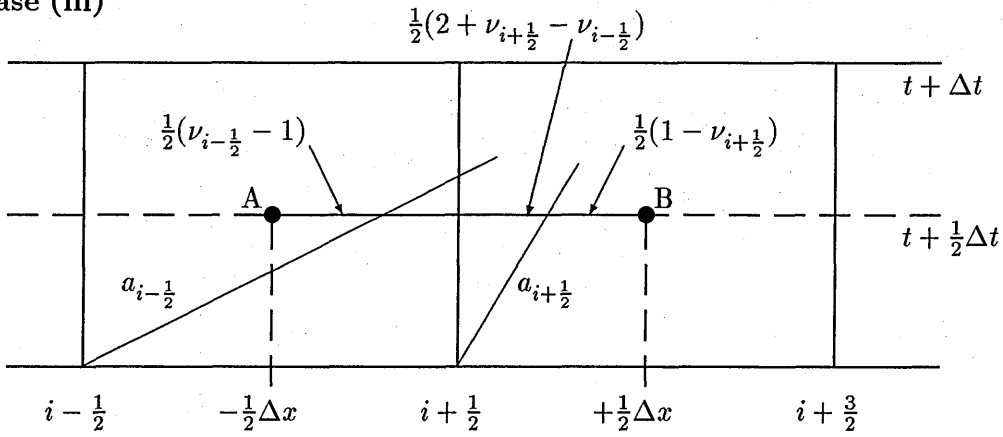


Figure 3.1: The integration range and wave structure for the second order scheme.

the Riemann problem at $i + \frac{1}{2}$. Thus (3.21) can be considered as the original **waf** integral (i.e taking only the central Riemann problem into account) *plus* an extra term for the wave coming from the Riemann problem on the left hand side.

Case (ii): $-2 \leq \nu_{i-\frac{1}{2}} \leq 1$, $-1 \leq \nu_{i+\frac{1}{2}} \leq 1$, $-1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Consider Fig. 3.1(ii). This case is the original **waf** situation, where the Lax-Wendroff scheme is reproduced in the constant wavespeed case. It is only necessary to consider the Riemann problem at boundary $i + \frac{1}{2}$. Evaluation of the integral in (3.19) gives:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(1 + \nu_{i+\frac{1}{2}})f_i + \frac{1}{2}(1 - \nu_{i+\frac{1}{2}})f_{i+1} \quad (3.23)$$

This can be written as

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}\phi_{i+\frac{1}{2}}(f_{i+1} - f_i) \quad (3.24)$$

with

$$\phi_{i+\frac{1}{2}} = \nu_{i+\frac{1}{2}} \quad (3.25)$$

Equation (3.24), (3.25) is the **waf** intercell flux (2.70) and reduces to the Lax-Wendroff flux (2.51) on the linear advection equation.

Case (iii): $1 \leq \nu_{i-\frac{1}{2}} \leq 2$, $-1 \leq \nu_{i+\frac{1}{2}} \leq 1$, $-1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Consider Fig. 3.1(iii). In this case, the waves from boundaries $i - \frac{1}{2}$ and $i + \frac{1}{2}$ intersect the integration range AB , and the wave from boundary $i + \frac{3}{2}$ does not. Evaluation of the integral in (3.19) gives:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(\nu_{i-\frac{1}{2}} - 1)f_{i-1} + \frac{1}{2}(2 + \nu_{i+\frac{1}{2}} - \nu_{i-\frac{1}{2}})f_i + \frac{1}{2}(1 - \nu_{i+\frac{1}{2}})f_{i+1} \quad (3.26)$$

This can be written as:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}\psi_{i+\frac{1}{2},L}(f_i - f_{i-1}) - \frac{1}{2}\phi_{i+\frac{1}{2}}(f_{i+1} - f_i) \quad (3.27)$$

with

$$\psi_{i+\frac{1}{2},L} = \nu_{i-\frac{1}{2}} - 1 \quad \phi_{i+\frac{1}{2}} = \nu_{i+\frac{1}{2}} \quad (3.28)$$

Note that in this case $\psi_{i+\frac{1}{2},L}$ and $\phi_{i+\frac{1}{2}}$ are the same as in cases (i) and (ii) respectively. This case acts as the explicitly constructed switching mechanism of Warming and Beam between cases (i) and (ii).

Other cases follow similarly. Cases (i), (ii) and (iii), and all other cases, can be summarized in one expression as:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}\psi_{i+\frac{1}{2},L}\Delta f_{i-\frac{1}{2}} - \frac{1}{2}\phi_{i+\frac{1}{2}}\Delta f_{i+\frac{1}{2}} - \frac{1}{2}\psi_{i+\frac{1}{2},R}\Delta f_{i+\frac{3}{2}} \quad (3.29)$$

where $\Delta f_{i+\frac{1}{2}} = f_{i+1} - f_i$ and

$$\psi_{i+\frac{1}{2},L} = \begin{cases} |\nu_{i-\frac{1}{2}}| - 1 & \text{if } \nu_{i-\frac{1}{2}} > 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.30)$$

$$\phi_{i+\frac{1}{2}} = \begin{cases} \nu_{i+\frac{1}{2}} & \text{if } -1 < \nu_{i+\frac{1}{2}} < 1 \\ \text{sgn}(\nu_{i+\frac{1}{2}}) & \text{otherwise} \end{cases} \quad (3.31)$$

$$\psi_{i+\frac{1}{2},R} = \begin{cases} -(|\nu_{i+\frac{3}{2}}| - 1) & \text{if } \nu_{i+\frac{3}{2}} < -1 \\ 0 & \text{otherwise} \end{cases} \quad (3.32)$$

The scheme (3.8), (3.29) — (3.32) reduces to either the Lax-Wendroff or the Warming-Beam scheme on the linear advection equation and is therefore second order accurate in space and time for that equation. It is stable provided $\nu_{max} \leq 2$.

A First Order Version of the Scheme

The unified extension of the Warming Beam and Lax-Wendroff schemes presented above is second order accurate and will suffer from unphysical oscillations in regions of high gradient, for example in the vicinity of shock waves. To avoid this problem, a TVD version of the scheme is developed in the next section. This will involve resorting to a first-order scheme near the high gradients. In this section, such a first order scheme is derived.

A first order scheme can be found from (3.19) by keeping x_1, x_2, t_1 , and t_2 as specified in (3.18), but performing exact integration in time and the midpoint rule in space. The resulting intercell flux can be written as:

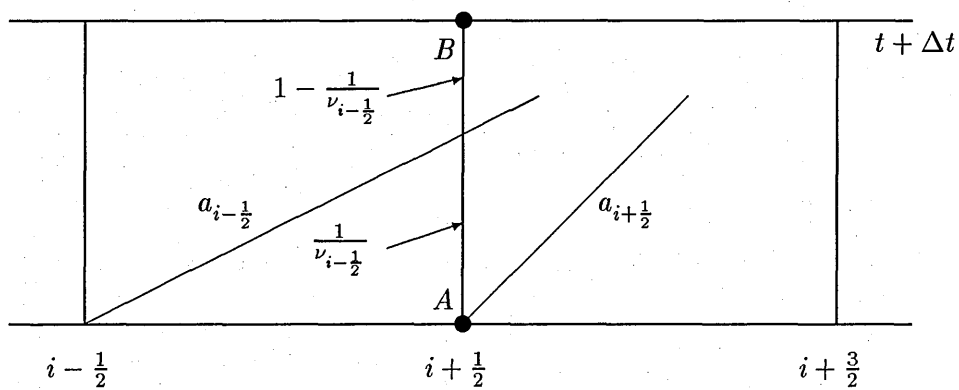
$$f_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_0^{\Delta t} f(\tilde{u}(0, t)) dt \quad (3.33)$$

where $\tilde{u}$ is again the combined solution of the three Riemann problems at interfaces $i - \frac{1}{2}$, $i + \frac{1}{2}$ and $i + \frac{3}{2}$. Note in this case the flux is found by integrating the solution of the Riemann problems along the t -axis at the interface $i + \frac{1}{2}$. Again it is assumed that the stability condition (3.16) is satisfied.

There are several possible cases. Two cases are considered here:

- (i) $1 \leq \nu_{i-\frac{1}{2}} \leq 2, 0 \leq \nu_{i+\frac{1}{2}} \leq 2, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$
- (ii) $-2 \leq \nu_{i-\frac{1}{2}} \leq 1, -1 \leq \nu_{i+\frac{1}{2}} \leq 1, -1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Case (i)



Case (ii)

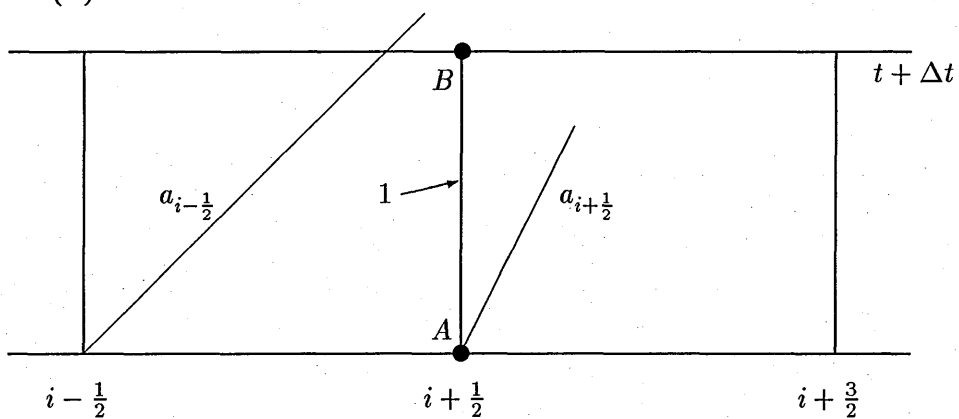


Figure 3.2: The integration range and wave structure for the first order scheme.

These are illustrated in Fig. 3.2.

Case (i): $1 \leq \nu_{i-\frac{1}{2}} \leq 2$, $0 \leq \nu_{i+\frac{1}{2}} \leq 2$, $-1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Consider Fig. 3.2(i). In this case, the wave from boundary $i - \frac{1}{2}$ intersects the integration range AB, and the waves from the other two Riemann problems do not. Evaluation of the integral (3.19) gives:

$$f_{i+\frac{1}{2}} = \frac{1}{\nu_{i-\frac{1}{2}}} f_i + \left(1 - \frac{1}{\nu_{i-\frac{1}{2}}}\right) f_{i-1} \quad (3.34)$$

This can be written as

$$f_{i+\frac{1}{2}} = f_i - \frac{1}{\nu_{i-\frac{1}{2}}} (\nu_{i-\frac{1}{2}} - 1) (f_i - f_{i-1}) \quad (3.35)$$

which is equation (3.21) with

$$\psi_{i+\frac{1}{2},L} = \frac{2}{\nu_{i-\frac{1}{2}}} (\nu_{i-\frac{1}{2}} - 1) \quad (3.36)$$

Case (ii): $-2 \leq \nu_{i-\frac{1}{2}} \leq 1$, $-1 \leq \nu_{i+\frac{1}{2}} \leq 1$, $-1 \leq \nu_{i+\frac{3}{2}} \leq 2$

Consider Fig. 3.2(ii). In this case, no waves intersect the integration range AB. Evaluation of the integral (3.19) gives:

$$\begin{aligned} f_{i+\frac{1}{2}} &= f_i \quad \text{if } \nu_{i+\frac{1}{2}} > 0 \\ f_{i+\frac{1}{2}} &= f_{i+1} \quad \text{if } \nu_{i+\frac{1}{2}} < 0 \end{aligned} \quad (3.37)$$

which is Godunov's intercell flux. Note this can be written in the form (3.24) with

$$\phi_{i+\frac{1}{2}} = \text{sgn}(\nu_{i+\frac{1}{2}}) \quad (3.38)$$

The two cases discussed above, and all other possible cases, can be summarized as the intercell flux (3.29) with:

$$\psi_{i+\frac{1}{2},L} = \begin{cases} \frac{1}{|\nu_{i-\frac{1}{2}}|} (|\nu_{i-\frac{1}{2}}| - 1) & \text{if } \nu_{i-\frac{1}{2}} > 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.39)$$

$$\phi_{i+\frac{1}{2}} = \text{sgn}(\nu_{i+\frac{1}{2}}) \quad (3.40)$$

$$\psi_{i+\frac{1}{2},R} = \begin{cases} -\frac{1}{|\nu_{i+\frac{3}{2}}|}(|\nu_{i+\frac{3}{2}}| - 1) & \text{if } \nu_{i+\frac{3}{2}} < -1 \\ 0 & \text{otherwise} \end{cases} \quad (3.41)$$

By substituting (3.27), (3.39) — (3.41) into the conservation form (3.8) and rearranging, it can be seen that the coefficients of the scheme are always positive when (3.16) holds. Thus the scheme is monotonic under this condition and by Theorem 6 it is also stable, at least in the linear case.

3.4 TVD Limiter Functions for the WAF-D2 Scheme

In this section, a TVD version of the WAF-D2 scheme presented in §3.3 is developed. The TVD analysis is performed by applying the method to the equation

$$u_t + f(u)_x = 0 \quad (3.42)$$

In the TVD analysis in [63] the flux function $f(u)$ was assumed linear, i.e $f(u) = au$ for some constant wavespeed a . The analysis performed here will apply to any flux function, linear or nonlinear.

3.4.1 TVD Limiter Functions for the Lax-Wendroff Component

Consider the WAF intercell flux introduced in Chapter 2, written in the form:

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}\phi_{i+\frac{1}{2}}(f_{i+1} - f_i) \quad (3.43)$$

It is used in this form in the WAF-D2 scheme, when $-1 \leq \nu_{i+\frac{1}{2}} \leq 1$. Assume, therefore, the timestep Δt is chosen such that $\nu_{max} \leq 1$ so that waves from other intercell boundaries do not reach the local integration range. Differencing the flux (3.43) across the cell i gives

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = \frac{1}{2}(\phi_{i+\frac{1}{2}} - 1)(f_{i+1} - f_i) + \frac{1}{2}(\phi_{i-\frac{1}{2}} + 1)(f_i - f_{i-1}) \quad (3.44)$$

To simplify the analysis, the speed of each of the waves is approximated via the Rankine-Hugoniot condition here, i.e

$$\lambda_{i+\frac{1}{2}} = \frac{f_{i+1} - f_i}{u_{i+1} - u_i} \quad (3.45)$$

Note that this is only an approximation *within the TVD analysis*. When the scheme is implemented, the speed of the single ray representing a rarefaction wave should still be taken as the true speed of the head of the rarefaction. Thus in the case of rarefaction waves, (3.45) is only approximately equal to the speed actually used, so the following TVD analysis only holds in some approximate sense. However, (3.45) is exactly what is used for discontinuous waves so the TVD analysis will hold in this case. Since it is discontinuous

waves that suffer most from spurious oscillations, the TVD analysis is correct where it is most needed. Using (3.45) in (3.44) gives:

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = -\frac{1}{2}\lambda_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)(u_{i+1} - u_i) + \frac{1}{2}\lambda_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1)(u_i - u_{i-1}) \quad (3.46)$$

Substituting (3.46) into the conservative formula (3.8) gives the scheme in full as:

$$u_i^{n+1} = u_i^n + \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)(u_{i+1} - u_i) - \frac{1}{2}\nu_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1)(u_i - u_{i-1}) \quad (3.47)$$

There are three cases to consider:

- (i) All wavespeeds positive
- (ii) All wavespeeds negative
- (iii) $\nu_{i+\frac{1}{2}} > 0$, $\nu_{i-\frac{1}{2}} < 0$

Case (i): All wavespeeds positive

In this case, since all wavespeeds are positive, limits for the function $\phi_{i+\frac{1}{2}}$ that give a TVD scheme are determined by applying the condition (2.57), i.e:

$$0 \leq C_i = \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} \leq 1 \quad (3.48)$$

Rearranging (3.47) gives:

$$C_i = \frac{1}{2}\nu_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1) - \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)\frac{1}{r_{i+\frac{1}{2}}} \quad (3.49)$$

where

$$r_{i+\frac{1}{2}} = \frac{\Delta_{upw}u}{\Delta_{loc}u} = \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n} \quad (3.50)$$

as before. Set general lower and upper limits L and U on the limiter functions ϕ so that $L \leq \phi_{i-\frac{1}{2}} \leq U$. These limits will be specified later. This gives the minimum value of C_i as

$$C_{min} = \frac{1}{2}\nu_{i-\frac{1}{2}}(L + 1) - \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)\frac{1}{r_{i+\frac{1}{2}}} \quad (3.51)$$

and the maximum value of C_i as

$$C_{max} = \frac{1}{2}\nu_{i-\frac{1}{2}}(U + 1) - \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)\frac{1}{r_{i+\frac{1}{2}}} \quad (3.52)$$

The scheme will be tvd provided $C_{min} \geq 0$ and $C_{max} \leq 1$. Since $\nu_{i-\frac{1}{2}} \geq 0$, $C_{min} \geq 0$ if and only if

$$\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq \frac{1}{r_{i+\frac{1}{2}}} + \frac{\nu_{i-\frac{1}{2}}}{\nu_{i+\frac{1}{2}}}(L+1) \quad (3.53)$$

Define a *compression parameter* as

$$\alpha_{i+\frac{1}{2}} = \frac{\nu_{upw}}{\nu_{loc}} \quad (3.54)$$

where ν_{upw} is the Courant number of the wave at the *upstream* boundary and ν_{loc} is the Courant number of the wave at the *local* boundary. In the present case, since $\nu_{i+\frac{1}{2}} \geq 0$,

$$\alpha_{i+\frac{1}{2}} = \frac{\nu_{i-\frac{1}{2}}}{\nu_{i+\frac{1}{2}}} \quad (3.55)$$

This parameter is new: it did not occur in the tvd region given in Chapter 2. The analysis there was based on the constant wavespeed case, so α was constant and equal to unity throughout the domain. Note that $\alpha_{i+\frac{1}{2}} \geq 1$ in a compression wave and $\alpha_{i+\frac{1}{2}} \leq 1$ in an rarefaction wave. Equation (3.53) becomes

$$\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq \frac{1}{r_{i+\frac{1}{2}}} + \alpha_{i+\frac{1}{2}}(L+1) \quad (3.56)$$

and $C_{min} \geq 0$ provided

$$\begin{aligned} \phi_{i+\frac{1}{2}} &\leq 1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} & \text{if } r_{i+\frac{1}{2}} > 0 \\ \phi_{i+\frac{1}{2}} &\geq 1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} & \text{if } r_{i+\frac{1}{2}} < 0 \end{aligned} \quad (3.57)$$

Since $\nu_{i+\frac{1}{2}} \geq 0$, $C_{max} \leq 1$ if and only if

$$\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \geq \frac{1}{r_{i+\frac{1}{2}}} + \alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{\nu_{i+\frac{1}{2}}} \quad (3.58)$$

which is satisfied provided

$$\begin{aligned} \phi_{i+\frac{1}{2}} &\geq 1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{\nu_{i+\frac{1}{2}}} \right] r_{i+\frac{1}{2}} & \text{if } r_{i+\frac{1}{2}} > 0 \\ \phi_{i+\frac{1}{2}} &\leq 1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{\nu_{i+\frac{1}{2}}} \right] r_{i+\frac{1}{2}} & \text{if } r_{i+\frac{1}{2}} < 0 \end{aligned} \quad (3.59)$$

Case (i) can now be summarized as follows. If

$$L \leq \phi_{i-\frac{1}{2}} \leq U \quad (3.60)$$

the scheme is TVD provided

$$1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{\nu_{i+\frac{1}{2}}} \right] r_{i+\frac{1}{2}} \leq \phi_{i+\frac{1}{2}} \leq 1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} \quad (3.61)$$

when $r_{i+\frac{1}{2}} > 0$ and

$$1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} \leq \phi_{i+\frac{1}{2}} \leq 1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{\nu_{i+\frac{1}{2}}} \right] r_{i+\frac{1}{2}} \quad (3.62)$$

when $r_{i+\frac{1}{2}} < 0$.

Case (ii): All wavespeeds negative

In this case, the analysis is performed on $\phi_{i-\frac{1}{2}}$. The TVD condition in this case is taken as (2.59), i.e:

$$0 \leq D_i = \frac{u_i^{n+1} - u_i^n}{u_{i+1}^n - u_i^n} \leq 1 \quad (3.63)$$

From (3.47):

$$D_i = \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1) - \frac{1}{2}\nu_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1) \frac{1}{r_{i-\frac{1}{2}}} \quad (3.64)$$

where

$$r_{i-\frac{1}{2}} = \frac{u_{i+1}^n - u_i^n}{u_i^n - u_{i-1}^n} \quad (3.65)$$

Assuming $L \leq \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq U$ the analysis follows in a similar fashion to Case (i).

Summary of cases (i) and (ii)

Cases (i) and (ii) can now be summarized in a single TVD region. Provided

$$L \leq \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq U \quad (3.66)$$

the scheme is TVD if $r_{i+\frac{1}{2}} > 0$ when:

$$1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{|\nu_{i+\frac{1}{2}}|} \right] r_{i+\frac{1}{2}} \leq \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq 1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} \quad (3.67)$$

and if $r_{i+\frac{1}{2}} < 0$ when

$$1 + \alpha_{i+\frac{1}{2}}(L+1)r_{i+\frac{1}{2}} \leq \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq 1 + \left[\alpha_{i+\frac{1}{2}}(U+1) - \frac{2}{|\nu_{i+\frac{1}{2}}|} \right] r_{i+\frac{1}{2}} \quad (3.68)$$

where

$$r_{i+\frac{1}{2}} = \frac{\Delta_{upw} u}{\Delta_{loc} u} = \begin{cases} \frac{\Delta u_{i-\frac{1}{2}}}{\Delta u_{i+\frac{1}{2}}} & \text{if } \nu_{i+\frac{1}{2}} > 0 \\ \frac{\Delta u_{i+\frac{3}{2}}}{\Delta u_{i+\frac{1}{2}}} & \text{if } \nu_{i+\frac{1}{2}} < 0 \end{cases} \quad (3.69)$$

and

$$\alpha_{i+\frac{1}{2}} = \frac{\nu_{upw}}{\nu_{loc}} = \begin{cases} \frac{\nu_{i-\frac{1}{2}}}{\nu_{i+\frac{1}{2}}} & \text{if } \nu_{i+\frac{1}{2}} > 0 \\ \frac{\nu_{i+\frac{3}{2}}}{\nu_{i+\frac{1}{2}}} & \text{if } \nu_{i+\frac{1}{2}} < 0 \end{cases} \quad (3.70)$$

Case (iii): $\nu_{i+\frac{1}{2}} > 0$, $\nu_{i-\frac{1}{2}} < 0$

The procedure for boundary $i + \frac{1}{2}$ would normally be to define

$$C_i = \frac{1}{2}\nu_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1) - \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1)\frac{1}{r_{i+\frac{1}{2}}} \quad (3.71)$$

as in (3.49), set arbitrary (but sensible) limits L and U on $\phi_{i-\frac{1}{2}}$ and find tvd regions for $\phi_{i+\frac{1}{2}}$ by setting

$$0 \leq C_i \leq 1 \quad (3.72)$$

The procedure for boundary $i - \frac{1}{2}$ would normally be to define

$$D_i = \frac{1}{2}\nu_{i+\frac{1}{2}}(\phi_{i+\frac{1}{2}} - 1) - \frac{1}{2}\nu_{i-\frac{1}{2}}(\phi_{i-\frac{1}{2}} + 1)\frac{1}{r_{i-\frac{1}{2}}} \quad (3.73)$$

as in (3.64), set arbitrary limits on $\phi_{i+\frac{1}{2}}$ and find tvd regions for $\phi_{i-\frac{1}{2}}$ from

$$0 \leq D_i \leq 1 \quad (3.74)$$

In this case (3.69) gives $r_{i+\frac{1}{2}} = \frac{1}{r_{i-\frac{1}{2}}}$. Writing $r = r_{i+\frac{1}{2}}$, it is easily shown from (3.71), (3.73) that

$$C_i r + D_i = 0 \quad (3.75)$$

Suppose $r \geq 0$. Since the wavespeeds at neighbouring cell boundaries are of opposite sign, this would occur at a sonic point in the centre of a rarefaction. From (3.75), C_i and D_i must be of opposite sign, so (3.72) and (3.74) cannot be satisfied simultaneously unless $C_i = D_i = 0$. This can only be achieved by setting

$$\phi_{i+\frac{1}{2}} = 1 \quad \phi_{i-\frac{1}{2}} = -1 \quad (3.76)$$

i.e taking the Godunov flux at both $i + \frac{1}{2}$ and $i - \frac{1}{2}$. Since conditions (3.72) and (3.74) are sufficient, but not necessary, conditions for a tvd scheme, it might be possible to use

conditions (2.54), (2.55) in Harten's theorem (Theorem 8) to derive tvd regions away from Godunov's scheme for this situation.

If $r \leq 0$, (3.75) combined with (3.74) gives

$$0 \leq C_i \leq \frac{1}{|r|} = \frac{1}{|r_{i+\frac{1}{2}}|} \quad (3.77)$$

So if $|r_{i+\frac{1}{2}}| \leq 1$, this does not impose any extra restriction over (3.72) on $\phi_{i+\frac{1}{2}}$, so the usual tvd region can be used; however, if $r_{i+\frac{1}{2}} \leq -1$, (3.77) must be taken into account, and new regions derived. A similar restriction holds for $\phi_{i-\frac{1}{2}}$, when (3.77) is replaced by

$$0 \leq D_i \leq |r| = \frac{1}{|r_{i-\frac{1}{2}}|} \quad (3.78)$$

This must be considered in the derivation of regions for $\phi_{i-\frac{1}{2}}$ when $r_{i-\frac{1}{2}} \leq -1$.

The simplest possible solution to all these cases is to return to Godunov's scheme whenever $\alpha_{i+\frac{1}{2}} \leq 0$. This will occur very infrequently, so using the first order scheme at these points should not, in general, damage the global accuracy of a calculation.

Design of limiter functions

When $\alpha \geq 0$, limiter functions are needed within the region (3.66)—(3.70). In this section it is shown that limiters given in Chapter 2 can be translated into limiters in the new region. For convenience, write $\phi = \phi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}})$, $\nu = \nu_{i+\frac{1}{2}}$ and $\alpha = \alpha_{i+\frac{1}{2}}$.

Firstly, the upper and lower limits U and L need to be fixed. When $\phi \equiv \text{sgn}(\nu)$, the waf flux (3.43) reduces to the Godunov intercell flux. Since for second order accuracy $\phi \equiv |\nu|$ and $|\nu| \leq 1$ here, $U = 1$ is a suitable choice for the upper limit. The lower limit L must satisfy $L \leq |\nu|$ to allow second order accuracy when possible, but need not be fixed here.

With these limits, the tvd region can be written in two parts as:

$$Z_L = \{(r, \phi) \text{ such that } r \leq 0, \phi \geq \phi_L, L \leq \phi \leq 1\} \quad (3.79)$$

$$Z_R = \{(r, \phi) \text{ such that } r \geq 0, \phi \geq \phi_R, L \leq \phi \leq 1\} \quad (3.80)$$

where

$$\phi_L = 1 + \alpha(L + 1)r \quad (3.81)$$

and

$$\phi_R = 1 - \frac{2}{|\nu|} [1 - \alpha |\nu|] r \quad (3.82)$$

Note the similarity with regions (2.72)—(2.75). The new region (3.79)—(3.82) is larger when $\alpha < 1$, i.e in rarefaction waves, and smaller when $\alpha > 1$, i.e in compression waves.

When $\alpha = 1$, (3.82) reduces to (2.75). Limiters lying within the region (2.72)—(2.75) have been given in Chapter 2; all that is needed is a way to translate them into functions in the region (3.79)—(3.82). This can be done by considering the slope

$$S = -\frac{2}{|\nu|} [1 - |\nu|] \quad (3.83)$$

of ϕ_R in (2.75). In the new region this becomes

$$S^\alpha = -\frac{2}{|\nu|} [1 - \alpha |\nu|] \quad (3.84)$$

A *stretching parameter* P between the regions can be defined as:

$$P = \frac{S^\alpha}{S} = \frac{\alpha |\nu| - 1}{|\nu| - 1} \quad (3.85)$$

Thus a new limiter function $\phi^\alpha(r, \nu, \alpha)$ can be defined from a limiter function $\phi(r, \nu)$ as:

$$\phi^\alpha(r, \nu, \alpha) = \phi(Pr, \nu) \quad (3.86)$$

For example, the new limiter analogous to the **MINMOD** limiter (2.79) is:

$$\phi^{\alpha \text{min}}(r, \nu, \alpha) = \begin{cases} 1 & \text{if } r \leq 0 \\ 1 + (\alpha |\nu| - 1)r & \text{if } 0 < r \leq \frac{1}{P} \\ |\nu| & \text{otherwise} \end{cases} \quad (3.87)$$

Other limiters, such as the **VAN LEER** limiter (2.80) and **SUPERBEE** limiter (2.81) can be translated in a similar fashion. Note that in the original **MINMOD** limiter (2.79), second order accuracy was always achieved when $r \approx 1$; however when $P \leq 1$, i.e. $\alpha \geq 1$ (a compression wave) this is not always the case in (3.87).

3.4.2 TVD Limiter Functions for the Warming-Beam Component

In this section, **TVD** limiter functions are developed for the Warming-Beam component of the **WAF-D2** scheme. In the derivation, it is assumed that all wavespeeds are of the same sign.

Case (i): All wavespeeds are positive.

The **TVD** analysis will be performed on the conservative scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] \quad (3.88)$$

with

$$f_{i+\frac{1}{2}} = f_i - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_i - f_{i-1}) \quad (3.89)$$

The stability region for (3.88), (3.89) is assumed to be $\nu_{max} \leq 2$. Limiter functions are only developed for the case $1 \leq \nu_{max} \leq 2$ here, since the scheme developed in the last section uses (3.89) only under this restriction. When combined with the regions derived in §3.4.1 they will provide TVD regions for the scheme (3.8), (3.29)—(3.32) whenever $\nu_{max} \leq 2$.

The condition used to develop limiter functions in §3.4.1 for the WAF scheme were derived from Harten's theorem [22] (Theorem 8), as discussed in Chapter 2. The theorem concerns only three point *centred* schemes, such as the Lax-Wendroff scheme. To derive a TVD condition for three-point, *one-sided* schemes, such as (3.88), (3.89), a new theorem is needed.

Theorem 9 Suppose u is a mesh function that is either periodic or of compact support. Any scheme on u that can be written in the form

$$u_i^{n+1} = u_{i-1}^n - C_{i-\frac{3}{2}}(u_{i-1}^n - u_{i-2}^n) + D_{i-\frac{1}{2}}(u_i^n - u_{i-1}^n) \quad (3.90)$$

is TVD provided

$$C_{i+\frac{1}{2}} \geq 0 \quad D_{i+\frac{1}{2}} \geq 0 \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1 \quad \forall i \quad (3.91)$$

Proof. This proof is very similar to the proof of Harten's theorem. Subtract (3.90) at i from (3.90) at $i+1$ to get:

$$u_{i+1}^{n+1} - u_i^{n+1} = D_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n) + (1 - C_{i-\frac{1}{2}} - D_{i-\frac{1}{2}})(u_i^n - u_{i-1}^n) + C_{i-\frac{3}{2}}(u_{i-1}^n - u_{i-2}^n) \quad (3.92)$$

Conditions (3.91) are exactly the conditions needed for the coefficients of jumps $u_{i+k+1}^n - u_{i+k}^n$ in (3.92) to be non-negative. Taking the absolute value of (3.93) and using the triangle inequality gives:

$$\begin{aligned} |u_i^{n+1} - u_i^n| &\leq D_{i+\frac{1}{2}} |u_{i+1}^n - u_i^n| + (1 - C_{i-\frac{1}{2}} - D_{i-\frac{1}{2}}) |u_i^n - u_{i-1}^n| \\ &\quad + C_{i-\frac{3}{2}} |u_{i-1}^n - u_{i-2}^n| \end{aligned} \quad (3.93)$$

Recall that the Total Variation of a mesh function u at time level n is defined as

$$TV(u^n) = \sum_i |u_{i+1}^n - u_i^n| \quad (3.94)$$

Summing (3.93) over i , assuming u to be periodic or of compact support and using (3.91) gives

$$TV(u^{n+1}) \leq \sum_i D_{i+\frac{1}{2}} |u_{i+1}^n - u_i^n| + \sum_i (1 - C_{i-\frac{1}{2}} - D_{i-\frac{1}{2}}) |u_i^n - u_{i-1}^n|$$

$$\begin{aligned}
& + \sum_i C_{i-\frac{3}{2}} |u_{i-1}^n - u_{i-2}^n| \\
& = \sum_i |(u_{i+1}^n - u_i^n)| = TV(u^n)
\end{aligned} \tag{3.95}$$

This completes the proof of the theorem.

Following directly from Theorem 9 is the Corollary:

Corollary 3 *Any scheme written in the form*

$$u_i^{n+1} = u_{i-1}^n - C_i (u_{i-1}^n - u_{i-2}^n) \tag{3.96}$$

is TVD provided

$$0 \leq C_i = \frac{u_i^{n+1} - u_{i-1}^n}{u_{i-2}^n - u_{i-1}^n} \leq 1 \tag{3.97}$$

Proof. Set $D_{i-\frac{1}{2}} \equiv 0$ in Theorem 9, write $C_{i-\frac{3}{2}}$ as C_i and rearrange.

Note that condition (3.97) can be obtained from the data compatibility relation (2.44) by setting $K = 1$.

Condition (3.97) is the condition that can be used to derive a TVD version of the scheme (3.88), (3.89). From (3.89):

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = - \left(\frac{1}{2} \psi_{i+\frac{1}{2}} - 1 \right) (f_i - f_{i-1}) + \frac{1}{2} \psi_{i-\frac{1}{2}} (f_{i-1} - f_{i-2}) \tag{3.98}$$

As in §3.4.1, the analysis is simplified by approximating each wavespeed $\lambda_{i+\frac{1}{2}}$ as the speed given by the Rankine-Hugoniot condition, i.e set:

$$\lambda_{i+\frac{1}{2}} = \frac{f_{i+1} - f_i}{u_{i+1} - u_i} \quad \forall i \tag{3.99}$$

From (3.98) and (3.99):

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = -\lambda_{i-\frac{1}{2}} \left(\frac{1}{2} \psi_{i+\frac{1}{2}} - 1 \right) (u_i - u_{i-1}) + \frac{1}{2} \lambda_{i-\frac{3}{2}} \psi_{i-\frac{1}{2}} (u_{i-1} - u_{i-2}) \tag{3.100}$$

Substituting (3.100) into the conservative formula (3.8) gives the scheme in full as:

$$\begin{aligned}
u_i^{n+1} &= u_{i-1}^n + \left(1 - \nu_{i-\frac{1}{2}} + \frac{1}{2} \nu_{i-\frac{1}{2}} \psi_{i+\frac{1}{2}} \right) (u_i^n - u_{i-1}^n) \\
&\quad - \frac{1}{2} \nu_{i-\frac{3}{2}} \psi_{i-\frac{1}{2}} (u_{i-1}^n - u_{i-2}^n)
\end{aligned} \tag{3.101}$$

Therefore

$$C_i = \frac{1}{2}\nu_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}} - \left(1 - \nu_{i-\frac{1}{2}} + \frac{1}{2}\nu_{i-\frac{1}{2}}\psi_{i+\frac{1}{2}}\right) \frac{1}{r_{i-\frac{1}{2}}} \quad (3.102)$$

where the flow parameter $r_{i-\frac{1}{2}}$ is defined by (3.69), i.e as:

$$r_{i-\frac{1}{2}} = \frac{\Delta_{upw}u}{\Delta_{loc}u} = \frac{u_{i-1} - u_{i-2}}{u_i - u_{i-1}} \quad (3.103)$$

Set

$$L \leq \psi_{i-\frac{1}{2}} \leq U \quad (3.104)$$

Since $\nu_{i-\frac{3}{2}} > 0$, this gives the minimum value C_{min} of C_i as:

$$C_{min} = \frac{1}{2}\nu_{i-\frac{3}{2}}L - \left(1 - \nu_{i-\frac{1}{2}} + \frac{1}{2}\nu_{i-\frac{1}{2}}\psi_{i+\frac{1}{2}}\right) \frac{1}{r_{i-\frac{1}{2}}} \quad (3.105)$$

and the maximum value C_{max} as:

$$C_{max} = \frac{1}{2}\nu_{i-\frac{3}{2}}U - \left(1 - \nu_{i-\frac{1}{2}} + \frac{1}{2}\nu_{i-\frac{1}{2}}\psi_{i+\frac{1}{2}}\right) \frac{1}{r_{i-\frac{1}{2}}} \quad (3.106)$$

For the scheme to be tvd, it is necessary to have $C_{min} \geq 0$ and $C_{max} \leq 1$. Since $\nu_{i-\frac{1}{2}} > 0$, $C_{min} \geq 0$ if and only if

$$\frac{\psi_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}}} \leq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) \frac{1}{r_{i-\frac{1}{2}}} + \alpha_{i-\frac{1}{2}}L \quad (3.107)$$

where

$$\alpha_{i-\frac{1}{2}} = \frac{\nu_{upw}}{\nu_{loc}} = \frac{\nu_{i-\frac{3}{2}}}{\nu_{i-\frac{1}{2}}} \quad (3.108)$$

This is satisfied if and only if:

$$\begin{cases} \psi_{i+\frac{1}{2}} \leq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \alpha_{i-\frac{1}{2}}Lr_{i-\frac{1}{2}} & \text{if } r_{i-\frac{1}{2}} > 0 \\ \psi_{i+\frac{1}{2}} \geq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \alpha_{i-\frac{1}{2}}Lr_{i-\frac{1}{2}} & \text{if } r_{i-\frac{1}{2}} < 0 \end{cases} \quad (3.109)$$

Since $\nu_{i-\frac{1}{2}} > 0$, $C_{max} \leq 1$ if and only if

$$\frac{\psi_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}}} \geq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) \frac{1}{r_{i-\frac{1}{2}}} + \left(\alpha_{i-\frac{1}{2}}U - \frac{2}{\nu_{i-\frac{1}{2}}} \right) \quad (3.110)$$

This is satisfied if and only if:

$$\begin{cases} \psi_{i+\frac{1}{2}} \geq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \left(\alpha_{i-\frac{1}{2}}U - \frac{2}{\nu_{i-\frac{1}{2}}} \right) r_{i-\frac{1}{2}} & \text{if } r_{i-\frac{1}{2}} > 0 \\ \psi_{i+\frac{1}{2}} \leq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \left(\alpha_{i-\frac{1}{2}}U - \frac{2}{\nu_{i-\frac{1}{2}}} \right) r_{i-\frac{1}{2}} & \text{if } r_{i-\frac{1}{2}} < 0 \end{cases} \quad (3.111)$$

The **TV**D region for the positive wavespeed case can be summarized as follows. If $L \leq \psi_{i+\frac{1}{2}} \leq U$ then the scheme is **TV**D provided:

$$\frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \left(\alpha_{i-\frac{1}{2}}U - \frac{2}{\nu_{i-\frac{1}{2}}} \right) r_{i-\frac{1}{2}} \leq \psi_{i+\frac{1}{2}} \leq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \alpha_{i-\frac{1}{2}}Lr_{i-\frac{1}{2}} \quad (3.112)$$

if $r_{i-\frac{1}{2}} > 0$ and:

$$\frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \alpha_{i-\frac{1}{2}}Lr_{i-\frac{1}{2}} \leq \psi_{i+\frac{1}{2}} \leq \frac{2}{\nu_{i-\frac{1}{2}}}(\nu_{i-\frac{1}{2}} - 1) + \left(\alpha_{i-\frac{1}{2}}U - \frac{2}{\nu_{i-\frac{1}{2}}} \right) r_{i-\frac{1}{2}} \quad (3.113)$$

if $r_{i-\frac{1}{2}} < 0$.

Case (ii): All wavespeeds are negative.

In this case, the stencil for the Warming-Beam scheme consists of the three points i , $i+1$ and $i+2$. Theorem 9 is replaced by

Theorem 10 *Suppose u is a mesh function that is either periodic or of compact support. Any scheme on u that can be written in the form*

$$u_i^{n+1} = u_{i+1}^n - C_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n) + D_{i+\frac{3}{2}}(u_{i+2}^n - u_{i+1}^n) \quad (3.114)$$

*is **TV**D provided*

$$C_{i+\frac{1}{2}} \geq 0 \quad D_{i+\frac{1}{2}} \geq 0 \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1 \quad \forall i \quad (3.115)$$

Proof. This proof is very similar to the proof of Theorem 9. Subtract (3.114) at $i-1$ from (3.114) at i and continue as before.

Following directly from Theorem 10 is the Corollary:

Corollary 4 *Any scheme written in the form*

$$u_i^{n+1} = u_{i+1}^n + D_i(u_{i+2}^n - u_{i+1}^n) \quad (3.116)$$

*is **TV**D provided*

$$0 \leq D_i = \frac{u_i^{n+1} - u_{i+1}^n}{u_{i+2}^n - u_{i+1}^n} \leq 1 \quad (3.117)$$

Proof. Set $C_{i+\frac{1}{2}} \equiv 0$ in Theorem 10, write $D_{i+\frac{3}{2}}$ as D_i and rearrange.

Note that condition (3.117) can be derived from the data compatibility relations (2.44) by setting $K = -2$. Condition (3.117) is the condition that can be used to derive a **TVB** version of the Warming-Beam scheme when the wavespeeds are negative. In this case the intercell flux is:

$$f_{i+\frac{1}{2}} = f_{i+1} - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_{i+2} - f_{i+1}) \quad (3.118)$$

Set $L \leq \psi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq U$. **TVB** regions for $\psi_{i-\frac{1}{2}}$ then follow in a similar fashion to the derivation in Case (i).

Summary

The **TVB** regions for both of the above cases can be summarized as follows. A general **TVB** region for the limiter $\psi = \psi(\nu, r, \alpha)$ can be defined as follows. Set

$$L \leq \psi \leq U \quad (3.119)$$

and take

$$\frac{2}{|\nu|}(|\nu| - 1) + \left(\alpha U - \frac{2}{|\nu|}\right)r \leq \psi \leq \frac{2}{|\nu|}(|\nu| - 1) + \alpha Lr \quad (3.120)$$

if $r > 0$ and:

$$\frac{2}{\nu}(|\nu| - 1) + \alpha Lr \leq \psi \leq \frac{2}{|\nu|}(|\nu| - 1) + \left(\alpha U - \frac{2}{|\nu|}\right)r \quad (3.121)$$

if $r < 0$. If $\nu_{i+\frac{1}{2}} > 0$ take

$$\psi_{i+\frac{1}{2}} = \psi(\nu_{i-\frac{1}{2}}, r_{i-\frac{1}{2}}, \alpha_{i-\frac{1}{2}}) \quad (3.122)$$

and if $\nu_{i+\frac{1}{2}} < 0$ take

$$\psi_{i+\frac{1}{2}} = -\psi(\nu_{i+\frac{3}{2}}, r_{i+\frac{3}{2}}, \alpha_{i+\frac{3}{2}}) \quad (3.123)$$

Design of limiter functions

Limiter functions will now be designed within the region (3.119), (3.120), (3.121). The first step is to set the upper and lower limits U and L . A suitable upper limit is the constant limiter for the first order scheme, i.e set $U = \frac{2}{|\nu|}(|\nu| - 1)$. The lower limit is arbitrary, provided $L \leq |\nu| - 1$ so that second order accuracy is allowed under some circumstances. The choice $L = 0$ is taken here. The **TVB** region (3.119), (3.120), (3.121) becomes

$$0 \leq \psi \leq \frac{2}{|\nu|}(|\nu| - 1) \quad (3.124)$$

$$\frac{2}{|\nu|}(|\nu| - 1) + \frac{2}{|\nu|}(\alpha(|\nu| - 1) - 1)r \leq \psi \leq \frac{2}{|\nu|}(|\nu| - 1) \quad (3.125)$$

if $r > 0$ and:

$$\psi = \frac{2}{|\nu|}(|\nu| - 1) \quad (3.126)$$

if $r < 0$. See Fig. 3.3. Note that the choice of L and U combined with (3.121) force the scheme to first order when $r \leq 0$.

The strategy for designing limiter functions in the region is to first set $\alpha \equiv 1$ in (3.124), (3.125), (3.126) and design limiters in the resulting region. These can be translated into functions within the region (3.124), (3.125), (3.126) by comparing slopes of the left-hand side boundary for $r \geq 0$, as in the case of the limiters for the Lax-Wendroff component. In this case the stretching parameter is:

$$Q = \frac{\alpha(|\nu| - 1) - 1}{|\nu| - 2} \quad (3.127)$$

As an example, a limiter function analogous to the MINMOD-type limiter (2.79) is

$$\psi^{\alpha\min}(r, \nu, \alpha) = \begin{cases} \frac{2}{|\nu|}(|\nu| - 1) & \text{if } r \leq 0 \\ \frac{2}{|\nu|}(|\nu| - 1) + \frac{1}{|\nu|}(|\nu| - 1)(\alpha(|\nu| - 1) - 1)r & \text{if } 0 < r \leq \frac{1}{Q} \\ |\nu| - 1 & \text{otherwise} \end{cases} \quad (3.128)$$

3.5 Extension to Non-Linear Systems

The extension of the WAF-D2 scheme to nonlinear systems is very simple. The situation is illustrated in Fig. 3.4.

To obtain the intercell flux $F_{i+\frac{1}{2}}$, first integrate the solution of the Riemann problem at boundary $i + \frac{1}{2}$, taking into account the fact that some of the waves may now have Courant numbers greater than one and therefore will not intersect the integration domain. To stabilize the scheme, extra terms for waves from the Riemann problems at $i - \frac{1}{2}$ and $i + \frac{3}{2}$ must then be added into the flux. As in the case of the WAF scheme, rarefaction waves that are not sonic are represented as single ray waves, with speed taken as the speed of the head of the rarefaction. The same procedure is also taken for sonic rarefactions. Linear superposition of waves is also assumed.

The integration of the solution of the central Riemann problem at $i + \frac{1}{2}$ at time $t = \frac{1}{2}\Delta t$ results in the flux

$$F_{i+\frac{1}{2}}^C = \sum_{k=1}^{N+1} W^{(k)} F_{i+\frac{1}{2}}^{(k)} \quad (3.129)$$

where

$$W^{(k)} = \frac{1}{2}(\phi_{i+\frac{1}{2}}^{(k)} - \phi_{i+\frac{1}{2}}^{(k-1)}) \quad (3.130)$$

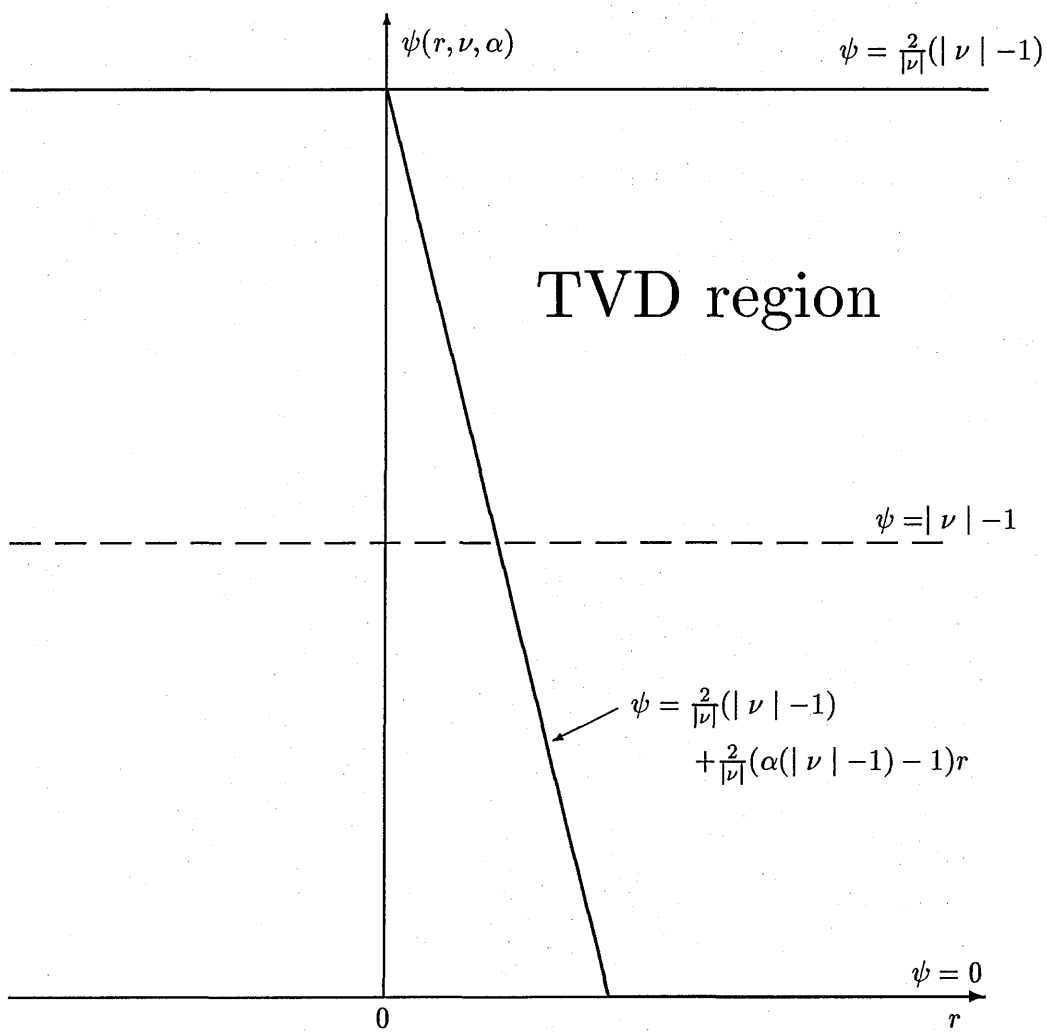


Figure 3.3: TVD region for the Warming Beam scheme when $1 \leq \nu \leq 2$.

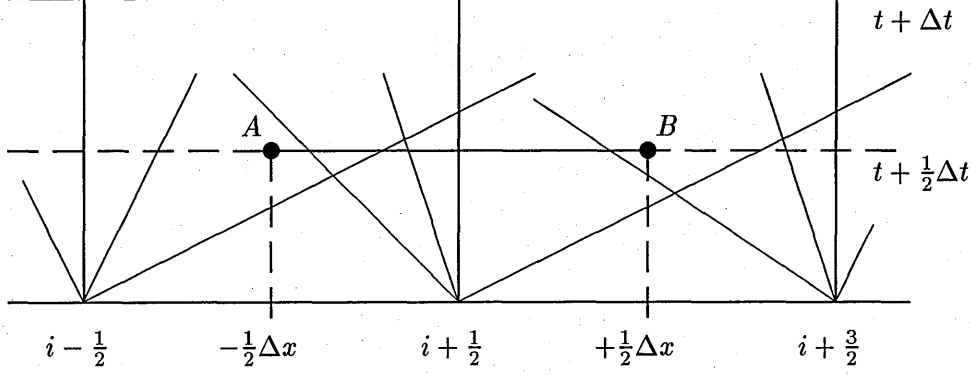


Figure 3.4: The integration for the wAF-D2 scheme on systems of equations.

with

$$\phi_{i+\frac{1}{2}}^{(k)} = \text{sgn}(\nu_{i+\frac{1}{2}}^{(k)}) \min \left\{ |\nu_{i+\frac{1}{2}}^{(k)}|, 1 \right\} \quad (3.131)$$

$$\phi_{i+\frac{1}{2}}^{(0)} = -1 \quad \phi_{i+\frac{1}{2}}^{(N+1)} = 1 \quad (3.132)$$

Note that the flux (3.129) can be written as

$$\mathbf{F}_{i+\frac{1}{2}}^C = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2} \sum_{k=1}^N \phi_{i+\frac{1}{2}}^{(k)} \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (3.133)$$

where $\Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} = \mathbf{F}_{i+\frac{1}{2}}^{(k+1)} - \mathbf{F}_{i+\frac{1}{2}}^{(k)}$.

Now consider the contribution from the Riemann problem at $i - \frac{1}{2}$. From the case of the scalar equation (3.15) in §3.3, it can be seen by generalizing (3.21), (3.22) that each wave k that intersects AB contributes

$$-\frac{1}{2}(|\nu_{i-\frac{1}{2}}^{(k)}| - 1) \Delta \mathbf{F}_{i-\frac{1}{2}}^{(k)} \quad (3.134)$$

The waves that do not intersect AB make no contribution. Thus the total contribution from the left is:

$$\mathbf{F}_{i+\frac{1}{2}}^L = -\frac{1}{2} \sum_{k=1}^N \psi_{i+\frac{1}{2},L}^{(k)} \Delta \mathbf{F}_{i-\frac{1}{2}}^{(k)} \quad (3.135)$$

with

$$\psi_{i+\frac{1}{2},L}^{(k)} = \begin{cases} |\nu_{i-\frac{1}{2}}^{(k)}| - 1 & \text{if } \nu_{i-\frac{1}{2}}^{(k)} > 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.136)$$

Now consider the contribution from the Riemann problem at $i + \frac{3}{2}$. From the case of the scalar equation (3.15) in §3.3, it can be seen that each wave k that intersects AB contributes

$$\frac{1}{2}(|\nu_{i+\frac{3}{2}}^{(k)}| - 1)\Delta\mathbf{F}_{i+\frac{3}{2}}^{(k)} \quad (3.137)$$

The waves that do not intersect AB make no contribution. Thus the total contribution from the right is:

$$\mathbf{F}_{i+\frac{1}{2}}^R = -\frac{1}{2} \sum_{k=1}^N \psi_{i+\frac{1}{2},R}^{(k)} \Delta\mathbf{F}_{i+\frac{3}{2}}^{(k)} \quad (3.138)$$

with

$$\psi_{i+\frac{1}{2},R}^{(k)} = \begin{cases} -(|\nu_{i+\frac{3}{2}}^{(k)}| - 1) & \text{if } \nu_{i+\frac{3}{2}}^{(k)} < -1 \\ 0 & \text{otherwise} \end{cases} \quad (3.139)$$

The intercell flux is now completely determined, as:

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_{i+\frac{1}{2}}^C + \mathbf{F}_{i+\frac{1}{2}}^L + \mathbf{F}_{i+\frac{1}{2}}^R \quad (3.140)$$

which can be written in full as:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2} \sum_{k=1}^N \psi_{i+\frac{1}{2},L}^{(k)} \Delta\mathbf{F}_{i-\frac{1}{2}}^{(k)} - \frac{1}{2} \sum_{k=1}^N \phi_{i+\frac{1}{2}}^{(k)} \Delta\mathbf{F}_{i+\frac{1}{2}}^{(k)} - \frac{1}{2} \sum_{k=1}^N \psi_{i+\frac{1}{2},R}^{(k)} \Delta\mathbf{F}_{i+\frac{3}{2}}^{(k)} \quad (3.141)$$

where $\psi_{i+\frac{1}{2},L}^{(k)}$, $\phi_{i+\frac{1}{2}}^{(k)}$ and $\psi_{i+\frac{1}{2},R}^{(k)}$ are defined as in (3.131), (3.136) and (3.139). This is our final intercell flux. On the linear advection equation, it reduces to the Lax-Wendroff flux for $|\nu| \leq 1$, and the Warming-Beam flux for $|\nu| \in [1, 2]$, and thus has an overall linear stability condition of $|\nu| \leq 2$. On non-linear systems, it provides a scheme that automatically switches between Lax-Wendroff and Warming-Beam components as the wave structures dictate. For a tvd scheme take

$$\psi_{i+\frac{1}{2},L}^{(k)} = \begin{cases} \psi(r_{i-\frac{1}{2}}^{(k)}, \nu_{i-\frac{1}{2}}^{(k)}, \alpha_{i-\frac{1}{2}}^{(k)}) & \text{if } \nu_{i-\frac{1}{2}}^{(k)} > 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.142)$$

$$\phi_{i+\frac{1}{2}}^{(k)} = \begin{cases} \phi(r_{i+\frac{1}{2}}^{(k)}, \nu_{i+\frac{1}{2}}^{(k)}, \alpha_{i+\frac{1}{2}}^{(k)}) \cdot \text{sgn}(\nu_{i+\frac{1}{2}}^{(k)}) & \text{if } |\nu_{i+\frac{1}{2}}^{(k)}| < 1 \\ \text{sgn}(\nu_{i+\frac{1}{2}}^{(k)}) & \text{otherwise} \end{cases} \quad (3.143)$$

$$\psi_{i+\frac{1}{2},R}^{(k)} = \begin{cases} -\psi(r_{i+\frac{3}{2}}^{(k)}, \nu_{i+\frac{3}{2}}^{(k)}, \alpha_{i+\frac{3}{2}}^{(k)}) & \text{if } \nu_{i+\frac{3}{2}}^{(k)} < -1 \\ 0 & \text{otherwise} \end{cases} \quad (3.144)$$

where ψ and ϕ are tvd limiter functions introduced in §3.4. All that is left is to find solutions (approximate or exact) to the Riemann problems at intercell boundaries. These

can be provided by any of the well-known Riemann solvers, such as the exact solvers of Gottlieb and Groth [20], Toro [61] or Pike [40], or approximate solvers such as those of Roe [43], Harten, Lax and van Leer [21] or Toro [64].

3.6 Numerical Experiments

The **WAF-D2** scheme has been tested under various conditions, and selected numerical results are now presented. The **MINMOD**-type limiter functions (3.87),(3.128) are used throughout.

To confirm both the derivation of the scheme in §3.3 and the **TVD** analysis in §3.4, results are first presented for the one dimensional inviscid Burgers equation

$$u_t + f(u)_x = 0 \quad f(u) = \frac{1}{2}u^2 \quad (3.145)$$

The computational domain is taken as the unit interval $x \in [0, 1]$ divided into 100 cells. The initial conditions are taken as

$$\begin{cases} u = 1.0 & \text{if } x \leq 0.2 \\ u = 0.0 & \text{otherwise} \end{cases} \quad (3.146)$$

and results are taken at time $t = 1.0$. The boundary conditions used are transmissive. Results are shown in Fig. 3.5 when $\nu_{max} = 0.8, 1.2, 1.6, 2.0$. Symbols represent the numerical solution, and the solid line represents the exact solution to the problem. Note that when $\nu_{max} = 2.0$ the scheme reproduces the exact solution.

Results are now presented for the Euler equations of Gas Dynamics. In all calculations, Riemann solver adaption of Toro [64] has been used, between the exact and linearized solvers discussed in that paper. Consider first the one dimensional Euler equations:

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}_x = 0 \quad (3.147)$$

where ρ is the density, u is the velocity, p is the pressure, $E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$ is the total energy and γ is the ratio of specific heats, taken as $\gamma = 1.4$ here. The computational domain used is the interval $x \in [0, 1]$ divided into 100 cells. Initial conditions consist of two constant states, left (L) and right (R):

$$\begin{aligned} \rho_L &= 1.0 & \rho_R &= 0.125 \\ u_L &= 0.0 & u_R &= 0.0 \\ p_L &= 1.0 & p_R &= 0.1 \end{aligned} \quad (3.148)$$

separated by a discontinuity at $x = 0.5$, and transmissive boundary conditions are used. This problem was first introduced by Sod [49]. Output is taken at time $t = 0.5$. Results taken with $\nu_{max} = 0.8, 1.2, 1.6, 2.0$ are shown in Figures 3.6, 3.7, 3.8 and 3.9 respectively. As before, symbols represent numerical solutions, and solid lines represent the exact solution. Note the appearance of oscillations behind the shock wave when $\nu_{max} \geq 1$. They are particularly noticeable for the internal energy when ν_{max} is just higher than one (see Fig. 3.7). For higher values of ν_{max} they are of the order of 1% of shock height in amplitude. Fig 3.10 shows the case $\nu_{max} = 1.2$ when *the first order scheme is used everywhere*. The shock still suffers from oscillatory behaviour, and therefore the oscillations are not necessarily due to a fault in the mechanism for keeping the scheme TVD. The oscillations are probably due to the assumption of linear interaction of waves, which will be inaccurate for shock waves.

The second test for the scheme on the one dimensional Euler equations is the blast wave problem of Woodard and Colella [82]. The computational domain is again taken as the interval $x \in [0, 1]$, now divided into 3000 cells. The initial conditions for this problem consist of three constant states left (L), middle (M) and right (R):

$$\begin{aligned} \rho_L &= 1.0 & \rho_M &= 1.0 & \rho_R &= 1.0 \\ u_L &= 0.0 & u_M &= 0.0 & u_R &= 0.0 \\ p_L &= 10^3 & p_M &= 10^{-2} & p_R &= 10^2 \end{aligned} \quad (3.149)$$

separated by discontinuities at $x = 0.1$ and $x = 0.9$. Reflective boundary conditions are used. There is no known exact solution to this problem; for reference solutions see the Woodard and Colella review paper [82]. Numerical results at time $t = 0.028$ for **WAF** and **WAF-D2** are shown in Figures 3.10 and 3.11 respectively. The equivalent results at time $t = 0.038$ are shown in Figures 3.12 and 3.13.

To illustrate the usefulness of the scheme in more than one space dimension, results are now presented for the two dimensional Euler equations:

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{bmatrix}_x + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}_y = 0 \quad (3.150)$$

where u and v are the x and y components of velocity, and the total energy is now given by $E = \frac{1}{2}\rho(u^2 + v^2) + \frac{p}{\gamma-1}$. The one dimensional scheme presented in this chapter is extended to solve these equations via first order operator splitting as described in Chapter 2, so the resulting two dimensional scheme is first order accurate in time and second order accurate in space.

Results of the reflection of shock waves over inclined ramps are presented. Such problems are classified by two parameters: the angle between the ramp and the direction of

propagation of the incident shock and the incident shock Mach number

$$M_S = \frac{S}{a_R} \quad (3.151)$$

where S is the speed of the shock and a_R is the sound speed ahead of the shock. Two main types of reflection are possible: *Regular Reflection* (RR) and *Mach Reflection* (MR). In Regular Reflection, two shock waves exist in the solution, the incident shock and a reflected shock, which meet at some point on the ramp. In Mach Reflection, the incident shock and a reflected shock meet at some point in the interior of the flow away from the boundary. This point is called the *Triple Point*, and is 'connected' to the ramp by a third shock wave called the *Mach Stem*. A slip line extends from the Triple Point into the region of smooth flow between the Mach Stem and reflected shock. Various types of Mach Reflection are possible. If the reflection is exactly as discussed above, it is called *Single Mach Reflection* (SMR); *Complex Mach Reflection* (CMR) is the same as SMR, except with a kink in the reflected shock; *Double Mach Reflection* (DMR) is the same as CMR, except a second Mach Stem appears between the kink in the reflected shock and the slip line. A full discussion of the theory behind shock reflection can be found in the book by Ben-Dor [3]. A detailed study of these problems can also be found in the report by Glaz *et. al.* [15].

