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Structural Integrity of Functionally Graded Composite Structure using Mindlin-Type Finite Elements

O. Oyekoya, D Mba[1], A. El-Zafrany

Summary

In this paper, two new Mindlin-type plate bending elements have been derived for the modelling of functionally graded plate subjected to various loading conditions such as tensile loading, in-plane bending and out-of-plane bending. The properties of the first Mindlin-type element (i.e. Average Mindlin element) are computed by using an average fibre distribution technique which averages the macro-mechanical properties over each element. The properties of the second Mindlin-type element (i.e. Smooth Mindlin element) are computed by using a smooth fibre distribution technique, which directly uses the macro-mechanical properties at Gaussian quadrature points of each element. There were two types of non-linearity considered in the modelling of the plate, which include finite strain and material degradation. The composite plate considered in this paper is functionally graded in the longitudinal direction only, but the FE code developed is capable of analysing composite plates with functional gradation in transverse and radial direction as well. This study was able to show that the structural integrity enhancement and strength maximisation of composite structures are achievable through functional gradation of material properties over the structure.

Introduction

Composite materials are often used in different engineering fields, especially in the aerospace field. The advantage of composite materials is the high stiffness-to-weight and strength-to-weight ratios. The limitations of composite materials are the following: the weakness of interfaces between layers may lead to de-lamination, extreme thermal loads may lead to de-bonding between matrix and fibre due to mismatch of mechanical properties, and residual stresses may be present due to difference in coefficients of thermal expansion of the fibre and the matrix. To overcome the limitations, functionally graded materials (FGMs) were proposed. The FGMs are made in such a way that the volume fractions of two or more materials are varied continuously along a certain dimension. The FGMs can be made as required for different applications. For example, thermal barrier plate structures can be made from a mixture of ceramic and metal for high temperature application. The advantage of the FGM plate is that its material properties vary continuously from one surface to the other, hence avoiding the interface problem that exists in homogeneous composites.

The FGM concept originated in Japan in 1984 during the space-plane project, in the form of a proposed thermal barrier material capable of withstanding a surface temperature of 2000 K and a temperature gradient of 1000 K across a cross section less than 10mm. In 2004, Chen et al [1] investigated the buckling behaviour of FGM rectangular plates subjected to non-linearly distributed in-plane edge loads. Chen et al [1] stated that a mesh-free method which approximates displacements based on scattered nodes (i.e. radial basis function and polynomial basis) was employed, in-order to avoid complicated numerical procedures that

[1] Corresponding Author. School of Engineering, Cranfield University, Cranfield, Bedfordshire, England, MK43 0AL

arises in the FEM from the use of elements. This FEM complication was dealt with in this paper. In 2000, Reddy [2] presented a theoretical formulation and finite element models based on third order shear deformation theory for the analysis of through-thickness functionally graded plates. The Navier solution for simply supported plates based on the linear third-order theory and non-linear static and dynamic finite element results based on the first-order theory were presented by Reddy [2] to show effects of volume fractions and modulus ratio of the constituents on deflections and transverse shear stresses.

Theory of Mindlin-Type Elements

This section defines the displacement equation, strain equation, stress equation, strain energy variation and generalised equation of equilibrium. The generalised equation of equilibrium is then linearised, in-order to obtain the Mindin-type element equation

Displacement Equations

The displacement components in the x, y and z directions at any point (x,y,z) inside the plate are given below [3, 4].

$$\begin{aligned} u(x,y,z) &= u^o(x,y) + z\theta_y(x,y) \\ v(x,y,z) &= v^o(x,y) - z\theta_x(x,y) \\ w(x,y,z) &\approx w(x,y) \end{aligned} \tag{1}$$

where u^o, v^o represent values at the midplane of the plate (z=0), and θ_x, θ_y are average slope angles defined below.

$$\begin{aligned} \theta_x(x,y) &= \frac{\partial w}{\partial y} - \bar{\gamma}_{yz} \\ \theta_y(x,y) &= -\left[\frac{\partial w}{\partial x} - \bar{\gamma}_{xz}\right] \end{aligned} \tag{2}$$

where $\bar{\gamma}_{yz}$ and $\bar{\gamma}_{yz}$ represent the average (over the thickness) of transverse shear strains.

