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Ph.D Thesis

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THE STABILITY AND POST-BUCKLING BEHAVIOUR
OF STIFFENED PLATES IN COMPRESSION

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SUMMARY

The investigation reported in this thesis was carried out to examine the behaviour of stiffened plates subjected to uniform end compression using energy methods.

The thesis begins with a short introduction, followed by a review of the relevant literature. The basic differential equations and energy expressions, using plate theory are then derived.

The theoretical analysis of the mechanics of local, lateral and overall instability of asymmetrically and symmetrically edge stiffened plates is then presented. The analysis is then extended to investigate the behaviour of infinitely wide panels with stiffeners of deformable cross section. The results obtained from this analysis are used to highlight the shortcomings of using an infinitely wide panel model to predict the behaviour of a panel of a finite size through comparison with existing work. This is then followed by the analysis of a simply supported panel of finite size stiffened by stiffeners of rigid cross section. Plate theory was used to model the skin and beam theory to model the stiffeners. The results obtained from the analysis compared favourably with those already existing. The method is then extended to investigate the post buckling behaviour of simply supported plates and stiffened panels.

The post-buckling behaviour of plates has received a considerable amount of attention since plates can carry loads beyond the critical load. The use of this additional strength is of great importance to the design of aerospace, naval and civil engineering structures since considerable weight savings can be achieved. In comparison the post-buckling of light gauge integrally stiffened panels has received far less attention. Most of the published literature concentrates on the interaction between the local and overall buckling modes and the sensitivity of panels to geometrical imperfections when the local and overall buckling loads are nearly coincident.

In all the above investigations, the overall buckling load for the panel is assumed to be equal to the Euler load for a pin-ended strut with the cross-section equal to the repeated panel cross-

section. This implies that the transverse edges of the panels are assumed to be free.

In general, stiffened panels are supported by a system of cross-frames. A consequence of this layout is that lateral support is provided on all boundaries of the panel and therefore the wide strut model is invalid.

In the present work the panels are assumed to be simply supported along all four edges. It is shown that the post-buckling behaviour of these panels is substantially different to that of the wide strut model.

The thesis finishes with a summary of the findings and a list of the main conclusions.

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NOTATION

A	area of cross section
C	$Et/(1 - \nu^2)$
D	flexural rigidity of plate
D_f	flexural rigidity of flange
D_w	flexural rigidity of web
D_s	flexural rigidity of skin
G	modulus of elasticity in shear
I	second moment of area about centroid
I_{Oy}	second moment of area about edge O - y
M_x, M_y, M_{xy}	bending stress resultants
N_x, N_y, N_{xy}	membrane stress resultants
P	load
P_{cr}	critical load
Q_x, Q_y	shear stress resultants
V	potential energy
b	half length of plate
d_f	width of flange
d_w	width of web
d_s	width of skin
h	thickness of strut
i	suffix relating to plate element
p_i	generalised force
q_i	displacement variable
r_i	generalised displacement of forces
t_f	thickness of flange
t_w	thickness of web
u, v_i, w_i	displacements in x, y_i, z_i derivatives

NOTATION (concluded)

w_0	initial imperfection
x, y_i, z_i	local coordinate for plate i
α	displacement variable
$\delta, \delta^2, \delta^T$	first, second and total variations
ϵ_i	generalised strain
ϵ_x, ϵ_y	strain in x and y directions
ϵ_{xy}	shear strain
θ	rotation
λ	load factor
ν	Poisson's ratio
σ_i	generalised stress
σ_x, σ_y	stress in x and y directions
σ_{xy}	shear stress
ϕ	stress function
Δ	small finite increment
$[D]$	elasticity matrix
$[N]$	stress resultant matrix
$\{M\}$	$[M_x, M_y, M_{xy}]^T$
$\{N\}$	$[N_x, N_y, N_{xy}]^T$
$\{\alpha\}$	displacement variable vector
$\{\epsilon\}$	$[\epsilon_x, \epsilon_y, \epsilon_{xy}]$
$\{\sigma\}$	$[\sigma_x, \sigma_y, \sigma_{xy}]$

1 INTRODUCTION

The design of many structures is governed primarily by the behaviour of members in buckling. In recent years, the continuing trend in the field of structural engineering for efficient use of materials, leading to lighter and thinner structures has served to promote the phenomenon of buckling to an importance not contemplated in earlier times.

In the aircraft industry where there is an obvious necessity for light weight, thin-walled structures, ie structures where the thickness is much less than the other dimensions of the structures, play an important role, their high strength to weight ratios make them ideally suited for aircraft construction.

Plates and plated sections fall in the category of thin-walled structures. In particular asymmetric and symmetric edge-stiffened plate elements, such as panels with 'L' and 'T' section integral stiffeners are frequently used as structural components in aircraft design. These stiffened elements, though representing a relatively small part of the total structure, substantially influence their strength, stiffness and stability.

The elastic instability of these sections when used as compressive members can be initiated by any one of a number of independent modes, for example, the local, the lateral or the overall mode. It is also possible for two or more of these modes to occur simultaneously and the resulting interaction can substantially affect the load carrying capacity of the member.

Therefore the theoretical work presented in this thesis attempts to describe the mechanics of the local, torsional and overall behaviour of thin-walled asymmetric and symmetric edge stiffened plates under pure compression using energy methods.

1.1 Literature Review

1.1.1 Plates

Plates are often used as structural components since they can take loads well in excess of the local buckling loads. An extensive amount of literature is available on the topic of buckling of flat plates and may be found in established texts by Bleich [1], Timoshenko [2,3], Bulson [4] and Allen and Bulson [5]. The post buckling properties of plates were first examined by Schuman and Black [6] in 1930, who conducted extensive tests on simply supported plates and arrived at the conclusion that wide plates under compression acted as though only part of their width is effective.

Von Karman [7] in 1932 published a theoretical study of simply supported plates and developed the effective width formula. The problem was analysed by assuming that the total load at collapse is taken by two strips at the edges. The total width of the two strips b_e is given by the formula:

$$\frac{b_e}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_e}}$$

where b_e is the effective width

σ_{cr} is the critical stress

σ_e is the axial edge stress .

Timoshenko [2] carried out an energy based post buckling analysis on a simply supported plate by assigning out-of-plane as well as in-plane displacements. The loaded were allowed uniform displacement and, the unloaded edges were kept straight and in their original position. The results obtained for the post buckling stiffness were about 25% greater than those obtained by later researchers. This can be attributed to the inaccuracy of in-plane displacement functions.

Marguerre [8] in 1939 carried out an analytical investigation on the post buckling behaviour of a square flat plate. The analysis consisted of prescribing an out-of-plane displacement function and deriving an airy stress function [9] through the solution to

Von Karman's compatibility equation. The total potential energy was then minimised with respect to the displacement variables leading to a system of simultaneous non-linear equations which could be solved to obtain the values of the displacement variables and hence the absolute magnitude of the deflections and stresses.

The first 'exact' solution to Von Karman's equations was obtained by Levy [10] by assuming a double sine series for the displacement function and solving Von Karman's compatibility equation to find the form of the stress function. Substituting both the displacement function and the stress function into Von Karman's equilibrium equation, Levy obtained a series of equations to find the deflection coefficients. Levy evaluated the first six deflection coefficients and showed from his six term solution, that Marguerre's three term solution was extremely accurate. In a subsequent paper [11] Levy adopted the solution to deal with the case of a clamped plate under normal pressure.

Hu et al [12] in 1946 investigated the post buckling behaviour of a square plate with an initial imperfection from flatness. Both the initial imperfection and the deflected shape were represented by Fourier series. Levy's method of solution was employed to obtain the unknown coefficients. The effective width was found to be markedly altered only at loads at or near the flat plate critical load. The ratio of post buckling stiffness was always lower than that of a flat plate at loads below the critical load and higher at loads above the critical load.

Coan [13] extended Levy's solution to deal with initially deformed simply supported plates with different edge stress conditions. A double Fourier series was used to represent the initial imperfection as well as the deflection. The conclusions obtained were the same as those of Hu et al. In a subsequent paper Coan et al [14] described the experimental set-up and procedure used to check the theoretical results.

Boley [15] in 1955 presented a procedure for the analysis of buckled plates. The method involved the use of successive approximations for the solution to Von Karman's compatibility and equilibrium equations. The boundary conditions studied were the same

as those of Marguerre and Levy, the results showed good agreement for the effective width of these two investigators.

Mayers and Budiansky [16] investigated the plastic behaviour of post buckled plates. The plate was assumed simply supported. The method of analysis involved the use of the theory of plasticity. The basic formulation of the problem followed Timoshenko's [2] method, ie out-of-plane as well as in-plane displacements were used. The accuracy of this method is dependent on the accuracy of the out-of-plane as well as the in-plane displacements, the numerical results obtained were higher than those obtained experimentally.

Yamaki [17] in 1959 carried out one of the most comprehensive theoretical analysis of the post buckling behaviour of plates. Imperfections as well as various boundary conditions for the unloaded edges were considered. The in-plane boundary conditions for the unloaded edges were either straight edges or free to wave in-plane. The analytical procedure consisted of assuming a displacement function and solving Von Karman's compatibility equation to obtain a stress function. The equilibrium equation was then solved using Galerkin's method. In a subsequent paper in 1960 Yamaki [18] predicted the collapse loads of square plates with various edge support and unloaded edge conditions using the Tresca yield criterion. Simple formulae for each condition investigated were presented. The formulae were compared with those of Timoshenko, Von-Karman and the empirical and experimental results obtained by Gerard [19], and shown to give results closer to Gerard's experimental results than the other investigators.

Using Galerkin's method Walker [20,21] in 1967 carried out an ambitious investigation on the post buckling behaviour of plates with arbitrary unloaded edge deflection conditions together with stress conditions of zero shear stress and edges free to wave. The loaded edges were considered simply supported with zero shear stresses and a uniform or linearly varying load system. Walker also carried out an experimental investigation concerning eccentrically loaded plates. However, the boundary conditions were practically the same as for a linearly varying end displacement rather than the theoretically assumed loading system. This is attributed to the

experimental loads being applied through rigid bars which tend to produce uniform or linearly varying end displacements. Graphs of load against end compression were presented, from which the collapse load was obtained by setting the maximum edge stress equal to yield stress.

Walker [22] in 1969 developed a solution to Von-Karman's equation using the Fourier coefficients obtained by Levy and using a perturbation technique. The method involves expanding the deformation magnitudes and loading parameter in a McLaurin's series about the buckling point. Formulae were developed for the deformation and maximum loading for a perfectly flat plate as well as plates with initial imperfections. These were identical to those obtained by Stein [23] who used a more precise and elaborate method. For the maximum load carrying capacity Walker assumed that collapse would occur when the maximum membrane stress reaches yield. The results obtained agreed well when compared with those of Yamaki [18].

Applying Marguerre's method of analysis, Rhodes and Harvey [25] investigated the post buckling behaviour of flat plates in compression where the unloaded edges were elastically restrained against rotation. The two term solution agreed closely with the four term solution obtained by Yamaki for extreme elastic restraint conditions of simply supported and clamped edges. Collapse was assumed to occur when the maximum edge stress reaches yield. The results showed closer agreement than those obtained by a rigorous plastic analysis by Graves-Smith [24] and Budansky [16] except for the cases where collapse quickly follows buckling.

In a subsequent paper Rhodes and Harvey [26] considered the post buckling behaviour of plates with a linear variation in edge displacement. Comparison with Walker's [21] results showed better agreement for load-deflection curves than Walker's own theoretical results, possible reason being the difference in the boundary conditions.

In recent years, due to the advent of powerful mainframe computers, numerical large deflection analysis of plates have provided realistic load displacement curves for simple plates and other complicated structural forms.

Roberts and Ashwell [27] in 1971, demonstrated the use of finite element mid-increment stiffness matrices for the post buckling analysis of imperfect structures. The method which was shown to reduce computation time was applied to the post buckling problem of a laterally loaded plate with clamped edges. The results showed good agreement with Levy's [11] fourier series solution. The method was also applied to the problem of an imperfect plate loaded along two opposite edges, the unloaded edges were allowed to move but constrained to remain straight by introducing stiffeners. They found that if the stiffeners were made too rigid, the solution became ill-conditioned and resulted in discrepancies of about two percent in the in-plane displacements, however the results were in good agreement with Levy's [10] perfect plate solution and Hu et al [12] solution to an imperfect plate.

Narayanan and Shanmugam [28] presented an approximate analysis on the compressive strength of biaxially loaded plates. The approximate analysis assumes that all edges of the plate are held straight and any induced membrane stresses are entirely due to the out-of-plane displacements. The out-of-plane displacements and the initial imperfection shape were assumed to be the same as that during buckling. An incremental energy equation was then derived relating increments in the out-of-plane displacements to the increments in stresses in the x and y directions, then using the stress-strain relationship two other equations relating increments in stresses to increments in displacements were derived. These three equations were solved numerically and, the stresses in the x and y directions at any section of the plate could then be obtained. Collapse analysis was carried out using a yield criterion which allowed for the upper bound energy solution. The authors presented numerous design interaction curves for various plate problems with imperfections and a wide range of plate width to thickness ratios.

Despite a number of existing texts in the field of theory and analysis of plates, a monograph which is entirely devoted to the geometrically non-linear problems of plates is by Cheun-Yuan Chia [29]. The author looks at the problems of static large deflection, post-buckling and non-linear vibration of isotropic,

anisotropic and laminated anisotropic plates based upon the Von Karman non-linear plate theory.

1.1.2 Stiffened Panels

Burge and Pritchard [30] in their book discuss the initial buckling and post buckling behaviour of flat plates and panels in compression. The authors discuss the various possible modes of buckling that may occur for z-section stiffened panels. Approximate methods for the failure of stiffened panels for various modes are discussed. Overall the book provides a good insight to the buckling behaviour of z-section stiffened panels.

Rothwell [31] in 1968 presented a technical note on the coupling between local and torsional buckling modes of z-section stiffened panels. In including the torsional mode Rothwell assumed the location of the neutral axis for in-plane flange bending to be based upon an approximate stress distribution, the stress distribution complied to the condition that during torsional buckling of the stiffener no additional moment is produced about the skin line. As Rothwell points out, this assumption excludes any coupling between the torsional mode and the overall mode. Rothwell found that for stiffeners with small flanges there was a marked reduction in the buckling stress indicating the influence of the torsional mode.

Wittrick [32] in 1968 developed a general approach for the determination of initial buckling of stiffened panels assumed to consist of a series of flat strips, connected together at the edges. Whatever the buckling mode it was assumed the deformation of the component was sinusoidal in the longitudinal direction.

For the in-plane deformation, non-linear theory of elasticity was used. All modes of buckling, ie overall, torsional and local were incorporated in one determined equation. This theory later being incorporated in the finite strip package VIPASA [34]. Williams and Wittrick [33] in 1972 presented numerical results for the initial buckling of stiffened panels. Angle, plain flat outstand integral and z-section stiffened panels were considered. The authors point out that the torsional buckling modes for flanged stiffeners contain a substantial amount of deformation of the stiffener cross-section

and, the assumption that the cross-section does not distort can lead to considerable over estimation of the buckling stress. The authors also compare their results for z-section stiffened panels with those obtained by Rothwell [31] and show that Rothwell's results were virtually the same as panels with angle integral stiffeners rather than z-section bonded stiffeners especially in the local buckling region where the z-section stiffened panels had a much higher buckling load. The reason for this is that the bonded flange of the z-section stiffener increases the stiffness of the portion of the skin between successive stiffeners.

Tvergaard [35] in 1973 investigated the imperfection sensitivity of wide unflanged integrally stiffened panels under pure compression. The panel was assumed to be infinitely wide and simply supported along the loaded edges, the stiffener cross-section being assumed rigid. Attention was directed to the imperfection sensitivity of panel design where the Euler and the local buckling modes occur simultaneously for a given end compression. It was concluded that these panels were highly sensitive to imperfections and might buckle well below the analytical buckling load.

Koiter and Pignataro [36] in 1974 analysed the same panel as Tvergaard [35]. They concluded that the high sensitivity to initial imperfections noted by Tvergaard occurred only for extremely small imperfections but was less serious for imperfections that are likely to occur in practice.

Thompson, Tulk, Walker [37] produced results of an experimental study of imperfection-sensitivity in the interaction buckling of stiffened plates by conducting small-scale experiments on panels manufactured from an epoxy plastic. Overall and local imperfections were varied for geometries covering the range of interaction between Euler and local plate buckling. Torsional buckling of thin torsionally weak stiffeners was also examined. A series of load-deflection and imperfection sensitivity curves were presented and, it was pointed out that for panels with slender stiffeners coupling between torsional and the overall mode could be very severe.

Williams, Wittrick, Plank [38] in 1974 presented a paper on the critical buckling loads of prismatic plate assemblies. They draw attention to the differences in critical stresses between an infinitely wide panel model simply supported along the loaded edges and, finite panels simply supported along all four edges. They point out that infinite panel models under-estimate the local buckling stress when compared with panels of finite size with a low number of stiffeners, however, the differences are negligible for a finite panel with eight or more stiffeners. It was also pointed out that critical stresses can be over-estimated by as much as 100% if it is assumed that the stiffener cross-section does not deform - especially for flanged stiffeners. Since the Euler type of mode is usually used to characterise the overall mode for infinite panel models, it was observed that this can under-estimate the critical stress for panels with angle stiffeners when compared to a finite panel with a high length to stiffener pitch ratios and, under-estimate the critical stress for a low length to stiffener pitch ratio, however, they suggest that this is acceptable for the range where the length to stiffener pitch ratio is between 8 and 32.

Horne and Narayanan [39] in 1977 presented detailed test results on a series of 'thick plate' welded panels loaded in longitudinal compression. Their investigations included the effects of intermittent and continuous plate/stiffener welds, the influence of local torsional and overall imperfections. The imperfections were introduced by straining plates beyond the yield point to induce permanent deformations. The authors observed that local - chequerboard types of imperfections of the order three times the limits specified by Merrison Workmanship rules [40] had no detrimental effect on the continuously welded panels and a slight detrimental effect in the intermittently welded panels. The effect of overall imperfections of the order five times the limits specified by Merrison rules had the greatest effect on slender panels than less slender panels. The authors considered three different types of stiffeners for stiffener failure tests, these being flats, bulb flats and angles. The authors observed that in panels with flat stiffeners torsional imperfections caused significant decrease in load carrying capacity. The effect of

overall stiffener imperfections caused an average drop in strength of 9% in panels with flat stiffeners and 17% in panels with bulb flats and angle stiffeners, the order of the torsional and overall imperfections being between three and four times the Merrison tolerances.

It was noted that the torsional failure stresses for flat, bulb flats and narrow angle stiffeners were less than the yield stress, however, the imposed imperfections were severe and not likely to occur in practice.

Overall, the paper outlines the shortcomings for various aspects in the Merrison Interim Design and Workmanship Rules.

Van Der Neut [41] in 1980 produced a report on the overall buckling of z-stiffened panels in compression. An infinitely wide panel is considered, the effects of interaction between overall and torsional modes is included in the analysis and an approximate method for the numerical computation for the critical buckling stress is presented. The method revolves around the restraint provided by the skin to the stiffeners and the solution to the equilibrium for the stiffeners. Both bonded and riveted stiffeners are considered. Comparison with the exact solution provided by Wittrick [32] showed good agreement.

Gunnink [42] in 1982 presented a report on the calculation of the buckling load of z-stiffened panels built up from orthotropic laminated plates. The author extends Van Der Neut's [41] method to calculate the overall buckling load of z-stiffened panels of isotropic material.

Bulson [5] in his text, also gives a very good account on the analysis of intermediate stiffeners on rectangular plates subjected to a variety of loading.

1.1.3 Sections and Edge Stiffeners

Chilver [43] in 1951 studied the buckling of channel sections analytically and obtained simultaneous equations to be solved for each individual plate element composing the section, the solution furnishing the value of the buckling stress.

Harvey [44] in 1952 carried out an analytical investigation on the stability of plain and lipped channels under end compression. Selecting a value of the edge restraint factor at the web-flange junction, the wavelength necessary for the minimum buckling stress could be obtained and used to find the ratio of the flange width to web width. For a given section, the method requires interpolative or iterative procedure to find the buckling load.

Bentham [45] in 1959 produced a general method of treating the initial post buckling behaviour of any system of plates under pure compression. Marguerre's method of analysis was used, the deflected form in the post buckling range was taken as that during buckling. A formula was obtained for the evaluation of the post buckling stiffness of each plate and, using the stiffness of the component plates, the overall stiffness of the section was obtained.

Numerical examples were presented for a U-section member, an infinite series of panels joined by hinges, a thin walled box section and a plate with one unloaded edge simply supported and the other free.

Bulson [46] reviewed and presented approximate solutions to the local buckling of thin walled structural sections. Particular attention was directed to the problems of reinforcing lips and the interaction of overall and local buckling. The problem of symmetrically edge stiffened plate was analysed through an exact method based on Chilver's [43] work. The exact analysis omitted the torsional contribution of the flange and an energy based solution which accounted for the torsional stiffness of the flange was presented. The interaction between local and torsional buckling which may reduce the buckling stress is pointed out. It should be stressed that Bulson's theory is limited to the evaluation of interaction of buckling modes which have the same wavelength. For the case of asymmetrically edge stiffened plates, the author recommends for conservative use, to include with the stiffener a strip of plate, width $30t$ and compute the moment of inertia of the stiffener about an axis through the resulting section. Using this concept, results for aluminium channel were compared with test results obtained by Dwight [47] which showed good agreement.

Kloppel and Schiedel [48] studied the behaviour of edge-stiffened plates, results for symmetric and asymmetric edge stiffeners for various geometrical configurations and axial loading were presented.

Rhodes [49] in his PhD thesis carried out a comprehensive investigation of thin walled lipped channel and trapezoidal section beams subjected to pure moment loading. The energy method was used to determine the local buckling loads for the sections. The semi-energy method employed by Marguerre [8] was adopted for the post buckling analysis. Algebraic polynomials were used to describe the transverse deflected forms of the plate elements. Collapse was assumed to occur when the average membrane stress in the beam reaches yield. The theoretical analysis was compared with experimental work and showed good agreement.

Loughlan [50] in his PhD thesis investigated the mode interaction of lipped channel columns under concentric and eccentric loading. The energy method was used for the local buckling analysis where the cross-sectional deflected shape was described using a series of algebraic polynomials. The semi-energy approach was utilized for the post buckling analysis, the method allowed for the change in the local buckled mode during loading which resulted in a continual shift in the section neutral axis and a reduced Euler load. The theory was also shown to predict both stable and unstable column behaviour for the case where the columns had critical loads far apart or coincident. It was shown that the post-local buckling paths eventually became unstable although they might initially be stable. Collapse was assumed to occur when the maximum edge stress at the component plate junctions reached yield stress. Comparison between theoretical and experimental work showed good agreement.

Rhodes and Loughlan [51] presented a simple design analysis for lipped channel columns based upon Loughlan's [50] rigorous theoretical approach. The effective width concept was used in the design method. The computed results agreed well with experimental results and it was found that the AISI expressions for effective width gave consistently accurate but conservative predictions.

Loughlan and Rhodes [52] presented some of the theoretical and experimental results obtained by Loughlan [50].

Loughlan and Upadhyaya [53] investigated the effect of local imperfections in isotropic plain channel columns. It was pointed out that channel columns are extremely sensitive to small imperfections particularly when the local buckling effects are predominant in the plate elements of the sections which are perpendicular to the axis of overall column bending.

Loughlan [54] investigated the ultimate load sensitivity of channel columns with column axis imperfection. It was shown that column axis imperfection reduces the ultimate load capability of perfect columns over the complete slenderness range covering stable, unstable and neutral behaviour. The largest reductions of the ultimate load were shown to occur in unstable equilibrium range especially for columns where the local and overall buckling stresses were coincident.

Lim [55] in his Phd looked at the behaviour of asymmetric edge stiffened plates. The buckling analysis was carried out using the energy method, the deflection being described by a series of polynomials along the width of the plate elements and a sine series along the length of the section. The location of the neutral axis for inplane bending of the component plates was based upon an approximate stress distribution over the cross section. The post buckling investigation was carried out using the semi-energy approach, taking into account the effect of imperfections. Comparison of theoretical results with experimental work showed good agreement. It was concluded that the effect of edge stiffeners is to delay the onset of buckling, after buckling occurs the stiffeners have no significant effect on the post buckling stiffness.

2 BASIC THEORY

2.1 Introduction

The energy, or variational methods of structural mechanics constitute a powerful and widely used approach. Forms of these methods have been tools for the analysis of engineering structures for more than a century. The application of energy methods to the derivation of forces and displacements in structures was developed by Castigliano [56] in the 1870's. The application of complementary energy for the analysis of non-linear elastic structures was developed by Engesser [56] in 1889. Since then a number of energy theorems have been formulated on the basis of these developments.

In this chapter, the basic energy theorems or variational principles employed in linear and non-linear analysis of structures are discussed for discrete systems, the theory later being extended to continuous systems [57].

To highlight the importance of the use of axial displacement functions for the analysis of asymmetric edge stiffened plates, where the position of the inplane axis of bending for the component plates is not so obvious, the theory is illustrated with reference to the linear and non-linear analysis of a pin-ended strut, where the axis of bending may be chosen at an arbitrary convenient position.

The chapter finishes with an outline of the derivation of the Von-Karman elastic large deflection equations, and a description of the derivation of various energy expressions for plates used in later chapters.

2.2 Energy Principles - Discrete Systems

Consider a discrete system where potential energy, V can be expressed as a function of displacements q_i and loads p_i .

If the system is subjected to a small variation in displacements δq_i so that its new configuration is $q_i + \delta q_i$ (loads p_i being assumed constant), the potential energy of the system in its new configuration can be expressed via a Taylor's series expansion as:

$$\begin{aligned}
V(p_i, q_i + \delta q_i) &= V(p_i, q_i) + \frac{\partial}{\partial q_i} V(p_i, q_i) \delta q_i \\
&\quad + \frac{1}{2!} \frac{\partial^2}{\partial q_i \partial q_j} V(p_i, q_i) \delta q_i \delta q_j \\
&\quad + \dots \text{etc}
\end{aligned} \tag{2.1}$$

Equation (2.1) can be written in a simplified form as:

$$\delta^T V = \delta V + \delta^2 V + \dots \text{etc} \tag{2.2}$$

where $\delta^T V$ is the total variation in potential energy expressed by:

$$\delta^T V = V(p_i, q_i + \delta q_i) - V(p_i, q_i) \tag{2.3}$$

δV and $\delta^2 V$ are the first and second variations of the potential energies given by:

$$\delta V = \frac{\partial}{\partial q_i} V(p_i, q_i) \delta q_i \tag{2.4}$$

$$\delta^2 V = \frac{1}{2} \frac{\partial^2}{\partial q_i \partial q_j} V(p_i, q_i) \delta q_i \delta q_j \tag{2.5}$$

2.2.1 Equilibrium

For the system to be in equilibrium, the potential energy is stationary with respect to displacements so that for all admissible values of δq_i , the first variation of the potential energy is zero, ie:

$$\delta V = \frac{\partial}{\partial q_i} V(p_i, q_i) \delta q_i = 0 \tag{2.6}$$

Equation (2.6) yields n equilibrium equations for $i = 1, n$.