Two examples of Mach Reflection will be considered here. In both cases, the ramp angle will be 25 degrees and a grid of 420×280 cells contained within a box of dimensions 0.7×0.4 is used, with the ramp starting at $x = 0.25$. A course version of this grid is shown in Fig. 3.15. In each case, results for the WAF and WAF-D2 scheme will be compared. The calculations were performed on a DEC 3000 AXP Model 400-400F workstation.

Fig. 3.16 shows the results when the incident shock wave has Mach number $M_s = 1.7$. This results in Single Mach Reflection. Fig. 3.16a shows the density profile given by the WAF scheme run with $\nu_{max} = 0.8$. The CPU time for this calculation was 239 minutes, 26.7 seconds. Fig. 3.16b shows the density profile computed by the WAF-D2 scheme with $\nu_{max} = 1.8$. The CPU time for this calculation was 130 minutes, 33.7 seconds — a saving of $\approx 45\%$ over the WAF computation. The results are virtually identical, except for the oscillations produced by the WAF-D2 scheme between the reflected shock and the solid wall. This is simply a manifestation of the oscillations observed earlier in the one-dimensional case. Fig. 3.16c shows a picture of an experimental data for this shock reflection, for which the author is indebted to Professor Takayama of Tohoku University, Sendai, Japan.

Fig. 3.17 shows the results when the incident shock wave has Mach number $M_s = 8.75$. This results in a Double Mach Reflection. Fig. 3.17a shows the density profile from the WAF scheme with $\nu_{max} = 0.8$, which took a CPU time of 356 minutes, 23.8 seconds to compute. Fig. 3.17b shows the density profile given by the WAF-D2 scheme with $\nu_{max} = 1.8$, which took a CPU time of 189 minutes, 17.8 seconds — a saving of $\approx 47\%$ over WAF. The results are virtually identical, and compare well with the equivalent result in the report by Glaz *et. al.* [15].

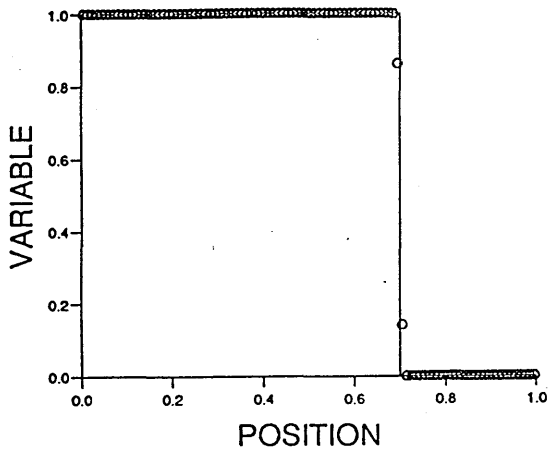
In the thesis by Fraser [12] a comparison of the WAF scheme with the Richtmyer two-step Lax-Wendroff method (see the book by Hirsch [25]), an artificial viscosity method,

was presented. In a comparison of Sod's problem, the **WAF** scheme was shown to cost approximately twice as much as the Richtmyer scheme. Timings in [64] indicate that if Riemann solver adaption is used in **WAF**, as has been done here, then the computational saving is almost 50% over **WAF** with exact Riemann solver everywhere; the experiments in this paper indicate that **WAF-D2** is approximately 50% cheaper than **WAF**. Thus **WAF-D2** with Riemann-solver adaption can be expected to be competitive with traditional artificial viscosity methods such as the Richtmyer scheme.

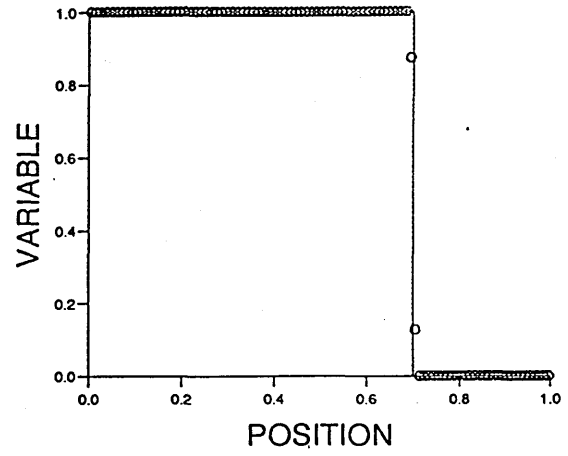
3.7 Conclusions

In this chapter, a conservative, large timestep **WAF**-type scheme for one dimensional conservation laws with linear stability limit of $\nu_{max} \leq 2$ has been presented. The scheme is essentially a unified extension of the Warming-Beam and Lax-Wendroff schemes with intercell fluxes based on solutions to Riemann problems at intercell boundaries. Wave interactions due to Courant numbers greater than unity are treated linearly. On scalar equations, the scheme is strictly **rvd**. The scheme can be extended to two or three space dimensions via space operator splitting. It has been tested successfully on two dimensional shock reflection problems, where savings in **CPU** time of 45% over the original **WAF** scheme were observed.

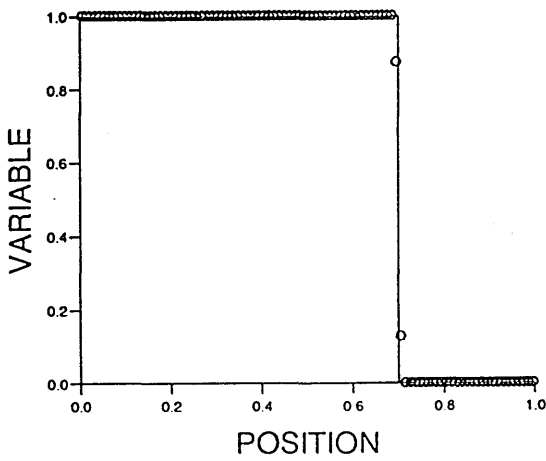
(a) Courant number coefficient: 0.8



(b) Courant number coefficient: 1.2



(c) Courant number coefficient: 1.6



(d) Courant number coefficient: 2.0

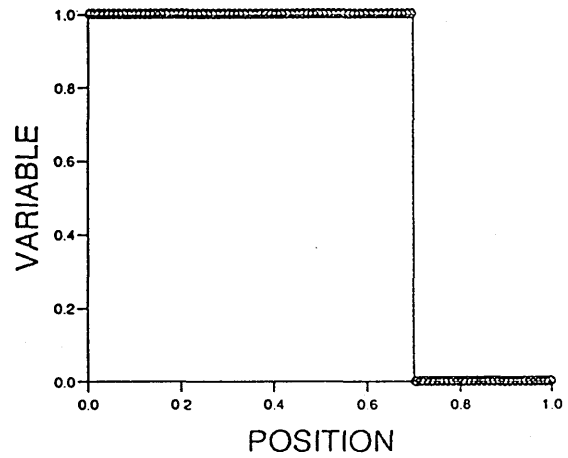


Figure 3.5: Results from the waf-d2 scheme applied to the inviscid Burgers equation.

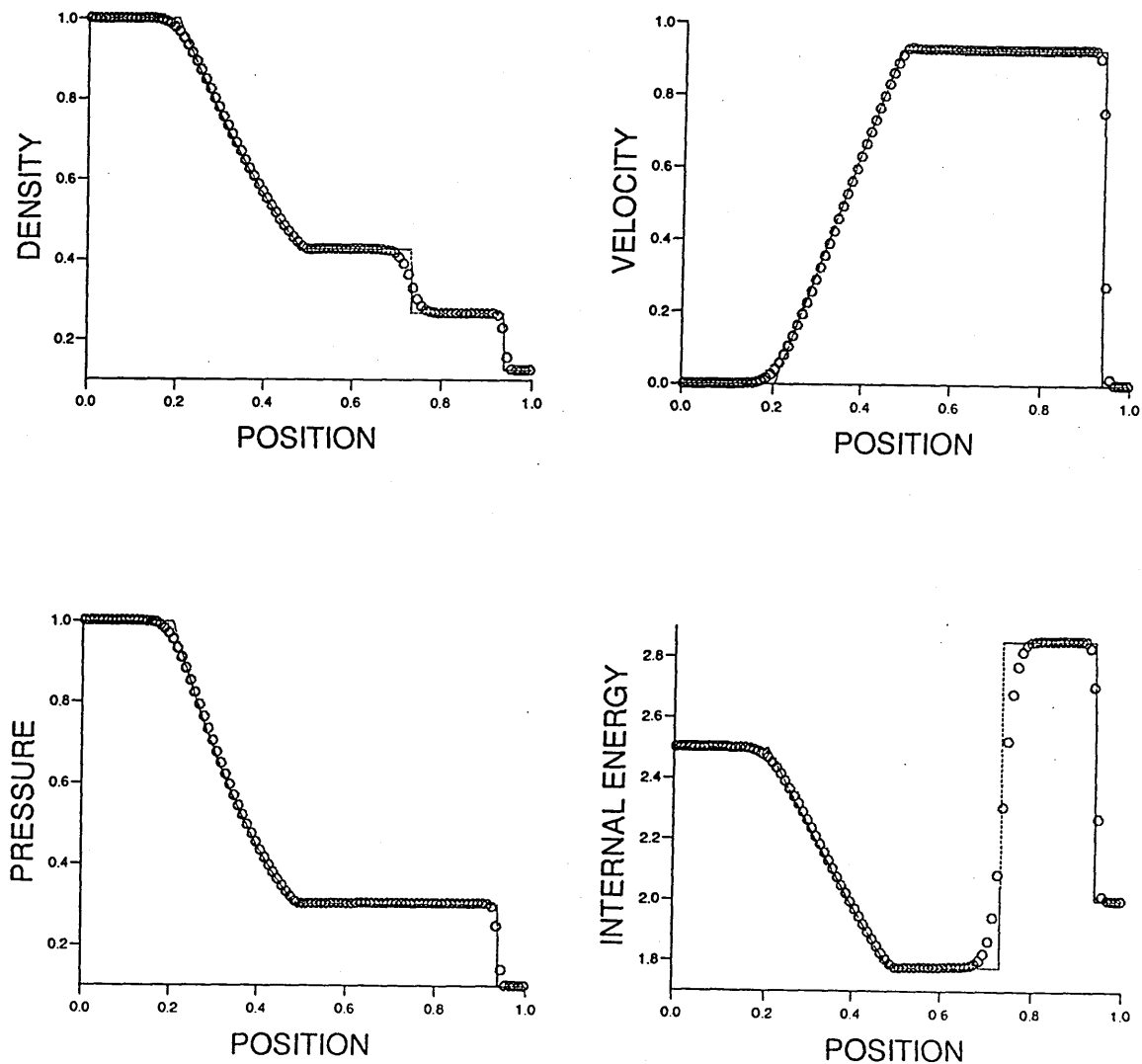


Figure 3.6: Results from the `waf-d2` scheme applied to Sod's problem. A Courant number coefficient of 0.8 was used. In this case the scheme reduces to the original `waf` scheme. The limiter (3.89) was used.

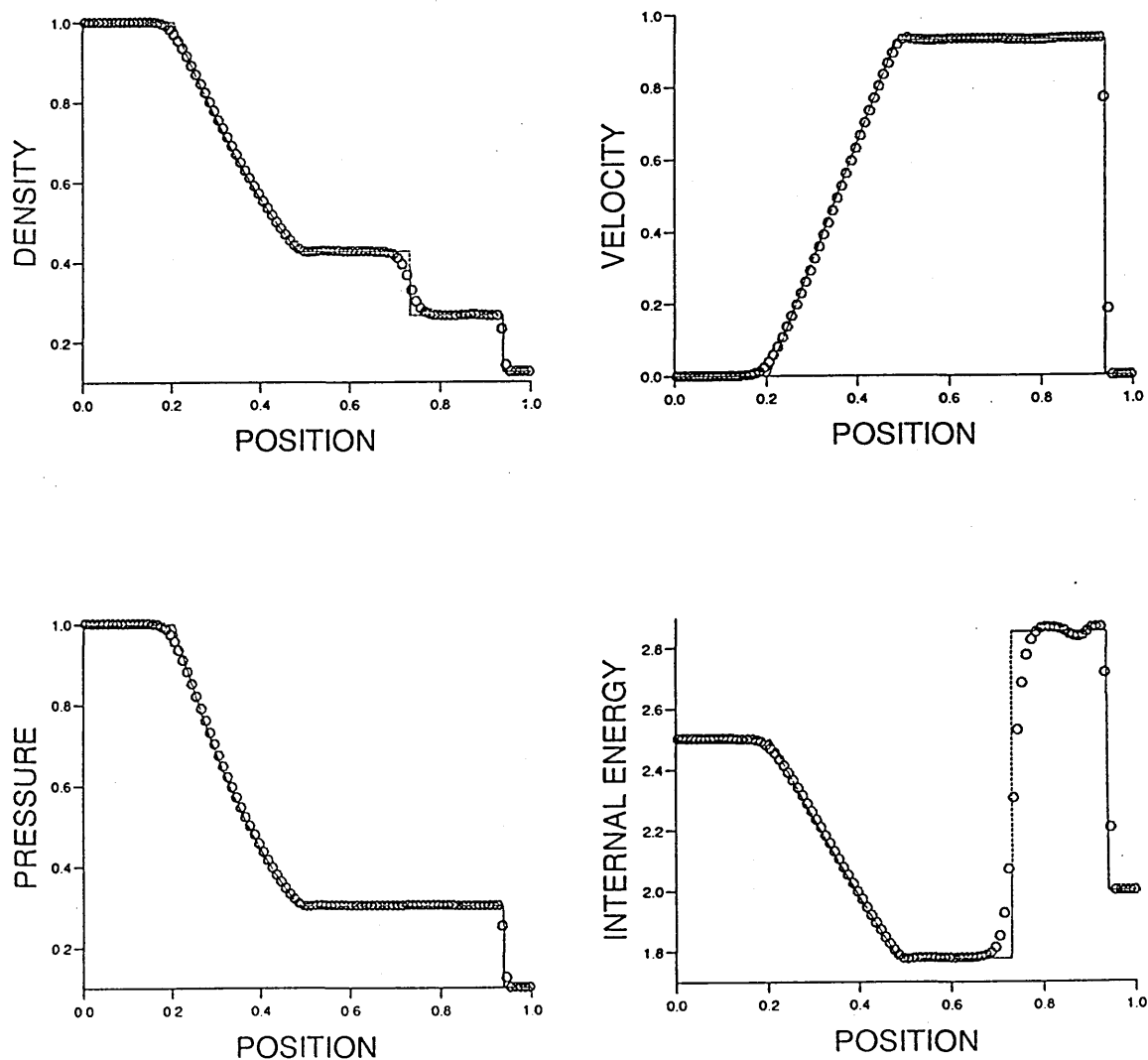


Figure 3.7: Results from the waf-d2 scheme applied to Sod's problem using limiters (3.89) and (3.128). A Courant number coefficient of 1.2 was used.

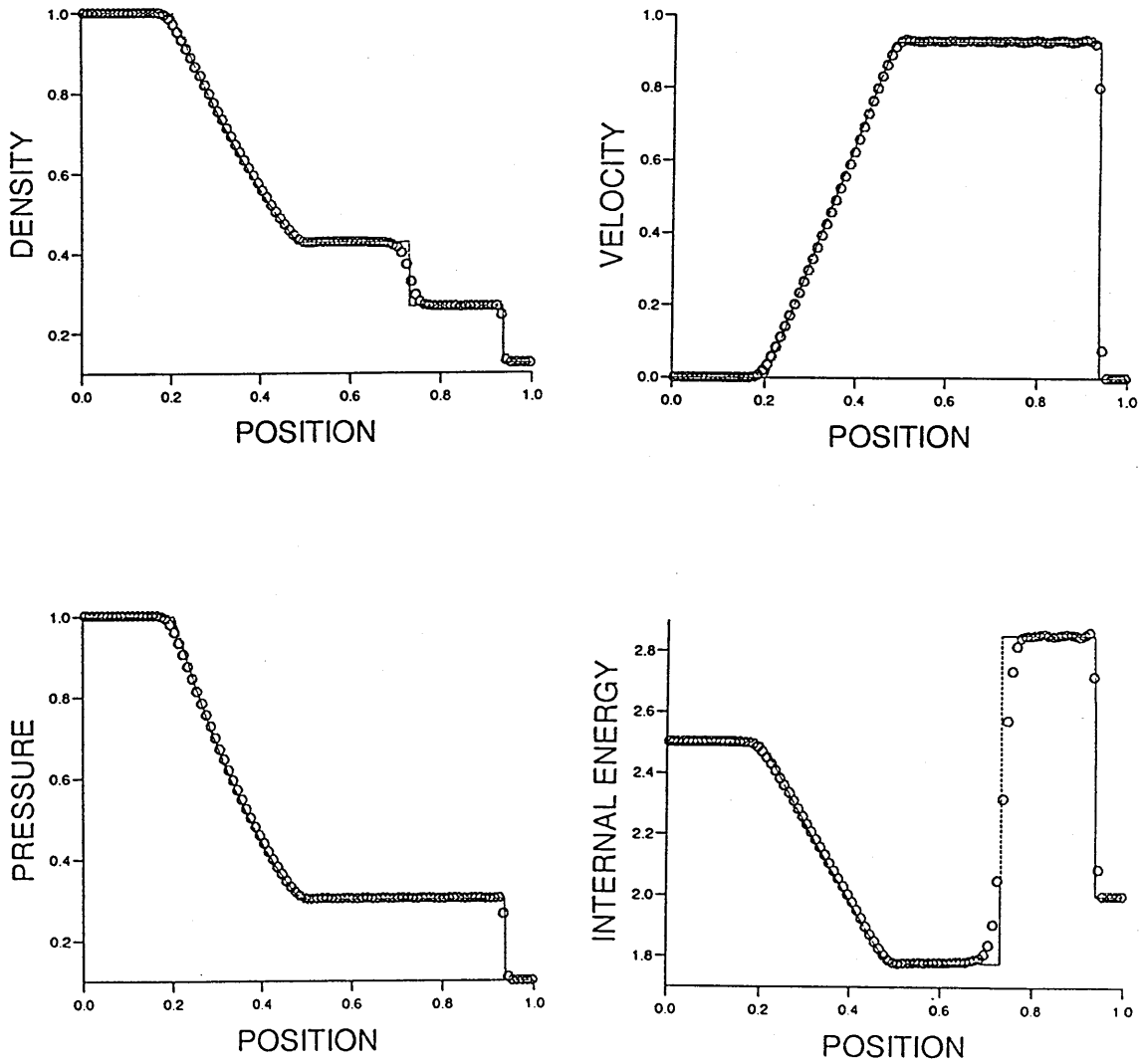


Figure 3.8: Results from the waf-d2 scheme applied to Sod's problem using limiters (3.89) and (3.128). A Courant number coefficient of 1.6 was used.

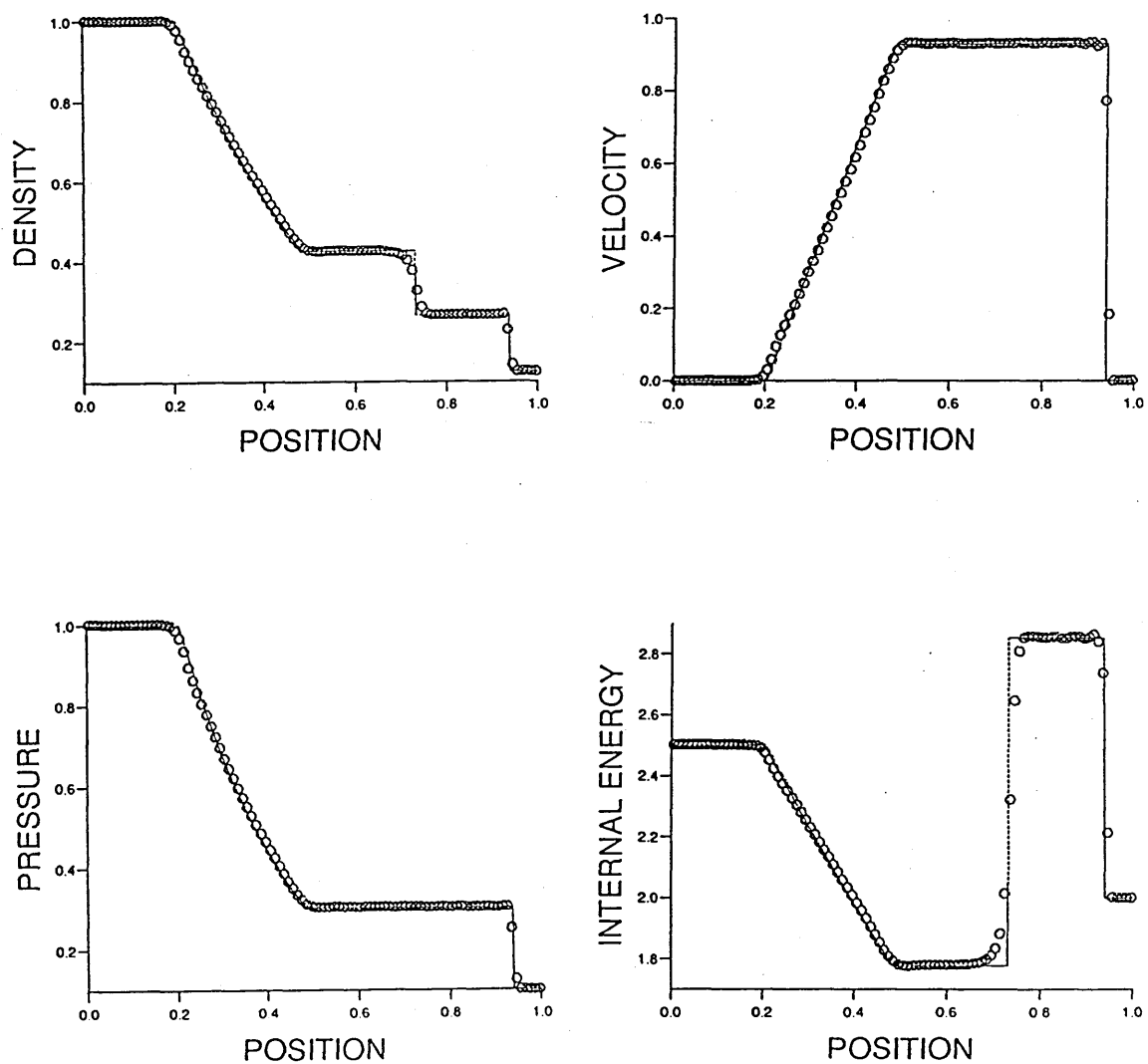


Figure 3.9: Results from the waf-d2 scheme applied to Sod's problem using limiters (3.89) and (3.128). A Courant number coefficient of 2.0 was used.

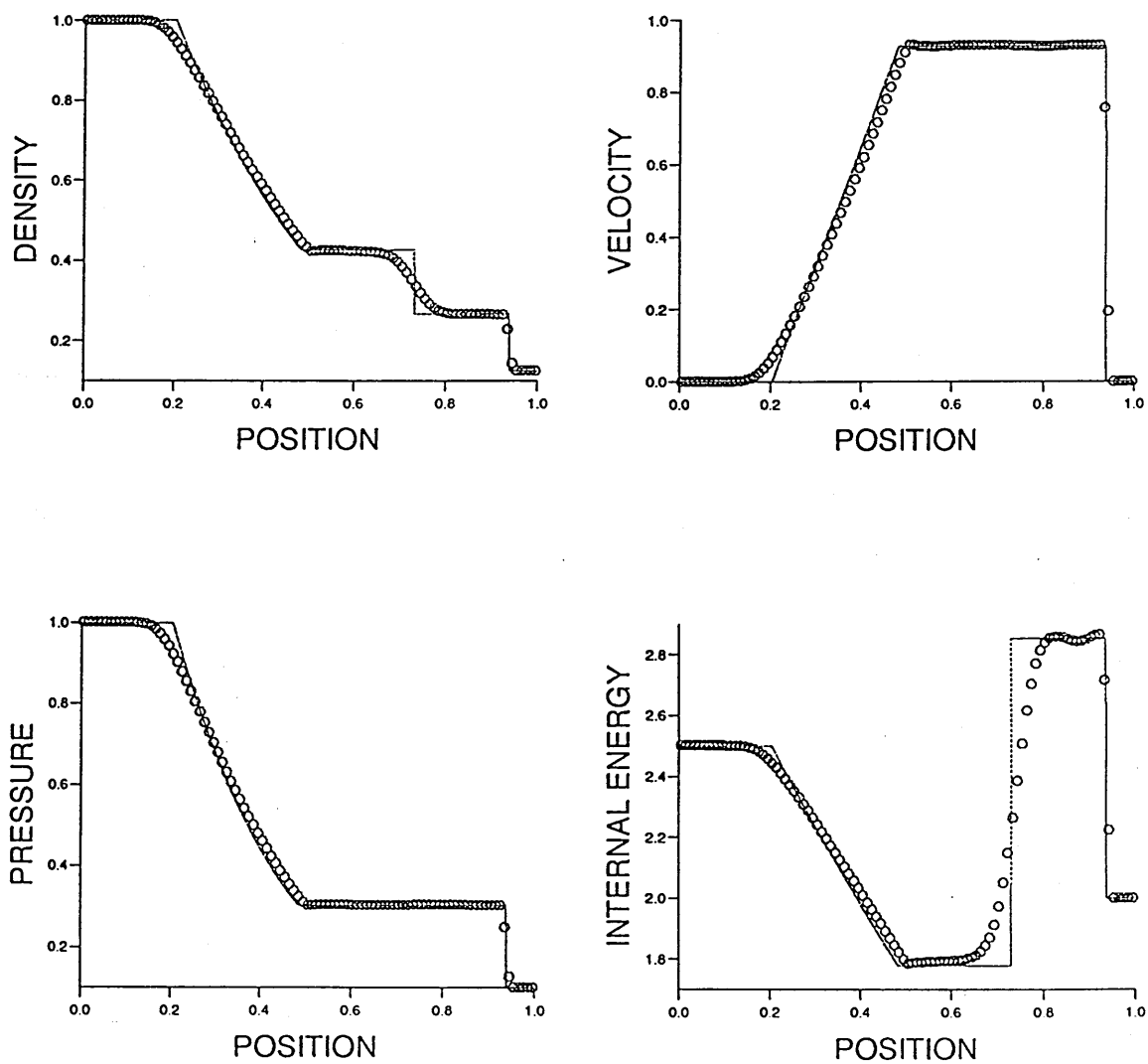


Figure 3.10: Results from the waf-d2 scheme applied to Sod's problem when the first order part of the scheme was used everywhere. A Courant number coefficient of 1.2 was used.

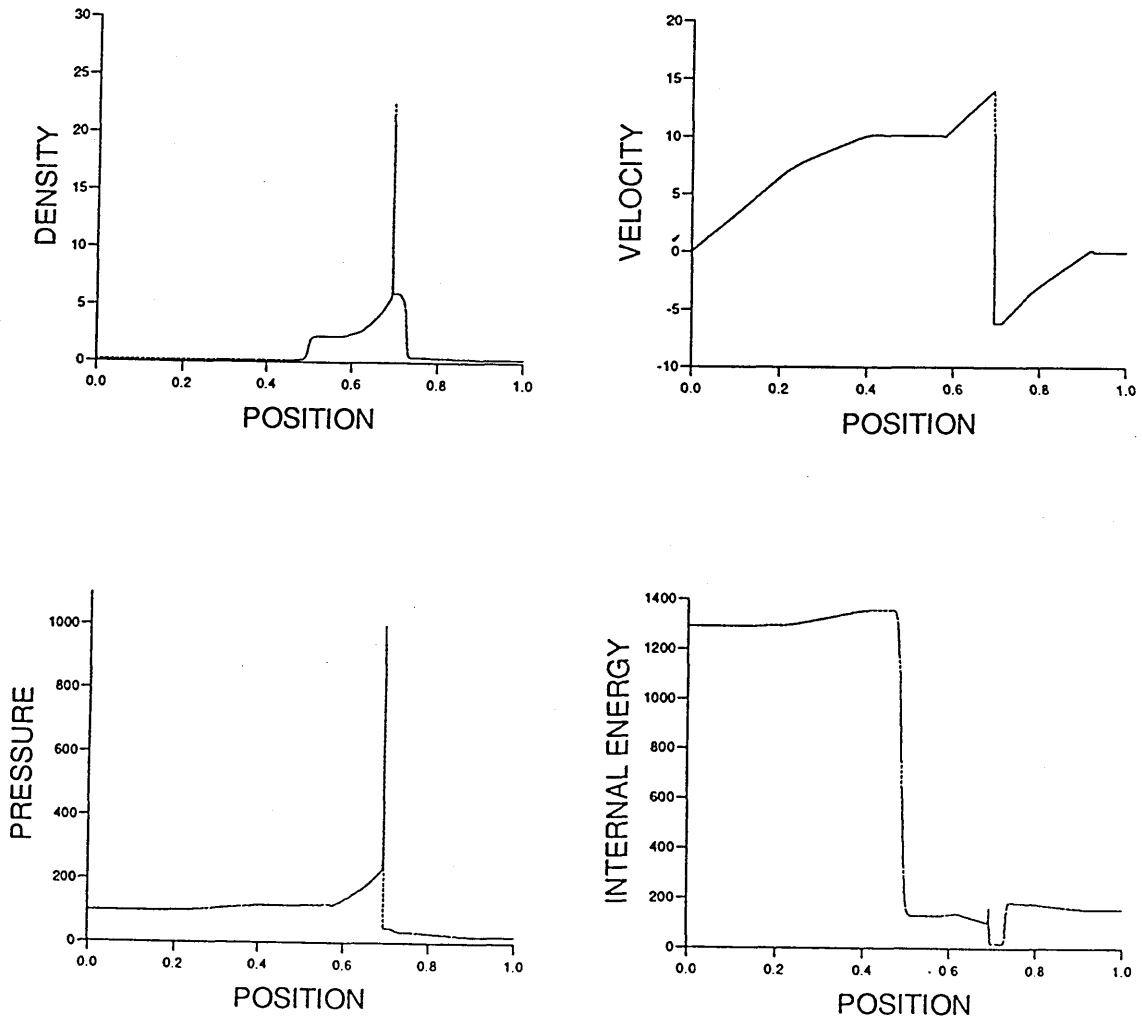


Figure 3.11: Results from the waf scheme applied to the Woodward and Colella blast wave problem using the limiter (3.89). Results were taken at time $t = 0.028$. A Courant number coefficient of 0.8 was used.

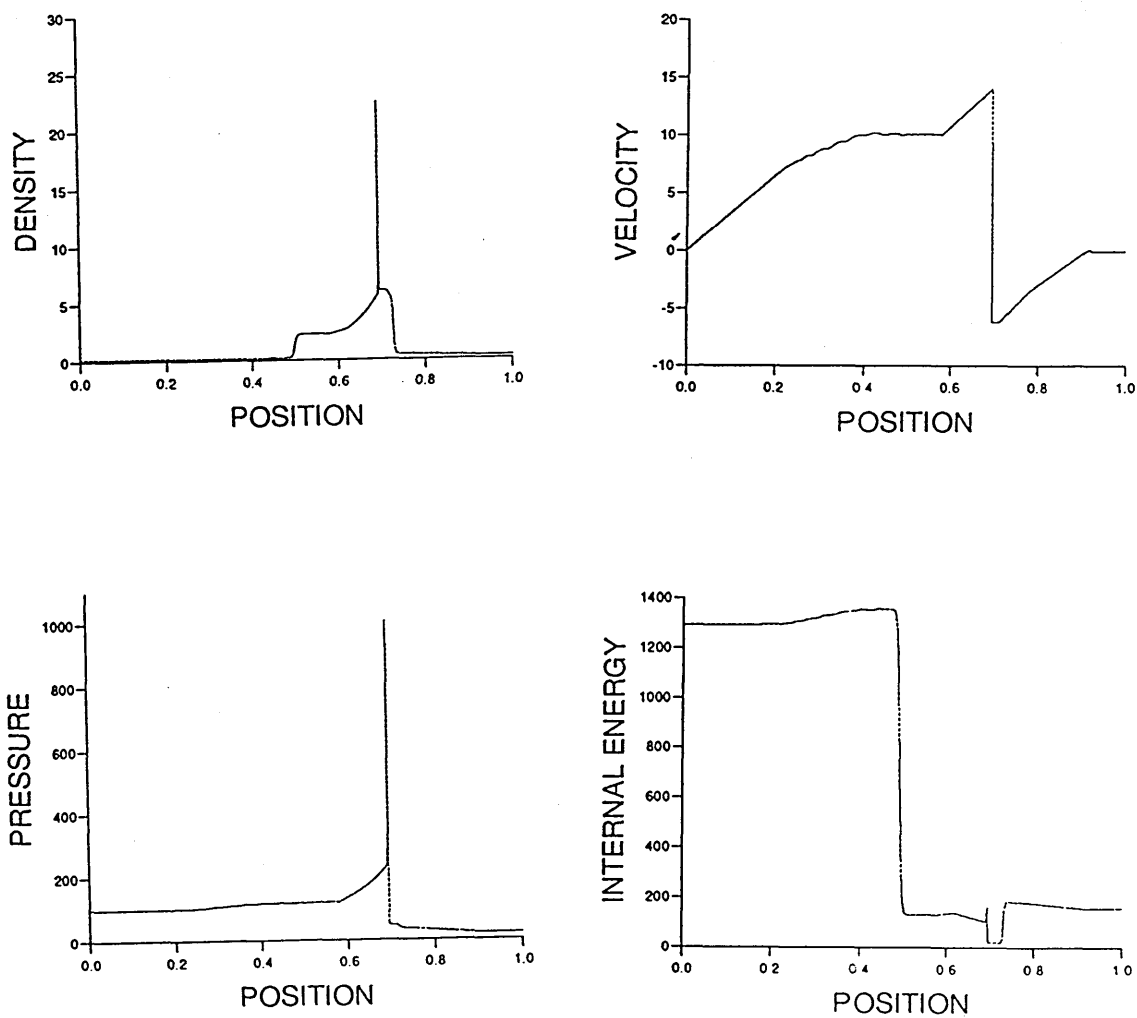


Figure 3.12: Results from the `waf-d2` scheme applied to the Woodward and Colella blast wave problem using the limiters (3.89) and (3.128). Results were taken at time $t = 0.028$. A Courant number coefficient of 1.8 was used.

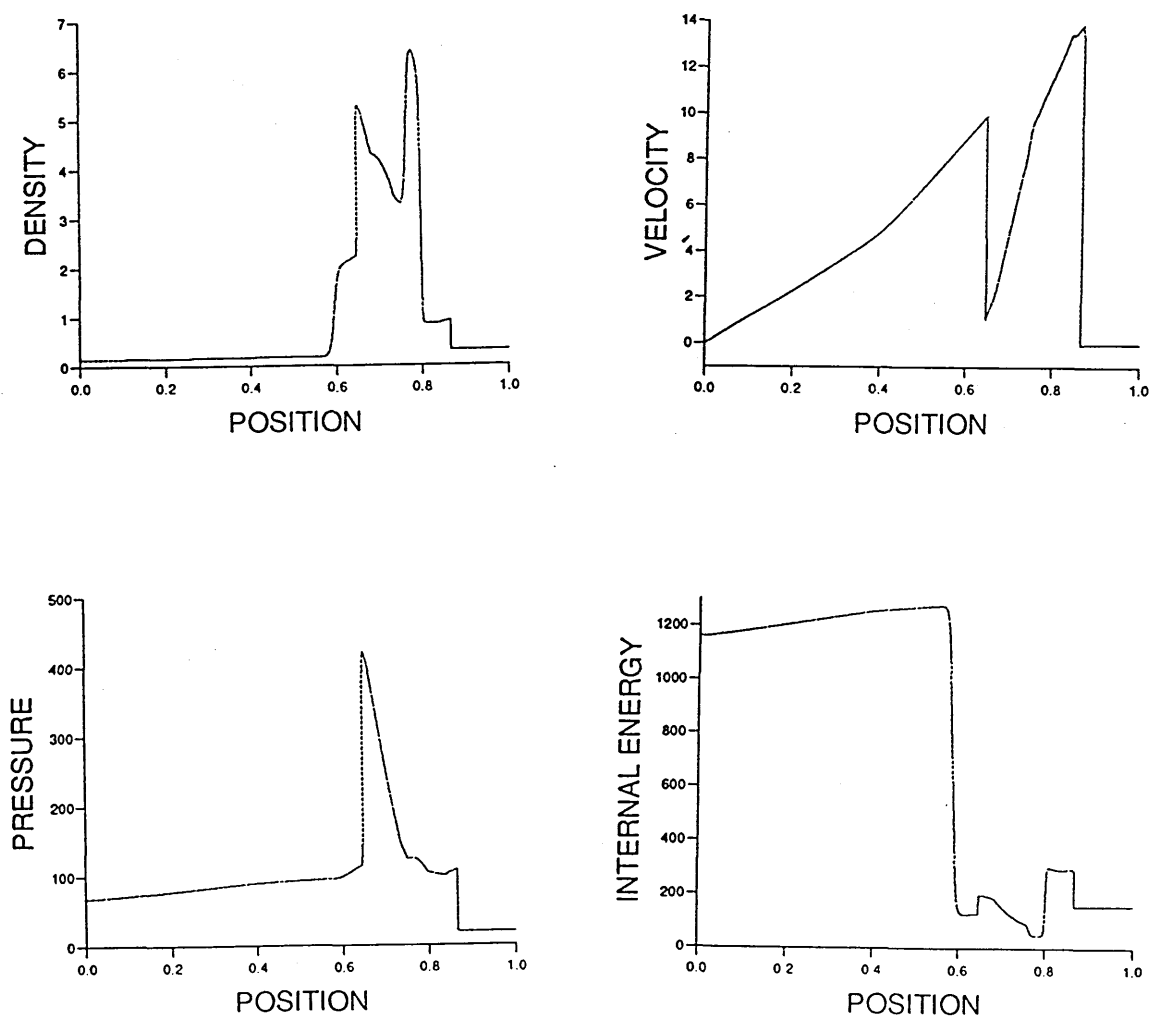


Figure 3.13: Results from the waf scheme applied to the Woodard and Colella blast wave problem using the limiter (3.89). Results were taken at time $t = 0.038$. A Courant number coefficient of 0.8 was used.

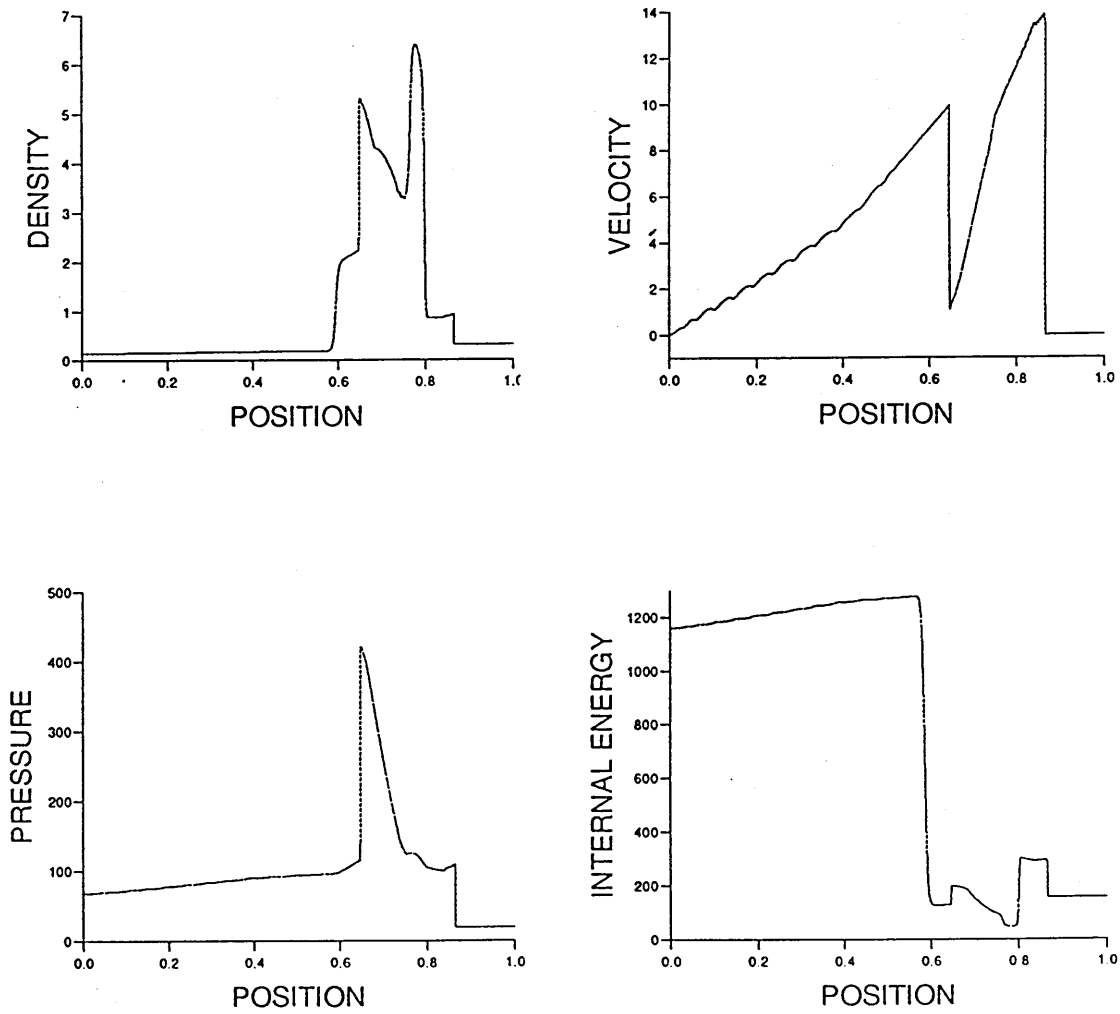


Figure 3.14: Results from the WAF-D2 scheme applied to the Woodward and Colella blast wave problem using the limiters (3.89) and (3.128). Results were taken at time $t = 0.038$. A Courant number coefficient of 1.8 was used.

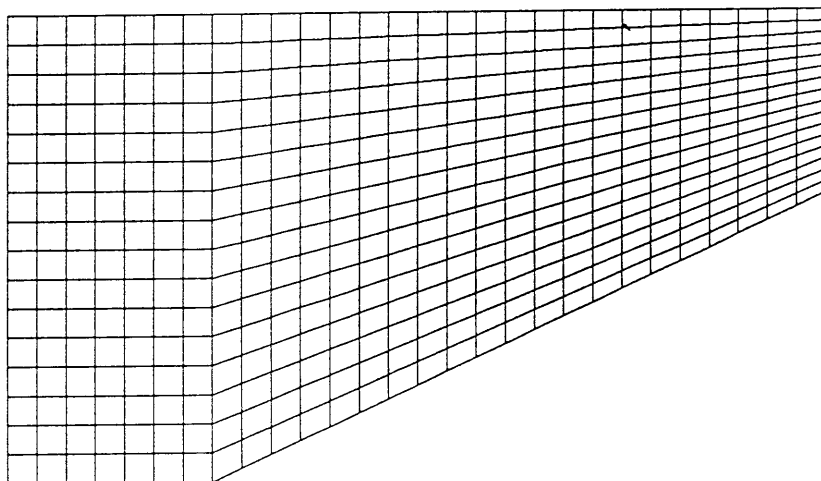
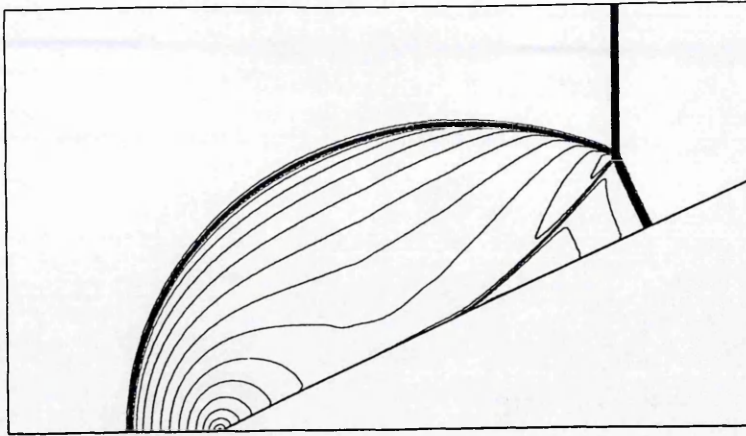
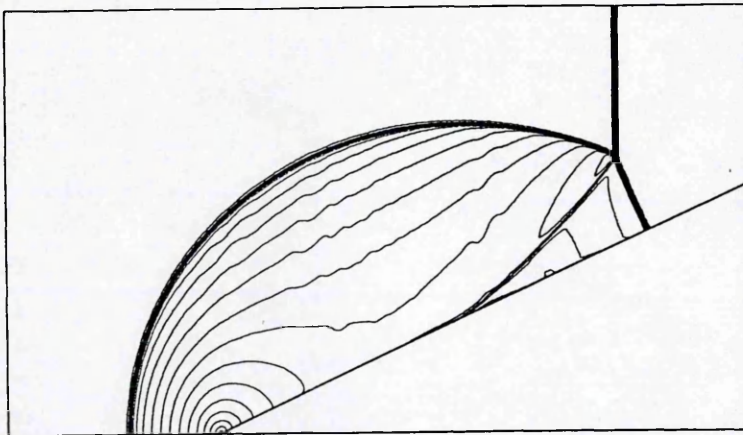


Figure 3.15: An illustration of the type of grid used in the shock reflection problems.



(a) Result from the **WAF** scheme using the limiter (3.89). A Courant number coefficient of 0.8 was used.

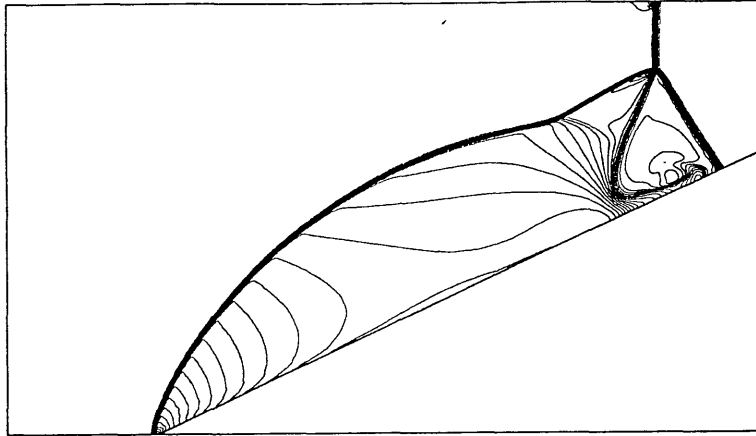


(b) Result from the **WAF-D2** scheme using the limiters (3.89) and (3.128). A Courant number coefficient of 1.8 was used.

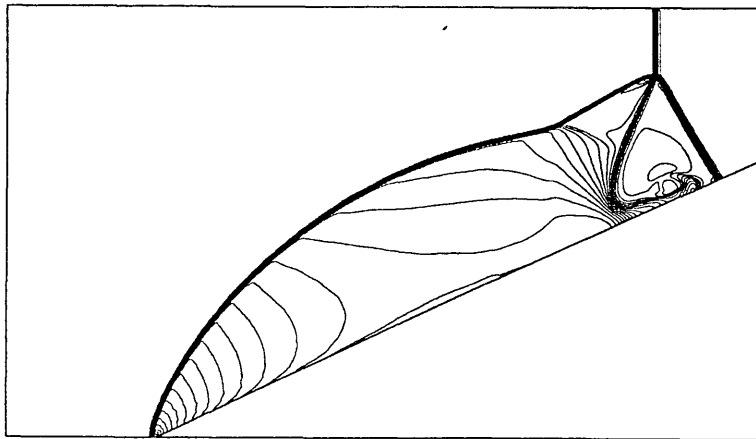


(c) Experimental result, courtesy of Professor Takayama of Tohoku University, Sendai, Japan.

Figure 3.16: Computed density profiles and an experimental result for the reflection of a Mach 1.7 shock over a 25 degree wedge.



(a) Result from the **WAF** scheme using the limiter (3.89). A Courant number coefficient of 0.8 was used.



(b) Result from the **WAF-D2** scheme using the limiters (3.89) and (3.128). A Courant number coefficient of 1.8 was used.

Figure 3.17: Computed density profiles for the reflection of a Mach 8.75 shock over a 25 degree wedge.

Chapter 4

A Riemann-Problem based extension of the Warming-Beam scheme

4.1 Introduction

The aim of this Chapter is to introduce a **tvD**, Riemann-Problem based extension of the Warming-Beam scheme to non-linear systems of hyperbolic conservation laws in one space dimension, *without* resorting to the Lax-Wendroff scheme for low Courant numbers, as is done for the **WAF-D2** scheme presented in the last Chapter. The first step will be to derive a **tvD** version of the Warming Beam scheme on scalar equations. The **tvD** analysis for $|\nu| \in [1, 2]$ has been presented in §3.4.2. The analysis for $|\nu| \in [0, 1]$ will be presented here, which, when combined with the analysis in §3.4.2, gives a **tvD** version of the Warming-Beam scheme for all $|\nu| \in [0, 2]$. The extension of this scheme to non-linear systems of equations will then be discussed. It will be shown that in regions of supersonic flow the update from one timelevel to the next using this extension depends only on locally upwind information — no downwind information is used. For certain Mach numbers, the **WAF-D2** scheme developed in Chapter 3 does not have this property. There are, however, at least two reasons why this should be desirable. Firstly, from a physical point of view, this is what should happen: for any given point in supersonic flow, information comes only from upstream. Any scheme that uses downwind information in the update of supersonic flow will allow the possibility of information propagating upstream, thus providing an unnecessary source of error. From a purely numerical point of view, it has been shown that one-sided schemes such as the Warming-Beam scheme have larger stability bounds and lower phase errors than centred schemes, such as the Lax-Wendroff scheme. See, for example, Warming-Beam paper [80]. The **WAF-D2** scheme takes advantage of the large stability bound of the Warming-Beam scheme, but will, at least in certain circumstances, lose the low phase error property because the Lax-Wendroff scheme is used for certain Courant numbers.

In [73], van Leer noted that Fromm's scheme [13] is the arithmetic average of the Lax-Wendroff and Warming-Beam schemes (note that van Leer's paper [73] preceded the Warming-Beam paper [80] by two years, so van Leer does not refer to the scheme by that name). He used this fact to average 'monotonic', but non-conservative, versions of the two schemes to obtain a 'monotonic', conservative version of the Fromm scheme. His criteria for monotonicity were, in fact, conditions (2.57), (2.59) used here to ensure a scheme is **TV****D**. The **TV****D** version of the Warming-Beam scheme used here will be conservative, but have a four point stencil in regions of high gradient, rather than the three points of van Leer's scheme. A weighted average of this scheme with the **WAF** scheme discussed in Chapter 2 will be used to obtain a one parameter class of **TV****D** schemes that includes a version of Fromm's scheme. Note that the two schemes averaged here will be conservative, and as a result all the schemes in this class will be conservative.

The rest of this Chapter is set out as follows. In §4.2, the **TV****D** analysis for the Warming-Beam scheme when $\nu \in [0, 1]$ is presented. In §4.3, the **TV****D** version of Warming-Beam scheme is extended systems of conservation laws. In §4.4, the one-parameter class of **TV****D** schemes is introduced. Some numerical experiments are presented in §4.5 and the chapter is concluded in §4.6.

4.2 TVD Analysis for the Warming-Beam Scheme

A **TV****D** analysis is now performed for the Warming-Beam scheme when $|\nu| \in [0, 1]$. In §3.4.2 the analysis was performed for the case $|\nu| \in [1, 2]$, so the work here completes the construction of a **TV****D** version of the scheme for the whole of $|\nu| \in [0, 2]$.

As in §3.4.2, the analysis is performed on the scalar conservation law

$$u_t + f(u)_x = 0 \quad (4.1)$$

where $f(u)$ is any function of u , not necessarily linear. The scheme is studied in conservation form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] \quad (4.2)$$

Consider first the case where the wavespeeds $a_{i+\frac{1}{2}}$ from Riemann problems at intercell boundaries are always positive. In this case, from §3.3, the Warming-Beam scheme has intercell flux

$$f_{i+\frac{1}{2}} = f_i - \frac{1}{2}(\nu_{i-\frac{1}{2}} - 1)(f_i - f_{i-1}) \quad (4.3)$$

As in Chapter 3, introduce a flux limiter $\psi_{i+\frac{1}{2}}$ by taking:

$$f_{i+\frac{1}{2}} = f_i - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_i - f_{i-1}) \quad (4.4)$$

Note that when $\psi_{i+\frac{1}{2}} = \nu_{i-\frac{1}{2}} - 1$ the Warming-Beam flux is reproduced, and thus the scheme is second order accurate in space and time; when $\psi_{i+\frac{1}{2}} = 0$, (4.4) reproduces

Godunov's intercell flux and the resulting scheme is first order accurate and monotonic. To obtain the full scheme, it is necessary to substitute (4.4) into the conservative formula (4.2). From (4.4):

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = \frac{1}{2}(2 - \psi_{i+\frac{1}{2}})(f_i - f_{i-1}) + \frac{1}{2}\psi_{i-\frac{1}{2}}(f_{i-1} - f_{i-2}) \quad (4.5)$$

Assume that the Rankine Hugoniot conditions hold across boundary $i + \frac{1}{2}$, i.e:

$$f_{i+1} - f_i = a_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n) \quad (4.6)$$

Thus

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = \frac{1}{2}a_{i-\frac{1}{2}}(2 - \psi_{i+\frac{1}{2}})(u_i^n - u_{i-1}^n) + \frac{1}{2}a_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}}(u_{i-1}^n - u_{i-2}^n) \quad (4.7)$$

Hence from (4.2):

$$u_i^{n+1} = u_i^n - \frac{1}{2}\nu_{i-\frac{1}{2}}(2 - \psi_{i+\frac{1}{2}})(u_i^n - u_{i-1}^n) - \frac{1}{2}\nu_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}}(u_{i-1}^n - u_{i-2}^n) \quad (4.8)$$

The aim here is to find limits on the functions $\psi_{i+\frac{1}{2}}$ such that the TVD condition

$$0 \leq C_i \leq \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} \leq 1 \quad (4.9)$$

always holds. From (4.8):

$$C_i = \frac{1}{2}\nu_{i-\frac{1}{2}}(2 - \psi_{i+\frac{1}{2}}) + \frac{1}{2}\nu_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}}r_{i-\frac{1}{2}} \quad (4.10)$$

where

$$r_{i-\frac{1}{2}} = \frac{\Delta_{upw}u}{\Delta_{loc}u} = \frac{u_{i-1}^n - u_{i-2}^n}{u_i^n - u_{i-1}^n} \quad (4.11)$$

is the flow parameter for boundary $i - \frac{1}{2}$ and is the ratio of the *upwind* jump $\Delta_{upw}u$ in u to the *local* jump $\Delta_{loc}u$ in u . Set lower and upper limits on $\psi_{i+\frac{1}{2}}$ as:

$$L \leq \psi_{i+\frac{1}{2}} \leq U \quad (4.12)$$

This gives a minimum value C_{min} of C_i as:

$$C_{min} = \frac{1}{2}\nu_{i-\frac{1}{2}}(2 - U) + \frac{1}{2}\nu_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}}r_{i-\frac{1}{2}} \quad (4.13)$$

and a maximum value C_{max} of C_i as:

$$C_{max} = \frac{1}{2}\nu_{i-\frac{1}{2}}(2 - L) + \frac{1}{2}\nu_{i-\frac{3}{2}}\psi_{i-\frac{1}{2}}r_{i-\frac{1}{2}} \quad (4.14)$$

For the scheme to be TVD, regions for $\psi_{i-\frac{1}{2}}$ must be found such that $C_{min} \geq 0$ and $C_{max} \leq 1$. Since $\nu_{i-\frac{3}{2}}$ is assumed positive here, $C_{min} \geq 0$ if and only if

$$\psi_{i-\frac{1}{2}}r_{i-\frac{1}{2}} \geq \frac{1}{\alpha_{i-\frac{1}{2}}}(U - 2) \quad (4.15)$$

where

$$\alpha_{i-\frac{1}{2}} = \frac{\nu_{upw}}{\nu_{loc}} = \frac{\nu_{i-\frac{3}{2}}}{\nu_{i-\frac{1}{2}}} \quad (4.16)$$

is the ratio of upwind Courant number to local Courant number. This is the compression parameter introduced in Chapter 3. If $r_{i-\frac{1}{2}} \geq 0$ (4.15) gives

$$\psi_{i-\frac{1}{2}} \geq \frac{1}{\alpha_{i-\frac{1}{2}}}(U-2)\frac{1}{r_{i-\frac{1}{2}}} \quad (4.17)$$

and if $r_{i-\frac{1}{2}} \leq 0$ (4.15) gives

$$\psi_{i-\frac{1}{2}} \leq \frac{1}{\alpha_{i-\frac{1}{2}}}(U-2)\frac{1}{r_{i-\frac{1}{2}}} \quad (4.18)$$

Since $\nu_{i-\frac{3}{2}} \geq 0$, $C_{max} \leq 1$ if and only if

$$\psi_{i-\frac{1}{2}} r_{i-\frac{1}{2}} \leq \frac{2}{\nu_{i-\frac{3}{2}}} - \frac{1}{\alpha_{i-\frac{1}{2}}}(2-L) \quad (4.19)$$

So if $r_{i-\frac{1}{2}} \geq 0$:

$$\psi_{i-\frac{1}{2}} \leq \left[\frac{2}{\nu_{i-\frac{3}{2}}} - \frac{1}{\alpha_{i-\frac{1}{2}}}(2-L) \right] \frac{1}{r_{i-\frac{1}{2}}} \quad (4.20)$$

and if $r_{i+\frac{1}{2}} \leq 0$:

$$\psi_{i-\frac{1}{2}} \geq \left[\frac{2}{\nu_{i-\frac{3}{2}}} - \frac{1}{\alpha_{i-\frac{1}{2}}}(2-L) \right] \frac{1}{r_{i-\frac{1}{2}}} \quad (4.21)$$

The tvd analysis for the case $a_{i+\frac{1}{2}} \geq 0$ for all i is now complete. In the case $a_{i+\frac{1}{2}} \leq 0$ for all i the intercell fluxes are taken as:

$$f_{i+\frac{1}{2}} = f_{i+1} - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_{i+2} - f_{i+1}) \quad (4.22)$$

with second order accuracy in space and time when $\psi_{i+\frac{1}{2}} = \frac{1}{2}(|\nu_{i+\frac{3}{2}}| - 1)$. The tvd condition in this case is:

$$0 \leq D_i \leq \frac{u_i^{n+1} - u_i^n}{u_{i+1}^n - u_i^n} \leq 1 \quad (4.23)$$

The restriction (4.12) is replaced by

$$L \leq \psi_{i-\frac{1}{2}} \cdot \text{sgn}(\nu_{i-\frac{1}{2}}) \leq U \quad (4.24)$$

and the analysis follows as before. Both cases can be summarized as follows.

Summary

Assume

$$L \leq \psi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq U \quad (4.25)$$

and define

$$s_{i+\frac{1}{2}} = \frac{1}{r_{i+\frac{1}{2}}} \quad (4.26)$$

This is the 'smoothness parameter' used by van Leer [73]. Define also the compression parameter

$$\alpha_{i+\frac{1}{2}} = \frac{\nu_{upw}}{\nu_{i+\frac{1}{2}}} \quad (4.27)$$

where ν_{upw} is the upwind Courant number

$$\nu_{upw} = \begin{cases} \nu_{i-\frac{1}{2}} & \text{if } \nu_{i+\frac{1}{2}} \geq 0 \\ \nu_{i+\frac{3}{2}} & \text{otherwise} \end{cases} \quad (4.28)$$

The scheme is **TV D** under the following conditions. If $s_{i+\frac{1}{2}} \geq 0$ then

$$\frac{1}{\alpha_{i+\frac{1}{2}}}(U-2)s_{i+\frac{1}{2}} \leq \psi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq \left[\frac{2}{|\nu_{upw}|} - \frac{1}{\alpha_{i+\frac{1}{2}}}(2-L) \right] s_{i+\frac{1}{2}} \quad (4.29)$$

and if $s_{i+\frac{1}{2}} \leq 0$ then

$$\left[\frac{2}{|\nu_{upw}|} - \frac{1}{\alpha_{i+\frac{1}{2}}}(2-L) \right] s_{i+\frac{1}{2}} \leq \psi_{i+\frac{1}{2}} \cdot \text{sgn}(\nu_{i+\frac{1}{2}}) \leq \frac{1}{\alpha_{i+\frac{1}{2}}}(U-2)s_{i+\frac{1}{2}} \quad (4.30)$$

If $\alpha_{i+\frac{1}{2}} \leq 0$ set $\psi_{i+\frac{1}{2}} = 0$.

Design of limiter functions

A specific limiter function $\psi = \psi_{i+\frac{1}{2}}(r_{i+\frac{1}{2}}, \alpha_{i+\frac{1}{2}}, \nu_{upw})$ needs to be chosen within the region (4.29), (4.30). An infinite number are, of course, possible. One is selected here for use in numerical experiments later.

