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as applied to a model hyperbolic conservation law

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Abstract

Oscillation-free versions of the weighted average flux (WAF) method for the model equation $u_t + au_x = 0$ are presented. Two approaches are discussed in detail namely, Courant Number amplification and flux, or weight, limiting. Extended TVD regions are derived and amplifying/limiting functions are constructed. Numerical experiments on and $u_t + au_x = 0$ are performed.

1. Introduction

A weighted average flux method or WAF for short, was presented in Reference 1 as applied to systems of hyperbolic conservation laws.

$$U_t + F(U)_x = 0 \quad (1)$$

where U is a vector of conserved variables and $F(U)$ is the corresponding vector of fluxes.

WAF advances the solution explicitly in time via the conservative formula

$$U_i^{n+1} = U_i^n - \frac{\Delta T}{\Delta x} (F_{i+1/2} - F_{i-1/2}) \quad (2)$$

Where Δx and ΔT specify the mesh on the x - t plane (see Fig.1) and the numerical intercell flux $F_{i+1/2}$ is defined as

$$F_{i+1/2} = \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} F(v_{i+1/2}^{n+1/2}) dx \quad (3)$$

With $v_{i+1/2}^{n+1/2}$ denoting the solution of the Riemann problem with piece-wise constant data U_i^n and U_{i+1}^n at time $\Delta T/2$.

A suitable approximation of the wave structure of the solution of the Riemann problem produces a corresponding approximation to $F_{i+1/2}$,

$$F_{i+1/2} = \sum_{k=1}^{K+1} W_k F_{i+1/2}^{(k)} \quad (4)$$

where

$$W_k = \frac{1}{2}(\nu_k - \nu_{k-1}), \quad k=1, \dots, K+1 \quad (5)$$

We call $F_{i+1/2}^{(k)}$ a partial flux and ν_k is the Courant number corresponding to the k -th wave of speed λ_k in the solution of the Riemann problem $RP(i, i+1)$, i.e.

$$\nu_k = \Delta T \lambda_k / \Delta x \quad (6)$$

We adopt the following convention

$$\nu_0 = -1 \text{ and } \nu_{K+1} = +1$$

There are K waves and K+1 weights.

Note that

$$\sum_{k=1}^{K+1} W_k = 1 \text{ and } W_k \geq 0 \quad (7)$$

Fig. 2 illustrates the geometric interpretation of the intercell flux of WAF for the model equation $u_t + au_x = 0$.

Note that the intercell flux $F_{i+1/2}$ in eq.(4) can also be expressed as

$$F_{i+1/2} = \frac{1}{2} (F_i^n + F_{i+1}^n) - \frac{1}{2} \sum_{k=1}^K \nu_k [F_{i+1/2}^{(k+1)} - F_{i+1/2}^{(k)}] \quad (8)$$

This expression serves to compare the present numerical flux to that of other methods. Also, it is worth remarking that terms in the summation imply, in a natural way, a flux difference splitting procedure. They represent a flux difference across each wave weighted by the respective Courant number.

An alternative version of WAF is obtained by defining $F_{i+1/2}$ in (3) as

$$F_{i+1/2} = F(\tilde{V}_{i+1/2}) \quad (9)$$

where $\tilde{V}_{i+1/2}$ is an integral average of the solution of the Riemann problem $RP(i, i+1)$ at the half-time level, i.e.

$$\tilde{V}_{i+1/2} = \frac{1}{\Delta x} \int_{\Delta x/2}^{\Delta x/2} V_{i+1/2}^{n+1/2} dx \quad (10)$$

Here $V_{i+1/2}^{n+1/2}$ represents values of the conserved variables U in (1), but there is no obvious reason to be preferred.

A suitable approximation gives

$$\tilde{V}_{i+1/2} = \sum_{k=1}^{K+1} \bar{W}_k V_{i+1/2}^k \quad (11)$$

where the weights $\bar{W}_k$ are as defined in (5).

If the variable vector $V_{i+1/2}^{n+1/2}$ in (10) or $V_{i+1/2}^k$ in (11) consists of the conserved variables in eq. (1), then this version of WAF is identical to the two step Richtmyer version of the Lax-Wendroff Method (RLW). In order to see this we integrate the system of conservation laws (1) over the rectangle

$$-\frac{\Delta x}{2} \leq x \leq \frac{\Delta x}{2}, \quad 0 \leq t \leq \Delta T/2$$

The result is

$$\int_{-\Delta x/2}^{\Delta x/2} U(x, \Delta T/2) dx = \int_{-\Delta x/2}^{\Delta x/2} U(x, 0) dx + \int_0^{\Delta T/2} F\left[U\left(-\frac{\Delta x}{2}, t\right)\right] dt - \int_0^{\Delta T/2} F\left[U\left(\frac{\Delta x}{2}, t\right)\right] dt$$

After performing the integration on the right-hand side terms, dividing through by Δx and using eq.(9) we obtain

$$\tilde{V}_{i+1/2} = \frac{1}{2} \left(U_i^n + U_{i+1}^n \right) - \frac{1\Delta T}{2\Delta x} \left(F_{i+1}^n - F_i^n \right) \quad (12)$$

Thus the integral average $\tilde{V}_{i+1/2}$ in (10) can be obtained exactly without reference to the solution of the Riemann-problem with data U_i^n, U_{i+1}^n . There are advantages in using the Riemann problem based flux (9) - (10). The extra information available, such as direction of wave propagation, can be profitably utilised to enhance stability, robustness and the shock capturing capabilities of the method.

It is interesting to note that the second version of WAF given by (9) - (10) is second order accurate in space and time. This follows trivially

from the fact that it is formally identical to the RLW scheme. It also follows that the original version of WAF given by (4) - (7) has second order accuracy in space and time for all linear systems of hyperbolic conservation laws. For specific problems one can construct other versions of WAF. For the Euler equations for instance, one can substitute $V_{i+1/2}^{n+1/2}$ in eq.(9) by the physical variables.

2. Extended TVD regions for a model equation.

The WAF method as presented in the previous section will produce overshoots and spurious oscillations in the vicinity of high gradients (e.g. shock waves). In this section we study in detail some procedures that produce oscillation free versions of WAF. To this purpose we consider the model equation

$$u_t + au_x = 0, \quad a = \text{constant} \quad (13)$$

which is usually called the linear advection (or convection) equation. It is the simplest hyperbolic (linear) partial differential equation.

At this stage it is necessary to define the concept of total variation.

The total variation, $TV(u^{n+1})$ of the solution is defined by

$$TV(u^{n+1}) = \sum_i |u_{i+1}^{n+1} - u_i^{n+1}| \quad (14)$$

A large class of difference schemes are those that are total variation diminishing, or TVD for short, i.e.

$$TV(u^{n+1}) \leq TV(u^n) \quad (15)$$

This discrete condition mimics the analytical constraint.

$$\frac{d}{dt} \int |u_x| dx \leq 0 \quad (16)$$

which is utilised in proofs of convergence of non-linear difference schemes. There is a theorem due to Harten (Harten, 1983) that says that if a scheme is TVD then it will not produce spurious oscillations.

2.1 WAF applied to the linear advection equation.

Consider the model eq. (13) with the speed 'a' a positive constant. When rewritten in conservation form eqn. (13) becomes

$$u_t + (au)_x = 0 \quad (17)$$

with flux function

$$F(u) = au \quad (18)$$

In order to advance the solution via formula (2) we require the intercell numerical flux $F_{i+1/2}$, which in turn requires the solution of the Riemann problem with data u_i^n, u_{i+1}^n , i.e. the initial value problem (IVP)

$$u_t + (au)_x = 0 \quad \left\{ \begin{array}{ll} u_i^n & x \leq x_{i+1/2} \\ u_{i+1}^n & x \geq x_{i+1/2} \end{array} \right\} \quad (19)$$

The solution of (18) is trivial, namely,

$$U(t,x) = \begin{cases} U_i^n & \text{if } \frac{x}{t} \leq a \\ U_{i+1}^n & \text{if } \frac{x}{t} \geq a \end{cases}$$

We wish to evaluate $F_{i+1/2}$ over a distance Δx at the half-time level. Fig. 2 illustrates the size of the two weights W_1 and W_2 . It is clearly seen that

$$W_1 = \frac{1}{2}(1+\nu), W_2 = \frac{1}{2}(1-\nu) \quad (20)$$

$$F_{i+1/2}^{(1)} = au_i^n, F_{i+1/2}^{(2)} = au_{i+1}^n \quad (21)$$

so the intercell flux (eq. 4) is

$$F_{i+1/2} = \frac{1}{2}(1+\nu)au_i^n + \frac{1}{2}(1-\nu)au_{i+1}^n \quad (22)$$

similarly

$$F_{i-1/2} = \frac{1}{2}(1+\nu)au_{i-1}^n + \frac{1}{2}(1-\nu)au_i^n$$

and so formula (2) gives, after rearranging

$$U_i^{n+1} = \frac{1}{2}(1+\nu)\nu u_{i-1}^n + (1-\nu^2)u_i^n - \frac{1}{2}(1-\nu)\nu u_{i+1}^n \quad (23)$$

which is the Lax-Wendroff method. Thus for the linear advection equation both versions of the WAF method are identical to the Lax-Wendroff method. Spurious oscillations are thus expected. Fig. 7a shows a comparison between the exact and the numerical solutions to eq. (13) with $a=1$, after 20 time steps. The initial condition is a squared wave.

Obviously the numerical solution is unacceptable; the next section is devoted to modifying WAF so as to make it TVD, and thus oscillation free.

2.2 Construction of TVD Regions

We begin by noting that $F_{i+1/2}$ in (22) is a weighted average involving an upwind weight $W_1 = \frac{1}{2}(1+\nu)$ and a downwind weight $W_2 = \frac{1}{2}(1-\nu)$. Here W_1 is responsible for stability while W_2 is responsible for higher (second order) accuracy, as well as spurious oscillations.

In the presence of high gradients or discontinuities we wish to reduce the influence of $\bar{W}_2$. In Ref. 1 we accomplished this by modifying the weights as follows (for $a > 0$)

$$\bar{W}_2 = W_2 B, \quad \bar{W}_1 = W_1 + (1-B) W_2 \quad (24)$$

A standard analysis of the new scheme led to a TVD region that was identical to that of flux limiters (Sweby 1984, Roe 1985). Here we explore a different way of producing an oscillation free WAF. Consider Fig. 2 for the evaluation of the intercell flux $F_{i+1/2}$. Clearly, one can alter the size of the downwind weight W_2 by a number of mechanisms. For instance, one can change the wave speed by multiplying a by a function A , namely

$$\bar{a} = aA \quad (25)$$

One could also perform the integration of the flux function (see eq. (3)) at a time other than $t = \Delta t/2$. The same effect can be achieved by altering the length of integration in eq. (3). All these procedures result in an alteration of the Courant Number in eq. (20) so that the modified weights are

$$\bar{W}_1 = \frac{1}{2}(1+\bar{\nu}), \quad \bar{W}_2 = \frac{1}{2}(1-\bar{\nu}) \quad (26)$$

where

$$\bar{\nu} = \frac{Aa\Delta t}{\Delta x} = A\nu \quad (27)$$

The modifying function A is yet to be found. The new weights (26) are much simpler than those given in Ref. 1. Such simplification is significant when programming the method for systems of conservation laws.