Mindlin-type elements are based on Lagrangian interpolation and for an n-node element, the mid-plane displacement components and average slope angle at any point (x, y) in the mid-plane of the plate can be interpolated as follows:

$$
\begin{aligned}
u^o(x,y) &= \sum_{i=1}^{n} N_i(x,y)\, u_i \\
v^o(x,y) &= \sum_{i=1}^{n} N_i(x,y)\, v_i \\
w(x,y) &= \sum_{i=1}^{n} N_i(x,y)\, w_i \\
\theta_x(x,y) &= \sum_{i=1}^{n} N_i(x,y)(\theta_x)_i \\
\theta_y(x,y) &= \sum_{i=1}^{n} N_i(x,y)(\theta_y)_i
\end{aligned}
\tag{3}
$$

Transverse shear strain components

These strain components are assumed infinitesimal and are represented by the equation shown below.

$$
\boldsymbol{\gamma}(x,y,z) = f_\gamma(z) \sum_{i=1}^{n} \begin{bmatrix} \dfrac{\partial N_i}{\partial x}\, w_i + N_i \left(\theta_y\right)_i \\ \dfrac{\partial N_i}{\partial y}\, w_i - N_i \left(\theta_x\right)_i \end{bmatrix} \tag{4}
$$

The shear strain vector equation above can be manipulated to obtain the variation of shear strain vector.

x-y strain components

These strain components can be obtained from Green's strain-displacement equations. They can be divided into two parts which include the infinitesimal component derived from the Cauchy's strain-displacement equation, and the additional non-linear terms in Green's equation. The total strain vector can be obtained in terms of the nodal parameters and shape function by substituting the above displacement and slope components into the Green's strain-displacement equations.

The total strain vector equation above can then be manipulated to obtain the variation of strain vector in terms of the nodal parameters and shape functions.

Strain Energy Variation

The variation of strain energy density at a point inside the L^{th} layer is given below.

$$
d\overline{U}^{(L)} = d\boldsymbol{\gamma}^t \boldsymbol{\tau}^{(L)} + d\boldsymbol{\varepsilon}^t \boldsymbol{\sigma}^{(L)} \tag{5}
$$

Generalised Equation of Equilibrium

The work done by actual loads can be expressed in terms of equivalent nodal loads as given below.

$$
dW = d\boldsymbol{\delta}_o^t \mathbf{F}_o + d\boldsymbol{\delta}_b^t \mathbf{F}_b \tag{6}
$$

Using the principle of virtual work, the generalised equation of equilibrium can be derived.

$$dU - dW = 0 \tag{7}$$

An approximate solution of this equation of equilibrium gives the expression for the residual vector.

Linearisation of Equations of Equilibrium and Derivation of Element Equations

In order to restore equilibrium, the residual vector must approach a value of zero. This equilibrium is achieved by employing the expressions below.

$$\boldsymbol{\delta}_{new} = \boldsymbol{\delta}_{old} + \Delta\boldsymbol{\delta}; \qquad \boldsymbol{\sigma}_{new}^{(L)} = \boldsymbol{\sigma}_{old} + \Delta\boldsymbol{\sigma}; \qquad \mathbf{A}_{new} = \mathbf{A}_{old} + \Delta\mathbf{A} \tag{8}$$

The combination of the above expressions and the residual vector expressions results in a final matrix equation, which is given below.

$$\sum_{e=1}^{N_e} \left\{ \left(\mathbf{K} + \mathbf{K}^{\sigma} \right) \begin{bmatrix} \Delta\boldsymbol{\delta}_o \\ \Delta\boldsymbol{\delta}_b \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{R}_o \\ \mathbf{R}_b \end{bmatrix} \tag{9}$$

Material properties for each composite layer are calculated at each point from fibre and matrix properties using different micromechanics equations. The integrations to obtain the stiffness matrices are carried out analytically through the element thickness. For average-type element, the material properties are averaged and considered constant over each element layer. For smooth-type element, the material properties are based on actual distributions of fibre to matrix ratios.

Progressive damage analysis

The load is applied incrementally. For each load increment, failure is assessed at each Gauss point, in each composite layer. If damage is detected in an element layer, the stiffness of that layer is reduced inside the element by the percentage of damage, and iterations are carried out to restore equilibrium. Non-linearity due to finite strains is also considered within the iterations. In the case of material non-linearity, during each load increment a check for failure is undertaken using an interactive failure criterion called Tsai-Hill criterion [5].

Finite Element Modelling

A rectangular plate made of a typical FGM with its midplane as shown in Figure 1 was considered. A 72 element mesh was employed for all the three validation case studies. The elements used in the validation exercise include 4-noded Average Mindlin element, 4-noded Smooth Mindlin element and 4-noded Ordinary Mindlin element. The boundary condition applied in the three case studies is that edge x=0 is a clamped edge. A load of 0.1kN was applied as an equivalent nodal loading at edge x=2 for all load cases.