2.2.2 Instability

Consider the system to be in an equilibrium configuration such that $\delta V = 0$. Equation (2.5) then becomes:

$$\delta^T V = \delta^2 V + \dots \text{etc} \tag{2.7}$$

If the system in its new configuration is in a state of stable equilibrium, the potential energy $V(p_i, q_i + \delta q_i)$ is a minimum and

the second variation $\delta^2 V$ which is a quadratic form in δq_i is positive definite for all admissible values of δq_i .

For unstable equilibrium $V(p_i, q_i + \delta q_i)$ is a maximum and $\delta^2 V$ is negative.

Critical conditions occur when $\delta^2 V$ changes from positive definite to a semi-positive definite indicating a possible transition from stable equilibrium to unstable equilibrium [57].

2.2.3 Non-linear Analysis

For a non-linear problem, the first variation of the potential energy, δV yields n unknown non-linear equations in the displacement variables $q_i (i = 1, n)$.

If Δ denotes a small but a finite increment in displacements and forces, then expanding the first variation of potential energy $\delta V(p_i + \Delta p_i, q_i + \Delta q_i)$ in a Taylor series about the (p_i, q_i) configuration yields:

$$\begin{aligned} \delta V(p_i + \Delta p_i, q_i + \Delta q_i) &= \delta V(p_i, q_i) + \frac{\partial}{\partial p_i} \delta V(p_i, q_i) \Delta p_i \\ &\quad + \frac{\partial}{\partial q_i} \delta V(p_i, q_i) \Delta q_i \\ &\quad + \dots \text{etc} \end{aligned} \quad (2.8)$$

Rearranging equation (2.8) and retaining only the first order terms in increments Δ , we get:

$$\begin{aligned} \delta V(p_i + \Delta p_i, q_i + \Delta q_i) - \delta V(p_i, q_i) &= \frac{\partial}{\partial p_i} \delta V(p_i, q_i) \Delta p_i \\ &\quad + \frac{\partial}{\partial q_i} \delta V(p_i, q_i) \Delta q_i \end{aligned} \quad (2.9)$$

If the system in configuration $(p_i + \Delta p_i, q_i + \Delta q_i)$ is equilibrium, then:

$$\delta V(p_i + \Delta p_i, q_i + \Delta q_i) = 0 \quad (2.10)$$

Rearranging equation (2.9) and using equation (2.10), equation (2.9) becomes:

$$\frac{\partial}{\partial q_i} \delta V(p_i, q_i) \Delta q_i = \frac{\partial}{\partial p_i} \delta V(p_i, q_i) \Delta p_i - \delta V(p_i, q_i) \quad (2.11)$$

Equation (2.11) provides a basis for a Newton-Raphson procedure for the solution of non-linear equilibrium equations. If the second term on the RHS of equation (2.11) is set to zero, then equation (2.11) represents the incremental equations of equilibrium.

2.3 Continuous Systems

The variational principles discussed in section 2.2 can be applied to continuous systems. The loads p_i and displacements q_i in the discrete system can be assumed analogous to the externally applied loads, and displacement coefficients which define the magnitude of displacements in continuous systems.

2.3.1 Equilibrium

Consider a system comprising a deformable body acted upon by external forces p_i , the corresponding displacements being r_i . If the system is in an equilibrium state then the first variation of the potential energy is zero. Assuming that the external forces are kept constant, the above statement can then be represented by the following equation [57]:

$$\delta V = -p_i \delta r_i + \int \sigma_i \delta \epsilon_i \quad d \text{ VOL} = 0 \quad (2.12)$$

where the repeated suffices imply summation, σ_i represents the internal stresses, $\delta \epsilon_i$ represents the first variation in the corresponding strains and the integration is over the volume of the body.

2.3.2 Instability

Noting that $\delta^2 V = \delta(\delta V)$, then from equation (2.12), the second variation of V is given by [57]:

$$\delta^2 V = -p_i \delta^2 r_i + \int \sigma_i \delta^2 \epsilon_i + \frac{1}{2} \delta \sigma_i \delta \epsilon_i \quad d \text{ VOL} \quad (2.13)$$

For stable equilibrium, the first variation is zero and corresponds to a minimum and the second variation is positive definite for all admissible variations in displacements.

Critical conditions occur when $\delta^2 V$ changes from positive definite to semi-positive definite indicating a possible transition from stable equilibrium to unstable equilibrium.

Since the second variation of any linear function vanishes, it is apparent that it is necessary to consider second order strains and displacements to completely define equation (2.13).

2.3.3 Non-linear Analysis

Assuming that σ_i and ϵ_i can be expressed as functions of displacement variables and r_i can be expressed as a linear function of displacement variables, then by considering equations (2.11) and (2.12) we get:

$$\int \sigma'_i \delta \epsilon_i + \sigma_i \delta \epsilon'_i \quad d \text{ VOL} - p'_i \delta r_i = p_i \delta r_i - \int \sigma_i \delta \epsilon_i \quad d \text{ VOL} \quad \dots (2.14)$$

in which the superscripts imply the operation $\frac{\partial f(x_i)}{\partial x_i} \Delta x_i$ with

respect to the applied loads and displacement variables.

Equation (2.14) provides a basis for a Newton-Raphson procedure for the analysis of non-linear equilibrium equations. If the last two terms on the RHS of equation (2.14) are assumed zero, then equation (2.14) represents the incremental equations of equilibrium. An alternative derivation of these equations by considering the LHS of equation (2.9) is presented by ref [58,59].

2.4 Application to a Pin-ended Strut

2.4.1 Linear Buckling Problem

Consider a pin-ended strut of length ℓ with a rectangular cross-section of width d and depth h shown in Fig 2.1. x, y and z represent the axis, u and w represent the displacements in x and z directions respectively. The first variation of the total potential energy can be expressed as:

$$\delta V = -p_i \delta r_i + \int \sigma_x \delta \epsilon_x \quad dx dy dz \quad (2.15)$$

where σ_x is the axial stress and $\delta \epsilon_x$ is the first variation of the axial strain. The relationship between σ_x and ϵ_x is given by:

$$\sigma_x = E\epsilon_x \quad (2.16)$$

where E is the Young's modulus of the material.

The non-linear expression for the strain in terms of displacements u and w is given by [2]:

$$\epsilon_x = u_x - zw_{xx} + \frac{1}{2} w_x^2 \quad (2.17)$$

where the suffices associated with u and w represent differentiation with respect to x . For stable equilibrium the first variation of the total potential energy is zero and corresponds to a minimum value, and the second variation, $\delta^2 V$, is a positive definite. The second variation of the total potential energy is given by:

$$\delta^2 V = -p_i \delta r_i + \int \sigma_x \delta^2 \epsilon_x + \frac{1}{2} \delta \sigma_x \delta \epsilon_x \quad dx dy dz \quad (2.18)$$

Using equation (2.16), the relationship between $\delta \sigma_x$ and $\delta \epsilon_x$ is given by:

$$\delta \sigma_x = E \delta \epsilon_x \quad (2.19)$$

Substituting equation (2.16) and equation (2.19) into equation (2.18) results in the following expression for $\delta^2 V$:

$$\delta^2 V = -p_i \delta r_i + E \int \epsilon_x \delta^2 \epsilon_x + \frac{1}{2} \delta \epsilon_x^2 \quad dx dy dz \quad (2.20)$$

Using equation (2.17) the expressions for the first variation of strain $\delta \epsilon_x$ and the second variation of strain $\delta^2 \epsilon_x$ are given by [57]:

$$\delta \epsilon_x = \delta u_x - z \delta w_{xx} + w_x \delta w_x \quad (2.21)$$

$$\delta^2 \epsilon_x = \frac{1}{2} \delta w_x^2 \quad (2.22)$$

In equation (2.22) it is assumed that the second variation of any linear function is zero and therefore $\delta^2 w$ and $\delta^2 u$ vanish.

Substituting equation (2.21) and equation (2.22) into equation (2.20) and omitting terms containing w , since prior to buckling w is zero, gives:

$$\delta^2 V = -p_i \delta^2 r_i + \frac{E}{2} \int \delta u_x (1.0) \delta u_x + 2 \delta u_x (-z) \delta w_{xx} + \delta w_{xx} (z^2) \delta w_{xx} + \delta w_x (u_x) \delta w_x \quad dx dy dz \quad (2.23)$$

For most problems r_i can be expressed as a linear function of the displacement variables, the first term in equation (2.23) vanishes.

Equation (2.23) can then be expressed in matrix form as:

$$\delta^2 V = \frac{1}{2} \int [\delta u_x, \delta w_{xx}] [D] \begin{bmatrix} \delta u_x \\ \delta w_{xx} \end{bmatrix} + [\delta w_x] [N] [\delta w_x] \quad dx dy dz \quad (2.24)$$

where the matrices $[D]$ and $[N]$ are given by:

$$[D] = E \begin{bmatrix} 1.0 & -z \\ -z & z^2 \end{bmatrix} \quad (2.25)$$

$$[N] = [Eu_x] \quad (2.26)$$

Prior and up to the point of buckling, the internal axial stress σ (tensile positive) can be related to the axial displacements u as:

$$\sigma = Eu_x \quad (2.27)$$

Equation (2.26) may then be written as:

$$[N] = [\sigma] \quad (2.28)$$

Now consider the strut to bend about an axis located at the edge 0 - y of the cross section as shown in Fig 2.2, the following integrals then apply in equation (2.24):

$$\int dy dz = A \quad (2.29)$$

$$\int z dy dz = A \frac{h}{2} \quad (2.30)$$

$$\int z^2 dy dz = I_{Oy} \quad (2.31)$$

where equations (2.30) and (2.31) represent the first and second moments of area of the cross-section about the edge 0 - y respectively.

For a strut buckling problem, the displacements δu and δw can be taken as:

$$\delta u = \delta A_u \cos\left(\frac{\pi x}{\ell}\right) \quad (2.32)$$

$$\delta w = \delta A_w \sin\left(\frac{\pi x}{\ell}\right) \quad (2.33)$$

Substituting equation (2.32) and equation (2.33) into equation (2.24) and using the integral equations (2.29) to (2.31), equation (2.24) may be written in a rearranged form as:

$$\begin{aligned} \delta^2 V = \frac{1}{2} \int \{ \delta \alpha \}^T [S_1]^T [D] [S_1] \{ \delta \alpha \} \\ + \{ \delta \alpha \}^T [S_2]^T [N] [S_2] \{ \delta \alpha \} \quad dx \end{aligned} \quad (2.34)$$

where

$$\{ \delta \alpha \} = \begin{Bmatrix} \delta A_u \\ \delta A_w \end{Bmatrix} \quad (2.35)$$

$$[S_1] = \begin{bmatrix} -\left(\frac{\pi}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) & 0 \\ 0 & -\left(\frac{\pi}{\ell}\right)^2 \sin\left(\frac{\pi x}{\ell}\right) \end{bmatrix} \quad (2.36)$$

$$[S_2] = \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{\pi}{\ell}\right) \cos\left(\frac{\pi x}{\ell}\right) \end{bmatrix} \quad (2.37)$$

$$[D] = E \begin{bmatrix} A & -A \frac{h}{2} \\ -A \frac{h}{2} & I_{Oy} \end{bmatrix} \quad (2.38)$$

$$[N] = \begin{bmatrix} 0 & 0 \\ 0 & A\sigma \end{bmatrix} \quad (2.39)$$

Equation (2.34) may be rewritten as:

$$\delta^2 V = \frac{1}{2} \{\delta\alpha\}^T \left[[K] + [K_g] \right] \{\delta\alpha\} \quad (2.40)$$

where

$$[K] = \int [S_1]^T [D] [S_1] \, dx \quad (2.41)$$

$$[K_g] = \int [S_2]^T [N] [S_2] \, dx \quad (2.42)$$

On carrying out the multiplications and integrations in equations (2.41) and (2.42), gives:

$$[K] = \begin{bmatrix} A \left(\frac{\pi}{\ell} \right)^2 \frac{\ell}{2} & -A \frac{h}{2} \left(\frac{\pi}{\ell} \right)^3 \frac{\ell}{2} \\ -A \frac{h}{2} \left(\frac{\pi}{\ell} \right)^3 \frac{\ell}{2} & I_0 \left(\frac{\pi}{\ell} \right)^4 \frac{\ell}{2} \end{bmatrix} \quad (2.43)$$

$$[K_g] = \begin{bmatrix} 0 & 0 \\ 0 & \sigma A \left(\frac{\pi}{\ell} \right)^2 \frac{\ell}{2} \end{bmatrix} \quad (2.44)$$

Taking $P = \sigma A$ in equation (2.44), equation (2.40) may then be written as:

$$\delta^2 V = \frac{1}{2} \{\delta\alpha\}^T \left[[K] + P[K_g] \right] \{\delta\alpha\} \quad (2.45)$$

where P is taken outside the second term in equation (2.40).

For compressive internal force P , equation (2.45) becomes:

$$\delta^2 V = \frac{1}{2} \{\delta\alpha\}^T \left[[K] - P[K_g] \right] \{\delta\alpha\} \quad (2.46)$$

Critical conditions are defined by the equation:

$$\delta^2 V = \frac{1}{2} \{\delta\alpha\}^T \left[[K] - P[K_g] \right] \{\delta\alpha\} = 0 \quad (2.47)$$

Equation (2.47) is a quadratic form in $\{\delta\alpha\}$ which changes from positive definite to semi-positive definite when the determinant of $[K] - P[K_g]$ vanishes [57]. Hence critical conditions occur when:

$$\text{DET} \left| [K] - P[K_g] \right| = 0 \quad (2.48)$$

In the present case where only two displacement functions are used to describe δu and δw . Equation (2.48) requires the evaluation of a 2×2 determinant which can be easily solved by hand, resulting in the following expression for the critical load:

$$P_{cr} = \frac{\pi^2 EI}{l^2} \left\{ I_{Oy} - A \left(\frac{h}{2} \right)^2 \right\} \quad (2.49)$$

If I is the second moment of area about an axis parallel to $O - y$ passing through the centroid of the cross-section, then I and I_{Oy} can be related through the use of the parallel axis theorem by the following equation:

$$I = I_{Oy} - A \left(\frac{h}{2} \right)^2 \quad (2.50)$$

Using equation (2.50), equation (2.49) becomes:

$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad (2.51)$$

Equation (2.51) represents the classical solution for the buckling load of an Euler strut.

2.4.2 Non-Linear Analysis

Consider an initially imperfect strut shown in Fig 2.3, where w_0 is the initial imperfection and P is the applied axial load, acting along the centroidal axes. If u and w are the total displacements, the expression for the axial strain given by equation (2.17) becomes:

$$\epsilon_x = u_x - z(w_{xx} - w_{0xx}) + \frac{1}{2} (w_x^2 - w_{0x}^2) \quad (2.52)$$

Equation (2.14) provides a basis for a Newton-Raphson method for the analysis of non-linear equilibrium equations and rewritten is here for convenience.

$$\begin{aligned} \int \sigma'_x \delta \epsilon_x + \sigma_x \delta \epsilon'_x \, dx dy dz &= p'_i \delta r_i + p_i \delta r_i \\ &- \int \sigma_x \delta \epsilon_x \, dx dy dz \end{aligned} \quad (2.53)$$

where σ_x is the axial stress and is related to the axial strain by the equation:

$$\sigma_x = E \epsilon_x \quad (2.54)$$

and therefore

$$\sigma'_x = E \epsilon'_x \quad (2.55)$$

Substituting for σ and σ' from equation (2.54) and equation (2.55) into equation (2.53) gives:

$$\begin{aligned} E \int \epsilon'_x \delta \epsilon_x + \epsilon_x \delta \epsilon'_x \, dx dy dz &= p'_i \delta r_i \\ &+ p_i \delta r_i - E \int \epsilon_x \delta \epsilon_x \, dx dy dz \end{aligned} \quad (2.56)$$

where $\delta \epsilon_x$, ϵ'_x , $\delta \epsilon'_x$ and p'_i are defined as follows:

$$\delta \epsilon_x = \delta u_x - z \delta w_{xx} + \delta w_x w_x \quad (2.57)$$

$$\epsilon'_x = \Delta w_x - z \Delta w_{xx} + w_x \Delta w_x \quad (2.58)$$

$$\delta \epsilon'_x = \delta w_x \Delta w_x \quad (2.59)$$

$$p'_i = \Delta p_i \quad (2.60)$$

Substituting equations (2.57) to (2.60) into equation (2.56) gives:

$$\begin{aligned} \int [\delta u_x, \delta w_{xx}, \delta w_x] [N] \begin{bmatrix} \Delta u_x \\ \Delta w_{xx} \\ \Delta w_x \end{bmatrix} dx dy dz &= \delta r_i \Delta p_i + \delta r_i p_i \\ - \int [\delta u_x, \delta w_{xx}, \delta w_x] [Q] dx dy dz & \end{aligned} \quad (2.61)$$

where the matrices $[N]$ and $[Q]$ are:

$$[N] = \begin{bmatrix} 1 & -z & w_x \\ -z & z^2 & -z w_x \\ w_x & -z w_x & u_x - z(w_{xx} - w_{0xx}) \\ & & + \frac{1}{2} (3w_x^2 - w_{0x}^2) \end{bmatrix} \quad (2.62)$$

$$[Q] = \begin{bmatrix} u_x - z(w_{xx} - w_{0xx}) + \frac{1}{2} (w_x^2 - w_{0x}^2) \\ -zu_x + z^2(w_{xx} - w_{0xx}) + \frac{z}{2} (w_x^2 - w_{0x}^2) \\ u_x w_x - zw_x(w_{xx} - w_{0xx}) + \frac{1}{2} w_x(w_x^2 - w_{0x}^2) \end{bmatrix} \quad (2.63)$$

Displacements u and w can be taken as follows:

$$u = A_{u1} \left(\frac{\ell}{2} - x \right) + A_{u2} \cos \left(\frac{\pi x}{\ell} \right) \quad (2.64)$$

$$w = A_w \sin \left(\frac{\pi x}{\ell} \right) \quad (2.65)$$

Substituting equations (2.64) and (2.65) into equation (2.61) gives:

$$\int \{\delta\alpha\}^T [S]^T [N] [S] \{\Delta\alpha\} \, dx dy dz = \{\delta\alpha\}^T \{\Delta P\} + \{\delta\alpha\}^T \{P\} - \int \{\delta\alpha\}^T [S] [Q] \, dx dy dz \quad (2.66)$$

where $\{\delta\alpha\}, \{\Delta\alpha\}, \{\Delta P\}, \{P\}$ and $[S]$ are given by:

$$\{\delta\alpha\} = \begin{Bmatrix} \delta A_{u1} \\ \delta A_{u2} \\ \delta A_w \end{Bmatrix} \quad (2.67)$$

$$\{\Delta\alpha\} = \begin{Bmatrix} \Delta A_{u1} \\ \Delta A_{u2} \\ \Delta A_w \end{Bmatrix} \quad (2.68)$$

$$\{\Delta P\} = \begin{Bmatrix} \ell \Delta P \\ 2 \Delta P \\ 2 \left(\frac{\pi}{\ell} \right) \frac{h}{2} \Delta P \end{Bmatrix} \quad (2.69)$$

$$\{P\} = \begin{Bmatrix} \ell P \\ 2 P \\ 2 \left(\frac{\pi}{\ell} \right) \frac{h}{2} P \end{Bmatrix} \quad (2.70)$$

$$[S] = \begin{bmatrix} -1 & -\left(\frac{\pi}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) & 0 \\ 0 & 0 & -\left(\frac{\pi}{\ell}\right)^2 \sin\left(\frac{\pi x}{\ell}\right) \\ 0 & 0 & \left(\frac{\pi}{\ell}\right) \cos\left(\frac{\pi x}{\ell}\right) \end{bmatrix} \quad (2.71)$$

On integration and simplifying, equation (2.66) becomes:

$$\{\delta\alpha\}^T [K_{inc}] \{\Delta\alpha\} = \{\delta\alpha\}^T \{\Delta P\} + \{\delta\alpha\}^T \{P\} - \{\delta\alpha\}^T [K_e] \quad (2.72)$$

where

$$[K_{inc}] = \int [S]^T [N] [S] \, dx dy dz \quad (2.73)$$

$$[K_e] = \int [S] [Q] \, dx dy dz \quad (2.74)$$

Since equation (2.72) is valid for all admissible variations in $\{\delta\alpha\}$ then:

$$[K_{inc}] \{\Delta\alpha\} = \{\Delta P\} + \{P\} - [K_e] \quad (2.75)$$

In which $[K_{inc}]$ is commonly known as the incremental or tangent stiffness matrix.

Equation (2.75) represents a set of linear simultaneous equations in the unknowns $\{\Delta\alpha\}$ and $\{\Delta P\}$. The solution procedure involves the application of a linear increment in displacements $\{\Delta\alpha\}$ or loads $\{\Delta P\}$, the equations are then solved for the unknown increments, the improved approximations of $\alpha + \Delta\alpha$ and $P + \Delta P$ is used as a starting point for further improvement cycle, and so on until equilibrium is satisfied which is evidenced by $\{P\} - [K_e]$ tending to zero.

Figure 2.4 shows some typical results for an imperfect strut with various degrees of imperfections.

2.5 General Equations for Large Deflections of Plates

Consider an element cut out of a plate by two pairs of planes parallel to the xz and yz planes as shown in Fig 2.5. The

element is acted upon by internal forces and moments as well as a distributed load q applied over its upper surface.

From consideration of force and moment equilibrium of the element in the x , y and z directions, the following equations are obtained:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad (2.76)$$

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \quad (2.77)$$

$$\frac{\partial M_{yx}}{\partial y} - \frac{\partial M_x}{\partial x} + Q_x = 0 \quad (2.78)$$

The expressions for the bending and twisting moments per unit length in terms of plate displacement w in the z direction are given by:

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2.79)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (2.80)$$

$$M_{xy} = -M_{yx} = D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \quad (2.81)$$

Differentiating equations (2.77) and (2.78) with respect to y and x respectively and substituting into equation (2.76) for the derivatives of q_x and q_y yields the following equation:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q \quad (2.82)$$

Consider now the forces acting on the middle plane of the plate element shown in Fig 2.5. Equilibrium of the element in the x and y directions yields:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (2.83)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (2.84)$$

The component of stress resultants N_x , N_y and N_{xy} in the z direction is given by:

$$N_x \cdot \frac{\partial^2 w}{\partial x^2} + N_y \cdot \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} \quad (2.85)$$

Adding equation (2.85) to the load $q dx dy$ acting on the element yields:

$$\begin{aligned} \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = & - \left(q + N_x \cdot \frac{\partial^2 w}{\partial x^2} + N_y \cdot \frac{\partial^2 w}{\partial y^2} \right. \\ & \left. + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} \right) \end{aligned} \quad (2.86)$$

Substituting into equation (2.86) from equations (2.79) to (2.81) for M_x , M_y and M_{xy} yields:

$$\begin{aligned} \nabla^4(w) &= \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \\ &= \frac{1}{D} \left[q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \end{aligned} \quad (2.87)$$

Assuming that q is zero and expressing the stress resultants N_x , N_y and N_{xy} by a stress function ϕ which automatically satisfies equations (2.83) and (2.84) such that:

$$\begin{aligned} \sigma_x = \frac{N_x}{t} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{N_y}{t} = \frac{\partial^2 \phi}{\partial x^2}, \\ \sigma_{xy} = \frac{N_{xy}}{t} = - \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \quad (2.88)$$

yields the following equation of equilibrium for the plate element in compact form:

$$\nabla^4(w) = \frac{t}{D} \left[\phi_{xx} w_{yy} + \phi_{yy} w_{xx} - 2 \phi_{xy} w_{xy} \right] \quad (2.89)$$

where x and y denote partial differentiation. Equation (2.89) was initially derived by Von Karman and is known as Von Karman's equilibrium equation.

Consider now the non-linear expressions for the membrane strains in terms of displacements u , v and w in the x , y and z directions respectively [2]:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad (2.90)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (2.91)$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \quad (2.92)$$

By differentiating these expressions it can be shown that:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \quad (2.93)$$

The two dimensional stress strain relationships for a linear elastic material are given by the equations:

$$\epsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) \quad (2.94)$$

$$\epsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad (2.95)$$

$$\epsilon_{xy} = \frac{1}{G} \sigma_{xy} = \frac{2(1 + \nu)}{E} \sigma_{xy} \quad (2.96)$$

Substituting equations (2.94) to (2.96) and equation (2.88) into equation (2.93) yields:

$$\begin{aligned} \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} &= \nabla^4(\phi) = \\ &= E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \right] \end{aligned} \quad (2.97)$$

Equation (2.97) was first derived by Von Karman and since it is derived from compatibility considerations it is known as Von Karman's compatibility equation.

2.6 Energy Equations for Plate Instability Analysis

A thin plate of thickness t , made of material having Young's modulus E and Poisson's ratio ν is shown in Fig 2.6. x, y, z represent the axis and u, v, w represent the corresponding displacements. The first variation of the total potential energy can be expressed as:

$$\delta V = -p_i \delta r_i + \int \sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_{xy} \delta \epsilon_{xy} \, d \text{VOL} \quad (2.98)$$

where σ_x , σ_y and σ_{xy} are the total internal stresses and $\delta \epsilon_x$, $\delta \epsilon_y$ and $\delta \epsilon_{xy}$ are the first variations of the corresponding strains.

The relationship between internal stresses and strains can be expressed in matrix form as:

$$\{\sigma\} = [D]\{\epsilon\} \quad (2.99)$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \quad \{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix}$$

$$[D] = \frac{E}{(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix}$$

... (2.100)

The non-linear expressions for the strains containing second order terms in displacements u, v and w are given by [60]:

$$\epsilon_x = u_x + \frac{1}{2} \left\{ u_x^2 + v_x^2 + w_x^2 \right\} - zw_{xx}$$

$$\epsilon_y = v_y + \frac{1}{2} \left\{ u_y^2 + v_y^2 + w_y^2 \right\} - zw_{yy}$$

$$\epsilon_z = u_y + v_x + \left\{ u_x u_y + v_x v_y + w_x w_y \right\} - 2zw_{xy}$$

... (2.101)

where the subscripts associated with u, v and w represent differentiation.

For stable equilibrium the first variation of the total potential energy is zero and corresponds to a minimum value, and the second variation $\delta^2 v$ is positive definite. The second variation of the total potential energy is given by:

$$\begin{aligned} \delta^2 v = & -p_i \delta^2 r_i + \int (\sigma_x \delta^2 \epsilon_x + \sigma_y \delta^2 \epsilon_y + \sigma_{xy} \delta^2 \epsilon_{xy}) \\ & + \frac{1}{2} (\delta \sigma_x \delta \epsilon_x + \delta \sigma_y \delta \epsilon_y + \delta \sigma_{xy} \delta \epsilon_{xy}) \quad d \text{ VOL} \\ & \dots (2.102) \end{aligned}$$

Since r_i can usually be expressed as a linear function of displacements u , v and w , the first term in equation (2.102) vanishes.