The modified intercell flux is now

$$\bar{F}_{i+1/2} = \frac{1}{2}(1+\bar{\nu})au_i^n + \frac{1}{2}(1-\bar{\nu})au_{i+1}^n \quad (28)$$

or

$$\bar{F}_{i+1/2} = \frac{1}{2}[au_i^n + au_{i+1}^n] - \frac{1}{2}\bar{\nu}[au_{i+1}^n - au_i^n] \quad (29)$$

For a general scalar conservation law

$$u_t + f_x = 0 \quad (30)$$

we could write

$$\bar{F}_{i+1/2} = \frac{1}{2} [f_i^n + f_{i+1}^n] - \frac{1}{2} \bar{\nu} [f_{i+1}^n - f_i^n] \quad (31)$$

In order to find the function A in (27) that produces an oscillation free scheme with flux $\bar{F}_{i+1/2}$ given by (28), or (29), we note that there are two obvious bounds for $\bar{\nu}$ (see e.g. (27)). These are

$$\bar{\nu} = -1 \Rightarrow \bar{W}_1 = 0, \bar{W}_2 = 1 \quad (\text{downwind differencing})$$

$$\bar{\nu} = +1 \Rightarrow \bar{W}_1 = 1, \bar{W}_2 = 0 \quad (\text{upwind differencing})$$

Thus we choose A such that

$$-\frac{1}{\bar{\nu}} \leq A \leq \frac{1}{\bar{\nu}} \quad (32)$$

Substitution of the modified fluxes into scheme (2) gives

$$u_i^{n+1} = u_i^n - \frac{1}{2} \nu \left\{ \left(u_{i+1}^n - u_{i-1}^n \right) + \nu \left[A_{i+1/2} \left(u_i^n - u_{i+1}^n \right) - A_{i-1/2} \left(u_{i-1}^n - u_i^n \right) \right] \right\} \quad (33)$$

which, after dividing through by $u_{i-1}^n - u_i^n$ and rearranging, produces

$$\frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} = \frac{1}{2} \nu^2 \left[\frac{1}{r_i} \left(\frac{1}{\bar{\nu}} - A_{i+1/2} \right) + A_{i-1/2} + \frac{1}{\bar{\nu}} \right] \quad (34)$$

with

$$r_i = \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n} \quad (35)$$

A simple sufficient condition for avoiding overshoots, or new extrema (new extrema increase $TV(u^{n+1})$) is that the new value u_i^{n+1} lies between the data values u_{i-1}^n and u_i^n , that is

$$0 \leq \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} \leq 1 \quad (36)$$

From eq. (34) it follows that

$$0 \leq \frac{1}{2}\nu^2 \left[\frac{1}{r_i} \left(\frac{1}{\nu} - A_{i+1/2} \right) + A_{i-1/2} + \frac{1}{\nu} \right] \leq 1$$

or

$$-\frac{1}{\nu} \leq \frac{1}{r_i} \left(\frac{1}{\nu} - A_{i+1/2} \right) + A_{i-1/2} \leq \frac{2-\nu}{\nu^2} \quad (37)$$

Restating constraint (32) we have

$$L \leq A_{i-1/2} \leq 1/\nu \quad (38)$$

with L in $[-1/\nu, 1]$.

We note here that if $L = 1$ the function A is an amplifying Courant number function. The problem is to choose ranges for $A_{i-1/2}$ and $A_{i+1/2}$ so that inequalities (37) - (38) are simultaneously satisfied. This is achieved by taking

$$-S_L \leq \frac{1}{r_i} \left(1/\nu - A_{i+1/2} \right) \leq \frac{2(1-\nu)}{\nu^2} \quad (39)$$

$$-1/\nu \leq L \leq A_{i-1/2} \leq 1/\nu \quad (40)$$

with

$$S_L = L + 1/\nu \quad (41)$$

The analysis leading to (39) - (41) is based on the assumption $a > 0$ in eq.(13). The case $a < 0$ is entirely analogous and the result is identical to that of $a > 0$, but ν must be replaced by $|\nu|$. Hence the general case is

$$-S_L \leq \frac{1}{r_i} \left(\frac{1}{|\nu|} - A_{i+1/2} \right) \leq S_R \quad (42)$$

$$-\frac{1}{|\nu|} \leq L \leq A_{i-1/2} \leq \frac{1}{|\nu|} \quad (43)$$

$$S_L = L + 1/|\nu|, S_R = \frac{2(1-|\nu|)}{|\nu|^2} \quad (44)$$

Now the fundamental inequality (37) becomes

$$-\frac{1}{|\nu|} \leq \frac{1}{r_i} \left(\frac{1}{|\nu|} - A_{i+1/2} \right) + A_{i-1/2} \leq \frac{2-|\nu|}{|\nu|^2} \quad (45)$$

It is clear that the choices (42) - (44) satisfy (45) automatically.

Next we analyse the bounds of inequality (42). For convenience subscripts are ignored. The lower bound satisfies

$$-S_L \leq \frac{1}{r} \left(\frac{1}{|\nu|} - A \right)$$

If $r > 0$ then $-S_L r \leq \frac{1}{|\nu|} - A$, or

$$A \leq \frac{1}{|\nu|} + S_L r = A_L \quad (46)$$

If $r < 0$ then $-S_L r \geq \frac{1}{|\nu|} - A$, or

$$A \geq \frac{1}{|\nu|} + S_L r = A_L \quad (47)$$

The upper inequality in (42) is

$$\frac{1}{r} \left(\frac{1}{|\nu|} - A \right) \leq S_R$$

If $r > 0$ then

$$A \geq \frac{1}{|\nu|} - S_R r = A_R \quad (48)$$

If $r < 0$ then

$$A \leq \frac{1}{|\nu|} - S_R r = A_R \quad (49)$$

For $L > -1/|\nu|$ in (44) there are two TVD regions R_L and R_R . These are illustrated in Fig.3. The horizontal bounds are $A = L$ and $A = 1/|\nu|$. Also there are two straight lines A_L and A_R with positive and negative slopes respectively; they intersect at $r = 0$. The TVD regions R_L and R_R are given by the sets

$$R_L = \left\{ (r, A) \text{ such that } r \leq 0, A \geq A_L, L \leq A \leq 1/|\nu| \right\} \quad (50a)$$

$$R_R = \left\{ (r, A) \text{ such that } r \geq 0, A \geq A_R, L \leq A \leq 1/|\nu| \right\} \quad (50b)$$

For the case $L = -1/|\nu|$ the region R_L coalesce to the single line $A = 1/|\nu|$, i.e. for $r < 0$ only upwind differencing is allowed in this case.

Having obtained regions where the amplifying function $A = A(r)$ gives a TVD scheme the task now is to design these functions.

3. Construction of amplifiers and numerical experiments.

There is an unlimited number of choices for $A(r)$. Here we consider functions of two types which, in analogy with flux limiters (eg. Roe 1985) we shall denote by families SUPERA and MINAM. The functions A in the former family will be the most compressive, i.e. discontinuities in the solution will be sharply resolved. The latter family will contain functions that compromise the resolution of discontinuities; they are an attempt to treat all features of the flow in a satisfactory manner. From experience with flux limiters we expect the MINAM functions to be more successful when extending these ideas to non-linear systems of conservations laws governing complicated phenomena.

The SUPERA Family

Here we shall consider only two members of SUPERA. These are given as follows:

$$\text{SUPERA1} = \begin{cases} 1/|\nu| & , r \leq 0 \\ 1/|\nu| - S_R r & , 0 \leq r \leq R_4 \\ -1/|\nu| & , r \geq R_4 \end{cases}$$

$$\text{SUPERA2} = \begin{cases} 1/|\nu| & , r \leq 0 \\ 1/|\nu| - S_R r & , 0 \leq r \leq R_1 \\ 1 & , R_1 \leq r \leq R_2 \\ 1 - S_R (r - R_2) & , R_2 \leq r \leq R_3 \\ -1/|\nu| & , r \geq R_3 \end{cases}$$

where

$$R_1 = |\nu|/2$$

$$R_2 = 2 - |\nu|/2$$

$$R_3 = (1 + 1/|\nu|)/S_R + 2 - |\nu|/2$$

$$R_4 = |\nu|/(1 - |\nu|)$$

Note that here we have taken $L = -1/|\nu|$ in eq.(44) and thus the left TVD region is the single line $A = 1/|\nu|$. The region R_R has maximum width in this case.

SUPERA1 is the most compressive function. It is the lower boundary of the TVD region R_R . SUPERA2 is less compressive but has the desirable property of passing through the point (1,1) in the $r - A$ plane. Fig.4 illustrates these two functions. They are coincident for some values of r .

The MINAM family

Here we consider three examples. Their common feature is that the lower boundary in eq. (44) is $L = 1$. They are all constructed so as to contain both the point (1,1) and (-1,1). This means that second order accuracy is preserved for differences in neighbouring states of comparable

magnitude. This is quantified by the variable r , which is defined by eq. (35).

The selected examples are the following:

$$\text{MINAM1} = \begin{cases} 1/|\nu| + S_L r, & \frac{|\nu| - 1}{|\nu| + 1} \leq r \leq 0 \\ 1/|\nu| - S_R r, & 0 \leq r \leq |\nu|/2 \\ 1, & \text{otherwise} \end{cases}$$

$$\text{MINAM2} = \begin{cases} 1/|\nu| + 2(1 - |\nu|)r, & -\frac{1}{2|\nu|} \leq r \leq 0 \\ 1/|\nu| - 2(1 - |\nu|)r, & 0 \leq r \leq \frac{1}{2|\nu|} \\ 1, & \text{otherwise} \end{cases}$$

$$\text{MINAM3} = \begin{cases} 1/|\nu| + \frac{(1 - |\nu|)}{|\nu|} r, & -1 \leq r \leq 0 \\ 1/|\nu| - \frac{(1 - |\nu|)}{|\nu|} r, & 0 \leq r \leq 1 \\ 1, & \text{otherwise} \end{cases}$$

These functions are illustrated in Fig.5. MINAM1 follows the boundaries of A_L and A_R (eqs. 46 and 48) until they intersect $A = 1$; it is the most compressive. The other two functions, however, are simpler to code via the statement.

$$\text{MINAM23} = 1/|\nu| - S_{23} r \text{ sign}(r) \text{ for } 0 \leq |r| \leq R_{23}$$

where the slopes S_{23} and the intersection points R_{23} are the obvious ones.

Numerical Experiments

We perform numerical experiments on the linear advection equation (13) with $a = 1$. We consider two initial conditions, namely a squared wave discretised by 20 computational cells and half a sine wave discretised by 30 cells. In all cases we took the spacing $\Delta x = 0.02$ and the time step size

given by the choice of Courant number $\nu = 3/4$. All computed profiles are displayed after 20 time steps.

Figs.6a,b show comparisons between the computed solutions by the unmodified fully second order WAF ($A \equiv 1$) and the exact solutions for the two initial conditions discussed above. Note the overshoots and spurious oscillations for the squared wave case. These are clearly unacceptable, although the test is severe, it contains two discontinuities. The second test (Fig.6b) is less severe; the solution is smooth except for two discontinuities in derivative. Note that even for this essentially smooth case the fully second order method is inaccurate; see for instance the tail of the wave in Fig.6b.

The results of Fig.6 illustrates the need for a modified version of WAF. The results that follow show the performance of the TVD WAF using various amplifying functions A .

Fig.7a,b show the results obtained when using SUPERA1. This is the most compressive amplifier. The performance of this function on the squared wave case is excellent; in fact it gives the exact solution for all even time steps. For odd time steps it gives a single intermediate point in any discontinuity, whose value is related to the Courant number ν . However, SUPERA1 is completely inadequate for the second test (Fig.7b). Smooth profiles tend to be squared. These results bring up the contradictory requirements on the functions A .

Figs.8a,b show the performance of SUPERA2. It performs very well on the two test problems. Discontinuities are resolved with two intermediate points for even and odd time steps. Smooth flows are also remarkably well represented; the solution near the discontinuities in derivative is very accurate; compare for instance with a fully second order unmodified method (fig.6b). There is a trend to square smooth regions, however; but this effect appears to be small.

The results that follow were obtained using functions in the MINAM family.

Figs.9a,b show the results obtained using MINAM1. Discontinuities

are now spread over a larger region of space and clipping of extrema is now visible in Fig.9b. These two features will be further exaggerated by MINAM2 and MINAM3 whose respective results are shown in Figs.10a,b and 11a,b.

Note that the members 1,2,3 of the MINAM family are obtained by spreading the branches of A around $r = 0$. The larger the spreading of the A branches the larger the spreading of discontinuities and the more severe the clipping of extrema. The limiting case is that in which both A-branches form the single line $A = 1/|\nu|$ for all r . This corresponds to upwind differencing throughout, which is the first order version of WAF.