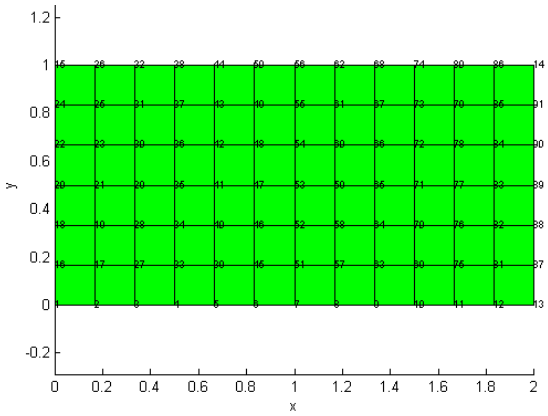


Figure 1: Mesh

Optimum Design

The optimum design criterion employed in this paper can be described as minimum deflection criterion.

The equation used for fibre distribution is as given below.

$$V_f(\xi) = V_1 + (V_2 - V_1)\,\xi^p$$

$$\text{where } \xi = \frac{x - x_1}{x_2 - x_1} \tag{13}$$

$$V_1, V_2 = \text{fibre ratio at } x_1 \text{ and } x_2$$

Nine fibre distribution cases were considered, where P could assume a value of 0.5, 1 and 2. Also V_1 could assume the value 0.5, 0.55 and 0.6. It must be noted that all cases have the same amount of fibre as in the P=0 case but different fibre distributions across the composite domain. The P=0 case represents the traditional composite case with uniform fibre distribution.

Figure 2 shows a displacement plot of out-of-plane bending cases with P=0 and P=2. This plot shows the most pronounced fibre distribution effect in all fibre distribution cases and all loading cases. The fibre distribution case P=2 and V_1=0.55 satisfies the minimum deflection criterion for the out-of-plane bending case with 59% w_{max}-deflection relative to the traditional composite case. This fibre distribution case also satisfies the minimum deflection criterion for the in-plane bending case with 84% v_{max}-deflection relative to the traditional composite case. But this fibre distribution case results in an adverse u_{max}-deflection result for the tension case with 125% u_{max}-deflection relative to the traditional case. Hence it can be deduced that the fibre distribution case with P=2 and V_1=0.55 gives the optimum design for most load cases. The prioritisation of the load cases need to be undertaken for a given design in determining whether this adverse result is a good enough trade-off.

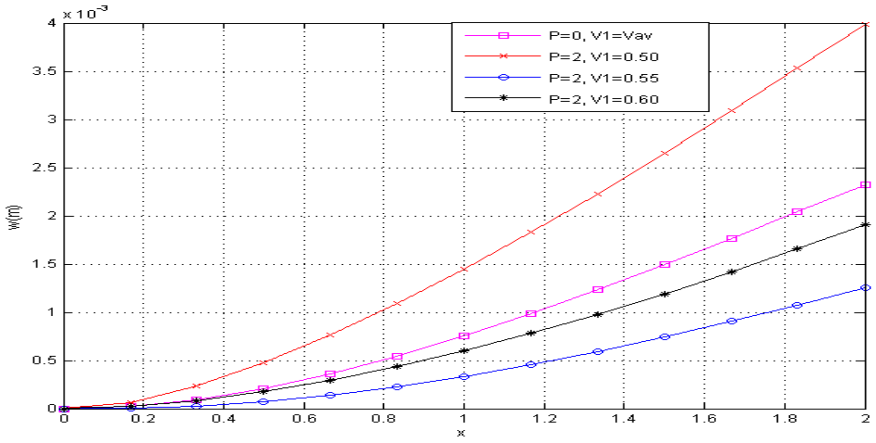


Figure 2: Displacement results for out-of-plane bending cases with P=0 and P=2

Conclusion

In this paper, two new Mindlin-type elements have been formulated and used in performing a finite strain analysis and progressive damage analysis of a functionally graded composite structure. Due to the accuracy of the Smooth Mindlin element, it was used to demonstrate the design optimisation of the functionally graded composite structure. A methodical approach was used in demonstrating the design optimisation process and an optimum fibre distribution was obtained for the load cases considered. Also this paper achieved its objective by presenting a detailed explanation of the functional graded technology from theoretical concept through to optimum design application. Future work recommendation would be to extend this work to cover non-linear dynamics and thermo-elasticity.

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Oyekoya, Oyedele O.; Mba, David; El-Zafrany, A.

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