The expressions for the first and second variations of the strains used in equation (2.102) are given by:

$$\begin{aligned} \delta \epsilon_x &= \delta u_x + \delta u_x u_x + \delta v_x v_x + \delta w_x w_x - z \delta w_{xx} \\ \delta \epsilon_y &= \delta v_y + \delta u_y u_y + \delta v_y v_y + \delta w_y w_y - z \delta w_{yy} \\ \delta \epsilon_{xy} &= \delta u_y + \delta v_y + \delta u_x u_y + u_x \delta u_y \\ &\quad + \delta v_x v_y + v_x \delta v_y \\ &\quad + \delta w_x w_y + w_x \delta w_y - 2z \delta w_{xy} \\ &\dots (2.103) \end{aligned}$$

$$\begin{aligned} \delta^2 \epsilon_x &= \frac{1}{2} \left\{ \delta u_x^2 + \delta v_x^2 + \delta w_x^2 \right\} \\ \delta^2 \epsilon_y &= \frac{1}{2} \left\{ \delta u_y^2 + \delta v_y^2 + \delta w_y^2 \right\} \\ \delta^2 \epsilon_{xy} &= \delta u_x \delta u_y + \delta v_x \delta v_y + \delta w_x \delta w_y \\ &\dots (2.104) \end{aligned}$$

Using equation (2.99), the relationship between the first variation of stresses and first variation of strains is given by:

$$\{\delta \sigma\} = [D] \{\delta \epsilon\} \quad (2.105)$$

In equation (2.102) the integration over the volume $dVOL$ can be replaced by $dx dy dz$. The following integrals then apply:

$$\int_{-t/2}^{t/2} z \, dz = 0$$

$$\int_{-t/2}^{t/2} z^2 \, dz = \frac{t^3}{12}$$

... (2.106)

Substituting in equation (2.102) for $\delta\epsilon$ and $\delta^2\epsilon$ from equations (2.103), (2.104) and expressing stresses σ , $\delta\sigma$ in terms of strains from equations (2.99) and (2.105) yields:

$$\begin{aligned} \delta^2 V = & \frac{1}{2} \iint \left\{ N_x (\delta u_x^2 + \delta v_x^2 + \delta w_x^2) \right. \\ & + N_y (\delta u_y^2 + \delta v_y^2 + \delta w_y^2) \\ & + N_{xy} (\delta u_x \delta u_y + \delta v_x \delta v_y + \delta w_x \delta w_y) \left. \right\} dx dy \\ & + \frac{C}{2} \iint \left\{ \delta u_x^2 + \delta v_y^2 + 2\nu \delta u_x \delta v_y \right. \\ & \quad \left. + \frac{(1-\nu)}{2} (\delta u_y + \delta v_x)^2 \right\} dx dy \\ & + \frac{D}{2} \iint \left\{ \delta w_{xx}^2 + \delta w_{yy}^2 + 2\nu \delta w_{xx} \delta w_{yy} \right. \\ & \quad \left. + 2(1-\nu) \delta w_{xy}^2 \right\} dx dy \end{aligned}$$

... (2.107)

where $C = \frac{Et}{1-\nu^2}$ and $D = \frac{Et^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate.

N_x , N_y and N_{xy} are the membrane forces per unit length and are related to the strains as follows:

$$\{N\} = \int_{-t/2}^{t/2} \{\sigma\} dz = \int_{-t/2}^{t/2} [D]\{\epsilon\} dz \quad (2.108)$$

where

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad (2.109)$$

The relationship between the bending and twisting moments M_x , M_y , M_{xy} and strains (to be used later) is given by:

$$\{M\} = \int_{-t/2}^{t/2} z\{\sigma\} dz = \int_{-t/2}^{t/2} z[D]\{\epsilon\} dz$$

where

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

... (2.110)

Equation (2.107) can be represented in matrix form as:

$$\delta^2 V = \frac{1}{2} \iint \begin{bmatrix} \delta u_x, \delta u_y, \delta v_x, \delta v_y, \delta w_x, \delta w_y \end{bmatrix} \begin{bmatrix} N_x & N_{xy} & 0 & 0 & 0 & 0 \\ N_{xy} & N_y & 0 & 0 & 0 & 0 \\ 0 & 0 & N_x & N_{xy} & 0 & 0 \\ 0 & 0 & N_{xy} & N_y & 0 & 0 \\ 0 & 0 & 0 & 0 & N_x & N_{xy} \\ 0 & 0 & 0 & 0 & N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \delta u_x \\ \delta u_y \\ \delta v_x \\ \delta v_y \\ \delta w_x \\ \delta w_y \end{bmatrix} dx dy$$

$$+ \frac{1}{2} \iint \begin{bmatrix} \delta w_{xx}, \delta w_{yy}, 2\delta w_{xy} \end{bmatrix} \begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \delta w_{xx} \\ \delta w_{yy} \\ 2\delta w_{xy} \end{bmatrix} dx dy$$

[...(2.111) continued]

$$+ \frac{1}{2} \iint [\delta u_x, \delta u_y, \delta v_x, \delta v_y] \begin{bmatrix} C & 0 & 0 & \nu C \\ 0 & \frac{C}{2}(1-\nu) & \frac{C}{2}(1-\nu) & 0 \\ 0 & \frac{C}{2}(1-\nu) & \frac{C}{2}(1-\nu) & 0 \\ \nu C & 0 & 0 & C \end{bmatrix} \begin{bmatrix} \delta u_x \\ \delta u_y \\ \delta v_x \\ \delta v_y \end{bmatrix} dx dy \quad \dots (2.111)$$

Equation (2.111) can now be expressed in simplified form as:

$$\begin{aligned} \delta^2 V &= \frac{1}{2} \iint [S_1]^T [D] [S_1] dx dy \\ &+ \frac{1}{2} \iint [S_2]^T [N] [S_2] dx dy \end{aligned} \quad (2.112)$$

where

$$\begin{aligned} [S_1]^T &= [\delta u_x, \delta u_y, \delta v_x, \delta v_y, \delta w_{xx}, \delta w_{yy}, 2\delta w_{xy}] \\ [S_2]^T &= [\delta u_x, \delta u_y, \delta v_x, \delta v_y, \delta w_x, \delta w_y] \end{aligned} \quad \dots (2.113)$$

$$[D] = \begin{bmatrix} C & 0 & 0 & \nu C & 0 & 0 & 0 \\ 0 & \frac{C}{2}(1-\nu) & \frac{C}{2}(1-\nu) & 0 & 0 & 0 & 0 \\ 0 & \frac{C}{2}(1-\nu) & \frac{C}{2}(1-\nu) & 0 & 0 & 0 & 0 \\ \nu C & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & \nu D & 0 \\ 0 & 0 & 0 & 0 & \nu D & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{D}{2}(1-\nu) \end{bmatrix}$$

and

[... (2.114) continued]

$$[N] = \begin{bmatrix} N_x & N_{xy} & 0 & 0 & 0 & 0 \\ N_{xy} & N_y & 0 & 0 & 0 & 0 \\ 0 & 0 & N_x & N_{xy} & 0 & 0 \\ 0 & 0 & N_{xy} & N_y & 0 & 0 \\ 0 & 0 & 0 & 0 & N_x & N_{xy} \\ 0 & 0 & 0 & 0 & N_{xy} & N_y \end{bmatrix}$$

... (2.114)

Plate stability problems can be analysed by assuming a suitable displacement function δu , δv and δw . For a general plate problem δu , δv and δw can be expressed by a series of displacement functions as:

$$\delta u = \delta\alpha_{u1}f_1 + \delta\alpha_{u2}f_2 + \dots \text{ etc}$$

$$\delta v = \delta\alpha_{v1}g_1 + \delta\alpha_{v2}g_2 + \dots \text{ etc} \quad (2.115)$$

$$\delta w = \delta\alpha_{w1}h_1 + \delta\alpha_{w2}h_2 + \dots \text{ etc}$$

where f , g and h are functions of x and y . Substituting equation (2.115) into equation (2.112) and rearranging gives:

$$\begin{aligned} \delta^2 V &= \frac{1}{2} \iint \{\delta\alpha\}^T [SD_1]^T [D] [SD_1] \{\delta\alpha\} \, dx dy \\ &+ \frac{1}{2} \iint \{\delta\alpha\}^T [SD_2]^T [N] [SD_2] \{\delta\alpha\} \, dx dy \end{aligned} \quad (2.116)$$

where $\{\delta\alpha\}^T$, $[SD_1]$ and $[SD_2]$ are:

$$\{\delta\alpha\}^T = \{\delta\alpha_{u1}, \delta\alpha_{u2}, \dots, \delta\alpha_{v1}, \delta\alpha_{v2}, \dots, \delta\alpha_{w1}, \delta\alpha_{w2}, \dots\}$$

$$\begin{aligned}
 [SD_1] &= \begin{bmatrix} f_{1x}, & f_{2x}, \dots, & 0, & 0, \dots, & 0, & 0, \dots, \\ f_{1y}, & f_{2y}, \dots, & 0, & 0, \dots, & 0, & 0, \dots, \\ 0, & 0, \dots, & g_{1x}, & g_{2x}, \dots, & 0, & 0, \dots, \\ 0, & 0, \dots, & g_{1y}, & g_{2y}, \dots, & 0, & 0, \dots, \\ 0, & 0, \dots, & 0, & 0, \dots, & h_{1xx}, & h_{2xx}, \dots, \\ 0, & 0, \dots, & 0, & 0, \dots, & h_{1yy}, & h_{2yy}, \dots, \\ 0, & 0, \dots, & 0, & 0, \dots, & 2h_{1xy}, & 2h_{2xy}, \dots \end{bmatrix} \\
 [SD_2] &= \begin{bmatrix} f_{1x}, & f_{2x}, \dots, & 0, & 0, & 0, & 0, \dots, \\ f_{1y}, & f_{2y}, \dots, & 0, & 0, & 0, & 0, \dots, \\ 0, & 0, \dots, & g_{1x}, & g_{2x}, & 0, & 0, \dots, \\ 0, & 0, \dots, & g_{1y}, & g_{2y}, & 0, & 0, \dots, \\ 0, & 0, \dots, & 0, & 0, & h_{1x}, & h_{2x}, \dots, \\ 0, & 0, \dots, & 0, & 0, & h_{1y}, & h_{2y}, \dots \end{bmatrix} \\
 &\dots (2.117)
 \end{aligned}$$

where the subscripts x and y associated with f , g and h represent differentiation.

Critical conditions occur when $\delta^2 V = 0$, therefore:

$$\begin{aligned}
 &\frac{1}{2} \iint \{\delta\alpha\}^T [SD_1]^T [D] [SD_1] \{\delta\alpha\} \, dx dy \\
 &+ \iint \{\delta\alpha\}^T [SD_2]^T [N] [SD_2] \{\delta\alpha\} \, dx dy = 0 \quad (2.118)
 \end{aligned}$$

For proportional loading, values of membrane forces can be expressed as λN_x , λN_y , λN_{xy} where λ is a load factor and N_x , N_y and N_{xy} are a base set of forces. Therefore taking λ outside the second term in equation (2.118) and rearranging gives:

$$\begin{aligned}
 &\frac{1}{2} \{\delta\alpha\}^T \left[\iint [SD_1]^T [D] [SD_1] \, dx dy \right. \\
 &\quad \left. + \lambda \iint [SD_2]^T [N] [SD_2] \, dx dy \right] = 0 \quad (2.119)
 \end{aligned}$$

Equation (2.119) can be written as:

$$\frac{1}{2} \{\delta\alpha\}^T \left[[K] + \lambda [K_g] \right] \{\delta\alpha\} = 0 \quad (2.120)$$

where

$$[K] = \iint [SD_1]^T [D] [SD_1] \, dx dy \quad (2.121)$$

$$[K_g] = \iint [SD_2]^T [N] [SD_2] \, dx dy \quad (2.122)$$

For compressive membrane stresses (negative), equation (2.120) becomes:

$$\frac{1}{2} \{\delta\alpha\}^T \left[[K] - \lambda [K_g] \right] \{\delta\alpha\} = 0 \quad (2.123)$$

This is a quadratic form in $\{\delta\alpha\}$ which changes from positive definite to semi-positive definite when the determinant of $[K] - \lambda [K_g]$ vanishes [57] [61] . Hence, critical conditions occur when:

$$\text{Det} \left| [K] - \lambda [K_g] \right| = 0 \quad (2.124)$$

Equation (2.124) can be treated as an eigenvalue problem of the form:

$$[K]\{\delta\alpha\} - \lambda [K_g]\{\delta\alpha\} = 0 \quad (2.125)$$

where λ is an eigenvalue, $\{\delta\alpha\}$ is the corresponding eigenvector and $[K]$ and $[K_g]$ are square-symmetric matrices.

The lowest eigenvalue represents the critical load and the corresponding eigenvectors represent the relative values of the constants which define the magnitude of the displacements.

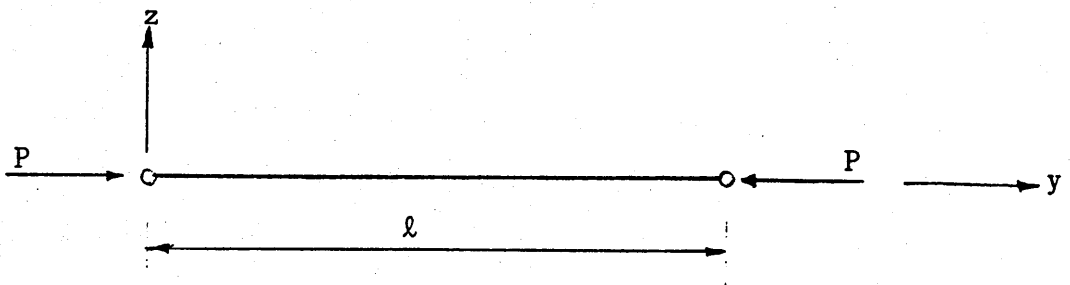


Fig 2.1 Euler Strut

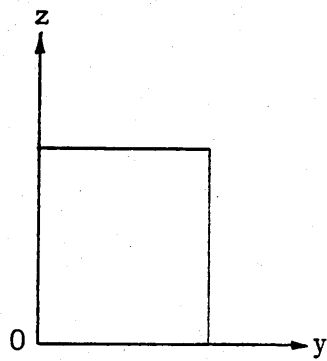


Fig 2.2 Strut Cross-section

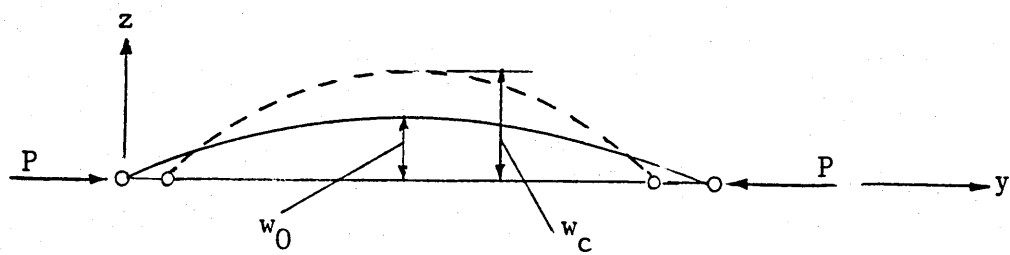


Fig 2.3 Imperfect Strut

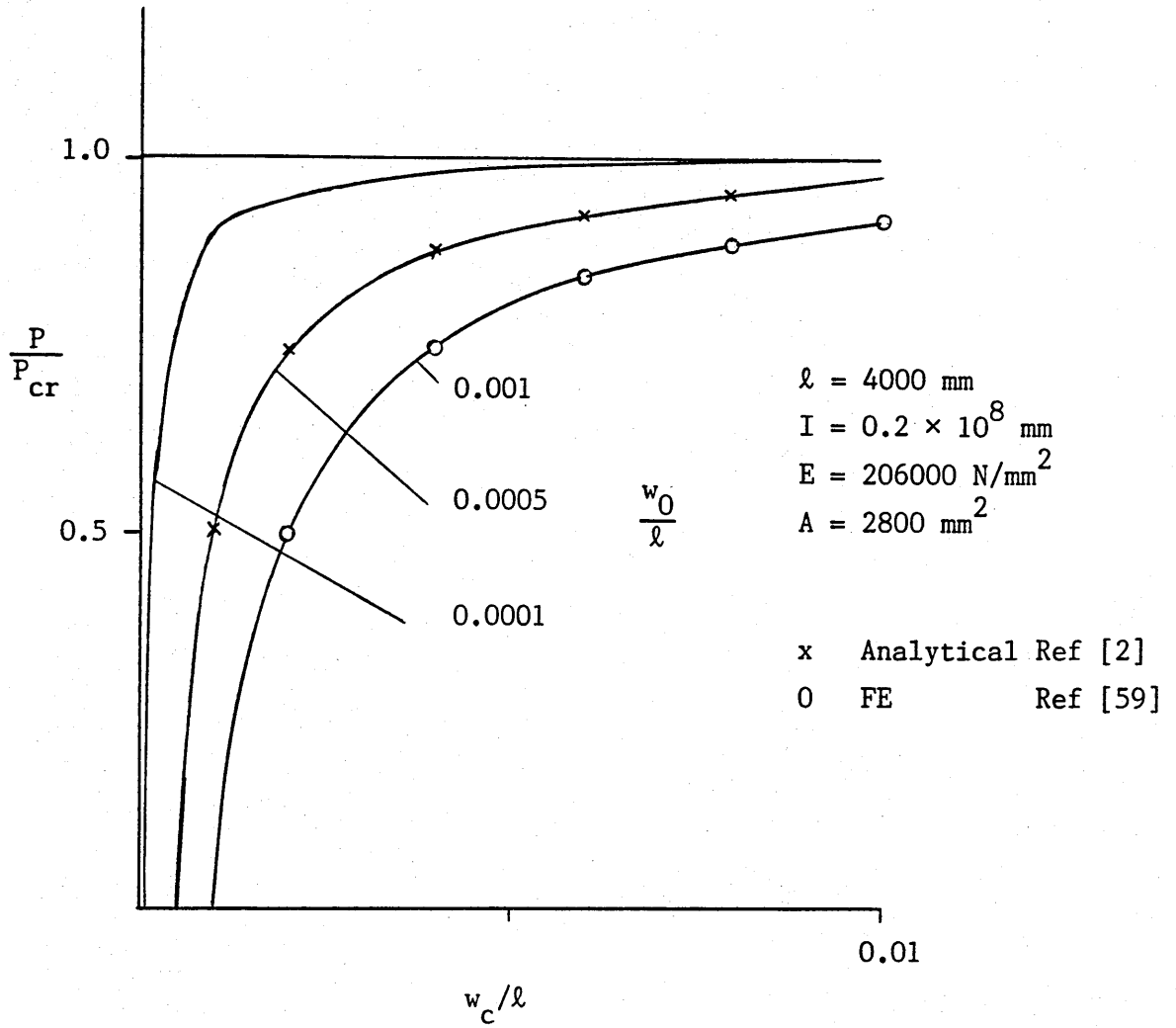


Fig 2.4 Imperfect Strut

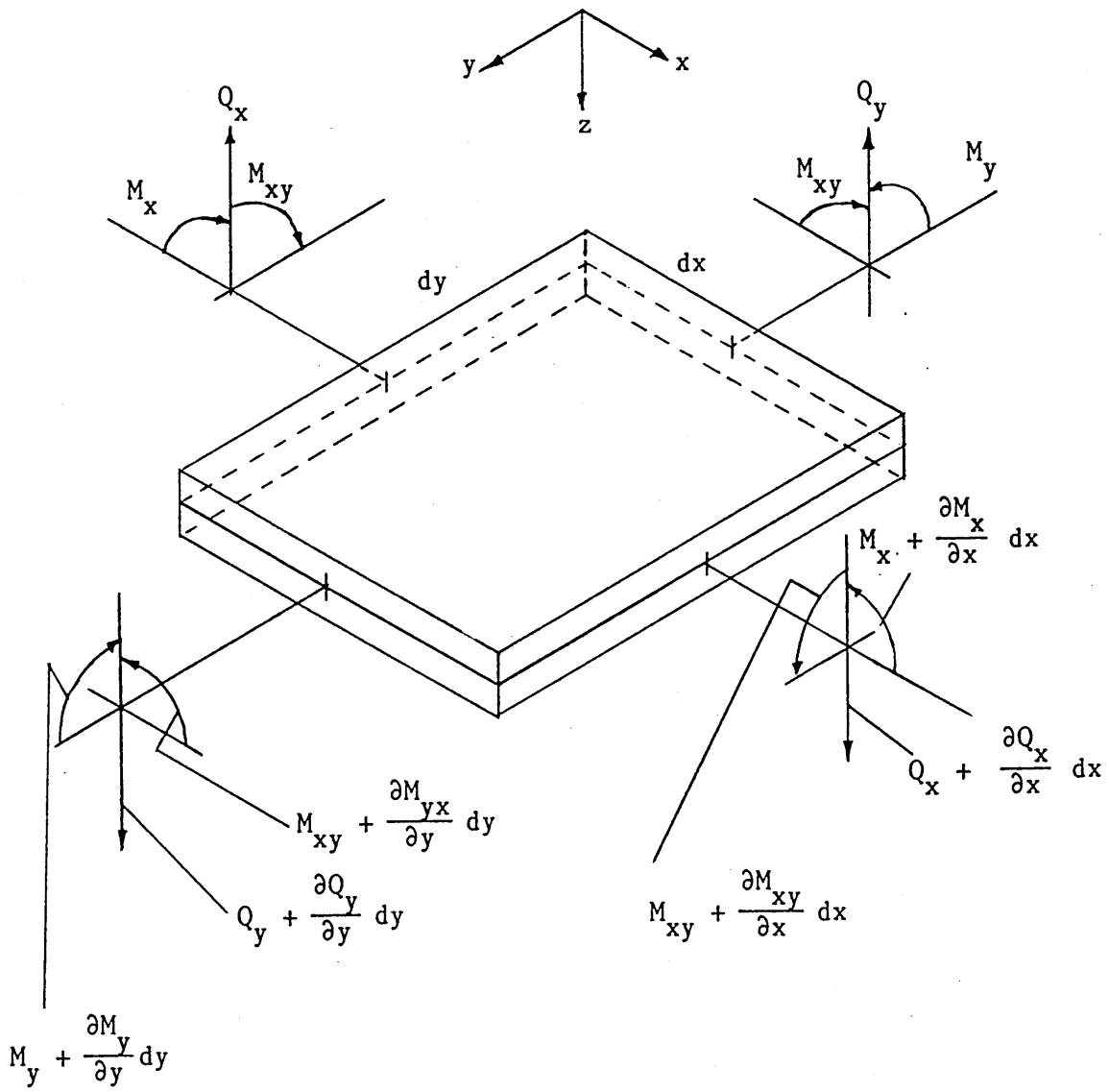


Fig 2.5

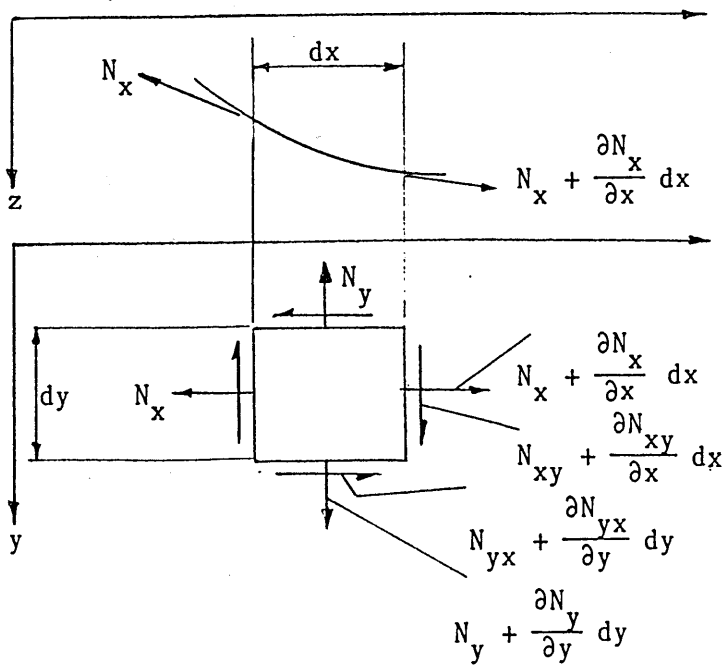


Fig 2.6

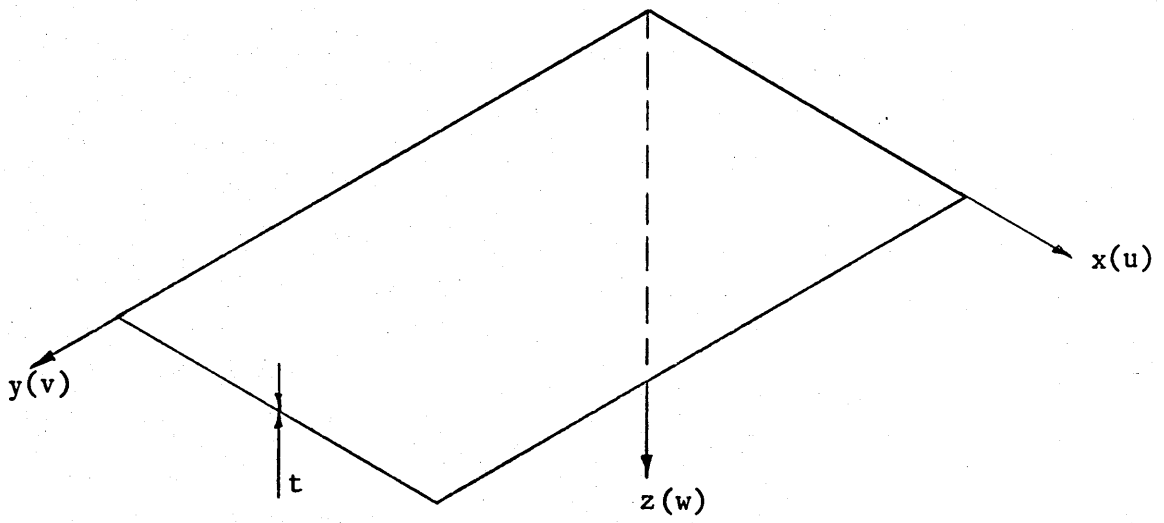


Fig 2.7 Thin plate - co-ordinate axis

3 INSTABILITY ANALYSIS OF STIFFENED PLATES

3.1 Introduction

Asymmetric and symmetric edge stiffened plate elements, such as panels with 'L' or 'T' section integral stiffeners are frequently used in aircraft structures. The elastic instability of these sections when used as load carrying members, can be initiated by any one of a number of independent buckling modes, for example, the local, the lateral or the overall mode. Coupling between the various modes, which reduces the buckling stress, is not taken into account by methods of analysis which investigate each possible mode independently. A method of analysis such as the energy method described in the previous chapter, can be used for determining the interaction between the various modes.

The lateral buckling mode is easily obtained for a symmetrically edge stiffened plate since the section properties - since the location of the neutral axis for in-plane bending of the stiffening plate, can be easily defined. The location of the neutral axis for in-plane bending of the stiffening plate is not so obvious for an asymmetrically edge stiffened plate, however, this problem may be overcome by providing additional in-plane degrees of freedom.

The analysis described in this chapter begins with the derivation of displacement functions, through the satisfaction of stress and displacement boundary conditions for an asymmetrically edge stiffened plate. The analysis is then extended to take account of symmetrically edge stiffened plates and later, infinite panels with stiffeners of deformable cross-sections. The analysis of a finite panel with stiffeners of rigid cross-sections is also included. The chapter finishes with a description of the energy approach and results for the instability analysis of stiffened plates.

3.2 Deflection Functions and Boundary Conditions

3.2.1 Asymmetrically Edge Stiffened Plate

Consider the asymmetrically edge stiffened plate shown in Fig 3.1. x, y_1, z_1 represent the local web axis and u_1, v_1, w_1 represent the corresponding displacements. The local axis system of the flange is represented by x, y_2, z_2 and the corresponding displacements are u_1, v_2, w_2 .