The first step in designing a limiter is to choose upper and lower limits U and L . These are restricted by the fact that it is necessary for some **TV D** ψ to exist for every value of $s_{i+\frac{1}{2}}$. This amounts to constraints on the slopes of the left and right hand sides in (4.29). The slope of the left hand side must be kept negative, i.e:

$$\frac{1}{\alpha_{i+\frac{1}{2}}}(U-2) \leq 0 \quad (4.31)$$

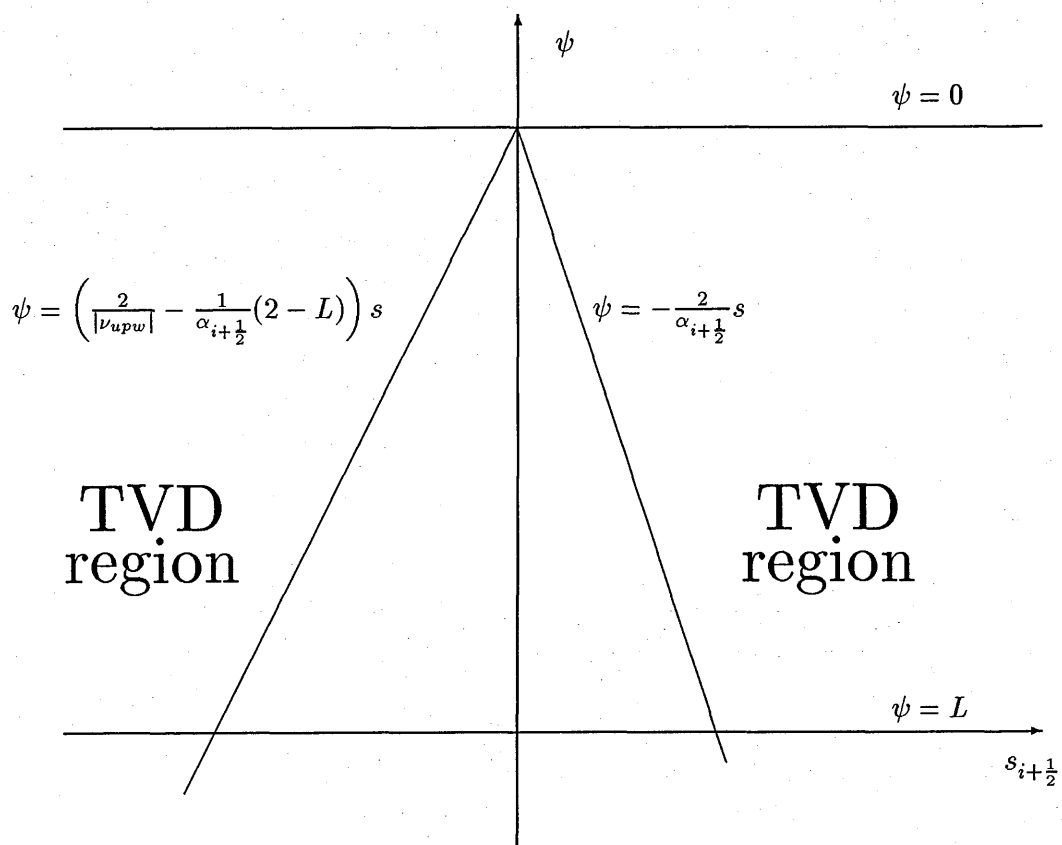


Figure 4.1: TVD region for the Warming Beam scheme when $0 \leq \nu_{max} \leq 1$.

The **tvD** region is only used when the local and upwind Courant numbers are of the same sign, i.e. $\alpha_{i+\frac{1}{2}} \geq 0$. Thus (4.31) implies

$$U \leq 2 \quad (4.32)$$

Since there is no need to be more dissipative than Godunov's scheme, set

$$U = 0 \quad (4.33)$$

here. The lower limit L is constrained by the fact that the right hand slope in (4.29) must be positive, i.e:

$$\frac{2}{|\nu_{upw}|} - \frac{1}{\alpha_{i+\frac{1}{2}}}(2 - L) \geq 0 \quad (4.34)$$

This gives a limit on L as:

$$L \geq \frac{2}{|\nu_{upw}|}(|\nu_{upw}| - \alpha_{i+\frac{1}{2}}) \quad (4.35)$$

Second order accuracy is required whenever possible, so ideally:

$$L \leq |\nu_{upw}| - 1 \quad (4.36)$$

However (4.35) and (4.36) can only be mutually satisfied when

$$\alpha_{i+\frac{1}{2}} \geq \frac{1}{2} |\nu_{upw}| (3 - |\nu_{upw}|) \quad (4.37)$$

Thus take the lower limit as

$$L = \begin{cases} |\nu_{upw}| - 1 & \text{if (4.37) holds} \\ \frac{2}{|\nu_{upw}|}(|\nu_{upw}| - \alpha_{i+\frac{1}{2}}) & \text{otherwise} \end{cases} \quad (4.38)$$

The **tvD** region with limits (4.33), (4.38) is shown in Fig. 4.1.

The simplest possible limiter satisfying (4.29), (4.30) with limits (4.33), (4.38) is:

$$\psi_{i+\frac{1}{2}} = \begin{cases} 0 & \text{if } s_{i+\frac{1}{2}} \leq 0 \text{ or } \alpha_{i+\frac{1}{2}} \leq 0 \\ \max \left\{ -\frac{s_{i+\frac{1}{2}}}{\alpha_{i+\frac{1}{2}}}, L \right\} & \text{otherwise} \end{cases} \quad (4.39)$$

4.3 Extension to Systems of Conservation Laws

In this section, a Riemann problem based extension of the Warming-Beam scheme to hyperbolic systems of conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \quad (4.40)$$

where $\mathbf{U}$ is a vector of N conserved variables and $\mathbf{F}(\mathbf{U})$ is a vector-valued flux function is considered. The scheme will be constructed in conservation form

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}] \quad (4.41)$$

where $\mathbf{U}_i^n$ is the integral average of $\mathbf{U}$ in cell i at timelevel n and $\mathbf{F}_{i+\frac{1}{2}}$ is an intercell flux. For the scalar case discussed in the last section, the intercell flux was taken as:

$$f_{i+\frac{1}{2}} = \begin{cases} f_i - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_i - f_{i-1}) & \text{if } a_{i+\frac{1}{2}} \geq 0 \\ f_{i+1} - \frac{1}{2}\psi_{i+\frac{1}{2}}(f_{i+2} - f_{i+1}) & \text{if } a_{i+\frac{1}{2}} < 0 \end{cases} \quad (4.42)$$

where $\psi_{i+\frac{1}{2}} = \psi(r_{i+\frac{1}{2}}, \alpha_{i+\frac{1}{2}}, \nu_{upw})$ for some limiter function ψ from §4.2 such as (4.39) if $|\nu_{upw}| \leq 1$, or $\psi(r_{upw}, \alpha_{upw}, \nu_{upw})$ for some limiter function ψ from §3.4.2 such as (3.128) if $1 \leq |\nu_{upw}| \leq 2$. Here r_{upw} and α_{upw} are the upwind flow and compression parameters, defined in an analogous fashion to ν_{upw} in (4.28). For systems of equations (4.40), the flux (4.42) can be interpreted as

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_{i+\frac{1}{2}}^{(GOD)} - \frac{1}{2} \sum_{k+} \psi_{i+\frac{1}{2}}^{(k+)} \Delta \mathbf{F}_{i-\frac{1}{2}}^{(k+)} - \frac{1}{2} \sum_{k-} \psi_{i+\frac{1}{2}}^{(k-)} \Delta \mathbf{F}_{i+\frac{3}{2}}^{(k-)} \quad (4.43)$$

Here $\mathbf{F}^{(GOD)}$ is the intercell flux for Godunov's scheme, i.e the flux evaluated on the time axis of the solution to the Riemann problem at boundary $i + \frac{1}{2}$. The ranges of parameters $k\pm$ have yet to be chosen, and the differences $\Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)}$ are jumps across waves in the solution to Riemann problems. These jumps are weighted by limiter functions $\psi_{i+\frac{1}{2}}$, which are taken as in the scalar case and evaluated on data from the solutions to local Riemann problems, i.e the flow parameter is taken as

$$r_{i+\frac{1}{2}}^{(k)} = \frac{\Delta_{upw}^{(k)} q}{\Delta_{loc}^{(k)} q} \quad (4.44)$$

for some variable q , and the compression parameter is taken as

$$\alpha_{i+\frac{1}{2}}^{(k)} = \frac{\nu_{upw}^{(k)}}{\nu_{loc}^{(k)}} \quad (4.45)$$

For $|\nu_{upw}^{(k)}| \leq 1$, ψ is taken from §4.2; for $|\nu_{upw}^{(k)}| \in [1, 2]$, ψ will be taken from §3.4.2. The only matter to be decided is which waves will be included in the summations in (4.43) and which will not. Since the TVD analysis has been based on having all waves of the same sign, a sensible choice is to take

$$\{k+\} = \left\{ k : \nu_{i+\frac{1}{2}}^{(k)} \geq 0 \text{ and } \nu_{i-\frac{1}{2}}^{(k)} \geq 0 \right\} \quad (4.46)$$

and

$$\{k-\} = \left\{ k : \nu_{i+\frac{1}{2}}^{(k)} \leq 0 \text{ and } \nu_{i+\frac{3}{2}}^{(k)} \leq 0 \right\} \quad (4.47)$$

Note that this choice ensures the scheme is always one-sided in supersonic flow i.e when all wavespeeds are of the same sign. For example, for constant supersonic flow from left to right, all wavespeeds, and therefore local Courant numbers, are positive, and (4.47) is an empty set. The flux (4.43) therefore depends only on the Godunov flux at boundary $i + \frac{1}{2}$, and the solution to the Riemann problem at the upwind boundary $i - \frac{1}{2}$. Thus no downwind information is included in the calculation of $\mathbf{F}_{i+\frac{1}{2}}$. This is not the case for the **WAF-D2** scheme of Chapter 3. Downwind information in the solution of the Riemann problem at boundary $i + \frac{1}{2}$ will only be discarded if all Courant numbers are greater than unity. For the Euler Equations of Gas Dynamics (see Chapter 2), this can be formulated as a condition on the Mach number $M = u/a$ as follows. In constant supersonic flow, the maximum Courant number is given by

$$\nu_{max} = (u + a) \frac{\Delta t}{\Delta x} \quad (4.48)$$

To avoid downwind information, it is necessary to have

$$(u - a) \frac{\Delta t}{\Delta x} \geq 1 \quad (4.49)$$

Dividing (4.49) by (4.48) gives

$$\frac{u - a}{u + a} \geq \frac{1}{\nu_{max}} \quad (4.50)$$

This can be written as

$$\frac{M - 1}{M + 1} \geq \frac{1}{\nu_{max}} \quad (4.51)$$

Assuming $\nu_{max} > 1$ this becomes a condition on M as:

$$M \geq \frac{\nu_{max} + 1}{\nu_{max} - 1} \quad (4.52)$$

Hence the stencil of the scheme for supersonic flow depends on the maximum Courant number used. As an example, if ν_{max} is taken as 1.8, a one-sided stencil is only used when $M \geq 3.5$. For lower Mach numbers, a four point, upwind-centred stencil is used. A similar analysis can be performed for the Shallow Water equations given in Chapter 2, using the Froude number $F = u^2/\phi$ instead of Mach number.

4.4 A One-parameter Family of Second Order TVD Schemes

Recall from Chapter 2 that the **WAF** scheme for one dimensional systems of conservation laws (4.39) has intercell flux

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2} \sum_{k=1}^N \phi_{i+\frac{1}{2}}^{(k)} \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (4.53)$$

where $\phi_{i+\frac{1}{2}}^{(k)}$ are limiter functions. This flux can be written in the form

$$\mathbf{F}_{i+\frac{1}{2}}^{(LW)} = \mathbf{F}_{i+\frac{1}{2}}^{(GOD)} - \frac{1}{2} \sum_{k=1}^N (\phi_{i+\frac{1}{2}}^{(k)} - 1) \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \quad (4.54)$$

where the superscript (LW) denotes the fact that this scheme reduces to the Lax-Wendroff on the linear advection equation. Compare (4.54) with the intercell flux for the extension of the Warming-Beam scheme presented in §4.3:

$$\mathbf{F}_{i+\frac{1}{2}}^{(WB)} = \mathbf{F}_{i+\frac{1}{2}}^{(GOD)} - \frac{1}{2} \sum_{k+} \psi_{i+\frac{1}{2}}^{(k+)} \Delta \mathbf{F}_{i-\frac{1}{2}}^{(k+)} - \frac{1}{2} \sum_{k-} \psi_{i+\frac{1}{2}}^{(k-)} \Delta \mathbf{F}_{i+\frac{3}{2}}^{(k-)} \quad (4.55)$$

A general class of **tvD** schemes can be constructed by taking a weighted average of the fluxes (4.54) and (4.55), i.e for some real number β take

$$\mathbf{F}_{i+\frac{1}{2}}^{(\beta)} = \beta \mathbf{F}_{i+\frac{1}{2}}^{(LW)} + (1 - \beta) \mathbf{F}_{i+\frac{1}{2}}^{(WB)} \quad (4.56)$$

This can be written in full as:

$$\begin{aligned} \mathbf{F}_{i+\frac{1}{2}}^{(\beta)} &= \mathbf{F}_{i+\frac{1}{2}}^{(GOD)} - \frac{1}{2} \beta \sum_{k=1}^N (\phi_{i+\frac{1}{2}}^{(k)} - 1) \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \\ &\quad - \frac{1}{2} (1 - \beta) \sum_{k+} \psi_{i+\frac{1}{2}}^{(k+)} \Delta \mathbf{F}_{i-\frac{1}{2}}^{(k+)} - \frac{1}{2} (1 - \beta) \sum_{k-} \psi_{i+\frac{1}{2}}^{(k-)} \Delta \mathbf{F}_{i+\frac{3}{2}}^{(k-)} \end{aligned} \quad (4.57)$$

The scheme generated by this flux will be referred to as the β -scheme. It will be assumed that the timestep Δt is chosen such that all Courant numbers are less than unity, since the **waf** scheme is only stable under this restriction. The limiters ψ should be taken in the region discussed in §4.2; the limiters ϕ should taken as in §3.4.1. For $\beta = 1$, (4.57) reduces to (4.54), and for $\beta = 0$ (4.57) reduces to (4.55). If $\beta = \frac{1}{2}$, (4.57) gives a scheme that is an extension of the Fromm scheme [13], at least in smooth regions of flow where the limiters have no effect. By using the conditions in Theorem 2, Chapter 2 it can be seen that the scheme reduces to a third order accurate scheme in smooth regions of flow on the linear advection equation if $\beta = \frac{1}{3}(2 - \nu_{i+\frac{1}{2}})$.

Note that β -scheme is **tvD** for any value of β , at least on the scalar equations. Suppose that $\{u_i^\beta\}$ is the solution obtained by applying one timestep of the β -scheme on the scalar equation with data $\{u_i\}$. For all i ,

$$u_i^{(\beta)} = \beta u_i^{(LW)} + (1 - \beta) u_i^{(WB)} \quad (4.58)$$

for all i , with the obvious meaning for $u_i^{(LW)}$ and $u_i^{(WB)}$. Assume all wavespeeds are positive. Since (4.54) and (4.55) give **tvD** schemes,

$$0 \leq \frac{u_i^{(LW)} - u_i}{u_{i-1} - u_i} \leq 1 \quad (4.59)$$

and

$$0 \leq \frac{u^{(WB)} - u_i}{u_{i-1} - u_i} \leq 1 \quad (4.60)$$

From (4.58):

$$\frac{u^{(\beta)} - u_i}{u_{i-1} - u_i} = \frac{\beta u_i^{(LW)} + (1 - \beta) u_i^{(WB)} - u_i}{u_{i-1} - u_i} \quad (4.61)$$

$$= \beta \left(\frac{u^{(LW)} - u_i}{u_{i-1} - u_i} \right) + (1 - \beta) \left(\frac{u^{(WB)} - u_i}{u_{i-1} - u_i} \right) \quad (4.62)$$

So from (4.59) and (4.60):

$$0 \leq \frac{u^{(\beta)} - u_i}{u_{i-1} - u_i} \leq \beta + (1 - \beta) = 1 \quad (4.63)$$

Thus the β -scheme is **tvD** provided **tvD** limiters ϕ, ψ are used in (4.57).

4.5 Numerical Experiments

Numerical results are now presented for the Riemann problem based extension of the Warming-Beam scheme discussed in §4.3. To confirm the **tvD** analysis of §4.2 and §3.4.2 results are first presented for the one dimensional inviscid Burgers equation

$$u_t + f(u)_x = 0 \quad f(u) = \frac{1}{2}u^2 \quad (4.64)$$

The computational domain is divided into 100 computational cells, initial conditions are taken as

$$u(x, 0) = \begin{cases} 1.0 & \text{if } x \leq 0.2 \\ 0.0 & \text{otherwise} \end{cases} \quad (4.65)$$

and results are taken at time $t = 1.0$. Boundary conditions are transmissive. Results are shown for maximum Courant numbers ν_{max} of 0.4, 0.8, 1.0, 1.4, 1.8, 2.0. In Fig 4.2, results are shown with the second order scheme used everywhere, i.e all limiters 'turned off'. Note that the spurious oscillations run ahead of waves when $\nu_{max} \leq 1$, due to the leading phase error of the scheme. In Fig. 4.3, results are shown for the case when limiters (3.128) and (4.39) are used. They show the validity of the **tvD** analysis. Note, however, that the exact solution is not obtained when $\nu_{max} = 2$, as is the case for the **WAFD2** scheme developed in Chapter 3 when applied to the inviscid Burger's equation.

The rest of the results shown here are for the one-dimensional Euler equations of Gas Dynamics

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix}_x = 0 \quad (4.66)$$

where ρ is the density, u is the velocity, p is the pressure, $E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$ is the total energy and $\gamma = 1.4$ is the ratio of specific heats. The computational domain is again taken as the interval $x \in [0, 1]$ divided into 100 cells. Transmissive boundary conditions are used. Initial conditions are taken as left (L) and right (R) states

$$\begin{aligned}\rho_L &= 1.0 & \rho_R &= 0.125 \\ u_L &= 0.0 & u_R &= 0.0 \\ p_L &= 1.0 & p_R &= 0.1\end{aligned}\tag{4.67}$$

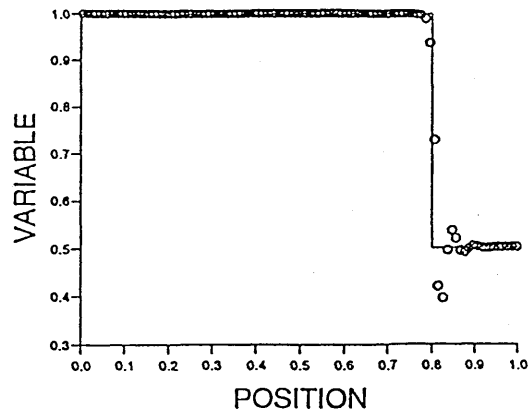
separated by a discontinuity at $x = 0.5$. Output is taken at time $t = 0.5$. This is the problem introduced by Sod [49]. Figures (4.4) — (4.8) show results when second order accuracy is applied everywhere, with maximum Courant numbers of 0.4, 0.8, 1.2, 1.6, 2.0 respectively. Note that when the maximum Courant number is 2.0, no spurious oscillations appear near the rarefaction wave. This is because the Courant numbers of rarefaction waves in local Riemann problems in this region are close to unity, so both leading and lagging phase errors are generated and mutually cancel. Figures (4.9) — (4.13) show results at the same Courant numbers when the limiters (3.128) and (4.39) are used. The scheme has also been tried on the blast wave problem of Woodard and Colella [82] (see Chapter 3). However, the scheme appears not to be robust enough to survive this test. The problem seems to be that because of the leading phase error of the scheme when $\nu \leq 1$, any small errors violating the tvd condition are propagated ahead of the strong shock waves in this problem and into the region where the pressure is 10^{-2} . The oscillation therefore produces negative pressures and densities and the code then crashes. More work is needed to improve the robustness of this scheme. The extension of Fromm's scheme and the third order scheme have not yet been tested numerically.

4.6 Conclusions

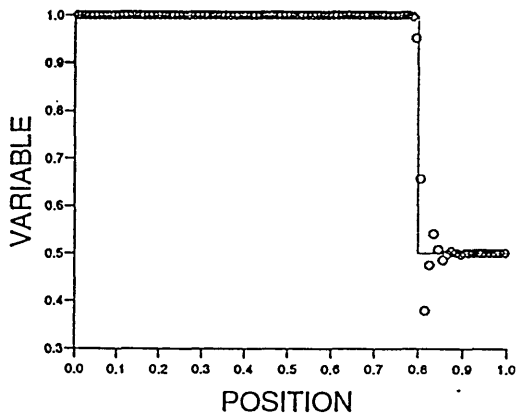
The Warming Beam scheme on the linear advection equation has been extended to produce a Riemann problem based, tvd method for the solution of non-linear systems of hyperbolic conservation laws in one space dimension that is second order accurate in space and time in smooth flow and linearly stable up to a maximum Courant number of two. Numerical experiments indicate that it can be used for problems containing relatively weak shocks, but the scheme appears not to be robust enough when strong shocks are present in the solution.

A one-parameter class of Riemann problem based, tvd schemes for non-linear systems that are second order accurate in smooth flow has also been suggested. It has been generated via weighted averages of the extension of the Warming-Beam scheme of §4.2 and the waf scheme. On the linear advection equation, it contains a scheme that reduces to Fromm's scheme in smooth flow and a scheme that is third order accurate in smooth flow, although these schemes have not yet been tested numerically.

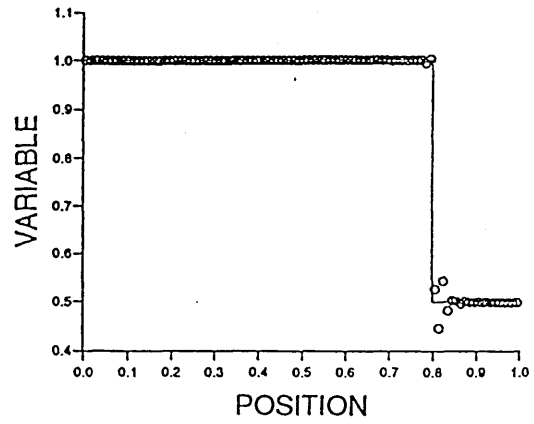
Courant number coefficient: 0.4



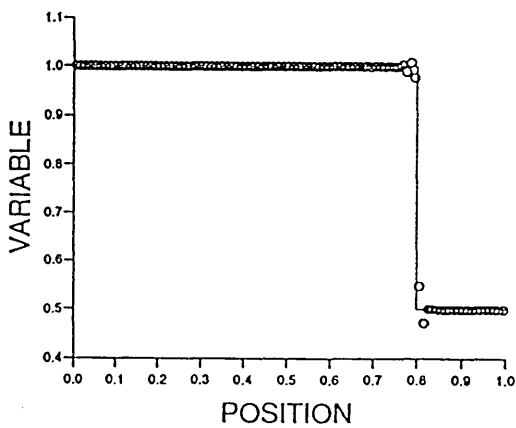
Courant number coefficient: 0.8



Courant number coefficient: 1.2



Courant number coefficient: 1.6



Courant number coefficient: 2.0

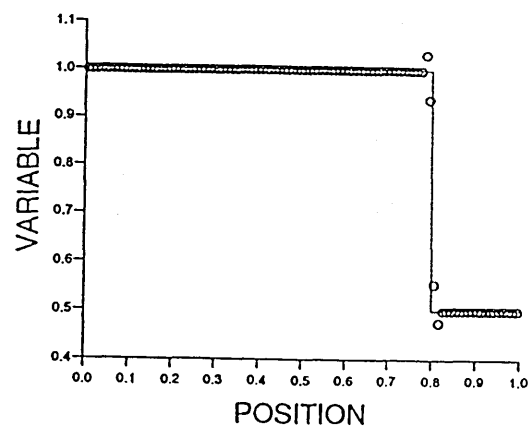
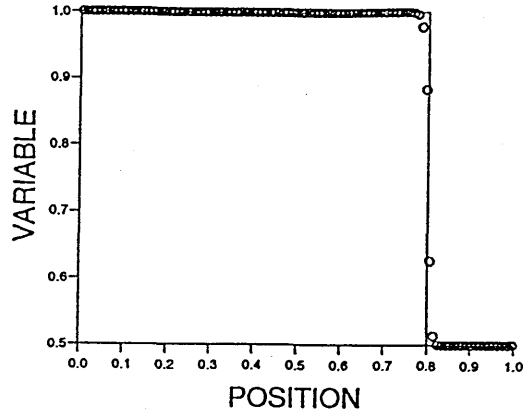
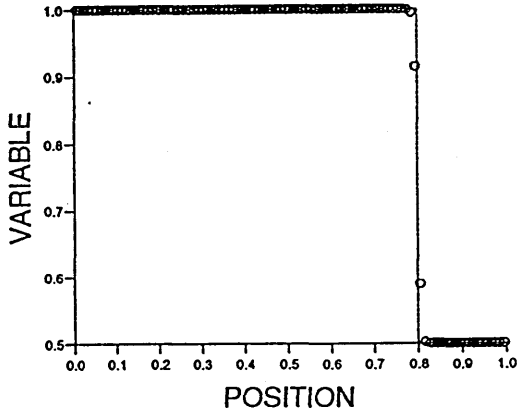


Figure 4.2: Results from the extension of the Warming Beam scheme applied to the inviscid Burgers equation with second order accuracy everywhere.

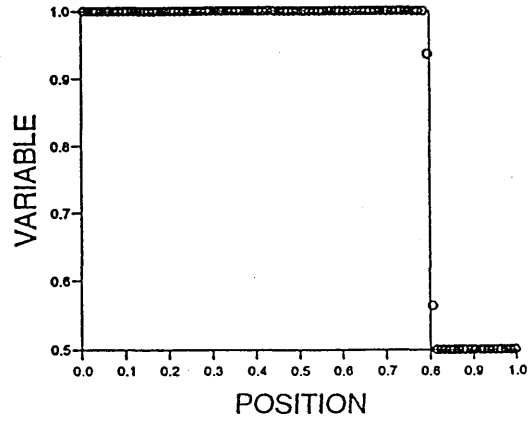
Courant number coefficient: 0.4



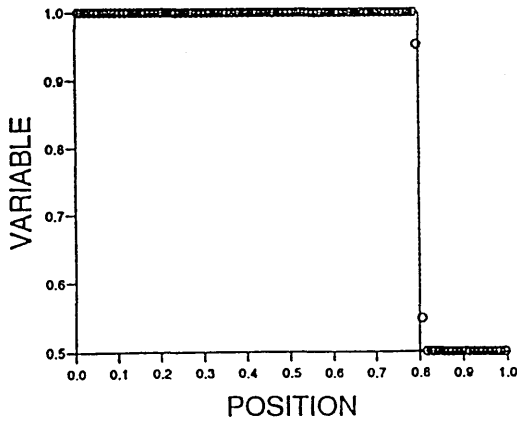
Courant number coefficient: 0.8



Courant number coefficient: 1.2



Courant number coefficient: 1.6



Courant number coefficient: 2.0

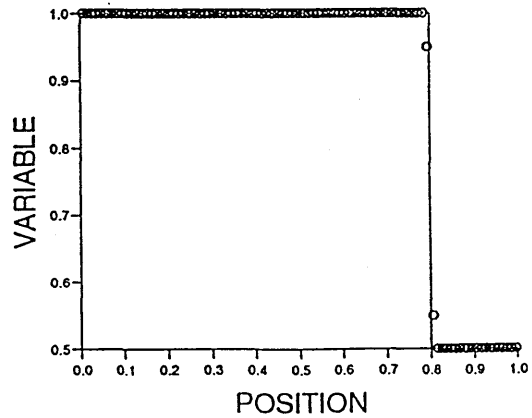


Figure 4.3: Results from the extension of the Warming Beam scheme applied to the inviscid Burgers equation using the limiters (3.128) and (4.39).

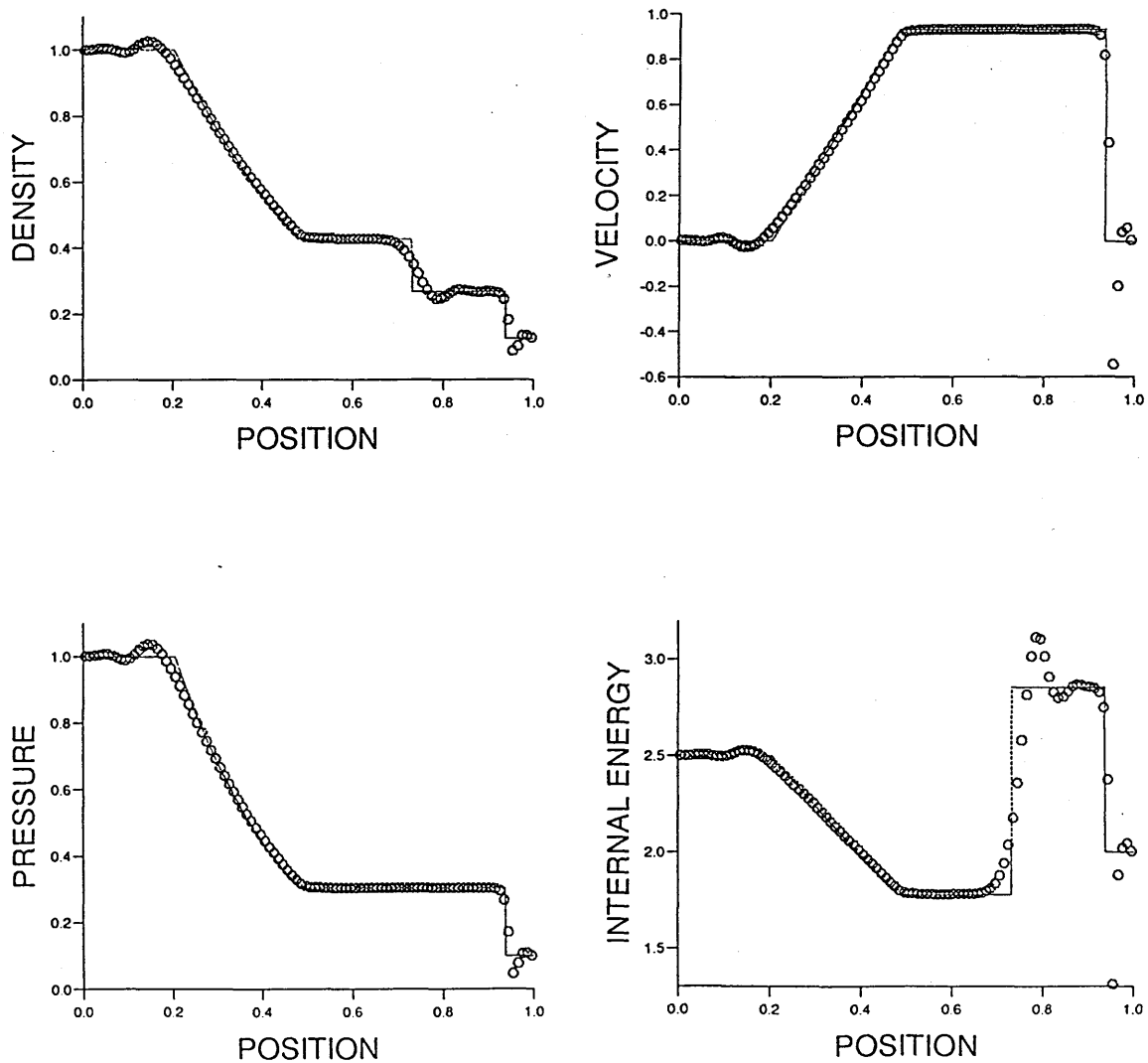


Figure 4.4: Results from the extension of the Warming Beam scheme applied to Sod's problem with second order accuracy everywhere. A Courant number coefficient of 0.4 was used.

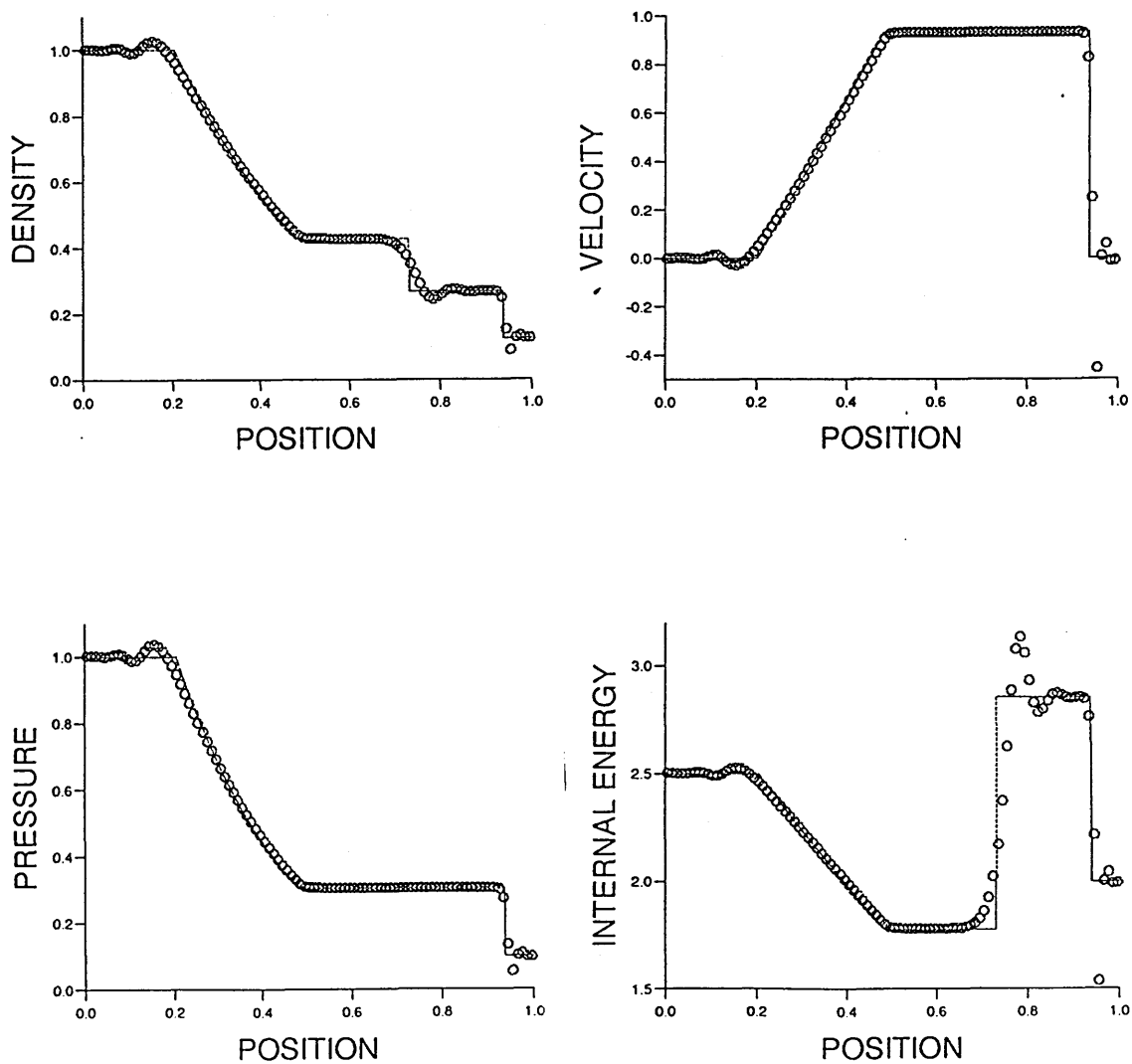


Figure 4.5: Results from the extension of the Warming Beam scheme applied to Sod's problem with second order accuracy everywhere. A Courant number coefficient of 0.8 was used.

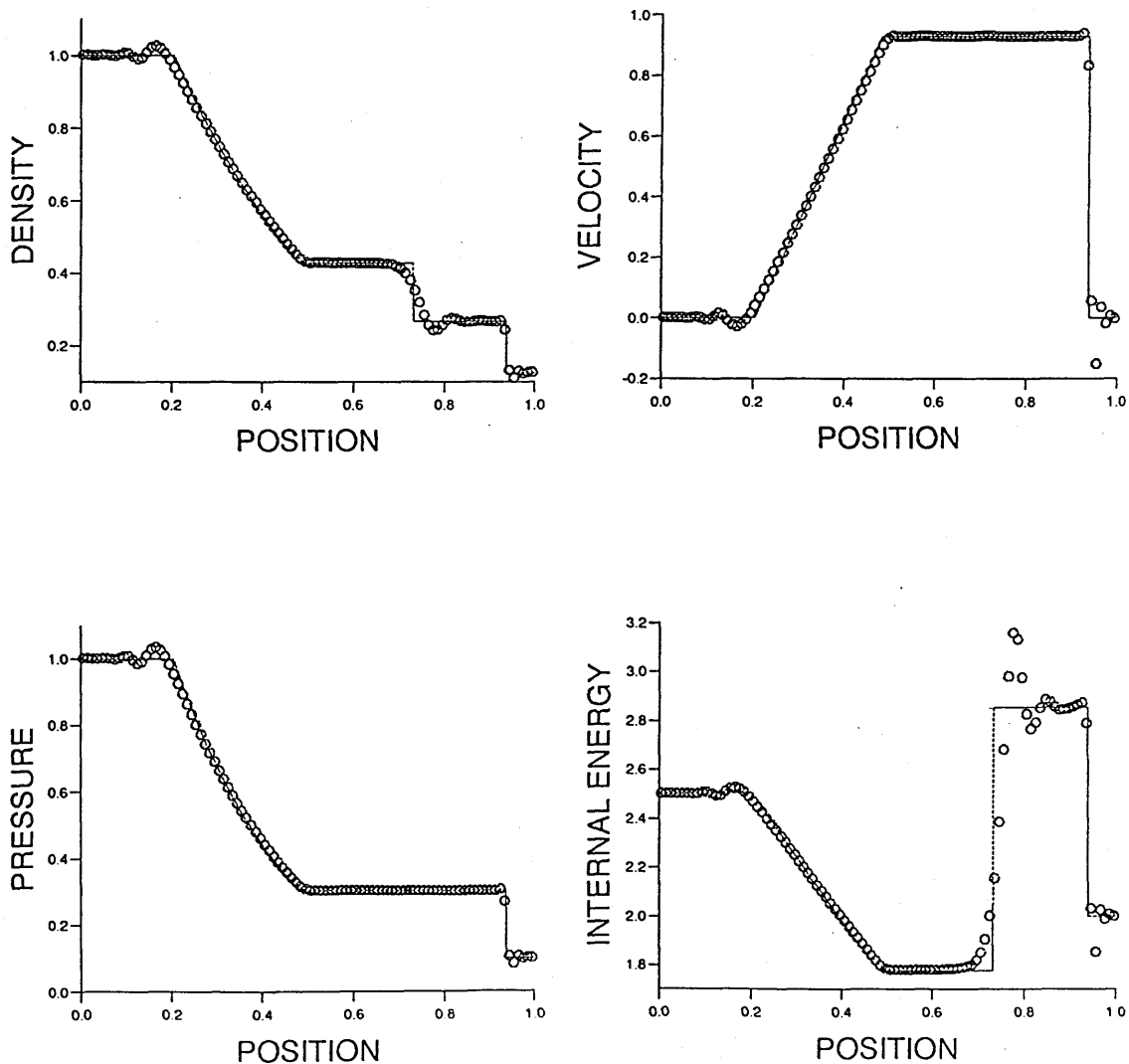


Figure 4.6: Results from the extension of the Warming Beam scheme applied to Sod's problem with second order accuracy everywhere. A Courant number coefficient of 1.2 was used.

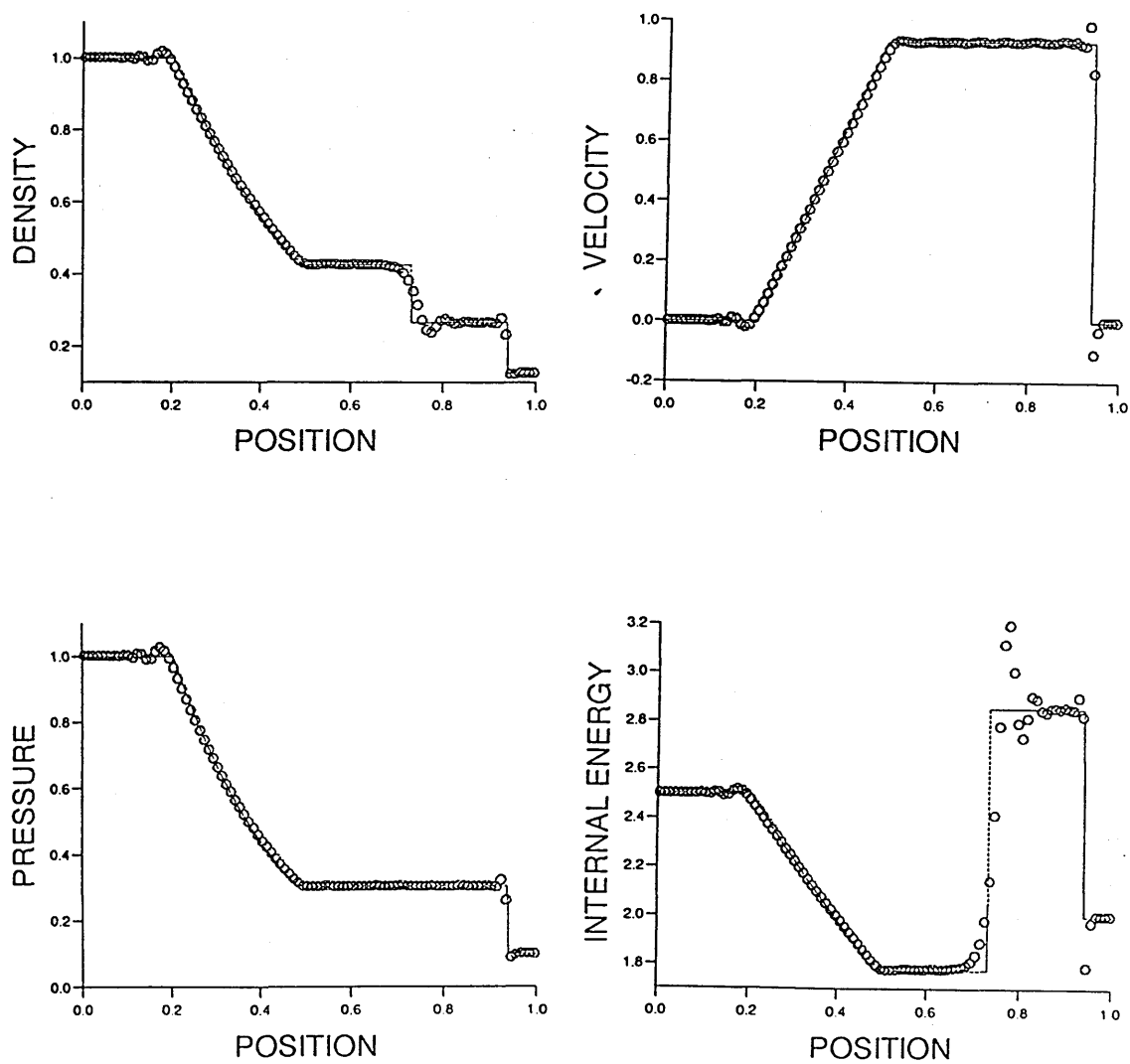


Figure 4.7: Results from the extension of the Warming Beam scheme applied to Sod's problem with second order accuracy everywhere. A Courant number coefficient of 1.6 was used.

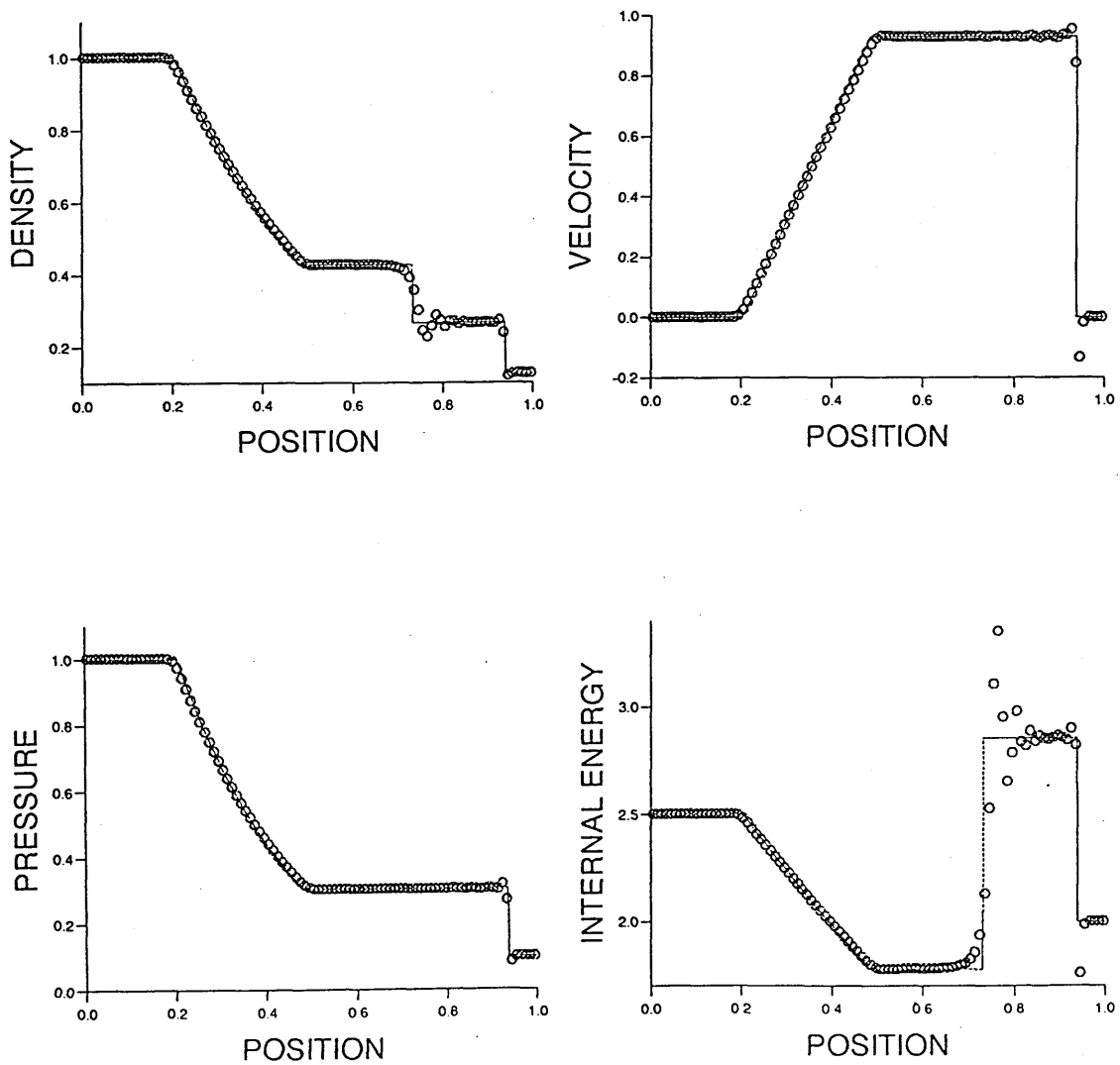


Figure 4.8: Results from the extension of the Warming Beam scheme applied to Sod's problem with second order accuracy everywhere. A Courant number coefficient of 2.0 was used.

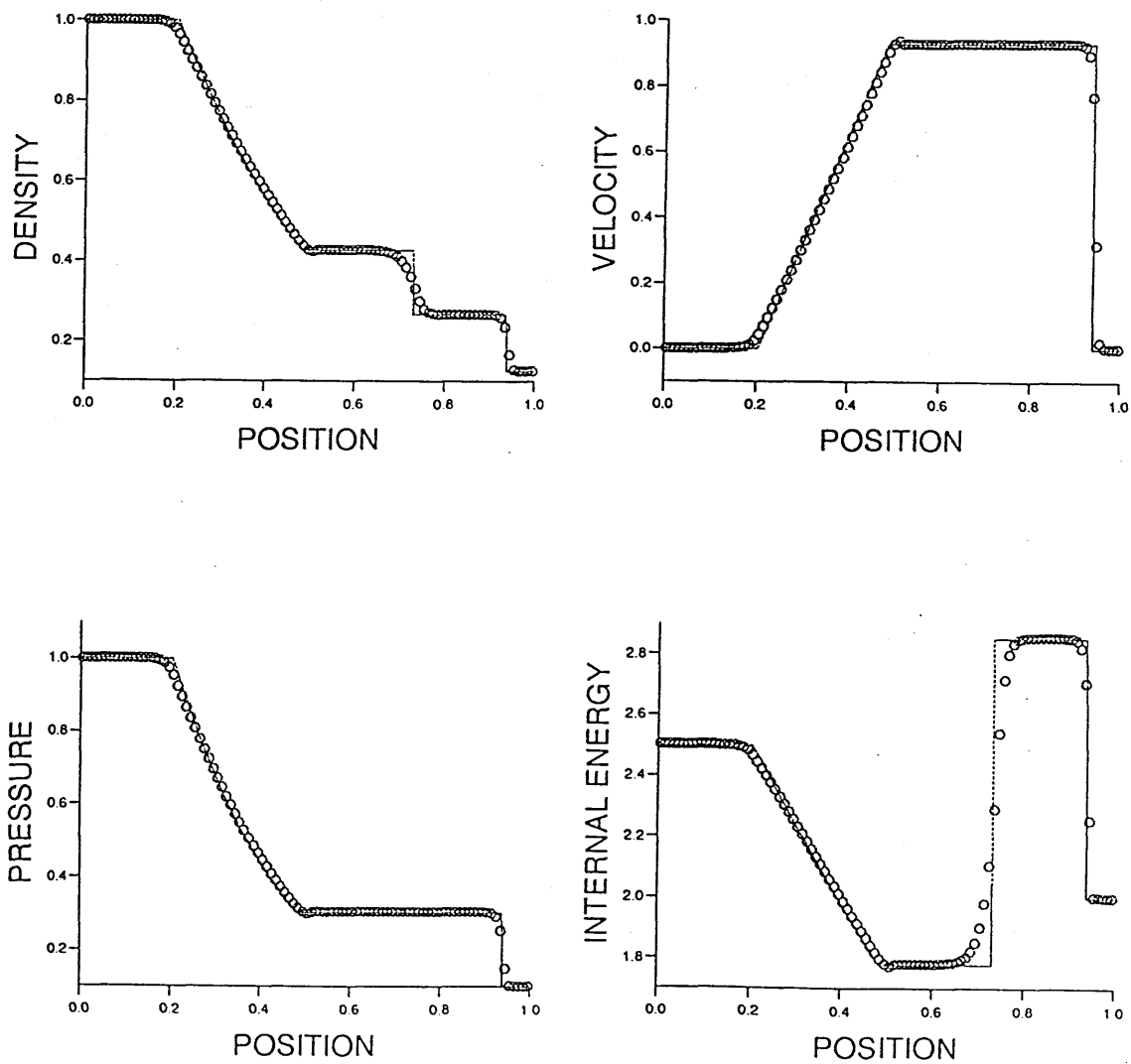


Figure 4.9: Results from the extension of the Warming Beam scheme applied to Sod's problem with the limiter (4.39). A Courant number coefficient of 0.4 was used.

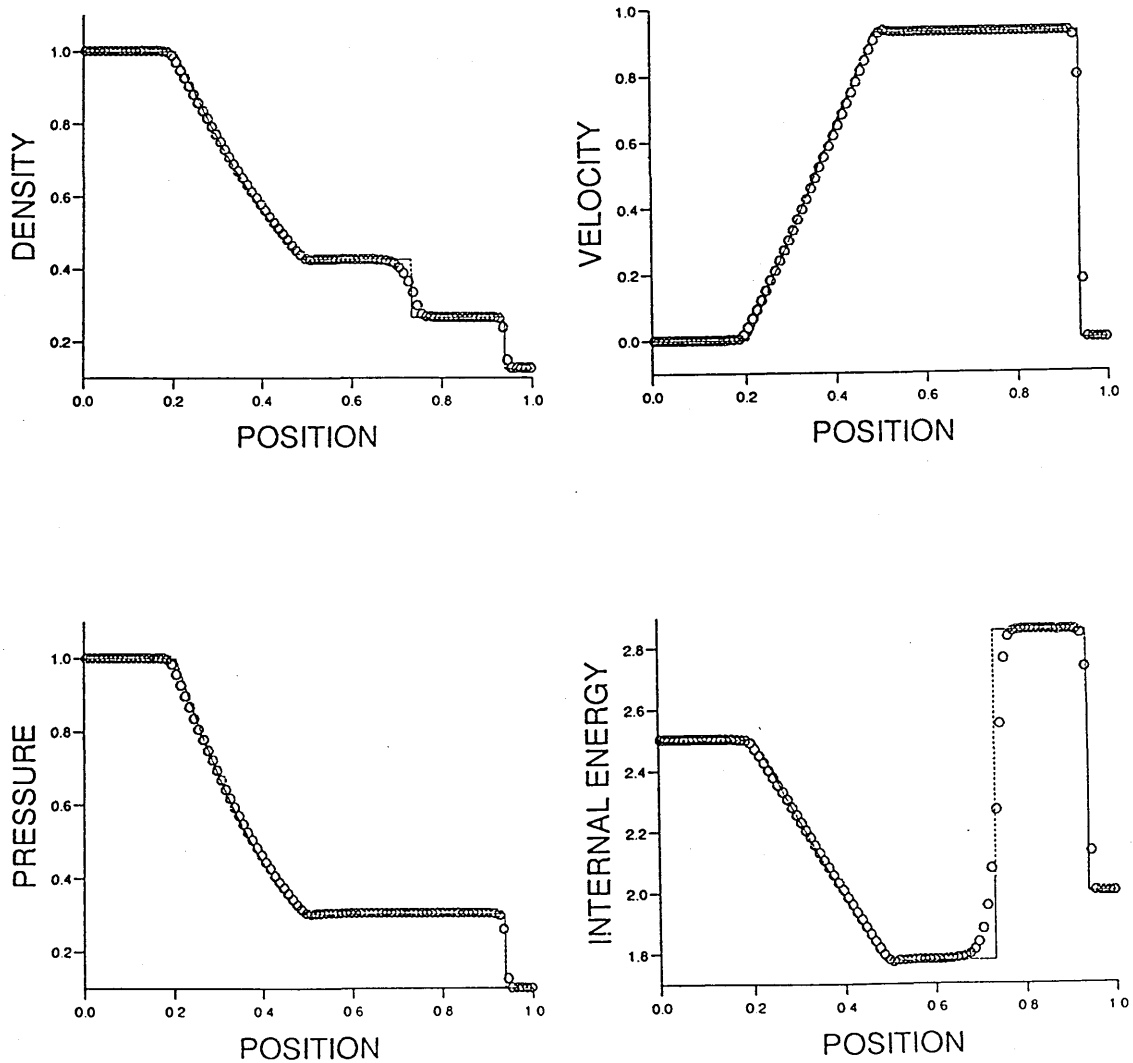


Figure 4.10: Results from the extension of the Warming Beam scheme applied to Sod's problem with the limiter (4.39). A Courant number coefficient of 0.8 was used.

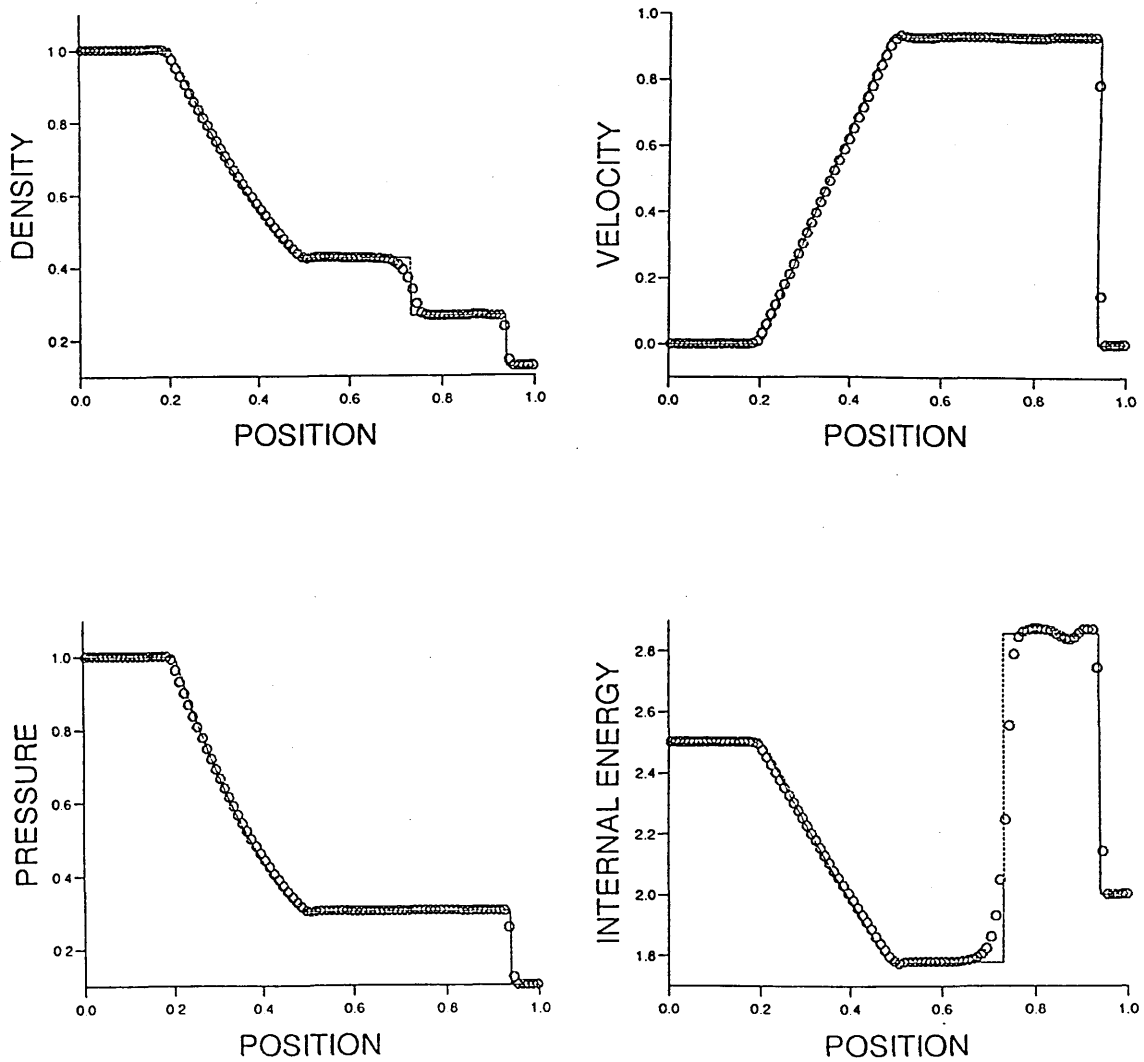


Figure 4.11: Results from the extension of the Warming Beam scheme applied to Sod's problem with with the limiters (3.128) and (4.39). A Courant number coefficient of 1.2 was used.

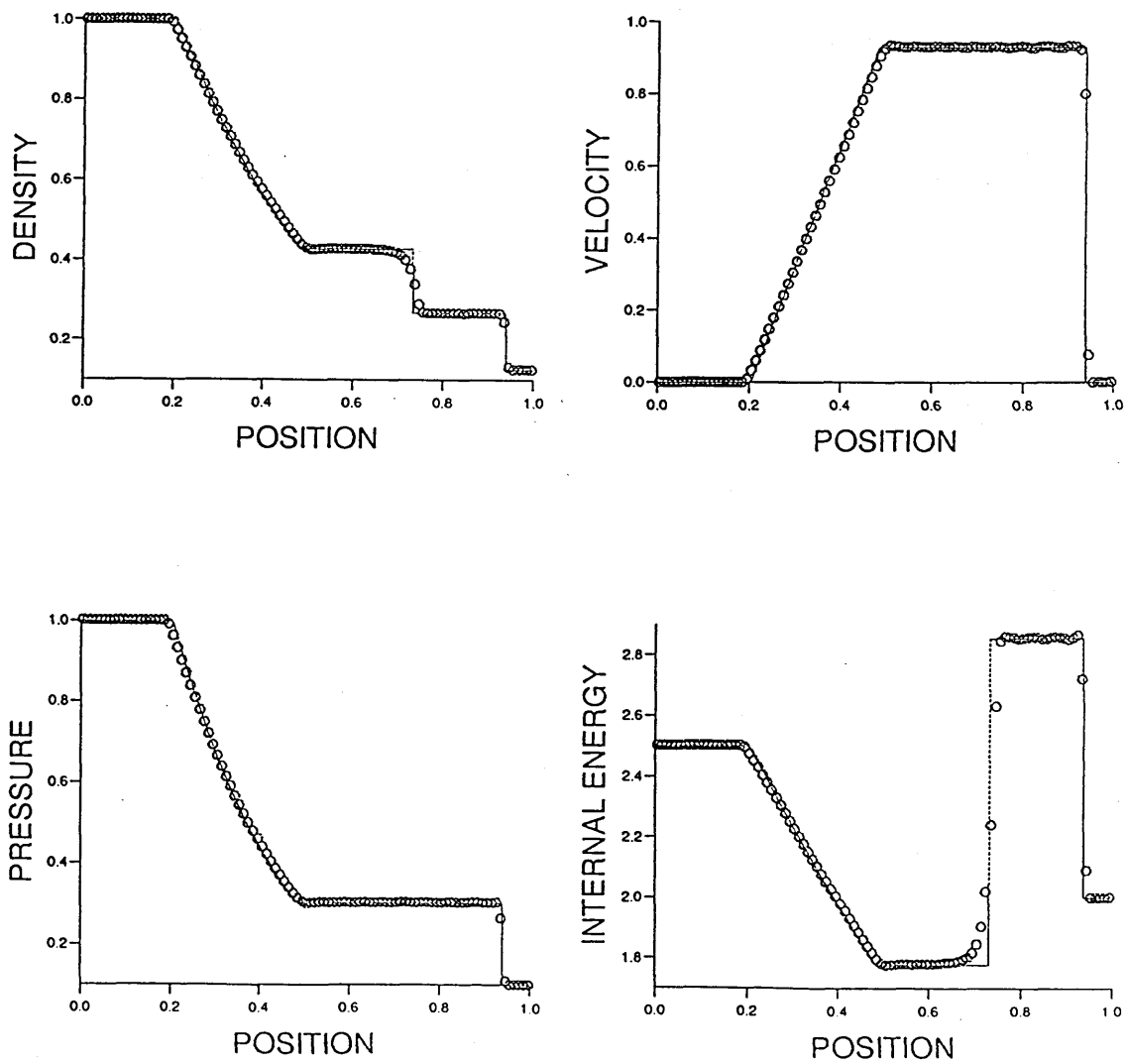


Figure 4.12: Results from the extension of the Warming Beam scheme applied to Sod's problem with with the limiters (3.128) and (4.39). A Courant number coefficient of 1.6 was used.