Figs. 12a,b show the results obtained by the first order method ($A = 1/|\nu|$ for all r). Clearly these results are very inaccurate, as expected from a first order scheme.

When extending the present TVD procedures to non-linear systems of conservation laws, members of the family MINAM tend to be more successful. This topic is currently being thoroughly investigated.

Amplifiers and Limiters

The approach to constructing TVD versions of WAF developed in this paper is novel and has the advantage of simplifying considerably the oscillation free flux. For applications of these ideas to non-linear systems such simplification gives the amplifier approach a clear advantage over the weight limiter approach, taken in Ref. 1 (Toro, 1989). As to the result the two approaches are entirely equivalent.

We can relate amplifiers A to limiters B by observing eqns. (24) and (26). It is seen that

$$A = \frac{1 - (1 - |\nu|) B}{|\nu|} \quad (51)$$

Hence given a limiter B we can immediately obtain an amplifier A. For instance if B is the minmod limiter (Roe, 1985)

$$B = \begin{cases} 0 & r \leq 0 \\ r & 0 \leq r \leq 1 \\ 1 & r \geq 0 \end{cases} \quad (52)$$

the corresponding amplifier is

$$A = \begin{cases} 1/|\nu| & r \leq 0 \\ 1/|\nu| - \frac{(1-|\nu|)}{|\nu|} r & 0 \leq r \leq 1 \\ 1 & r \geq 0 \end{cases} \quad (53)$$

This is in fact like the positive- r branch of MINAM3. The function MINAM3 has, by virtue of its negative- r branch, an advantage over A given by (52) near turning points, where r goes through a change of sign. Clipping of extrema is less severe with MINAM3 than with A given by (53). But the advantage shown by numerical experiments is less significant than we had hoped for, by including the negative- r TVD region R_L (Fig.3).

From amplifiers we can construct limiters. An interesting example is the limiter B associated with SUPERA2, namely

$$B = \begin{cases} 0 & r \leq 0 \\ \frac{2}{|\nu|} r & , 0 \leq r \leq R_1 \\ 1 & , R_1 \leq r \leq R_2 \\ 1 + \frac{2}{|\nu|} (r - R_2) & , R_2 \leq r \leq R_3 \\ \frac{2}{1-|\nu|} & , r \geq R_3 \end{cases} \quad (54)$$

where R_1 , R_2 and R_3 are given with the definition of SUPERA2.

4. Conclusions

An alternative approach to constructing TVD versions of WAF as applied to the model equation $u_t + au_x = 0$ has been presented. The derived TVD regions include one for which the monitoring parameter r takes on negative values. This allows the possibility of reducing clipping of extrema although in practice we found the benefit is small.

We have constructed five amplifying functions that produce oscillation free versions of WAF. Numerical experiments on the model equation confirm the theory.

A relationship between amplifiers and limiters B is established. This is useful when designing these functions.

The analysis presented in this paper is strictly applicable to the linear model equation. But preliminary results obtained, on empirical basis, to non-linear systems of conservation laws of hyperbolic type, are very satisfactory.

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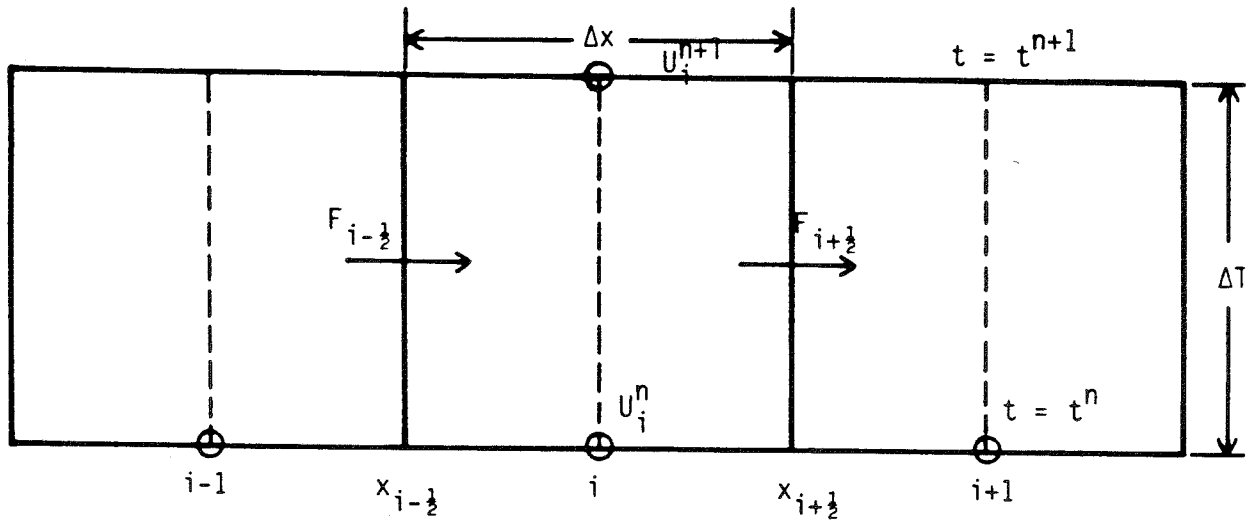


Fig. 1 Computing grid on the x - t plane. Cell i has spatial and temporal dimensions Δx and ΔT respectively. U_i^n is data at cell i at time t^n and U_i^{n+1} is the updated solution at the new time $t^{n+1} = t^n + \Delta T$. $F_{i+1/2}$ is the intercell flux corresponding to (U_i^n, U_{i+1}^n) ; likewise $F_{i-1/2}$.

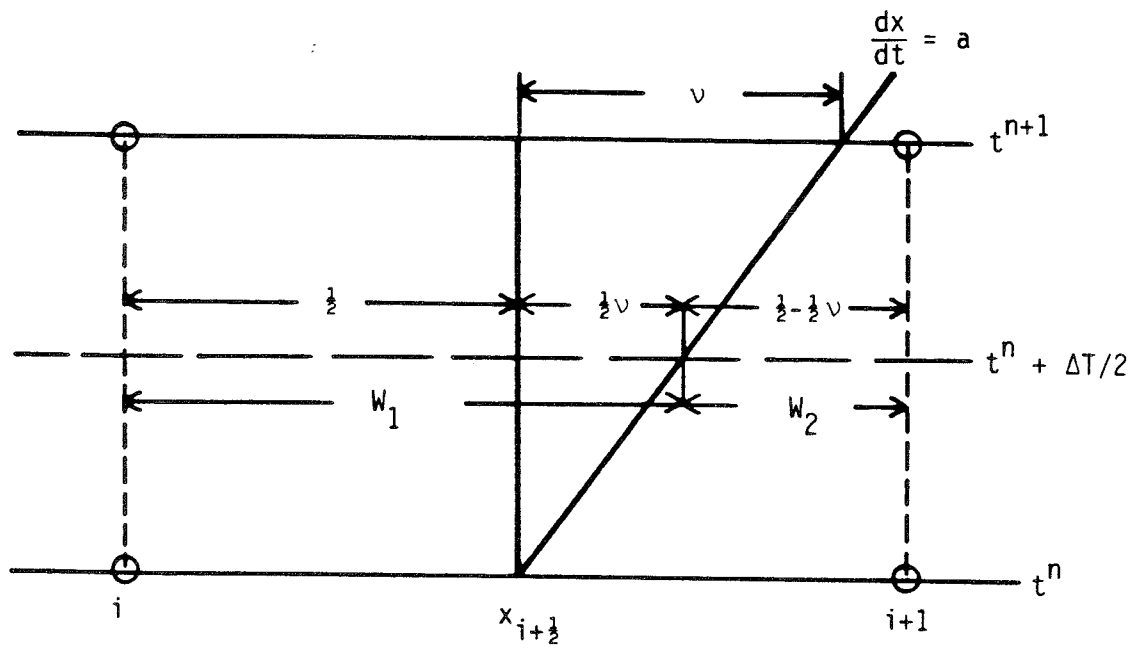


Fig. 2 Weights W_1 and W_2 given by the solution of the Riemann problem for $u_t + au_x = 0$ ($a > 0$) with data (U_i^n, U_{i+1}^n) at time $t = t^n + \Delta T/2$.

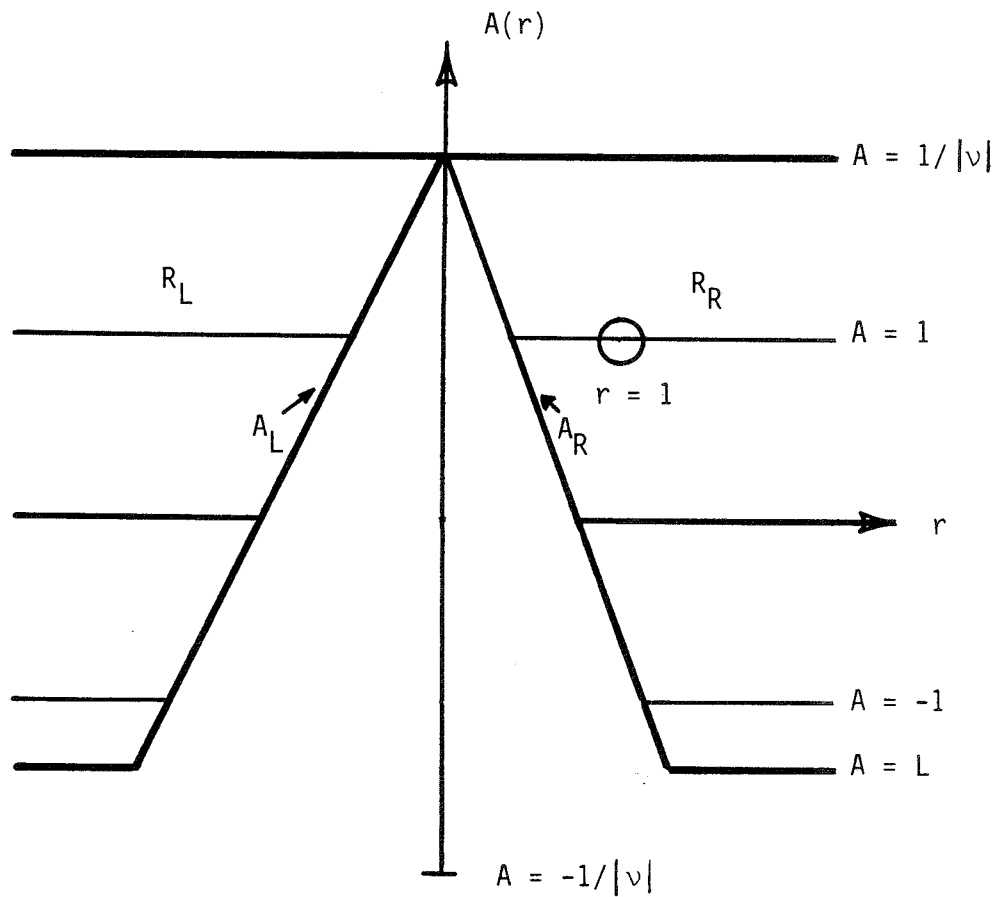


Fig. 3 TVD regions on the r - A plane for the WAF method as applied to the model equation $u_t + au_x = 0$. R_L lies between $A = 1/|v|$ and $A = L$ and to the left of A_L . R_R lies between $A = 1/|v|$ and $A = L$ and to the right of A_R .