The displacements δw_1 and δw_2 can be expressed as a sum of two independent displacement functions as:

$$\delta w_i = \delta w_{ai} + \delta w_{bi} \quad (3.1)$$

where

$$\delta w_{ai} = Y_{ai}(y_i) \sin \frac{\pi x}{2b} \quad (3.2)$$

and

$$\delta w_{bi} = Y_{bi}(y_i) \sin \frac{\pi x}{2b} \quad (3.3)$$

in which

$$Y_{ai} = \sum_{n=1, N}^N Y_{ain}(y_i) \quad (3.4)$$

$$Y_{bi} = \sum_{m=1, M}^M Y_{bim}(y_i) \quad (3.5)$$

where M, N are the number of displacement terms used in the solution and i is a particular plate element. δw_{ai} defines a lateral buckling mode and δw_{bi} defines a local buckling mode.

The deflection functions Y_{ai} and Y_{bi} are given the following polynomial forms:

$$Y_{ain} = \sum_{j=1, J}^J \delta A_{nj} \left(\frac{y_1}{dw} \right)^{r_{nj}} \quad (3.6)$$

$$Y_{a2n} = \sum_{k=1, K}^K \delta C_{nk} \left(\frac{y_2}{df} \right)^{s_{nk}} \quad (3.7)$$

$$Y_{b1m} = \sum_{p=1, P}^P \delta B_{mp} \left(\frac{y_1}{dw} \right)^{g_{mp}} \quad (3.8)$$

$$Y_{b2m} = \sum_{q=1, Q}^Q \delta D_{mq} \left(\frac{y_2}{df} \right)^{h_{mq}} \quad (3.9)$$

where J , K and P , Q define the number of terms considered in the n^{th} and m^{th} polynomial. r_{nj} , s_{nk} and g_{mp} , h_{mq} are arbitrarily chosen integers subject to certain restrictions (see later). The coefficients δA_{nj} , δC_{nk} and δB_{mp} , δD_{mq} are used to satisfy displacement and stress boundary conditions at the plate edges in terms of δA_{n1} and δB_{m1} .

The boundary conditions to be satisfied by the displacement functions δw_{ai} and δw_{bi} with reference to Fig 3.1 are as follows:

at $y_1 = 0$

- (a) local deflection of the web is zero
- (b) there is moment equilibrium with the elastic restraint

at $y_1 = dw$ and $y_2 = 0$

- (c) local deflection of the flange is zero
- (d) the slopes are equal at the junction
- (e) there is moment equilibrium about the junction
- (f) there is equilibrium of shear at the junction

at $y_2 = df$

- (g) there is zero moment at the free edge
- (h) shear stress is zero.

An additional boundary condition for δw_{bi} is as follows:

at $y_1 = dw$ and $y_2 = 0$

- (i) the local deflection of the web at the web-flange junction is zero.

From equation (2.110), after integration with respect to z , the expression for the moment M_y is given by:

$$M_y = -D[w_{yy} + \nu w_{xx}] \quad (3.10)$$

The expression for shear V_y is given by [3]:

$$V_y = -D[w_{yyy} - (2 - \nu)w_{xxy}] \quad (3.11)$$

The rotational restraint R applied along the edge $y = 0$ of a plate is defined as [49]:

$$R = \frac{M_y}{w_y} \left(\frac{d}{D} \right) \quad (3.12)$$

where w_y and d represent the rotation and the width of the plate in the y direction respectively.

Considering first the displacement function δw_{ai} then conditions (a) to (h) are met by δw_{ai} by the satisfaction of the following equations:

$$(\delta w_{a1})_{y_1=0} = 0 \quad (3.13)$$

$$R_{an}(\delta w_{a1})_{y_1=0} = -d_w(\delta w_{a1_{yy}} + \nu \delta w_{a1_{xx}})_{y_1=0} \quad (3.14)$$

$$(\delta w_{a2})_{y_2=0} = 0 \quad (3.15)$$

$$(\delta w_{a1})_{y_1=dw} = (\delta w_{a2})_{y_2=0} \quad (3.16)$$

$$D_w(\delta w_{a1_{yy}} + \nu \delta w_{a1_{xx}})_{y_1=dw} = n_f D_f(\delta w_{a2_{yy}} + \nu \delta w_{a2_{xx}})_{y_2=0} \quad \dots (3.17)$$

$$D_w \left(\delta w_{a1_{yyy}} + (2 - \nu) \delta w_{a1_{yxx}} \right)_{y_1=dw} = n_f D_f \left(\delta w_{a2_{yyy}} + (2 - \nu) \delta w_{a2_{yxx}} \right)_{y_2=0} \quad (3.18)$$

$$D_f \left(\delta w_{a2_{yy}} + \nu \delta w_{a2_{xx}} \right)_{y_2=df} = 0 \quad (3.19)$$

$$D_f \left(\delta w_{a2_{yyy}} - (2 - \nu) \delta w_{a2_{yxx}} \right)_{y_2=df} = 0 \quad (3.20)$$

In equations (3.17) and (3.18) n_f represents the number of flanges at the web flange junction, ie $n_f = 1$ for asymmetrically stiffened plate and $n_f = 2$ for a symmetrically stiffened plate.

Using equations (3.2) and (3.4) and noting that equations (3.13) and (3.15) are automatically satisfied if it is assumed that

$r_{nj} = r_{nk} \neq 0$, then the above equations may be written as follows:

$$R_{an} \left(Y_{aln_y} \right)_{y_1=0} = -dw \left(Y_{aln_{yy}} \right)_{y_1=0} \quad (3.21)$$

$$\left(Y_{aln_y} \right)_{y_1=dw} = \left(Y_{a2n_y} \right)_{y_2=0} \quad (3.22)$$

$$D_w \left(Y_{aln_{yy}} - \nu \left(\frac{\pi}{2b} \right)^2 Y_{aln} \right)_{y_1=dw} = n_f D_f \left(Y_{a2n_y} \right)_{y_2=0} \quad (3.23)$$

$$\begin{aligned} D_w \left(Y_{aln_{yyy}} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{aln_y} \right)_{y_1=dw} &= \\ &= n_f D_f \left(Y_{a2n_{yyy}} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{a2n_y} \right)_{y_2=0} \end{aligned} \quad (3.24)$$

$$\left(Y_{a2n_{yy}} - \nu \left(\frac{\pi}{2b} \right)^2 Y_{a2n} \right)_{y_2=df} = 0 \quad (3.25)$$

$$\left(Y_{a2n_{yyy}} - 2(1 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{a2n_y} \right)_{y_2=df} = 0 \quad (3.26)$$

The boundary conditions at the plate edges for w_{ai} and the relevant equations (3.20) to (3.26) necessary to satisfy these conditions, also apply to w_{bi} . The additional boundary condition, for w_{bi} stated by condition (i) can be expressed as:

$$\left(Y_{blm} \right)_{y_1=dw} = 0 \quad (3.27)$$

3.2.2 Satisfaction of Boundary Conditions

If $r_{nj} = S_{nk} \neq 0$ results in six boundary conditions that need to be satisfied by w_{a1} and w_{a2} . Providing J and K in equations (3.6) and (3.7) are such that $J + K = 7$, the coefficients δA_{nj} and δC_{nk} can be expressed in terms of δA_{n1} through the satisfaction of the six boundary conditions.

Therefore if $J = 4$ and $K = 3$, Y_{aln} and Y_{a2n} can be expressed as:

$$Y_{aln} = \delta A_{n1} \left(\frac{y_1}{dw} \right) + \delta A_{n2} \left(\frac{y_1}{dw} \right)^2 + \delta A_{n3} \left(\frac{y_1}{dw} \right)^3 + \delta A_{n4} \left(\frac{y_1}{dw} \right)^{h_n} \quad (3.28)$$

$$Y_{a2n} = \delta C_{n1} \left(\frac{y_2}{df} \right) + \delta C_{n2} \left(\frac{y_2}{df} \right)^2 + \delta C_{n3} \left(\frac{y_2}{df} \right)^{g_n} \quad (3.29)$$

In which r_{n1} , r_{n2} , r_{n3} and S_{n1} , S_{n2} have been assigned fixed values and $r_{n4} = h_n$, $S_{n3} = g_n$.

Substituting equations (3.28) and (3.29) into equations (3.20) to (3.26) and solving for the coefficients δA_{n2} , δA_{n3} , δA_{n4} , δC_{n1} , δC_{n2} , δC_{n3} in terms of δA_{n1} (see Appendix 1), yields the final form of equations (3.28) and (3.29) as:

$$Y_{aln} = \delta A_{n1} \left[F_{n1} \left(\frac{y_1}{dw} \right) + F_{n2} \left(\frac{y_1}{dw} \right)^2 + F_{n3} \left(\frac{y_1}{dw} \right)^3 + F_{n4} \left(\frac{y_1}{dw} \right)^{f_n} \right] \quad (3.30)$$

$$Y_{a2n} = \delta A_{n1} \left[G_{n1} \left(\frac{y_2}{df} \right) + G_{n2} \left(\frac{y_2}{df} \right)^2 + G_{n3} \left(\frac{y_2}{df} \right)^{g_n} \right] \quad (3.31)$$

where F_1, F_2, F_3, F_4 and G_1, G_2, G_3 are constants involving f_n and g_n and are defined in Appendix 1.

The restrictions applied to f_n and g_n are:

$$f_n > 3, \quad g_n > 3$$

Similarly seven boundary conditions need to be satisfied by δw_{b1} and δw_{b2} , therefore if $P = Q = 4$ in equations (3.8) and (3.9), Y_{b1m} and Y_{b2m} can be expressed as:

$$Y_{b1m} = \delta B_{m1} \left(\frac{y_1}{dw} \right) + \delta B_{m2} \left(\frac{y_1}{dw} \right)^2 + \delta B_{m3} \left(\frac{y_1}{dw} \right)^3 + \delta B_{m4} \left(\frac{y_1}{dw} \right)^{h_m} \quad (3.32)$$

$$Y_{b2m} = \delta D_{m1} \left(\frac{y_2}{df} \right) + \delta D_{m2} \left(\frac{y_2}{df} \right)^2 + \delta D_{m3} \left(\frac{y_2}{df} \right)^3 + \delta D_{m4} \left(\frac{y_2}{df} \right)^{j_m} \quad (3.33)$$

After satisfaction of the appropriate boundary conditions, yields the following equations for Y_{b1m} and Y_{b2m} (see Appendix 1):

$$Y_{b1m} = \delta B_{m1} \left[H_{m1} \left(\frac{y_1}{dw} \right) + H_{m2} \left(\frac{y_1}{dw} \right)^2 + H_{m3} \left(\frac{y_1}{dw} \right)^3 + H_{m4} \left(\frac{y_1}{dw} \right)^{h_m} \right] \quad (3.34)$$

$$Y_{b2m} = \delta B_{m1} \left[J_{m1} \left(\frac{y_2}{df} \right) + J_{m2} \left(\frac{y_2}{df} \right)^2 + J_{m3} \left(\frac{y_2}{df} \right)^3 + J_{m4} \left(\frac{y_2}{df} \right)^{j_m} \right] \quad (3.35)$$

The restrictions applied to h_m and j_m are:

$$h_m > 3 \quad \text{and} \quad j_m > 3$$

Using equations (3.2) and (3.3), displacements δw_{ai} and δw_{bi} take the following form:

$$\delta w_{a1} = \sum_{n=1, N}^N \delta A_n \left[F_{n1} \left(\frac{y_1}{dw} \right) + F_{n2} \left(\frac{y_1}{dw} \right)^2 + F_{n3} \left(\frac{y_1}{dw} \right)^3 + F_{n4} \left(\frac{y_1}{dw} \right)^{f_n} \right] \sin \frac{\pi x}{2b} \quad \dots (3.36)$$

$$\delta w_{a2} = \sum_{n=1, N}^N \delta A_n \left[G_{n1} \left(\frac{y_2}{df} \right) + G_{n2} \left(\frac{y_2}{df} \right)^2 + G_{n3} \left(\frac{y_2}{df} \right)^{g_n} \right] \sin \frac{\pi x}{2b} \quad \dots (3.37)$$

$$\delta w_{b1} = \sum_{m=1, M}^N \delta B_m \left[H_{m1} \left(\frac{y_1}{dw} \right) + H_{m2} \left(\frac{y_1}{dw} \right)^2 + H_{m3} \left(\frac{y_1}{dw} \right)^3 + H_{m4} \left(\frac{y_1}{dw} \right)^{h_m} \right] \sin \frac{\pi x}{2b} \quad \dots (3.38)$$

$$\delta w_{b2} = \sum_{n=1, M}^M \delta B_m \left[J_{m1} \left(\frac{y_2}{df} \right) + J_{m2} \left(\frac{y_2}{df} \right)^2 + J_{m3} \left(\frac{y_2}{df} \right)^3 + J_{m4} \left(\frac{y_2}{df} \right)^{J_m} \right] \sin \frac{\pi x}{2b} \quad \dots (3.39)$$

Having solved for the boundary conditions the total displacement fields for the various plate elements will now be defined.

The total out of plane displacement for the web δw_1 is given by:

$$\delta w_1 = \delta w_{a1} + \delta w_{b1} \quad (3.40)$$

The inplane displacement δv_1 can be taken to represent the overall mode of buckling and is defined as:

$$\delta v_1 = \delta A_v \sin \frac{\pi x}{2b} \quad (3.41)$$

The axial displacement δu_1 can be taken as:

$$\delta u_1 = \delta A_{u1} \cos \frac{\pi x}{2b} - y_1 \delta v_{1x} \quad (3.42)$$

The second term in equation (3.42) arises from the assumption that transverse sections which are plane before bending remain plane after bending - see Fig 3.2.

The total out of plane displacement of the flange now becomes:

$$\delta w_2 = -\delta v_2 + \delta w_{a2} + \delta w_{b2} \quad (3.43)$$

The inplane displacement δv_2 is defined by:

$$\delta v_2 = \left(\delta w_{a1} \right)_{y_1=dw} \quad (3.44)$$

and hence the axial displacement δu_2 is given by:

$$\delta u_2 = \delta A_{u1} \cos \frac{\pi x}{2b} - dw \delta v_{1x} - y_2 \left(\delta w_{a1} \right)_{y_1=dw} \quad (3.45)$$

Having stated earlier the indices f_n , g_n , h_m and j_m in the expression for δw_{ai} and δw_{bi} are arbitrary but subject to certain restrictions, then a variety of deflected forms can be specified for the section simply by choosing a different value for indices.

3.2.3 Semi-Infinite Panel

Consider a panel which is assumed to be infinitely wide and of finite length. The panel is stiffened by stiffeners of deformable cross-sections at constant pitch. If the axially applied compressive stress is constant then it can be assumed that the buckling mode is sinusoidal in the longitudinal direction. The analysis is confined to a part of the structure comprising two stiffeners and two skins, over a half wavelength of $2b$ as shown in Fig 3.3. The out of plane displacements are assumed to be composed of three independent displacement functions so that:

$$\delta w_4 = \delta w_{a4} + \delta w_{b4} + \delta w_{c4} \quad (3.46)$$

$$\delta w_5 = \delta w_{a5} + \delta w_{b5} + \delta w_{c5} \quad (3.47)$$

where it is assumed that:

$$\delta w_{a4} = \delta w_{a5} \quad (3.48)$$

$$\delta w_{b4} = \delta w_{b5} \quad (3.49)$$

In equations (3.46), (3.47), δw_{a4} and δw_{a5} are associated with the lateral buckling mode of the stiffeners developed in sections 3.2.1 and 3.2.2 and, δw_{b4} and δw_{b5} are associated with the local buckling mode developed in sections 3.2.1 and 3.2.2. These modes for a panel are depicted in Fig 3.4a and Fig 3.4b. The displacements δw_{c4} and δw_{c5} are associated with an alternating up and down stiffener mode shown in Fig 3.4c. In the following development of the skin displacement functions δw_{a4} and δw_{b5} , each skin is treated as an infinitely long plate restrained elastically along the longitudinal edges. The skins are then connected to stiffeners through the satisfaction of stress and displacement boundary conditions. The displacement functions of the stiffeners can be

taken as those developed in sections 3.2.1 and 3.2.2 for asymmetrically and symmetrically edge stiffened plates.

The boundary conditions along the longitudinal edges of the skin plate for displacements δw_{a4} and δw_{b4} , δw_{b5} are as follows:

- (a) Out of plane displacement is zero at $y_4 = y_5 = 0$ and $y_4 = y_5 = ds$.
- (b) There is moment equilibrium with the applied elastic restraint at $y_4 = y_5 = 0$ and $y_4 = y_5 = ds$.

The additional condition for δw_{4b} and δw_{5b} is as follows:

- (c) The slope is zero at $y_4 = y_5 = \frac{ds}{2}$.

Consider first the displacement δw_{a4} and following the procedure in section 3.2.1 δw_{a4} can be expressed as:

$$\delta w_{a4} = Y_{a4}(y_4) \sin \frac{\pi x}{2b} \quad (3.50)$$

where

$$Y_{a4}(y_4) = \sum_{n=1}^{n=N} Y_{a4n}(y_4) \quad (3.51)$$

and

$$Y_{a4n} = \sum_{\ell=1, L}^L E_{ne} \left(\frac{y_4}{ds} \right)^{k_{ne}} \quad (3.52)$$

Boundary conditions (a) and (b) can be expressed as:

$$\left(\delta w_{a4} \right)_{y_4=0} = 0 \quad (3.53)$$

$$\left(\delta w_{a4} \right)_{y_4=ds} = 0 \quad (3.54)$$

$$D_s \frac{R_{1n}}{ds} \left(\delta w_{a4} \right)_{y_4=0} = \left(\delta w_{a4} + \nu \delta w_{a4_{xx}} \right)_{y_4=0} \quad (3.55)$$

$$D_s \frac{R_{2n}}{ds} \left(\delta w_{a4y} \right)_{y_4=ds} = \left(\delta w_{a4yy} + \nu \delta w_{a4xx} \right)_{y_4=ds} \quad (3.56)$$

If $k_{ne} \neq 0$ then equation (3.53) is automatically satisfied, then using equations (3.50) and (3.51) equations (3.54), (3.55) and (3.56) can be expressed as:

$$\left(Y_{a4n} \right)_{y_4=ds} = 0 \quad (3.57)$$

$$D_s \frac{R_{1n}}{ds} \left(Y_{a4ny} \right)_{y_4=0} = \left(Y_{a4nyy} \right)_{y_4=0} \quad (3.58)$$

$$D_s \frac{R_{2n}}{ds} \left(Y_{a4ny} \right)_{y_4=ds} = \left(Y_{a4nyy} - \nu \left(\frac{\pi}{2b} \right)^2 Y_{a4n} \right)_{y_4=ds} \quad (3.59)$$

The boundary conditions between the skin and stiffeners are as follows:

- (d) Slopes are equal at skin stiffener junctions.
- (e) There is shear equilibrium about the skin stiffener junctions.

The boundary conditions (d) and (e) can be written as:

$$\left(Y_{a1ny} \right)_{y_1=0} = \left(Y_{a4ny} \right)_{y_4=0} \quad (3.60)$$

$$\left(Y_{a6ny} \right)_{y_6=0} = \left(Y_{a4ny} \right)_{y_4=ds} \quad (3.61)$$

$$\begin{aligned} \frac{D_w}{D_s} \left(Y_{a1nyyy} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{a1ny} \right)_{y_1=0} &= \\ &= -2 \left(Y_{a4nyyy} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{a4ny} \right)_{y_4=0} \\ &\dots (3.62) \end{aligned}$$

$$\begin{aligned}
\frac{D}{D_s} \left(Y_{a6n_{yyy}} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{a6n_y} \right)_{y_6=0} &= \\
&= 2 \left(Y_{a4n_{yyy}} - (2 - \nu) \left(\frac{\pi}{2b} \right)^2 Y_{4an_y} \right)_{y_4=ds} \quad (3.63)
\end{aligned}$$

The deflection function Y_{a4n} is given the following form:

$$\begin{aligned}
Y_{a4n} = & \left[\delta E_{n1} \left(\frac{y_4}{ds} \right) + \delta E_{n2} \left(\frac{y_4}{ds} \right)^2 + \delta E_{n3} \left(\frac{y_4}{ds} \right)^3 \right. \\
& \left. + \delta E_{n4} \left(\frac{y_4}{ds} \right)^{a_n} + \delta E_{n5} \left(\frac{y_4}{ds} \right)^{b_n} + \delta E_{n6} \left(\frac{y_4}{ds} \right)^{c_n} \right] \\
& \dots (3.64)
\end{aligned}$$

The displacements Y_{aln} from section 3.2.1, equation (3.30) is given by:

$$Y_{aln} = \delta A_n \left[F_{n1} \left(\frac{y_1}{dw} \right) + F_{n2} \left(\frac{y_1}{dw} \right)^2 + F_{n3} \left(\frac{y_1}{dw} \right)^3 + F_{n4} \left(\frac{y_1}{dw} \right)^{f_n} \right] \quad (3.65)$$

therefore displacement Y_{a6n} can be taken as:

$$\begin{aligned}
Y_{a6n} = & \delta L_n \left[F_{n1} \left(\frac{y_6}{dw} \right) + F_{n2} \left(\frac{y_6}{dw} \right)^2 + F_{n3} \left(\frac{y_6}{dw} \right)^3 + F_{n4} \left(\frac{y_6}{dw} \right)^{f_n} \right] \\
& \dots (3.66)
\end{aligned}$$

Therefore using equations (3.64) to (3.66) and solving equations (3.57) to (3.63), the coefficients $\delta E_{n1} - \delta E_{n6}$ and δL_n can all be expressed in terms of δA_n (see Appendix 2) yielding the final form for Y_{a4n} and Y_{a6n} as follows:

$$\begin{aligned}
Y_{a4n} = & \delta A_n \left[K_{n1} \left(\frac{y_4}{ds} \right) + K_{n2} \left(\frac{y_4}{ds} \right)^2 + K_{n3} \left(\frac{y_4}{ds} \right)^3 \right. \\
& \left. + K_{n4} \left(\frac{y_4}{ds} \right)^{a_n} + K_{n5} \left(\frac{y_4}{ds} \right)^{b_n} + K_{n6} \left(\frac{y_4}{ds} \right)^{c_n} \right] \quad (3.67)
\end{aligned}$$

$$Y_{a6n} = \delta A_n \left[\bar{F}_{n1} \left(\frac{y_6}{dw} \right) + \bar{F}_{n2} \left(\frac{y_6}{dw} \right)^2 + \bar{F}_{n3} \left(\frac{y_6}{dw} \right)^3 + \bar{F}_{n4} \left(\frac{y_6}{dw} \right)^f \right] \quad (3.68)$$

Equations (3.57) to (3.63) also apply to Y_{b4m} with the additional condition (c) which can be expressed as:

$$\left(Y_{b4m} \right)_{y_5 = \frac{ds}{2}} = 0 \quad (3.69)$$

the form of Y_{b4m} can be taken as:

$$Y_{b4m} = \left[\delta P_{m1} \left(\frac{y_4}{ds} \right) + \delta P_{m2} \left(\frac{y_4}{ds} \right)^2 + \delta P_{m3} \left(\frac{y_4}{ds} \right)^3 + \delta P_{m4} \left(\frac{y_4}{ds} \right)^d_m + \delta P_{m5} \left(\frac{y_4}{ds} \right)^l_m + \delta P_{m6} \left(\frac{y_4}{ds} \right)^k_m + \delta P_{m7} \left(\frac{y_4}{ds} \right)^l_m \right] \quad (3.70)$$

The displacement Y_{b1m} from equation (3.34) is given by:

$$Y_{b1m} = \delta B_m \left[H_{m1} \left(\frac{y_1}{dw} \right) + H_{m2} \left(\frac{y_1}{dw} \right)^2 + H_{m3} \left(\frac{y_1}{dw} \right)^3 + H_{m4} \left(\frac{y_1}{dw} \right)^h_m \right] \quad (3.71)$$

therefore Y_{b6m} can be expressed as:

$$Y_{b6m} = \delta R_m \left[H_{m1} \left(\frac{y_6}{dw} \right) + H_{m2} \left(\frac{y_6}{dw} \right)^2 + H_{m3} \left(\frac{y_6}{dw} \right)^3 + H_{m4} \left(\frac{y_6}{dw} \right)^h_m \right] \quad (3.72)$$

Using equations (3.70) to (3.72) and solving equations (3.57) to (3.63) and (3.69) yields the final form of Y_{b4m} and Y_{b6m} as follows (see Appendix 2):

$$Y_{b4m} = \delta B_m \left[I_{m1} \left(\frac{y_4}{ds} \right) + I_{m2} \left(\frac{y_4}{ds} \right)^2 + I_{m3} \left(\frac{y_4}{ds} \right)^3 + I_{m4} \left(\frac{y_4}{ds} \right)^d_m + I_{m5} \left(\frac{y_4}{ds} \right)^e_m + I_{m6} \left(\frac{y_4}{ds} \right)^k_m + I_{m7} \left(\frac{y_4}{ds} \right)^l_m \right] \quad \dots (3.73)$$

$$Y_{b6m} = \delta B_m \left[\bar{H}_{m1} \left(\frac{y_4}{dw} \right) + \bar{H}_{m2} \left(\frac{y_4}{dw} \right)^2 + \bar{H}_{m3} \left(\frac{y_4}{dw} \right)^3 + \bar{H}_{m4} \left(\frac{y_4}{dw} \right)^{h_m} \right] \quad (3.74)$$

The restrictions imposed upon indices a_n , b_n and c_n are as follows:

$$a_n > 3, \quad b_n > 3, \quad c_n > 3 \quad (3.75)$$

The restrictions imposed upon the indices d_m , e_m , k_m , ℓ_m are as follows:

$$d_m > 3, \quad e_m > 3, \quad k_m > 3, \quad \ell_m > 3 \quad (3.76)$$

The boundary conditions for δw_{c4} are as follows:

$$\left(\delta w_{c4} \right)_{y_4=0} = 0 \quad (3.77)$$

$$\left(\delta w_{c4} \right)_{y_4=0, ds} = 0 \quad (3.78)$$

$$\left(\delta w_{c4} \right)_{yy_{y_4=0, ds}} = 0 \quad (3.79)$$

Expressing δw_{c4} as:

$$\delta w_{c4} = Y_{c4}(y_4) \sin \frac{\pi x}{2b} \quad (3.80)$$

where

$$Y_{c4} = \sum_{r=1, R}^R Y_{c4r}(y_4) \quad (3.81)$$

$$Y_{c4r} = \sum_{s=1, S}^S D_{rs} \left(\frac{y_4}{ds} \right)^{g_{rs}} \quad (3.82)$$

Boundary condition (3.77) is automatically satisfied if $g_{rs} \neq 0$, boundary conditions (3.78) and (3.79) can then be expressed as:

$$\left(Y_{c4r} \right)_{yy_{y_4=0, ds}} = 0 \quad (3.83)$$

$$\left(Y_{c4r} \right)_{yy=0,ds} = 0 \quad (3.84)$$

Expressing Y_{c4r} as:

$$Y_{c4r} = \left[\delta D_{r1} \left(\frac{y_4}{ds} \right)^{g_{r1}} + \delta D_{r2} \left(\frac{y_4}{ds} \right)^{g_{r2}} + \delta D_{r3} \left(\frac{y_4}{ds} \right)^{g_{r3}} \right] \quad (3.85)$$

Using equation (3.85) to solve equations (3.83) and (3.84) coefficients δD_{r2} , δD_{r3} can be expressed in terms of δD_{r1} resulting in the following expression for Y_{c4r} (see Appendix 2):

$$Y_{c4r} = \delta D_r \left[U_{r1} \left(\frac{y_4}{ds} \right)^{g_r} + U_{r2} \left(\frac{y_4}{ds} \right)^{h_r} + U_{r3} \left(\frac{y_4}{ds} \right)^{j_r} \right] \quad (3.86)$$

where $\delta D_r = \delta D_{r1}$.