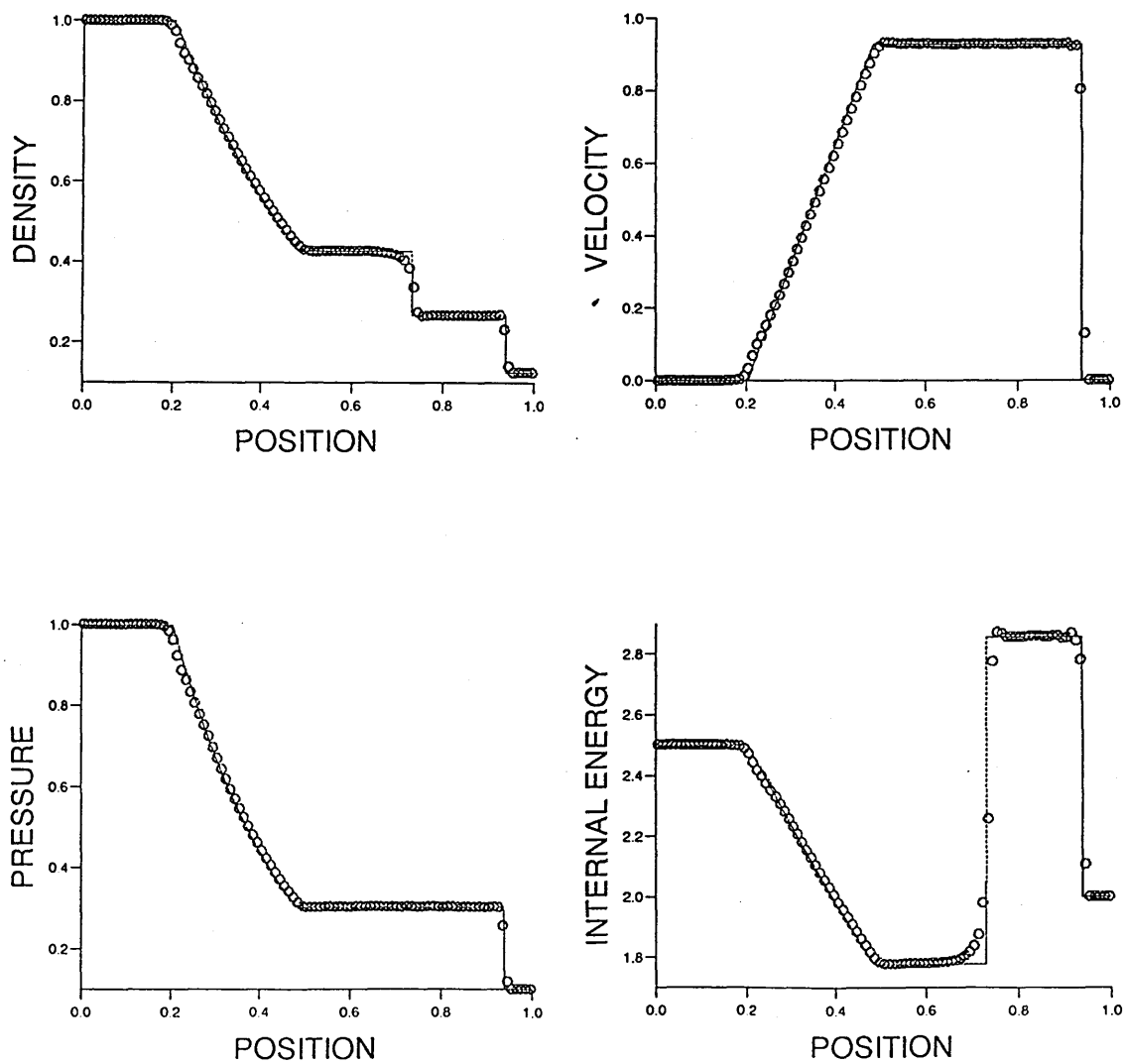


Figure 4.13: Results from the extension of the Warming Beam scheme applied to Sod's problem with the limiters (3.128) and (4.39). A Courant number coefficient of 2.0 was used.

Chapter 5

Multidimensional WAF-Type Schemes for Linear Advection

5.1 Introduction

In this chapter the **waf** approach in one space dimension, as discussed in Chapter 2, is extended to generate two and three dimensional finite volume schemes for the linear advection equation. The aim is to illustrate the approach rather than provide an exhaustive account of all possibilities. Therefore, after defining an intercell flux for the general case, the discussion will be restricted to regular, Cartesian grids, data will be assumed constant within each cell and only Courant numbers less than unity will be considered. Applications to curvilinear or unstructured grids are also possible, as well as **muscl** type schemes where the data is reconstructed or sloped and 'large timestep' schemes that are stable for Courant numbers are greater than unity, but these are not considered here. Schemes that can be generated from combinations of exact integration and the midpoint rule in space and time will be presented and their accuracy and stability will be discussed.

The chapter is organised as follows. In §5.2, some two dimensional schemes are derived and their accuracy is discussed. In §5.3 two three dimensional schemes are derived: one will be first order accurate and the other will be second order accurate in space and time. The stability of the schemes generated in §5.2 and §5.3 is then discussed in §5.4.

5.2 The WAF Approach in Two Space Dimensions

In two space dimensions the linear advection equation can be written in conservative form as

$$u_t + f_x + g_y = 0 \quad (5.1)$$

where $f = au$ and $g = bu$ are flux components, a and b are the speeds of propagation, assumed constant and positive here, and (x, y) are the two space coordinates.

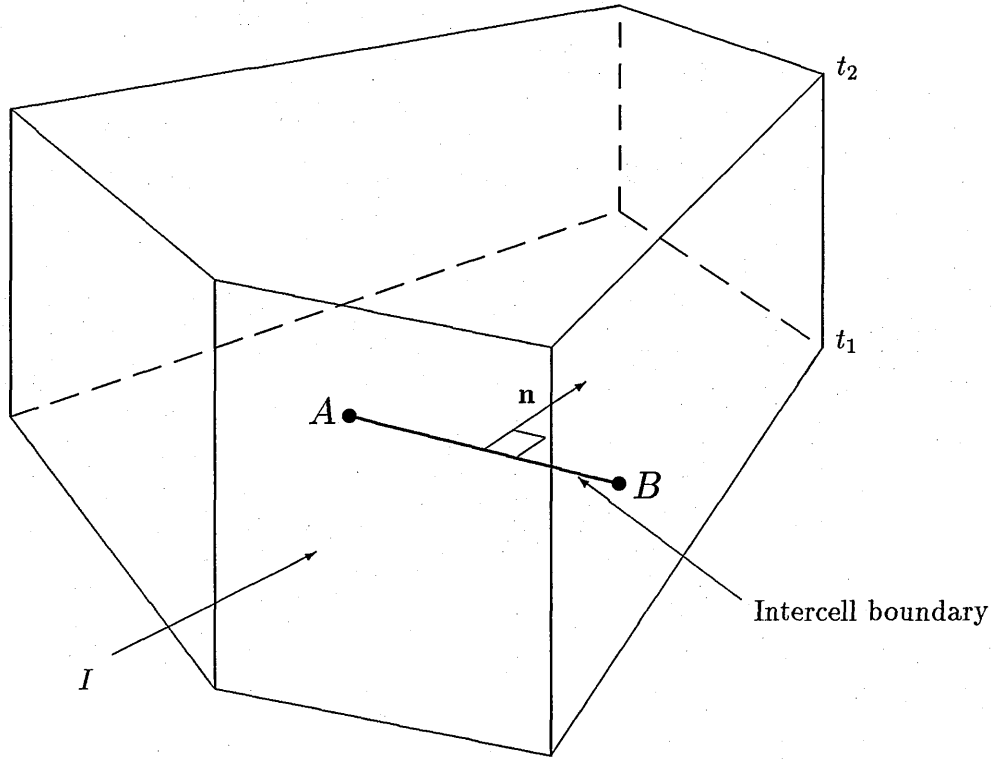


Figure 5.1: The general integration range.

For a general two-dimensional grid (structured or unstructured), a **WAF** intercell flux $\mathbf{e}_{AB}$ for some intercell boundary AB can be defined as

$$\mathbf{e}_{AB} = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I \mathbf{e} \cdot \mathbf{n} \, dx \, dy \, dt \quad (5.2)$$

where I is some space integration range in (x, y) , $V(I)$ is the area of I , $\mathbf{e} = (au, bu)$ is the general flux function and $\mathbf{n}$ is the unit outward facing vector normal to the intercell boundary AB . See Fig. 5.1. The integrand in (5.2) is the component of flux perpendicular to the boundary AB evaluated on the exact solution of the linear advection equation (5.1) with some initial conditions yet to be specified. The choice of these initial conditions, the grid to be used, the space integration range I , the time limits t_1 and t_2 and the method of integration used in space and time (e.g. exact integration, midpoint rule, etc) will determine the actual scheme.

Some schemes resulting from the general flux (5.2) on regular, Cartesian grids will be considered in detail here. Let Δx , Δy be the mesh spacing and Δt be the timestep. The schemes will be generated in the two dimensional conservative form

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j}] - \frac{\Delta t}{\Delta y} [g_{i,j+\frac{1}{2}} - g_{i,j-\frac{1}{2}}] \quad (5.3)$$

where

$$u_{i,j}^n = \frac{1}{\Delta x \Delta y} \int_{\Delta_{i,j}} u(x, y, t^n) dx dy \quad (5.4)$$

is the integral average of u in cell $\Delta_{(i,j)}$ at time level n and $f_{i+\frac{1}{2},j}$ and $g_{i,j+\frac{1}{2}}$ are intercell fluxes in the x and y directions respectively. When applied to Cartesian grids, the formula (5.2) gives intercell fluxes $f_{i+\frac{1}{2},j}$ in the x -direction for the conservative form (5.3) as

$$f_{i+\frac{1}{2},j} = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I f(u^*(x, y, t)) dx dy dt \quad (5.5)$$

where $u^*(x, y, t)$ is the exact solution to the initial value problem defined by (5.1) with initial data now chosen to be the data u^n at time level n kept piecewise constant within cells, i.e:

$$u^*(x, y, t) = u^n(x - at, y - bt) \quad (5.6)$$

The fluxes $g_{i,j+\frac{1}{2}}$ in the y -direction are defined similarly. Define the directional Courant numbers as $\nu_x = a \frac{\Delta t}{\Delta x}$ and $\nu_y = b \frac{\Delta t}{\Delta y}$. Time limits of $t_1 = 0$ and $t_2 = \Delta t$ are taken here, where Δt is chosen such that $\nu_x \leq 1$ and $\nu_y \leq 1$. For the flux $f_{i+\frac{1}{2},j}$ the space integration range I is taken as:

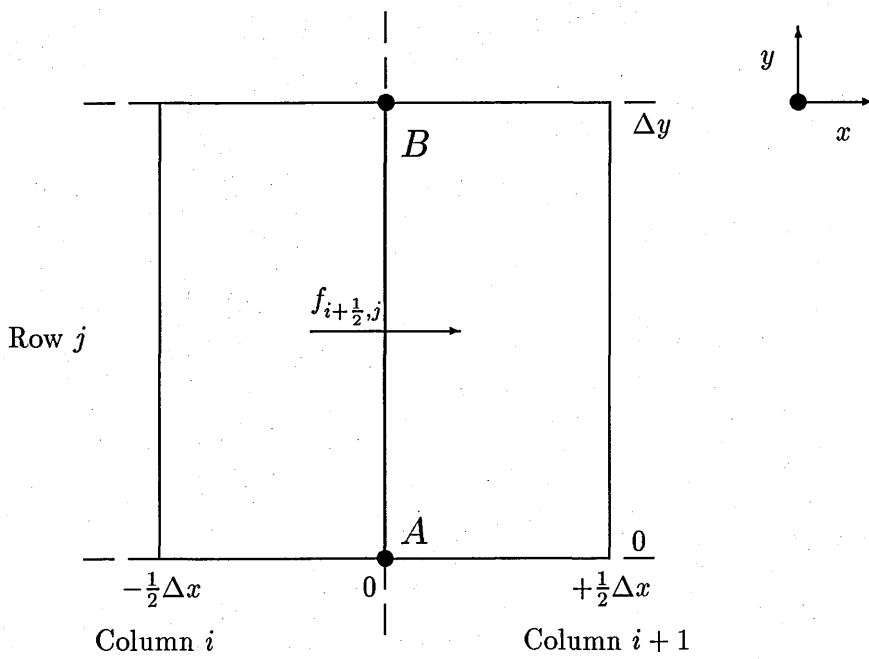
$$I = \left[-\frac{1}{2}\Delta x, \frac{1}{2}\Delta x \right] \times [0, \Delta y] \quad (5.7)$$

where, in local coordinates, $x = 0$ lies on the boundary and $y = 0, y = \Delta y$ lie on the ends of the boundary. For the flux $g_{i,j+\frac{1}{2}}$ the obvious changes are made. See Fig. 5.2. Since a and b are assumed positive here, this choice of Courant numbers and integration range and the choice of piecewise constant data in cells restricts the number of states that influence $f_{i+\frac{1}{2},j}$ to four: $u_{i,j}^n, u_{i+1,j}^n, u_{i,j-1}^n$ and $u_{i+1,j-1}^n$. These are laid out in the four quadrants of a square and form the initial conditions of a two dimensional Riemann problem.

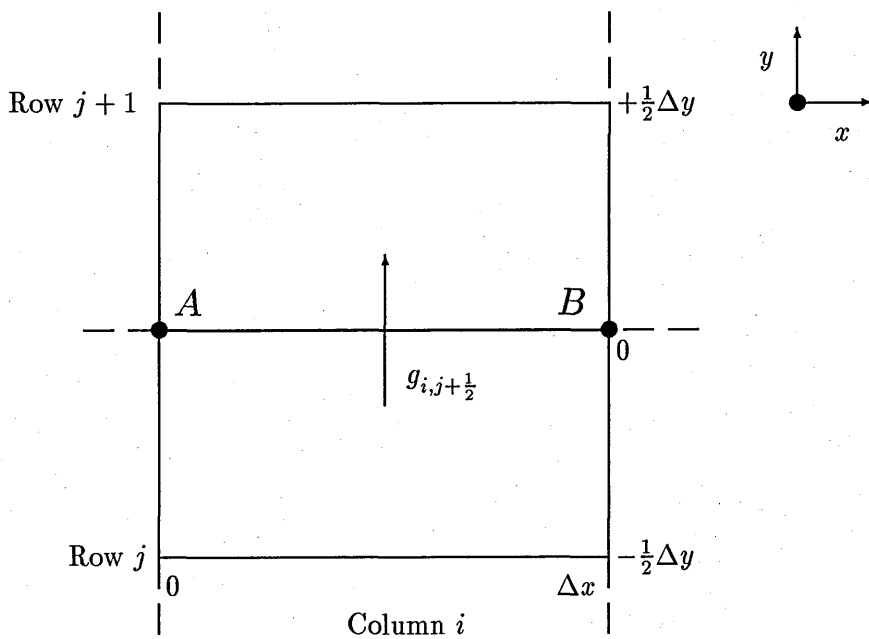
5.2.1 Exact integration in all Directions

Consider first the case of exact integration in time and both space directions in (5.5). Assume that $\nu_x \leq \frac{1}{2}$ and $\nu_y \leq \frac{1}{2}$ to restrict the number of waves entering the integration range. The evaluation of the integral (5.5) is, infact, not as hard as it might seem: the exact solution to the two dimensional Riemann problem for (5.1) is given by (5.6) for all times, with any states in the four quadrants. See Fig. 5.3. The point A_1 moves to the point A_2 and the point B moves out of the integration range. At any given point (x, y) in I and at any $t \in [t_1, t_2]$, the flux function $f(u) = au$ can be calculated from (5.6). To obtain the intercell flux, this flux function is integrated exactly through space and time. This gives the intercell flux $f_{i+\frac{1}{2},j}$ as:

$$\begin{aligned} f_{i+\frac{1}{2},j} = & \frac{1}{12} [6 + 6\nu_x - 3\nu_y - 4\nu_x\nu_y] f_{i,j} + \frac{1}{12} [6 - 6\nu_x - 3\nu_y + 4\nu_x\nu_y] f_{i+1,j} \\ & + \frac{1}{12} [3\nu_y + 4\nu_x\nu_y] f_{i,j-1} + \frac{1}{12} [3\nu_y - 4\nu_x\nu_y] f_{i+1,j-1} \end{aligned} \quad (5.8)$$



(a) Space integration range for the fluxes $f_{i+\frac{1}{2},j}$.



(b) Space integration range for the fluxes $g_{i,j+\frac{1}{2}}$.

Figure 5.2: The two-dimensional space integration ranges.

where $f_{i,j} = f(u_{i,j}^n) = au_{i,j}^n$. The intercell fluxes $g_{i,j+\frac{1}{2}}$ follow by symmetry as

$$\begin{aligned} g_{i,j+\frac{1}{2}} &= \frac{1}{12} [6 + 6\nu_y - 3\nu_x - 4\nu_x\nu_y] g_{i,j} + \frac{1}{12} [6 - 6\nu_y - 3\nu_x + 4\nu_x\nu_y] g_{i,j+1} \\ &\quad + \frac{1}{12} [3\nu_x + 4\nu_x\nu_y] f_{i-1,j} + \frac{1}{12} [3\nu_x - 4\nu_x\nu_y] f_{i-1,j+1} \end{aligned} \quad (5.9)$$

where $g_{i,j} = g(u_{i,j}^n) = bu_{i,j}^n$. Substituting both $f_{i+\frac{1}{2},j}$ and $g_{i,j+\frac{1}{2}}$ into the conservative formula (5.3) gives the full scheme as:

$$\begin{aligned} u_{i,j}^{n+1} &= \left[1 - \nu_x^2 - \nu_y^2 + \frac{2}{3} \nu_x \nu_y (\nu_x + \nu_y) \right] u_{i,j}^n \\ &\quad + \frac{1}{12} \nu_y [6(\nu_y - 1) + \nu_x(3 - 4\nu_y)] u_{i,j+1}^n \\ &\quad + \frac{1}{12} \nu_x [6(\nu_x - 1) + \nu_y(3 - 4\nu_x)] u_{i+1,j}^n \\ &\quad + \frac{1}{12} \nu_x \nu_y (4\nu_x - 3) u_{i+1,j-1}^n + \frac{1}{12} \nu_x \nu_y (4\nu_y - 3) u_{i-1,j+1}^n \\ &\quad + \frac{1}{12} \nu_y [6(1 + \nu_y) - \nu_x(3 + 4\nu_y + 8\nu_x)] u_{i,j-1}^n \\ &\quad + \frac{1}{12} \nu_x [6(1 + \nu_x) - \nu_y(3 + 4\nu_x + 8\nu_y)] u_{i-1,j}^n \\ &\quad + \frac{1}{12} \nu_x \nu_y [6 + 4(\nu_x + \nu_y)] u_{i-1,j-1}^n \end{aligned} \quad (5.10)$$

This scheme has an eight point stencil with both upwind and downwind components, as shown in Fig. 5.4. Note that when $\nu_x = 0$ or $\nu_y = 0$, the scheme reduces to the one dimensional Lax-Wendroff scheme. Scheme (5.10) appears to be new.

In appendix A, schemes for the two dimensional linear advection equation (5.1) of the form

$$u_{i,j}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta} u_{i+\alpha,j+\beta}^n \quad (5.11)$$

are shown to be p^{th} order accurate in space and time provided

$$\sum_{\alpha,\beta} \alpha^q \beta^r A_{\alpha,\beta} = (-\nu_x)^q (-\nu_y)^r \quad (5.12)$$

for all integer pairs (q, r) such that $q \geq 0$, $r \geq 0$ and $q + r \leq p$. Thus schemes with the stencil shown in Fig. 5.4 are first order accurate in space and time provided

$$\sum_{\alpha,\beta} A_{\alpha,\beta} = 1 \quad (5.13)$$

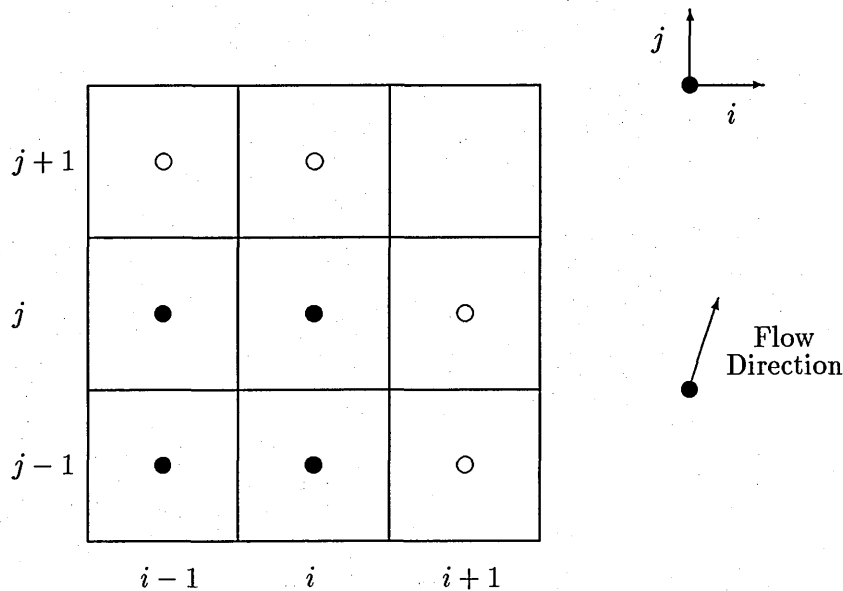


Figure 5.4: The stencil for the second order schemes: full circles denote upwind points, empty circles denote downwind points.

$$A_{1,0} + A_{1,-1} - A_{-1,1} - A_{-1,0} - A_{-1,-1} = -\nu_x \quad (5.14)$$

$$A_{0,1} + A_{-1,1} - A_{1,-1} - A_{0,-1} - A_{-1,-1} = -\nu_y \quad (5.15)$$

If the coefficients also satisfy

$$A_{1,0} + A_{1,-1} + A_{-1,1} + A_{-1,0} + A_{-1,-1} = \nu_x^2 \quad (5.16)$$

$$A_{0,1} + A_{-1,1} + A_{1,-1} + A_{0,-1} + A_{-1,-1} = \nu_y^2 \quad (5.17)$$

$$A_{-1,-1} - A_{1,-1} - A_{-1,1} = \nu_x \nu_y \quad (5.18)$$

then the scheme is second order accurate in space and time. The coefficients for scheme (5.10) satisfy all the conditions (5.13)–(5.18) so (5.10) is second order accurate in space and time. The scheme is therefore not monotonic and will suffer from spurious oscillations in regions of high gradient.

5.2.2 The Midpoint Rule in Space

Schemes generated by using exact integration in time and the midpoint rule in space in the general form (5.5) will now be considered. Assume that $\nu_x \leq \frac{1}{2}$ and $\nu_y \leq \frac{1}{2}$. There are three cases to consider: (i) the midpoint rule parallel to the boundary and exact integration perpendicular to the boundary; (ii) the midpoint rule perpendicular to the boundary and exact integration parallel to the boundary; (iii) the midpoint rule in both directions.

(i) *The Midpoint Rule Parallel to the Boundary*

Consider exact integration in time, the midpoint rule parallel to the boundary and exact integration perpendicular to the boundary. The integral form of the intercell flux (5.5) in this case is:

$$f_{i+\frac{1}{2},j} = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_0^{\Delta t} \int_{I \cap \{y=\frac{1}{2}\Delta y\}} f(u^*(x, \frac{1}{2}\Delta y, t)) dx dt \quad (5.19)$$

where the integration is over the plane $y = \frac{1}{2}\Delta y$. See Fig. 5.5(a). This gives the intercell flux $f_{i+\frac{1}{2},j}$ as:

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(1 + \nu_x)f_{i,j} + \frac{1}{2}(1 - \nu_x)f_{i+1,j} \quad (5.20)$$

This is the flux for the finite volume **waf** scheme put forward by Speares [50]. The flux $g_{i,j+\frac{1}{2}}$ follows similarly and the resulting scheme can be written in full as

$$\begin{aligned} u_{i,j}^{n+1} = & (1 - \nu_x^2 - \nu_y^2)u_{i,j}^n \\ & + \frac{1}{2}\nu_x(1 + \nu_x)u_{i-1,j}^n - \frac{1}{2}\nu_x(1 - \nu_x)u_{i+1,j}^n \\ & + \frac{1}{2}\nu_y(1 + \nu_y)u_{i,j-1}^n - \frac{1}{2}\nu_y(1 - \nu_y)u_{i,j+1}^n \end{aligned} \quad (5.21)$$

The flux (5.20) is the one-dimensional **waf** flux and is the most obvious flux for a two dimensional extension of the scheme. However, the coefficients of the scheme satisfy conditions (5.13)–(5.17) above, but condition (5.18) is only satisfied when $\nu_x = 0$ or $\nu_y = 0$, i.e when the flow is aligned to the grid. Hence the scheme is, in general, only first order accurate. Since at least one of the coefficients in (5.21) is negative for any choice of (ν_x, ν_y) , the scheme is not monotonic under any conditions.

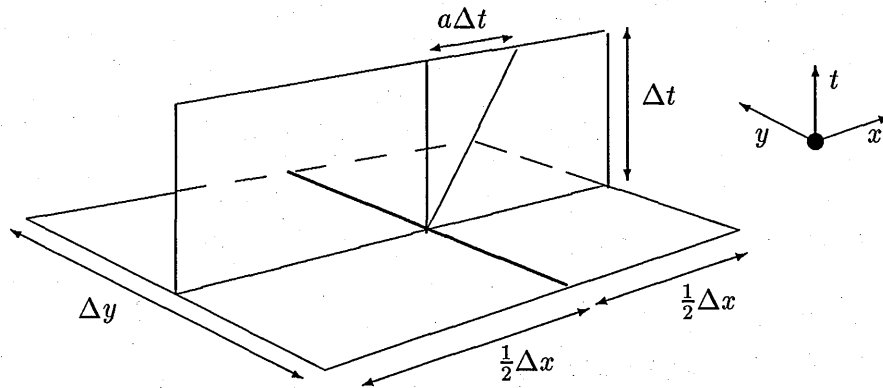
(ii) *The Midpoint Rule Perpendicular to the Boundary*

Consider exact integration in time, the midpoint rule perpendicular to the boundary and exact integration parallel to the boundary. The integral form of the intercell flux (5.5) simplifies in this case to:

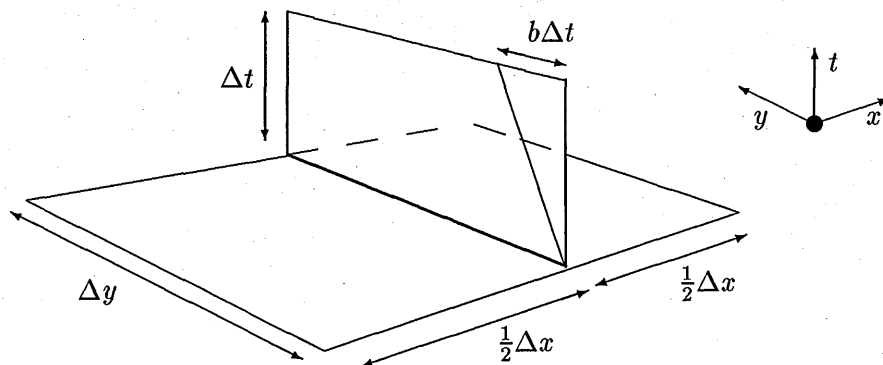
$$f_{i+\frac{1}{2},j} = \frac{1}{\Delta t} \frac{1}{\Delta y} \int_0^{\Delta t} \int_{I \cap \{x=0\}} f(u^*(0, y, t)) dy dt \quad (5.22)$$

where the integration is over the plane $x = 0$. See Fig. 5.5(b). This gives the intercell flux $f_{i+\frac{1}{2},j}$ as:

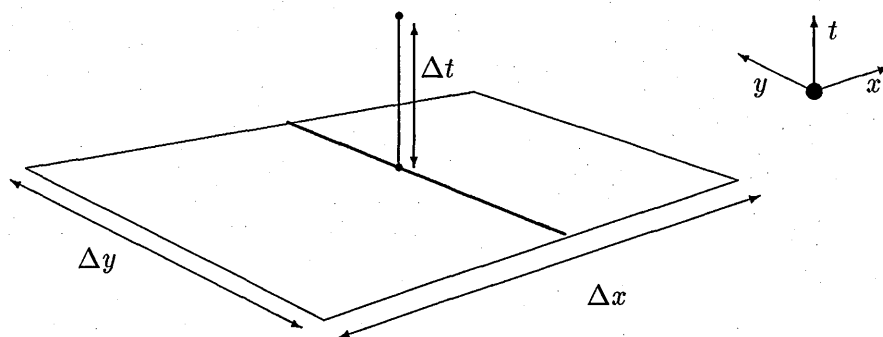
$$f_{i+\frac{1}{2},j} = \frac{1}{2}(2 - \nu_y)f_{i,j} + \frac{1}{2}\nu_y f_{i,j-1} \quad (5.23)$$



(a) Two dimensional integration range: exact integration in time, exact integration perpendicular to the boundary and the midpoint rule parallel to the boundary.



(b) Two dimensional integration range: exact integration in time, exact integration parallel to the boundary and the midpoint rule perpendicular to the boundary.



(c) Two dimensional integration range: exact integration in time and the midpoint rule in both space directions.

Figure 5.5: Combinations of exact integration in time and the midpoint rule in space

The flux $g_{i,j+\frac{1}{2}}$ follows similarly as:

$$g_{i,j+\frac{1}{2}} = \frac{1}{2}(2 - \nu_x)f_{i,j} + \frac{1}{2}\nu_x f_{i-1,j} \quad (5.24)$$

The scheme resulting from these fluxes is:

$$\begin{aligned} u_{i,j}^{n+1} = & (1 - \nu_x)(1 - \nu_y)u_{i,j}^n + \nu_y(1 - \nu_x)u_{i,j-1}^n \\ & + \nu_x(1 - \nu_y)u_{i-1,j}^n + \nu_x\nu_y u_{i-1,j-1}^n \end{aligned} \quad (5.25)$$

The coefficients in (5.25) satisfy conditions (5.13), (5.14), (5.15) and (5.18) above, but not (5.16) or (5.17), so the scheme is first order accurate. It is identical to the first order Corner Transport Upwind (CTU) scheme of Colella [8] on the linear equation (5.1). Since all coefficients of (5.25) are positive when $\nu_x \leq 1$ and $\nu_y \leq 1$, the scheme is monotonic under these restrictions although the scheme is derived under the assumption of $\nu_x \leq \frac{1}{2}$, $\nu_y \leq \frac{1}{2}$.

(iii) The Midpoint Rule in Both Directions

Consider exact integration in time and the midpoint rule in both space directions. The integral form of the intercell flux (5.4) simplifies in this case to:

$$f_{i+\frac{1}{2},j} = \frac{1}{\Delta t} \int_0^{\Delta t} f(u^*(0, \frac{1}{2}\Delta y, t)) dt \quad (5.26)$$

i.e the integration is along the line $x = 0$, $y = \frac{1}{2}\Delta y$. See Fig. 5.5(c). This gives the intercell flux $f_{i+\frac{1}{2},j}$ as:

$$f_{i+\frac{1}{2},j} = f_{i,j} \quad (5.27)$$

The flux $g_{i,j+\frac{1}{2}}$ follows similarly and the scheme can be written in full as

$$u_{i,j}^{n+1} = (1 - \nu_x - \nu_y)u_{i,j}^n + \nu_x u_{i-1,j}^n + \nu_y u_{i,j-1}^n \quad (5.28)$$

This is simply a finite volume version of Godunov's method. It satisfies conditions (5.13), (5.14) and (5.15) above, but not (5.16), (5.17) and (5.18). It is therefore first order accurate in space and time, as is already well known. All the coefficients of (5.28) are positive provided $\nu_x + \nu_y \leq 1$, so the scheme is monotonic under this restriction.

5.2.3 The Midpoint Rule in Time

Consider the midpoint rule in time combined with exact integration in both space directions. Assume that $\nu_x \leq 1$ and $\nu_y \leq 1$. The integral form of the intercell flux (5.4) simplifies in this case to:

$$f_{i+\frac{1}{2},j} = \frac{1}{\Delta x \Delta y} \int_I f(u^*(x, y, \frac{1}{2}\Delta t)) dx dy \quad (5.29)$$

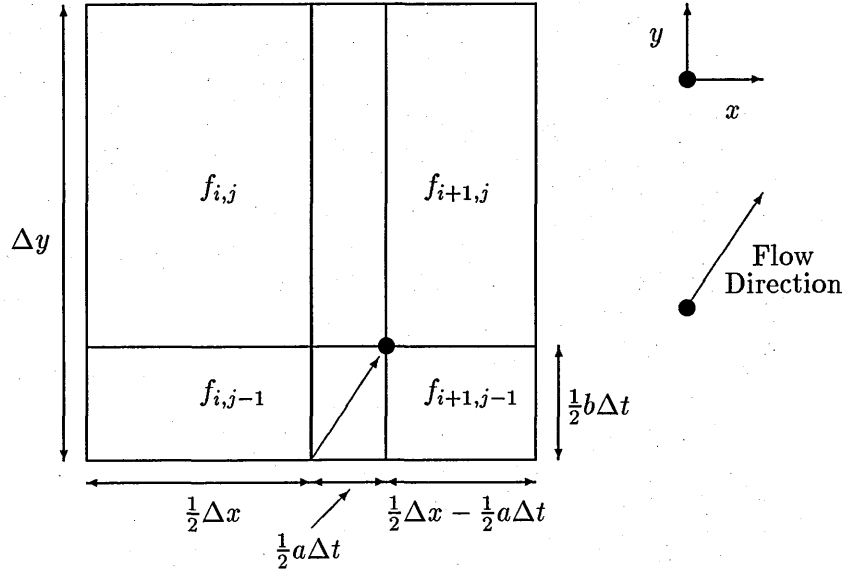


Figure 5.6: Two dimensional integration range: midpoint rule in time and exact integration in both space directions.

i.e the integration is over the plane $t = \frac{1}{2}\Delta t$. See Fig. 5.6. This gives intercell flux:

$$\begin{aligned}
 f_{i+\frac{1}{2},j} &= \frac{1}{4}(1+\nu_x)(2-\nu_y)f_{i,j} + \frac{1}{4}(1-\nu_x)(2-\nu_y)f_{i+1,j} \\
 &\quad + \frac{1}{4}(1+\nu_x)\nu_y f_{i,j-1} + \frac{1}{4}(1-\nu_x)\nu_y f_{i+1,j-1}
 \end{aligned} \tag{5.30}$$

The flux $g_{i,j+\frac{1}{2}}$ follows similarly as:

$$\begin{aligned}
 g_{i,j+\frac{1}{2}} &= \frac{1}{4}(1+\nu_y)(2-\nu_x)g_{i,j} + \frac{1}{4}(1-\nu_y)(2-\nu_x)g_{i,j+1} \\
 &\quad + \frac{1}{4}(1+\nu_y)\nu_x f_{i-1,j} + \frac{1}{4}(1-\nu_y)\nu_x f_{i-1,j+1}
 \end{aligned} \tag{5.31}$$

Substituting both $f_{i+\frac{1}{2},j}$ and $g_{i,j+\frac{1}{2}}$ into the conservative scheme (5.5) gives the scheme in full form as:

$$\begin{aligned}
 u_{i,j}^{n+1} &= \left[1 - \frac{1}{2}\nu_x^2(2-\nu_y) - \frac{1}{2}\nu_y^2(2-\nu_x) \right] u_{i,j}^n \\
 &\quad - \frac{1}{4}\nu_x(1-\nu_x)(2-\nu_y)u_{i+1,j}^n - \frac{1}{4}(2-\nu_x)\nu_y(1-\nu_y)u_{i,j+1}^n
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\nu_x\nu_y(1-\nu_x)u_{i+1,j-1}^n - \frac{1}{4}\nu_x\nu_y(1-\nu_y)u_{i-1,j+1}^n \\
& + \frac{1}{4}\left[\nu_x(1+\nu_x)(2-\nu_y) - 2\nu_x\nu_y^2\right]u_{i-1,j}^n + \frac{1}{4}\left[\nu_y(1+\nu_y)(2-\nu_x) - 2\nu_x^2\nu_y\right]u_{i,j-1}^n \\
& + \frac{1}{4}\nu_x\nu_y(2+\nu_x+\nu_y)u_{i-1,j-1}^n
\end{aligned} \tag{5.32}$$

Note that the scheme has the eight point compact stencil shown in Fig. 5.4, i.e the same stencil as the scheme (5.10). Note that when $\nu_x = 0$ or $\nu_y = 0$, the scheme reduces to the one dimensional Lax-Wendroff scheme and that when $\nu_x = \nu_y = 1$, the scheme reduces to

$$u_{i,j}^{n+1} = u_{i-1,j-1}^n \tag{5.33}$$

which reproduces the exact solution under the given initial conditions. The scheme satisfies conditions (5.13)–(5.18) and is therefore second order accurate in space and time and is not monotonic. The scheme (5.32) appears to be new and forms the basis of finite volume schemes developed in Chapter 6 for non-linear systems of hyperbolic conservation laws.

5.2.4 The Midpoint Rule in Both Space and Time

Finally, consider the schemes generated by using the midpoint rule in time and combinations of exact integration and the midpoint rule in space. Assume that $\nu_x \leq 1$ and $\nu_y \leq 1$. As in §5.2.2, there are three cases to consider: (i) the midpoint rule parallel to the boundary and exact integration perpendicular to the boundary; (ii) the midpoint rule perpendicular to the boundary and exact integration parallel to the boundary; (iii) the midpoint rule in both directions. In cases (i) and (ii), the integral is an integral in two dimensions: one space dimension, where exact integration is used, and one in time, where the midpoint rule is used. In this situation, the resulting scheme is the same as when exact integration in time is used. The same result holds for (iii). Thus the three cases here will result in the same schemes as cases (i), (ii), (iii) in §5.2.2.

5.2.5 Summary

After choosing a particular space integration range I , placing restrictions on the Courant numbers and taking initial conditions for the integrand in (5.2) and piecewise constant data from the previous timelevel, it has still been possible to produce several schemes, two of which are second order accurate in space and time and have good stability properties. Other choices of I , Courant number restrictions and initial conditions will produce other schemes, some of which may have higher accuracy and even better stability properties.

5.3 Generation of Three-Dimensional Schemes

The **WAF** approach will now be extended to generate finite volume schemes for the linear advection equation in three space dimensions:

$$u_t + f_x + g_y + h_z = 0 \quad (5.34)$$

where $f = au$, $g = bu$ and $h = cu$ are the flux functions. The wavespeeds a , b and c will be assumed constant and positive here. A general **WAF** intercell flux in three space dimensions can be defined in an analogous way to the two-dimensional flux (5.2). The flux $\mathbf{e}_A$ across boundary A in three space dimensions is:

$$\mathbf{e}_A = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I \mathbf{e} \cdot \mathbf{n} \, dx \, dy \, dz \, dt \quad (5.35)$$

where I is the three dimensional space integration range, $V(I)$ is the volume of I , $\mathbf{e} = (au, bu, cu)$ is the general flux function and $\mathbf{n}$ is the outward facing unit vector normal to the boundary.

As in the two dimensional case, the discussion will be restricted to the simple case of regular, Cartesian grids. Let Δx , Δy and Δz be the grid spacings in the x , y and z directions respectively. The schemes will be developed in conservative form:

$$\begin{aligned} u_{i,j,k}^{n+1} = & u_{i,j,k}^n - \frac{\Delta t}{\Delta x} \left[f_{i+\frac{1}{2},j,k} - f_{i-\frac{1}{2},j,k} \right] \\ & - \frac{\Delta t}{\Delta y} \left[g_{i,j+\frac{1}{2},k} - g_{i,j-\frac{1}{2},k} \right] - \frac{\Delta t}{\Delta z} \left[h_{i,j,k+\frac{1}{2}} - h_{i,j,k-\frac{1}{2}} \right] \end{aligned} \quad (5.36)$$

where $f_{i+\frac{1}{2},j,k}$, $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ are the intercell fluxes in the x , y and z directions respectively. From (5.35), fluxes $f_{i+\frac{1}{2},j,k}$ for the conservative form (5.36) can be written as

$$f_{i+\frac{1}{2},j,k} = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I f(u^*(x, y, z, t)) \, dx \, dy \, dz \, dt \quad (5.37)$$

where u^* is the exact solution to equation (5.34) with the data u^n at time level n as initial conditions, i.e:

$$u^*(x, y, z, t) = u^n(x - at, y - bt, z - ct) \quad (5.38)$$

The fluxes $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ follow similarly. As the space integration range for $f_{i+\frac{1}{2},j,k}$, the natural extension of the two dimensional space integration range (5.7) is used, which can be written as

$$I = \left[-\frac{1}{2}\Delta x, \frac{1}{2}\Delta x \right] \times [0, \Delta y] \times [0, \Delta z] \quad (5.39)$$

See Fig. 5.7.

The integration ranges for $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ follow by symmetry. Define the directional Courant numbers as $\nu_x = a \frac{\Delta t}{\Delta x}$, $\nu_y = b \frac{\Delta t}{\Delta y}$ and $\nu_z = c \frac{\Delta t}{\Delta z}$. In the discussion, it will be assumed that all three are less than unity.

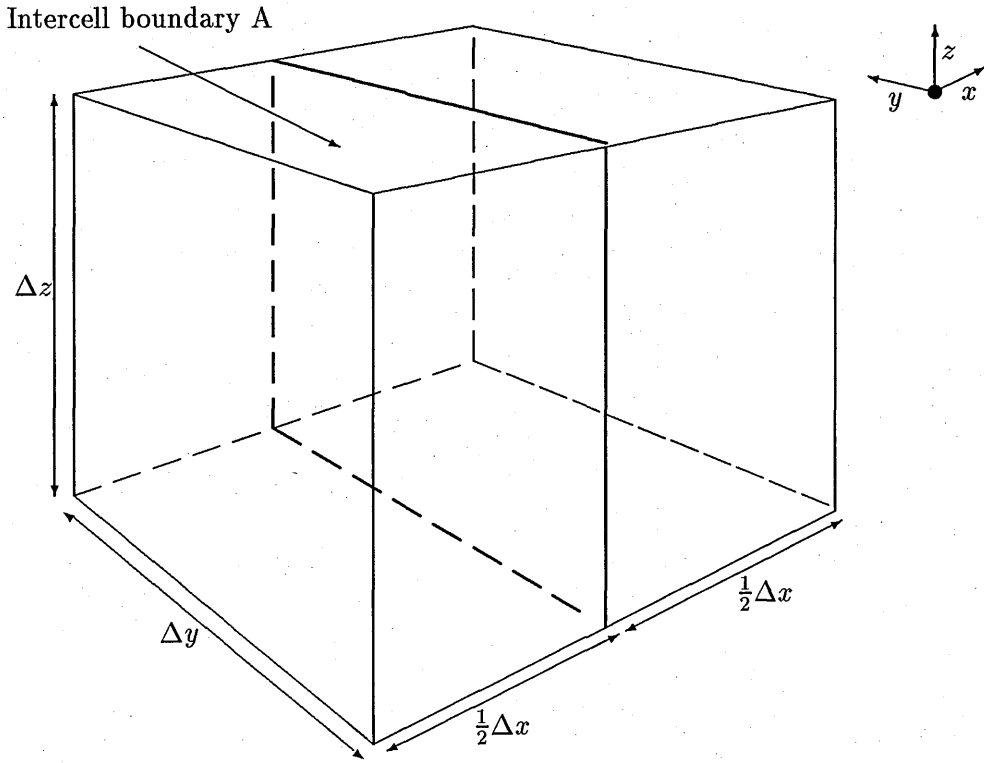


Figure 5.7: The three dimensional space integration range for $f_{i+\frac{1}{2},j,k}$.

The choice of integration range and Courant numbers restricts the number of states influencing $f_{i+\frac{1}{2},j,k}$ to a maximum of eight: $u_{i,j,k}$, $u_{i+1,j,k}$, $u_{i,j-1,k}$, $u_{i+1,j-1,k}$, $u_{i,j,k-1}$, $u_{i+1,j,k-1}$, $u_{i,j-1,k-1}$ and $u_{i+1,j-1,k-1}$. These are arranged in the eight corners of a cube and can be considered as the initial conditions for a three dimensional Riemann problem for (5.34).

5.3.1 A First Order Scheme

A three dimensional version of the first order upwind scheme (5.25) for the linear equation (5.34) is now derived. Following the derivation of (5.25), this is done via (5.37) by using the midpoint rule in time, the midpoint rule in space perpendicular to the boundary and exact integration in space in both directions parallel to the boundary. See Fig. 5.8. The resulting intercell flux is:

$$f_{i+\frac{1}{2},j,k} = \frac{1}{\Delta y \Delta z} \int_{I \cap \{x=0\}} f(u(0, y, z, \frac{1}{2}\Delta t)) dy dz \quad (5.40)$$

This can be written explicitly as:

$$\begin{aligned} f_{i+\frac{1}{2},j,k} = & \frac{1}{4}(2 - \nu_y)(2 - \nu_z)f_{i,j,k} + \frac{1}{4}\nu_y(2 - \nu_z)f_{i,j-1,k} \\ & + \frac{1}{4}(2 - \nu_y)\nu_z f_{i,j,k-1} + \frac{1}{4}\nu_y\nu_z f_{i,j-1,k-1} \end{aligned} \quad (5.41)$$

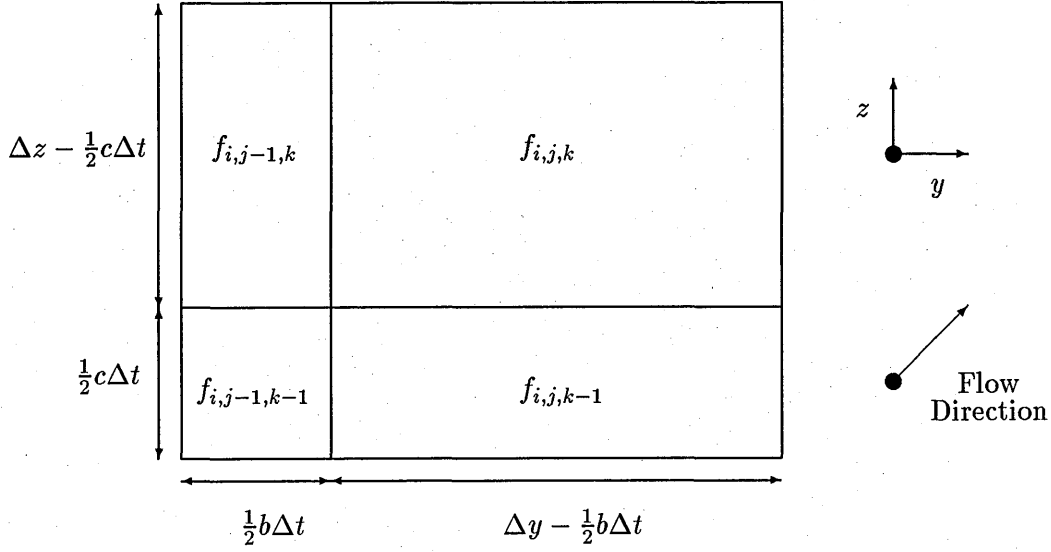


Figure 5.8: Integration range for $f_{i+\frac{1}{2},j,k}$ for the first order scheme in three dimensions: midpoint rule in time, midpoint rule perpendicular to the boundary and exact integration in both directions parallel to the boundary.

The other fluxes, $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$, follow by symmetry as:

$$g_{i,j+\frac{1}{2},k} = \frac{1}{4}(2-\nu_x)(2-\nu_z)f_{i,j,k} + \frac{1}{4}\nu_x(2-\nu_z)f_{i-1,j,k} + \frac{1}{4}(2-\nu_x)\nu_z f_{i,j,k-1} + \frac{1}{4}\nu_x\nu_z f_{i-1,j,k-1} \quad (5.42)$$

$$h_{i,j,k+\frac{1}{2}} = \frac{1}{4}(2-\nu_y)(2-\nu_x)f_{i,j,k} + \frac{1}{4}\nu_y(2-\nu_x)f_{i,j-1,k} + \frac{1}{4}(2-\nu_y)\nu_x f_{i-1,j,k} + \frac{1}{4}\nu_y\nu_x f_{i-1,j-1,k} \quad (5.43)$$

Substituting $f_{i+\frac{1}{2},j,k}$, $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ into the conservative formula (5.36) gives the scheme in its full form as:

$$u_{i,j,k}^{n+1} = \left[1 - \frac{1}{4}\nu_x(2-\nu_y)(2-\nu_z) - \frac{1}{4}(2-\nu_x)\nu_y(2-\nu_z) - \frac{1}{4}(2-\nu_x)(2-\nu_y)\nu_z \right] u_{i,j,k}^n + \nu_x \left[(1-\nu_y)(1-\nu_z) - \frac{1}{4}\nu_y\nu_z \right] u_{i-1,j,k}^n + \frac{1}{4}(4-3\nu_x)\nu_y\nu_z u_{i,j-1,k-1}^n$$

$$\begin{aligned}
& +\nu_y \left[(1-\nu_x)(1-\nu_z) - \frac{1}{4}\nu_x\nu_z \right] u_{i,j-1,k}^n + \frac{1}{4}\nu_x(4-3\nu_y)\nu_z u_{i-1,j,k-1}^n \\
& +\nu_z \left[(1-\nu_x)(1-\nu_y) - \frac{1}{4}\nu_x\nu_y \right] u_{i,j,k-1}^n + \frac{1}{4}\nu_x\nu_y(4-3\nu_z)u_{i-1,j-1,k}^n \\
& +\frac{3}{4}\nu_x\nu_y\nu_z u_{i-1,j-1,k-1}^n
\end{aligned} \tag{5.44}$$

This scheme has an eight point, upwind stencil. In Appendix A, schemes of the form

$$u_{i,j,k}^{n+1} = \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma} u_{i+\alpha,j+\beta,k+\gamma}^n \tag{5.45}$$

for the three dimensional linear advection equation (5.34) are shown to be p^{th} order accurate in space and time provided:

$$\sum_{\alpha,\beta,\gamma} \alpha^q \beta^r \gamma^s A_{\alpha,\beta,\gamma} = (-\nu_x)^q (-\nu_y)^r (-\nu_z)^s \tag{5.46}$$

for all integer triples (q, r, s) such that $q \geq 0, r \geq 0, s \geq 0$ and $q + r + s \leq p$. Hence the conditions for such a scheme with stencil $(\alpha, \beta, \gamma) \in [-1, 1] \times [-1, 1] \times [-1, 1]$ to be first order accurate are:

$$\sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma} = 1 \tag{5.47}$$

$$\sum_{\beta,\gamma} (A_{+1,\beta,\gamma} - A_{-1,\beta,\gamma}) = -\nu_x \tag{5.48}$$

$$\sum_{\alpha,\gamma} (A_{\alpha,+1,\gamma} - A_{\alpha,-1,\gamma}) = -\nu_y \tag{5.49}$$

$$\sum_{\alpha,\beta} (A_{\alpha,\beta,+1} - A_{\alpha,\beta,-1}) = -\nu_z \tag{5.50}$$

For second order accuracy, the scheme must also satisfy

$$\sum_{\beta,\gamma} (A_{+1,\beta,\gamma} + A_{-1,\beta,\gamma}) = \nu_x^2 \tag{5.51}$$

$$\sum_{\alpha,\gamma} (A_{\alpha,+1,\gamma} + A_{\alpha,-1,\gamma}) = \nu_y^2 \tag{5.52}$$

$$\sum_{\alpha,\beta} (A_{\alpha,\beta,+1} + A_{\alpha,\beta,-1}) = \nu_z^2 \tag{5.53}$$

$$\sum_{\gamma} (A_{+1,+1,\gamma} - A_{+1,-1,\gamma} - A_{-1,+1,\gamma} + A_{-1,-1,\gamma}) = \nu_x \nu_y \tag{5.54}$$

$$\sum_{\beta} (A_{+1,\beta,+1} - A_{+1,\beta,-1} - A_{-1,\beta,+1} + A_{-1,\beta,-1}) = \nu_x \nu_z \tag{5.55}$$

$$\sum_{\alpha} (A_{\alpha,+1,+1} - A_{\alpha,+1,-1} - A_{\alpha,-1,+1} + A_{\alpha,-1,-1}) = \nu_y \nu_z \quad (5.56)$$

The scheme satisfies (5.47)—(5.50) and (5.54)—(5.56), but not (5.51)—(5.53) and is therefore first order accurate.

By setting $\nu_x = \nu_y = \nu_z = \nu$ in (5.44) it can be seen that not all the coefficients are positive if $\nu \geq \frac{2}{3}$ (see, for example, the coefficient of $u_{i-1,j,k}^n$). Thus a sufficient condition for the scheme to remain monotonic is

$$\max\{\nu_x, \nu_y, \nu_z\} \leq \frac{2}{3} \quad (5.57)$$

5.3.2 A Second Order Scheme

Working from the experience of the two dimensional case, the midpoint rule in time and exact integration in all three space directions will now be used to generate a scheme. The flux in the x direction becomes

$$f_{i+\frac{1}{2},j,k} = \frac{1}{V(I)} \int_I f(u^*(x, y, z, \frac{1}{2}\Delta t)) dx dy dz \quad (5.58)$$

where u^* is the exact solution of (5.34) with initial conditions $\{u_{i,j,k}^n\}$, the solution at time level n . The space integration range (5.39) is used, illustrated in Fig. 5.9 divided into the regions occupied by different fluxes at time $\frac{1}{2}\Delta t$. Performing the integral gives the flux as:

$$\begin{aligned} f_{i+\frac{1}{2},j,k} = & \frac{1}{8}(1+\nu_x)(2-\nu_y)(2-\nu_z)f_{i,j,k} + \frac{1}{8}(1-\nu_x)(2-\nu_y)(2-\nu_z)f_{i+1,j,k} \\ & + \frac{1}{8}(1+\nu_x)\nu_y(2-\nu_z)f_{i,j-1,k} + \frac{1}{8}(1-\nu_x)\nu_y(2-\nu_z)f_{i+1,j-1,k} \\ & + \frac{1}{8}(1+\nu_x)(2-\nu_y)\nu_z f_{i,j,k-1} + \frac{1}{8}(1-\nu_x)(2-\nu_y)\nu_z f_{i+1,j,k-1} \\ & + \frac{1}{8}(1+\nu_x)\nu_y\nu_z f_{i,j-1,k-1} + \frac{1}{8}(1-\nu_x)\nu_y\nu_z f_{i+1,j-1,k-1} \end{aligned} \quad (5.59)$$

The fluxes $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ follow by symmetry as:

$$\begin{aligned} g_{i,j+\frac{1}{2},k} = & \frac{1}{8}(1+\nu_y)(2-\nu_x)(2-\nu_z)g_{i,j,k} + \frac{1}{8}(1-\nu_y)(2-\nu_x)(2-\nu_z)g_{i,j+1,k} \\ & + \frac{1}{8}(1+\nu_y)\nu_x(2-\nu_z)g_{i-1,j,k} + \frac{1}{8}(1-\nu_y)\nu_x(2-\nu_z)g_{i-1,j+1,k} \end{aligned} \quad (5.60)$$

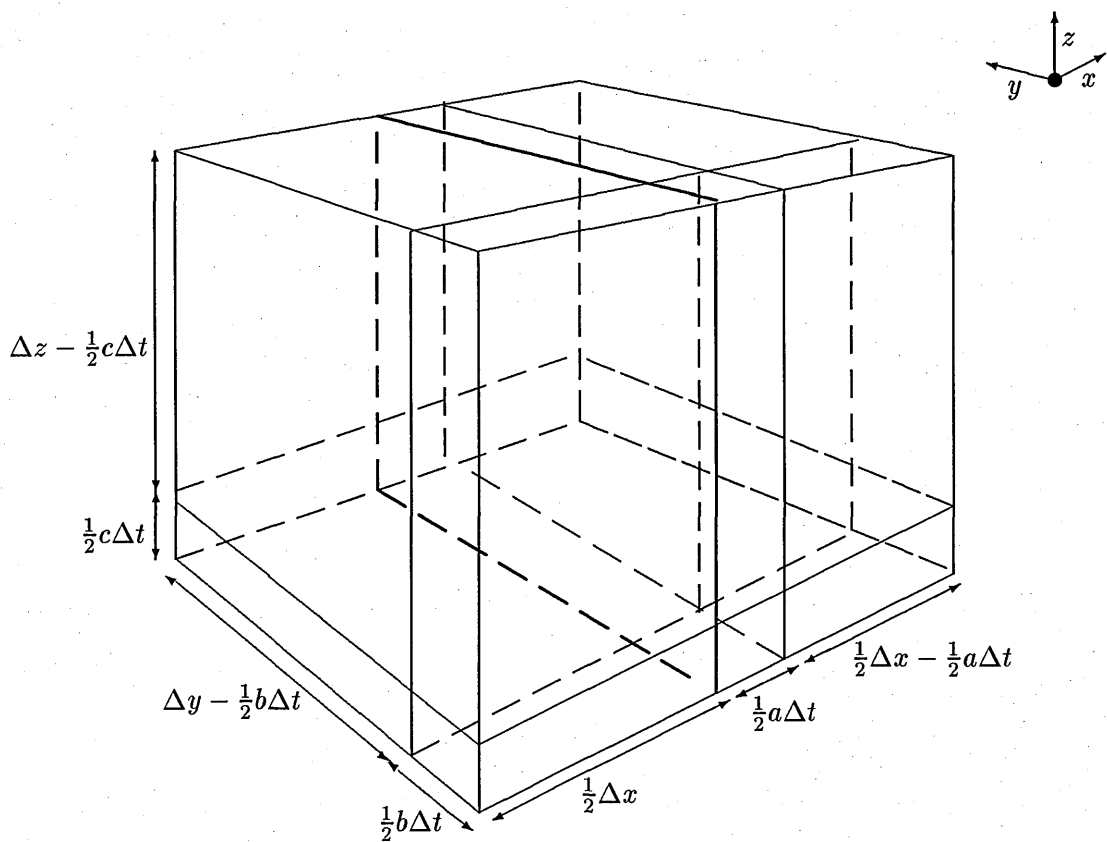


Figure 5.9: Integration for the 3D scheme: midpoint rule in time, exact integration in all space directions. Status of the space integration range at time $t = \frac{1}{2}\Delta t$.

$$\begin{aligned}
& + \frac{1}{8}(1 + \nu_y)(2 - \nu_x)\nu_z g_{i,j,k-1} + \frac{1}{8}(1 - \nu_y)(2 - \nu_x)\nu_z g_{i,j+1,k-1} \\
& + \frac{1}{8}(1 + \nu_y)\nu_x \nu_z g_{i-1,j,k-1} + \frac{1}{8}(1 - \nu_y)\nu_x \nu_z g_{i-1,j+1,k-1} \\
h_{i,j,k+\frac{1}{2}} = & \frac{1}{8}(1 + \nu_z)(2 - \nu_x)(2 - \nu_y)h_{i,j,k} + \frac{1}{8}(1 - \nu_z)(2 - \nu_x)(2 - \nu_y)h_{i,j,k+1} \\
& + \frac{1}{8}(1 + \nu_z)(2 - \nu_x)\nu_y h_{i,j-1,k} + \frac{1}{8}(1 - \nu_z)(2 - \nu_x)\nu_y h_{i,j-1,k+1} \\
& + \frac{1}{8}(1 + \nu_z)\nu_x(2 - \nu_y)h_{i-1,j,k} + \frac{1}{8}(1 - \nu_z)\nu_x(2 - \nu_y)h_{i-1,j,k+1} \\
& + \frac{1}{8}(1 + \nu_z)\nu_x \nu_y h_{i-1,j-1,k} + \frac{1}{8}(1 - \nu_z)\nu_x \nu_y h_{i-1,j-1,k+1}
\end{aligned} \tag{5.61}$$

Substituting $f_{i+\frac{1}{2},j,k}$, $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$ into the conservative formula (5.36) gives the final scheme as:

$$\begin{aligned}
u_{i,j,k}^{n+1} = u_{i,j,k}^n & - \frac{1}{4} \left[\nu_x^2(2 - \nu_y)(2 - \nu_z) + (2 - \nu_x)\nu_y^2(2 - \nu_z) + (2 - \nu_x)(2 - \nu_y)\nu_z^2 \right] u_{i,j,k}^n \\
& + \frac{1}{8}\nu_x \left[(1 + \nu_x)(2 - \nu_y)(2 - \nu_z) - 2\nu_y^2(2 - \nu_z) - 2(2 - \nu_y)\nu_z^2 \right] u_{i-1,j,k}^n \\
& + \frac{1}{8}\nu_y \left[(2 - \nu_x)(1 + \nu_y)(2 - \nu_z) - 2\nu_x^2(2 - \nu_z) - 2(2 - \nu_x)\nu_z^2 \right] u_{i,j-1,k}^n \\
& + \frac{1}{8}\nu_z \left[(2 - \nu_x)(2 - \nu_y)(1 + \nu_z) - 2(2 - \nu_x)\nu_y^2 - 2\nu_x^2(2 - \nu_y) \right] u_{i,j,k-1}^n \\
& + \frac{1}{8}\nu_x \nu_y \left[(2 - \nu_z)(2 + \nu_x + \nu_y) - 2\nu_z^2 \right] u_{i-1,j-1,k}^n \\
& + \frac{1}{8}\nu_x \nu_z \left[(2 - \nu_y)(2 + \nu_x + \nu_z) - 2\nu_y^2 \right] u_{i-1,j,k-1}^n \\
& + \frac{1}{8}\nu_y \nu_z \left[(2 - \nu_x)(2 + \nu_y + \nu_z) - 2\nu_x^2 \right] u_{i,j-1,k-1}^n \\
& + \frac{1}{8}\nu_x \nu_y \nu_z (3 + \nu_x + \nu_y + \nu_z) u_{i-1,j-1,k-1}^n \\
& - \frac{1}{8}\nu_x(1 - \nu_x)(2 - \nu_y)(2 - \nu_z) u_{i+1,j,k}^n - \frac{1}{8}\nu_x(1 - \nu_x)\nu_y(2 - \nu_z) u_{i+1,j-1,k}^n
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{8}\nu_x(1-\nu_x)(2-\nu_y)\nu_z u_{i+1,j,k-1}^n - \frac{1}{8}\nu_x(1-\nu_x)\nu_y\nu_z u_{i+1,j-1,k-1}^n \\
& - \frac{1}{8}(2-\nu_x)\nu_y(1-\nu_y)(2-\nu_z)u_{i,j+1,k}^n - \frac{1}{8}\nu_x\nu_y(1-\nu_y)(2-\nu_z)u_{i-1,j+1,k}^n \\
& - \frac{1}{8}(2-\nu_x)\nu_y(1-\nu_y)\nu_z u_{i,j+1,k-1}^n - \frac{1}{8}\nu_x\nu_y(1-\nu_y)\nu_z u_{i-1,j+1,k-1}^n \\
& - \frac{1}{8}(2-\nu_x)(2-\nu_y)\nu_z(1-\nu_z)u_{i,j,k+1}^n - \frac{1}{8}(2-\nu_x)\nu_y\nu_z(1-\nu_z)u_{i,j-1,k+1}^n \\
& - \frac{1}{8}\nu_x(2-\nu_y)\nu_z(1-\nu_z)u_{i-1,j,k+1}^n - \frac{1}{8}\nu_x\nu_y\nu_z(1-\nu_z)u_{i-1,j-1,k+1}^n \quad (5.62)
\end{aligned}$$

This scheme has a twenty point compact stencil. The coefficients satisfy conditions (5.47)–(5.56) and the scheme is therefore second order accurate in space and time and not monotonic. Note that if any one of the Courant numbers are zero, the scheme reduces to the two dimensional second order scheme (5.32).