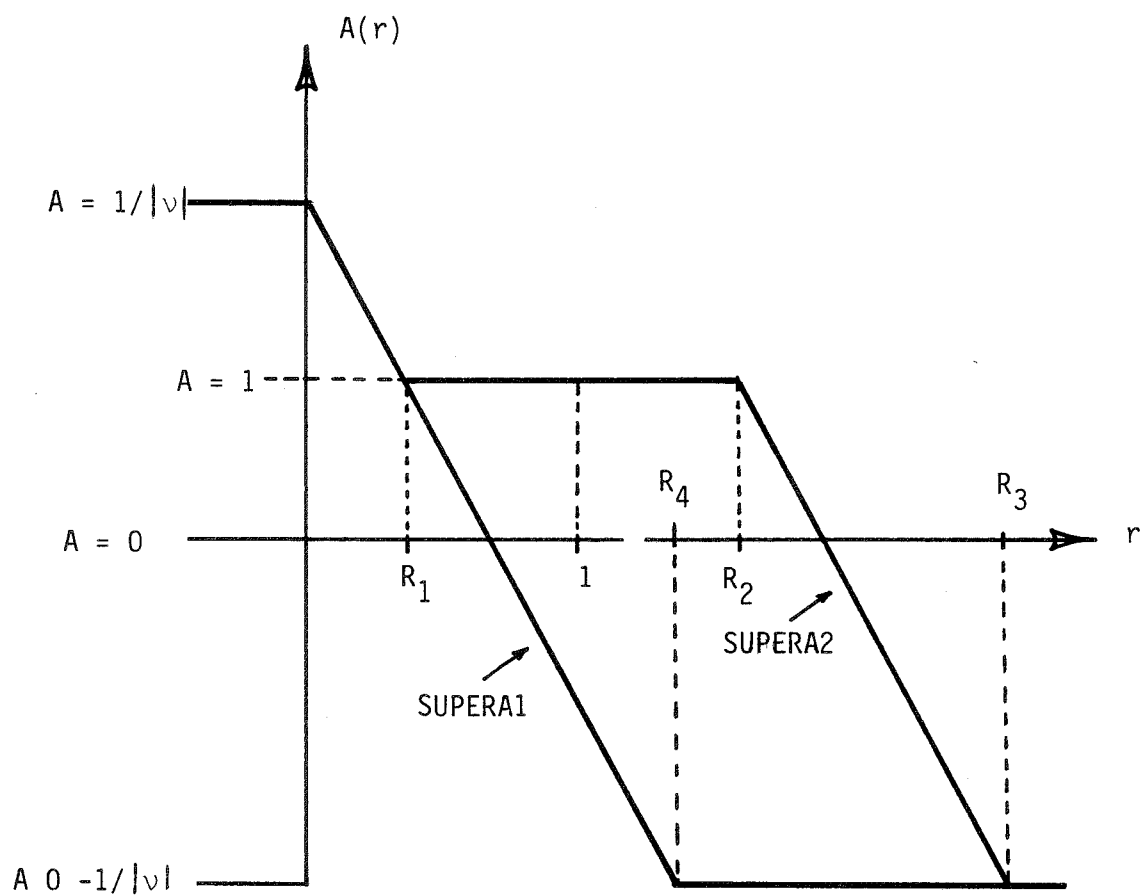


Fig. 4 Amplifiers of the type SUPERA. The function $SUPERA1$ is R_L (collapsed to a line) together with the lower boundary of R_R . $SUPERA2$ is as $SUPERA1$ except for values of r in the interval $[R_1, R_3]$; it passes through $(1, 1)$.

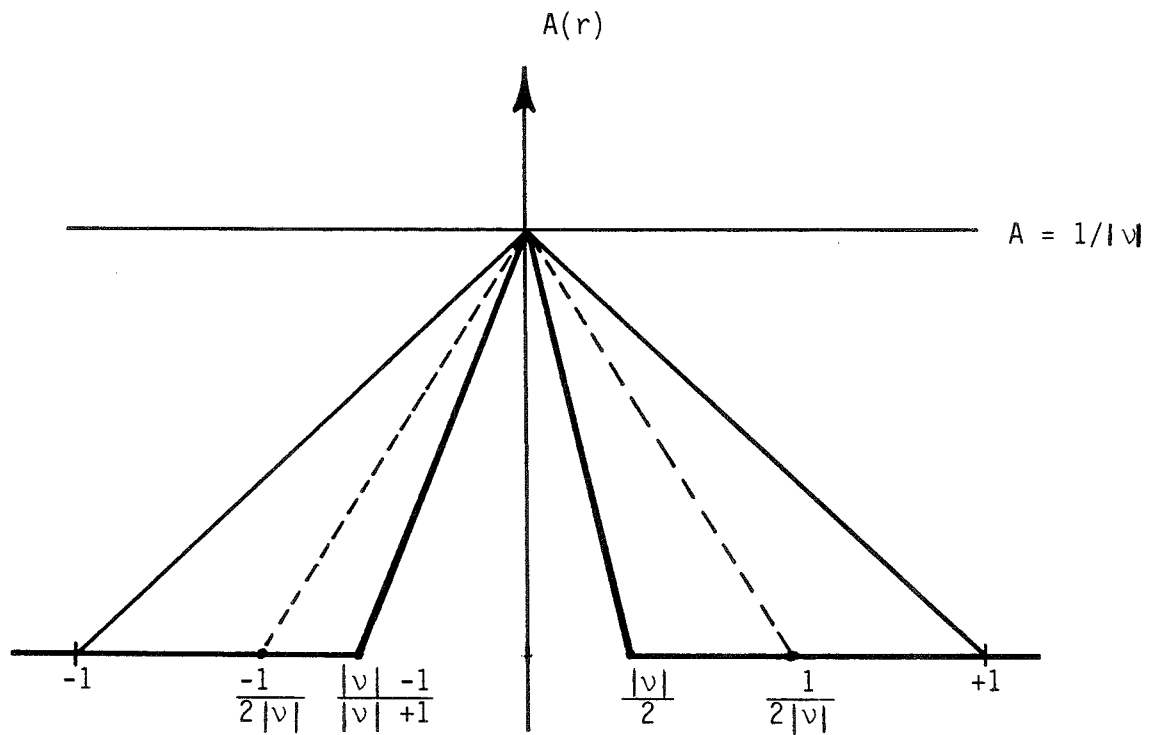
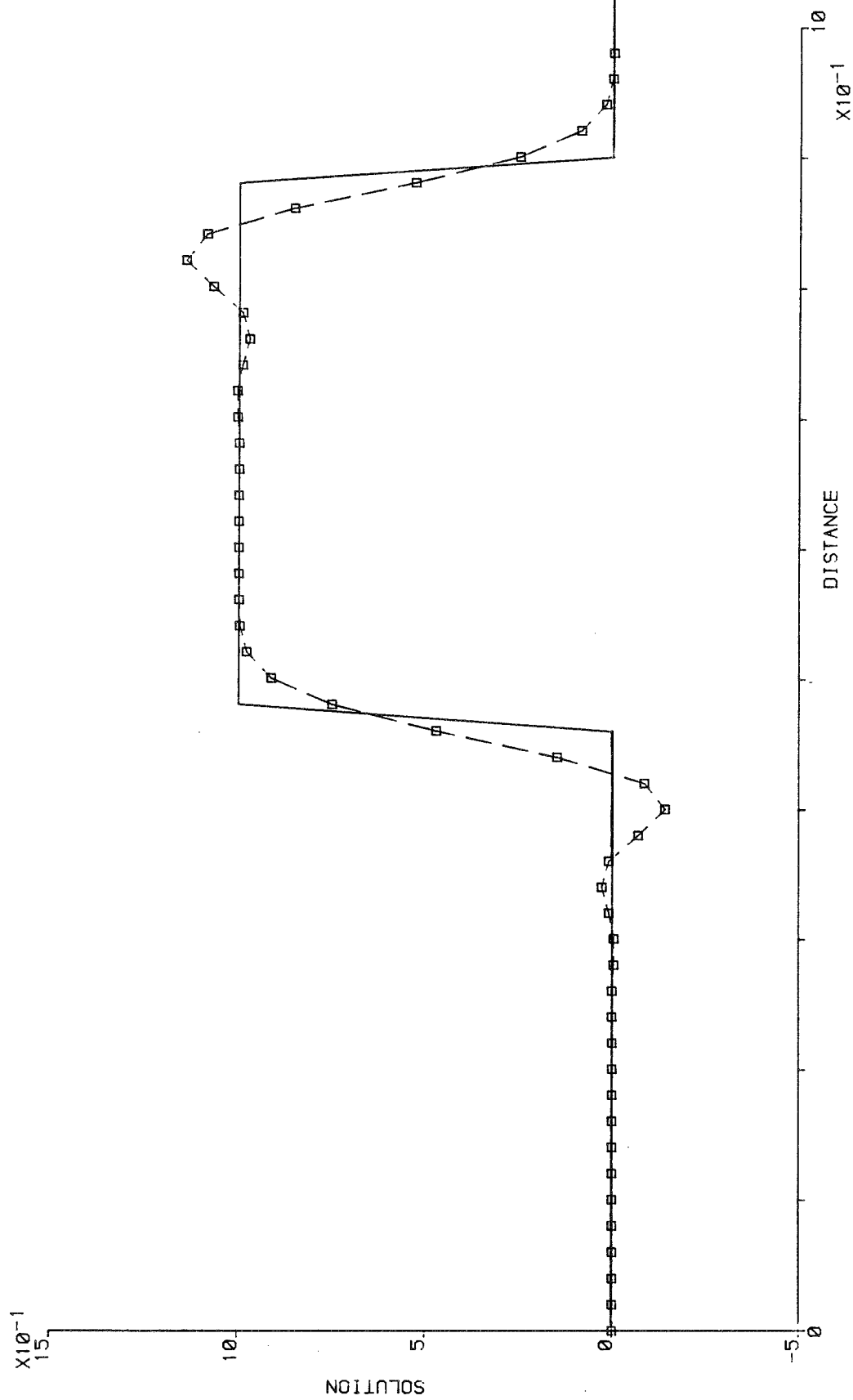
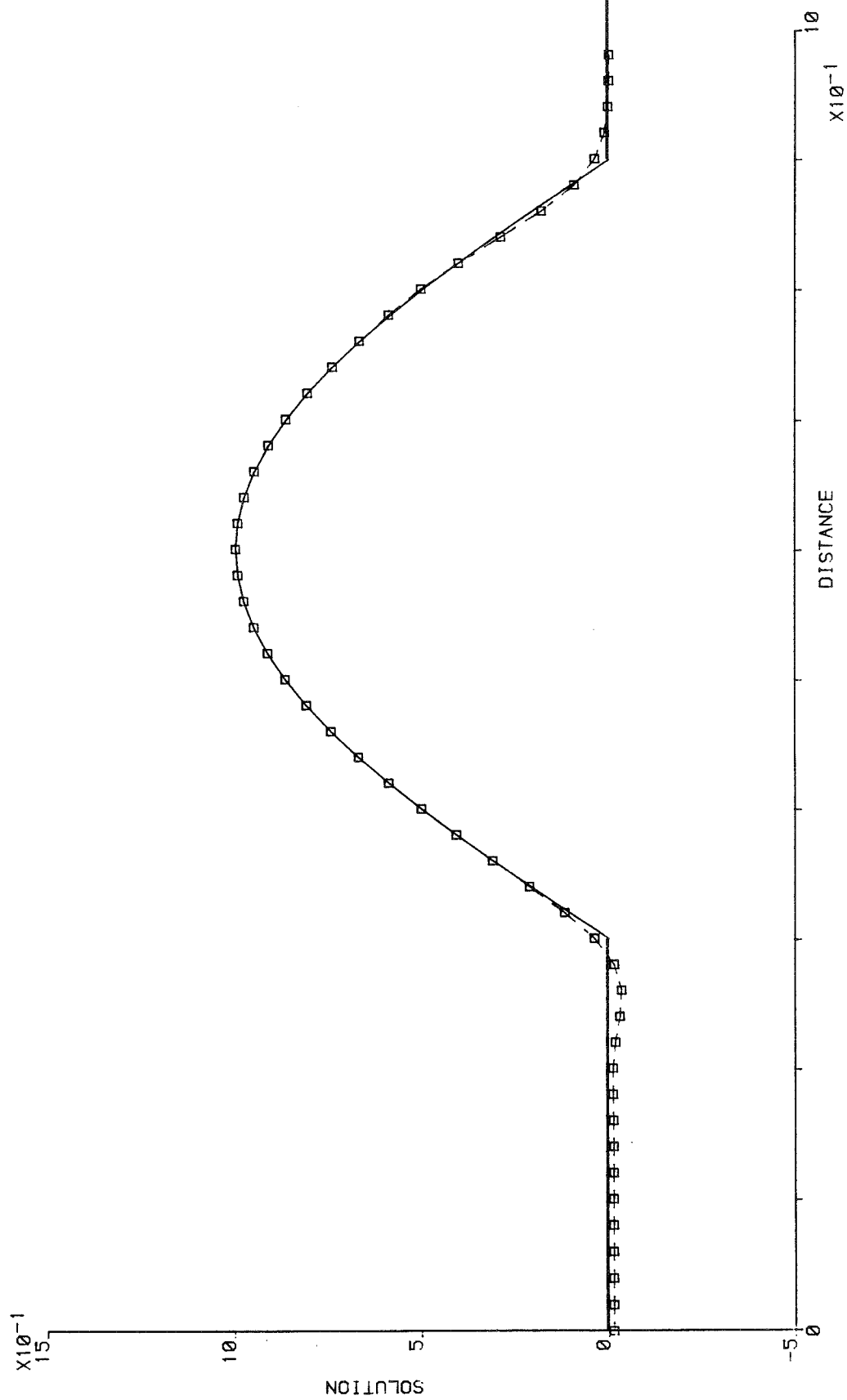