The final expression for displacements δw_{ai} , δw_{bi} and δw_{ci} for the various plate elements are as follows:

$$\delta w_{a1} = \sum_{n=1, N}^N \delta A_n \left[F_{n1} \left(\frac{y_1}{dw} \right) + F_{n2} \left(\frac{y_1}{dw} \right)^2 + F_{n3} \left(\frac{y_1}{dw} \right)^3 + F_{n4} \left(\frac{y_1}{dw} \right)^n \right] \sin \frac{\pi x}{2b} \quad \dots (3.87)$$

$$\delta w_{a2} = \sum_{n=1, N}^N \delta A_n \left[G_{n1} \left(\frac{y_2}{df} \right) + G_{n2} \left(\frac{y_2}{df} \right)^2 + G_{n3} \left(\frac{y_2}{df} \right)^n \right] \sin \frac{\pi x}{2b} \quad \dots (3.88)$$

$$\delta w_{a3} = \delta w_{a2} \quad (3.89)$$

$$\begin{aligned} \delta w_{a4} = \sum_{n=1, N}^N \delta A_n \left[K_{n1} \left(\frac{y_4}{ds} \right) + K_{n2} \left(\frac{y_4}{ds} \right)^2 + K_{n3} \left(\frac{y_4}{ds} \right)^3 \right. \\ \left. + K_{n4} \left(\frac{y_4}{ds} \right)^{a_n} + K_{n5} \left(\frac{y_4}{ds} \right)^{b_n} + K_{n6} \left(\frac{y_4}{ds} \right)^{c_n} \right] \sin \frac{\pi x}{2b} \quad \dots (3.90) \end{aligned}$$

$$\delta w_{a5} = \delta w_{a4}$$

$$\delta w_{a6} = \sum_{n=1, N}^N \delta A_n \left[\bar{F}_{n1} \left(\frac{y_6}{dw} \right) + \bar{F}_{n2} \left(\frac{y_6}{dw} \right)^2 + \bar{F}_{n3} \left(\frac{y_6}{dw} \right)^3 + \bar{F}_{n4} \left(\frac{y_6}{dw} \right)^{f_n} \right] \sin \frac{\pi x}{2b} \quad (3.91)$$

$$\delta w_{a7} = \sum_{n=1, N}^N \delta A_n \left[\bar{G}_{n1} \left(\frac{y_7}{df} \right) + \bar{G}_{n2} \left(\frac{y_7}{df} \right)^2 + \bar{G}_{n3} \left(\frac{y_7}{df} \right)^{g_n} \right] \sin \frac{\pi x}{2b} \quad (3.92)$$

$$\delta w_{a8} = \delta w_{a7}$$

$$\delta w_{b1} = \sum_{m=1, M}^M \delta B_m \left[H_{m1} \left(\frac{y_1}{dw} \right) + H_{m2} \left(\frac{y_1}{dw} \right)^2 + H_{m3} \left(\frac{y_1}{dw} \right)^3 + H_{m4} \left(\frac{y_1}{dw} \right)^{h_m} \right] \sin \frac{\pi x}{2b} \quad (3.93)$$

$$\delta w_{b2} = \sum_{m=1, M}^M \delta B_m \left[J_{m1} \left(\frac{y_2}{df} \right) + J_{m2} \left(\frac{y_2}{df} \right)^2 + J_{m3} \left(\frac{y_2}{df} \right)^3 + J_{m4} \left(\frac{y_2}{df} \right)^{j_m} \right] \sin \frac{\pi x}{2b} \quad (3.94)$$

$$\delta w_{b3} = \delta w_{b2} \quad (3.95)$$

$$\delta w_{b4} = \sum_{m=1, M}^M \delta B_m \left[I_{m1} \left(\frac{y_4}{ds} \right) + I_{m2} \left(\frac{y_4}{ds} \right)^2 + I_{m3} \left(\frac{y_4}{ds} \right)^3 + I_{m4} \left(\frac{y_4}{ds} \right)^{d_m} + I_{m5} \left(\frac{y_4}{ds} \right)^{e_m} + I_{m6} \left(\frac{y_4}{ds} \right)^{k_m} + I_{m7} \left(\frac{y_4}{ds} \right)^{l_m} \right] \sin \frac{\pi x}{2b} \quad (3.96)$$

$$\delta w_{b5} = \delta w_{b4} \quad (3.97)$$

$$\delta w_{b6} = \sum_{m=1, M}^M \delta B_m \left[\bar{H}_{m1} \left(\frac{y_6}{dw} \right) + \bar{H}_{m2} \left(\frac{y_6}{dw} \right)^2 + \bar{H}_{m3} \left(\frac{y_6}{dw} \right)^3 + \bar{H}_{m4} \left(\frac{y_6}{dw} \right)^{h_m} \right] \sin \frac{\pi x}{2b} \quad (3.98)$$

$$\delta w_{b7} = \sum_{m=1, M}^M \delta B_m \left[\bar{J}_{m1} \left(\frac{y_7}{df} \right) + \bar{J}_{m2} \left(\frac{y_7}{df} \right)^2 + \bar{J}_{m3} \left(\frac{y_7}{df} \right)^3 + \bar{J}_{m4} \left(\frac{y_7}{df} \right)^{j_m} \right] \sin \frac{\pi x}{2b} \quad (3.99)$$

$$\delta w_{b8} = \delta w_{b7} \quad (3.100)$$

$$\delta w_{c4} = \sum_{r=1, R}^R \delta D_r \left[U_{r1} \left(\frac{y_4}{ds} \right)^{g_r} + U_{r2} \left(\frac{y_4}{ds} \right)^{h_r} + U_{r3} \left(\frac{y_4}{ds} \right)^{j_r} \right] \quad (3.101)$$

$$\delta w_{c5} = -\delta w_{c4} \quad (3.102)$$

The total out-of-plane displacements δw_i for the various plate elements can now be defined as:

$$\begin{aligned} \delta w_1 &= \delta w_{a1} + \delta w_{b1} \\ \delta w_2 &= \delta w_{a2} + \delta w_{b2} - \left(\delta v_1 \right)_{y_1=dw} \\ \delta w_3 &= \delta w_{a3} + \delta w_{b3} + \left(\delta v_1 \right)_{y_1=dw} \\ \delta w_4 &= \delta w_{a4} + \delta w_{b4} + \delta w_{c4} \\ \delta w_5 &= \delta w_{a5} + \delta w_{b5} + \delta w_{c5} \\ \delta w_6 &= \delta w_{a6} + \delta w_{b6} \end{aligned}$$

[... (3.103) continued]

$$\delta w_7 = \delta w_{a7} + \delta w_{b7} - (\delta v_6)_{y_6=dw}$$

$$\delta w_8 = \delta w_{a8} + \delta w_{b8} - (\delta v_6)_{y_6=dw}$$

... (3.103)

The total in-plane displacements δv_i are:

$$\delta v_1 = A_v \sin \frac{\pi x}{2b}$$

$$\delta v_2 = (\delta w_{a1})_{y_1=dw}$$

$$\delta v_3 = -(\delta w_{a1})_{y_1=dw}$$

$$\delta v_4 = 0$$

$$\delta v_5 = 0$$

$$\delta v_6 = \delta v_1 + (\delta w_{c5})_{y_5=ds}$$

$$\delta v_7 = (\delta w_{a6})_{y_6=dw}$$

$$\delta v_8 = -(\delta w_{a1})_{y_1=dw}$$

... (3.104)

The total axial displacements δu_1 for each plate element are:

$$\delta u_1 = \delta A_{u1} \cos \frac{\pi x}{2b} - y_1 \delta v_{1x}$$

$$\delta u_2 = \delta A_{u1} \cos \frac{\pi x}{2b} - (y_1 \delta v_{1x})_{y_1=dw} - y_2 \delta v_{2x}$$

$$\delta u_3 = \delta A_{u1} \cos \frac{\pi x}{2b} - (y_1 \delta v_{1x})_{y_1=dw} - y_3 \delta v_{3x}$$

[... (3.105) continued]

$$\begin{aligned}
\delta u_4 &= \delta A_{u1} \cos \frac{\pi x}{2b} \\
\delta u_5 &= \delta A_{u1} \cos \frac{\pi x}{2b} \\
\delta u_6 &= \delta A_{u1} \cos \frac{\pi x}{2b} - y_6 \delta v_{6x} \\
\delta u_7 &= \delta A_{u1} \cos \frac{\pi x}{2b} - \left(y_6 \delta v_{6x} \right)_{y_6=dw} - y_7 \delta v_{7x} \\
\delta u_8 &= \delta A_{u1} \cos \frac{\pi x}{2b} - \left(y_6 \delta v_{6x} \right)_{y_6=dw} - y_8 \delta v_{8x} \\
&\dots (3.105)
\end{aligned}$$

All displacements for the various plate elements have now been defined, a variety of deflected forms for δw_i can be specified for the section simply by choosing different values for the indices as well as the rotational restraints at the web-skin junctions.

3.2.4 Finite Panel

Consider a panel of length $2b$ and width $2d$ shown in Fig 3.5. The panel is assumed simply supported on all four sides and stiffened by stiffeners of non-deformable cross-sections. With reference to Fig 3.5, the out of plane displacement of the skin δw is assumed to be represented in the form of a double sine series as:

$$\delta w = \sum_{m=1, M}^M \sum_{n=1, N}^N \delta A_{mn} \sin m \frac{\pi y}{2d} \sin n \frac{\pi x}{2b} \quad (3.106)$$

The in-plane displacements of the skin δu , δv are assumed as:

$$\delta u = \sum_{n=1, N}^N \delta B_n \cos n \frac{\pi x}{2b} \quad (3.107)$$

$$\delta v = 0 \quad (3.108)$$

Consider now a stiffener placed at a distance $y = s$ as shown in Fig (3.5). The coordinate system for a typical stiffener and the skin are shown in Fig (3.6). The $x_s y_s z_s$ axis pass through the shear center of the stiffener cross section denoted by O , where O_y and O_z are parallel to the principal axes of the section which pass through the centroid denoted by C . $u_s v_s w_s$ define displacements of the shear center in the x_s , y_s and z_s directions and θ_s denotes the rotation of the stiffener cross section about the shear center.

The displacements u_s and v_s of the stiffener can be represented in terms of the skin displacements through the following compatibility expressions:

$$\begin{aligned} u_s &= (u - h_{cz} w_x - h_{cy} v_x)_{y=s} \\ v_s &= (v - h_s w_y)_{y=s} \end{aligned} \quad (3.109)$$

where h_{cy} , h_{cz} represent the distance to the centroid with respect to the x and y axis and h_s represents the distance to the shear center as shown in Fig (3.6).

The displacement w_s can be represented as:

$$w_s = (w)_{y=s} \quad (3.110)$$

The rotation θ_s about the shear center can be represented in terms of the skin displacements as:

$$\theta_s = (w_y)_{y=s} \quad (3.111)$$

The displacements for the various plate elements have now been defined, a variety of deflected forms δu , δw can be specified by considering a number of terms in series representing δu and δw .

3.3 Energy Equations for Stiffened Plates

3.3.1 Plate - Plate Stiffened Sections

The energy equations described in the previous chapter for the stability analysis of plate problems can be extended to investigate the buckling of stiffened plates by taking into account the bending and membrane energies of the various plate elements comprising the structure. If the plate structure is also elastically restrained against rotation, then the bending energy of the restraint has also to be taken into account. The first variation of the bending energy of the rotational restraint U_r is given by:

$$\delta U_r = \int M \delta \theta \quad dx \quad (3.112)$$

In which M is the rotational restraint moment and θ is the angle of rotation.

The relationship between M and θ is given by:

$$M = R\theta \quad (3.113)$$

where R is the rotational restraint.

The second variation of the rotational restraint energy is given by:

$$\delta^2 U_r = \int M \delta^2 \theta + \frac{1}{2} \delta M \delta \theta \quad dx \quad (3.114)$$

The first term in equation vanishes since θ can be expressed as a linear function of displacement variables.

Using equations (3.113), δM is given by:

$$\delta M = R \delta \theta \quad (3.115)$$

Substituting for δM into equation (3.114) yields:

$$\delta^2 U_r = \frac{1}{2} \int R \delta \theta^2 \quad dx \quad (3.116)$$

If the restraint is say applied about the edge $y = 0$ of the plate then $\delta \theta$ is given by:

$$\delta \theta = \left(\delta w_y \right)_{y=0} \quad (3.117)$$

The displacement δw can be expressed in the form of a series as:

$$\delta w = \delta \alpha_{w1} h_1 + \delta \alpha_{w2} h_2 \dots \quad (3.118)$$

Then following the procedure outlined in section (2.6), equation (3.116) can be expressed in matrix form as:

$$\delta^2 U_r = \frac{1}{2} \int \{ \delta \alpha_w \} [S_{1r}]^T [K_r] [S_{1r}] \{ \delta \alpha_w \} \quad dx \quad (3.119)$$

where

$$[S_{1r}] = [h_{1y}, h_{2y}, \dots]$$

$$[K_r] = [R]$$

... (3.120)

For a plate stability problem the second variation of the total potential energy is given by equation (2.116) which can be expressed as:

$$\delta^2 V = \delta^2 U_b + \delta^2 U_m \quad (3.121)$$

where

$$\delta^2 U_b = \frac{1}{2} \int \{ \delta \alpha \}^T [SD_1]^T [D] [SD_1] \{ \delta \alpha \} \quad dx dy$$

and

$$\delta^2 U_m = \frac{1}{2} \int \{ \delta \alpha \}^T [SD_2] [N] [SD_2] \{ \delta \alpha \} \quad dx dy \quad \dots (3.122)$$

The second variation of the total potential energy of an elastically restrained plated structure comprised of n plate elements can be expressed as:

$$\delta^2 V = \sum_{i=1, N}^N \left[\delta^2 U_b^i + \delta^2 U_m^i + \delta^2 U_r \right] \quad (3.123)$$

For critical conditions, the second variation of the total potential energy vanishes so that:

$$\sum_{i=1, N}^N \left[\delta^2 U_b^i + \delta^2 U_m^i + \delta^2 U_r \right] = 0 \quad (3.124)$$

Following the procedure outlined in section 2.6, equation (3.124) can be reduced to the form of equation (2.129) to obtain the critical load.

3.3.2 Plate - Beam Stiffened Sections

Consider a panel of length $2b$, width $2d$ shown in Fig 3.5. The panel is assumed to be simply supported along all four edges and, stiffened by stiffeners of non-deformable cross-section which are allowed to warp.

The complete non-linear expressions for the strains occurring in thin walled bars of open cross section have been derived by Roberts [57], which are briefly outlined here.

A thin walled bar of open cross-section is shown in Fig (3.7). The $x_s y_s z_s$ axis pass through the shear center of the cross-section denoted by O and Oy and Oz are parallel to the principal axes of the section which pass through the centroid C . u_s , v_s and w_s define displacement of the shear center in the x_s , y_s and z_s directions and θ_s denotes the rotation of the section about the shear center.

Due to translation and rotation, a set of axes, originally coincident with the fixed $x_s y_s z_s$ axes but assumed to move with the cross-section to a new position defined by $\bar{x}_s \bar{y}_s \bar{z}_s$ as shown schematically in Fig (3.8). Displacements in the $\bar{x}_s \bar{y}_s \bar{z}_s$ directions are defined by $\bar{u}_s$, $\bar{v}_s$ and $\bar{w}$ and rotation about the $\bar{x}$ axis by $\bar{\theta}_s$. The total strain $\bar{\epsilon}_x$ in the $\bar{x}_s$ direction is given by:

$$\begin{aligned} \bar{\epsilon}_x = & -\bar{y}_s \bar{v}_{sxx} - \bar{z}_s \bar{w}_{sxx} + \alpha_w \bar{\theta}_{sxx} + \frac{\bar{\rho}}{2} \bar{\theta}_s^2 \\ & + u_{sx} + \frac{1}{2} u_{sx}^2 + \frac{1}{2} v_{sx}^2 + \frac{1}{2} w_{sx}^2 \end{aligned} \quad (3.125)$$

in which α_w defines section properties relating to warping, $\bar{\rho}$ defines the distance from the shear center to a point on the cross-section relative to $\bar{x}_s \bar{y}_s \bar{z}_s$ axis.

The first two terms in the above expression represent bending strains about the $\bar{z}_s$ and $\bar{y}_s$ axis respectively, where $\bar{y}_s$ and $\bar{z}_s$ are measured from the centroid. The third term represents strain due to restraint in warping, which can be neglected in this case since the stiffeners are assumed free to warp. The fourth term represents a second order strain produced by stretching of an element due to rotation about the $\bar{x}$ axis. The remaining terms represent strains produced by stretching of an element due to displacements u , v and w .

The second order strain in u can be neglected since its contribution can assumed to be small in comparison with other terms.

From Fig (3.8) displacements $\bar{u}_s$, $\bar{w}_s$ and $\bar{\theta}_s$ can be related to v_s , w_s and θ_s by the following equations, terms containing u being neglected.

$$\begin{aligned}\bar{u}_s &= v_s \cos \theta + w_s \sin \theta \approx v_s + w_s(\theta) \\ \bar{w}_s &= w_s \cos \theta - v_s \sin \theta \approx w_s - v_s(\theta) \\ \bar{\theta}_s &\approx \theta_s - w_{sx} v_{sx} + v_{sx} w_{sx}\end{aligned}\quad (3.126)$$

Substituting above equations into equation (3.125) and noting that $d\bar{x}$ differs from dx by the magnitude of the axial strain which is small compared with unity we can take $d\bar{x} \approx dx$ and therefore, the total axial strain ϵ_x in the x direction is given by:

$$\begin{aligned}\epsilon_x &= -y\{u_{sxx} + w_{sxx}\theta_s\} - z\{w_{sxx} - v_{sxx}\theta_s\} \\ &\quad + u_{sx} + \frac{1}{2}v_{sx}^2 + \frac{1}{2}w_{sx}^2 + \rho\frac{\theta_s^2}{z_{sx}}\end{aligned}\quad (3.127)$$

in which ρ represents the distance from the shear center to a point on the cross-section relative to the $x_s y_s z_s$ axis, and since the cross-section is assumed to be non-deformable then $\rho = \bar{\rho}$, $y = \bar{y}$ and $z = \bar{z}$. Third order terms in $\bar{\theta}_x^2$ have been neglected.

Shear deformation of the bar due to non-uniform bending has been neglected, however shear deformation due to St Venant torsion need to be considered. For a narrow rectangular strip subjected to rotation $\bar{\theta}_s$ about $\bar{x}_s$ axis, the shear strain $\bar{\epsilon}_{xy}$ at a distance ξ for, the centerline is given by:

$$\bar{\epsilon}_{xy} = 2\xi\bar{\theta}_x \quad (3.128)$$

where ξ represents direction perpendicular to the wall of a cross-section as shown in Fig (3.9).

Substituting equation (3.126) into equation (3.128), the shear strain ϵ_{xy} relative to the x_s axis is given by:

$$\epsilon_{xy} = 2\xi\{\theta_{sx} - w_{sxx}v_{sx} + v_{sxx}w_{sx}\} \quad (3.129)$$

When considering the strain energy associated with St Venant torsion, thin walled open cross-sections of any shape can be approximated as single or several disconnected narrow strips.

For a thin walled bar of open cross-section, the first variation of the total potential energy δV_s is given by:

$$\delta V_s = -p_i \delta r_i + \int \sigma_x \delta \epsilon_x + \sigma_{xy} \delta \epsilon_{xy} \quad d \text{ vol} \quad (3.130)$$

where σ_x is the axial stress (tensile positive), $\delta \epsilon_x$ is the first variation of the axial strains, σ_{xy} is the shear stress and $\delta \epsilon_{xy}$ is the first variation of the shear strains. The relationship between the stresses and strains are given by:

$$\begin{aligned} \sigma_x &= E \epsilon_x \\ \sigma_{xy} &= G \epsilon_{xy} \end{aligned} \quad (3.131)$$

where E is Young's modulus and G is the shear modulus.

For stable equilibrium the first variation of the total potential energy is zero and corresponds to a minimum, and the second variation is positive definite.

Critical conditions occur when the second variation of the total potential energy $\delta^2 V_s$ vanishes indicating a possible transition from stability to instability, $\delta^2 V_s$ is given by:

$$\begin{aligned} \delta^2 V_s &= -p_i \delta^2 r_i + \int \sigma_x \delta^2 \epsilon_x + \frac{1}{2} \delta \sigma_x \delta \epsilon_x \\ &\quad + \sigma_{xy} \delta^2 \epsilon_{xy} + \frac{1}{2} \delta \sigma_{xy} \delta \epsilon_{xy} \quad d \text{ vol} \end{aligned} \quad (3.132)$$

For most problems r_i can be expressed as a linear function of displacement variables, and therefore it can be assumed $\delta^2 r_i = 0$.

Using equations (3.131), the second variation of the total potential energy for a thin walled bar under axial compression, ie with $\sigma_{xy} = 0$, can be written as:

$$\delta^2 V_s = \int -\sigma_x \delta^2 \epsilon_x + \frac{E}{2} \delta \epsilon_x \delta \epsilon_x + \frac{G}{2} \delta \epsilon_{xy} \delta \epsilon_{xy} \quad d \text{ vol} \quad (3.133)$$

From equations (3.127), (3.129), the first and second variations of strains are given by:

$$\begin{aligned}
\delta \epsilon_x &= -y\{\delta v_{sxx} + \delta w_{sxx} \theta_s + w_{sxx} \delta \theta_s\} \\
&\quad -z\{\delta w_{sxx} \delta v_{sxx} \theta_s + v_{sxx} \delta \theta_s\} \\
&\quad + \delta u_{sx} + v_{sx} \delta v_{sx} + w_{sx} \delta w_{sx} + \rho^2 \theta_{sx} \delta \theta_{sx} \\
\delta \epsilon_{xy} &= 2\xi\{\delta \theta_{sx} - \delta w_{sxx} v_{sx} - w_{sxx} \delta v_{sx} + \delta v_{sxx} w_{sx} + v_{sxx} \delta w_{sx} \\
&\hspace{15em} (3.134)
\end{aligned}$$

$$\begin{aligned}
\delta^2 \epsilon_x &= -y \delta w_{sxx} \delta \theta_s - z \delta v_{sxx} \delta \theta_s \\
&\quad + \frac{1}{2} \delta v_{sx}^2 + \frac{1}{2} \delta w_{sx}^2 + \frac{1}{2} \rho^2 \delta \theta_{sx}^2 \\
\delta^2 \epsilon_{xy} &= 2\xi\{\delta w_{sxx} \delta v_{sx} + \delta v_{sxx} \delta w_{sx}\} \\
&\hspace{15em} (3.135)
\end{aligned}$$

In the above equations it is assumed that displacements u_s, v_s, w_s and θ_s can be expressed as linear functions of displacement variables, and since the second variation of a linear function is zero, therefore $\delta^2 u, \delta^2 v, \delta^2 w$ are zero. Furthermore, prior and up to the point of buckling, it can be assumed that v, w and θ are zero and therefore terms such as θ, v_{xx} , etc can be neglected in equations (3.134) and (3.135).

Substituting the expressions for the first and second variations of strains into equation (3.133), the second variation of the total potential energy can be written as:

$$\begin{aligned}
\delta^2 V_s &= \iint -\sigma_x \left(-y \delta w_{sxx} \delta \theta_s - z \delta v_{sxx} \delta \theta_s \right. \\
&\quad \left. + \frac{1}{2} \delta v_{sx}^2 + \frac{1}{2} \delta w_{sx}^2 + \frac{1}{2} \rho^2 \delta \theta_{sx}^2 \right) \\
&\quad + \frac{E}{2} \left(\delta u_{sx} - y \delta v_{sxx} - z \delta w_{sxx} \right)^2 \\
&\quad + \frac{G}{2} \left(4\xi^2 \delta \theta_{sx}^2 \right) dA dx \hspace{10em} (3.136)
\end{aligned}$$

In the above equation the limit of integration over the volume $d \text{ vol}$ has been replaced by $dA dx$, where A is the area of cross-section. On integrating with respect to A , the following integrations then apply:

$$\int y \, dA = \int z \, dA = 0$$

$$\int y^2 \, dA = I_0 \int z^2 \, dA = I_y$$

$$\int \rho^2 \, dA = I_0 \int 4\xi^2 \, dA = \sum_i h_i t_i^3 / 3 = J \quad (3.137)$$

where I_z, I_y are the second moments of area of the cross-section about axis through the centroid parallel to the y and z axis respectively, I_0 is the polar second moment of area about the shear center. The last integral above is the well known St Venant torsion constant denoted by J , and for a thin walled bar of open cross-section, the cross-section can be approximated as a single of several disconnected narrow rectangular strips. Due to the assumption the cross-section does not distort, the last expression can also be obtained through plate theory when considering energy associated with twist.

Integrating over the area, $\delta^2 V_s$ can then be written as:

$$\begin{aligned} \delta^2 V_s = & -\frac{1}{2} \sigma_x \int A (\delta v_{sx}^2 + \delta w_{sx}^2) + I_0 \delta \theta_{sx}^2 \, dx \\ & + \frac{1}{2} EA \int \delta u_{sx}^2 \, dx \\ & + \frac{1}{2} E \int I_z \delta v_{sxx}^2 + I_y \delta w_{sxx}^2 \, dx \\ & + \frac{1}{2} GJ \int \delta \theta_{sx}^2 \, dx \end{aligned} \quad (3.138)$$

Using equations (3.109) to (3.111), stiffener displacements u_s, v_s, w_s, θ_s can be represented in terms of plate displacements u, v, w , $\delta^2 V_s$ can then be expressed in matrix form as:

$$\delta^2 V_s = \frac{1}{2} \int [s_1]^T [N_s] [s_1] + [s_2]^T [D_s] [s_2] \, dx \quad (3.139)$$

where

$$[s_1]^T = [v_x, w_{yx}, w_x]$$

$$[N_s] = -\sigma_x \begin{bmatrix} A & -Ah_s & 0 \\ -Ah_s & (Ah_s^2 + I_0) & 0 \\ 0 & 0 & A \end{bmatrix}$$

$$[S_2]^T = [u_x, w_{xx}, v_{xx}, w_{yxx}, w_{yx}]$$

$$[D_s] = E \begin{bmatrix} A & -Ah_{cz} & -Ah_{cy} & 0 & 0 \\ -Ah_{cz} & Ah_{cz}^2 + I_y & Ah_{cz}h_{cy} & 0 & 0 \\ -Ah_{cy} & Ah_{cy}h_{cz} & Ah_{cy}^2 + I_z & -I_z h_s & 0 \\ 0 & 0 & -I_z h_s & I_z h_s^2 & 0 \\ 0 & 0 & 0 & 0 & \frac{J}{2(1+\nu)} \end{bmatrix}$$

(3.140)

Using expressions for u, v, w given by equations (106) to (108) and following the procedure outlined in section (2.6) for instability analysis of plates, $[S_1] [S_2]$ can be expressed as:

$$\begin{aligned} [S_1] &= \{\delta\alpha\} [SA_1] \\ [S_2] &= \{\delta\alpha\} [SA_2] \end{aligned} \quad (3.141)$$

$\delta^2 V_s$ can then be expressed as:

$$\begin{aligned} \delta^2 V_s &= \frac{1}{2} \int \{\delta\alpha\}^T [SA_1]^T [N_s] [SA_1] \{\delta\alpha\} \\ &\quad + \{\delta\alpha\}^T [SA_2]^T [D_s] [SA_2] \{\delta\alpha\} \, dx \end{aligned} \quad (3.142)$$

Equation (3.142) can be written in a more compact form as:

$$\delta^2 V_s = \delta^2 U_{sb} + \delta^2 U_{sm} \quad (3.143)$$

where $\delta^2 U_{sb}$ and $\delta^2 U_{sm}$ represent the second variation of stiffener bending and membrane energy respectively and are given by:

$$\begin{aligned}\delta^2 U_{sm} &= \frac{1}{2} \int \{\delta\alpha\}^T [SA_1]^T [N_s] [SA_1] \{\delta\alpha\} dx \\ \delta^2 U_{sb} &= \frac{1}{2} \int \{\delta\alpha\}^T [SA_2]^T [D_s] [SA_2] \{\delta\alpha\} dx\end{aligned}\quad (3.144)$$

For a stiffened plate section stiffened by n stiffeners, the second variation of the total potential energy can be expressed as:

$$\delta^2 V = \delta^2 U_b + \sum_{i=1, N}^N \delta^2 U_{sb}^i + \delta^2 U_m + \sum_{i=1, N}^N \delta^2 U_{sm}^i \quad (3.145)$$

where $\delta^2 U_b$ and $\delta^2 U_m$ represent the second variations of skin bending and membrane energy respectively and obtained through the procedure outlined in section (2.6).