5.4 Stability

The aim of this section is to establish the stability of the schemes derived in the last two sections. Consider first the two dimensional schemes derived in §5.2. The stability of a scheme of the form (5.11) for the linear advection equation (5.1) can be found via von Neumann stability analysis. The application of this technique in one space dimension was described in Chapter 2. In two space dimensions, it consists of performing a Fourier analysis and looking for restrictions on the coefficients $A_{i+\alpha,j+\beta}$ to ensure all Fourier modes are stable. Consider an error of the form

$$\epsilon_{i,j}^n = S^n e^{I\theta i} e^{I\phi j} \quad (5.63)$$

where S is the amplification coefficient, θ and ϕ are phase angles and $I = \sqrt{-1}$. Note that S^n in (5.63) means S to the power n , **not** S at timelevel n . S is assumed a constant in time for any given scheme. Due to the linearity of the problem, errors $\epsilon_{i,j}^n$ in cell $\Delta_{i,j}$ at time level n will be evolved in time via (5.11), i.e

$$\epsilon_{i,j}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta} \epsilon_{i+\alpha,j+\beta}^n \quad (5.64)$$

Substituting (5.63) into (5.64) gives:

$$\begin{aligned}
S^{n+1} e^{I\theta i} e^{I\phi j} &= \sum_{\alpha,\beta} A_{\alpha,\beta} S^n e^{I\theta(i+\alpha)} e^{I\phi(j+\beta)} \\
&= S^n e^{I\theta i} e^{I\phi j} \sum_{\alpha,\beta} A_{\alpha,\beta} e^{I(\alpha\theta+\beta\phi)} \quad (5.65)
\end{aligned}$$

Dividing throughout by $S^n e^{I\theta i} e^{I\phi j}$ gives

$$S = \sum_{\alpha, \beta} A_{\alpha, \beta} e^{I(\alpha\theta + \beta\phi)} \quad (5.66)$$

Using the standard relation

$$e^{I\eta} = \cos \eta + I \sin \eta \quad (5.67)$$

gives

$$S = \sum_{\alpha, \beta} A_{\alpha, \beta} \cos(\alpha\theta + \beta\phi) + I \cdot \sum_{\alpha, \beta} A_{\alpha, \beta} \sin(\alpha\theta + \beta\phi) \quad (5.68)$$

For stability, this expression for S must satisfy

$$|S| \leq 1 \quad (5.69)$$

i.e the Fourier mode (θ, ϕ) must not be amplified. It is more convenient to use the equivalent condition

$$S^2 \leq 1 \quad (5.70)$$

To obtain usable stability conditions, it is necessary to convert the condition (5.70) into conditions on the directional Courant numbers ν_x and ν_y . However, if this is attempted, the expressions for S^2 become, in general, extremely complex and rigorously deriving clear stability regions for schemes is very difficult.

A simple way to obtain a good guess at the stability condition(s) for a scheme is now described. By computing S^2 for a large number of quadruples $(\nu_x, \nu_y, \theta, \phi)$, a function $st2D(\nu_x, \nu_y)$ can be defined to be the ratio of pairs (θ, ϕ) that satisfy (5.70) for given (ν_x, ν_y) , to the total number tested. The scheme is only stable for the pair (ν_x, ν_y) if $st2D(\nu_x, \nu_y) = 1$, i.e all Fourier modes are stable. By studying a two dimensional plot of $st2D$ against ν_x and ν_y a good indication of the stability region of a scheme can be obtained. This approach obviously does not give a rigorous derivation of stability conditions, but is a relatively easy method of obtaining an indication of the conditions and should be quite reliable if the number of points tested in the calculation of $st2D$ is sufficiently large.

Fig. 5.10 shows contour plots of $st2D(\nu_x, \nu_y)$ against (ν_x, ν_y) when $0 \leq \nu_x \leq 2$ and $0 \leq \nu_y \leq 2$ for four of the schemes derived §5.2.

Fig. 5.10a is the plot for the scheme (5.10) derived using exact integration in all directions in §5.2.1. It suggests that the scheme is stable whenever

$$\sqrt{(\nu_x^2 + \nu_y^2)} \leq 1 \quad (5.71)$$

Recall that the scheme is also second order accurate in space and time and a two-dimensional extension of the Lax-Wendroff scheme. It is much more stable than the "traditional" Lax-Wendroff schemes [29] in two dimensions, which are restricted by the condition

$$\sqrt{(\nu_x^2 + \nu_y^2)} \leq \frac{1}{\sqrt{8}} \quad (5.72)$$

See Hirsch [25].

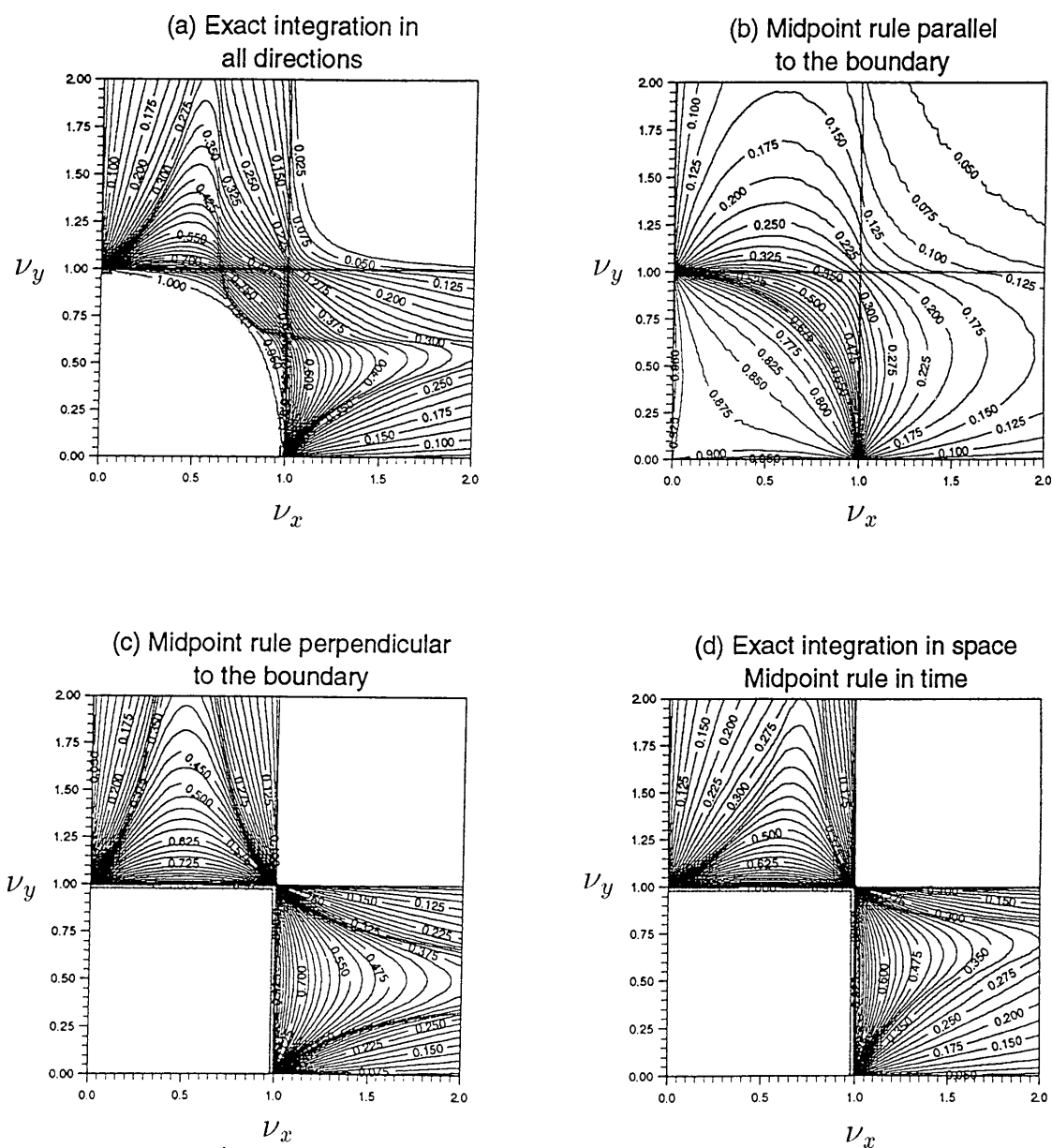


Figure 5.10: Stability diagrams for the two dimensional schemes.

Fig. 5.10b is the plot for the scheme (5.21) derived in §5.2.2.(i) using exact integration in time and perpendicular to the boundary, and the midpoint rule parallel to the boundary. It suggests that the scheme is unstable unless the flow is aligned to the grid, when the appropriate Courant number must be less than unity. The lack of stability of this scheme is due to the fact that it does not take into account the multidimensional nature of the problem.

Fig. 5.10c is the plot for the first order scheme (5.25) derived using exact integration in time, exact integration parallel to the boundary and the midpoint rule perpendicular to the boundary in §5.2.2(ii). As mentioned before, it is Colella's cru scheme [8]. Fig. 5.10c suggests that this scheme is stable whenever

$$\max \{\nu_x, \nu_y\} \leq 1 \quad (5.73)$$

Colella proved this condition analytically.

Fig. 5.10d is the plot for the second order accurate scheme (5.32) derived using the midpoint rule in time and exact integration in both space dimensions in §5.2.3. It suggests the scheme also has the stability condition (5.73).

A plot of the function $st2D(\nu_x, \nu_y)$ for the finite volume Godunov scheme (5.28) is not shown. The scheme is stable provided

$$\nu_x + \nu_y \leq 1 \quad (5.74)$$

This is a well known result.

Consider now the three dimensional schemes derived in §5.3. The stability analysis in three dimensions is essentially the same as in two dimensions. In this case, the test error (5.63) becomes:

$$\epsilon_{i,j,k}^n = S^n e^{I\theta i} e^{I\phi j} e^{I\zeta k} \quad (5.75)$$

with real phase angles θ , ϕ and ζ . The amplification factor has the form

$$S = \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma} \cos(\alpha\theta + \beta\phi + \gamma\psi) + I \cdot \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma} \sin(\alpha\theta + \beta\phi + \gamma\psi) \quad (5.76)$$

and the scheme is stable provided

$$S^2 \leq 1 \quad (5.77)$$

As in the two dimensional case, the expressions for S^2 in terms of the directional Courant numbers (now ν_x , ν_y and ν_z) are too complex to rigorously prove the precise stability conditions easily. However, a three dimensional equivalent to the function $st2D(\nu_x, \nu_y)$ can be defined, which is called $st3D(\nu_x, \nu_y, \nu_z)$ here. Information on the stability of a three dimensional scheme can then be found by studying $st3D$ for fixed values of one of the Courant numbers, e.g ν_z , i.e by plotting two-dimensional 'slices' of the three-dimensional function.

From the experience of the equivalent scheme (5.32) in two space dimensions, it was hoped that the stability condition for the three dimensional second order scheme (5.62) would be

$$\max \{\nu_x, \nu_y, \nu_z\} \leq 1 \quad (5.78)$$

However, after studying slices of $st3D(\nu_x, \nu_y, \nu_z)$ for (5.62) with fixed ν_z it is clear that this is not the case. Six of these 'slices' are shown in Fig. 5.11, for $\nu_z = 0.0, 0.25, 0.5, 0.75, 1.0, 1.25$. Note that ν_z is written CFL in the figure. When $\nu_z = 0.0$, the scheme (5.8) reduces to the two dimensional scheme (5.18) and the stability condition is therefore the same as for that scheme in this case. However, as ν_z increases, an area around the point $\nu_x = 1, \nu_y = 1$ is lost to the unstable region. When $\nu_z \geq 1$, the stability region vanishes, but this was expected. The author has not yet been able to work out the exact stability region; however, it was found that by insisting on the condition (5.78) and adding the condition

$$\nu_x + \nu_y + \nu_z \leq 2 \quad (5.79)$$

then any points satisfying both appear to be stable. The factor S^2 has been calculated for many thousands of points (ν_x, ν_y, ν_z) within the region (5.78), (5.79) and none have been found to be unstable. Studying the slices in Fig. 5.11 indicate that conditions (5.78), (5.79) are sufficient conditions, but not necessary for stability.

Reasons for this 'loss' of stability can be seen by studying the coefficients of (5.62) when all the directional Courant numbers equal one. Recall that in this situation, the two dimensional scheme (5.32) reduces to

$$u_{i,j}^{n+1} = u_{i-1,j-1}^n \quad (5.80)$$

which is the exact solution and therefore must be stable. However, the three dimensional scheme (5.62) becomes:

$$\begin{aligned} u_{i,j,k}^{n+1} = & \frac{1}{4}u_{i,j,k}^n - \frac{1}{4}u_{i-1,j,k}^n - \frac{1}{4}u_{i,j-1,k}^n - \frac{1}{4}u_{i,j,k-1}^n \\ & + \frac{1}{4}u_{i-1,j-1,k}^n + \frac{1}{4}u_{i-1,j,k-1}^n + \frac{1}{4}u_{i,j-1,k-1}^n + \frac{3}{4}u_{i-1,j-1,k-1}^n \end{aligned} \quad (5.81)$$

which is *not* the exact solution. Some difference in stability is thus expected.

A similar study of the first order scheme (5.44) indicates that it probably has the same stability condition (5.79) as the second order scheme (5.62) since the slices look the same. Recall from §5.2.2 that a sufficient condition for the first order scheme to be monotonic was found to be

$$\max\{\nu_x, \nu_y, \nu_z\} \leq \frac{2}{3} \quad (5.82)$$

When all Courant numbers are the same, this agrees with the empirical stability condition (5.79). This indicates that the mononicity condition for the first order scheme and the stability conditions for both first and second order schemes may be the same. More work is needed to establish the conditions precisely. Note that condition (5.82) is not too severe. Consider, for example, the *rou* scheme for the linear advection equation in three space dimensions, i.e when a, b and c are all positive take

$$f_{i+\frac{1}{2},j,k} = f_{i,j,k} \quad g_{i,j+\frac{1}{2},k} = g_{i,j,k} \quad h_{i,j,k+\frac{1}{2}} = h_{i,j,k} \quad (5.83)$$

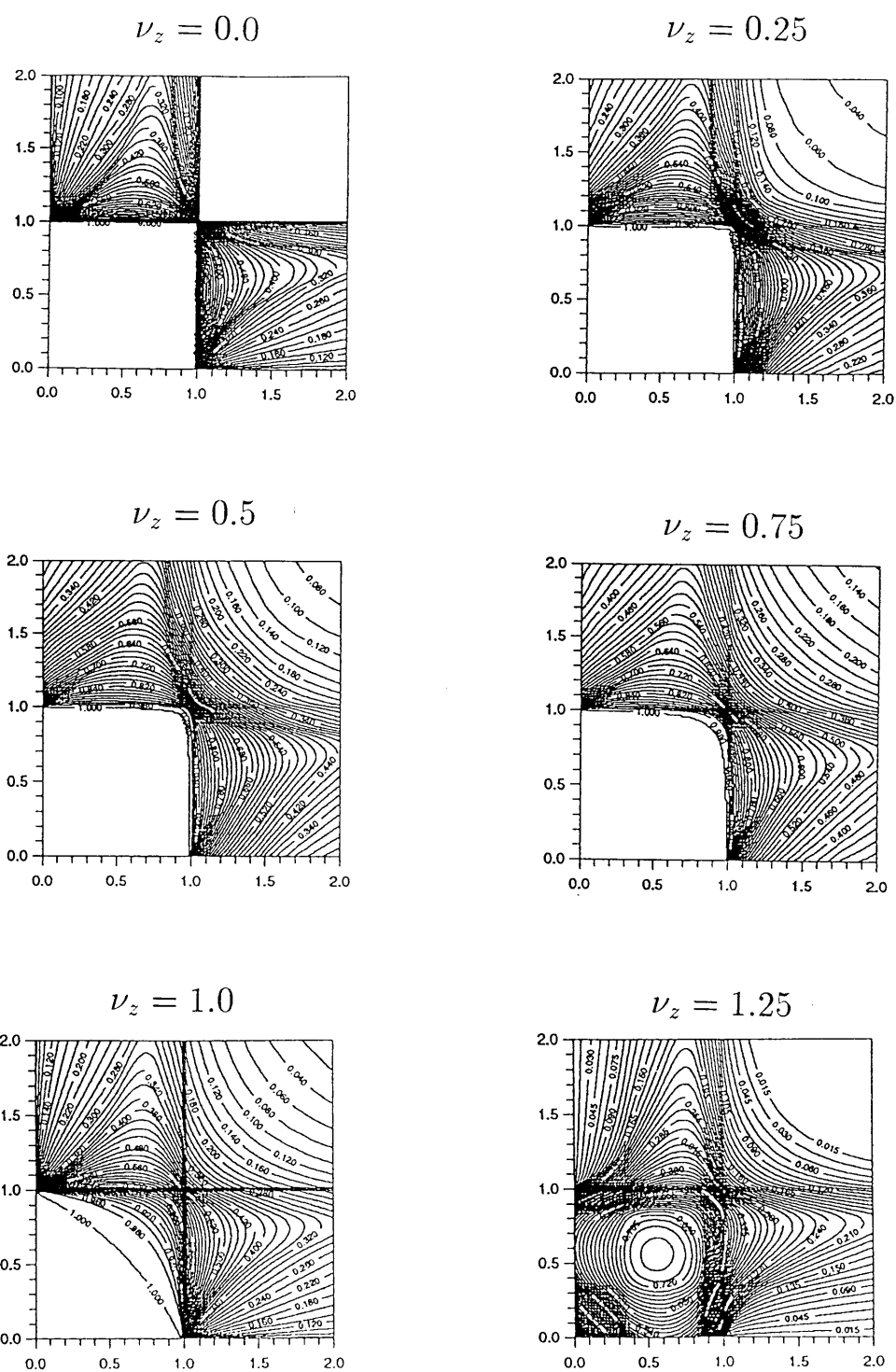


Figure 5.11: Stability diagrams for the three dimensional second order scheme.

Substituting these fluxes into the conservative formula (5.36) gives

$$u_{i,j,k}^{n+1} = (1 - \nu_x - \nu_y - \nu_z)u_{i,j,k}^n + \nu_x u_{i-1,j,k}^n + \nu_y u_{i,j-1,k}^n + \nu_z u_{i,j,k-1}^n \quad (5.84)$$

Studying the coefficients of this scheme shows that the scheme is only monotonic if $\nu_x + \nu_y + \nu_z \leq 1$, so the equivalent of (5.82) is

$$\max\{\nu_x, \nu_y, \nu_z\} \leq \frac{1}{3} \quad (5.85)$$

Thus the Courant number coefficient for which the `FOU` scheme in three dimensions is monotonic is only half the Courant number coefficient for which the new first order scheme remains monotonic.

Chapter 6

Construction of Schemes for Non-linear Systems in Two and Three Space Dimensions

6.1 Introduction

In Chapter 5, various finite volume WAF-type schemes on Cartesian grids were derived for the two-dimensional linear advection equation

$$u_t + f_x + g_y = 0 \quad (6.1)$$

where $f = au$ and $g = bu$ are linear flux functions and a and b are the speeds of propagation in the x and y directions respectively, assumed constant and positive. The schemes were all derived in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j}] - \frac{\Delta t}{\Delta y} [g_{i,j+\frac{1}{2}} - g_{i,j-\frac{1}{2}}] \quad (6.2)$$

Two schemes were found with an apparent stability restriction

$$\max\{\nu_x, \nu_y\} \leq 1 \quad (6.3)$$

where $\nu_x = a \frac{\Delta t}{\Delta x}$ and $\nu_y = b \frac{\Delta t}{\Delta y}$ are the directional Courant numbers, Δt is the timestep and $\Delta x, \Delta y$ are the constant mesh spacings. One of these schemes was a first order scheme with intercell flux $f_{i+\frac{1}{2},j}$ as (5.23), i.e:

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(2 - \nu_y)f_{i,j} + \frac{1}{2}\nu_y f_{i,j-1} \quad (6.4)$$

This results in a scheme that is identical to Colella's first order cru scheme [8] on the linear advection equation (6.1). The other scheme found with stability limit (6.3) is second

order accurate in space and time. It has intercell flux $f_{i+\frac{1}{2},j}$ as (5.31), i.e

$$\begin{aligned} f_{i+\frac{1}{2},j} = & \frac{1}{4}(1+\nu_x)(2-\nu_y)f_{i,j} + \frac{1}{4}(1-\nu_x)(2-\nu_y)f_{i+1,j} \\ & + \frac{1}{4}(1+\nu_x)\nu_y f_{i,j-1} + \frac{1}{4}(1-\nu_x)\nu_y f_{i+1,j-1} \end{aligned} \quad (6.5)$$

Equation (6.3) places a restriction on the timestep Δt but is the least restrictive possible for schemes with the stencils of those resulting from (6.4) and (6.5).

In §4.3 fluxes equivalent to (6.4) and (6.5) were found for the three dimensional linear advection equation

$$u_t + f_x + g_y + h_z = 0 \quad (6.6)$$

where $f = au$, $g = bu$ and $h = cu$, with the wavespeeds a , b and c assumed constant and positive. In this case there are three directional Courant numbers $\nu_x = a\frac{\Delta t}{\Delta x}$, $\nu_y = b\frac{\Delta t}{\Delta y}$ and $\nu_z = c\frac{\Delta t}{\Delta z}$, and the conservation form is

$$\begin{aligned} u_{i,j,k}^{n+1} = & u_{i,j,k}^n - \frac{\Delta t}{\Delta x} [f_{i+\frac{1}{2},j,k} - f_{i-\frac{1}{2},j,k}] \\ & - \frac{\Delta t}{\Delta y} [g_{i,j+\frac{1}{2},k} - g_{i,j-\frac{1}{2},k}] - \frac{\Delta t}{\Delta z} [h_{i,j,k+\frac{1}{2}} - h_{i,j,k-\frac{1}{2}}] \end{aligned} \quad (6.7)$$

The flux equivalent to (6.4) was found to be

$$\begin{aligned} f_{i+\frac{1}{2},j,k} = & \frac{1}{4}(2-\nu_y)(2-\nu_z)f_{i,j,k} + \frac{1}{4}\nu_y(2-\nu_z)f_{i,j-1,k} \\ & + \frac{1}{4}(2-\nu_y)\nu_z f_{i,j,k-1} + \frac{1}{4}\nu_y\nu_z f_{i,j-1,k-1} \end{aligned} \quad (6.8)$$

Substituting (6.8) into (6.7) gives a first order scheme that is monotonic under the sufficient (but perhaps not necessary) condition

$$\max\{\nu_x, \nu_y, \nu_z\} \leq \frac{2}{3} \quad (6.9)$$

The three dimensional flux equivalent to (6.5) was found to be

$$\begin{aligned} f_{i+\frac{1}{2},j,k} = & \frac{1}{8}(1+\nu_x)(2-\nu_y)(2-\nu_z)f_{i,j,k} + \frac{1}{8}(1-\nu_x)(2-\nu_y)(2-\nu_z)f_{i+1,j,k} \\ & + \frac{1}{8}(1+\nu_x)\nu_y(2-\nu_z)f_{i,j-1,k} + \frac{1}{8}(1-\nu_x)\nu_y(2-\nu_z)f_{i+1,j-1,k} \\ & + \frac{1}{8}(1+\nu_x)(2-\nu_y)\nu_z f_{i,j,k-1} + \frac{1}{8}(1-\nu_x)(2-\nu_y)\nu_z f_{i+1,j,k-1} \\ & + \frac{1}{8}(1+\nu_x)\nu_y\nu_z f_{i,j-1,k-1} + \frac{1}{8}(1-\nu_x)\nu_y\nu_z f_{i+1,j-1,k-1} \end{aligned} \quad (6.10)$$

When substituted into the conservation formula (6.7) this flux gives a scheme that is second order accurate in space and time. In §5.4 it was found that condition (6.9) appears to be sufficient to ensure the schemes resulting from (6.8) and (6.10) are stable. This has not been proven rigorously.

The aim of this chapter is to explore possible combined extensions of the two dimensional schemes with intercell fluxes (6.4), (6.5) to nonlinear, time-dependent hyperbolic systems of conservation laws

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0 \quad (6.11)$$

where $\mathbf{U}$ is a vector of conserved variables, and $\mathbf{F}$, $\mathbf{G}$ are vector-valued flux functions. The schemes can, of course, be equally applied as space marching schemes for the three dimensional steady supersonic Euler equations, or other such systems. However, for clarity of notation, it is assumed that the time dependent two dimensional system (6.11) is being solved. Only schemes on Cartesian grids will be considered in this chapter — general body-fitted grids of quadrilaterals are explored in Chapter 7. The aim is to use extensions of (6.5) in regions of smooth flow to obtain second order accuracy, but blend in (6.4) in regions of high gradient to avoid the unphysical oscillations inherent in second-order schemes. This will be done by incorporating one-dimensional limiter functions into the second order flux function.

Two extensions will be considered, and each will lead to schemes that are based on solutions to one-dimensional grid-aligned Riemann problems. Both will be developed in conservative form

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+\frac{1}{2},j} - \mathbf{F}_{i-\frac{1}{2},j}] - \frac{\Delta t}{\Delta y} [\mathbf{G}_{i,j+\frac{1}{2}} - \mathbf{G}_{i,j-\frac{1}{2}}] \quad (6.12)$$

where, as in the linear case,

$$\mathbf{U}_{i,j}^n = \frac{1}{V(\Delta_{i,j})} \int_{\Delta_{i,j}} \mathbf{U}(x, y, t^n) dx dy \quad (6.13)$$

is the integral average of the solution $\mathbf{U}(x, y, t)$ over the cell $\Delta_{i,j}$ indexed by i and j at time level n , $V(\Delta_{i,j})$ is the area of $\Delta_{i,j}$ and $\mathbf{F}_{i+\frac{1}{2},j}$ and $\mathbf{G}_{i,j+\frac{1}{2}}$ are the intercell fluxes. Both extensions will be tested numerically on the two dimensional Shallow Water equations discussed in Chapter 2.

The first approach to developing an extension will be to manipulate equation (6.5) algebraically to obtain a two-step formulation of the second-order flux calculation. To extend the scheme to non-linear systems, linear waves will be re-interpreted as waves from one-dimensional grid aligned Riemann problems moving perpendicular to cell boundaries. The resulting scheme will have similarities with Colella's finite volume scheme [8].

The second scheme will be constructed from an extension of the two dimensional **war** approach discussed in Chapter 5. Ideally, this would be done by taking the intercell flux as an integral average over exact solutions to two dimensional Riemann problems, i.e initial value problems with initial data $\{\mathbf{U}_{i,j}, \mathbf{U}_{i+1,j}, \mathbf{U}_{i,j+1}, \mathbf{U}_{i+1,j+1}\}$ kept constant in their respective cells. Some theoretical and numerical results have been obtained for solutions

to these problems, for example Zhang [86] or Shultz-Rinne *et. al.* [48]. However, there do not appear to be any exact or approximate solution techniques available at the present time that are sufficiently general to allow the construction of the proposed scheme and even if such solution techniques were available the proposed scheme would probably be too impractical for general use. To circumvent this problem, an approximate solution will be constructed from solutions to one-dimensional grid-aligned Riemann problems by giving waves in the solutions to these problems components of velocity parallel to the boundaries from which they originate. Any wave interactions, for example near where two intercell boundaries meet, will be treated in a linear fashion. It will be shown that the flux (6.5) can be rewritten as the integral average of the flux component perpendicular to the boundary over these approximate solutions at time $t = \frac{1}{2}\Delta t$. This approach of extending (6.5) is similar to the construction of LeVeque's scheme [32] in that a similar wave model is used on the linear advection equation, although LeVeque employs a **MUSCL**-type reconstruction of data to achieve second order accuracy, whereas the scheme presented here achieves this accuracy from the **WAF**-type averaging of the solutions to Riemann problems with piecewise constant data.

The two approaches described above will also be used to extend the second order flux (6.10) to three dimensional systems of conservation laws

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y + \mathbf{H}_z = 0 \quad (6.14)$$

As in the two dimensional case, only Cartesian grids will be considered. The schemes will be developed in conservation form

$$\begin{aligned} \mathbf{U}_{i,j,k}^{n+1} = & \mathbf{U}_{i,j,k}^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+\frac{1}{2},j,k} - \mathbf{F}_{i-\frac{1}{2},j,k}] \\ & - \frac{\Delta t}{\Delta y} [\mathbf{G}_{i,j+\frac{1}{2},k} - \mathbf{G}_{i,j-\frac{1}{2},k}] - \frac{\Delta t}{\Delta z} [\mathbf{H}_{i,j,k+\frac{1}{2}} - \mathbf{H}_{i,j,k-\frac{1}{2}}] \end{aligned} \quad (6.15)$$

where

$$\mathbf{U}_{i,j,k}^n = \frac{1}{V(\Delta_{i,j,k})} \int_{\Delta_{i,j,k}} \mathbf{U} dx dy dz \quad (6.16)$$

with notation as before. These three dimensional schemes have not yet been tested numerically.

The rest of the chapter will be arranged as follows. In §6.2 and §6.3 the two approaches briefly described above will be discussed in detail in two and three dimensions. In §6.4 some results of testing the two two dimensional schemes on the Shallow Water equations are presented. In §6.5, the chapter is summarized and conclusions are drawn.

6.2 A Simple Extension

6.2.1 The Two Dimensional Case

Consider the second order intercell flux (6.5) for the linear advection equation (6.1). It can be rewritten as:

$$\begin{aligned} f_{i+\frac{1}{2},j} = & \frac{1}{2}(1+\nu_x) \left[\frac{1}{2}(2-\nu_y)f_{i,j} + \frac{1}{2}\nu_y f_{i,j-1} \right] \\ & + \frac{1}{2}(1-\nu_x) \left[\frac{1}{2}(2-\nu_y)f_{i+1,j} + \frac{1}{2}\nu_y f_{i+1,j-1} \right] \end{aligned} \quad (6.17)$$

Note that the terms in square brackets are CTU fluxes discussed in Chapter 5. (6.17) can be rearranged as:

$$\begin{aligned} f_{i+\frac{1}{2},j} = & \frac{1}{2}(1+\nu_x)f \left(u_{i,j} - \frac{\frac{1}{2}\Delta t}{\Delta y} \left[g'_{i,j+\frac{1}{2}} - g'_{i,j-\frac{1}{2}} \right] \right) \\ & + \frac{1}{2}(1-\nu_x)f \left(u_{i+1,j} - \frac{\frac{1}{2}\Delta t}{\Delta y} \left[g'_{i+1,j+\frac{1}{2}} - g'_{i+1,j-\frac{1}{2}} \right] \right) \end{aligned} \quad (6.18)$$

where $g'_{i,j+\frac{1}{2}} = bu_{i,j}$. Hence

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(1+\nu_x)f(u_{i,j}^{n+\frac{1}{2}}) + \frac{1}{2}(1-\nu_x)f(u_{i+1,j}^{n+\frac{1}{2}}) \quad (6.19)$$

where

$$u_{i,j}^{n+\frac{1}{2}} = u_{i,j}^n - \frac{\frac{1}{2}\Delta t}{\Delta y} \left[g'_{i,j+\frac{1}{2}} - g'_{i,j-\frac{1}{2}} \right] \quad (6.20)$$

and

$$g'_{i,j+\frac{1}{2}} = g_{i,j} = bu_{i,j} \quad (6.21)$$

is the FOU intercell flux for boundary $(i, j + \frac{1}{2})$. Equations (6.19), (6.20), (6.21) suggest a very simple extension of the flux (6.5) to non-linear systems. The calculation of the intermediate states $u_{i,j}^{n+\frac{1}{2}}$ in (6.20), (6.21) can be interpreted for non-linear systems as using any method that reduces to the first-order upwind method on the one dimensional linear advection equation, such as Godunov's method. The solution U^n at time level n will be updated by half a timestep along strips perpendicular to the boundary $(i + \frac{1}{2}, j)$ i.e strips along which j is constant, to obtain an intermediate state $U_{i,j}^{n+\frac{1}{2}}$. It is this stage that introduces modelling of cross derivative terms into the scheme. These terms were missing from the finite volume scheme of Speares [50] and their absence caused the scheme to be unstable (see §5.2.4). Equation (6.19) is interpreted here as using the one dimensional WAF scheme to calculate the final intercell flux $F_{i+\frac{1}{2},j}$ based on $U_{i,j}^{n+\frac{1}{2}}$.

The scheme with flux (6.19), (6.20), (6.21), being second order accurate, will suffer from spurious oscillations in regions of high gradients. These are unphysical, and therefore undesirable. In one space dimension, one way to overcome this problem and maintain second order accuracy in smooth flow is to design schemes that satisfy certain Total Variation Diminishing (TVD) criteria. This has been discussed fully for the one dimensional WAF method in Chapter 2, but will be briefly summarized here. The WAF method applied to the linear advection equation in one space dimension has intercell flux

$$f_{i+\frac{1}{2}} = \frac{1}{2}(1 + \phi_{i+\frac{1}{2}})f_i + \frac{1}{2}(1 - \phi_{i+\frac{1}{2}})f_{i+1} \quad (6.22)$$

The functions $\phi_{i+\frac{1}{2}} = \phi_{i+\frac{1}{2}}(r_{i+\frac{1}{2}}, \nu)$ are the limiters for the one dimensional WAF scheme (see Toro, [67]). When $\phi \equiv \nu$ (6.22) gives the intercell flux of the second order Lax-Wendroff scheme, and when $\phi \equiv \text{sgn}(\nu)$ it gives the intercell flux for Godunov's scheme, which does not suffer from unphysical oscillations. In general, the limiters depend on the (local) Courant number ν and a flow parameter $r_{i+\frac{1}{2}}$ defined as the *upwind* jump in u divided by the *local* jump in u , i.e:

$$r_{i+\frac{1}{2}} = \frac{\Delta_{upw} u}{\Delta_{loc} u} = \begin{cases} \frac{u_i - u_{i-1}}{u_{i+1} - u_i} & \text{if } \nu > 0 \\ \frac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i} & \text{if } \nu < 0 \end{cases} \quad (6.23)$$

Examples of limiter functions ϕ can be found in Chapter 2.

Goodman and LeVeque [19] have shown that in two space dimensions any scheme that is strictly TVD is at most first order accurate. Since the aim is to preserve the second order accuracy of the scheme with flux (6.19), (6.20), (6.21) in regions of smooth flow, there is no point in constructing a strictly TVD version of the scheme. As an attempt to limit any oscillations in regions of high gradient the 'corrector step' flux (6.19) is replaced by the one-dimensional TVD WAF intercell flux

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(1 + \phi_{i+\frac{1}{2},j})f(u_{i,j}^{n+\frac{1}{2}}) + \frac{1}{2}(1 - \phi_{i+\frac{1}{2},j})f(u_{i+1,j}^{n+\frac{1}{2}}) \quad (6.24)$$

The limiters $\phi_{i+\frac{1}{2},j}$ are taken as one dimensional limiters $\phi_{i+\frac{1}{2}}$, such as those presented in Chapter 2, computed from a flow parameter $r_{i+\frac{1}{2}}$ based on data along strip j . Note that when $\phi_{i+\frac{1}{2},j} = \text{sgn}(\nu_x)$ equations (6.24), (6.20), (6.21) give

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(2 - \nu_y)f_{i,j} + \frac{1}{2}\nu_y f_{i,j-1} \quad (6.25)$$

which is the intercell flux for the first order scheme (6.4). The scheme will therefore be monotonic provided $\nu_x \leq 1$ and $\nu_y \leq 1$. Tests of the scheme using (6.24) on the linear advection equation have shown that it fails to produce non-oscillatory results when discontinuities that are not aligned to the grid exist. The reason for this is that the one dimensional flow parameters fail to properly detect these flow features. A replacement for the one dimensional flow parameter is suggested in the next section, but has not yet been

tested in this scheme. The fluxes $g_{i,j+\frac{1}{2}}$ can be written in an analogous form to (6.22), (6.19), (6.20).

The scheme described above is based on the two schemes resulting from fluxes (6.4) and (6.5), each of which has stability condition

$$\max\{\nu_x, \nu_y\} \leq 1 \quad (6.26)$$

It is thus reasonable to assume that the new scheme is also stable under this restriction.

Despite the lack of monotonicity, the scheme has been shown to work reasonably well on non-linear systems, as will be seen in §6.4. For this reason, an algorithm for the complete scheme on non-linear systems of the form (6.11) is now given.

An Algorithm For Nonlinear Systems

1. Set up a regular, Cartesian grid of $M_i \times M_j$ cells with grid spacings Δx and Δy in the x and y directions respectively, add two strips of ghost cells for each boundary and impose the required initial conditions. Set the time counter t equal to zero.
2. Choose the timestep Δt as:

$$\Delta t = \nu_{max} \frac{\min\{\Delta x, \Delta y\}}{\lambda_{max}} \quad (6.27)$$

where $\lambda_{max} = \max_{i,j}\{\lambda_{i,j}\}$, $\lambda_{i,j}$ is the largest absolute characteristic speed in cell $\Delta_{i,j}$, and $\nu_{max} \in (0, 1]$ is some user-defined Courant number coefficient. If $t + \Delta t$ is greater than the required output time, reduce Δt as necessary.

3. For each $i \in [1, M_i]$
 - (a) Set boundary conditions for strip i as follows. For transmissive boundaries, set primitive variables in each ghost cell equal to those in the cell that is inside the flow domain, on the same strip of cells perpendicular to the boundary and the same number of cells from the boundary. Reflective boundaries are treated identically, except the component of velocity perpendicular to the boundary is given the opposite sign.
 - (b) For each $j \in [0, M_j]$, calculate a first order flux $G'_{i,j+\frac{1}{2}}$ using some suitable cheap, robust method.
 - (c) Update strip i of cells by $\frac{1}{2}\Delta t$ to obtain the intermediate state:

$$U_{i,j}^{n+\frac{1}{2},i} = U_{i,j}^n - \frac{\frac{1}{2}\Delta t}{\Delta y} [G'_{i,j+\frac{1}{2}} - G'_{i,j-\frac{1}{2}}] \quad (6.28)$$

This step is effectively a j -sweep of Godunov's method with timestep $\frac{1}{2}\Delta t$, as would be used in an operator split scheme.

4. For each $j \in [1, M_j]$,

- (a) Set boundary conditions for strip j with data $U_{i,j}^{n+\frac{1}{2},i}$, as in 3(a).
- (b) For each $i \in [-1, M_i + 1]$, solve the Riemann problem at boundary $i + \frac{1}{2}, j$ using data $U_{i,j}^{n+\frac{1}{2},i}$ and any suitable Riemann solver.
- (c) Calculate a strip of **WAF** intercell fluxes $F_{i+\frac{1}{2},j}$ from these solutions, as described in Chapter 2, and store them for later use.

This step is effectively an i -sweep of the **WAF** method, as would be used in an operator split scheme, based on the data $U_{i,j}^{n+\frac{1}{2},i}$, without the update at the end.

5. For each $j \in [0, M_j]$

- (a) Set boundary conditions for strip j , as in 3(a).
- (b) For each $i \in [0 : M_i]$, calculate a first order flux $F'_{i+\frac{1}{2},j}$ via some suitable cheap robust method.
- (c) Update strip j of cells by $\frac{1}{2}\Delta t$ to obtain the intermediate state:

$$U_{i,j}^{n+\frac{1}{2},j} = U_{i,j}^n - \frac{\frac{1}{2}\Delta t}{\Delta x} [F'_{i+\frac{1}{2},j} - F'_{i-\frac{1}{2},j}] \quad (6.29)$$

This step is effectively an i -sweep of Godunov's method with timestep $\frac{1}{2}\Delta t$, as would be used in an operator split scheme.

6. For each $i \in [1, M_i]$,

- (a) Set boundary conditions for strip i with data $U_{i,j}^{n+\frac{1}{2},j}$, as in 3(a).
- (b) For each $j \in [-1, M_j + 1]$, solve the Riemann problem at boundary $i, j + \frac{1}{2}$ using data $U_{i,j}^{n+\frac{1}{2},j}$ and any suitable Riemann solver.
- (c) Calculate a strip of **WAF** intercell fluxes $G_{i,j+\frac{1}{2}}$ from these solutions, as described in Chapter 2, and store them for later use.

This step is effectively a j -sweep of the **WAF** method, as would be used in an operator split scheme, based on the data $U_{i,j}^{n+\frac{1}{2},j}$, without the update at the end.

7. Update the solution U^n at timelevel n to the new timelevel $n + 1$ using the conservative formula:

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{\Delta x} [F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j}] - \frac{\Delta t}{\Delta y} [G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}}] \quad (6.30)$$

and increment the time counter t by Δt .

8. If the required output has not been reached, return to step 2.

9. Output solution.

The ratio of Riemann problems solved per timestep to the number of intercell boundaries for this algorithm is 2. For an operator-split **WAF** scheme that is second order accurate in space and time, the value is $\frac{3}{2}$, since three sweeps are required in the Strang splitting. Thus the scheme presented above appears at first to be significantly more expensive than an operator split scheme of equivalent theoretical accuracy. However, when applying the operator split scheme, the full **WAF** scheme, including the limiting and calculation of all fluxes, must be applied on each of the three sweeps. In the case of the scheme suggested above, the full calculation of a one-dimensional **WAF** flux is only required once per intercell boundary per timestep, in stages 4 and 6 of the above listing, which is roughly equivalent in computational cost to just two sweeps in the operator split scheme. On the predictor stages (stages 3 and 5) the calculation of limiter functions and multiple flux calculations are avoided, since only first order accuracy is required. Also, since they are only predictor sweeps, a relatively cheap Riemann solver can be used, provided it is sufficiently robust. The Harten-Lax-van Leer (**HLL**) scheme [23], [66] scheme is recommended, since it has proved suitable in numerical experiments performed by the author. However, it is not suggested that this is the optimum choice of schemes: a full exploration of the use of the other available Riemann solvers, and even replacing Riemann solver technology by, for example, a flux vector splitting scheme in the predictor stages, is needed to find the cheapest alternative that is sufficiently robust. The final answer will probably be problem dependent, i.e depend on, for example, the strength of shocks in the flow being calculated. As far as the corrector stages are concerned, Riemann solver adaption, as put forward by Toro [65] is recommended, between the Selective Curve Fitting (**scf**) solver [59] and the Two Rarefaction (**tr**) solver [66].

The scheme presented above has similarities with Colella's finite volume scheme [8], and the first order versions of the two schemes are, in fact, identical (assuming the same Riemann solvers are used). The difference lies in the addition of the second order terms. Essentially, in steps 3 and 5 in the above algorithm, Colella adds second order terms into the calculation of the predicted state $U^{n+\frac{1}{2}}$ by reconstructing, or 'sloping', the data U^n in a direction perpendicular to the direction of the boundary for which the fluxes are being calculated, i.e $\frac{\delta U}{\delta x}$ terms are added to $U^{n+\frac{1}{2}}$ in step 3 and $\frac{\delta U}{\delta y}$ terms are added to $U^{n+\frac{1}{2}}$ in step 5. In the corrector steps 4 and 6 the calculation of **WAF** fluxes are then 'replaced' by the calculation of Godunov fluxes, i.e first order fluxes. The resulting scheme is second order accurate in space and time, as is the scheme presented here. To avoid oscillatory behaviour the slopes added into Colella's scheme are limited in the way slopes are limited in one-dimensional **MUSCL** schemes. Colella reported that his scheme, as described above, did not achieve satisfactory results in all situations. He found that moderately strong shocks oblique to the grid were either smeared or suffered from oscillatory behaviour. He argued that the first order accuracy of the calculations of gradients parallel to the boundary (i.e the equivalent of our first order steps 3 and 5 in the above algorithm) were not sufficiently accurate to correctly model the fluid flow in these situations. He resolved the problem by sloping the data parallel to, as well as perpendicular to, boundaries in the calculation of $U^{n+\frac{1}{2}}$, making the step second order accurate. He reported that this did alleviate the

problem. It will be shown in the numerical results in §6.4 that the algorithm presented above does suffer from oscillatory behaviour when computing discontinuities oblique to the grid. A similar idea to Colella's could be used, by replacing steps 3 and 5 above with second-order accurate **war** steps. However, this would change the stencil of the scheme on the linear advection equation and the resulting scheme would therefore not be an extension of the scheme this chapter has set out to extend. It should be remembered that when fully limited, the scheme presented above does reduce to a first order monotonic scheme, and thus the problem lies not in the general approach, but the mechanism by which the scheme detects high gradients in the flow. At present, only one-dimensional detection mechanisms along strips of cells are used. On hindsight, these are bound to be insufficient — they can not be expected to react properly to multi-dimensional flow features. More work is needed to determine a suitable parameter that takes into account more multidimensional information. Such a parameter is suggested in the next section for the scheme developed there. It has not yet been tested on the scheme developed in this section.

6.2.2 An Alternative Formulation of the Two Dimensional Case

It is important to note here that the formulation of the scheme presented in §6.2.1 is certainly not unique. One of many alternative formulations is now given. Consider again the intercell flux (6.5) for the second order scheme on the linear advection equation:

$$\begin{aligned} f_{i+\frac{1}{2},j} &= \frac{1}{4}(1+\nu_x)(2-\nu_y)f_{i,j} + \frac{1}{4}(1-\nu_x)(2-\nu_y)f_{i+1,j} \\ &\quad + \frac{1}{4}(1+\nu_x)\nu_y f_{i,j-1} + \frac{1}{4}(1-\nu_x)\nu_y f_{i+1,j-1} \end{aligned} \quad (6.31)$$

This can be rewritten as:

$$\begin{aligned} f_{i+\frac{1}{2},j} &= \frac{a}{2}(2-\nu_y) \left[\frac{1}{2}(1+\nu_x)u_{i,j}^n + \frac{1}{2}(1-\nu_x)u_{i+1,j}^n \right] \\ &\quad + \frac{a}{2}\nu_y \left[\frac{1}{2}(1+\nu_x)u_{i,j-1}^n + \frac{1}{2}(1-\nu_x)u_{i+1,j-1}^n \right] \end{aligned} \quad (6.32)$$

Define

$$\hat{u}_{i+\frac{1}{2},j} = \frac{1}{2}(1+\nu_x)u_{i,j}^n + \frac{1}{2}(1-\nu_x)u_{i+1,j}^n \quad (6.33)$$

Then

$$f_{i+\frac{1}{2},j} = \frac{a}{2}(2-\nu_y)\hat{u}_{i+\frac{1}{2},j} + \frac{a}{2}\nu_y\hat{u}_{i+\frac{1}{2},j-1} \quad (6.34)$$

$$= a \left[\hat{u}_{i+\frac{1}{2},j} - \frac{1}{2}\nu_y \left(\hat{u}_{i+\frac{1}{2},j} - \hat{u}_{i+\frac{1}{2},j-1} \right) \right] \quad (6.35)$$

$$= a \left[\hat{u}_{i+\frac{1}{2},j} - \frac{\frac{1}{2}\Delta t}{\Delta y} \left(g'_{i,j+\frac{1}{2}} - g'_{i,j-\frac{1}{2}} \right) \right] \quad (6.36)$$

where $g'_{i,j+\frac{1}{2}} = b\hat{u}_{i,j+\frac{1}{2}}$ is the FOU flux at boundary $(i, j + \frac{1}{2})$ evaluated on data $\hat{u}$. Hence

$$f_{i+\frac{1}{2},j} = f\left(u_{i+\frac{1}{2},j}^{n+\frac{1}{2}}\right) \quad (6.37)$$

$$u_{i+\frac{1}{2},j}^{n+\frac{1}{2}} = \hat{u}_{i+\frac{1}{2},j} - \frac{\frac{1}{2}\Delta t}{\Delta y} [g'_{i,j+\frac{1}{2}} - g'_{i,j-\frac{1}{2}}] \quad (6.38)$$

$$g'_{i,j+\frac{1}{2}} = g(\hat{u}_{i,j}) = b\hat{u}_{i,j}$$

$$\hat{u}_{i+\frac{1}{2},j} = \frac{1}{2}(1 + \nu_x)u_{i,j} + \frac{1}{2}(1 - \nu_x)u_{i+1,j} \quad (6.39)$$

One dimensional limiter functions $\phi_{i+\frac{1}{2},j} = \phi_{i+\frac{1}{2},j}(r_{i+\frac{1}{2},j}, \nu_{i+\frac{1}{2},j})$ can be incorporated into this version of the scheme quite simply, by replacing (6.39) with a TVD 'weighted average state' step:

$$\hat{u}_{i+\frac{1}{2},j} = \frac{1}{2}(1 + \phi_{i+\frac{1}{2},j})u_{i,j} + \frac{1}{2}(1 - \phi_{i+\frac{1}{2},j})u_{i+1,j} \quad (6.40)$$

It was hoped that the algorithm (6.40), (6.38), (6.39) would not suffer as badly from the oscillations affecting the first version of the algorithm near oblique discontinuities, due to the fact that the flow parameters $r_{i+\frac{1}{2},j}$ depend on the data at timelevel n and not some intermediate state. However, numerical experiments have shown this not to be the case. To find a non-oscillatory version of the scheme, it does appear necessary to find multi-dimensional flow parameters such as the one suggested in the next section.

The extension of the algorithm (6.40), (6.38), (6.39) to nonlinear systems follows very similar lines to the previous extension. It has not yet been tested numerically, and will thus not be discussed further here, other than to say that it is potentially cheaper than the scheme of §6.2.2, since comparing (6.38) with (6.19) shows that fewer flux calculations are required.

6.2.3 The Three Dimensional Case

The three dimensional equivalent of the scheme discussed in the last section is now considered. The scheme for the three dimensional linear advection equation (6.6) is determined by the flux (6.10) which is repeated here for convenience:

$$\begin{aligned} f_{i+\frac{1}{2},j,k} &= \frac{1}{8}(1 + \nu_x)(2 - \nu_y)(2 - \nu_z)f_{i,j,k} + \frac{1}{8}(1 - \nu_x)(2 - \nu_y)(2 - \nu_z)f_{i+1,j,k} \\ &+ \frac{1}{8}(1 + \nu_x)\nu_y(2 - \nu_z)f_{i,j-1,k} + \frac{1}{8}(1 - \nu_x)\nu_y(2 - \nu_z)f_{i+1,j-1,k} \\ &+ \frac{1}{8}(1 + \nu_x)(2 - \nu_y)\nu_z f_{i,j,k-1} + \frac{1}{8}(1 - \nu_x)(2 - \nu_y)\nu_z f_{i+1,j,k-1} \\ &+ \frac{1}{8}(1 + \nu_x)\nu_y\nu_z f_{i,j-1,k-1} + \frac{1}{8}(1 - \nu_x)\nu_y\nu_z f_{i+1,j-1,k-1} \end{aligned} \quad (6.41)$$

The approach used to extend the two dimensional scheme in §5.2.2 can be applied, quite easily, to extend (6.41) to non-linear systems. It is easy to see that (6.41) can be written as

$$f_{i+\frac{1}{2},j,k} = \frac{1}{2}(1 + \nu_x)f\left(u_{i,j,k}^{n+\frac{1}{2}}\right) + \frac{1}{2}(1 - \nu_x)f\left(u_{i+1,j,k}^{n+\frac{1}{2}}\right) \quad (6.42)$$

where

$$\begin{aligned} u_{i,j,k}^{n+\frac{1}{2}} &= \frac{1}{4}(2 - \nu_y)(2 - \nu_z)u_{i,j,k} + \frac{1}{4}\nu_y(2 - \nu_z)u_{i,j-1,k} \\ &\quad + \frac{1}{4}(2 - \nu_y)\nu_z u_{i,j,k-1} + \frac{1}{4}\nu_y\nu_z u_{i,j-1,k-1} \end{aligned} \quad (6.43)$$

As in the two-dimensional case, spurious oscillations can be at least partially controlled by replacing (6.42) with a one dimensional **WAF** flux described in Chapter 2, i.e by taking

$$f_{i+\frac{1}{2},j,k} = \frac{1}{2}(1 + \phi_{i+\frac{1}{2},j,k})f(u_{i,j,k}^{n+\frac{1}{2}}) + \frac{1}{2}(1 - \phi_{i+\frac{1}{2},j,k})f(u_{i+1,j,k}^{n+\frac{1}{2}}) \quad (6.44)$$

The calculation of $u_{i,j,k}^{n+\frac{1}{2}}$ in (6.43) can be done in two ways. The first is to treat (6.43) as a finite volume scheme, and re-write it in conservation form

$$u_{i,j,k}^{n+\frac{1}{2}} = u_{i,j,k}^n - \frac{\Delta t}{\Delta y} \left[g'_{i,j+\frac{1}{2},k} - g'_{i,j-\frac{1}{2},k} \right] - \frac{\Delta t}{\Delta z} \left[h'_{i,j,k+\frac{1}{2}} - h'_{i,j,k-\frac{1}{2}} \right] \quad (6.45)$$

with

$$g'_{i,j+\frac{1}{2},k} = \frac{1}{8}(4 - \nu_z)g_{i,j,k} + \frac{1}{8}\nu_z g_{i,j,k} \quad (6.46)$$

$$h'_{i,j,k+\frac{1}{2}} = \frac{1}{8}(4 - \nu_y)h_{i,j,k} + \frac{1}{8}\nu_y h_{i,j,k} \quad (6.47)$$

Alternatively, note that (6.43) is, in fact, the scheme resulting from first order operator splitting of the one dimensional **FOU** scheme, i.e:

$$u_{i,j,k}^{n+\frac{1}{2}} = L_j^{(\frac{1}{2}\Delta t)} \left(L_k^{(\frac{1}{2}\Delta t)} \left(u_{i,j,k}^n \right) \right) = L_k^{(\frac{1}{2}\Delta t)} \left(L_j^{(\frac{1}{2}\Delta t)} \left(u_{i,j,k}^n \right) \right) \quad (6.48)$$

where the operators $L_j^{(\Delta t)}$ and $L_k^{(\Delta t)}$ can be written in conservation form as

$$L_j^{(\Delta t)} \left(u_{i,j,k}^n \right) = u_{i,j,k}^n - \frac{\Delta t}{\Delta y} \left[g_{i,j+\frac{1}{2},k}^{os} - g_{i,j-\frac{1}{2},k}^{os} \right] \quad (6.49)$$

with $g_{i,j+\frac{1}{2},k}^{os} = g_{i,j,k} = bu_{i,j,k}^n$ and

$$L_k^{(\Delta t)} \left(u_{i,j,k}^n \right) = u_{i,j,k}^n - \frac{\Delta t}{\Delta z} \left[h_{i,j,k+\frac{1}{2}}^{os} - h_{i,j,k-\frac{1}{2}}^{os} \right] \quad (6.50)$$

with $h_{i,j,k+\frac{1}{2}}^{os} = h_{i,j,k} = cu_{i,j,k}^n$.

The scheme can be extended to non-linear systems

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y + \mathbf{H}(\mathbf{U})_z = 0 \quad (6.51)$$

via the conservative formula

$$\begin{aligned} \mathbf{U}_{i,j,k}^{n+1} = & \mathbf{U}_{i,j,k}^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+\frac{1}{2},j,k} - \mathbf{F}_{i-\frac{1}{2},j,k}] \\ & - \frac{\Delta t}{\Delta y} [\mathbf{G}_{i,j+\frac{1}{2},k} - \mathbf{G}_{i,j-\frac{1}{2},k}] - \frac{\Delta t}{\Delta z} [\mathbf{H}_{i,j,k+\frac{1}{2}} - \mathbf{H}_{i,j,k-\frac{1}{2}}] \end{aligned} \quad (6.52)$$

where

$$\mathbf{U}_{i,j,k}^n = \frac{1}{V(\Delta_{i,j,k})} \int_{\Delta_{i,j,k}} \mathbf{U} dx dy dz \quad (6.53)$$

The intermediate state $\mathbf{U}^{n+\frac{1}{2}}$ for the flux $\mathbf{F}_{i+\frac{1}{2},j,k}$ is calculated from either an extension of the finite volume scheme (6.45), (6.46), (6.47) or the operator split scheme (6.48), (6.49), (6.50). An extension of the (6.45), (6.46), (6.47) has yet to be developed. However, the operator split scheme (6.48), (6.49) (6.50) can be reinterpreted as operator split Godunov's scheme with any exact or approximate Riemann solver or any other first order scheme that reduces to the FOU scheme on the linear advection equation. The calculation of intermediate states $\mathbf{U}^{n+\frac{1}{2}}$ for the fluxes $\mathbf{G}_{i,j+\frac{1}{2},k}$ and $\mathbf{H}_{i,j,k+\frac{1}{2}}$ follow in an analogous fashion. Although the calculation of each of the three intermediate states involves two sweeps of the one-dimensional scheme, only five sweeps are needed altogether: since the order of the sweeps in each calculation is arbitrary, two of the intermediate states can share the sweep performed on data $\mathbf{U}^n$.

To ensure the scheme described above remains stable, the timestep Δt should be chosen as follows. First compute a maximum wavespeed λ_{max} for the whole domain as

$$\lambda_{max} = \max_{i,j,k} \{ |\lambda_{i,j,k}| \} \quad (6.54)$$

where $\lambda_{i,j,k}$ are characteristic speeds based on the data $\mathbf{U}^n$. The timestep can then be computed as

$$\Delta t = \nu_{max} \frac{\min\{\Delta x, \Delta y, \Delta z\}}{\lambda_{max}} \quad (6.55)$$

where $\nu_{max} \in (0, \frac{2}{3}]$ is some user-defined Courant number coefficient.

6.3 A WAF-Type Extension

Since the schemes for the linear advection equation presented in Chapter 5 were constructed via the WAF approach, it seems reasonable to try constructing intercell fluxes for non-linear systems in a similar manner. The aim of this section is to discuss such extensions.

6.3.1 Extension of the Two Dimensional Scheme

A WAF-type intercell flux for the system (6.11) can be defined as

$$F_{i+\frac{1}{2},j} = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I \mathbf{F}(\mathbf{U}^*(x, y, t)) dx dy dt \quad (6.56)$$

for some space integration range I . Here $\mathbf{U}^*(x, y, t)$ represents the solution to some initial value problem for the system (6.11). The obvious choice for these initial value problems are two dimensional Riemann problems, which have as initial data the states $\mathbf{U}_{i,j}$, $\mathbf{U}_{i+1,j}$, $\mathbf{U}_{i,j+1}$, $\mathbf{U}_{i+1,j+1}$ constant in their respective cells. However, as explained earlier, no sufficiently general exact solutions to these problems seem to exist at present. Therefore, rather than use (6.56) directly, the flux (6.5) for the linear advection equation is reinterpreted as the integral of the component of flux prepending to the boundary $(i + \frac{1}{2}, j)$ on the combined solution to one dimensional grid-aligned Riemann problems at nearby intercell boundaries, as follows.

Consider, then, the case of the linear advection equation (6.1). Assume the wavespeeds a and b are constant, but of any sign. The flux (6.5) can be written as:

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(f_{i,j} + f_{i+1,j}) - \sum_{l=1}^7 \zeta^{(l)} \Delta^{(l)} f \quad (6.57)$$

where the index l refers to nearby intercell boundaries, labeled as in Fig. 6.1. By comparing (6.57) with (6.5) it can be easily seen that there is no unique solution for the set of coefficients $\{\zeta^{(l)}\}_{l=1}^7$. However, a sensible set can be chosen as follows.

Consider a wave from the Riemann problem aligned to boundary $(i + \frac{1}{2}, j)$. It will have a component of velocity perpendicular to the boundary at which it was generated. This velocity is the wavespeed from the solution to the Riemann problem at that boundary and always equals the constant speed a in the present case. The wave can be made truly two dimensional by also giving it a component of velocity parallel to the boundary. For this velocity, the component of velocity parallel to the boundary in the state behind the wave can be used. This will always be the constant b in the present case. With these two components of velocity, the wave will sweep over an area in space as time progresses. Waves from boundaries $(i, j + \frac{1}{2})$ are treated in an analogous fashion, with the roles of a and b reversed. See Fig. 6.2. Recall the scheme was originally derived from the general WAF intercell flux

$$f_{i+\frac{1}{2},j} = \frac{1}{t_2 - t_1} \frac{1}{V(I)} \int_{t_1}^{t_2} \int_I f(u^*(x, y, t)) dx dy dt \quad (6.58)$$

via the midpoint rule in time. At time $t = \frac{1}{2}\Delta t$, part (or all) of the area swept out by each wave will intersect the space integration range I defined in §5.2. The weight η for each wave jump in (6.57) is set equal to the area of this intersection divided by the area of the space integration range.