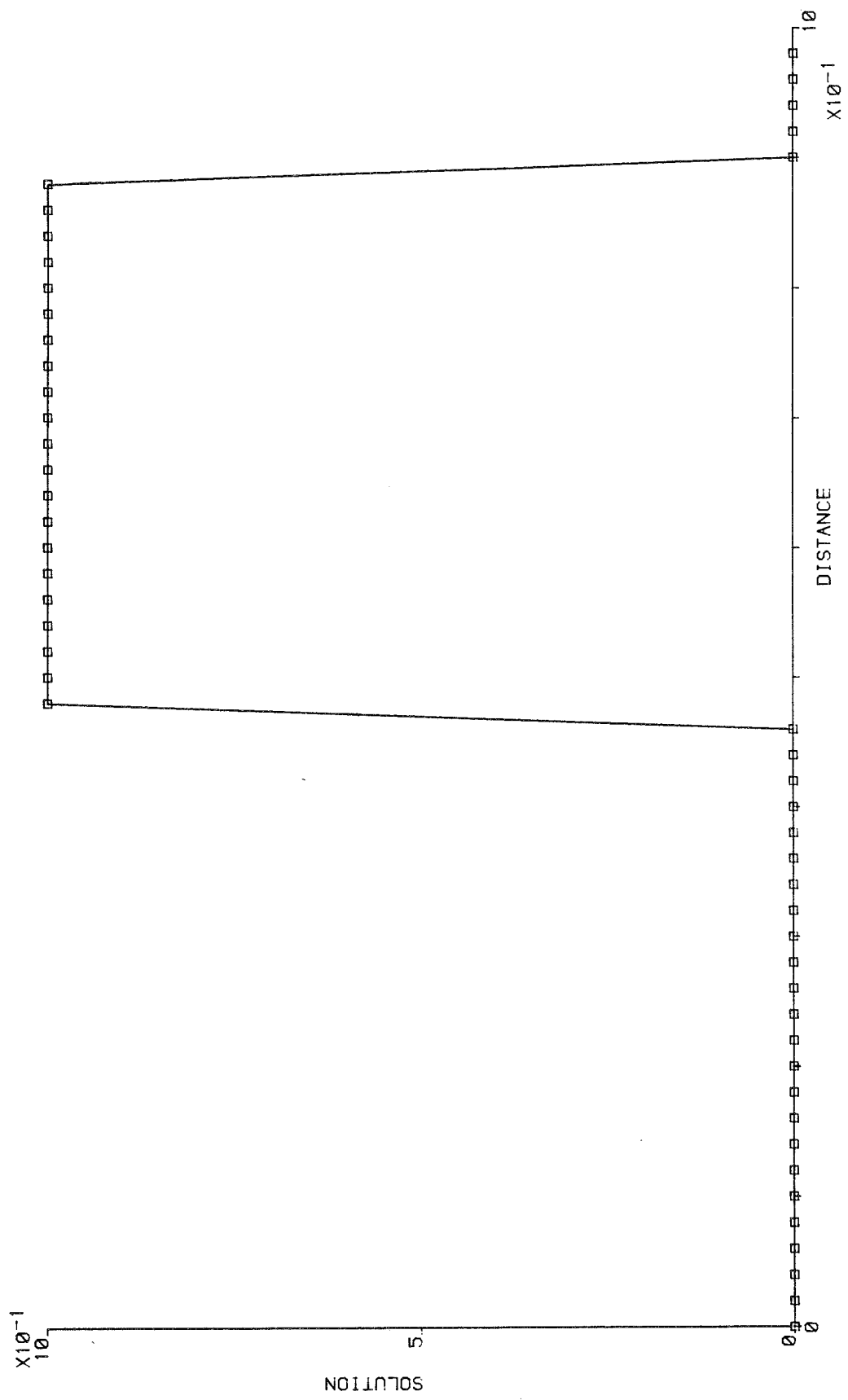
Fig. 5 Amplifiers of the type MINAM. These functions have two branches emanating from $r = 0$. MINAM1 has the steepest gradients near $r = 0$ (thick line) and is asymmetric. MINAM2 (broken line) and MINAM3 (full line) are symmetric about $r = 0$ by construction. All amplifiers in MINAM pass through $(-1,1)$ and $(1,1)$.



THE LINEAR ADVECTION EQUATION
 FIGURE 6a : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH $\lambda = 1$.

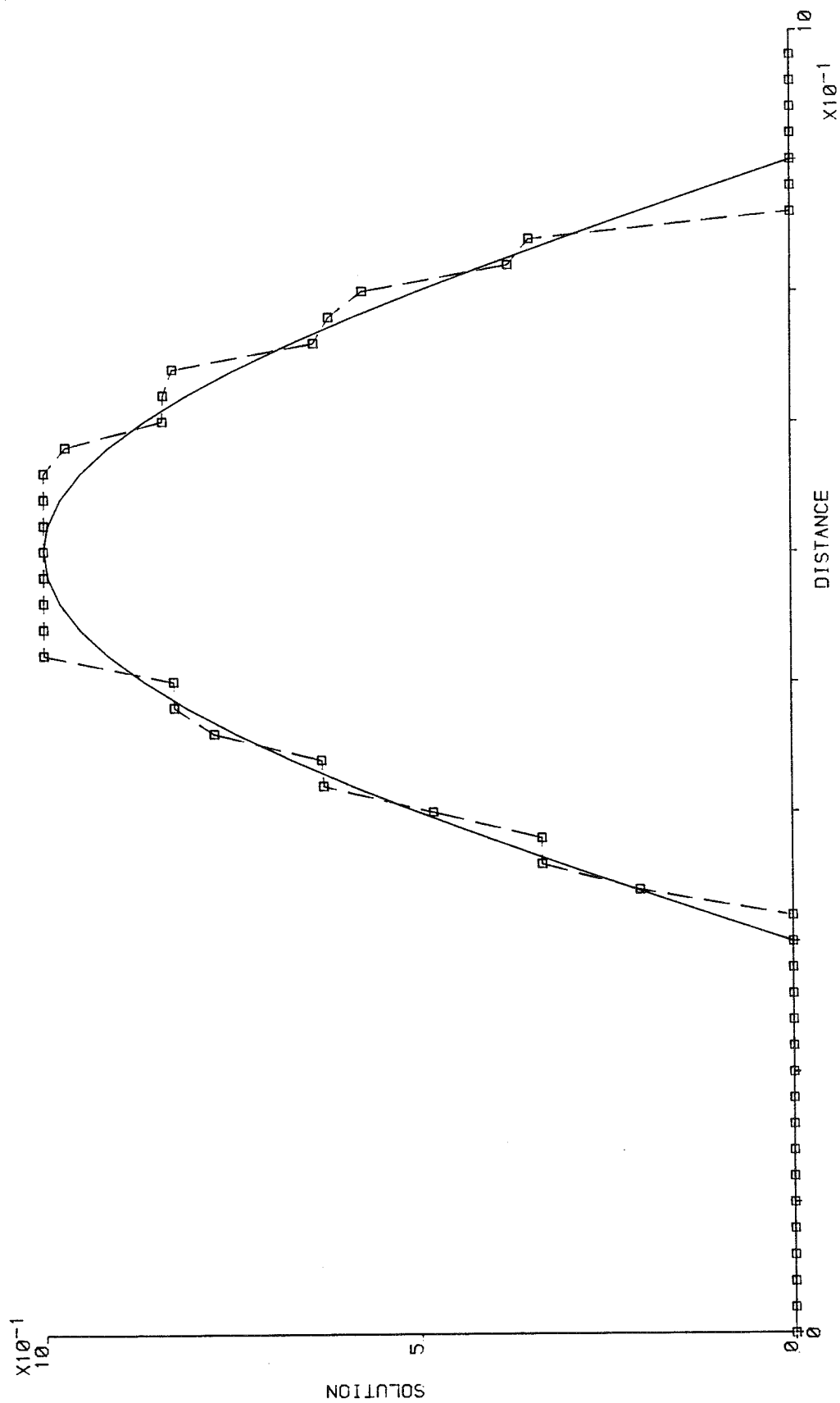


THE LINEAR ADVECTION EQUATION
 FIGURE 6b : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH $A = 1$.



THE LINEAR ADVECTION EQUATION

FIGURE 7a : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH SUPERA1.



THE LINEAR ADVECTION EQUATION

FIGURE 7b : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH SUPERA1.

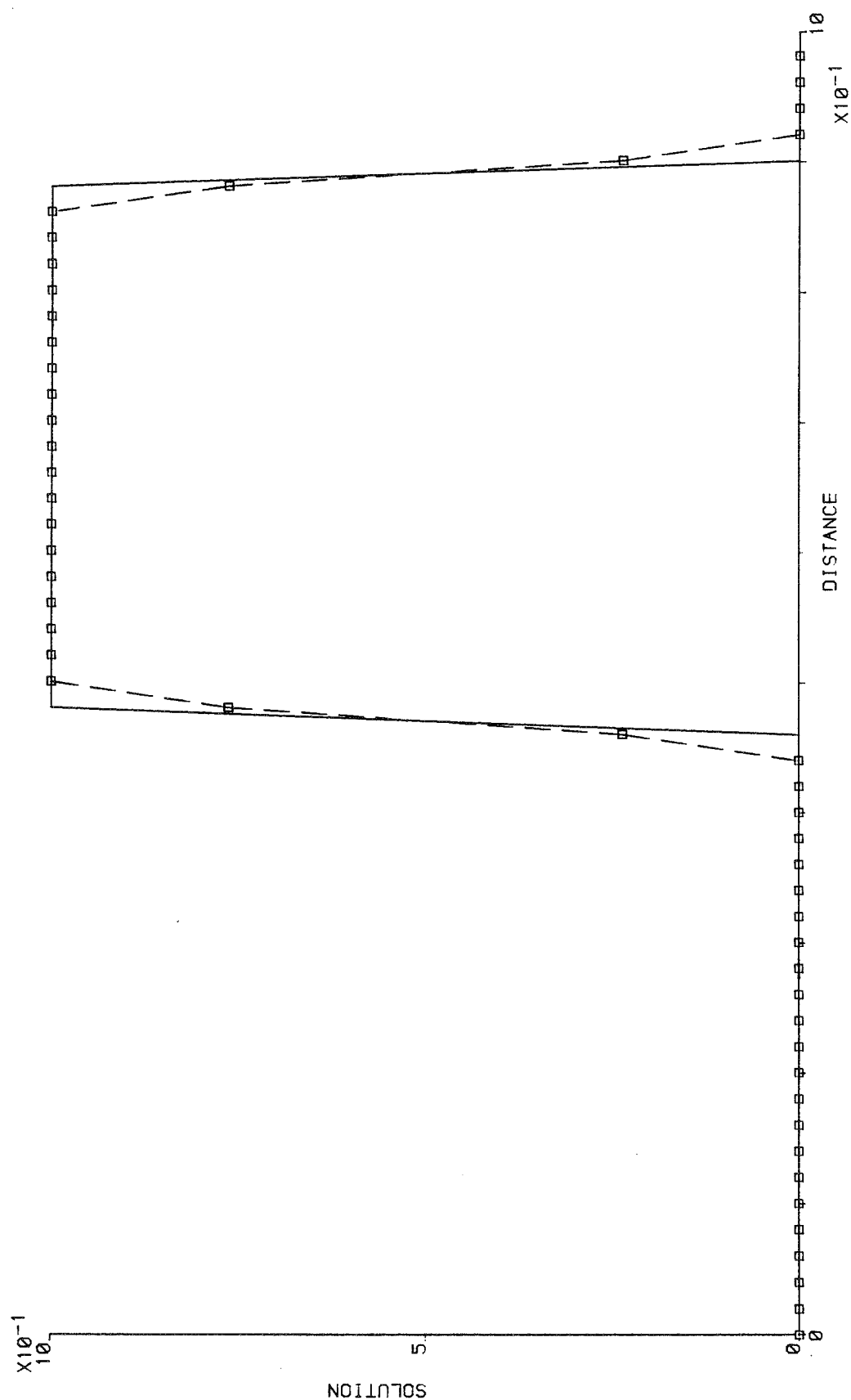


FIGURE 8a: EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH SUPERA2.