For critical conditions, the second variation of the total potential energy vanishes, and therefore:

$$\delta^2 U_b + \sum_{i=1, N}^N \delta^2 U_{sb}^i + \delta^2 U_m + \sum_{i=1, N}^N \delta^2 U_{sm}^i = 0 \quad (3.146)$$

Equation (3.146) can be expressed in the form of equation (2.124) to obtain the critical load.

3.4 Results

3.4.1 Edge Stiffened Plates

(i) Local lateral instability of symmetrically edge stiffened plates

The problem of a symmetrically edge stiffened plate under uniform axial compression is depicted in Fig 3.10. The root of the web is assumed to be elastically restrained against rotation by a restraint of stiffness R .

For comparative reasons, the overall mode is excluded for this problem. The inplane displacement δv_1 for the web can therefore be taken as:

$$\delta v_1 = 0 \quad . \quad (3.147)$$

Using equation (3.147), the displacements δw_1 , δu_1 , δw_2 , δu_2 and δv_2 for the various plate elements are represented by equations (3.40), (3.42) to (3.45), and the displacements δw_3 , δv_3 , δu_3 can be taken as:

$$\begin{aligned} \delta w_3 &= \delta w_2 \\ \delta v_3 &= -\delta v_2 \\ \delta u_3 &= \delta A_{u1} \cos \frac{\pi x}{2b} - y_3 \delta v_{3x} \quad . \end{aligned} \quad (3.148)$$

If σ is the uniformly applied axial stress, the membrane forces per unit length for the various plate elements can be taken as:

$$\begin{aligned} N_{xi} &= \sigma t_i \\ N_{yi} &= 0 \\ N_{xyi} &= 0 \end{aligned} \quad (3.149)$$

where the subscript i refers to a particular plate element and t_i its thickness.

Substituting for the displacements and membrane forces into equation (3.124) and solving for the lowest eigenvalue yields the buckling load.

Fig 3.14 shows the relationship between a non-dimensional buckling coefficient K and web plate aspect ratio $2b/dw$ for zero elastic restraint value. The buckling coefficient K is given by:

$$K = \frac{\sigma_{cr} t w d w^2}{\pi^2 D_w} \quad (3.150)$$

The results are compared against those presented by Bulson [4] which are based upon the assumption that flanges have negligible torsional stiffness. This assumption leads to minor differences between the two sets of results for aspect ratios greater than 2.5 where the buckling mode is predominantly torsional for all cross-section geometries considered. For low values of aspect ratios, where the buckling mode is predominantly local for stiffened plates with large flanges, the assumption that flanges have negligible torsional stiffness leads to a considerable underestimation of the local buckling load and leads to the false conclusion that local buckling load is the same as for a plate simply supported along all four edges.

For large values of aspect ratios, all plates irrespective of the flange size buckle into only a single half-wave and the minimum value of the buckling coefficient occurs when the value of the aspect ratio is infinite.

Fig (3.15) shows the relationship between the non-dimensional buckling coefficient given by equation (3.150) and the aspect ratio for infinite value of elastic restraint - providing built in conditions at the root of the web. The results obtained from the energy solution are compared with those presented by Bulson [4], in which the effect of the torsional stiffness of the flange is included for the lateral buckling of the stiffened plate. Bulson's results show that the lateral buckling mode can occur over a number of half wavelengths as the aspect ratio is increased. There is good agreement between Bulson's exact solutions and the present energy solution over the region where the mode of buckling is predominantly lateral. However, for low values of aspect ratios where the buckling mode is local, Bulson again excludes the effect of the torsional stiffness of the flanges, which consequently leads to the assumption that the local buckling load of the stiffened plate is the same as for a plate built along one edge,

simply supported along the loaded and the stiffened edges. This assumption, as shown in Fig 3.15 underestimates the local buckling load.

Fig 3.16 shows the relationship between the non-dimensional buckling coefficient K and aspect ratio for a stiffened plate where the flange width and the flange thickness is made negligibly small. The results are compared with the exact solutions by Bulson [4] for a plate elastically restrained at $y = 0$, free at $y = dw$ and simply supported along the loaded edges $x = 0$ and $x = 2b$. It can be seen that the present energy based results are in good agreement with Bulson's results for various values of restraint coefficient ϵ considered. Where the restraint coefficient ϵ is the ratio of edge moment to edge slope and can be defined in terms of the elastic restraint R and the web plate parameters as:

$$\epsilon = \frac{Rdw}{D_w} \quad (3.151)$$

Note that when $\epsilon = \infty$ the edge is clamped and when $\epsilon = 0$ the edge is simply supported.

(ii) Local lateral and overall instability of asymmetrical edge stiffened plates

The problem of an asymmetrically edge stiffened plate under uniform axial compression is shown in Fig 3.11. The displacements δu_1 , δv_1 , δw_1 , δu_2 , δv_2 and δw_2 for the individual plate elements are given by equations (3.40) to (3.45). The membrane forces for the various plate elements are given by equation (3.149). Substituting for the displacements and membrane forces into equation (3.124) and solving for the lowest eigenvalue yields the critical load.

Fig 3.17 shows the relationship between a non-dimensional buckling coefficient K given by equation (3.150) and the web plate aspect ratio. The results for an asymmetrically stiffened plate with a df/dw ratio of 0.2 are compared against Lim's [55] results also based upon an energy solution. The slight differences in the two sets of results can be attributed to Lim's assumption for the location of neutral axis for inplane bending of the component plates. Lim assumes that the location of the neutral axis for the component plates can be determined from a stress distribution shown in Fig 3.13, resulting from the flexural

torsional buckling of the section. The following assumptions are then used to derive explicit expressions for the location of neutral axis $\bar{y}'_w$ for the web and $\bar{y}'_f$ for the flange.

- (a) The net axial load due to the stress distribution is zero.
- (b) The moment about any axis normal to the plane of the web is zero.

It should be noted that assumption (b) is only valid for the case where there is zero bending about an axis normal to the plane of the web and, does not hold for the case where the section buckles in an overall or flexural torsional manner.

With reference to Fig 3.13 and assumptions (a) and (b), $\bar{y}'_w$ and $\bar{y}'_f$ can be expressed as (see Appendix 3):

$$\bar{y}'_w = \frac{dw}{3}$$

$$\bar{y}'_f = 2df^2/(dw + 4df) \quad . \quad (3.152)$$

Fig 3.18 shows the relationship between the non-dimensional buckling coefficient K and the df/dw ratio for various values of web aspect ratio. The results are compared against those presented by Lim [55] and Klöppel [48]. It can be seen that there is good agreement between the three sets of results for case when $2b/dw = 1$. However, Lim's analysis underestimates the value of K for the torsional mode at various web aspect ratios. These differences can be attributed to the assumption (b) used by Lim in leading to equations (3.152). Conversely, Klöppel's results are substantially higher for the torsional mode at various aspect ratios. These differences may be attributed to Klöppel's analysis in which, the effect of the overall mode is excluded and the location of the neutral axis for inplane flange bending $\bar{y}'_f$, is taken at the flange web junction. The latter assumption, greatly overestimates the inplane bending stiffness of the flange.

Figs 3.19 and 3.20 show the relationship between $\bar{y}'_i/d_i$ and $\bar{y}_i/d_i$ with df/dw where, the subscript i refers to the particular plate element, $\bar{y}_i$ refers to the location of axis of bending predicted

by the present analysis (see Appendix 3) and, $\bar{y}'$ refers to the location of bending axis given by equations (3.152).

It can be seen from Fig 3.19 that for values of df/dw less than 0.3 there is close agreement in the positions of the neutral axis predicted by the present analysis and those given by equations (3.152). The buckling mode in this region is predominantly lateral as shown in Fig 3.19a. For values of df/dw greater than 0.3, the present analysis shows a sharp drop in the distance of the neutral axis from plate edges of the various plate elements, indicating a transition from the lateral buckling mode to the local buckling mode as shown in Fig 3.19a. Fig 3.20 shows results for the case where the web aspect ratio $2b/dw = 25$. It can be seen that there is a progressive increase in the differences between the position of the neutral axis predicted by the present analysis and those given by equation (3.152) as the df/dw ratio is increased. Fig 3.19b shows that the buckling mode over the range of df/dw considered is predominantly lateral.

Fig 3.21 shows the relationship between the non-dimensional web buckling coefficient K (see equation 3.150) and the web aspect ratio $2b/dw$ for various values of df/dw . It can be seen that as df/dw is increased up to 0.2 the value of K for which local buckling occurs, also increases. For these stiffened plate geometries the flange restrains the web from buckling and buckling can be said to be induced by the web. As the df/dw ratio is increased above 0.2 the value of K for local buckling starts to drop. For these stiffened plate geometries the web restrains the flange from buckling and it can be said that the buckling is induced by the flange. It should be noted that the minimum value of K for flange induced local buckling occurs at a greater value of half wavelength than for web induced local buckling. It can also be seen that as the half wavelength $2b$ is increased, all curves for the various stiffened plate geometries show a transition from the local buckling mode to the lateral/overall buckling mode (note that as the web aspect ratio $2b/dw$ is increased, local buckling can occur over several half wavelengths and lateral/overall buckling occurs when the value of K for which lateral/overall buckling occurs falls below the minimum value of K for which local buckling occurs).

Fig 3.22 depicts typical buckling modes at mid span for some of the

stiffened plate geometries considered in Fig 3.21 Flange induced local buckling is represented by the deflected shape where the maximum deflection of the flange is greater than the maximum deflection of the web. This can be seen quite clearly for the stiffened plate with the df/dw ratio of 0.5 and web aspect ratios $2b/dw$, of 1.0, 2.0 and 4.0. Similarly, web induced local buckling is represented by the mode shapes where the maximum deflection of the web is greater than the maximum deflection of the flange. This can be seen for the case where $df/dw = 0.1$ and $2b/dw = 1.0$.

Fig 3.23 shows the relationship between the web buckling coefficient K and $2b/dw$ for the case where $t_f/t_w = 3.0$. Note that for the stiffened plate with $df/dw = 0.5$, flange induced local buckling occurs at nearly the same half wavelength as the lateral buckling mode. This results in a large amount of interaction between the two modes which causes a substantial reduction in the value of K at this point. However, it should be noted that this is of no practical significance since there is no length of stiffened plate for which buckling occurs at this half wavelength. The buckling modes at mid span for some of the stiffened plate sections considered in Fig 3.23 are shown in Fig 3.24. From examination of Fig 3.21 in which the case $t_f/t_w = 1$ is treated and, Fig 3.23, it can be seen that for stiffened plates with df/dw ratios of greater than 0.1, failure through the lateral buckling mode occurs at a smaller half wavelength if the flange thickness is greater than the web thickness.

Fig 3.25 shows the variation of the web buckling coefficient K versus the web aspect ratio where $dw/t_w = 150$ and $t_f/t_w = 3.0$. From examination of Figs 3.25 and 3.23 in which the case $t_w/dw = 50$, $t_f/t_w = 3.0$ is treated. It can be seen that failure through the lateral buckling mode occurs at an increasingly longer half wavelength if the web thickness ratio dw/t_w is increased.

Fig 3.26 shows the variation between the web buckling coefficient and K and the web aspect ratio $2b/dw$. For $df/dw = 0.1$ and various values of elastic restraint ϵ (see equation (3.151)) applied at the root of the web. It can be seen that for the case where $\epsilon = 0$ the buckling mode is local for low values of $2b/dw$. As the aspect ratio is increased, the buckling mode changes from local to lateral when the

value of K , which represent the lateral buckling mode falls below the minimum value of K , at which the local buckling mode occurs. It should be noted that the lateral buckling mode where $\epsilon = 0$ occurs over a single half wavelength.

For the cases where $\epsilon = 1$ and $\epsilon = 3$, the buckled mode is local for low values of $2b/dw$, however, as the aspect ratio is increased, the buckling mode changes from local to lateral which can now occur over several half wavelengths. As the aspect ratio is further increased the buckling mode changes from lateral to overall when the value of K , which represents the overall buckling mode is lower than the minimum value of K , which represents the lateral buckling mode.

For the case where $\epsilon = \infty$ which effectively provides built-in condition at the root of the web, the buckling mode is local for low values of $2b/dw$ and changes to the overall mode for large values of aspect ratio. The lateral buckling mode for this case does not occur for any value of $2b/dw$ since the value of K at which lateral buckling occurs is always higher than the minimum value of K at which local buckling occurs. Figs 3.27 and 3.28 show the variation of K and $2b/dw$ for $df/dw = 0.3$, $df/dw = 0.5$ respectively. Again it can be seen that the effect of the elastic restraint is to reduce the influence of the lateral mode. Fig 3.29 shows some typical buckled modes at mid span over a single half wavelength for cases $\epsilon = \infty$ considered in Figs 3.26, 3.27 and 3.28.

Fig 3.30 and Fig 3.31 show the variation of the web buckling coefficient K versus the web aspect ratio $2b/dw$. The results from the case where the overall mode is included (ie $\delta v_1 \neq 0$) are compared against the case where the overall mode is excluded (ie $\delta v = 0$). It can be seen that by neglecting the δv_1 degree of freedom results in an overestimation in the value of K for lateral buckling. The differences being the greatest for stiffened plates with large flanges and, where the thickness of the flange is greater than the thickness of the web.

Figs 3.32 and 3.33 show the variation of the web buckling coefficient K and the df/dw ratio. The results from the two cases where the overall mode is included and excluded are compared against each other as well as the results obtained by Klöppel [48], wherever possible.

It can be seen that Klöppel's analysis overestimates the value of K even when compared with the present analysis where the overall mode is excluded. The differences being the greatest for stiffened plates with large web aspect ratios. Similarly the differences in results from the two cases where the overall mode is included and excluded are the greatest for plates with large aspect ratios with wide and thick flanges.

3.4.2 Semi-Infinite Panels

(i) Local instability

The problem of an infinite panel subjected to uniform axial compression is depicted in Fig 3.12. The displacement fields δw_i , δu_i and δv_i for the various plate elements are given in section 3.2.3. For local instability analysis of panels with unflanged stiffeners the displacement δw_{ai} is only considered and the energy contribution of the other displacements can be omitted. For the cases where the panels are stiffened by flanged stiffeners, the energy contribution of displacement δw_{bi} need only be considered.

If σ is the uniformly applied axial stress, the membrane force per unit length for the various plate elements can be taken as:

$$\begin{aligned} N_{xi} &= \sigma t_i \\ N_{yi} &= 0 \\ N_{xyi} &= 0 \end{aligned} \quad (3.153)$$

Substituting for the displacements and membrane forces into equation (3.124) and solving for the lowest eigenvalue yields the buckling load.

Fig 3.34 shows the variation of $K_{s_{min}}$ with dw/ds ratio for integrally stiffened panels with unflanged stiffeners. Results are presented for various ts/tw ratios. The value for $K_{s_{min}}$ was calculated iteratively for a given dw/ds and ts/tw ratio by varying the half wavelength $2b$ until a minimum value for the skin buckling coefficient K_s was obtained, where

$$K_s = \left(\frac{\sigma_{cr}}{E} \right) \left(\frac{ds}{ts} \right)^2 \quad (3.154)$$

in which σ_{cr} is the critical buckling stress.

The results are compared against those presented in ESDU data sheets [62], and those obtained through the use of the finite strip package VIPASA [34]. It can be seen that the results obtained through the present analysis compare favourably with those predicted by VIPASA. However, there are substantial differences between the present results and those presented in ESDU data sheets, which are based upon an analytical approach suggested by Cox [63] requiring the solution to the equilibrium equations for each individual plate element connected together through rotational restraints - which should provide 'exact' answers. It is therefore assumed that computational or iterative errors may be the cause of discrepancies in the ESDU curves. It should be noted that all curves for low values of dw/ds represent skin induced buckling and for high values of dw/ds represent stiffener induced buckling. The change being the most dramatic for high values of ts/tw .

Figs 3.35 and 3.36 show the variation of $K_{s_{min}}$ with dw/ds for integral stiffened panels with flanged stiffeners. Both symmetrically flanged and asymmetrically flanged stiffeners are considered. The results from the current analysis are compared with those obtained through the use of the finite strip package VIPASA [34] and those presented in ESDU data sheets [62]. It can be seen that the results obtained from the current analysis compares favourably with those obtained through the use of VIPASA. Any discrepancies between the results from the present energy analysis and those presented in ESDU data sheets can again be attributed to computational or iterative errors in the ESDU results.

(ii) Local lateral and overall instability

The problem of an infinite panel subjected to uniform axial compression is depicted in Fig 3.12. The assumed displacement fields δw_i , δu_i and δv_i for the various plate elements are given in section 3.2.3. The expressions for the membrane force per unit length are given by equation (3.153).

Substituting for the displacements and membrane forces into equation (3.129) and solving for the lowest eigenvalue yields the buckling load.

Fig 3.37 shows the variation of a non-dimensional critical stress factor $\sigma_{cr} \times 692/E$ with the df/dw ratio for an integral panel with asymmetrically flanged stiffeners. The results from the present upper-bound energy analysis are compared with results based upon an 'exact' theory for an infinite panel, presented by Williams and Wittrick [33]. It can be seen that the two sets of results are in good agreement over the range of df/dw values considered. The buckling mode to the left of the sudden change in slope is predominantly lateral and to the right is predominantly local. The sudden discontinuity in the slope suggests that these two modes occur at different half wavelengths and indicates little or no interaction between the two modes.

Figs 3.38, 3.39 and 3.40 shows the variation of a non-dimensional critical stress factor $\sigma_{cr} \times 10^3/E$ and the aspect ratio $2b/ds$. The results for infinite integral panels with unflanged and asymmetrically flanged stiffeners are compared with results presented by Williams and Wittrick³³ for a simply supported finite size panel stiffened by 12 stiffeners.

Consider first Fig 3.38, which shows results for an integral panel with unflanged stiffeners. The curves shown have two distinct portions corresponding to the local and overall buckling modes. The results from the infinite panel analysis compare favourably with those for a finite panel over the local buckling region. However, large differences occur between the two sets of results at the transition point where the buckling mode changes from local to the overall mode. It can be seen that the infinite panel model overpredicts the buckling load; however this is of no practical significance since it is apparent that there is no length of panel for which buckling would actually occur with a wavelength in this region. As the aspect ratio is further increased in the overall buckling region, the differences between the two sets of results become progressively smaller and are negligible at the maximum value of aspect ratio considered. It should be noted that further increase in the aspect ratio results in divergence in the two sets of results for overall buckling [33].

Fig 3.39 shows the results for an integral panel with asymmetrically flanged stiffeners with df/dw ratio of 0.3. The curve has three distinct portions relating to the local, lateral and the overall mode. It can be seen that the two sets of results show good agreement over the local buckling region. However, the infinite panel overestimates the critical stress in the middle portion of the curve which represents the lateral and the overall buckling modes. As the aspect ratio is further increased in the lateral buckling region, the curves converge and the differences between them become negligible at the maximum level of aspect ratio considered.

It should be noted that the differences in the results from the infinite panel model and the finite size panel [33] in the middle portion of the curves is of no practical significance as there is no length of panel at which buckling would occur at this half wavelength.

Fig 3.40 shows the results for an integral panel with asymmetrically flanged stiffeners which have a df/dw ratio of 0.8. Due to the large size of flange, the local mode in this case is induced by the flange. It can also be seen that the lateral buckling mode occurs at nearly the same half wavelength as the overall buckling mode resulting in a large amount of interaction between the two modes. Comparing the results from the infinite panel model and the finite size panel [33], it can be seen that the infinite panel model overpredicts the buckling load. However for large values of aspect ratios considered here, which are practically more significant, the differences in the two sets of results for overall buckling are negligible.

Fig 3.41 shows the results for an integral panel with asymmetrically flanged stiffeners which has a relatively thick skin and very thin stiffeners such that $ds/ts = 20$, $tw/ts = 0.28$ and $tf = tw$. The infinite panel model again overpredicts the buckling stress in the middle portion of the curve which represents the lateral buckling mode and the transition from lateral buckling to overall buckling. It can be seen that the finite panel will buckle in the lateral mode if $2b/ds$ ratio is 3.0, however, the infinite panel model predicts that the buckling mode at this point is still local due to its overestimation of the lateral buckling stress.

It should be noted that this is a rather unusual panel, having a heavy skin and light stiffeners and is unlikely to be chosen in practice since it buckles in the lateral mode at very small values $2b/ds$.

The differences between the results obtained from the present energy analysis for an infinite panel and the results presented by Williams and Wittrick [33] for a finite size panel with 12 stiffeners can be attributed to the following factors.

- (1) The influence of the finite panel longitudinal simple support edge conditions during the lateral buckling mode.
- (2) The overall mode for a finite panel involves bowing of the skin with rotation of stiffener cross-section and therefore providing greater interaction between the lateral and the overall modes.

The overall mode for an infinite panel involves a rigid body translation of the panel cross-section perpendicular to the plane of the skin with no rotation of the stiffener cross-section.

Fig 3.42 shows the relationship between a critical stress factor $\sigma/E \times 10^3$ and the skin aspect ratio $2b/ds$. The results for an integral panel with asymmetrically flanged stiffeners are compared against results for an integral panel of equal mass with symmetrically flanged stiffeners. It can be seen that panels with asymmetrically flanged stiffeners are more effective against lateral buckling than panels with symmetrically flanged stiffeners. This is evident for the case where $df/dw = df'/dw = 0.1$. It can also be seen that panels with large asymmetrically flanged stiffeners are prone to flange induced buckling which occurs at a lower buckling stress than panels with symmetrically flanged stiffeners. This is notable for the case $df/dw = df'/dw = 0.5$.

Fig 3.43a shows the relationship between $\sigma_{cr}/E \times 10^3$ and $2b/ds$ for integral panels stiffened by asymmetrically flanged stiffeners with df/dw ratios of 0.1, 0.3 and 0.5. Figs 3.43b, 3.43c and 3.43d depict the corresponding buckled modes of deformation at mid span for various values of $2b/ds$. It can be seen that for $2b/ds$ ratio of 0.5, the buckled mode is predominantly local for all stiffener geometries considered. For the case where $2b/ds$ ratio is 8, the buckling mode is

predominantly lateral involving some deformation of the stiffener cross-section and rotation of adjacent stiffeners in the opposite direction. Note that for stiffeners with df/dw ratios of 0.3 and 0.5, lateral buckling of the stiffeners is also coupled with flexural deformation of the stiffeners, resulting in the deformation of the skin stiffener junction.

For $2b/ds$ ratio of 16, the buckled modes represent coupling between the lateral and overall modes involving large distortion of the stiffener cross-section and rotation of adjacent stiffeners in the same direction as well as an overall displacement.

For $2b/ds$ ratio of 32, the buckling mode is predominantly overall involving no deformation of the panel cross-section.

Fig 3.44a shows the relationship between a non-dimensional buckling stress factor $\sigma/E \times 10^3$ and the skin aspect ratio $2b/ds$, for panels stiffened by symmetrically flanged stiffeners where $t_f = t_w = 2t_s$. Figs 3.44b, 3.44c and 3.44d represent the buckling modes for the panels considered in Fig 3.44a at various values of $2b/ds$. It can be seen that due to the thick stiffeners, distortion of the cross-section in the local and the lateral mode is completely confined to the skin. It should be noted that for the case $df/dw = 0.25$, $2b/ds = 32$, the overall mode comprises flexural displacement of alternative stiffeners.

3.4.3 Finite Panel

This section presents results for 11 panels under uniaxial loading. The panels are referred to as I1, I2..., T1, T2..., L1, L2..., where I, T and L refer to panels with I, T and L section stiffeners respectively. Panels I1-I3 are analysed for comparative purposes and have a different stiffener layout than other panels. The stiffener layout for all panels is shown schematically wherever results are presented.

All panels are assumed to be simply supported along all four edges and the stiffeners are allowed to warp.

Full details of the panel cross-section are presented in Table 1 and Fig 3.45.

Table 1
DETAILS OF PANELS

Panel	2d(M)	t_s (M)	$\frac{d_w}{d'_s}$	$\frac{d_f}{d_w}$	$\frac{t_w}{t_s}$	$\frac{t_f}{t_w}$	N
I1	0.4	0.001	0.1	-	1	-	3
I2	0.4	"	0.25	-	1	-	9
I3	0.4	"	0.5	-	1	-	9
I4	0.42	"	0.25	-	1	-	8
I5	"	"	"	-	3	-	8
T1	"	"	"	$\frac{1}{3}$	1	1	8
T2	"	"	"	$\frac{1}{2}$	1	1	8
T3	"	"	"	$\frac{1}{2}$	3	1	8
L1	"	"	"	$\frac{1}{3}$	1	1	8
L2	"	"	"	$\frac{1}{2}$	1	1	8
L3	"	"	"	$\frac{1}{2}$	3	1	8

In Table 1, N refers to the number of stiffeners on a particular panel and d'_s represents the stiffener pitch and is given by:

$$d'_s = \frac{2d}{(N+1)} \quad \text{for panels I1-I3}$$

$$d'_s = \frac{2d}{(N-1)} \quad \text{for panels I4-L3}$$

Fig 3.46 shows the variation of a buckling stress factor K with the aspect ratio α and $\bar{\alpha}$ for panel I1 which is stiffened by three stiffeners of rectangular cross-section. K , α and $\bar{\alpha}$ are given by:

$$K = \sigma_{cr}(1-\nu^2)/E$$

and

$$\alpha = b/d \quad \text{for } d \geq b$$

$$\bar{\alpha} = d/b \quad \text{for } d \leq b$$

where σ_{cr} refers to the critical buckling stress, $2b$ and $2d$ are the lengths of the panel in the x and y directions respectively.

The results in Fig 3.46 are compared against those obtained through an "exact" finite strip analysis by Iraki [62]. It can be seen that the two sets of results are in very good agreement.