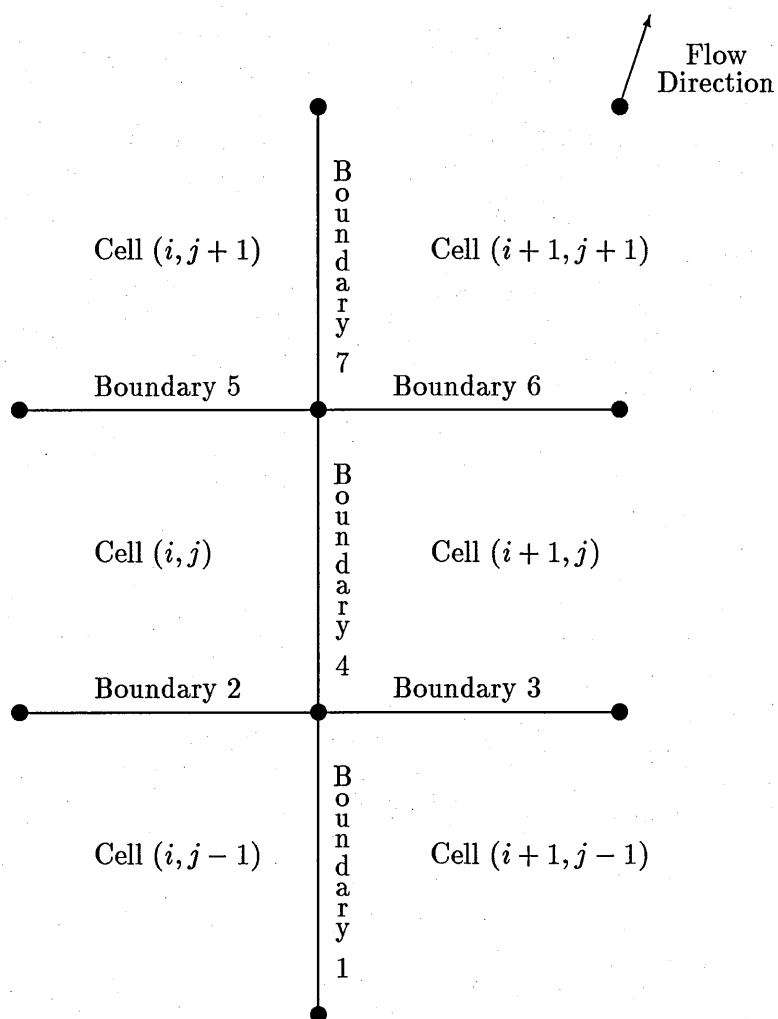
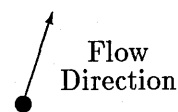
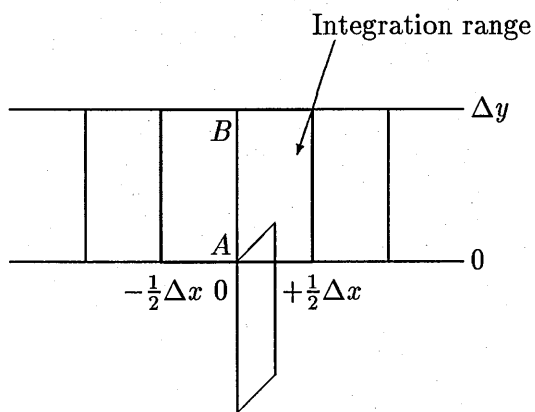


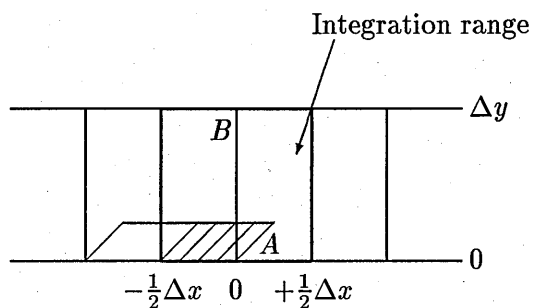
Figure 6.1: Labeling of the boundaries in the two dimensional wave model for $f_{i+\frac{1}{2},j}$. Boundary 4 is the boundary across which the flux in being calculated.



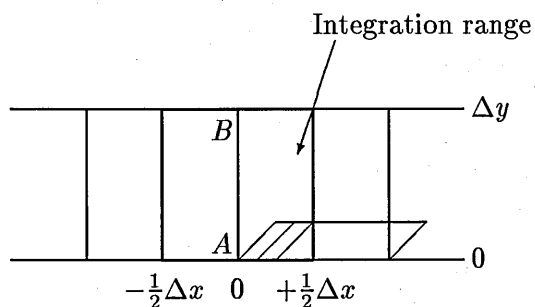
BOUNDARY 1



BOUNDARY 2



BOUNDARY 3



BOUNDARY 4

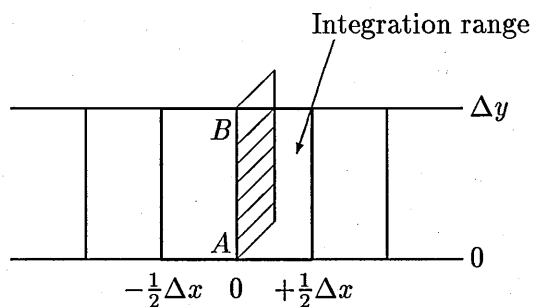


Figure 6.2: Weights for the two-dimensional wave model.

These weights can easily be shown to be:

$$\begin{aligned}
 \zeta^{(1)} &= \frac{1}{8}\nu_x \max\{\nu_y, 0\} & \zeta^{(7)} &= -\frac{1}{8}\nu_x \min\{\nu_y, 0\} \\
 \zeta^{(2)} &= \frac{1}{8}(2 + \nu_x) \max\{\nu_y, 0\} & \zeta^{(6)} &= \frac{1}{8}(2 - \nu_x) \min\{\nu_y, 0\} \\
 \zeta^{(3)} &= \frac{1}{8}(2 - \nu_x) \max\{\nu_y, 0\} & \zeta^{(5)} &= \frac{1}{8}(2 + \nu_x) \min\{\nu_y, 0\} \\
 \zeta^{(4)} &= \frac{1}{8}\nu_x(4 - |\nu_y|)
 \end{aligned} \tag{6.59}$$

Substituting the weights (6.59) into (6.57) gives the flux in full as:

$$f_{i+\frac{1}{2},j} = \frac{1}{4}(1 + \nu_x)(2 - |\nu_y|)f_{i,j} + \frac{1}{4}(1 - \nu_x)(2 - |\nu_y|)f_{i+1,j}$$

$$\begin{aligned}
& + \frac{1}{4}(1 + \nu_x)\max\{\nu_y, 0\}f_{i,j-1} + \frac{1}{4}(1 - \nu_x)\max\{\nu_y, 0\}f_{i+1,j-1} \\
& - \frac{1}{4}(1 + \nu_x)\min\{\nu_y, 0\}f_{i,j+1} - \frac{1}{4}(1 - \nu_x)\min\{\nu_y, 0\}f_{i+1,j+1}
\end{aligned} \quad (6.60)$$

This is a generalization of (6.5) to wavespeeds a, b of any sign. The flux $g_{i,j+\frac{1}{2}}$ can be written in a similar form to (6.57) as

$$g_{i,j+\frac{1}{2}} = \frac{1}{2}(g_{i,j} + g_{i,j+1}) - \sum_{l=1}^7 \eta^{(l)} \Delta^{(l)} g \quad (6.61)$$

where the boundaries are now labelled as in Fig. 6.3. Following the same procedure as above to derive the coefficients gives:

$$\begin{aligned}
\eta^{(1)} &= \frac{1}{8}\nu_y\max\{\nu_x, 0\} & \eta^{(7)} &= -\frac{1}{8}\nu_y\min\{\nu_x, 0\} \\
\eta^{(2)} &= \frac{1}{8}(2 + \nu_y)\max\{\nu_x, 0\} & \eta^{(6)} &= \frac{1}{8}(2 - \nu_y)\min\{\nu_x, 0\} \\
\eta^{(3)} &= \frac{1}{8}(2 - \nu_y)\max\{\nu_x, 0\} & \eta^{(5)} &= \frac{1}{8}(2 + \nu_y)\min\{\nu_x, 0\} \\
\eta^{(4)} &= \frac{1}{8}\nu_y(4 - |\nu_x|)
\end{aligned} \quad (6.62)$$

In the linear problem studied in this section, the wave model described above is essentially the same as the wave model of LeVeque [32] but the scheme he used is different from the scheme presented here.

The scheme resulting from (6.57),(6.59),(6.61),(6.62) is second order accurate, and will therefore suffer from spurious oscillations in regions of high gradient. In one dimension, this problem can be avoided by making the scheme satisfy **tvD** criteria (see Chapter 2). As mentioned earlier, Goodman and LeVeque [19] have shown that any two dimensional **tvD** scheme is at most first order accurate. Since the aim here is to preserve second order accuracy at least in regions of smooth flow, the derivation of a **tvD** version of the above scheme has not been attempted. However, spurious oscillations can at least be partially removed by incorporating one-dimensional **tvD** limiter functions into the fluxes (6.57), (6.61). One way to achieve this is to replace the weights (6.59) by

$$\begin{aligned}
\zeta^{(1)} &= \frac{1}{8}\phi_{i+\frac{1}{2},j}^{(1)}\max\{\nu_y, 0\} & \zeta^{(7)} &= -\frac{1}{8}\phi_{i+\frac{1}{2},j}^{(7)}\min\{\nu_y, 0\} \\
\zeta^{(2)} &= \frac{1}{8}(2 + \phi_{i+\frac{1}{2},j}^{(2)})\max\{\nu_y, 0\} & \zeta^{(6)} &= \frac{1}{8}(2 - \phi_{i+\frac{1}{2},j}^{(6)})\min\{\nu_y, 0\} \\
\zeta^{(3)} &= \frac{1}{8}(2 - \phi_{i+\frac{1}{2},j}^{(3)})\max\{\nu_y, 0\} & \zeta^{(5)} &= \frac{1}{8}(2 + \phi_{i+\frac{1}{2},j}^{(5)})\min\{\nu_y, 0\} \\
\zeta^{(4)} &= \frac{1}{8}\phi_{i+\frac{1}{2},j}^{(4)}(4 - |\nu_y|)
\end{aligned} \quad (6.63)$$

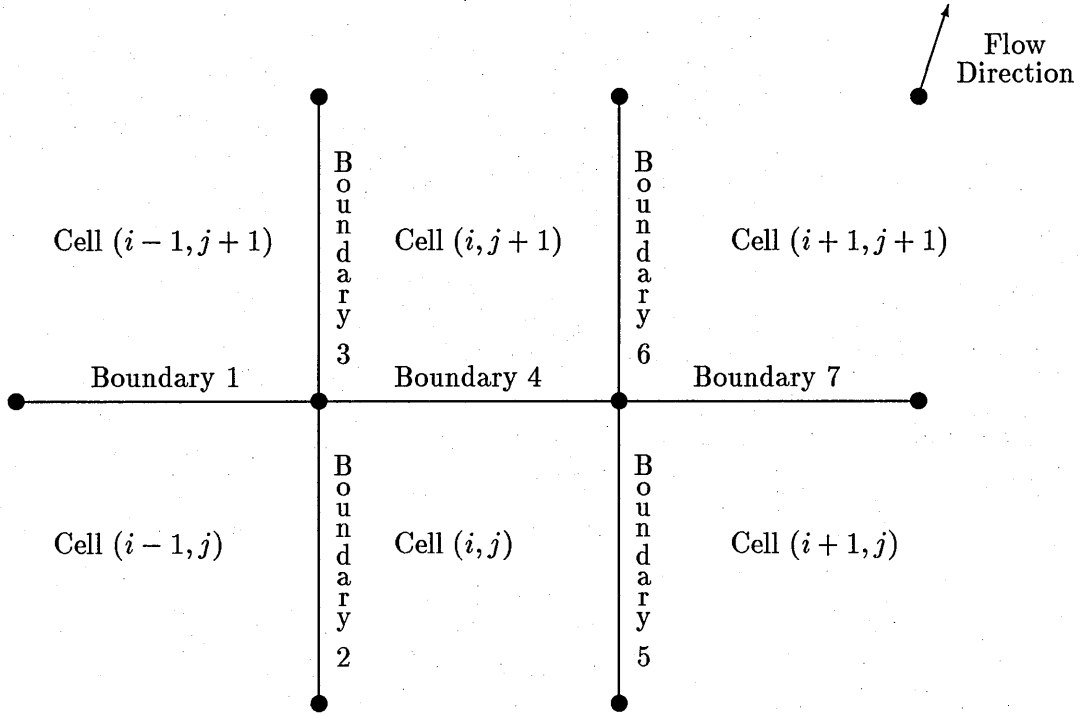


Figure 6.3: Labeling of the boundaries the two dimensional wave model for $g_{i,j+\frac{1}{2}}$. Boundary 4 is the boundary across which the flux is being calculated.

where the limiter $\phi_{i+\frac{1}{2},j}^{(l)}$ is the one dimensional limiter for the contribution of boundary l ($l = 1, \dots, 7$) to the flux at boundary $(i+\frac{1}{2}, j)$. Note that when $\phi^{(l)} \equiv \nu_x$, (6.63) reduces to (6.59) and is second order accurate in space and time and when $\phi^{(l)} \equiv \text{sgn}(\nu_x)$ (6.63), (6.57) reduces to:

$$f_{i+\frac{1}{2},j} = \frac{1}{4}(4 - |\nu_y|)f_{i,j} + \frac{1}{4}\max\{\nu_y, 0\}f_{i,j-1} - \frac{1}{4}\min\{\nu_y, 0\}f_{i,j+1} \quad (6.64)$$

for positive ν_x , and

$$f_{i+\frac{1}{2},j} = \frac{1}{4}(4 - |\nu_y|)f_{i+1,j} + \frac{1}{4}\max\{\nu_y, 0\}f_{i+1,j-1} - \frac{1}{4}\min\{\nu_y, 0\}f_{i+1,j+1} \quad (6.65)$$

for negative ν_x . This is a generalization of the first order flux (6.4) to wavespeeds a, b that are constant but of any sign. The scheme resulting from fluxes (6.64) and (6.65) is monotonic provided $|\nu_x| \leq 1$, $|\nu_y| \leq 1$, and will thus not produce oscillations in regions of high gradients under this restriction. Hence it is suitable for use as the first order component of a scheme using limiter functions.

It was hoped that using the MINMOD or VAN LEER type limiters (2.79), (2.80) directly from the one dimensional case in (6.57), (6.63) would ensure the resulting scheme remained oscillation-free in all situations. However, this was only partially successful. The strategy worked well provided the flow was approximately aligned to the grid, but shocks moving

oblique to the grid lines suffered from overshoots of a few percent of shock height, though much smaller than the overshoots in the second order scheme without limiters. It was found that the situation can be greatly improved by replacing the flow parameter (6.23) by

$$r_{i+\frac{1}{2},j}^{2D} = (1 - |\nu_y|)r_{i+\frac{1}{2},j} + \max\{\nu_y, 0\}r_{i+\frac{1}{2},j-1} + \min\{\nu_y, 0\}r_{i+\frac{1}{2},j+1} \quad (6.66)$$

where $r_{i+\frac{1}{2},j}$ is the one-dimensional flow parameter $r_{i+\frac{1}{2}}$ from (6.23) along strip j . In most situations, this parameter appears to result in solutions without significant overshoots. There can be, however, small *undershoots* in certain situations. Further experimentation may provide an improvement on (6.66).

The essential components of the scheme constructed here have been the generalizations of the second order scheme (5.32) and the first order scheme (5.25) to the case of speeds a and b of any sign. These are both stable provided the condition

$$\max\{|\nu_x|, |\nu_y|\} \leq 1 \quad (6.67)$$

is satisfied. It is thus reasonable to assume that the scheme presented here also has this stability condition.

The extension of the flux (6.57), (6.59) to nonlinear systems can be done as follows. Equation (6.57) is replaced by

$$\mathbf{F}_{i+\frac{1}{2},j} = \frac{1}{2}(\mathbf{F}_{i,j} + \mathbf{F}_{i+1,j}) - \sum_{l=1,7} \sum_{m=1,N} \zeta^{(l,m)} \Delta^{(l,m)} \mathbf{F} \quad (6.68)$$

where $\Delta^{(l,m)} \mathbf{F}$ denotes the jump in $\mathbf{F}$ across the m^{th} wave from the Riemann problem at boundary l in Fig. 6.4, and $\zeta^{(l,m)}$ is the associated coefficient, defined via (6.63) as follows. First define, for each boundary l a Courant number $\nu_P^{(l,m)}$ for the m^{th} wave from the grid aligned Riemann problem at that boundary and moving perpendicular to that boundary as $\nu_P^{(l,m)} = a^{(l,m)} \frac{\Delta t}{\Delta d}$, where $a^{(l,m)}$ is the wavespeed of the m^{th} wave in the solution to the Riemann problem, and Δd is Δx or Δy , depending on the orientation of the boundary. Similarly define a Courant number $\nu_S^{(l,m)}$ for flow moving parallel to the boundary as $\nu_S^{(l,m)} = v^{(l,m)} \frac{\Delta t}{\Delta d}$ where $v^{(l,m)}$ is the component of velocity parallel to the boundary on the state immediately *behind* the m^{th} wave in the solution to the Riemann problem and Δd is the length of the boundary. The weights $\zeta_{i+\frac{1}{2},j}^{(l,m)}$ are derived from (6.63) and are defined as:

$$\begin{aligned} \zeta_{i+\frac{1}{2},j}^{(1,m)} &= \frac{1}{8} \phi_{i+\frac{1}{2},j}^{(1,m)} \max\{\nu_P^{(1,m)}, 0\} & \zeta_{i+\frac{1}{2},j}^{(7,m)} &= -\frac{1}{8} \phi_{i+\frac{1}{2},j}^{(7,m)} \min\{\nu_P^{(7,m)}, 0\} \\ \zeta_{i+\frac{1}{2},j}^{(2,m)} &= \frac{1}{8} (2 + \phi_{i+\frac{1}{2},j}^{(2,m)}) \max\{\nu_S^{(2,m)}, 0\} & \zeta_{i+\frac{1}{2},j}^{(6,m)} &= \frac{1}{8} (2 - \phi_{i+\frac{1}{2},j}^{(6,m)}) \min\{\nu_S^{(6,m)}, 0\} \\ \zeta_{i+\frac{1}{2},j}^{(3,m)} &= \frac{1}{8} (2 - \phi_{i+\frac{1}{2},j}^{(3,m)}) \max\{\nu_S^{(3,m)}, 0\} & \zeta_{i+\frac{1}{2},j}^{(5,m)} &= \frac{1}{8} (2 + \phi_{i+\frac{1}{2},j}^{(5,m)}) \min\{\nu_S^{(5,m)}, 0\} \\ \zeta_{i+\frac{1}{2},j}^{(4,m)} &= \frac{1}{8} \phi_{i+\frac{1}{2},j}^{(4,m)} (4 - |\nu_P^{(4,m)}|) \end{aligned} \quad (6.69)$$

where $\phi_{i+\frac{1}{2},j}^{(l,m)} = \phi(r,\nu)$ is a one dimensional limiter function computed using the flow parameter (6.66) for the m^{th} wave at boundary $i + \frac{1}{2}$, and $\nu_S^{(l,m)}$ for $l = 1, 4, 7$, $\nu_P^{(l,m)}$ otherwise.

The fluxes $G_{i,j+\frac{1}{2}}$ follow in an analogous manner. Assuming the stability of the scheme transfers from the linear case (6.67), the timestep is taken from

$$\Delta t = \nu_{max} \frac{\min\{\Delta x, \Delta y\}}{\lambda_{max}} \quad (6.70)$$

where λ_{max} is the maximum absolute characteristic speed over the whole domain at timelevel n and $\nu_{max} \in (0, 1]$ is some user defined Courant number coefficient.

6.3.2 Extension of the Three Dimensional Scheme

The second order flux (6.41) for three space dimensions can be extended for use on non-linear systems of equations in a similar manner to that discussed for the two dimensional scheme in the last section. All that is needed is a suitable wave decomposition. The equivalent of (6.57) is

$$f_{i+\frac{1}{2},j,k} = \frac{1}{2}(f_{i,j,k} + f_{i+1,j,k}) - \sum_{l=1}^{12} \zeta^{(l)} \Delta^{(l)} f \quad (6.71)$$

where $\Delta^{(l)} f$ is the flux jump across the l th boundary and $\zeta^{(l)}$ is the corresponding coefficient. There are twelve boundaries involved in the present case (constant, positive wavespeeds) which are labeled as in Fig. 6.4. The flux is sought for boundary 1.

Writing the flux (6.71) out in full and comparing it with (6.41), eight conditions (from enforcing equality of coefficients) are found governing twelve "variables" (the coefficients $\zeta^{(l)}$). The set of possible coefficients $\{\zeta^{(l)}\}$ is therefore not unique. To choose a set of coefficients, recall that in the two dimensional case the weight for each boundary equalled the area of the intersection of the integration range with the region swept out by a wave from that boundary at time $t = \frac{1}{2}\Delta t$, divided by the area of the integration range. To calculate the weights $\zeta^{(l)}$ for the three dimensional case, the equivalent principle is applied, using *volumes* instead of *areas* — each weight should be the volume of the intersection of the integration range with the region swept out by the wave from the boundary at time $t = \frac{1}{2}\Delta t$ divided by the volume of the integration range. As an example, Fig. 6.5 shows the intersection of the volume swept out by the wave from boundary 1 with the integration range.

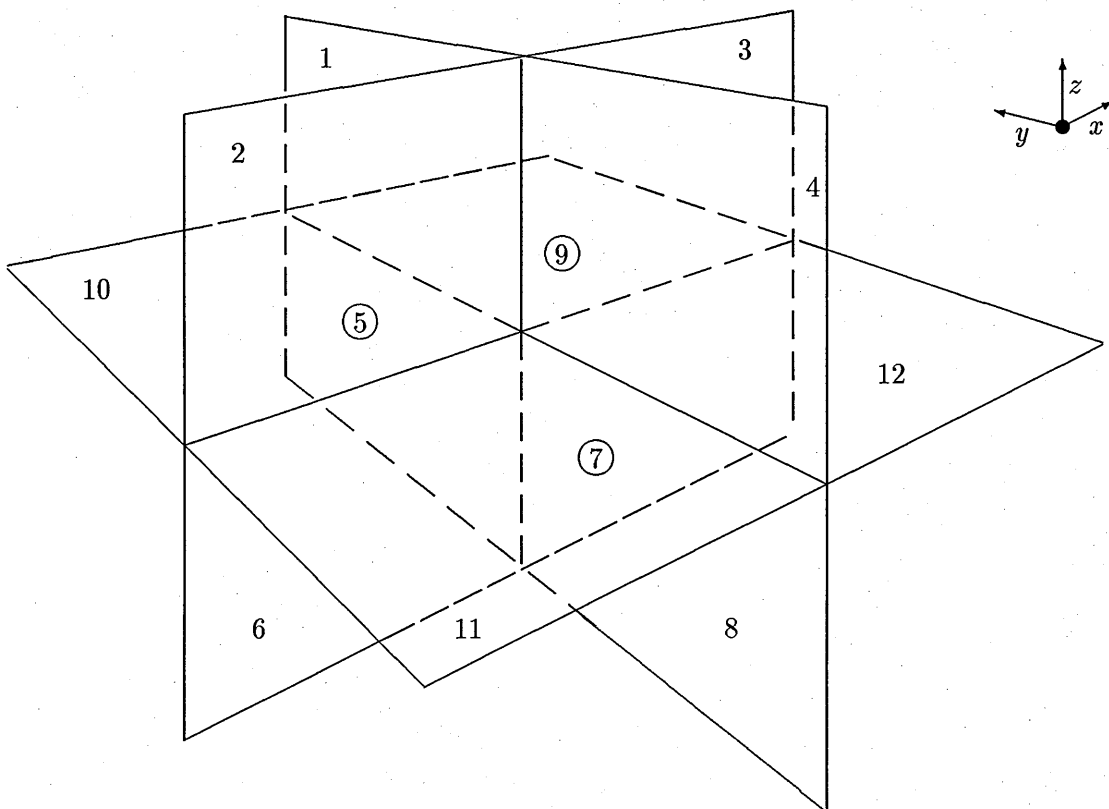


Figure 6.4: Labeling of the twelve boundaries contributing to the three dimensional second order flux. Boundary 1 is the boundary across which the flux is being calculated. The flow is moving *into* the page, and *upwards*. Circles indicate the boundary 'behind'.

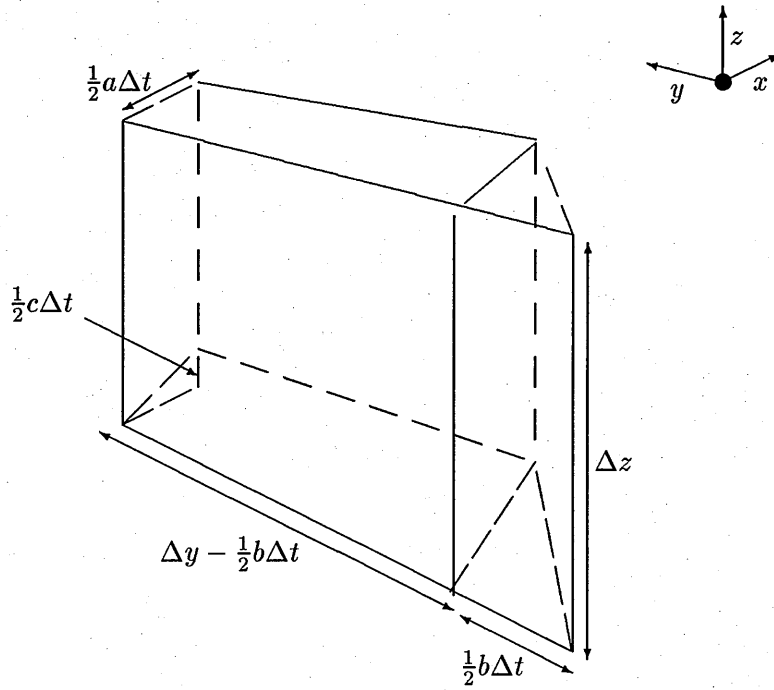


Figure 6.5: Integration for the weights in the 3D wave model — the volume for boundary 1. The flow is moving *into* the page, and *upwards*.

The coefficients derived via this approach are:

$$\begin{aligned}
 \zeta^{(1)} &= \frac{1}{24} \nu_x (12 - 3\nu_y - 3\nu_z + \nu_y \nu_z) & \zeta^{(2)} &= \frac{1}{48} \nu_y (12 + 6\nu_x - 3\nu_z - 2\nu_x \nu_y) \\
 \zeta^{(3)} &= \frac{1}{48} \nu_y (12 - 6\nu_x - 3\nu_z + 2\nu_x \nu_z) & \zeta^{(4)} &= \frac{1}{24} \nu_x \nu_y (3 - \nu_z) \\
 \zeta^{(5)} &= \frac{1}{24} \nu_x \nu_z (3 - \nu_y) & \zeta^{(6)} &= \frac{1}{48} \nu_y \nu_z (3 + 2\nu_x) \\
 \zeta^{(7)} &= \frac{1}{48} \nu_y \nu_z (3 - 2\nu_x) & \zeta^{(8)} &= \frac{1}{24} \nu_x \nu_y \nu_z \\
 \zeta^{(9)} &= \frac{1}{48} \nu_z (12 - 6\nu_x - 3\nu_y + 2\nu_x \nu_y) & \zeta^{(10)} &= \frac{1}{48} \nu_z (12 + 6\nu_x - 3\nu_y - 2\nu_x \nu_y) \\
 \zeta^{(11)} &= \frac{1}{48} \nu_y \nu_z (3 + 2\nu_x) & \zeta^{(12)} &= \frac{1}{48} \nu_y \nu_z (3 - 2\nu_x)
 \end{aligned} \tag{6.72}$$

In the general case (variable wavespeeds of any sign) there would be 23 boundaries involved. This is not discussed here, since it follows easily from the two dimensional case.

The second order scheme can be made virtually oscillation-free by taking the flux (6.71), (6.72) and replacing each ν_x by a one-dimensional limiter ϕ , as in the two dimensional case. Similarly for $g_{i,j+\frac{1}{2},k}$ and $h_{i,j,k+\frac{1}{2}}$. The flow parameter will again have to be replaced by a more multi-dimensional flow parameter. A natural extension of the one

dimensional flow parameter would be:

$$\begin{aligned}
 r_{i+\frac{1}{2},j,k}^{3D} = & (1 - \nu_y)(1 - \nu_z)r_{i+\frac{1}{2},j,k} + (1 - \nu_y)\nu_z r_{i+\frac{1}{2},j,k-1} \\
 & + \nu_y(1 - \nu_z)r_{i+\frac{1}{2},j-1,k} + \nu_y\nu_z r_{i+\frac{1}{2},j-1,k-1}
 \end{aligned} \tag{6.73}$$

where $r_{i+\frac{1}{2},j,k}$ is the one-dimensional flow parameter $r_{i+\frac{1}{2}}$ defined by (6.23) along the one dimensional strip (j, k) . Other choices are, of course, possible.

The extension of the flux (6.71), (6.72) to non-linear systems follows trivially from the two dimensional case. Fluxes are replaced by their vector-valued equivalents, jumps in flux are interpreted as jumps across waves in the solution of grid-aligned Riemann problems and Courant numbers for the calculation of weights are taken from the solutions to Riemann problems at boundaries, or from components of velocity parallel to the boundary on states within cells, depending on the orientation of the boundary. The timestep Δt for the scheme is taken as in (6.54),(6.55) for the other extension of the three dimensional scheme.

6.4 Numerical Experiments

Numerical results from the schemes discussed in the last two section are now presented for the two dimensional Shallow Water equations

$$\begin{pmatrix} \phi \\ \phi u \\ \phi v \end{pmatrix}_t + \begin{pmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \\ \phi uv \end{pmatrix}_x + \begin{pmatrix} \phi v \\ \phi uv \\ \phi v^2 + \frac{1}{2}\phi^2 \end{pmatrix}_y = 0 \tag{6.74}$$

where $\phi = gH$ is the geopotential, H is the total depth, $g \approx 9.81$ is the acceleration due to gravity and u and v are the x and y components of velocity respectively. Riemann solvers for the one dimensional counterpart of (6.74) can be found in [66]. Results from the schemes presented in the last two sections will be compared with those from a code based on an operator split version of the one dimensional **waf** scheme. Since both of the finite volume schemes under consideration are second order accurate in space and time, the second order splitting of Strang [53], as discussed in Chapter 2, is used.

Two test problems are considered, both of which are radially symmetric about the centre of the domain. They can thus be compared with numerical solutions to the one-dimensional Shallow Water equations with a geometric source term:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}, r) \tag{6.75}$$

where

$$\mathbf{U} = \begin{pmatrix} \phi \\ \phi w \end{pmatrix} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \phi w \\ \phi w^2 + \frac{1}{2}\phi^2 \end{pmatrix} \quad \mathbf{S}(\mathbf{U}, r) = - \begin{pmatrix} \frac{\phi w}{r} \\ \frac{\phi w^2}{r} \end{pmatrix} \tag{6.76}$$

Here $w = \sqrt{u^2 + v^2}$ is the radial velocity, and r is the distance from the centre of the domain. Numerical solutions to (6.75), (6.76) have been taken from a code by Mr Ed Boden, to whom the author is indebted. The code is based on time operator splitting, in which each timestep consists of two stages. Firstly, the hyperbolic system

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \quad (6.77)$$

is solved with data $\mathbf{U}^n$ as initial conditions for a time Δt to find an intermediate solution $\bar{\mathbf{U}}$. This is followed by the solution of the system of ordinary differential equations

$$\mathbf{U}_t = \mathbf{S}(\mathbf{U}, r) \quad (6.78)$$

with data $\bar{\mathbf{U}}$ as initial conditions for a time Δt to obtain the solution $\mathbf{U}^{n+1}$ at the next timelevel. In the code used here, the one dimensional **WAF** method, as described in Chapter 2, with the van Leer limiter (2.80) was used for the solution of (6.77) and a simple Euler step was used for the solution of (6.78). In all solutions of (6.75), (6.76) presented below, 2000 computational cells were used.

For both initial value problems considered here, equations (6.74) are solved within the domain $[0, 2] \times [0, 2]$ divided into 101×101 computing cells. The **VAN LEER** limiter is used in all calculations. Both problems are 'circular dam-break problems', i.e the initial conditions consist of two states separated by a circle. Here the circle has radius $r = 0.35$ and is centred on the point (1,1). Any cells cut by this circle are given a weighted average of the two states, based on the relative areas of the part of the cell lying within the circle and the rest of the cell.

The first problem considered is an 'explosion' problem. For initial conditions, both components of velocity are set to zero everywhere and ϕ is set to 1.0 inside the circle and 0.1 outside the circle. The Courant number coefficient is set at 0.8, and results are taken at times $t = 0.2$ and $t = 0.5$. By time $t = 0.2$ a circular shock wave has formed and is 'expanding' away from the centre (1,1). Within this circular shock is a circular rarefaction wave which 'drains' fluid from the original deep region to 'feed' the shock. By time $t = 0.5$ the shocks have grown dramatically in strength and a small dip in geopotential has formed in the central deep region. For each solution, two cross sections are shown: a section in the plane $y = 1$, called x -strips here, and a section in the plane $x = y$, called diagonal strips here. Strips from two dimensional solutions will always be represented by symbols in the figures; solutions to the one dimensional equation (6.75),(6.76) are represented by solid lines. Figures 6.6 and 6.7 show the x strips and diagonal strips obtained using the operator split scheme. Figures 6.8 and 6.9 show the equivalent results for the scheme of §6.2, and figures 6.10 and 6.11 show the same for the scheme of §6.3. Note that the scheme of §6.2 fails to resolve the dip in geopotential at $x = 1, y = 1$ at time $t = 1.0$ as well as the other schemes.

Figures 6.12 and 6.13 show two dimensional plots of the explosion problem computed by the operator split scheme and the scheme of §6.2 respectively. A grid of 400×400 cells was used on each of these calculations and output was taken at time $t = 0.5$.

The second problem considered is a 'shock focusing' problem. In the initial conditions, both components of velocity are again set to zero everywhere, but now $\phi = 0.1$ inside the circle and $\phi = 1.0$ outside the circle. The Courant number coefficient is again taken as 0.8 and results are taken at times $t = 0.4$ and $t = 1.0$. At time $t = 0.0$, a circular shock forms at the initial discontinuity and grows in strength as its radius diminishes in time, so the shock is being focused on the centre (1,1). At the same time, a circular rarefaction wave travels outward from the initial discontinuity. By time $t = 0.4$, the shock has almost diminished into a singularity at the centre and is beginning to interact with the tail of the rarefaction. At time $t = 1.0$, the initial circular shock has passed through the singularity and a circular shock is expanding outward from the centre and interacting fully with the rarefaction wave. x strips and diagonal strips for the operator split scheme on this problem are shown in figures 6.14 and 6.15 respectively. The equivalent results for the scheme of §6.2 are shown in figures 6.16 and 6.17, and figures 6.18 and 6.19 show the same results for the scheme of §6.3. Note the loss of monotonicity of the scheme of §6.2 along diagonal strips. This problem can only be cured by the use of a multidimensional flow parameter, such as (6.66), and is currently under investigation. In figure 6.18 it can be seen that the scheme of §6.3 has become unstable for this problem. However, results for the same test with a Courant number coefficient of 0.5 are shown in figures 6.20 and 6.21, where the instability is not evident. The scheme has been tested with a Courant number coefficient of 0.8 and the limiters turned off, i.e. second order accuracy everywhere and the instability was again not evident (results not shown). Since the instability did not occur with a Courant number coefficient of 0.8 when second order accuracy was used everywhere but did occur when the monotonicity controls were in place, the problem must lie in the first order part of the scheme. Further work is needed to establish why this problem arises. It is interesting to see that none of the schemes are able to accurately resolve the velocity profile near (1,1) at time $t = 1.0$.

Figures 6.22 and 6.23 show two dimensional plots of the shock focusing problem computed by the operator split scheme and the scheme of §6.2 respectively. A grid of 400×400 cells was used on each of these calculations and output was taken at time $t = 1.0$.

6.5 Conclusions

In this chapter, two new finite volume schemes for the solution of two-dimensional hyperbolic systems of conservation laws on Cartesian grids have been presented. Both are based on solutions to grid-aligned Riemann problems and are different extensions of the same scheme for the linear advection equation. This scheme is a **waf**-type scheme derived in chapter 5. In smooth flow, the new schemes are second order accurate in space and time, and achieve atleast partial monotonicity in regions of high gradient by the use of one-dimensional **rpd** limiter functions. Numerical experiments on the Shallow Water equations have shown both schemes are suitable for the solution problems containing bore waves, although the first order version of the scheme of §6.3 appears to lack robustness when Courant number coefficients greater than one half are used. Some details of the

application of the schemes in three space dimensions have also been given.

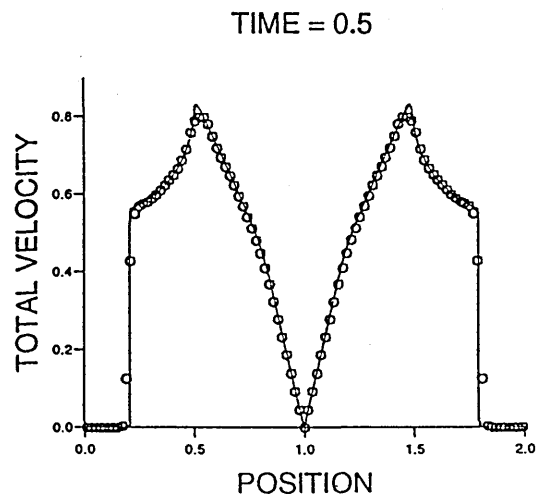
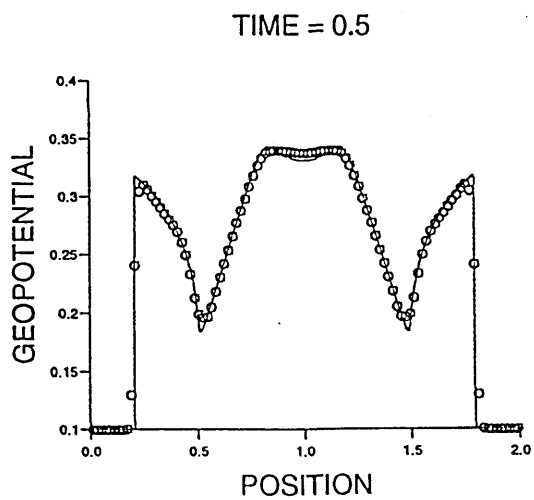
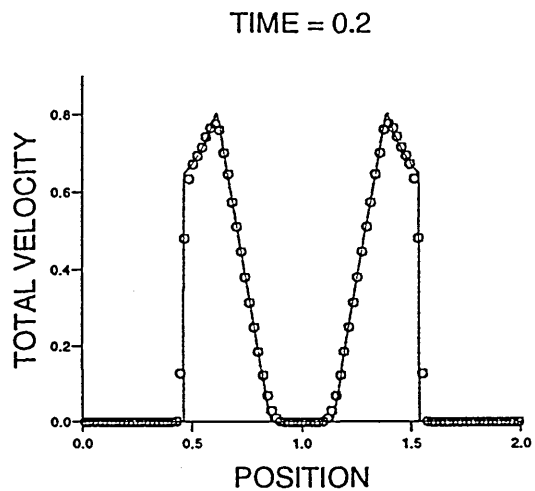
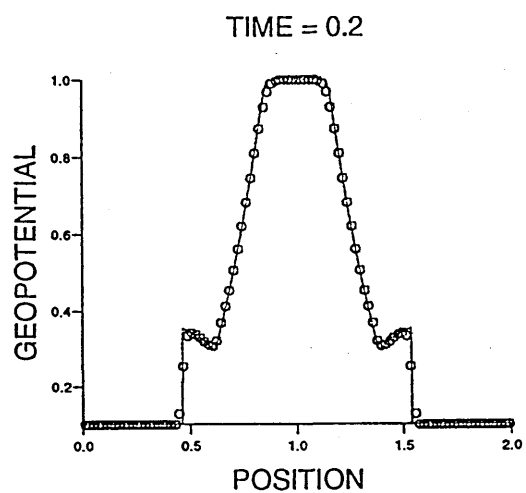


Figure 6.6: Results from the explosion test: x -strips from the operator split scheme with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

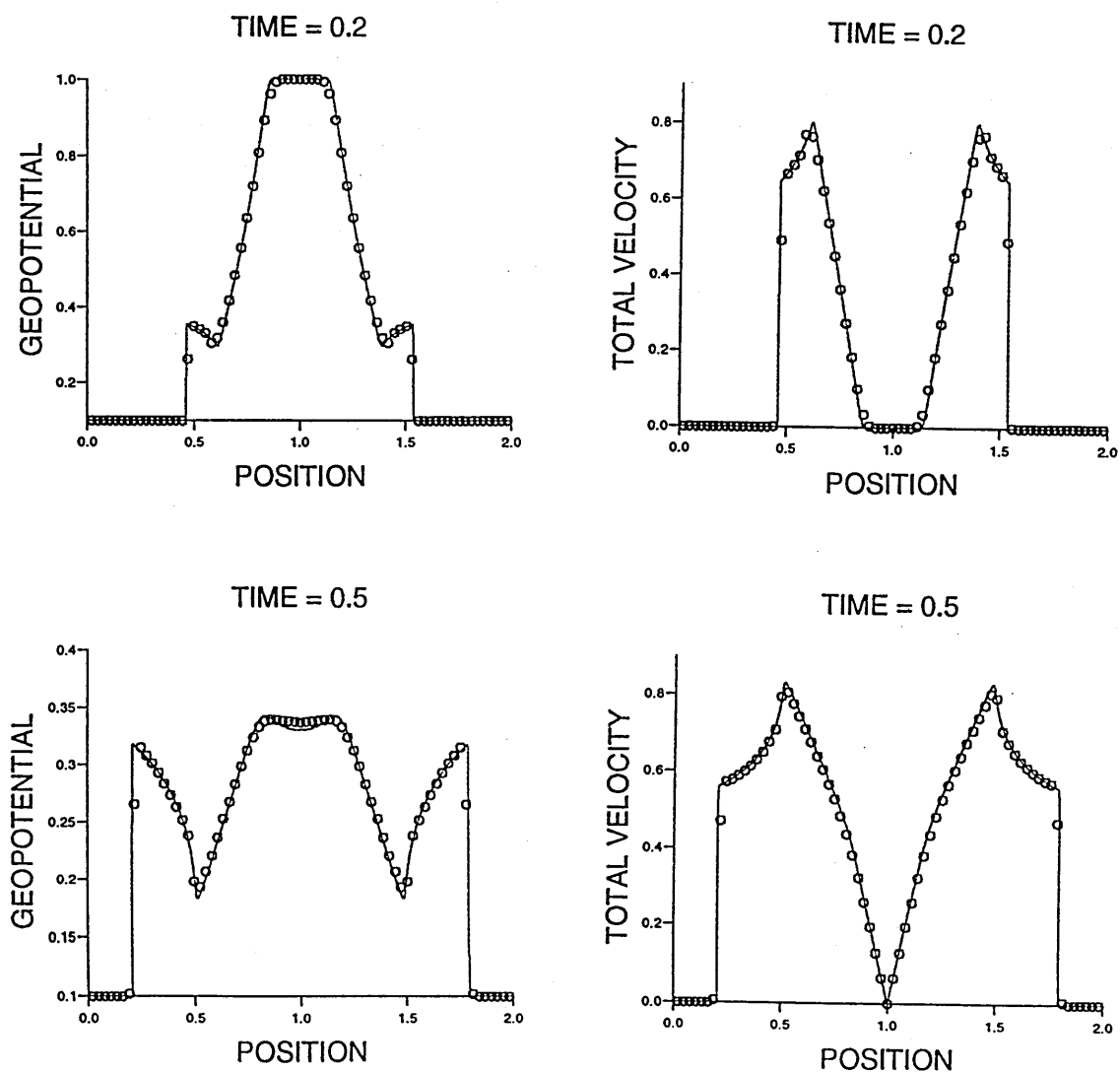


Figure 6.7: Results from the explosion test: diagonal strips from the operator split scheme with the `VAN LEER` limiter. A Courant number coefficient of 0.8 was used.

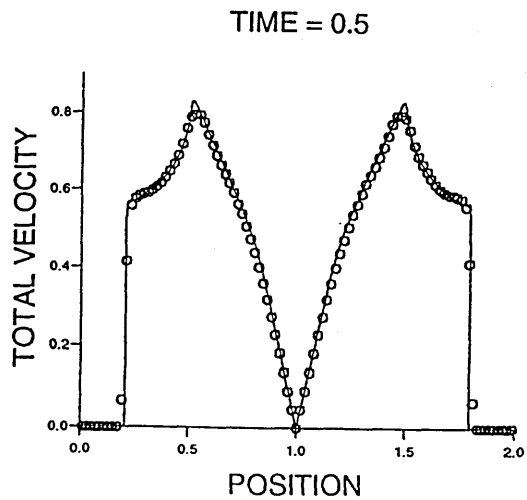
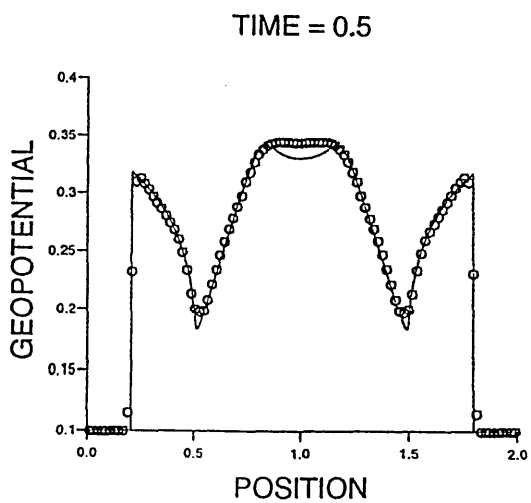
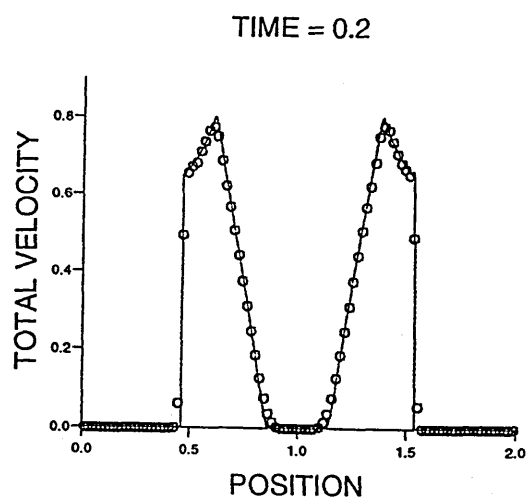
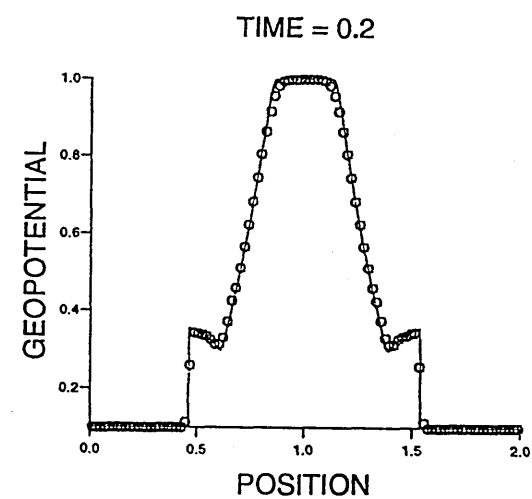


Figure 6.8: Results from the explosion test: x -strips from the scheme of §6.2 with the VAN LEEU limiter. A Courant number coefficient of 0.8 was used.

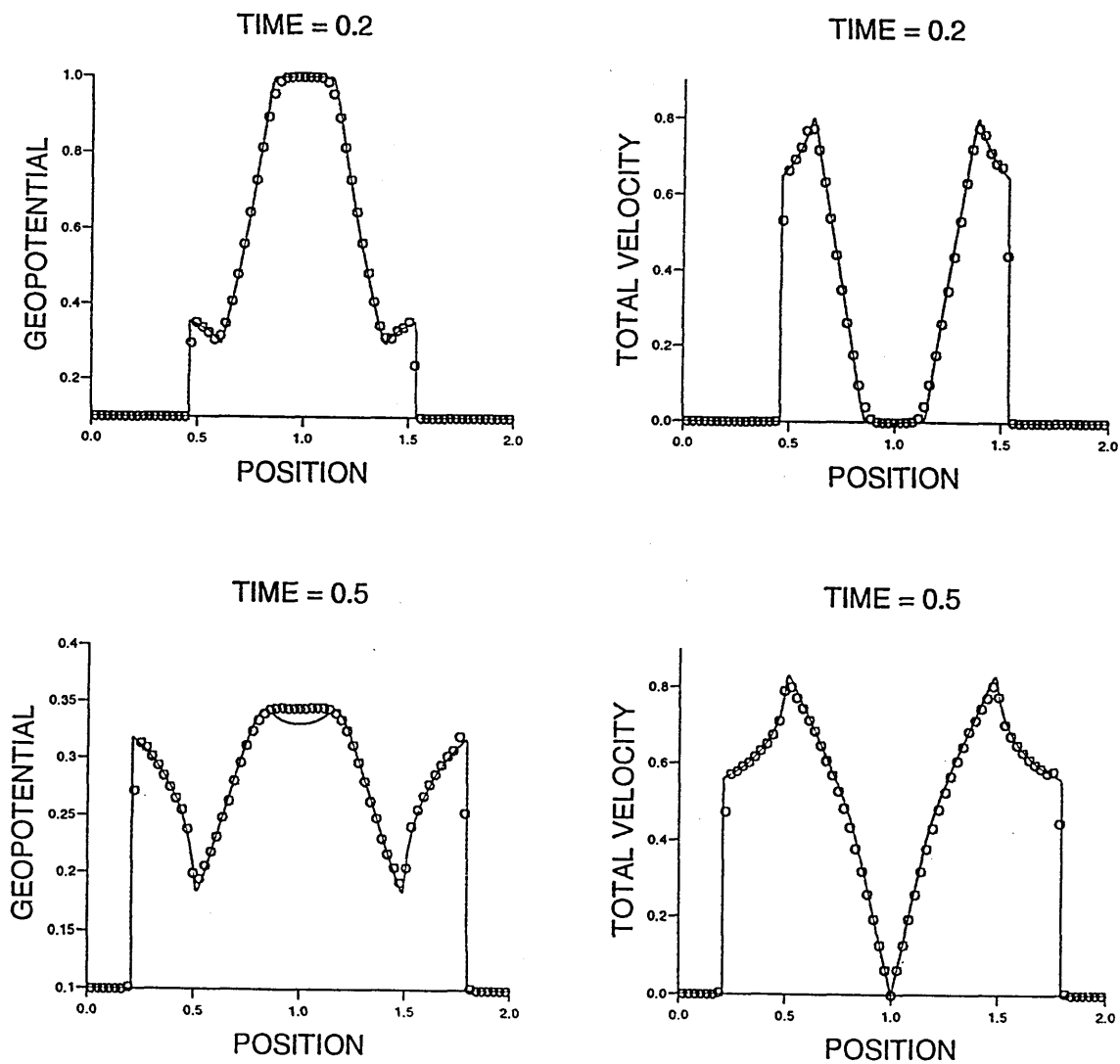


Figure 6.9: Results from the explosion test: diagonal strips from the scheme of §6.2 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

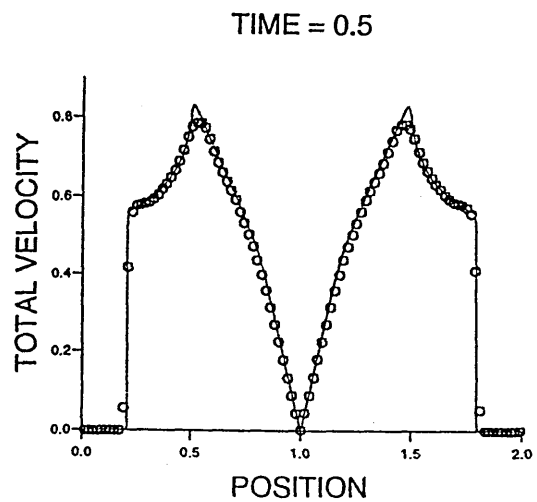
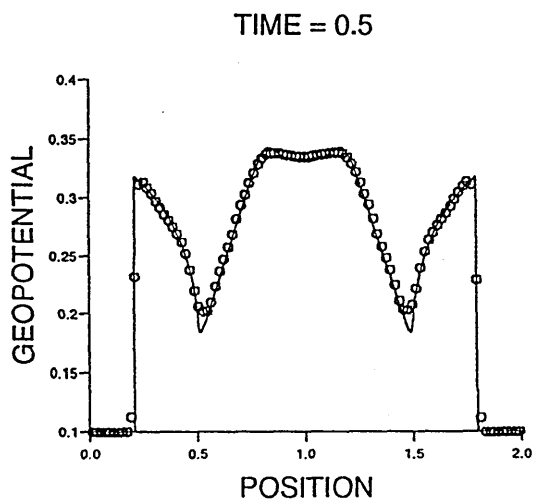
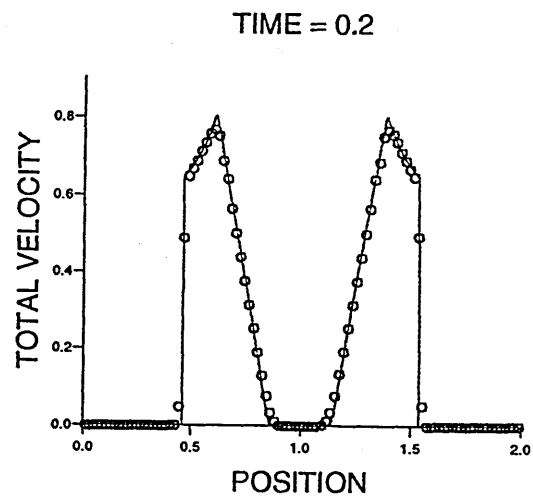
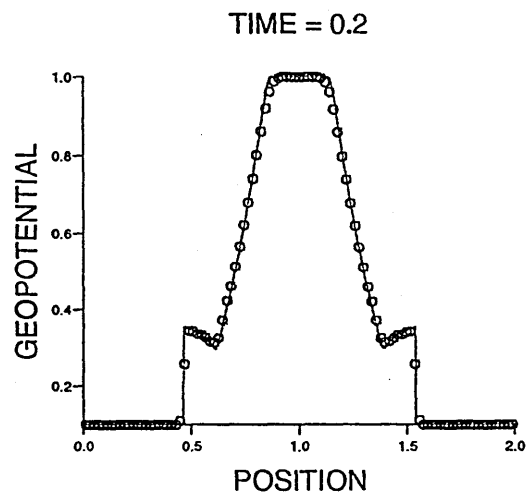


Figure 6.10: Results from the explosion test: x -strips from the scheme of §6.3 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

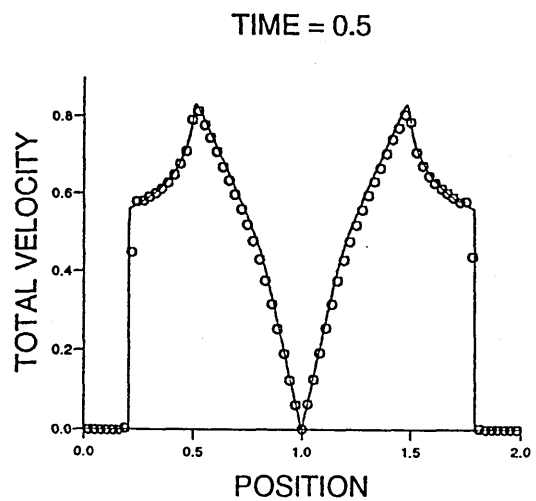
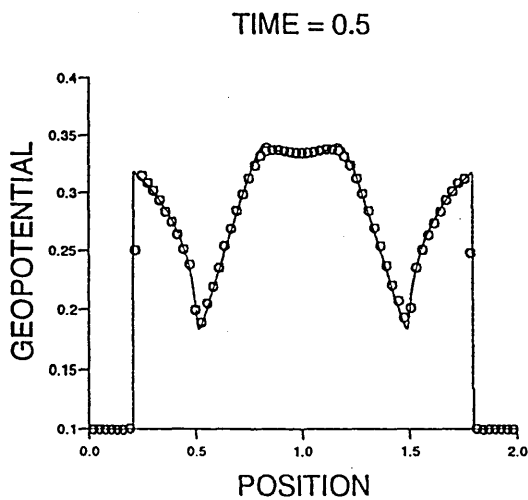
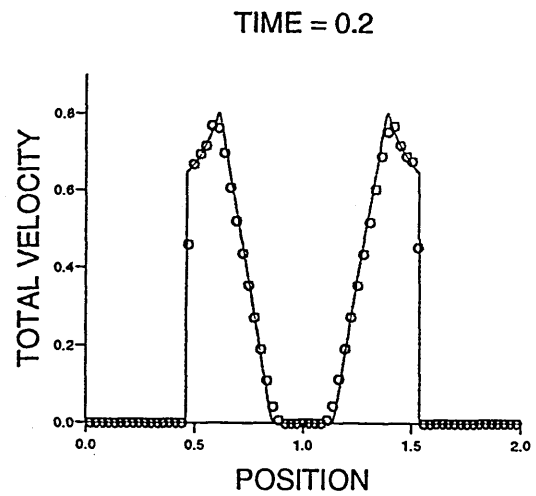
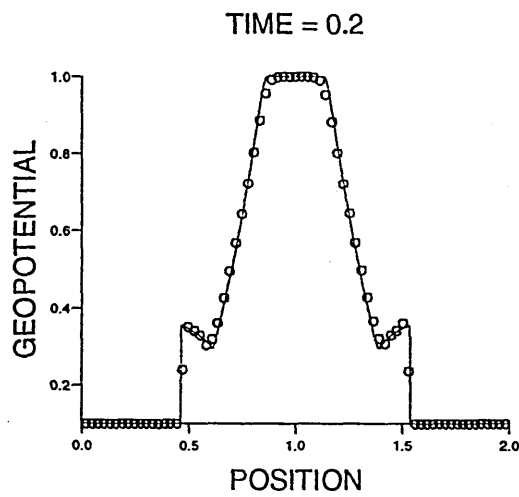
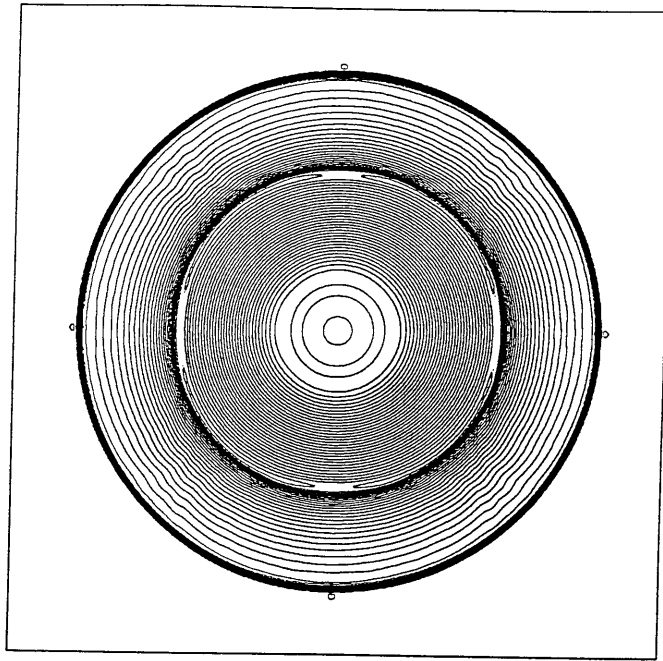
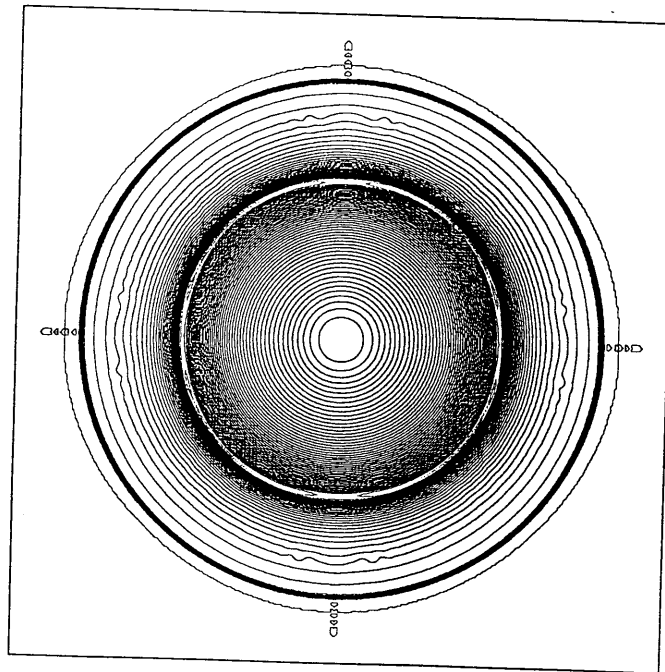


Figure 6.11: Results from the explosion test: diagonal strips from the scheme of §6.3 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

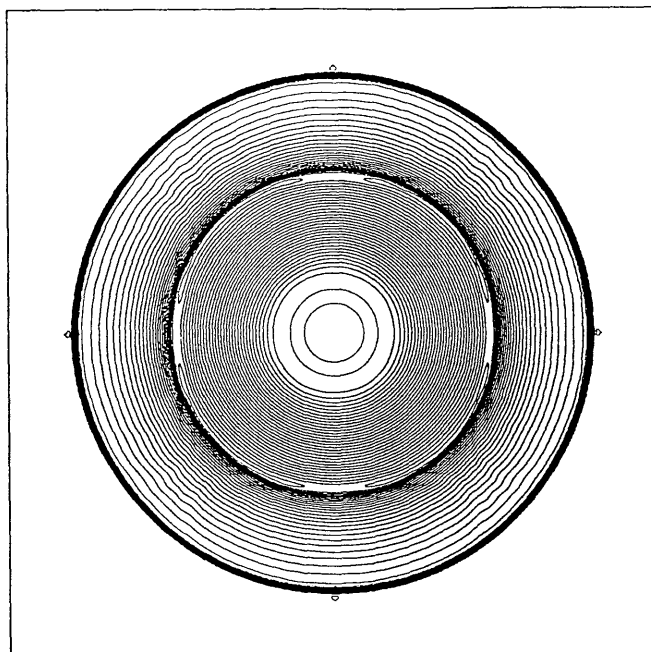


(a) Geopotential.