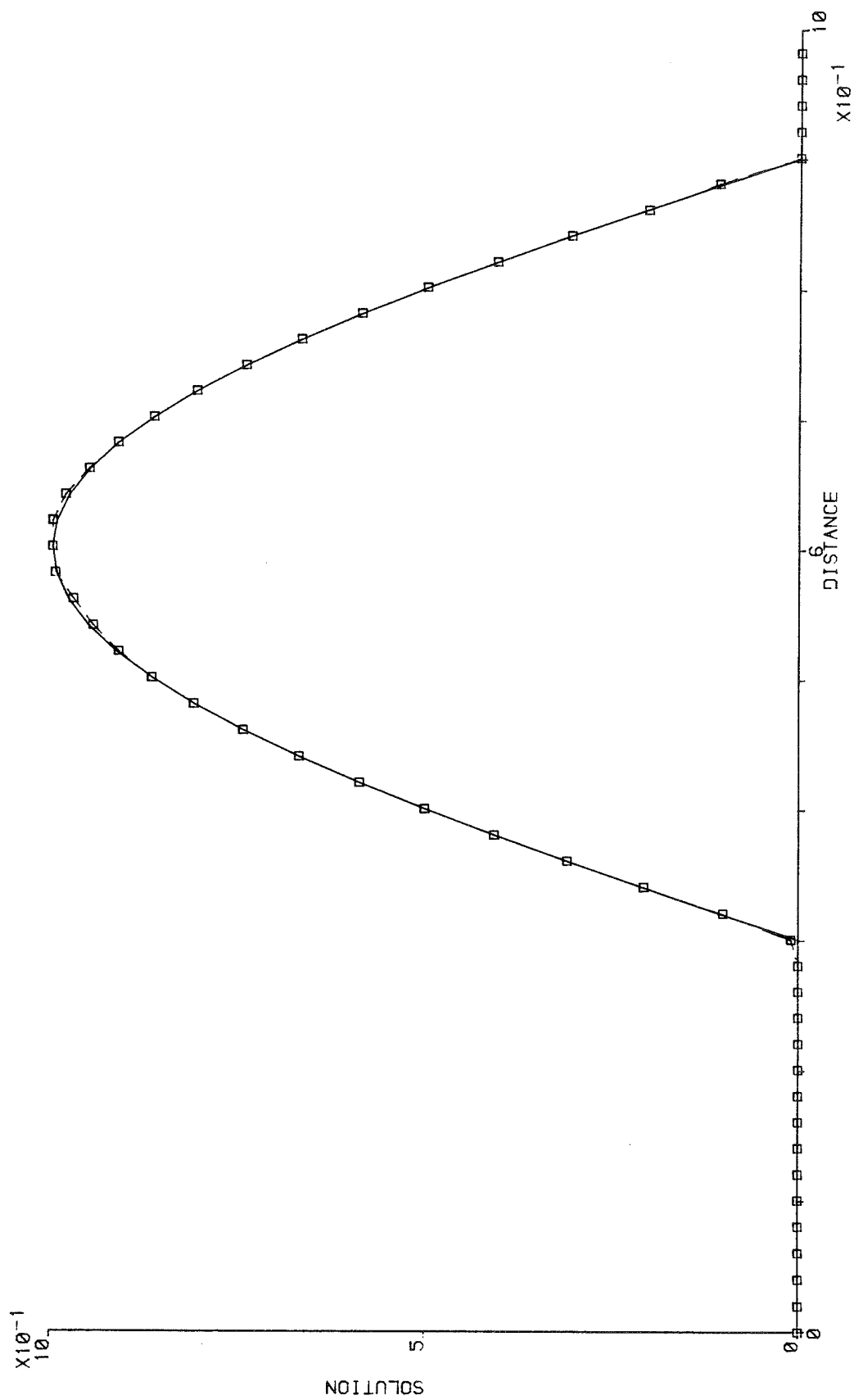


FIGURE 8b : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH SUPERA2.

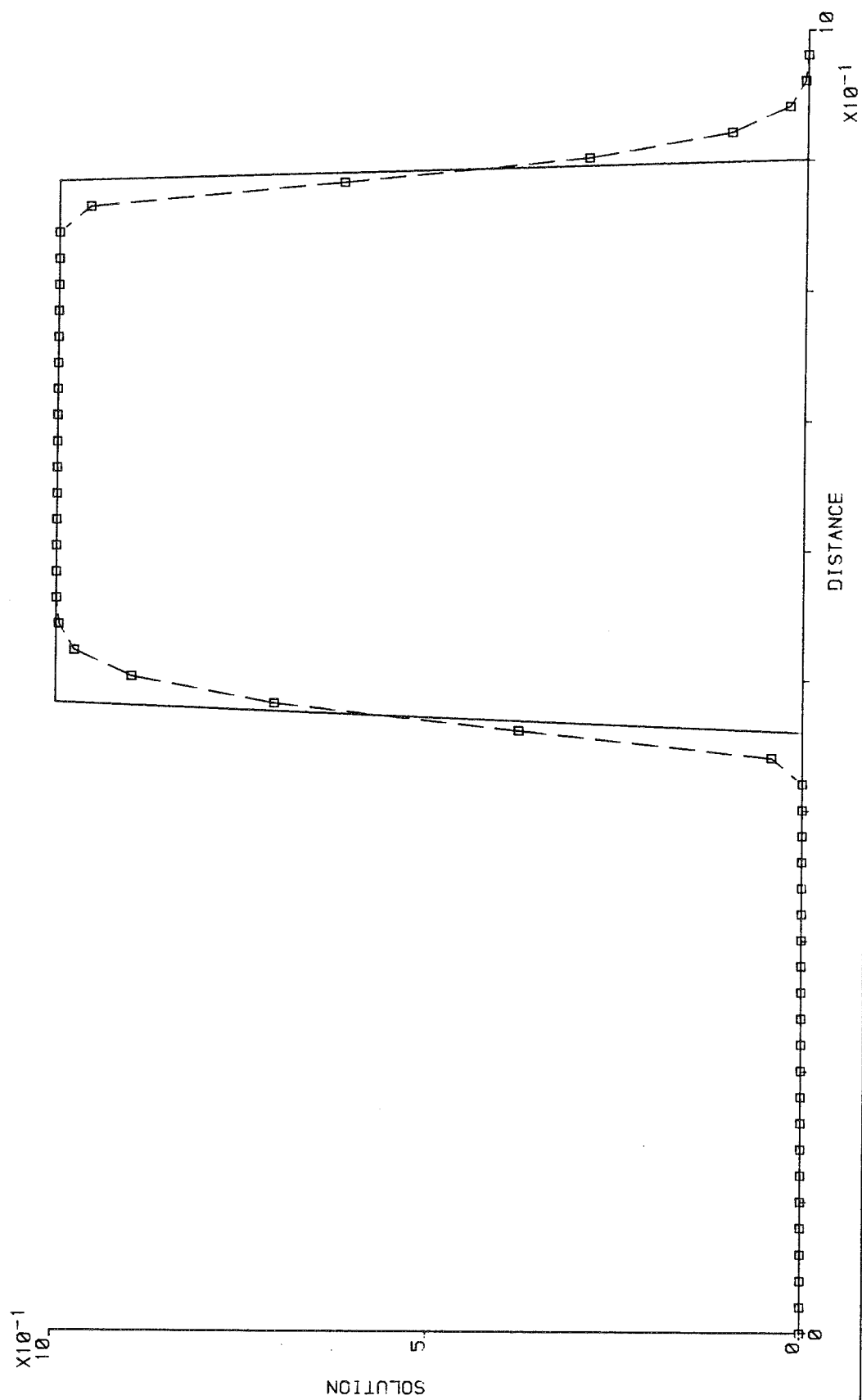
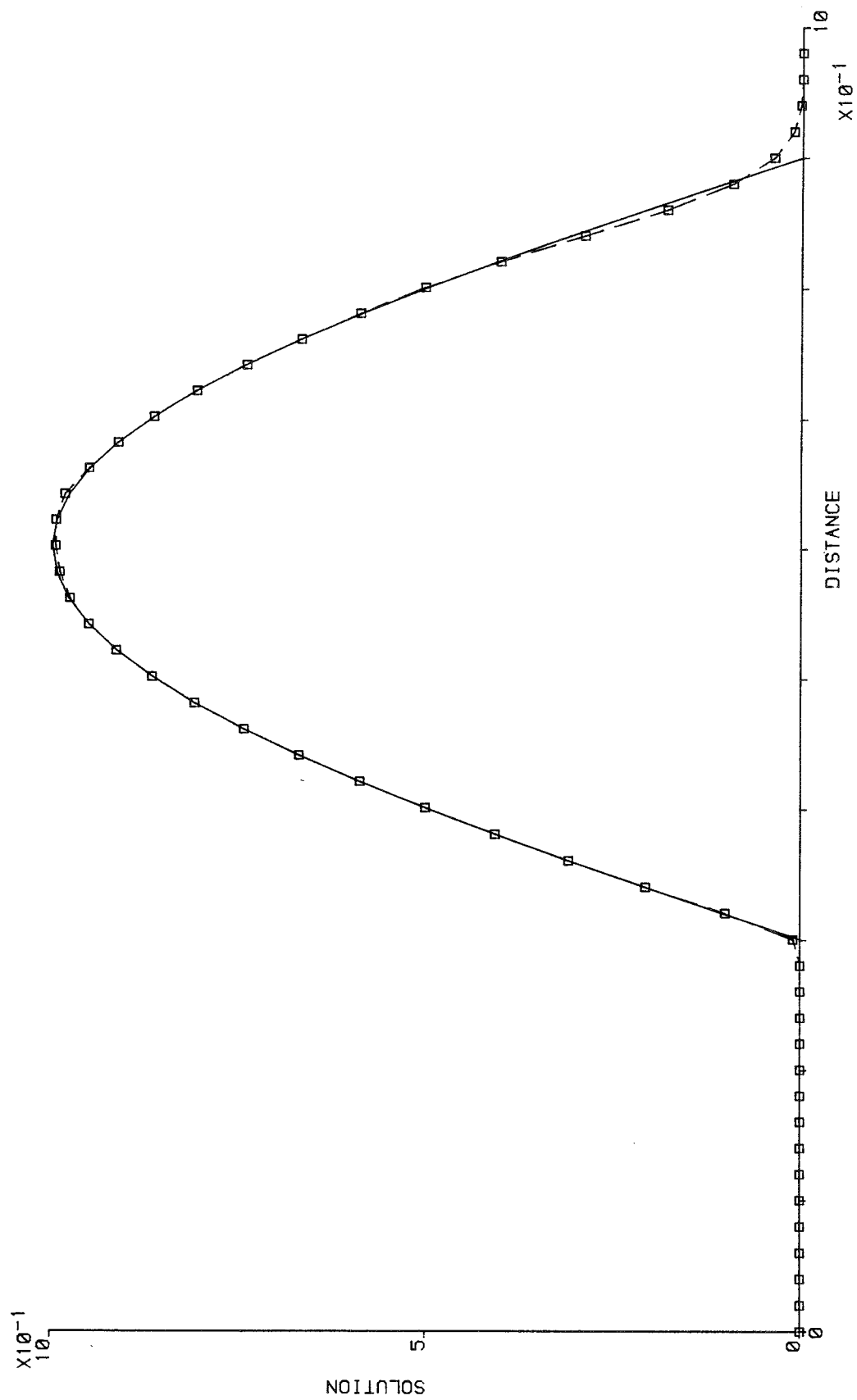
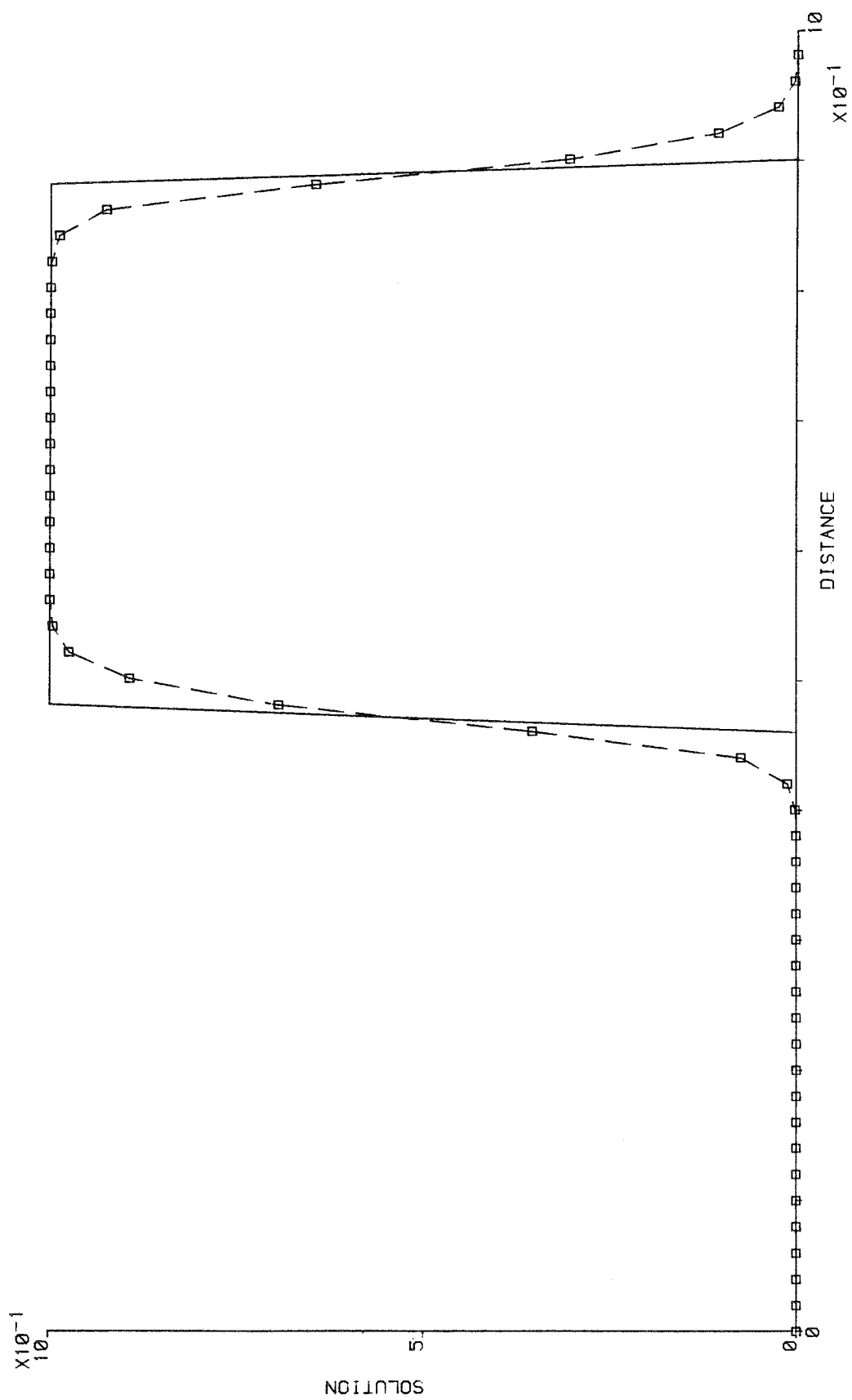