Both sets of results show that the maximum number of half waves over which local buckling can occur is three - this being the case where α is about 0.7. It should be pointed out here that the local buckling mode includes the torsional deformation of the stiffeners. For larger values of α , overall buckling occurs. Depending on the aspect ratio, the overall buckling mode may itself occur over several half-wavelengths. Fig 3.47 shows the results for panel I2. This panel is stiffened by nine stiffeners of rectangular cross-section. The cross-section dimensions of the stiffeners are the same as those for panel I1. It can be again seen that the present results are in good agreement with the "exact" finite strip results [62].

Consider now the results for panel I3 shown in Fig 3.48. This panel is stiffened by nine stiffeners of rectangular cross-section. The height of the stiffeners is twice that for panel I1 and I2. It is clearly evident that the present "upper bound" results from the energy method and the "exact" finite strip results are in good agreement over the region where overall buckling occurs. However for values of α where local buckling occurs, the present analysis

overpredicts the buckling stress factor - this remains to be the case even if the number of terms representing the buckled configuration is increased in the present model. Furthermore, the reason for these discrepancies is not readily obvious since the finite strip analysis is presented in the German language. However a closer examination of the finite strip equations suggests that in modelling the stiffeners, the following energy expressions have been effectively neglected when compared to the present analysis - see equation (3.138).

$$EI_y \int_0^{2b} \delta v_{sxx}^2 dx + \sigma A_s \int_0^{2b} \delta \theta_x^2 dx$$

Where the first term represents the second variation of bending energy due to displacement v_s and the last term represents the second variation of the "membrane" energy due to rotation θ . The contribution of these terms is greater for panel I3 than for panels I1 and I2. This is because the size of the stiffeners for panel I3 is larger than those for panels I1 and I2.

This explains why the results from the two methods are in better agreement for panels I1 and I2 and not for panel I3.

Fig 3.49 shows the variation of the critical stress σ_{cr} with the aspect ratio α and $\bar{\alpha}$ for panels L1, T1 and I4. Panels L1 and T1 in this case have the same cross-sectional area. The local buckling for panels shown in this figure involves the torsional deformation of the stiffeners. It can be seen that L-section stiffeners are more effective than T-section stiffeners since the local buckling stress is higher for panel L1 when compared with T1. The reason for this being that L-section stiffeners have a higher torsional stiffness when compared with T-section stiffeners of equal cross-section and thickness.

Fig 3.50 shows the variation of the critical stress with the aspect ratio for panels L2, T2 and I4.

Panel L2 is of particular interest here since it highlights the difference between skin induced and stiffener induced local buckling. For values of α less than 0.2, local buckling is largely confined to the skin with little or no torsional deformation of the skin. This

is shown in Fig 3.51a. For values of α greater than 0.2, local buckling is induced by the torsional mode of the stiffeners as shown in Fig 3.51b. This type of buckling occurs at a lower critical stress and over longer half wavelengths than for skin induced buckling. For values of α greater than 0.8, overall buckling occurs. This mode shape is shown in Fig 3.51c.

Fig 3.52 shows the results for panels L3, T3 and I5. These panels are stiffened by relatively thick stiffeners which are three times the thickness of the skin. The results consequently show that local buckling for all panels is induced by the skin. Panels L3 and T3 have the same cross-sectional area. It can be seen that the local buckling load is almost identical for these panels since it is confined to the skin and involves little or no torsional deformation of the stiffeners.

Fig 3.53 and Fig 3.54 show the results for panels L1, L2, L3 and T1, T2, T3 respectively. Both sets of results show that the local buckling stress increases as the flange width of the stiffeners is made larger. The rate of increase is particularly greater for L-section stiffened panels than it is for T-section stiffened panels. This results from the greater torsional stiffness provided by L-section stiffeners when compared with T-section stiffeners of equal cross-sectional area and thickness.

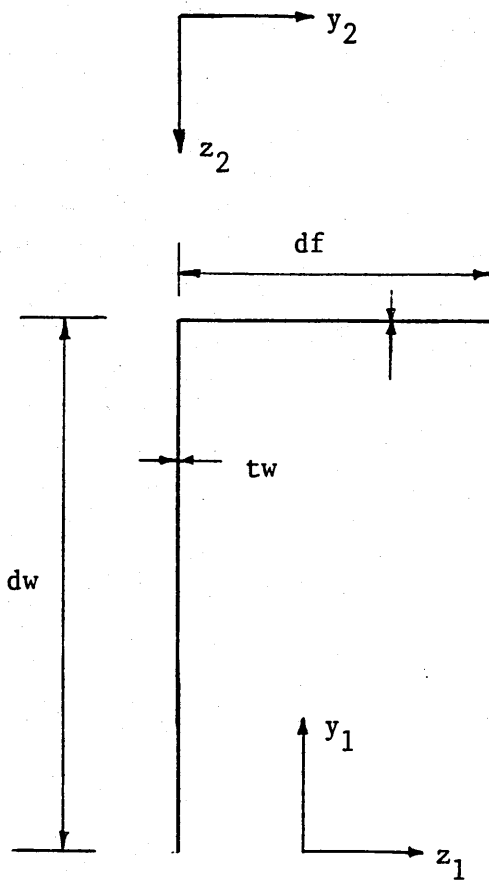


Fig 3.1 Asymmetrically Stiffened Plate

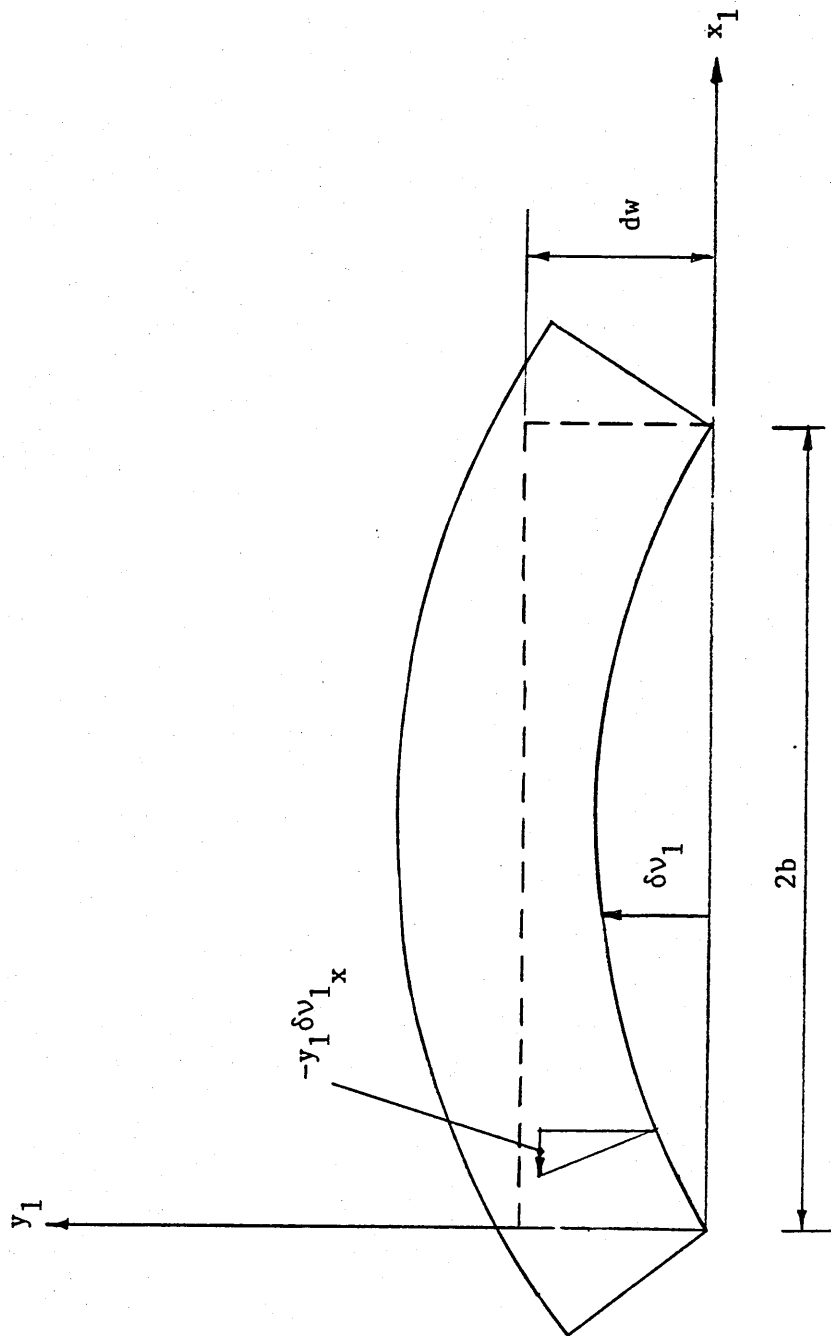


Fig 3.2 In-Plane Displacements

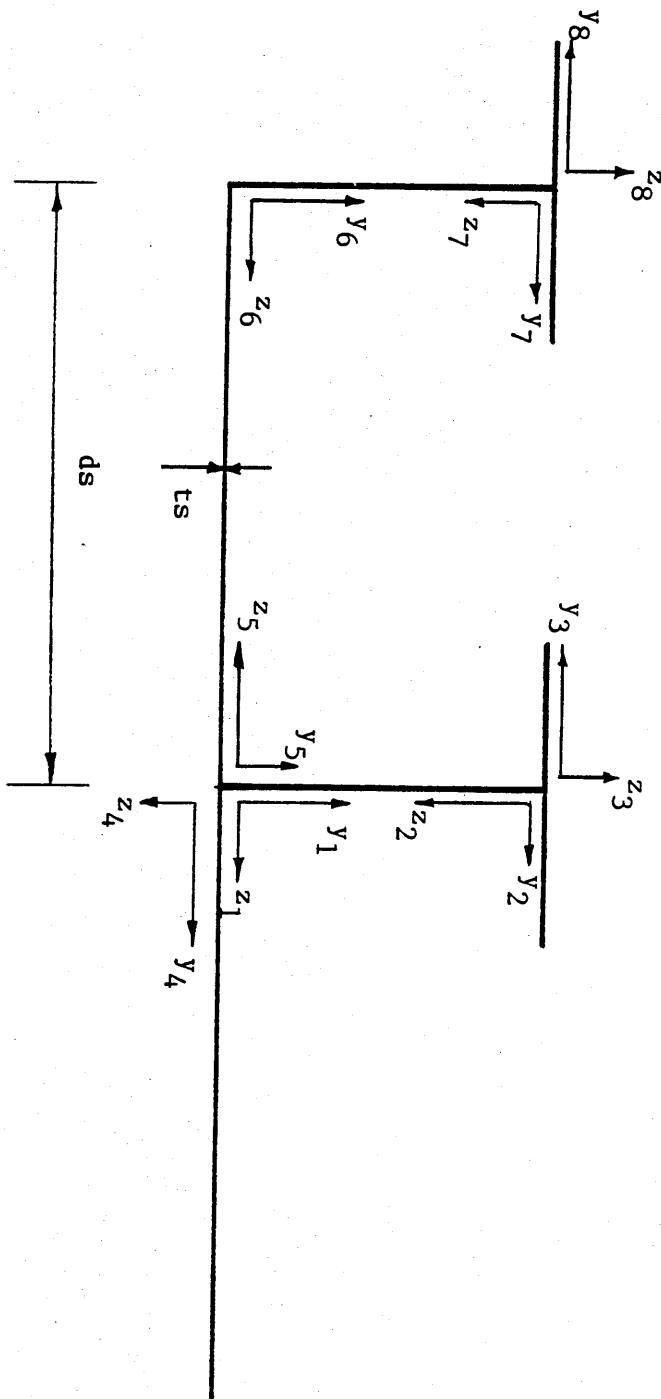


Fig 3.3 Semi-Infinite Panel

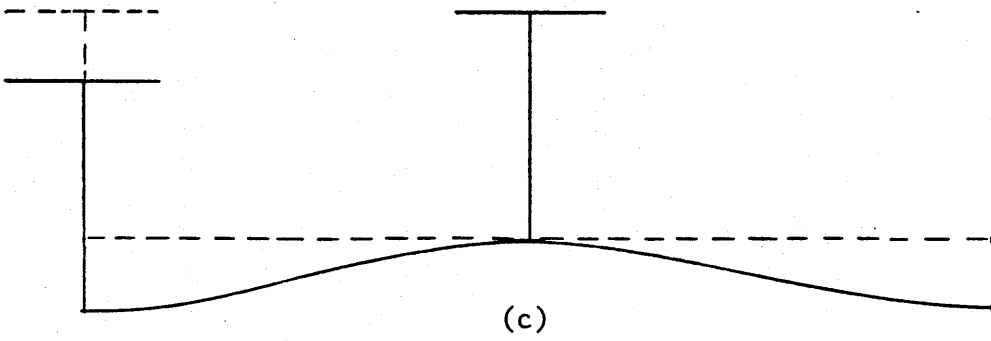
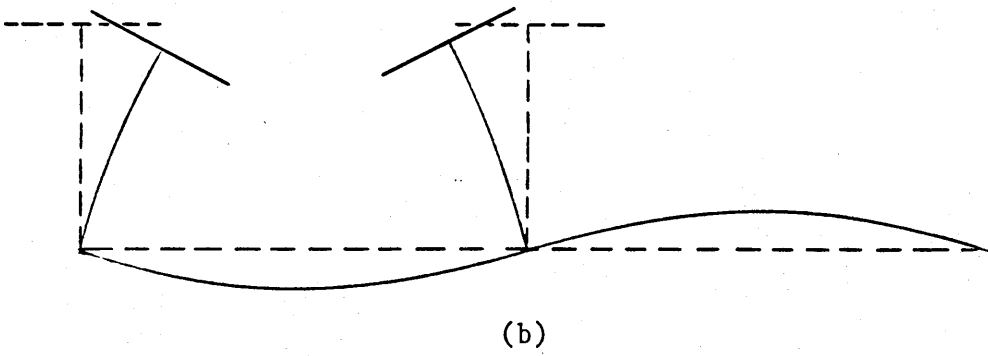
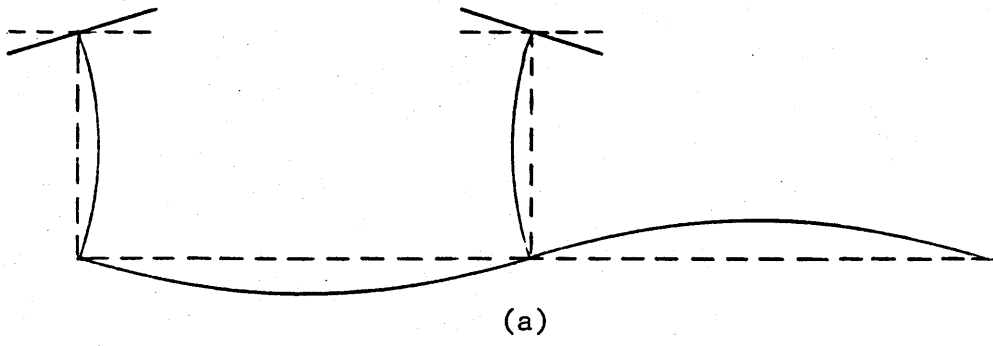


Fig 3.4 Displacement Functions

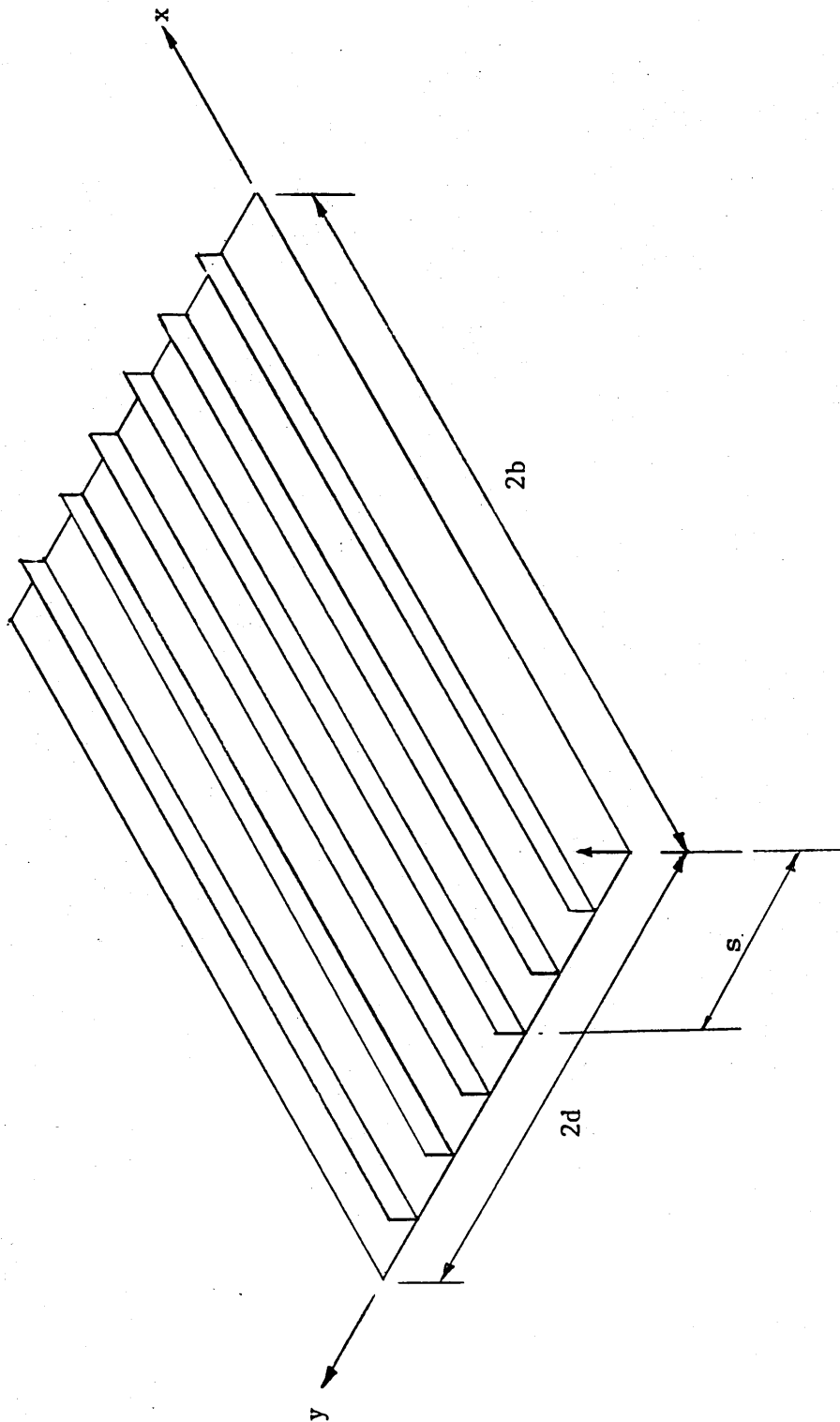


Fig 3.5 Stiffened Panel

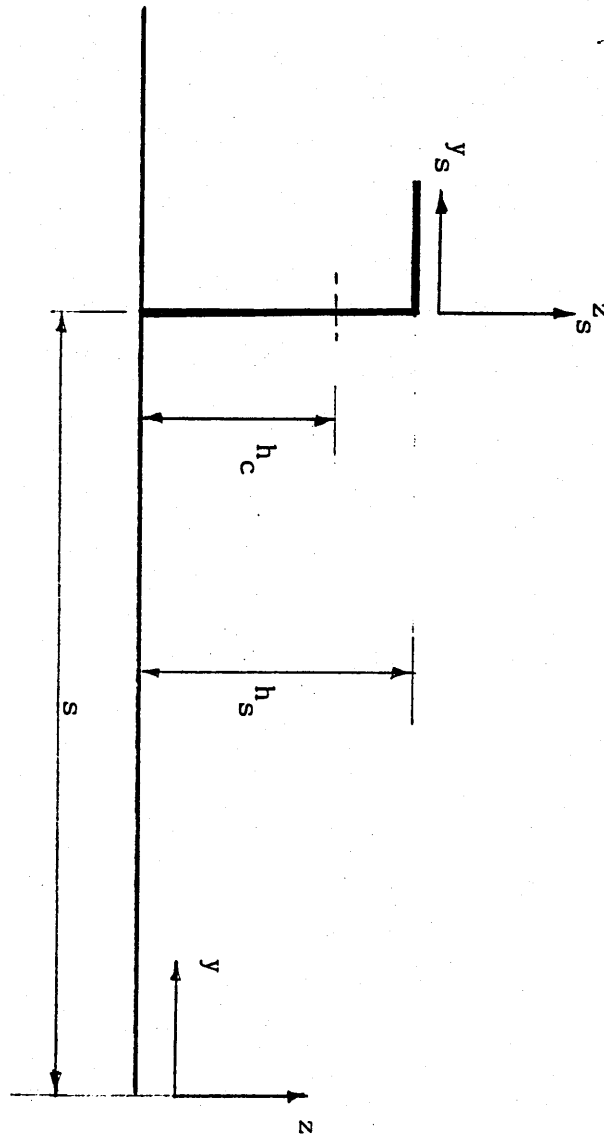


Fig 3.6 Coordinate axis

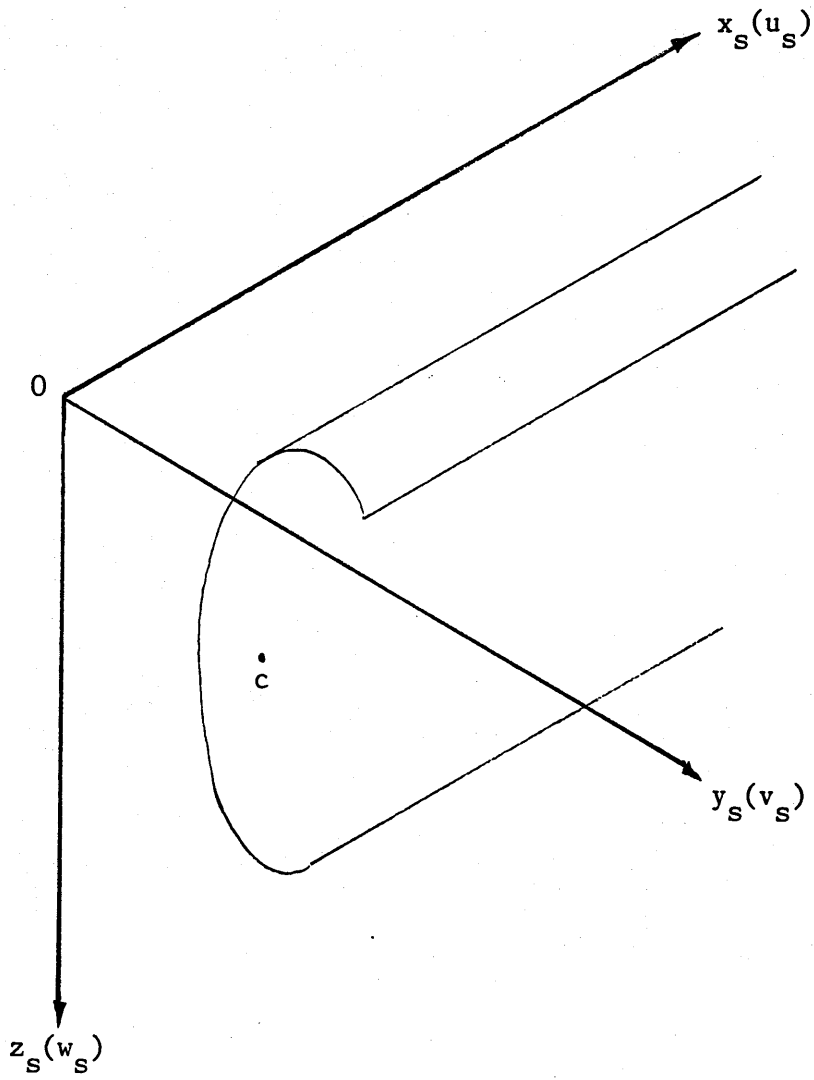


Fig 3.7 Coordinate axis

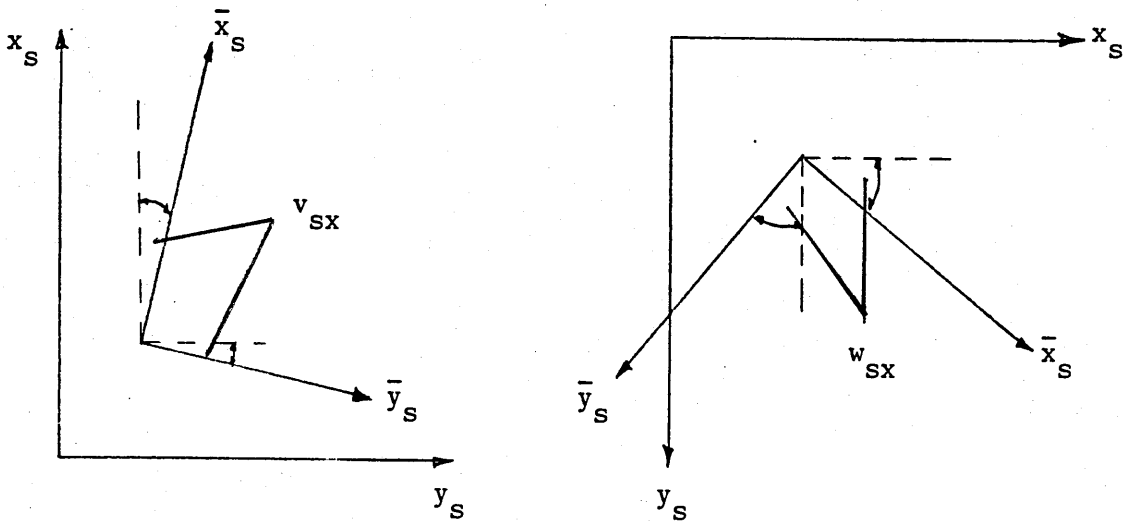
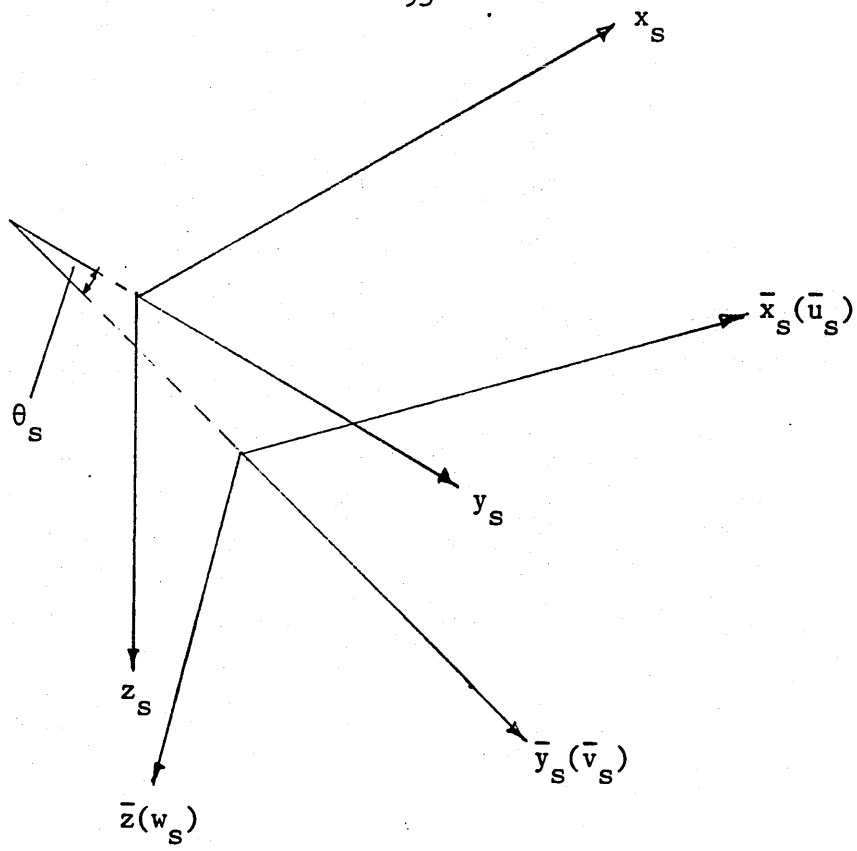