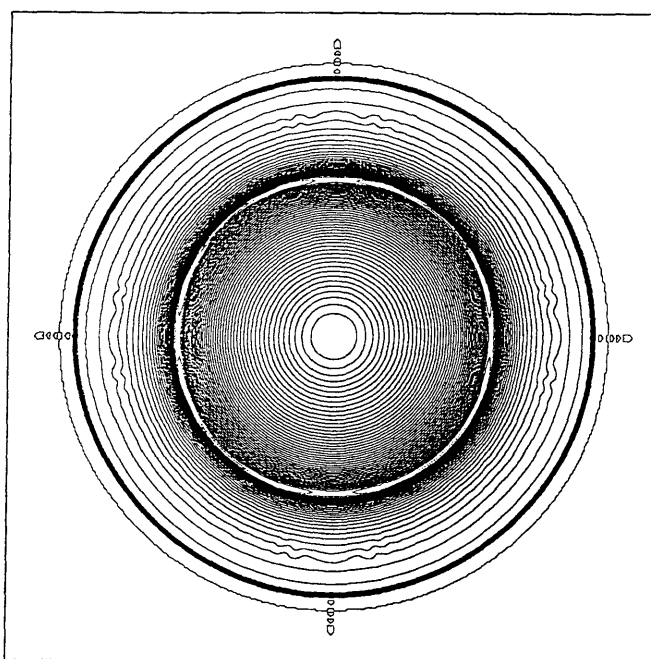


(b) Froude number.

Figure 6.12: Two dimensional results from the operator split scheme for the explosion problem. A grid of 400×400 cells and Courant number coefficient of 0.8 were used.



(a) Geopotential.



(b) Froude number.

Figure 6.13: Two dimensional results from the scheme of §6.2 scheme for the explosion problem. A grid of 400×400 cells and Courant number coefficient of 0.8 were used.

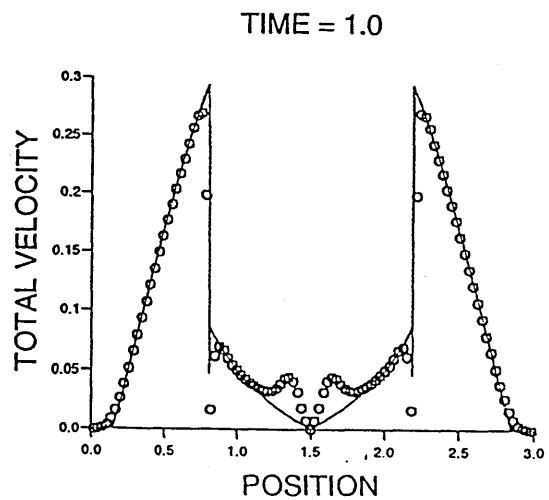
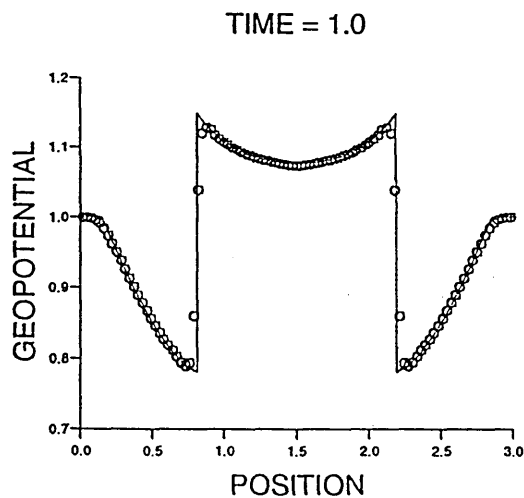
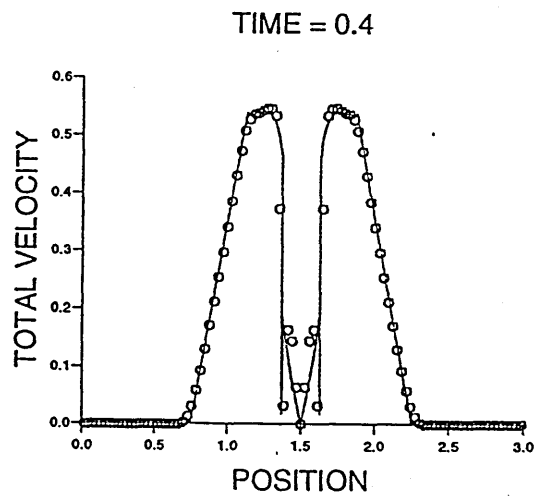
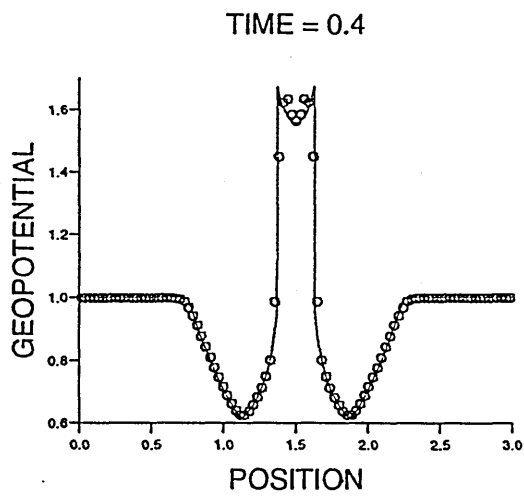


Figure 6.14: Results from the shock focusing test: x -strips from the operator split scheme with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

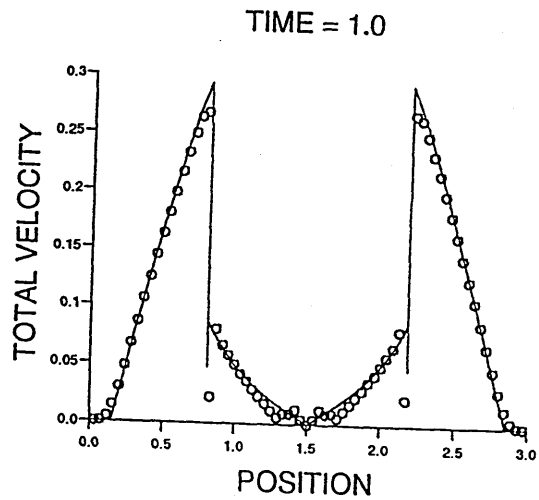
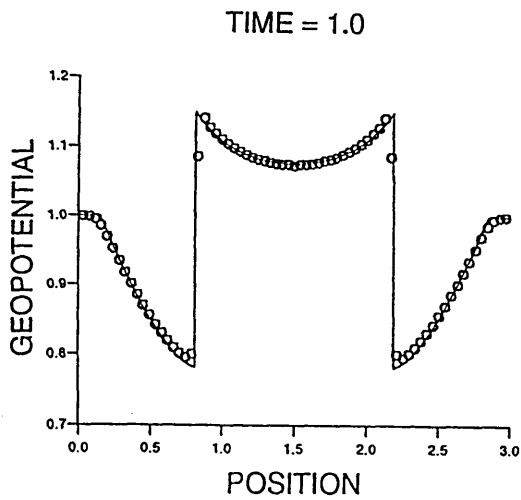
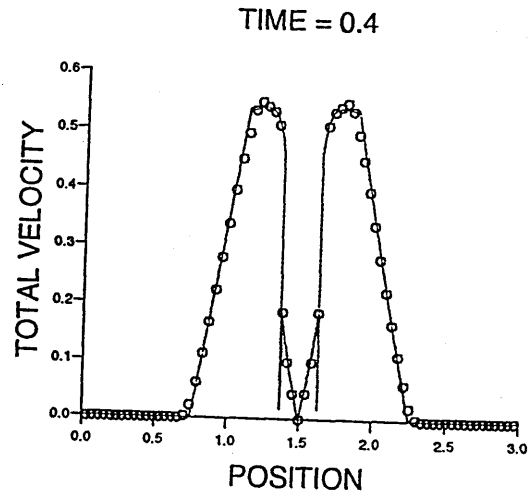
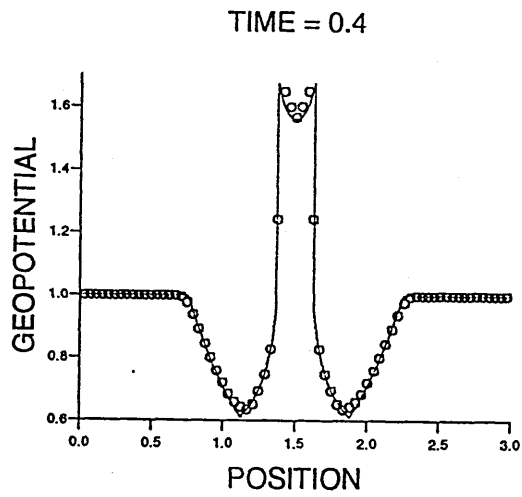


Figure 6.15: Results from the shock focusing test: diagonal strips from the operator split scheme with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

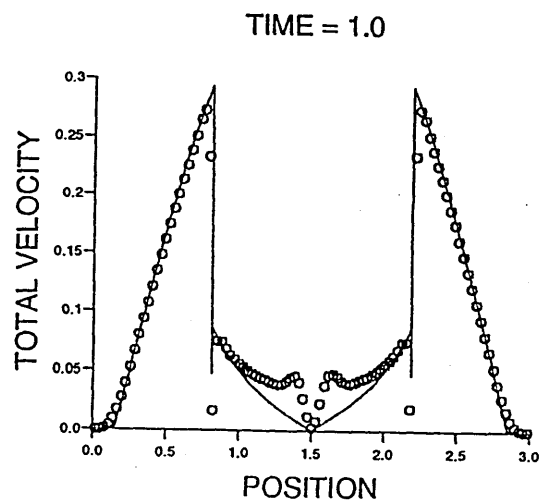
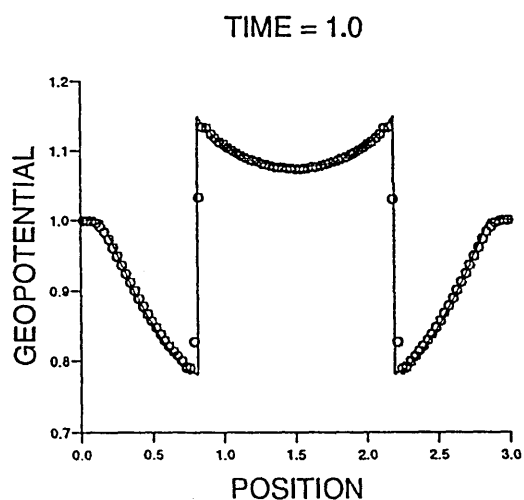
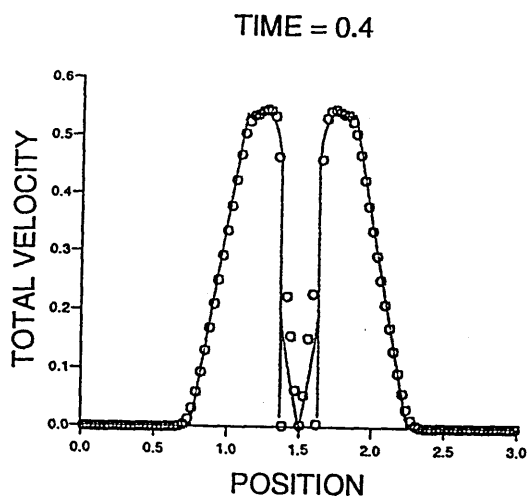
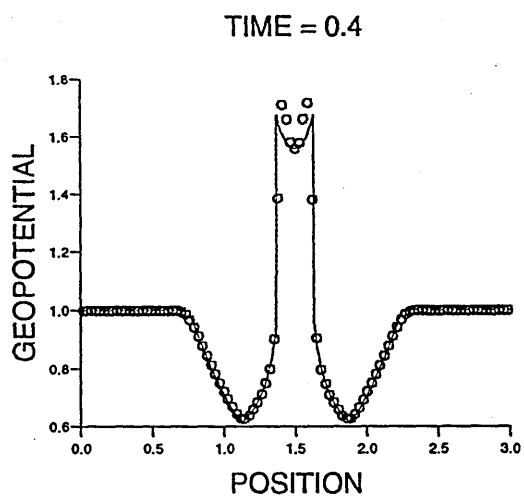


Figure 6.16: Results from the shock focusing test: x -strips from the scheme of §6.2 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

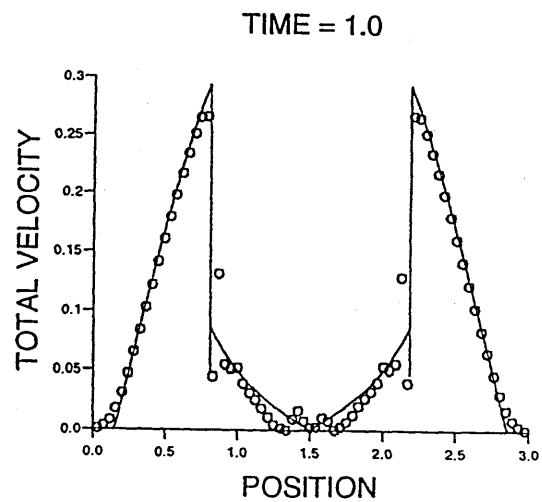
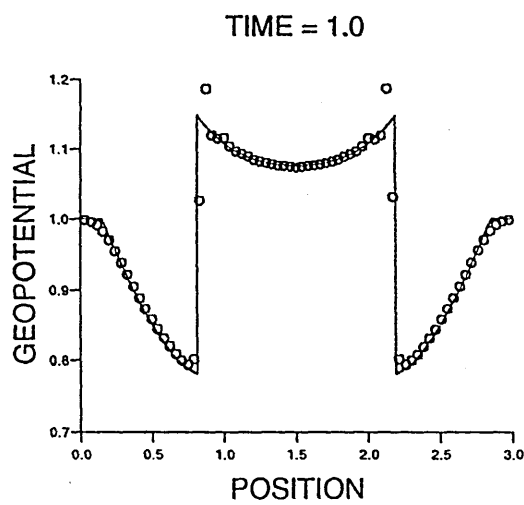
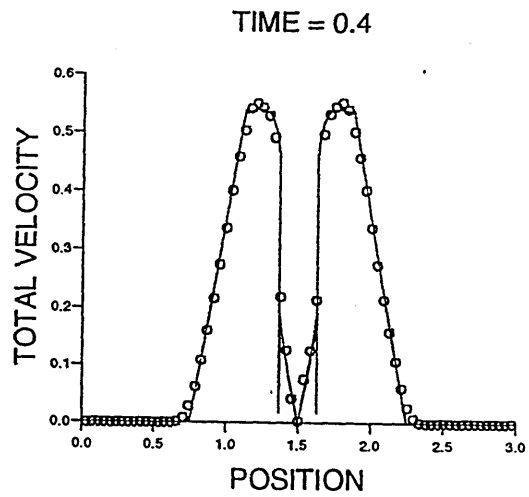
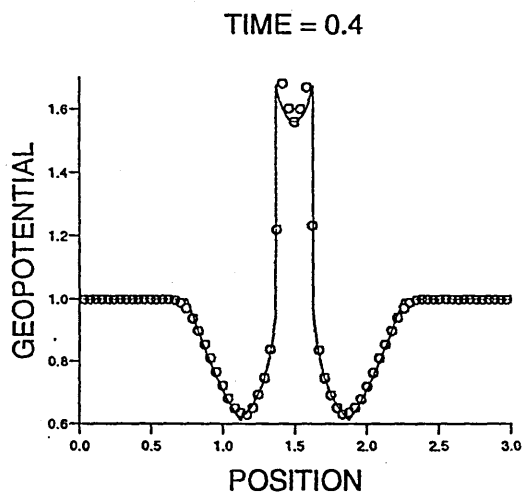


Figure 6.17: Results from the shock focusing test: diagonal strips from the scheme of §6.2 with the *VAN LEER* limiter. A Courant number coefficient of 0.8 was used.

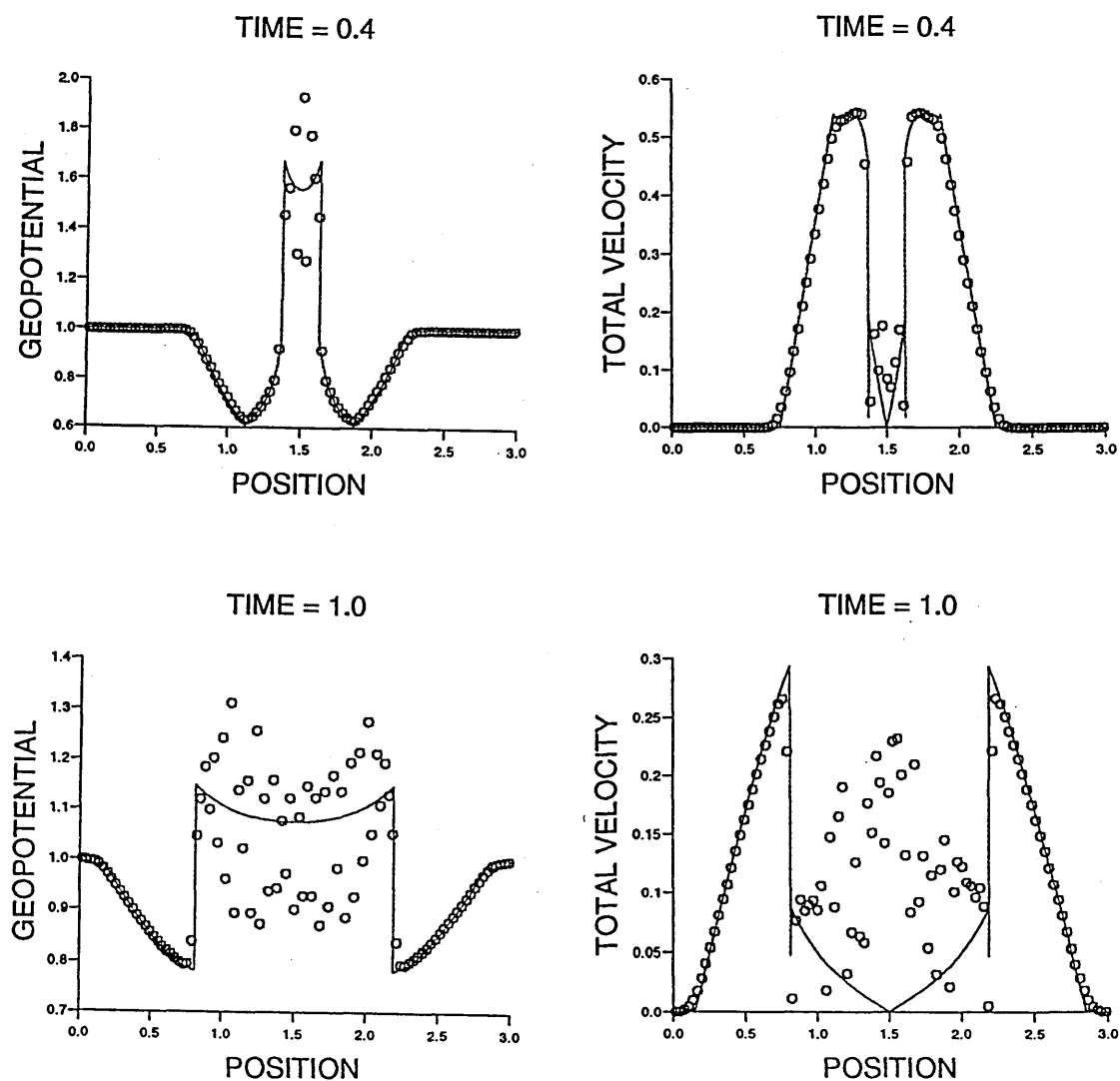


Figure 6.18: Results from the shock focusing test: x -strips from the scheme of §6.3 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

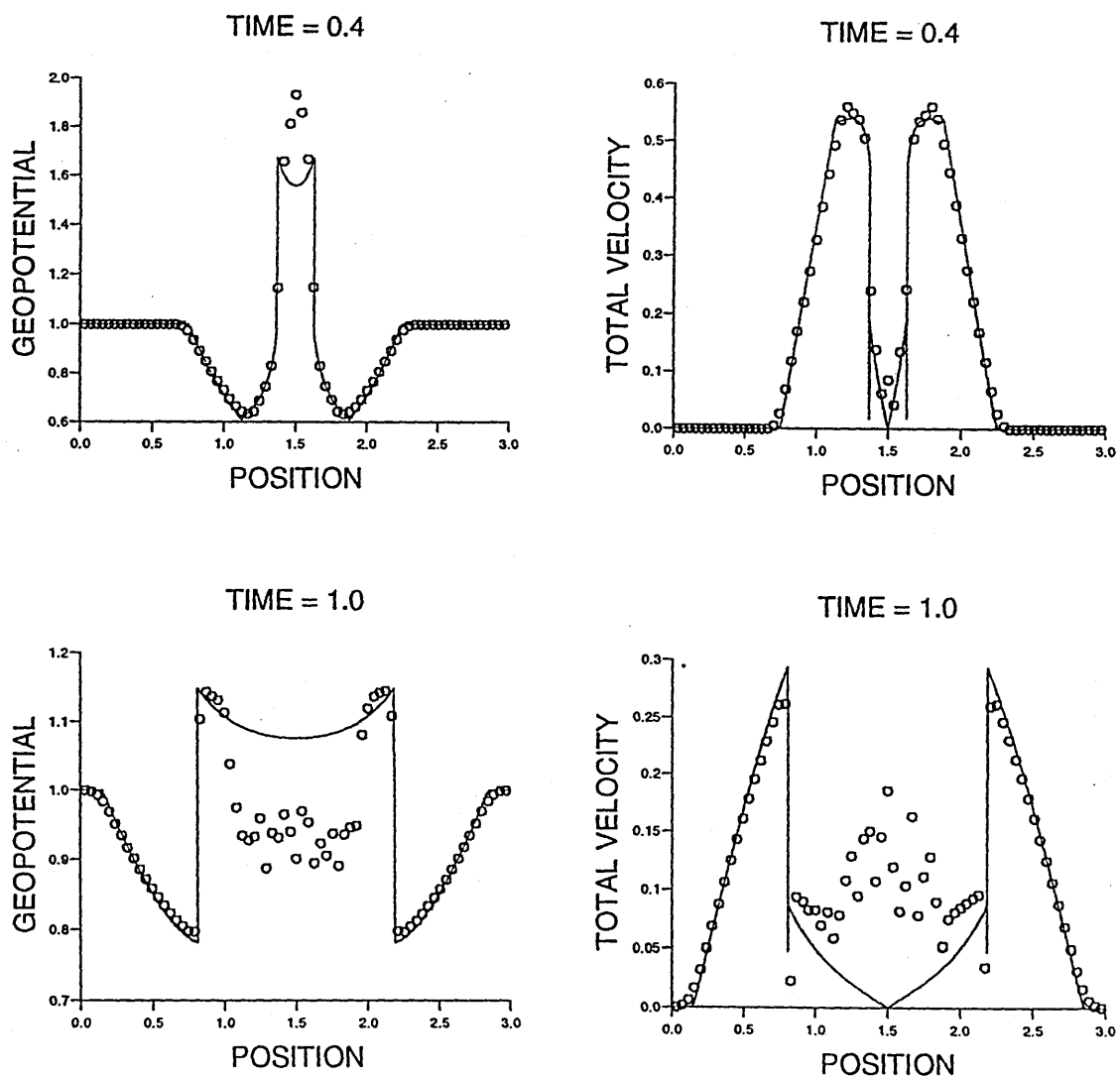


Figure 6.19: Results from the shock focusing test: diagonal strips from the scheme of §6.3 with the VAN LEER limiter. A Courant number coefficient of 0.8 was used.

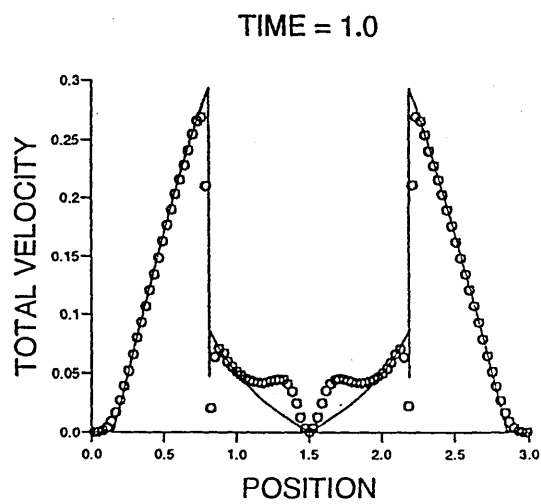
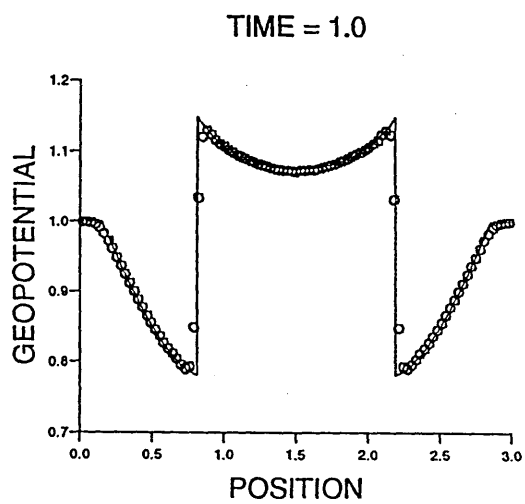
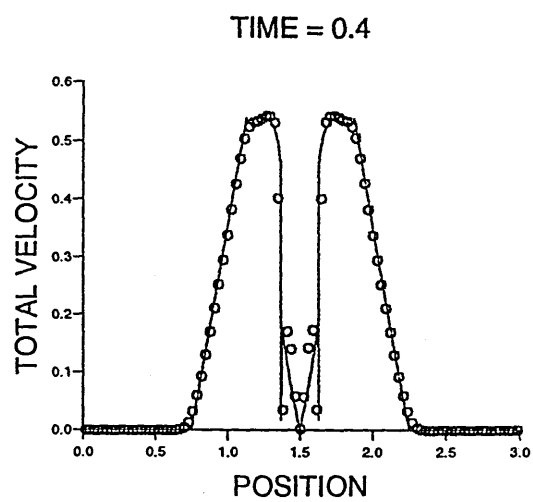
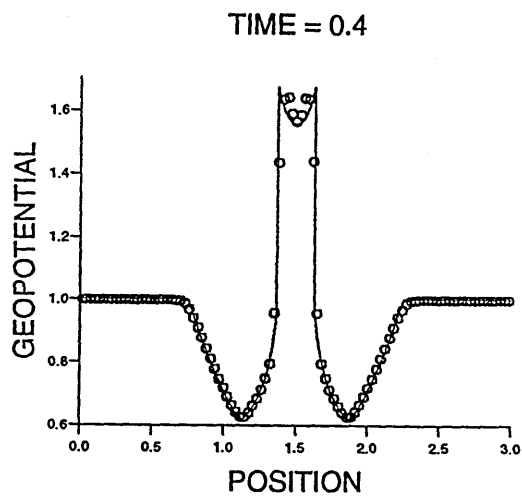


Figure 6.20: Results from the shock focusing test: x -strips from the scheme of §6.3 with the VAN LEER limiter. A Courant number coefficient of 0.5 was used.

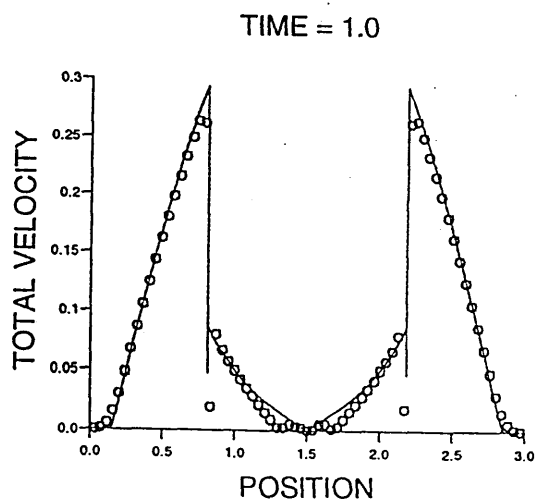
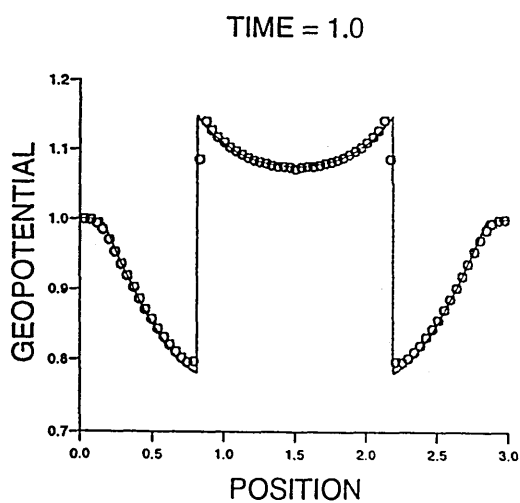
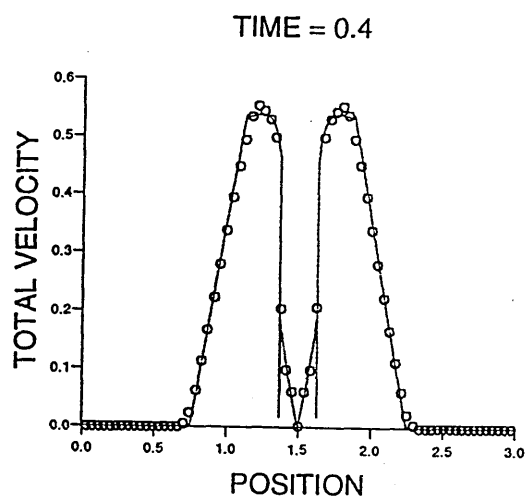
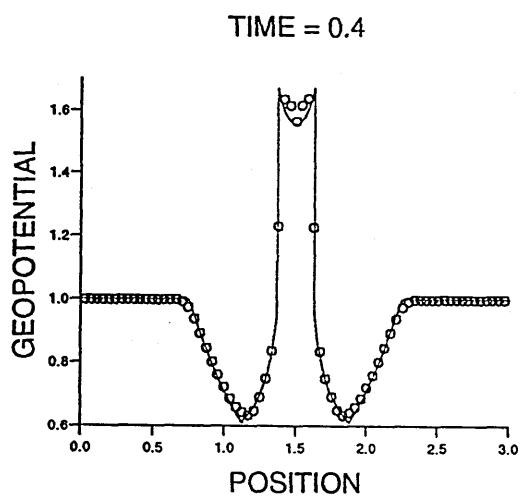
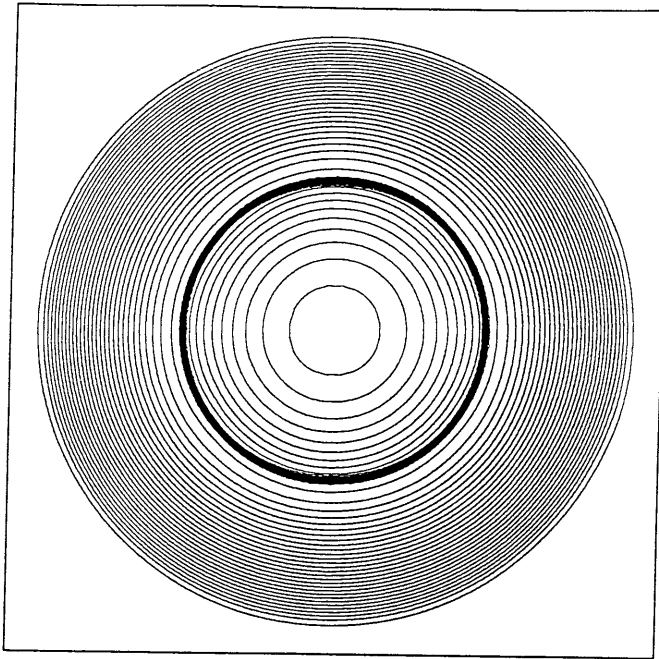
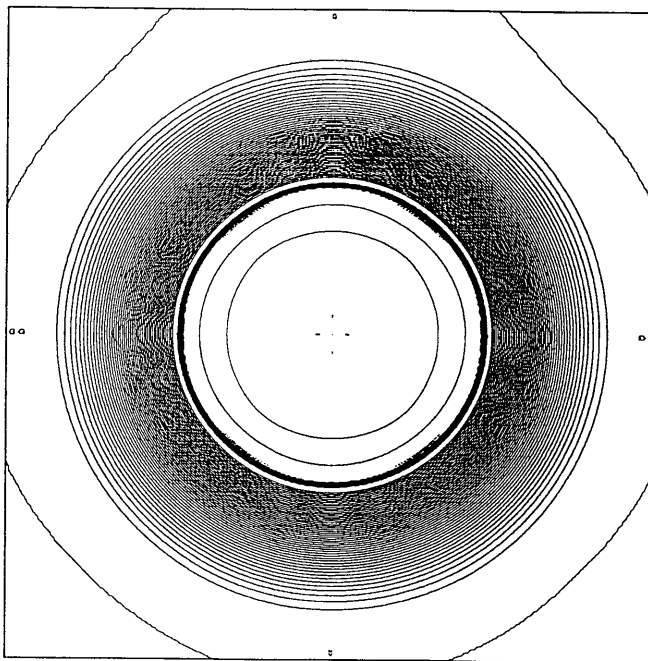


Figure 6.21: Results from the shock focusing test: diagonal strips from the scheme of §6.3 with the `VAN LEER` limiter. A Courant number coefficient of 0.5 was used.

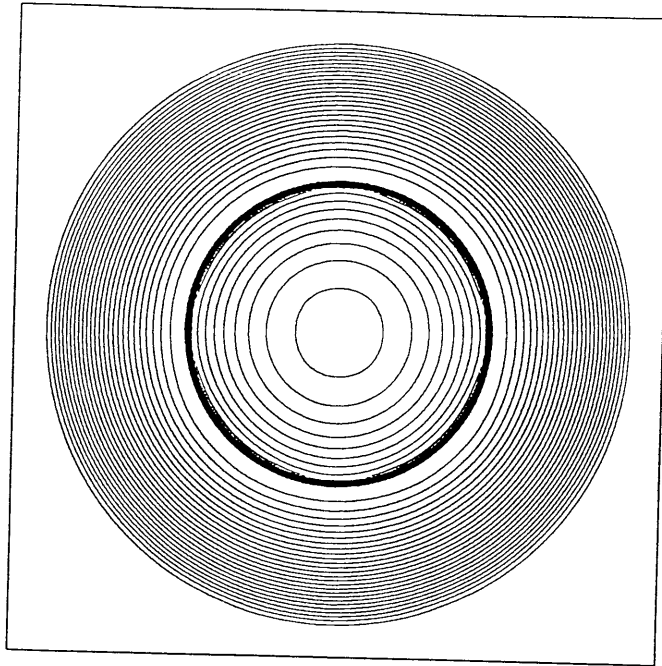


(a) Geopotential.

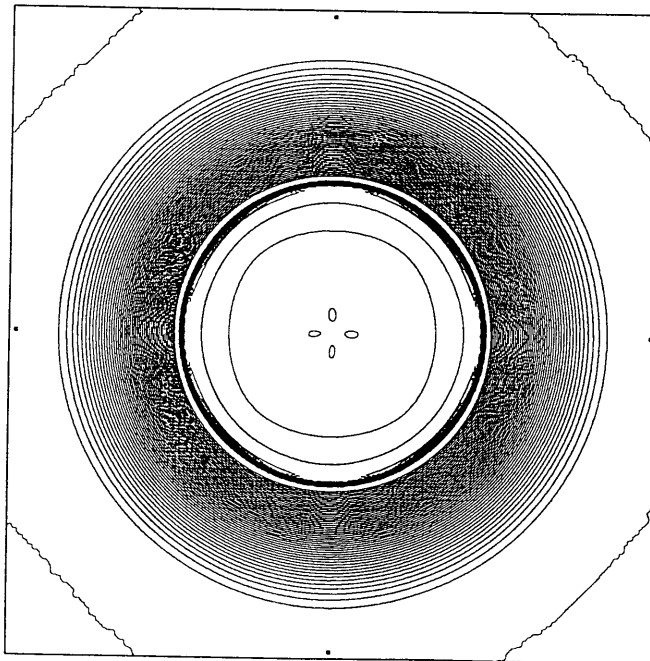


(b) Froude number.

Figure 6.22: Two dimensional results from the operator split scheme for the shock focusing problem. A grid of 400×400 cells and Courant number coefficient of 0.8 were used.



(a) Geopotential.



(b) Froude number.

Figure 6.23: Two dimensional results from the scheme of §6.2 scheme for the shock focusing problem. A grid of 400×400 cells and Courant number coefficient of 0.8 were used.

Chapter 7

Finite Volume Schemes for Curvilinear Grids

7.1 Introduction

In general, numerical methods are needed to simulate flows around arbitrarily complex two and three dimensional bodies and Cartesian grids cannot be used directly. One possibility is to extend methods for Cartesian grids, such as those presented in Chapter 6, for use in the so-called 'Cartesian Boundary Technique' (see, e.g. Quirk [41]), in which solid bodies blank out areas of a background Cartesian grid and the resulting cut cells are treated specially. Alternatively, the numerical methods could be adapted for use on unstructured grids.

In this chapter, a third alternative will be considered. The two finite volume schemes for Cartesian grids in two space dimensions presented in Chapter 6 will be extended for use on structured, body-fitted grids of quadrilaterals, i.e grids that completely cover the flow domain, do not extend beyond the flow domain, and comprise only of four-sided cells that can be labelled via two integers as $\{\Delta_{i,j}\}$. Such grids are called *curvilinear grids* here. Many numerical methods for the generation of these grids exist and fall into two main categories: differential and algebraic. Differential methods consist of those based on solving differential equations, such as those by Thompson *et. al* [57],[58]. Algebraic methods consist of those that make use only of algebraic expressions involving the coordinates of the grid nodes, such as the multisurface method of Eiseman [11]. A general discussion of numerical methods for grid generation can be found in Kutler [28].

As in Chapter 6, the schemes considered here will be written in conservative form, which on the grids studied here can be written as:

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{V(\Delta_{i,j})} \left[\mathbf{E}_{i+\frac{1}{2},j} \cdot \mathbf{l}_{i+\frac{1}{2},j} - \mathbf{E}_{i-\frac{1}{2},j} \cdot \mathbf{l}_{i-\frac{1}{2},j} \right] \quad (7.1)$$

$$-\frac{\Delta t}{V(\Delta_{i,j})} \left[\mathbf{E}_{i,j+\frac{1}{2}} \cdot \mathbf{l}_{i,j+\frac{1}{2}} - \mathbf{E}_{i,j-\frac{1}{2}} \cdot \mathbf{l}_{i,j-\frac{1}{2}} \right]$$

One of the resulting schemes will be used to study the problem of Mach reflection of bore waves in shallow water, by reflecting bore waves from solid boundaries. Results will be compared against those from a second order accurate operator split **WAF** code. The extension of the three dimensional schemes for Cartesian grids presented in Chapter 6 will not be considered here, although the extension of these schemes follows directly from the two dimensional cases.

The remainder of this chapter is set out as follows. In §7.2 and §7.3, the schemes of §6.2 and §6.3 will be extended to curvilinear grids; in §7.4, numerical results are presented and in §7.5 the chapter is summarized and conclusions are drawn.

7.2 Extension of the Scheme of §6.2

In this section, the finite volume scheme presented in §6.2 for the solution of two dimensional hyperbolic systems of conservation laws on Cartesian grids is extended for use on curvilinear grids. The extension of the scheme is almost a trivial matter, given the discussion in Chapter 2 of how the operator splitting technique is applied to the one-dimensional **WAF** scheme. However, a detailed algorithm will now be given.

1. Set up a curvilinear grid of $M_i \times M_j$ cells within the flow domain and add two strips of ghost cells along each boundary by reflecting points on the interior of the grid across the boundary. Compute and store the area $V(\Delta_{i,j})$ of each cell, the lengths $l_{i+\frac{1}{2},j}$, $l_{i,j+\frac{1}{2}}$ of the cell boundaries, the angles $\theta_{i+\frac{1}{2},j}$, $\theta_{i,j+\frac{1}{2}}$ those boundaries make with the x -direction, and suitable integration lengths $l_{i+\frac{1}{2},j}^L$, $l_{i+\frac{1}{2},j}^R$, $l_{i,j+\frac{1}{2}}^L$, $l_{i,j+\frac{1}{2}}^R$ representing some characteristic cell lengths perpendicular to the boundaries, as for the operator split **WAF** scheme described in Chapter 2. Impose the required initial conditions.
2. Choose a timestep Δt based on the maximum absolute characteristic speed λ_{max} on the data and the minimum side length:

$$\Delta t = \nu_{max} \frac{\min_{i,j} \{l_{i+\frac{1}{2},j}, l_{i,j+\frac{1}{2}}\}}{\lambda_{max}} \quad (7.2)$$

where $\nu_{max} \in (0, 1]$ is some user-defined Courant number coefficient. If this timestep takes the solution past the required solution time, reduce it as necessary.

3. For each $i \in [1, M_i]$
 - (a) Set boundary conditions at either end of strip i . For transmissive boundaries, place in each ghost cell the value of each primitive variable in the cell within the flow domain that shares the same strip perpendicular to the boundary, and is

the same number of cells from that boundary. Reflective boundaries are treated in the same way, except the sign of the component of velocity perpendicular to the boundary is changed.

- (b) For each $j \in [0, M_j]$, compute components of velocity in cells $\Delta_{i,j}$, $\Delta_{i,j+1}$ perpendicular and parallel to boundary $(i, j + \frac{1}{2})$, using the boundary angle $\theta_{i,j+\frac{1}{2}}$. The perpendicular velocity and other primitive variables from the two cells define an 'augmented' one dimensional Riemann problem. Solve this using any suitable robust, accurate Riemann solver, such as the **HLL** solver. Determine the component of velocity perpendicular to the boundary and on the boundary from the position of the slip line, and compute a Godunov flux from the resulting state on the boundary. Using $\theta_{i,j+\frac{1}{2}}$, compute x and y components of the momentum flux to give a final intercell flux $\mathbf{E}_{i,j+\frac{1}{2}}$.

- (c) Update strip i of cells by $\frac{1}{2}\Delta t$ to obtain an intermediate state $\mathbf{U}_{i,j}^{n+\frac{1}{2},i}$ by using the conservative formula

$$\mathbf{U}_{i,j}^{n+\frac{1}{2},i} = \mathbf{U}_{i,j}^n - \frac{\frac{1}{2}\Delta t}{V(\Delta_{i,j})} [\mathbf{E}_{i,j+\frac{1}{2}} \cdot \mathbf{l}_{i,j+\frac{1}{2}} - \mathbf{E}_{i,j-\frac{1}{2}} \cdot \mathbf{l}_{i,j-\frac{1}{2}}] \quad (7.3)$$

This step is effectively a j -sweep of Godunov's method, as would be used in an operator split scheme, with a timestep $\frac{1}{2}\Delta t$.

- 4. For each $j \in [1, M_j]$,

- (a) Set boundary conditions at each end of strip j , as in 3(a).
- (b) For each $i \in [-1 : M_i + 1]$, solve the one dimensional Riemann problem perpendicular to boundary $(i + \frac{1}{2}, j)$ based on the data $\mathbf{U}^{n+\frac{1}{2},i}$, in an analogous manner to the solution of Riemann problems step 3(b).
- (c) For each $i \in [0, M_i]$, compute a one dimensional **WAF** flux, as is done in the operator split case described in Chapter 2.
- (d) Transform the flux back into $x - y$ coordinates by computing x and y components of momentum and store the resulting flux $\mathbf{E}_{i+\frac{1}{2},j}^{WAF}$ for later use.

This step is effectively an i -sweep of the **WAF** method as would be used in an operator split scheme, without the update at the end.

- 5. For each $j \in [1, M_j]$,

- (a) Set boundary conditions at each end of strip j , as in 3(a).
- (b) For each $i \in [0, M_i]$, compute a one dimensional Godunov intercell flux $\mathbf{E}_{i+\frac{1}{2},j}$ in an analogous fashion to step 3(b).
- (c) Update strip j of cells by $\frac{1}{2}\Delta t$ to obtain an intermediate state $\mathbf{U}_{i,j}^{n+\frac{1}{2},j}$ by using the conservative formula

$$\mathbf{U}_{i,j}^{n+\frac{1}{2},j} = \mathbf{U}_{i,j}^n - \frac{\frac{1}{2}\Delta t}{V(\Delta_{i,j})} [\mathbf{E}_{i+\frac{1}{2},j} \cdot \mathbf{l}_{i+\frac{1}{2},j} - \mathbf{E}_{i-\frac{1}{2},j} \cdot \mathbf{l}_{i-\frac{1}{2},j}] \quad (7.4)$$

This step is effectively an i -sweep of Godunov's method, as would be used in an operator split scheme, with timestep $\frac{1}{2}\Delta t$.

6. For each $i \in [1, M_i]$
 - (a) Set boundary conditions at each end of strip i , as in 3(a).
 - (b) For each $j \in [-1, M_j + 1]$, solve the one dimensional Riemann problem perpendicular to boundary $(i, j + \frac{1}{2})$ based on the data $U^{n+\frac{1}{2},j}$, in an analogous manner to the solution of Riemann problems in the step 3(b).
 - (c) For each $j \in [0, M_j]$, compute a one dimensional waf flux, as is done in the operator split case described in Chapter 2.
 - (d) Transform the flux back into $x - y$ coordinates by computing x and y components of momentum and store the resulting flux $E_{i,j+\frac{1}{2}}^{WAF}$ for later use.

This step is effectively a j -sweep of the waf method as would be used in an operator split scheme, without the update at the end.

7. Update the data U^n at time level n to data U^{n+1} at the next timelevel by using the fluxes $E_{i+\frac{1}{2},j}^{WAF}$, $E_{i,j+\frac{1}{2}}^{WAF}$ in the two dimensional conservative form (7.1).
8. Increment the time variable by Δt . If the required solution time has not yet been reached, go to step 2.
9. Output data.

The main fault of the scheme described above is that it will suffer from the operator splitting error triggered by 'irregular' quadrilaterals discussed in Chapter 2, i.e constant data will not be preserved. This is because predictor states $U^{n+\frac{1}{2},i}$, $U^{n+\frac{1}{2},j}$ are calculated from sweeps identical to those used in operator split calculations. The errors are thus introduced into the predictor states, from which the intercell fluxes are calculated. These fluxes therefore never know about the constant data. One possible way to rectify this problem is to replace the Godunov sweeps used to calculate the predictor states with non-conservative steps. The overall scheme would still be conservative, since the conservation form (7.1) is still used. A non-conservative predictor step has not yet been derived. An example of an existing finite volume scheme with a non-conservative predictor step is the MUSCL-Hancock scheme described by Quirk in [41].

7.3 Extension of the Scheme of §6.3

In this section, the scheme described in §6.3 for the solution of two dimensional hyperbolic systems of conservation laws on Cartesian grids is extended for use on curvilinear grids. As before, intercell fluxes are taken as an integral average at the half timestep level, over some fixed space integration range, of the component of flux perpendicular to the boundary on

the combined solutions to nearby grid-aligned Riemann problems. Waves from the one dimensional, grid-aligned Riemann problems are modelled as in the Cartesian case, i.e they have a component of velocity perpendicular to the boundary from which they originate, taken as a wavespeed in the solution to the Riemann problem, and a component of velocity parallel to the boundary, taken from the state immediately behind the wave.

Consider the boundary $(i + \frac{1}{2}, j)$. Only waves from boundaries labelled 1—7 in Fig. 7.1 will be taken into account when computing an intercell flux for this boundary (labelled 4 in Fig 7.1). It may be possible, under extreme circumstances, such as Courant number coefficient close to unity or a grid cell that is far from rectangular, that waves from other boundaries may reach the space integration range (defined below), but their contribution will be small, and their omission should not significantly affect the global accuracy of the scheme. The space integration range I chosen is also illustrated in Fig. 7.1. It is defined as the area between the straight line joining the midpoints of boundaries 2 and 5 and the straight line joining the midpoints of boundaries 3 and 6 intersected with the union of cells $\Delta_{i,j}$ and $\Delta_{i+1,j}$. On Cartesian grids, this reduces to the integration range (5.7) defined in §5.2.

Define angles $\Theta_{i+\frac{1}{2},j}^{\pm}$ to be those between the normal to the boundary $(i + \frac{1}{2}, j)$ and the lines joining the midpoints of this boundary and the midpoints of boundaries $(i + \frac{1}{2} \pm 1, j)$, as shown in Fig. 7.2. Define $(x_{i+\frac{1}{2},j+\frac{1}{2}}, y_{i+\frac{1}{2},j+\frac{1}{2}})$ to be the intersection of boundaries $(i + \frac{1}{2}, j)$ and $(i, j + \frac{1}{2})$, and $(x_{i+\frac{1}{2},j}^m, y_{i+\frac{1}{2},j}^m)$ to be the midpoint of boundary $(i + \frac{1}{2}, j)$. Simple geometric arguments show that

$$\tan \theta_{i+\frac{1}{2},j}^+ = \sqrt{\frac{(x^p - x_{i+\frac{1}{2},j}^m)^2 + (y^p - y_{i+\frac{1}{2},j}^m)^2}{(x^p - x_{i+\frac{3}{2},j}^m)^2 + (y^p - y_{i+\frac{3}{2},j}^m)^2}} \quad (7.5)$$

where

$$x^p = \frac{1}{1+g^2} \left[x_{i+\frac{3}{2},j}^m + g(y_{i+\frac{3}{2},j}^m - y_{i+\frac{3}{2},j+\frac{1}{2}}) + g^2 x_{i+\frac{3}{2},j}^m \right] \quad (7.6)$$

$$y^p = g(x^p - x_{i+\frac{3}{2},j+\frac{1}{2}}) + y_{i+\frac{3}{2},j+\frac{1}{2}} \quad (7.7)$$

$$g = \frac{y_{i+\frac{1}{2},j+\frac{1}{2}} - y_{i+\frac{1}{2},j-\frac{1}{2}}}{x_{i+\frac{1}{2},j+\frac{1}{2}} - x_{i+\frac{1}{2},j-\frac{1}{2}}} \quad (7.8)$$

If $x_{i+\frac{1}{2},j+\frac{1}{2}} - x_{i+\frac{1}{2},j-\frac{1}{2}}$ is small, then

$$y^p = \frac{1}{1+h^2} \left[y_{i+\frac{3}{2},j}^m + h(x_{i+\frac{3}{2},j}^m - x_{i+\frac{3}{2},j+\frac{1}{2}}) - h^2 y_{i+\frac{3}{2},j}^m \right] \quad (7.9)$$

$$x^p = h(y^p - y_{i+\frac{3}{2},j+\frac{1}{2}}) + x_{i+\frac{3}{2},j+\frac{1}{2}} \quad (7.10)$$

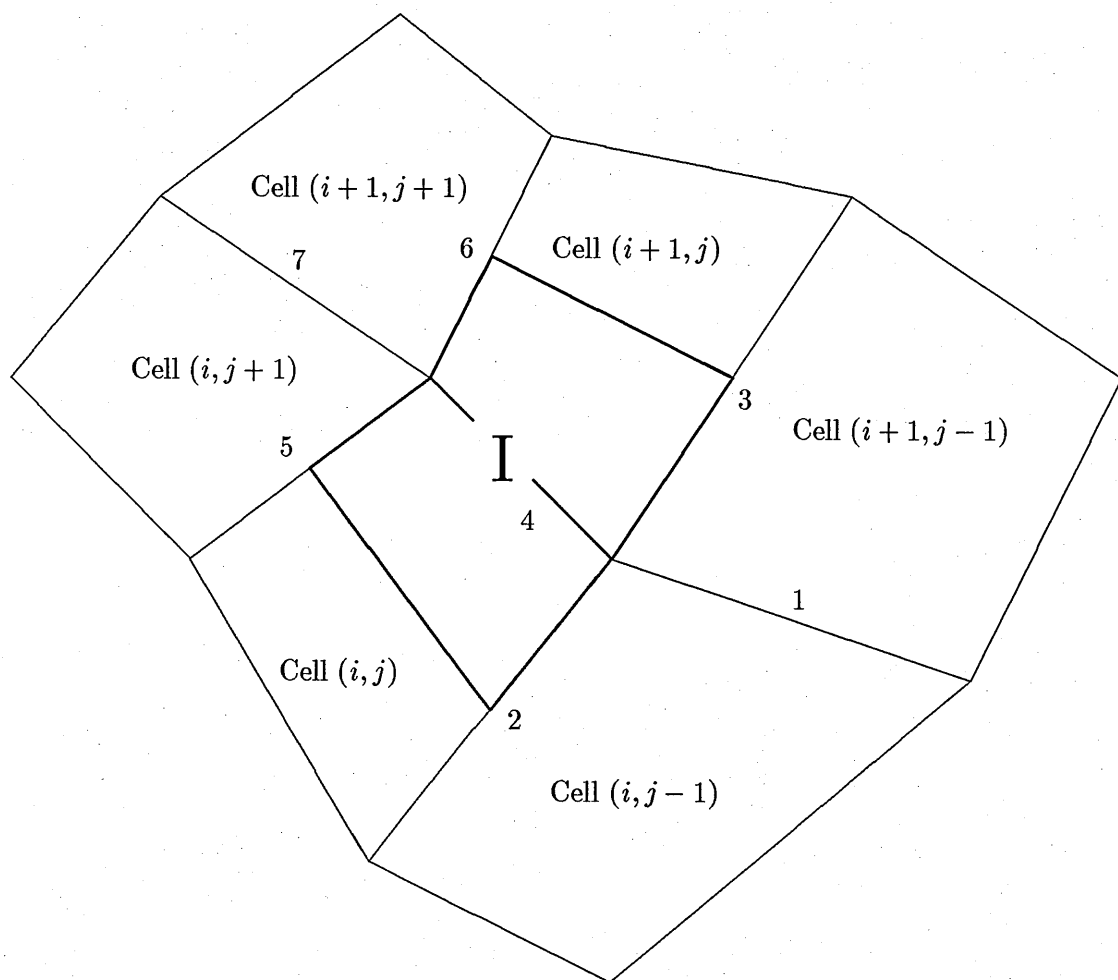


Figure 7.1: Illustration of the integration range for the scheme of §7.3. The numbers indicate the labelling of boundaries in the calculation of the intercell flux.

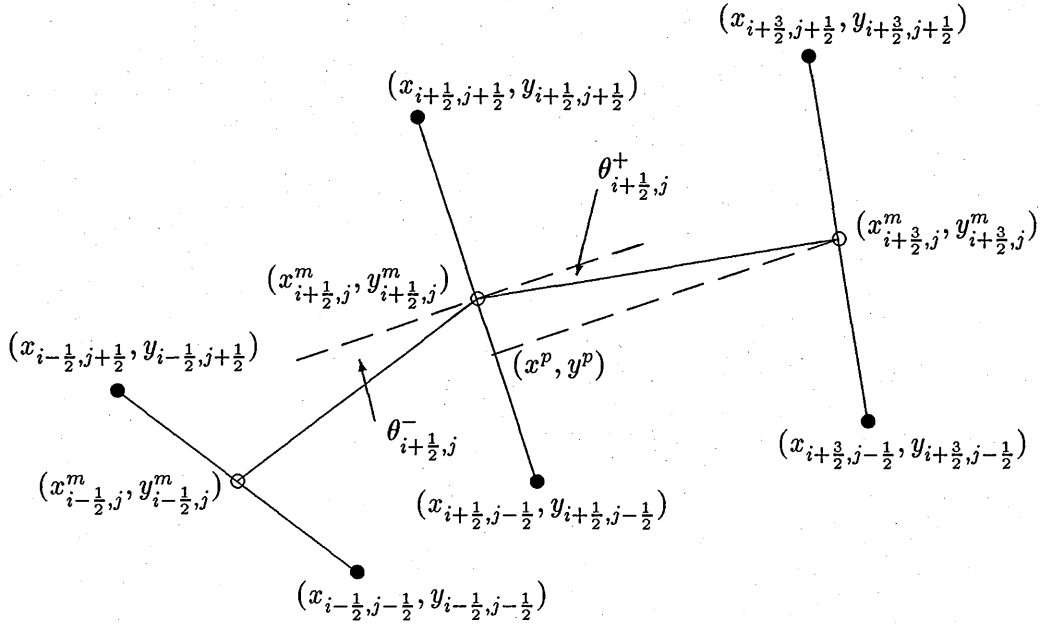


Figure 7.2: Definition of angles $\theta_{i+\frac{1}{2},j}^+$ for the scheme of §7.3. Full circles denote grid nodes; empty circles denote midpoints of boundaries.

$$h = \frac{x_{i+\frac{1}{2},j+\frac{1}{2}} - x_{i+\frac{1}{2},j-\frac{1}{2}}}{y_{i+\frac{1}{2},j+\frac{1}{2}} - y_{i+\frac{1}{2},j-\frac{1}{2}}} \quad (7.11)$$

Similarly for $\theta_{i+\frac{1}{2},j}^-$, $\theta_{i,j+\frac{1}{2}}^\pm$. Define also angles $\eta_{i+\frac{1}{2},j}^{L\pm}$ to be the angles between the normal to the boundary $(i+\frac{1}{2},j)$ and the lines between the midpoint of boundary $(i+\frac{1}{2},j)$ and the midpoints of boundaries $(i,j \pm \frac{1}{2})$ and angles $\eta_{i+\frac{1}{2},j}^{R\pm}$ to be the angles between the normal to the boundary $(i+\frac{1}{2},j)$ and the lines between the midpoint of boundary $(i+\frac{1}{2},j)$ and the midpoints of boundaries $(i+1,j \pm \frac{1}{2})$. See Fig. 7.3. Tangents of these angles η can be found from simple geometric arguments. Denote by $a^{(l)}$ the component of velocity of a wave from boundary l perpendicular to that boundary, and by $b^{(l)}$ the component of velocity parallel to that boundary assigned to the wave in the wave model described above. Define, as in §6.2, the weight $\zeta_{i+\frac{1}{2},j}^{(l)}$ for the wave to be the area of the intersection of the region the wave sweeps over at time $t = \frac{1}{2}\Delta t$ with the integration range, divided by the area of the integration range. Careful analysis shows that:

$$\zeta_{i+\frac{1}{2},j}^{(1)} = \frac{a^{(1)}\Delta t^2}{8V(I)} \max \left\{ b^{(1)} - a^{(1)} \tan \eta_{i+\frac{1}{2},j-1}^{K+}, 0 \right\} \quad (7.12)$$

$$\zeta_{i+\frac{1}{2},j}^{(2)} = \frac{a^{(2)}\Delta t^2}{8V(I)} \left[\frac{2l^{(2)}}{\Delta t} + b^{(2)} - a^{(2)} \tan \theta_{i,j-\frac{1}{2}}^+ \right] \quad (7.13)$$

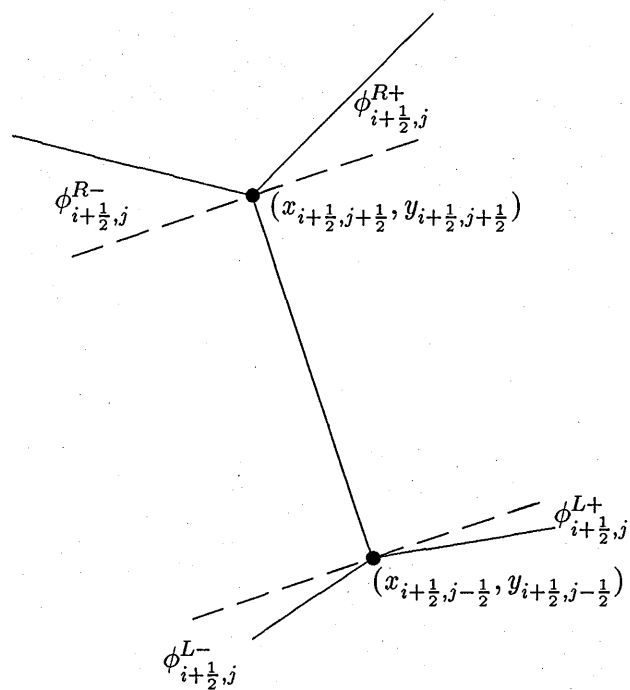


Figure 7.3: Definition of angles $\phi_{i+\frac{1}{2},j}^{R\pm}$, $\phi_{i+\frac{1}{2},j}^{L\pm}$ for the scheme of §7.3.

$$\zeta_{i+\frac{1}{2},j}^{(3)} = \frac{a^{(3)}\Delta t^2}{8V(I)} \left[\frac{2l^{(3)}}{\Delta t} - b^{(3)} + a^{(3)} \tan \theta_{i+1,j-\frac{1}{2}}^+ \right] \quad (7.14)$$

$$\begin{aligned} \zeta_{i+\frac{1}{2},j}^{(4)} = \frac{a^{(4)}\Delta t^2}{8V(I)} & \left[\frac{4l^{(4)}}{\Delta t} - \max \left\{ b^{(4)}, a^{(4)} \tan \eta_{i+\frac{1}{2},j}^{K-} \right\} \right. \\ & \left. - \min \left\{ b^{(4)}, a^{(4)} \tan \eta_{i+\frac{1}{2},j}^{K+} \right\} \right] \end{aligned} \quad (7.15)$$

$$\zeta_{i+\frac{1}{2},j}^{(5)} = \frac{a^{(5)}\Delta t^2}{8V(I)} \left[\frac{2l^{(5)}}{\Delta t} + b^{(5)} - a^{(5)} \tan \theta_{i,j+\frac{1}{2}}^- \right] \quad (7.16)$$

$$\zeta_{i+\frac{1}{2},j}^{(6)} = \frac{a^{(6)}\Delta t^2}{8V(I)} \left[\frac{2l^{(6)}}{\Delta t} - b^{(6)} + a^{(6)} \tan \theta_{i+1,j+\frac{1}{2}}^- \right] \quad (7.17)$$

$$\zeta_{i+\frac{1}{2},j}^{(7)} = -\frac{a^{(7)}\Delta t^2}{8V(I)} \min \left\{ b^{(7)} - a^{(7)} \tan \eta_{i+\frac{1}{2},j+1}^{K-}, 0 \right\} \quad (7.18)$$

$$(7.19)$$

where

$$K = K(a^{(l)}) = \begin{cases} L & \text{if } a^{(l)} < 0 \\ R & \text{otherwise} \end{cases} \quad (7.20)$$

and

$$V(I) = \frac{1}{2}(V(\Delta_{i,j}) + V(\Delta_{i+1,j})) \quad (7.21)$$

is the area of the integration range I , equal to the mean of the areas of the cells either side of the boundary. The intercell flux can now be defined as

$$\mathbf{E}_{i+\frac{1}{2},j} = \frac{V(\Delta_{i,j})\mathbf{E}_{i,j} + V(\Delta_{i+1,j})\mathbf{E}_{i+1,j}}{V(\Delta_{i,j}) + V(\Delta_{i+1,j})} - \sum_{l=1\dots 7, k=1\dots N} \zeta_{i+\frac{1}{2},j}^{(l,k)} \Delta \mathbf{E}^{(l,k)} \quad (7.22)$$

where the summation takes place over all seven boundaries and all N waves, $\mathbf{E}$ always denotes the component of flux perpendicular to the boundary $(i + \frac{1}{2}, j)$, $\zeta_{i+\frac{1}{2},j}^{(l,k)}$ is the coefficient for the k^{th} wave from the l^{th} boundary defined as the coefficient $\zeta_{i+\frac{1}{2},j}^{(l)}$ from one of (7.12) — (7.18) using the components of velocity for the wave and $\Delta \mathbf{E}^{(l,k)}$ is the

jump in $\mathbf{E}$ across that wave. The flux $\mathbf{E}_{i+\frac{1}{2},j}$ is transformed back into $x-y$ coordinates by finding components of the momentum flux in that frame. The calculation of fluxes $\mathbf{E}_{i,j+\frac{1}{2}}$ follows similarly. Given a complete set of such fluxes, the solution $\mathbf{U}^n$ can be updated to the next timelevel using the conservative form (7.1).