FIGURE 9a : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAMI.

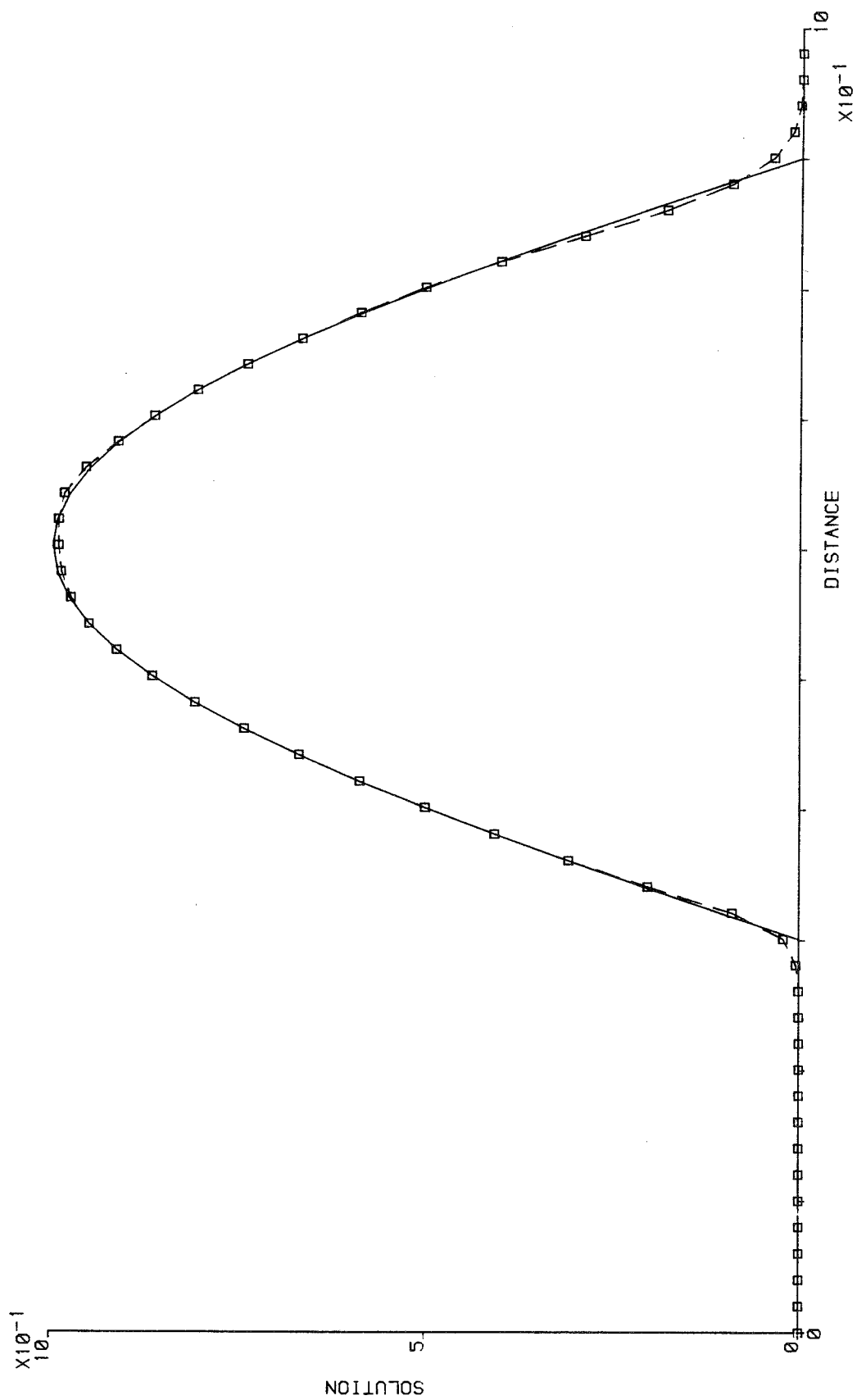


THE LINEAR ADVECTION EQUATION
 FIGURE 9b : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAM1.

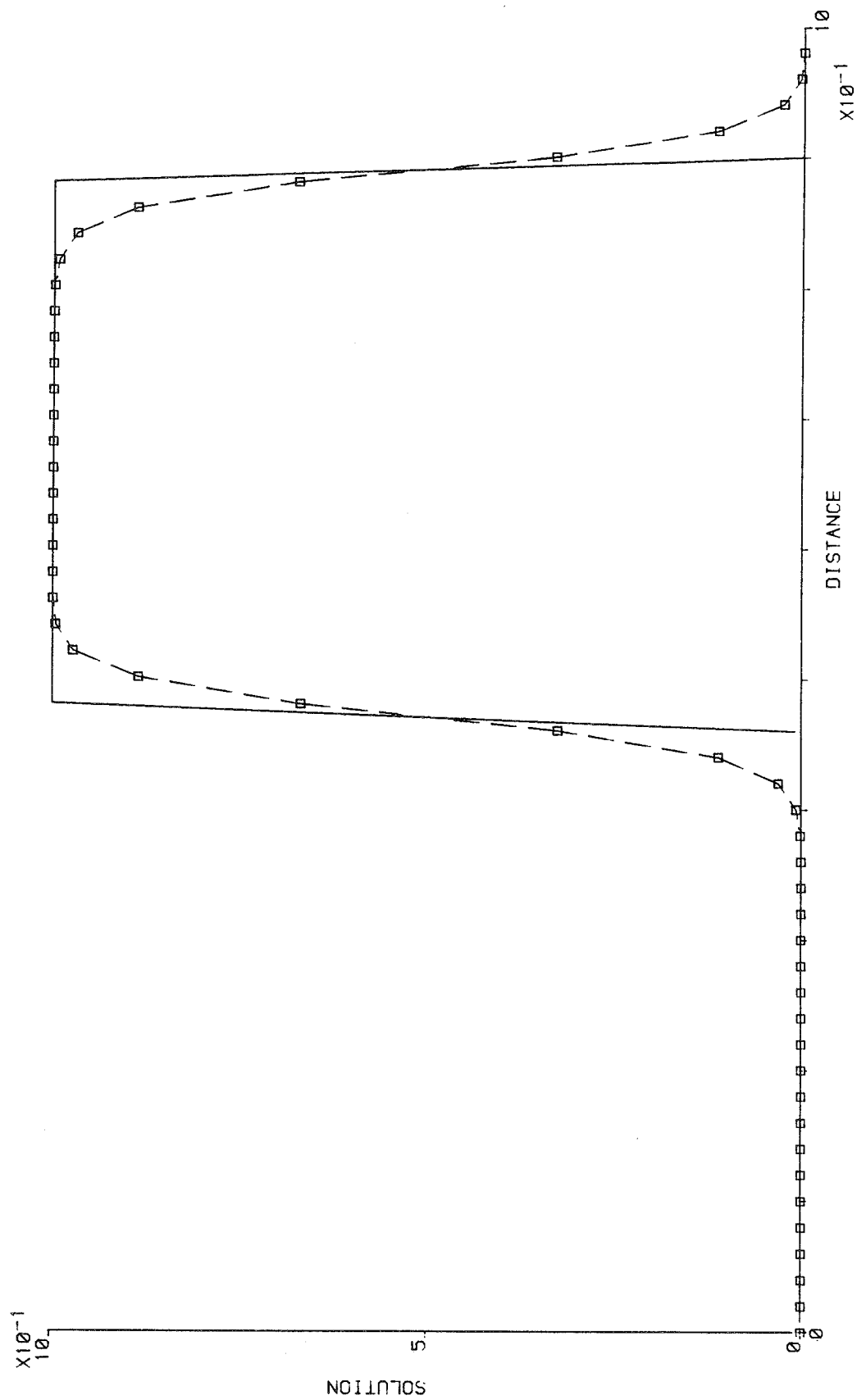


THE LINEAR ADVECTION EQUATION

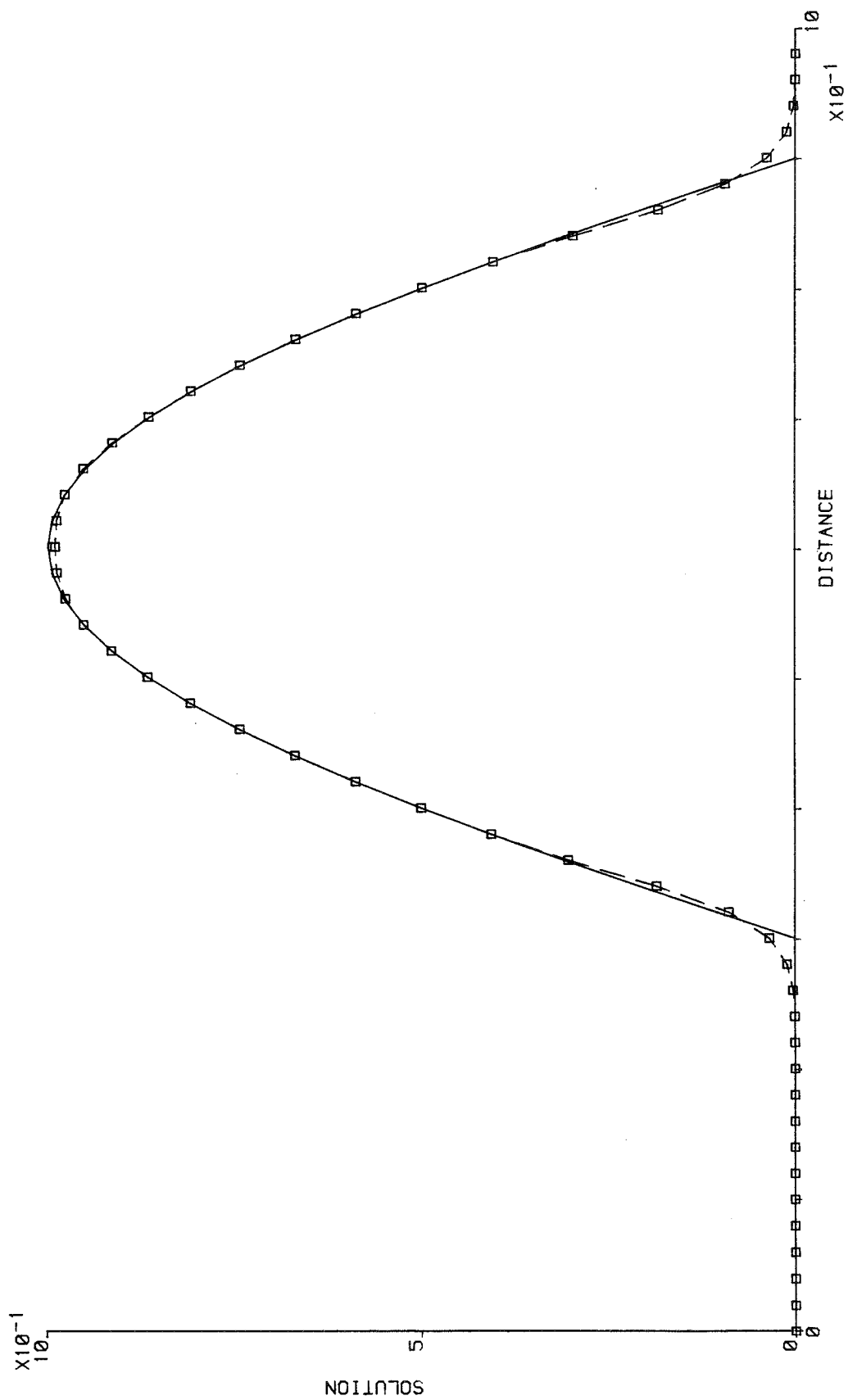
FIGURE 10a: EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAM2.



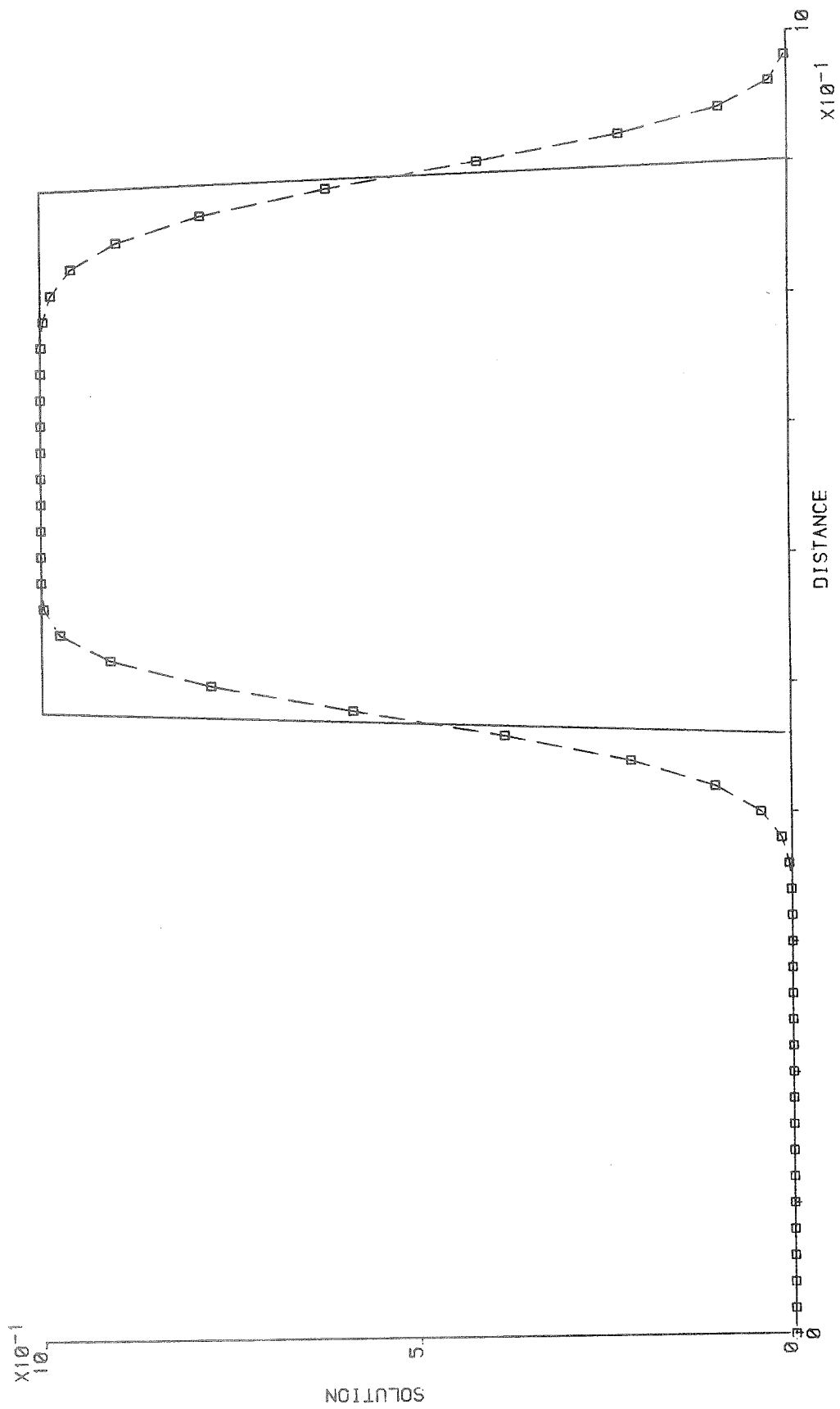
THE LINEAR ADVECTION EQUATION
 FIGURE 10b: EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAM2.



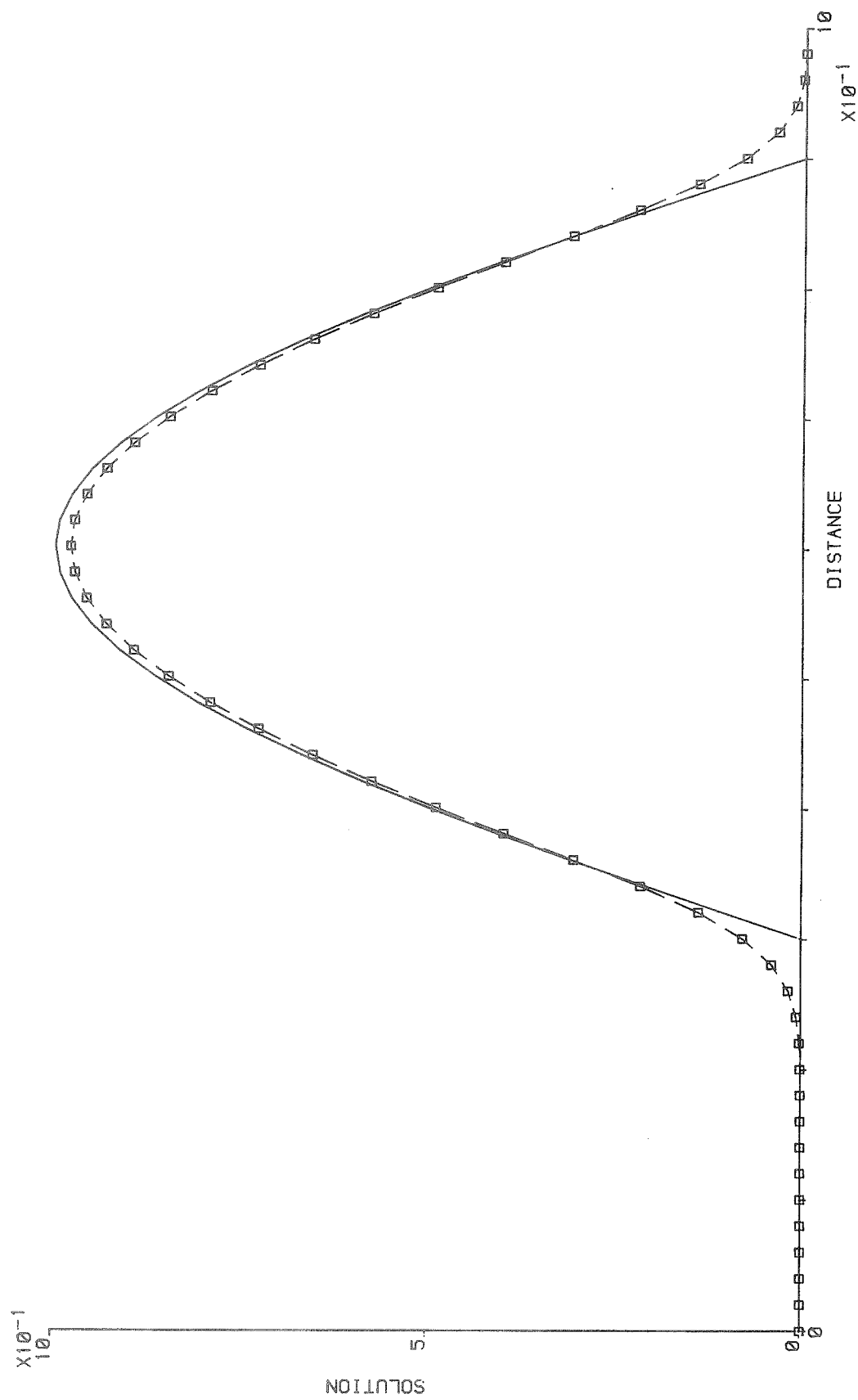
THE LINEAR ADVECTION EQUATION
 FIGURE 11a: EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAM3.



THE LINEAR ADVECTION EQUATION
 FIGURE 11b : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF METHOD WITH MINAM3.



THE LINEAR ADVECTION EQUATION
 FIGURE 12a : EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF 1ST ORDER METHOD.



THE LINEAR ADVECTION EQUATION
FIGURE 12b: EXACT (FULL-LINE) AND COMPUTED (SYMBOL) SOLUTIONS; WAF 1ST ORDER METHOD.