To avoid the large number of flux calculations inherent in (7.22), and thus improve the efficiency of the scheme, an alternative intercell flux can be constructed from a weighted average of primitive variables. Denote by $\mathbf{P}$ the vector of primitive variables for the system being solved. A weighted average state for boundary $(i + \frac{1}{2}, j)$ can be defined as:

$$\mathbf{P}_{i+\frac{1}{2},j} = \frac{V(\Delta_{i,j})\mathbf{P}_{i,j} + V(\Delta_{i+1,j})\mathbf{P}_{i+1,j}}{V(\Delta_{i,j}) + V(\Delta_{i+1,j})} - \sum_{l=1 \dots 7, k=1 \dots N} \zeta_{i+\frac{1}{2},j}^{(l,k)} \Delta^{(l,k)} \mathbf{P} \quad (7.23)$$

where $\Delta^{(l,k)} \mathbf{P}$ is the jump in $\mathbf{P}$ across wave k from boundary l . The intercell flux $\mathbf{E}_{i+\frac{1}{2},j}$ can then be calculated as:

$$\mathbf{E}_{i+\frac{1}{2},j} = \mathbf{E}(\mathbf{P}_{i+\frac{1}{2},j}) \quad (7.24)$$

Fluxes $\mathbf{E}_{i,j+\frac{1}{2}}$ follow similarly. For the linear advection equation on Cartesian grids, this scheme is identical to the previous one, so it has the same accuracy and stability on that equation. However, on nonlinear systems the two schemes are different, although it is assumed here that they have the same accuracy and stability properties.

The schemes presented above, being second order accurate in space and time, will suffer from spurious oscillations in regions of high gradient. These can be damped by the use of one-dimensional limiter functions as follows. For $l = 1, 4, 7$, $a^{(l,k)}$ for each wave k is replaced in $\zeta^{(l,k)}$ by a function $\Phi^{(l,k)}$ defined from a one dimensional limiter function $\phi^{(l,k)}$ as:

$$\Phi^{(l,k)} = \phi^{(l,k)} \frac{l^{(K)}_{i+\frac{1}{2},j}}{\Delta t} \quad (7.25)$$

where K is defined as in (7.20) and $l^{(K)}_{i+\frac{1}{2},j}$ is an integration length. The limiters $\phi^{(l,k)}$ are computed using the flow parameter for boundary $(i + \frac{1}{2}, j)$ and a Courant number $\nu^{(l,k)}_{i+\frac{1}{2},j} = a^{(l,k)} \frac{\Delta t}{l^{(K)}_{i+\frac{1}{2},j}}$. For $l = 2, 3, 5, 6$ take

$$\Phi^{(l,k)} = \phi^{(l,k)} \frac{l^{(l)}}{\Delta t} \quad (7.26)$$

where $l^{(l)}$ is the length of side l , and the Courant number for the limiter is taken as $\nu^{(l,k)}_{i+\frac{1}{2},j} = b^{(l,k)} \frac{\Delta t}{l^{(l)}}$. Note that when $\phi^{(l,k)} \equiv \nu^{(l,k)}$, the scheme reduces to the original second order scheme.

The main advantage of the scheme presented in this section over the scheme of §7.2 is that it will preserve constant data, since all wavejumps $\Delta \mathbf{E}^{(l,k)}$ (or $\Delta \mathbf{P}^{(l,k)}$) vanish in this case and the intercell flux is the flux on that constant state. However, this scheme has three main disadvantages relative to the scheme of the last section. Firstly, for each intercell

flux, N limiters must be computed for each of 7 boundaries, i.e. $7N$ in total, as opposed to just N for the other scheme. It should be possible to cure this problem by rethinking the way the scheme is limited. The second problem is that the scheme presented here requires significantly more memory: tangents for the angles $\theta_{i+\frac{1}{2},j}^{\pm}$, $\theta_{i,j+\frac{1}{2}}^{\pm}$, $\eta_{i+\frac{1}{2},j}^{L\pm}$, $\eta_{i+\frac{1}{2},j}^{R\pm}$, $\eta_{i,j+\frac{1}{2}}^{L\pm}$ and $\eta_{i,j+\frac{1}{2}}^{R\pm}$ must be stored. A third disadvantage of the scheme is that there is a significant amount of logic involved in deciding which waves need to be used in the calculation of a given flux and which do not. This adds to the computation time. Numerical experiments using the primitive variable version of the scheme described above have indicated that it can be approximately twice as expensive, in terms of CPU time, as the scheme of §7.2.

7.4 Numerical Experiments

It is well known in the field of Gas Dynamics that when a shock wave is reflected obliquely from a solid boundary, two main types of reflection can occur: regular reflection and Mach reflection. In regular reflection, just two waves exist — the incident shock and a reflected shock. These meet at some point on the boundary. In Mach reflection, the incident and reflected shocks meet at a point within the flowfield called the *Triple point*, and a third shock, called the *Mach stem*, joins this point to the boundary. There are various types of Mach reflection. Some of these have been discussed in §3.6. See the book by Ben-Dor [3] for a discussion of Mach Reflection applied to the Euler equations.

It seems to be less well known that Mach reflection of bore waves can occur in shallow water — the only recent papers in which the phenomena is discussed appears to be Krehl *et. al.* [27] and Takayama *et. al.* [56]. In this section, the scheme of §7.2 will be used to illustrate the phenomena. Since experimental data was not available to the author at the time of writing, the results will be compared against some from a code based on the one dimensional `war` scheme extended to two dimensions via space operator splitting. The Strang splitting (see Chapter 2) was used, since it is of the same theoretical accuracy as the finite volume scheme being tested.

A bore wave travelling in the x direction in two space dimensions is completely determined by the state (ϕ_R, u_R, v_R) ahead of the bore and the Froude number

$$F_S = \frac{S^2}{\phi_R} \quad (7.27)$$

of the bore, where S is the speed of the bore. Conditions that determine the state (ϕ_L, u_L, v_L) behind the bore can be found from the one dimensional Shallow Water equations, and can be summarized as follows. Define

$$r = \frac{1}{2} \left(-1 + \sqrt{1 + 8(\sqrt{F_R} - \sqrt{F_S})} \right) \quad (7.28)$$

where $F_R = u_R^2/\phi_R$ is the Froude number of the pre-bore state. The post-bore state is then determined as:

$$\phi_L = r \cdot \phi_R \quad (7.29)$$

$$u_L = S + \frac{1}{r}(u_R - S) \quad (7.30)$$

$$v_L = v_R \quad (7.31)$$

where $S = \sqrt{\phi_R F_S}$ from (7.27). A bore with a Froude number of 8.0 traveling into a pre-bore state

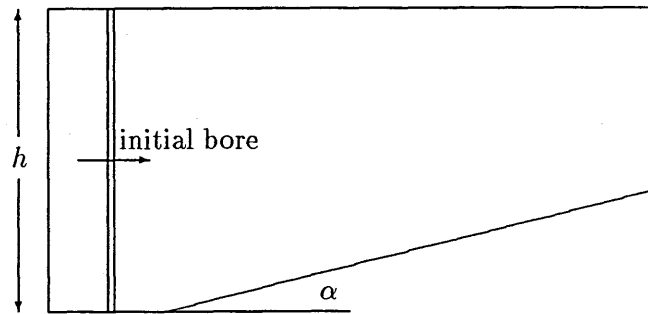
$$(\phi_R, u_R, v_R) = (1.0, 0.0, 0.0) \quad (7.32)$$

is considered here. It will be reflected from a boundaries at 25, 40 and 55 degrees to the direction of propagation of the incident bore. The initial setup and type of grid used are shown in Fig. 7.4. In all calculations, the Courant number coefficient was set at 0.8. The Selective Curve Fitting (scf) [59] and Two Rarefaction (tr) Riemann solver were used adaptively [64] everywhere except in the predictor stages of the scheme of §7.2, where the HLL solver [21], [66] was used. Plots of geopotential ϕ and Froude number $(u^2 + v^2)/\phi$ are shown for each reflection. All runs were performed on a DEC 3000 AXP Model 400-400F workstation.

Results from the operator splitting scheme and the scheme of §7.2 with a ramp angle of 25 degrees are shown in figures (7.5) and (7.6) respectively. The grid used in both computations had 600×300 grid cells. The operator split scheme took a CPU time of 8 hours, 16 minutes, 24.6 seconds; the finite volume scheme took a CPU time of 6 hours, 1 minute, 31.4 seconds. Both schemes indicate that this reflection is a Single Mach Reflection. Note the appearance of a small 'glitch' at the base of the Mach stem in the operator split result. This has not appeared in the finite volume result and its cause is unknown. It has been reported (Bill Speares, private communication) that similar effects have been observed when studying Mach reflection in the Euler equations of Gas Dynamics using an operator split code on curvilinear grids, and therefore may be a feature of such schemes.

The results for the two schemes when the ramp angle is 40 degrees are shown in figures 7.7 and 7.8. The grid used for these calculations had 400×400 cells. The operator split scheme took a CPU time of 5 hours, 21 minutes, 20.5 seconds; the finite volume scheme took a CPU time of 4 hours, 5 minutes, 28.0 seconds. The schemes indicate that this reflection is a Complex Mach Reflection, since a kink appears in the reflected bore. Note that the glitch in the operator split result is more pronounced than in the last case and the difference between this solution and the finite volume solution is now significant. The glitch in the operator split result is large enough to have deformed the Mach stem near the wall.

Results from the two schemes when the ramp angle is 55 degrees are shown in figures 7.9 and 7.10. The grid used for these calculations had 400×600 cells. The operator split scheme took a CPU time of 10 hours, 33 minutes, 2.5 seconds; the finite volume scheme took a CPU time of 7 hours, 16 minutes, 25.3 seconds. Again, both schemes seem to suggest a complex Mach reflection, although this reflection is probably close to the boundary between Mach and regular reflection, because the Mach stem is very short in the finite volume case. The Mach stem in the operator split case has been completely obliterated by the glitch.



$h = 0.5$ if $\alpha = 25$ degrees

$h = 1.5$ if $\alpha = 40$ degrees

$h = 2.5$ if $\alpha = 55$ degrees

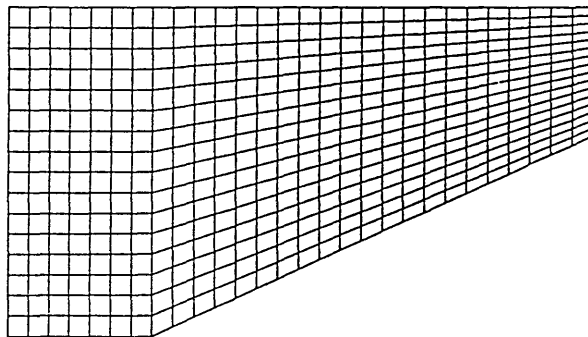


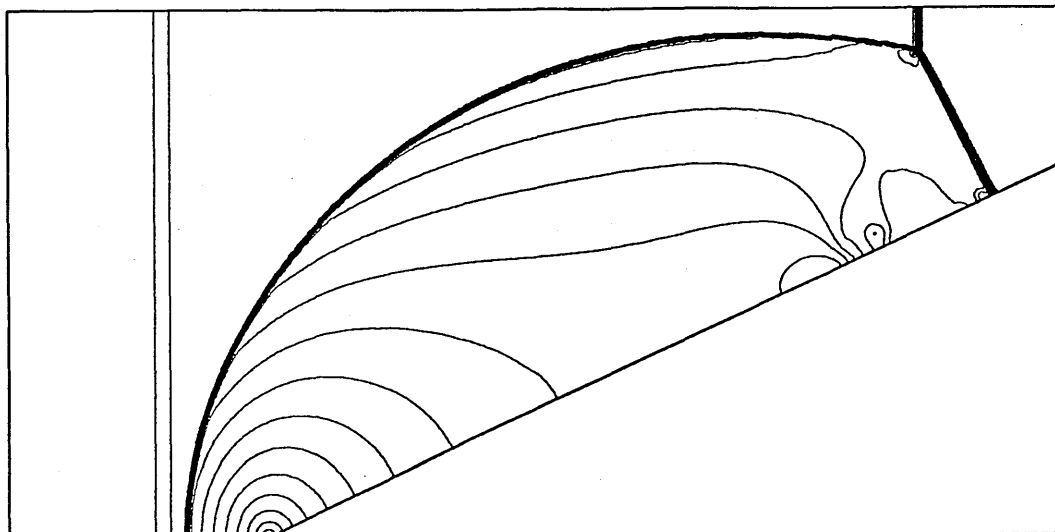
Figure 7.4: The flow domain, initial conditions and type of grid used for the bore reflection problems.

The question of why 'glitches' have appeared in the operator split results but not in the finite volume results needs to be solved. The only way to determine which scheme is at fault may be to seek experimental results or numerical solutions from radically different schemes.

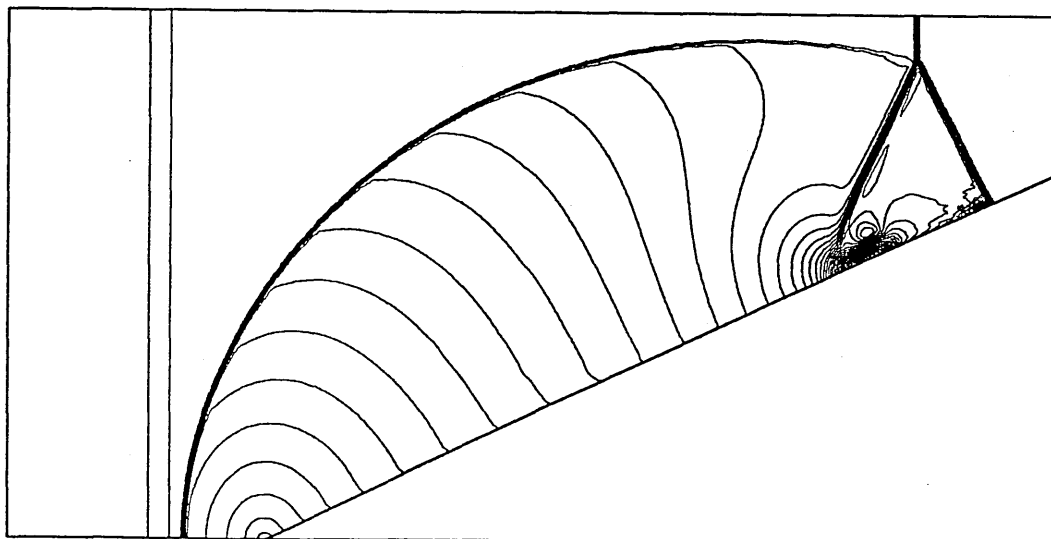
The CPU times given above suggest that the finite volume scheme is approximately 20% cheaper than space operator split WAF with Strang splitting.

7.5 Summary

Two new finite volume schemes for the solution of hyperbolic systems of conservation laws on curvilinear grids in two space dimensions have been presented. They are extensions of schemes for Cartesian grids presented in Chapter 6. Only one of the schemes presented in this chapter has been tested numerically and has been used to demonstrate the phenomena of Mach reflection of bore waves in shallow water. Numerical results were compared with those from a second order operator split version of the one dimensional WAF scheme. The results were significantly different in some situations and further work is needed to establish why this is the case.

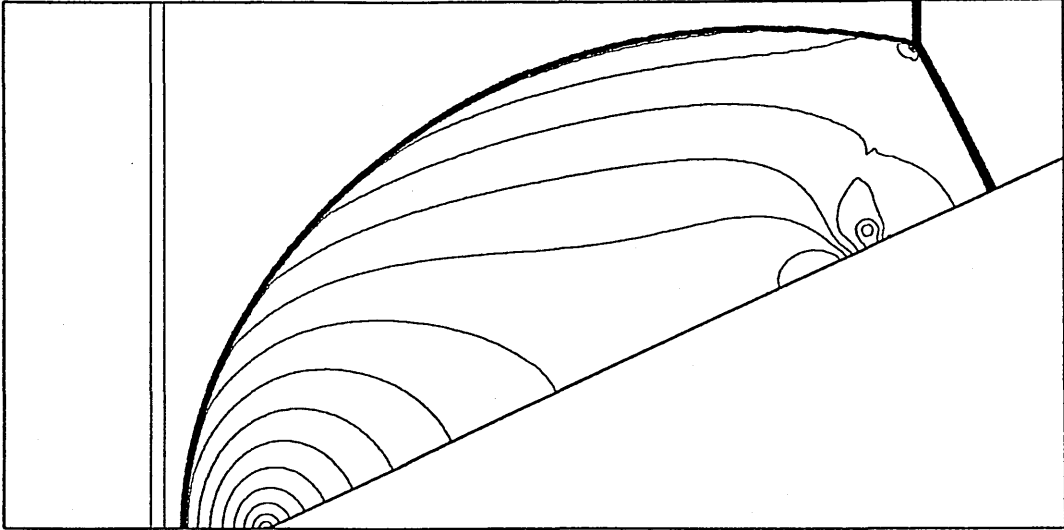


(a) Geopotential.

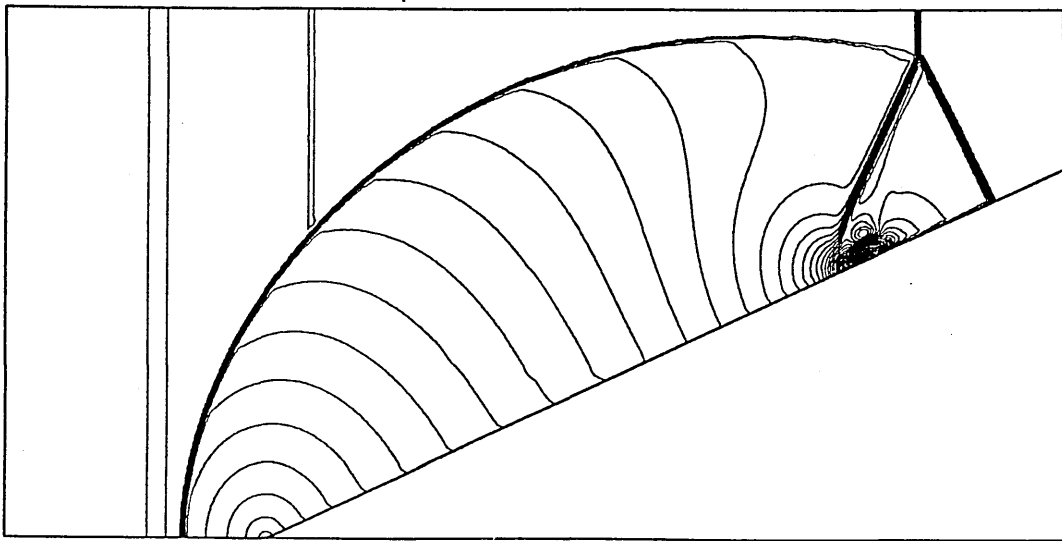


(b) Froude number.

Figure 7.5: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 25 degree ramp. Results obtained from the operator split scheme with the `VAN LEER` limiter on a 600×300 grid. A Courant number coefficient of 0.8 was used.

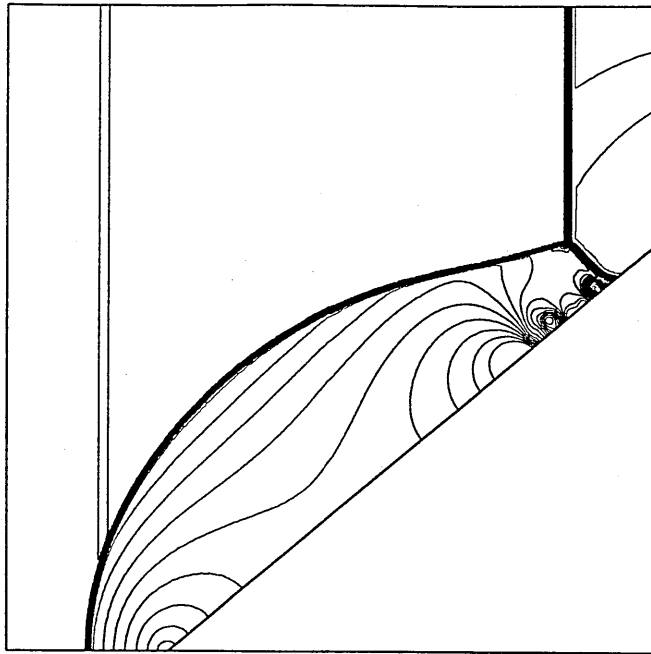


(a) Geopotential.

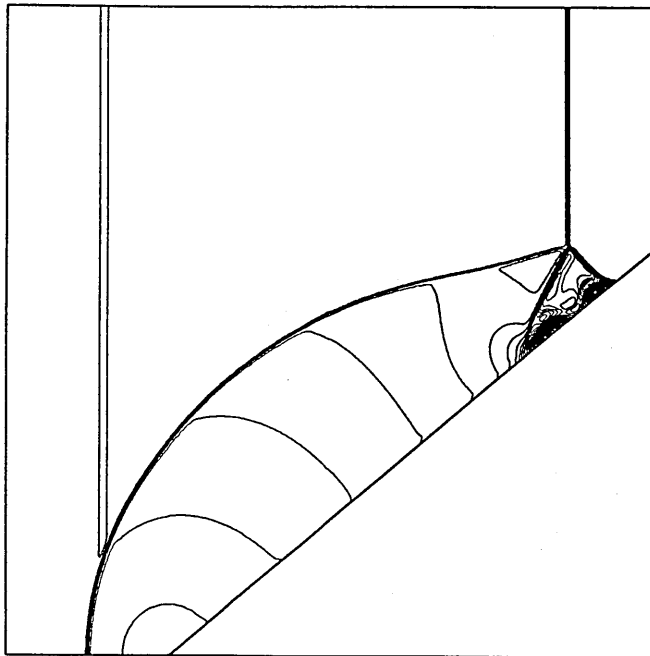


(b) Froude number.

Figure 7.6: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 25 degree ramp. Results obtained from the scheme of §7.2 with the `VAN LEER` limiter on a 600×300 grid. A Courant number coefficient of 0.8 was used.

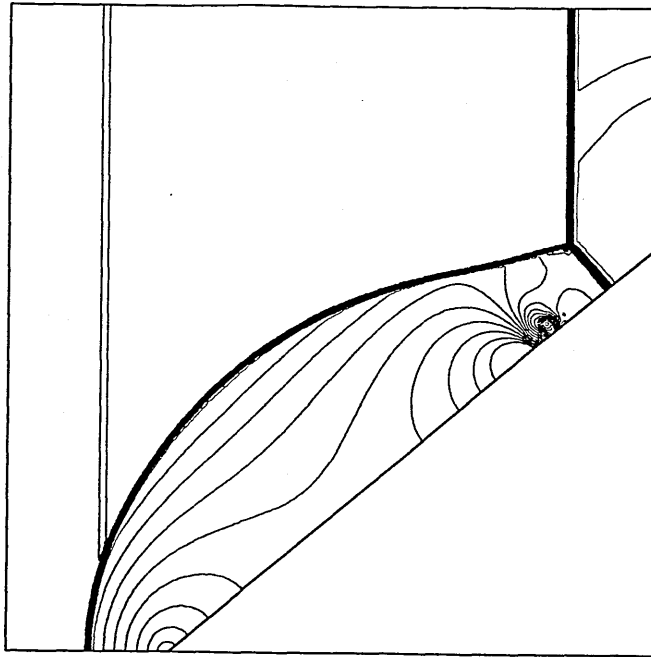


(a) Geopotential.

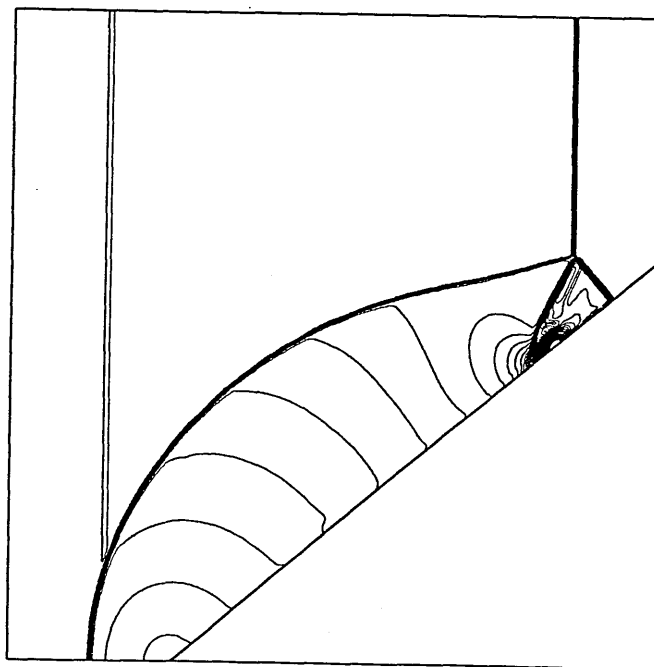


(b) Froude number.

Figure 7.7: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 40 degree ramp. Results obtained from the operator split scheme with the `VAN LEER` limiter on a 400×400 grid. A Courant number coefficient of 0.8 was used.

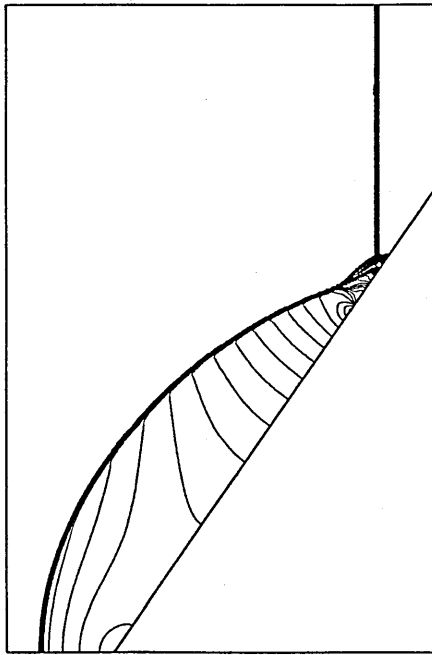


(a) Geopotential.

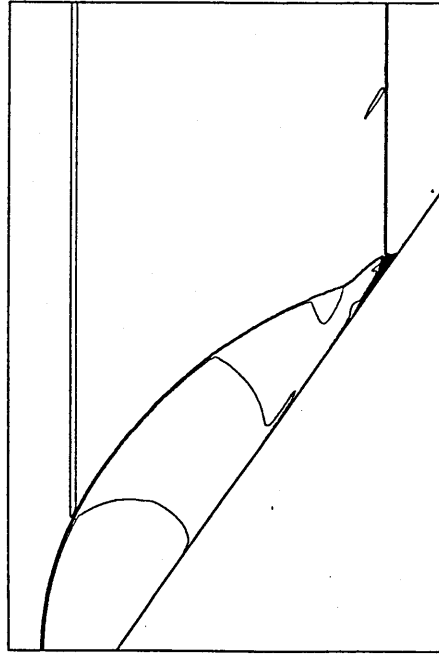


(b) Froude number.

Figure 7.8: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 40 degree ramp. Results obtained from the scheme of §7.2 with the VAN LEER limiter on a 400×400 grid. A Courant number coefficient of 0.8 was used.

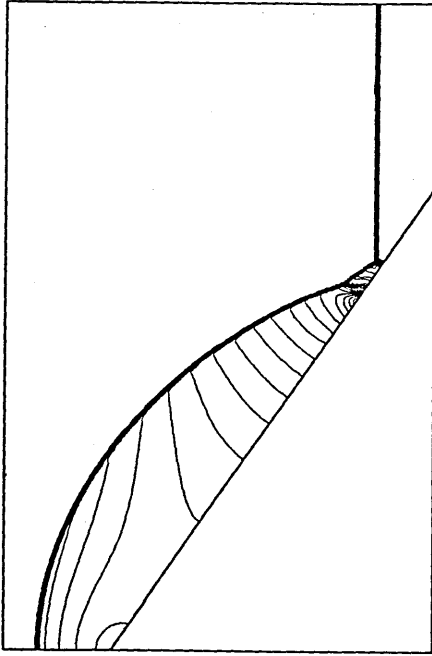


(a) Geopotential.

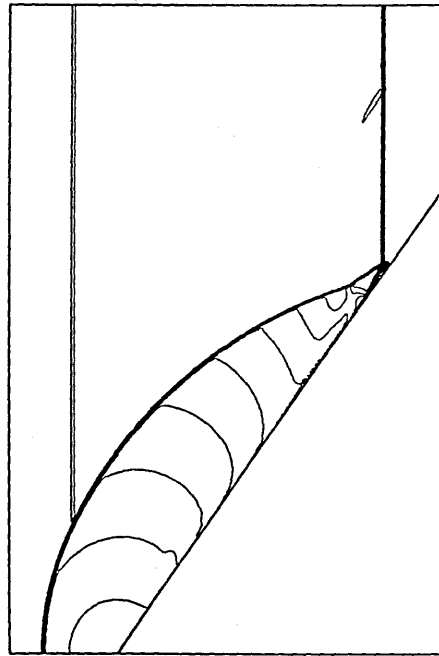


(b) Froude number.

Figure 7.9: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 55 degree ramp. Results obtained from the operator split scheme with the VAN LEER limiter on a 400×600 grid. A Courant number coefficient of 0.8 was used.



(a) Geopotential.



(b) Froude number.

Figure 7.10: Plots of geopotential and Froude number for the reflection of a bore with Froude number $F_S = 8.0$ over a 55 degree ramp. Results obtained from the scheme of §7.2 with the **VAN LEER** limiter on a 400×600 grid. A Courant number coefficient of 0.8 was used.

Chapter 8

Summary and Future Work

In this thesis, various numerical methods for the solution of hyperbolic conservation laws in one, two and three space dimensions have been developed. The main aim has been to show the potential of the **waf** approach of Toro [62], [67] to generate accurate, upwind-biased schemes with good stability properties. The thesis is now summarized, and the scope for future work is reviewed.

Chapters 1 and 2 introduced the aims of the thesis and background material, and need not be discussed further here. In chapter 3, a **tvd** 'large timestep' scheme was developed via the **waf** approach that is stable for timesteps up to twice the size of those allowed by the stability restrictions on the classical **waf** scheme. It is essentially a Riemann-problem based extension of the Warming-Beam and Lax-Wendroff schemes to non-linear systems. The method was extended to two space dimensions by the use of space operator splitting and numerical experiments on the Euler equations of Gas Dynamics have shown the resulting scheme to be a robust, accurate method. The only fault of the scheme appears to be that when applied to non-linear systems, it suffers from small amplitude oscillations behind shock waves, even when forced to first order accuracy everywhere. This is probably due to the assumption of linear interaction of waves, which is obviously inaccurate near shock waves. More work is needed to eliminate this problem from the method, possibly by solving extra Riemann problems to model wave interactions more accurately, although this would impair the cost-effectiveness of the method. Other future work based on Chapter 3 could be to extend the method to allow even higher timesteps, though the negative effect of the assumption of linear interaction of waves would then be enhanced.

In Chapter 4, an alternative extension of the Warming-Beam scheme to non-linear systems by the use of local Riemann problems was developed. This scheme was not derived via the **waf** approach. Numerical experiments with this scheme proved disappointing, since they suggested that this extension of the Warming-Beam scheme is not robust enough for the calculation of strong shock waves. The problem appears to be that the leading phase error of the scheme for Courant numbers less than unity forces any errors in the solution to move *ahead* of the shock and into the relatively low pressure region, causing unphysical negative densities and pressures. Further investigation is needed to increase

the robustness of this scheme. A one-parameter class of **TVD** methods that are second order accurate in smooth flow was created by averaging this scheme with the original **WAF** scheme. The class includes an extension of Fromm's method [13] and a third-order accurate method. None of the resulting new schemes have been tested numerically yet.

In Chapter 5, the **WAF** approach was extended to construct finite volume methods for the solution of the linear advection equation in two and three space dimensions on Cartesian grids. In two dimensions, two of the methods found proved to be second order accurate in space and time, and have good stability properties. The potential for further work based on the ideas presented in this Chapter are immense. Firstly, the stability conditions on the second order schemes derived in the Chapter need to be proven analytically. Different integration ranges and integration schemes can be tried to explore whether more stable schemes can be derived. The ideas of Chapter 3 could be imported to develop large timestep finite volume schemes in two and three space dimensions, most notably finite volume extensions of the Warming-Beam scheme. Extensions to unstructured grids are also possible: all that is needed is a suitable integration range. One possibility may be to simply join the ends of each intercell boundary to the centres of the cells it separates. This is a particularly interesting choice, since it applies to grids of any shaped cells, even grids of mixed shapes. Another direction future work could take is the generation of finite volume versions of the Random Flux scheme [60], by randomly sampling the integration range to find the intercell flux. Alternatively, one could develop **MUSCL**-type schemes by reconstructing data prior to integrating to find the intercell flux. This approach has the potential of producing schemes of higher than second order accuracy.

In Chapter 6, one of the second order accurate schemes found in Chapter 5 was extended to non-linear systems. Two extensions were considered. One involved simply rewriting the calculation of intercell flux as a two-step procedure. The resulting extension proved to be a robust and accurate scheme, that is competitive with space operator split **WAF** on the Shallow Water equations. However, numerical experiments revealed that the mechanism employed to control spurious oscillations does not always work. More work is needed to rectify this fault. Further numerical experiments are also needed to establish whether the scheme does, in fact, offer any improvements in accuracy. The second extension was developed more in the spirit of the original **WAF** scheme by integrating over the solutions to grid aligned Riemann problems. However, it proved to lack the robustness of the first extension when computing the interactions of strong shocks. More work is needed to establish why this is the case. Both extensions need to be tested on a wider variety of problems, in particular on the solution of various standard problems for the Euler equations. The convergence of these schemes needs to be assessed, and the three dimensional versions need to be implemented and fully tested. Future work could involve studying the effect of replacing the **WAF** flux with a **MUSCL** flux in the scheme of §6.2, or even using the flux from the Random Flux scheme [60]. The extension of the other second order accurate scheme found in Chapter 5 to non-linear systems of equations could also prove an interesting area of research. There is also potential for treating source terms via some kind of integral average approach.

In Chapter 7, the two schemes discussed in Chapter 6 were extended for use on general

structured grids of quadrilaterals. The methods were tested on the problem of Mach reflection in shallow water. No experimental results were available for comparison, so the numerical results were compared against results from an operator split **WAF** code. Some significant differences between the results were found, so both codes need to be fully validated and the differences between the results need to be carefully assessed. Future work must include the development of non-conservative predictor steps for the scheme of §7.2, to prevent the scheme suffering from the operator splitting error, i.e to ensure it preserves constant data. Following that, rigorous testing of both schemes presented in Chapter 7 on the Euler equations is needed, to establish the true accuracy and relative cost of the two schemes.

Appendix A

On the Accuracy of Multidimensional Advection Schemes

In [44] Roe provided, in a simple form, conditions which determine the accuracy of a numerical scheme for the solution of the one-dimensional linear advection equation

$$u_t + au_x = 0 \quad (\text{A.1})$$

where a is a constant wavespeed, by considering a general scheme for (A.1) in the form

$$u_i^{n+1} = \sum_{\alpha} A_{\alpha} u_{i+\alpha}^n \quad (\text{A.2})$$

where $u_i^n = u(i\Delta x, n\Delta t)$, $\{A_{\alpha}\}$ is a finite set of constant, non-zero coefficients, Δx is the constant mesh spacing and Δt is the timestep. Define the Courant number as $\nu = a \frac{\Delta t}{\Delta x}$. Roe proved the following two theorems

Theorem 11 *If u_i^n is a polynomial of degree p in i , scheme (A.2) will give the exact solution to (A.1) if and only if*

$$\sum_{\alpha} \alpha^q A_{\alpha} = (-\nu)^q \quad (\text{A.3})$$

for all integers q such that $0 \leq q \leq p$.

Theorem 12 *If any scheme meets the conditions of Theorem 11, the leading term in its pointwise error is:*

$$\frac{a\Delta x^p}{\nu(p+1)!} \left[(-\nu)^{p+1} - \sum_{\alpha} \alpha^{p+1} A_{\alpha} \right] \frac{\partial^{p+1}}{\partial x^{p+1}} u \quad (\text{A.4})$$

These two theorems enabled the following definition:

Definition 1 Any scheme of the form (A.2) for the one dimensional linear advection equation (A.1) that satisfies the conditions in Theorem 11 is called p^{th} order accurate.

The aim here is to note equivalent conditions for p^{th} order accuracy of multidimensional advection schemes. In two space dimensions the linear advection equation can be written as

$$u_t + au_x + bu_y = 0 \quad (\text{A.5})$$

where a and b are constant wavespeeds. Schemes for (A.5) on regular, Cartesian grids in the form

$$u_{i,j}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta} u_{i+\alpha,j+\beta}^n \quad (\text{A.6})$$

will be considered, where $u_{i,j}^n = u(i\Delta x, j\Delta y, n\Delta t)$, Δx and Δy define the mesh spacing in the x and y directions and $\{A_{\alpha,\beta}\}$ form a finite set of constant, non-zero coefficients. Define the directional Courant numbers $\nu_x = a\frac{\Delta t}{\Delta x}$ and $\nu_y = b\frac{\Delta t}{\Delta y}$. In two dimensions, Theorem 11 becomes

Theorem 13 If $u_{i,j}^n$ is a polynomial of degree p in i and j then scheme (A.6) is the exact solution of (A.5) if and only if

$$\sum_{\alpha,\beta} \alpha^q \beta^r A_{\alpha,\beta} = (-\nu_x)^q (-\nu_y)^r \quad (\text{A.7})$$

for all integer pairs (q, r) such that $q \geq 0$, $r \geq 0$ and $q + r \leq p$.

Proof Choose a basis for the set of two-dimensional polynomials as:

$$\{x^q y^r\}_{q,r} \quad (\text{A.8})$$

for integers q, r . It is only necessary to consider the performance of the scheme on the general basis element, i.e set

$$u_{\alpha,\beta}^n = (\alpha\Delta x)^q (\beta\Delta y)^r \quad (\text{A.9})$$

Without loss of generality, only the origin $(0, 0)$ is considered. Performing one timestep of (A.6) at $(0, 0)$ gives:

$$u_{0,0}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta} (\alpha\Delta x)^q (\beta\Delta y)^r \quad (\text{A.10})$$

The exact solution is:

$$u_{0,0}^{n+1} = u(-a\Delta t, -b\Delta t, n\Delta t) = (-a\Delta t)^q (-b\Delta t)^r \quad (\text{A.11})$$

Therefore the scheme gives the exact solution if and only if:

$$\sum_{\alpha,\beta} A_{\alpha,\beta} (\alpha\Delta x)^q (\beta\Delta y)^r = (-a\Delta t)^q (-b\Delta t)^r \quad (\text{A.12})$$

i.e

$$\sum_{\alpha,\beta} \alpha^q \beta^r A_{\alpha,\beta} = (-\nu_x)^q (-\nu_y)^r \quad (\text{A.13})$$

for all (q, r) such that $q \geq 0$, $r \geq 0$ and $q + r \leq p$. This is the required result, and the theorem is proved.

Theorem 14 *If the scheme (A.6) meets the conditions in Theorem 13, then the leading terms in the pointwise error have the form:*

$$\frac{1}{\Delta t} \frac{\Delta x^q}{q!} \frac{\Delta y^{p+1-q}}{(p+1-q)!} \cdot D_q \cdot u_{(qx)((p+1-q)y)}^n \quad (\text{A.14})$$

where

$$D_q = (-\nu_x)^q (-\nu_y)^{p+1-q} - \sum_{\alpha,\beta} \alpha^q \beta^{p+1-q} A_{q,p+1-q} \quad (\text{A.15})$$

for some integer q such that $0 \leq q \leq p+1$, and

$$u_{(qx)(ry)}^n = \frac{\partial}{\partial x^q \partial y^r} u^n \quad (\text{A.16})$$

Proof The pointwise error is defined as the difference between the exact solution and the numerical solution divided by the timestep. Without loss of generality, consider the origin $(0,0)$. The pointwise error here is:

$$P(0,0) = \frac{1}{\Delta t} \left[u(0,0,(n+1)\Delta t) - u_{0,0}^{n+1} \right] \quad (\text{A.17})$$

$$= \frac{1}{\Delta t} \left[u(-a\Delta t, -b\Delta t, n\Delta t) - \sum_{\alpha,\beta} A_{\alpha,\beta} u_{\alpha,\beta}^n \right] \quad (\text{A.18})$$

Expanding both terms in brackets as Taylor series gives:

$$P(0,0) = \frac{1}{\Delta t} \left[\sum_{q,r} \frac{(-a\Delta t)^q}{q!} \frac{(-b\Delta t)^r}{r!} u_{(qx)(ry)}^n - \sum_{\alpha,\beta} A_{\alpha,\beta} \left(\sum_{q,r} \frac{(\alpha\Delta x)^q}{q!} \frac{(\beta\Delta y)^r}{r!} u_{(qx)(ry)}^n \right) \right] \quad (\text{A.19})$$

where the derivatives $u_{(qx)(ry)}^n$ are taken at the origin. Rearranging gives

$$P(0,0) = \frac{1}{\Delta t} \left[\sum_{q,r} \frac{(-a\Delta t)^q}{q!} \frac{(-b\Delta t)^r}{r!} \left(1 - \frac{1}{\nu_x^q} \frac{1}{\nu_y^r} \sum_{\alpha,\beta} (-\alpha)^q (-\beta)^r A_{\alpha,\beta} \right) u_{(qx)(ry)}^n \right] \quad (\text{A.20})$$

If the conditions in Theorem 13 hold, then the terms in (A.20) with $q + r \leq p$ vanish, and the first terms in the pointwise error have the form

$$\frac{1}{\Delta t} \frac{(-a\Delta t)^q}{q!} \frac{(-b\Delta t)^r}{(r)!} \left(1 - \frac{1}{\nu_x^q} \frac{1}{\nu_y^r} \sum_{\alpha, \beta} (-\alpha)^q (-\beta)^r A_{\alpha, \beta} \right) u_{(qx)(ry)} \quad (\text{A.21})$$

where $q + r = p + 1$. This can be written as (A.14), (A.15) by using the identity $r = p + 1 - q$ and rearranging. Hence the theorem is proved.

Theorems 13 and 14 enable the following definition:

Definition 2 Any scheme of the form (A.6) for the two dimensional linear advection equation (A.5) that satisfies the conditions in Theorem 13 is called p^{th} order accurate.

Corollary 5 Any scheme of the form (A.6) for the two dimensional linear advection equation (A.5) must satisfy

$$N = \sum_{i=1}^{p+1} i \quad (\text{A.22})$$

conditions for p^{th} order accuracy, and therefore must have a stencil of at least N points.

Proof Follows easily from Theorem 13.

In three space dimensions the linear advection equation can be written as

$$u_t + au_x + bu_y + cu_z = 0 \quad (\text{A.23})$$

where a , b and c are constant wavespeeds. Schemes for (A.5) of the form

$$u_{i,j,k}^{n+1} = \sum_{\alpha, \beta, \gamma} A_{\alpha, \beta, \gamma} u_{i+\alpha, j+\beta, k+\gamma}^n \quad (\text{A.24})$$

will be considered, where $u_{i,j,k}^n = u(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$, Δx , Δy and Δz define the mesh spacing in the x , y and z directions respectively, and $\{A_{\alpha, \beta, \gamma}\}$ form a finite set of constant, non-zero coefficients. Define directional Courant numbers as $\nu_x = a\frac{\Delta t}{\Delta x}$, $\nu_y = b\frac{\Delta t}{\Delta y}$ and $\nu_z = c\frac{\Delta t}{\Delta z}$. In this case Theorem 11 becomes

Theorem 15 If $u_{i,j,k}^n$ is a polynomial of degree p in i , j and k then scheme (A.24) is the exact solution of (A.23) if and only if

$$\sum_{\alpha, \beta, \gamma} \alpha^q \beta^r \gamma^s A_{\alpha, \beta, \gamma} = (-\nu_x)^q (-\nu_y)^r (-\nu_z)^s \quad (\text{A.25})$$

for all integer triples (q, r, s) such that $q \geq 0$, $r \geq 0$, $s \geq 0$ and $q + r + s \leq p$.

Theorem 16 *If the scheme (A.24) meets the conditions in Theorem 15, then the leading terms in the pointwise error have the form:*

$$\frac{1}{\Delta t} \frac{\Delta x^q}{q!} \frac{\Delta y^r}{r!} \frac{\Delta z^{p+1-q-r}}{(p+1-q-r)!} \cdot D_{q,r} \cdot u_{(qx)(ry)((p+1-q-r)z)}^n \quad (\text{A.26})$$

where

$$D_{q,r} = (-\nu_x)^q (-\nu_y)^r (-\nu_z)^{p+1-q-r} - \sum_{\alpha,\beta,\gamma} \alpha^q \beta^r \gamma^{p+1-q-r} A_{q,r,p+1-q-r} \quad (\text{A.27})$$

for some integer pair (q, r) such that $q \geq 0$, $r \geq 0$, $0 \leq q + r \leq p + 1$ and

$$u_{(qx)(ry)(sz)}^n = \frac{\partial}{\partial x^q \partial y^r \partial z^s} u^n \quad (\text{A.28})$$

The proof of Theorems 15 and 16 follows easily from the two dimensional case. They lead to the definition

Definition 3 *Any scheme of the form (A.24) for the three dimensional linear advection equation (A.23) that satisfies the conditions in Theorem 15 is called p^{th} order accurate.*

Corollary 6 *Any scheme of the form (A.24) for the two dimensional linear advection equation (A.23) must satisfy*

$$N = \sum_{j=1}^{p+1} \left(\sum_{i=1}^j i \right) \quad (\text{A.29})$$

conditions for p^{th} order accuracy, and therefore must have a stencil of at least N points.

Proof Follows easily from Theorem 15.

Bibliography

- [1] J.D Anderson. *Modern Compressible Flow*. Mc Graw-Hill, 1990.
- [2] M. Ben-Artzi and J. Falcovitz. A Second Order Godunov-Type Scheme for Compressible Fluid Dynamics. *J. Comput. Phys.*, 55:1–32, 1985.
- [3] G. Ben-Dor. *Shock Wave Reflection Phenomena*. Springer-Verlag, 1992.
- [4] J.P. Boris and D.L. Book. Flux-Corrected Transport. I. SHASTA, A Fluid Transport Algorithm That Works. *J. Comput. Phys.*, 11:38–69, 1973.
- [5] A. J. Chorin. Random Choice Solutions of Hyperbolic Systems. *J. Comput. Phys.*, 22:517–533, 1976.
- [6] P. Colella. Glimm’s Method for Gas Dynamics. *SIAM J. Sci. Stat. Comput.*, 3(1):76–110, 1982.
- [7] P. Colella. A Direct Eulerian MUSCL Scheme for Gas Dynamics. *SIAM J. Sci. Stat. Comput.*, 6:104–117, 1985.
- [8] P. Colella. Multidimensional Upwind Methods for Hyperbolic Conservation Laws. *J. Comput. Phys.*, 87:171–200, 1990.
- [9] P. Colella and P.R. Woodard. The Piecewise Parabolic Method (PPM) method for Gas Dynamical Simulation. *J. Comput. Phys.*, 54:174–201, 1984.
- [10] R. Courant, E. Isaacson, and M. Rees. On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences. *Comm. Pure. Appl. Math.*, pages 243–255, 1952.
- [11] P.R. Eiseman. A Multi-Surface Method of Grid Generation. *J. Comput. Phys.*, 33:118–150, 1979.
- [12] I.D. Fraser. An Assessment of the Performance of Several Modern Explicit Methods Applied to the Euler Equations. MSc Thesis, College of Aeronautics, Cranfield Institute of Technology, U.K, 1992.
- [13] J.E Fromm. A Method for Reducing Dispersion in Convective Difference Schemes. *J. Comput. Phys.*, 3:176–189, 1968.

- [14] P. Glaister. Shock Capturing For Steady Supersonic Two Dimensional Isentropic Flow. *Int. J. Numer. Meth. Fluids.*, 13:883–894, 1991.
- [15] H.M. Glaz, P. Colella, I.I. Glass, and R.L. Deschambault. A Detailed Numerical, Graphical and Experimental Study of Oblique Shock Wave Reflections. Technical Report 285, Institute for Aerospace Science, University of Toronto (UTIAS), 1986.
- [16] J. Glimm. Solution in the Large for Nonlinear Hyperbolic Systems of Equations. *Comm. Pure. Appl. Math.*, 18:697–715, 1965.
- [17] S.K Godunov. A Difference Method for the Numerical Calculation of Discontinuous Solutions of Hydrodynamic Equations. *Mat. Sb.*, 47:271–306, 1959.
- [18] S.K Godunov. A Finite Difference Method for the Computation of Discontinuous Solutions of the Equations of Fluid Dynamics. *Mat. Sb.*, 47:357–393, 1959.
- [19] J.B. Goodman and R.J LeVeque. On the Accuracy of Stable Schemes for 2D Scalar Conservation Laws. *Math. Comp.*, 45(21):15–21, 1985.
- [20] J.J Gottlieb and C.P.T. Groth. Assessment of Riemann Solvers for Unsteady One-Dimensional Inviscid Flows of Perfect Gases. *J. Comput. Phys.*, 78:437–458, 1988.
- [21] A. Harten. High Resolution Schemes for Hyperbolic Conservation Laws. *J. Comput. Phys.*, 49:357–393, 1983.
- [22] A. Harten. On a Class of High Resolution Total Variation Stable Finite Difference Schemes. *SIAM J. Numer. Anal.*, 21(1):1–23, 1984.
- [23] A. Harten, P.D. Lax, and B van Leer. On Upstream Differencing and Godunov-Type Schemes for Hyperbolic Conservation Laws. *SIAM Review*, 25(1):35–61, 1983.
- [24] A. Harten and S. Osher. Uniformly High-Order Accurate Nonoscillatory Schemes I. *SIAM J. Numer. Anal.*, 24(2):279–309, 1987.
- [25] C. Hirsch. *Numerical Computation of Internal and External Flows*. Wiley, 1988.
- [26] A. Jameson. Transonic Aerofoil Calculations Using The Euler Equations. In *Numerical Methods in Aeronautical Fluid Dynamics*. Academic Press, New York, 1982.
- [27] P. Krehl and M van der Geest. The Discovery of the Mach Reflection Effect and its Demonstration in the Auditorium. *Shock Waves*, 1:3–15, 1991.
- [28] P. Kutler. A Perspective of Theoretical and Applied Computational Fluid Dynamics. *AIAA Journal*, 23(3):328–341, 1985.
- [29] P. Lax and P. Wendroff. Systems of Conservation Laws. *Comm. Pure Appl. Math.*, pages 217–237, 1960.
- [30] P.D. Lax. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, 1972.

- [31] R.J. LeVeque. A Large Timestep Generalization of Godunov's Method for Systems of Conservation Laws. *SIAM J. Numer. Anal.*, 22(6):1051–1073, 1985.
- [32] R.J. LeVeque. High Resolution Finite Volume Methods on Arbitrary Grids via Wave Propagation. *J. Comput. Phys.*, 78:36–63, 1988.
- [33] R.J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhauser Verlag, 1992.
- [34] R.J. LeVeque. Simplified Multidimensional Flux Limiter Methods. In M.J. Baines and K.W. Morton, editors, *Numerical Methods in Fluid Dynamics 4: Proceedings of the 1992 International Conference on Numerical Methods in Fluids*, pages 173–189, Reading, 1993.
- [35] R.W. MacCormack. The Effect of Viscosity in Hypervelocity Impact Cratering. Technical Report 69-354, AIAA Journal, Cincinnati, Ohio, 1969.
- [36] G. Moretti. Computation of Flows with Strong Shocks. *Ann. Rev. Fluid Mech.*, 19:313–337, 1987.
- [37] K.W. Morton. Node-cell mappings for the cell vertex method. In M.J. Baines and K.W. Morton, editors, *Numerical Methods in Fluid Dynamics 4: Proceedings of the 1992 International Conference on Numerical Methods in Fluids*, pages 1–18, Reading, 1993.
- [38] K.W. Morton and E. Suli. Finite Volume Methods and their Analysis. *IMA J. Numer. Anal.*, 11:241–260, 1991.
- [39] S. Osher and F. Solomon. Upwind Difference Schemes for Hyperbolic Conservation Laws. *Math. Comp.*, 38,158:339–374, 1982.
- [40] J. Pike. Riemann Solvers for Perfect and Near-Perfect Gases. *AIAA Journal*, 31(10):1801–1808, 1993.
- [41] J.J. Quirk. An Alternative to Unstructured Grids for Computing Gas Dynamic Flows Around Arbitrarily Complex Two Dimensional Bodies. *Computers and Fluids*, 23(1):125–142, 1994.
- [42] A. Rizzi and B. Engquist. Selected Topics in the Theory and Practice of Computational Fluid Dynamics. *J. Comput. Phys.*, 72:1–69, 1987.
- [43] P.L. Roe. Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes. *J. Comput. Phys.*, 43:357–372, 1981.
- [44] P.L. Roe. Numerical Algorithms for the Linear Wave Equation. Technical Report 81047, Royal Aircraft Establishment, Bedford, England, 1981.
- [45] P.L. Roe. The Use of the Riemann Problem in Finite Difference Schemes. In *Proceedings of the Seventh International Conference on Numerical Methods in Fluid Dynamics*, pages 354–359, 1981.

- [46] P.L. Roe. Discrete Models for the Numerical Analysis of Time-Dependent Multidimensional Gas Dynamics. *J. Comput. Phys.*, 63:458–476, 1986.
- [47] M. Schleicher. Ein Einfaches und Effizientes Verfahren zur Loesung des Riemann-Problems. *Z. Flugwiss. Weltraumforsch.*, 17:265–269, 1993.
- [48] C. W. Shultz-Rinne, J. P. Collins, and H. M. Glaz. Numerical Solution of the Riemann Problem for Two Dimensional Gas Dynamics. *SIAM J. Sci. Stat.*, 14(6):1394–1414, 1993.
- [49] G.A. Sod. A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws. *J. Comput. Phys.*, 27:1–31, 1978.
- [50] W. Speares. A Finite Volume Approach to the Weighted Average Flux Method. MSc thesis, College of Aeronautics, Cranfield Institute of Technology, U.K., 1991.
- [51] J.L Steger and R.F. Warming. Flux Vector Splitting of the Inviscid Gasdynamic Equations with Applications to Finite Difference Methods. *J. Comput. Phys.*, 40:263–293, 1981.
- [52] J.J. Stoker. *Water Waves*. Interscience Publishers, Inc., New York, 1957.
- [53] G. Strang. On the Construction and Comparison of Difference Schemes. *SIAM J. Numer. Anal.*, 5(3):506–517, 1968.
- [54] P.K. Sweby. High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws. *SIAM J. Numer. Anal.*, 21:995–1011, 1984.
- [55] P.K. Sweby and M.J. Baines. On Convergence of Roe's Scheme for the General Non-linear Wave Equation. *J. Comput. Phys.*, 56(1):135–148, 1984.
- [56] K. Takayama, Y. Miura, M. Olim, T. Saito, and E.F. Toro. Mach Reflection of Water Waves in Conjunction with Tsunami at the Okushir Island. In *Symposium on Shock Waves*, Tokyo, Japan, Jan. 1994.
- [57] J.F. Thompson, F.C. Thames, and C.W. Mastin. Automatic Generation of Body-Fitted Curvilinear Coordinate System for Field Containing Any Number of Arbitrary Two-Dimensional Bodies. *J. Comput. Phys.*, 15:299–319, 1974.
- [58] J.F. Thompson, F.C. Thames, and C.W. Mastin. TOMCAT — A Code for Numerical Generation of Boundary-Fitted Curvilinear Coordinate Systems on Fields Containing Any Number of Arbitrary Two-Dimensional Bodies. *J. Comput. Phys.*, 24:274–302, 1977.
- [59] E.F. Toro. Curve-Fitting Type Riemann Solvers for the Time-Dependent Euler Equations. (*submitted*).
- [60] E.F. Toro. A New Numerical Technique for Quasi-Linear Hyperbolic Systems of Conservation Laws. Technical Report 8708, Cranfield CoA, 1986.

- [61] E.F. Toro. A Fast Riemann Solver with Constant Covolume Applied to the Random Choice Method. *Int. J. Numer. Meth. Fluids*, 9:1145–1164, 1989.
- [62] E.F. Toro. A Weighted Average Flux Method for Hyperbolic Conservation Laws. *Proc. Roy. Soc. Lond.*, A423:401–418, 1989.
- [63] E.F. Toro. TVD regions for the Weighted Average Flux Method as Applied to Model Hyperbolic Conservation Laws. Technical Report 8907, CoA, Cranfield Institute of Technology, U.K., 1989.
- [64] E.F. Toro. A Linearized Riemann Solver for the Time-Dependent Euler Equations of Gas Dynamics. *Proc. Roy. Soc. Lond.*, A434:683–693, 1991.
- [65] E.F. Toro. Some Aspects of Shock Capturing Methods for Gas Dynamics. Technical Report 9112, Cranfield CoA, 1991.
- [66] E.F. Toro. Riemann Problems and the WAF Method for Solving Two-Dimensional Shallow Water Equations. *Phil. Trans. Roy. Soc. Lond.*, A338:43–68, 1992.
- [67] E.F. Toro. The Weighted Average Flux Method Applied to the Time-Dependent Euler Equations. *Phil. Trans. Roy. Soc. Lond.*, A341:499–530, 1992.
- [68] E.F. Toro and S.J. Billett. A WAF Type Extension of the Warming-Beam and Lax-Wendroff Schemes. *IMA J. Numer. Anal.* (to appear).
- [69] E.F. Toro and S.J. Billett. A Unified, Riemann Problem Based Extension of the Warming-Beam and Lax-Wendroff Methods. In H. Daiguji, editor, *Proceedings of the Fifth International Symposium on Computational Fluid Dynamics*, volume III, pages 243–248, Tohoku University, Sendai, Japan, September 1993.
- [70] E.F. Toro and S.J. Billett. Multidimensional WAF-Type Schemes for Model Conservation Laws. Technical Report 9312, Cranfield CoA, 1993.
- [71] E.F. Toro and C. Chou. A Linearized Riemann Solver for the Steady Supersonic Euler Equations. *Int. J. Numer. Meth. Fluids*, 16:173–186, 1993.
- [72] E.F. Toro and P.L. Roe. A Hybridised High-Order Random Choice Method for Quasi-Linear Hyperbolic Systems. In Gronig, editor, *Proc. 16th Intern. Symp. on Shock Tubes and Waves*, pages 701–708, Aachen, Germany, July 1987.
- [73] B. van Leer. Towards the Ultimate Conservative Difference Scheme II. Monotonicity and Conservation Combined in a Second Order Scheme. *J. Comput. Phys*, 14:361–370, 1974.
- [74] B. van Leer. Towards the Ultimate Conservative Difference Scheme IV. A New Approach to Numerical Convection. *J. Comput. Phys*, 23:276–299, 1977.
- [75] B. van Leer. Towards the Ultimate Conservative Difference Scheme V. A Second Order Sequel to Godunov's Method. *J. Comput. Phys*, 32:101–136, 1979.

- [76] B. van Leer. Flux-Vector Splitting for the Euler Equations. Technical Report 23665, ICASE, NASA Langley Research Center, 1982.
- [77] B. van Leer. On the Relation between the Upwind-Differencing Schemes of Godunov, Engquist-Osher and Roe. *SIAM J. Sci. Stat. Comput.*, 5(1):1-20, 1985.
- [78] B. van Leer. Progress in Multi-Dimensional Upwind Differencing. Technical Report CR-189708/ICASE 92-43, NASA, Sept. 1992.
- [79] J. von Neumann and R.D. Richtmyer. A Method for the Numerical Calculation of Hydrodynamic Shocks. *J. Appl. Phys.*, 21:232-257, 1950.
- [80] R.F. Warming and R.W. Beam. Upwind Second Order Difference Schemes with Applications in Aerodynamic Flows. *AIAA Journal*, 24:1241-1249, 1976.
- [81] N.P. Weatherill. Numerical Grid Generation. Lecture series 1990-06, von Karman Institute for Fluid Dynamics, 1990.
- [82] P. Woodard and P. Colella. The Numerical Simulation of Two-Dimensional Fluid Flow with Strong Shocks. *J. Comput. Phys.*, 54:115-173, 1984.
- [83] N.N. Yanenko. *The Method of Fractional Steps*. Springer Verlag, New York, 1971.
- [84] S.T. Zalesak. Fully Multidimensional Flux-Corrected Transport Algorithms for Fluids. *J. Comput. Phys.*, 31:335-362, 1979.
- [85] G-C. Zha and E. Bilgen. Numerical Solution of Euler equations by a New Flux Vector Splitting Scheme. *Int. J. Numer. Meth. Fluids*, 17:115-144, 1993.
- [86] T. Zhang and Y. Zheng. Conjecture on the Structure of Solutions of the Riemann Problem for Two Dimensional Gas Dynamics. *SIAM J. Math. Anal.*, 21(3):593-630, 1990.