Fig 3.8 Translation and rotation of axis

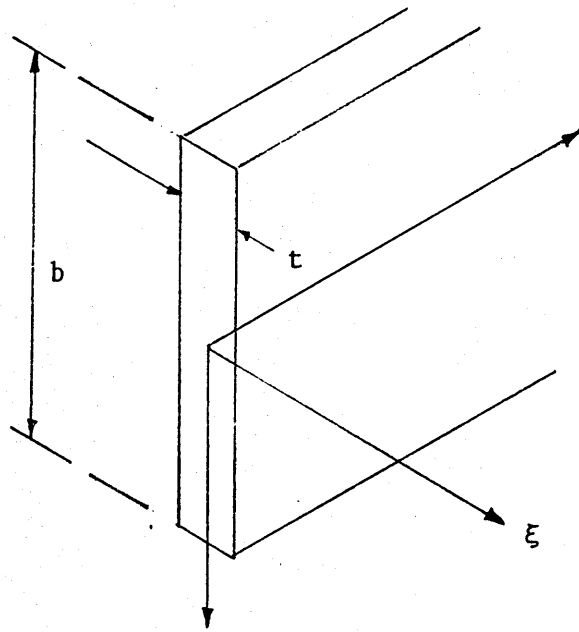


Fig 3.9 St Venant torsion for a narrow rectangular strip

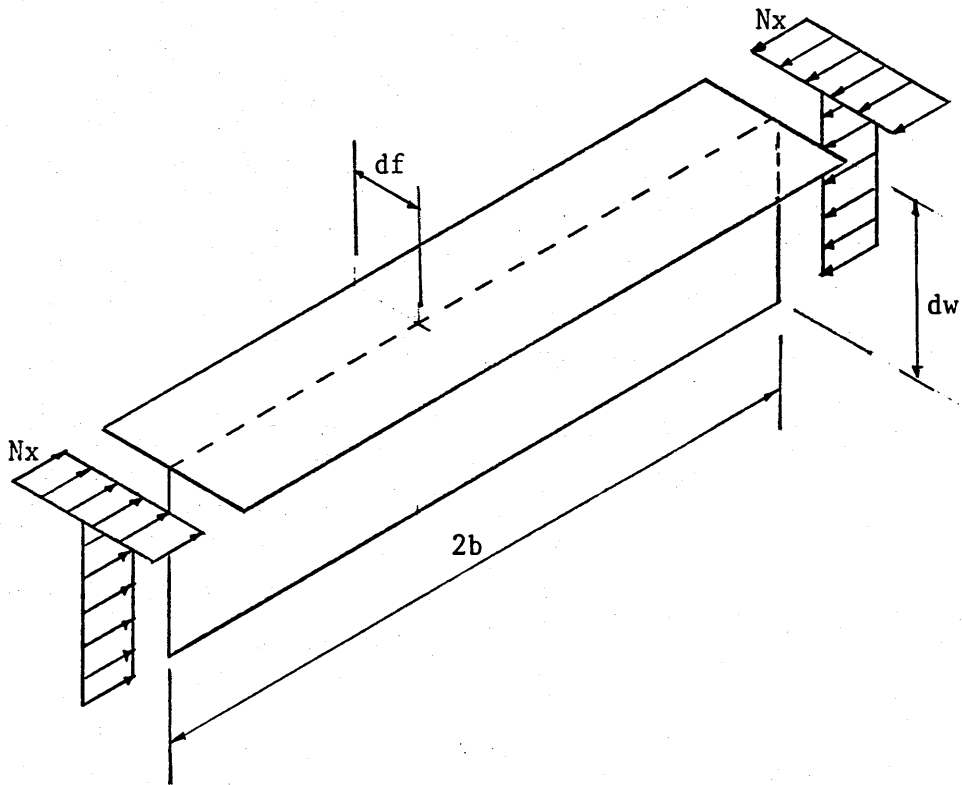


Fig 3.10 Symmetrically edge stiffened plate under pure compression

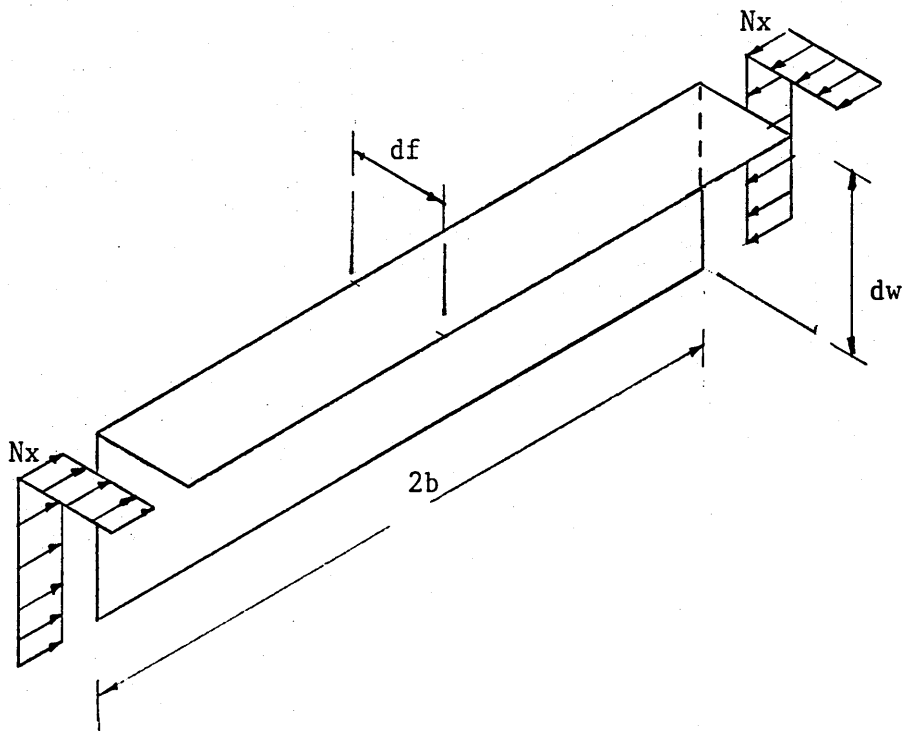


Fig 3.11 Asymmetrically edge stiffened plate under pure compression

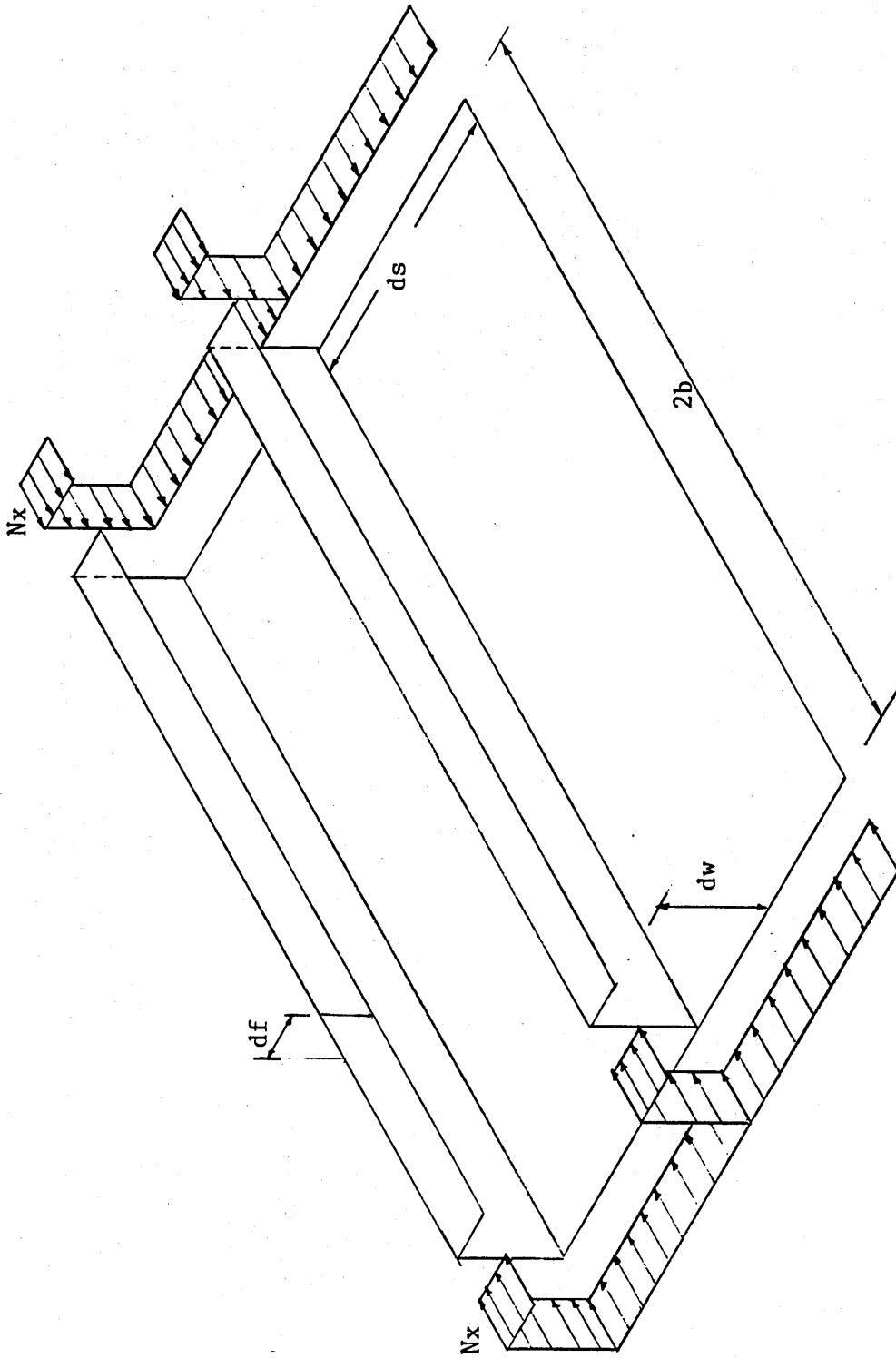


Fig 3.12 Semi-infinite panel under uniform compression

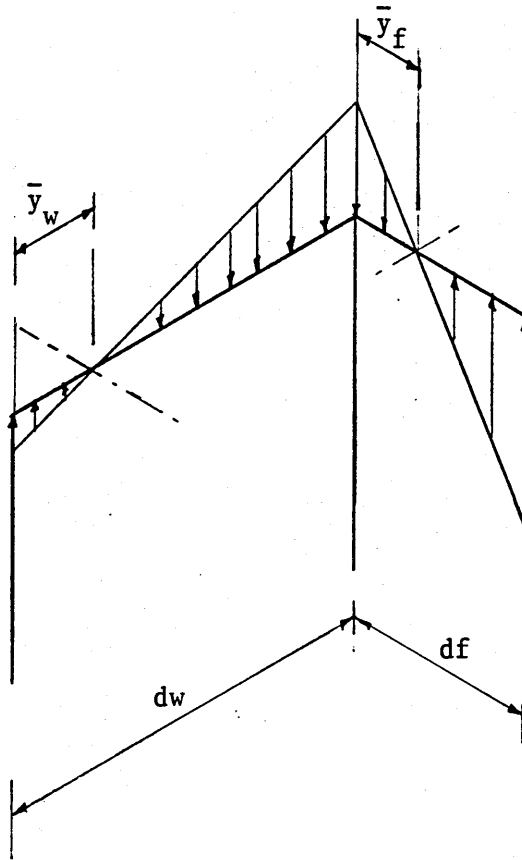


Fig 3.13 Stress distribution used to position neutral axis

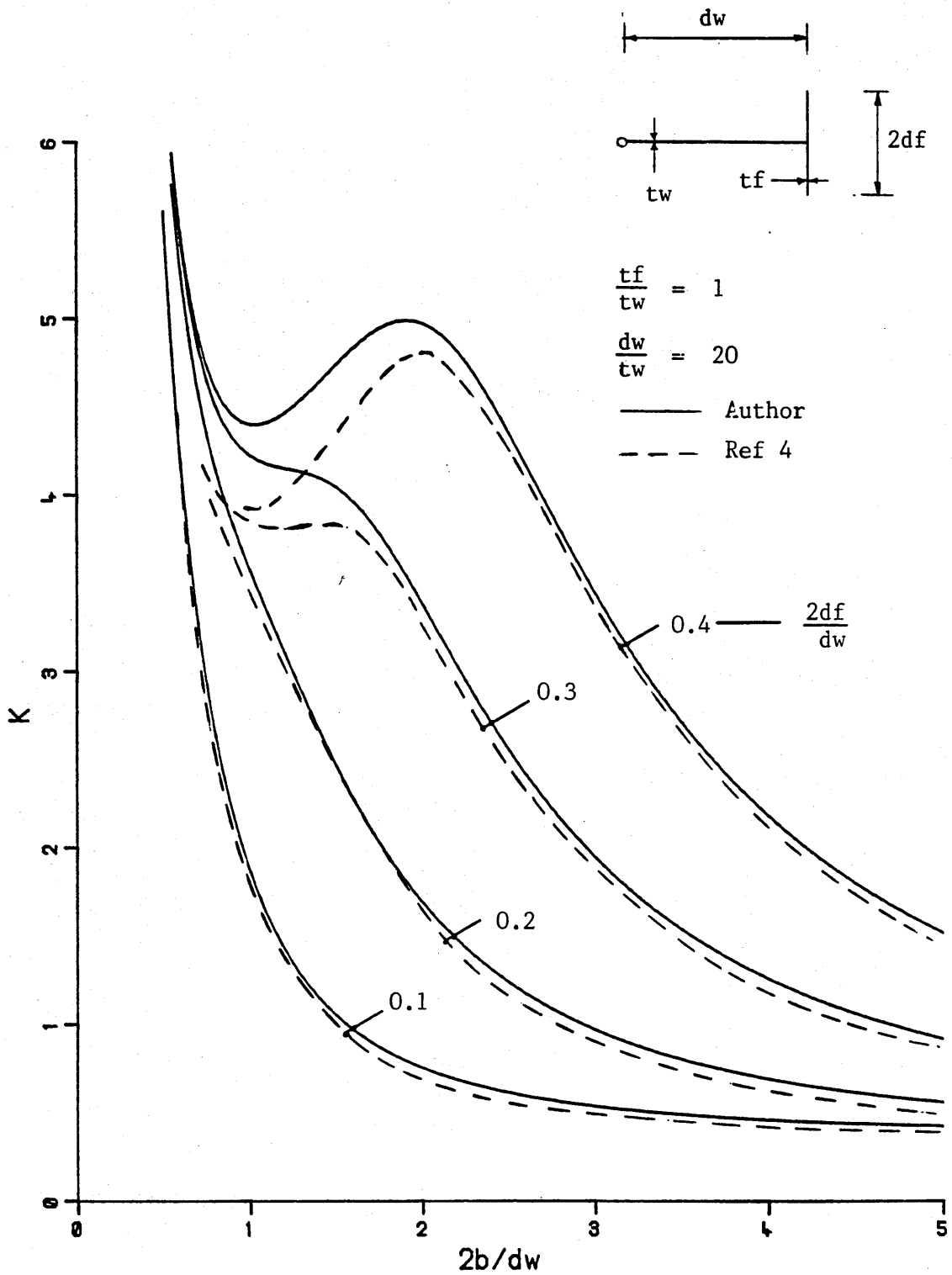


Fig 3.14 Variation of buckling coefficient K with web aspect ratio $2b/dw$

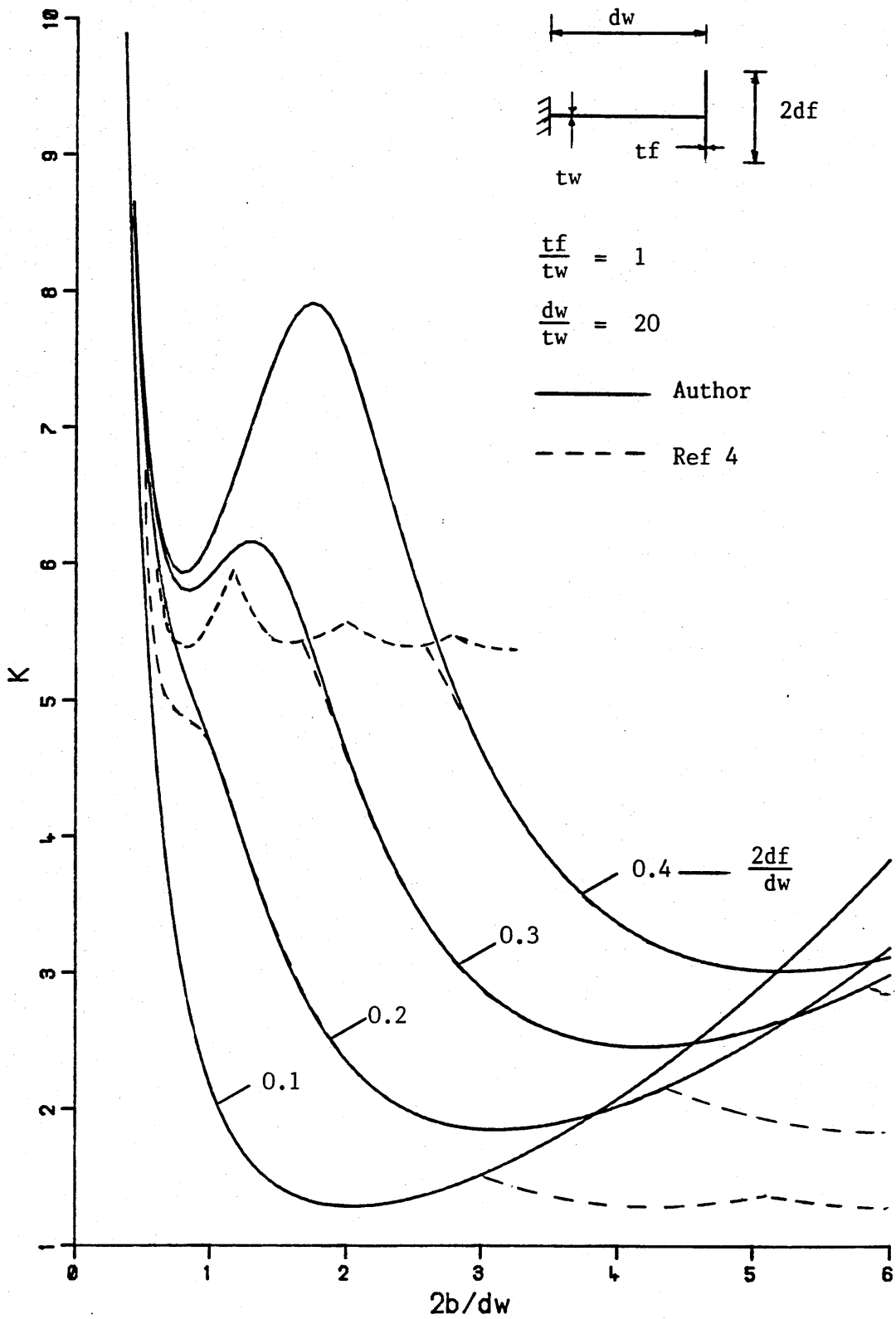


Fig 3.15 Variation of buckling coefficient K with web aspect ratio $2b/dw$

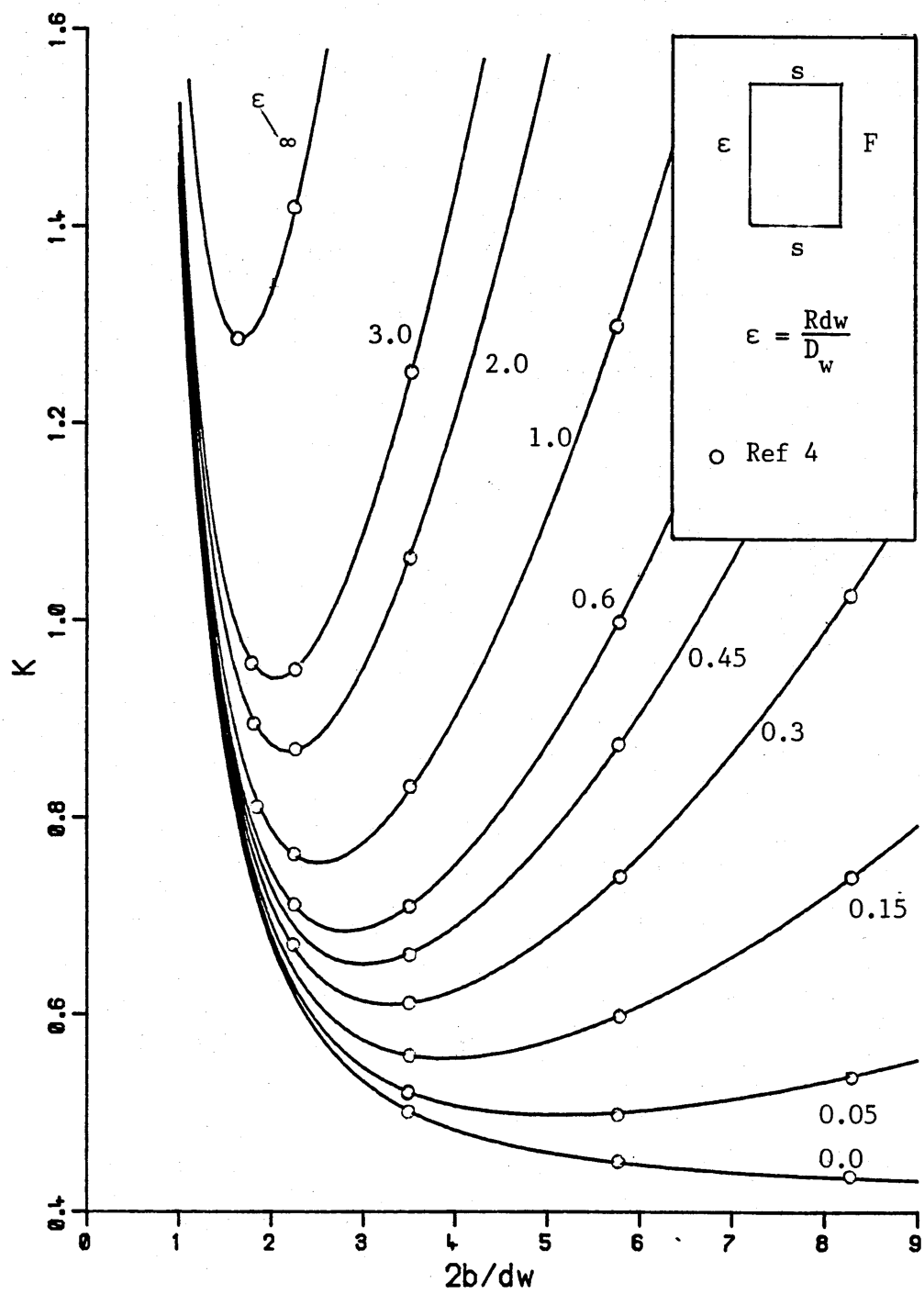


Fig 3.16 Elastically restrained plate

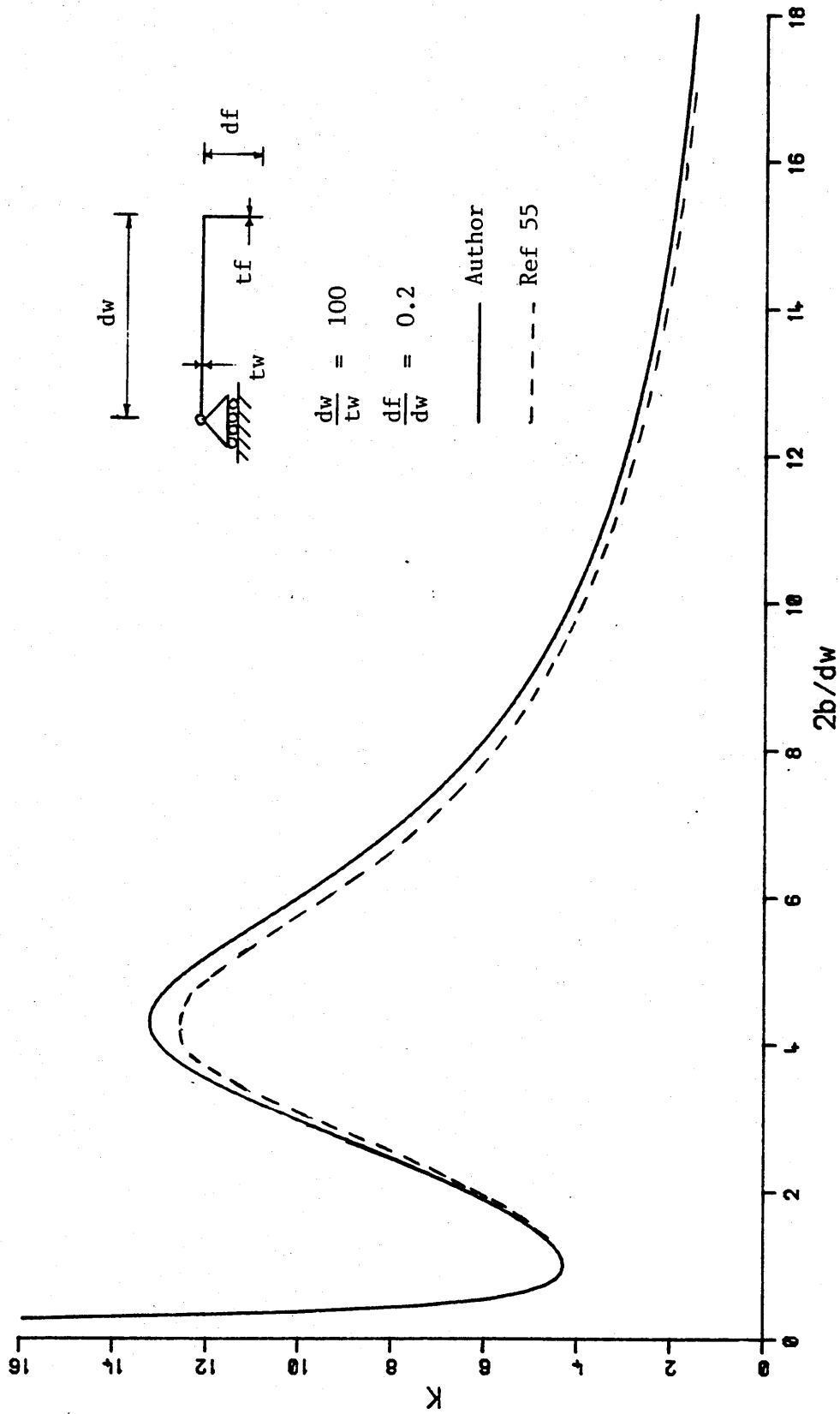


Fig 3.17 Variation of buckling coefficient K with web aspect ratio $2b/dw$

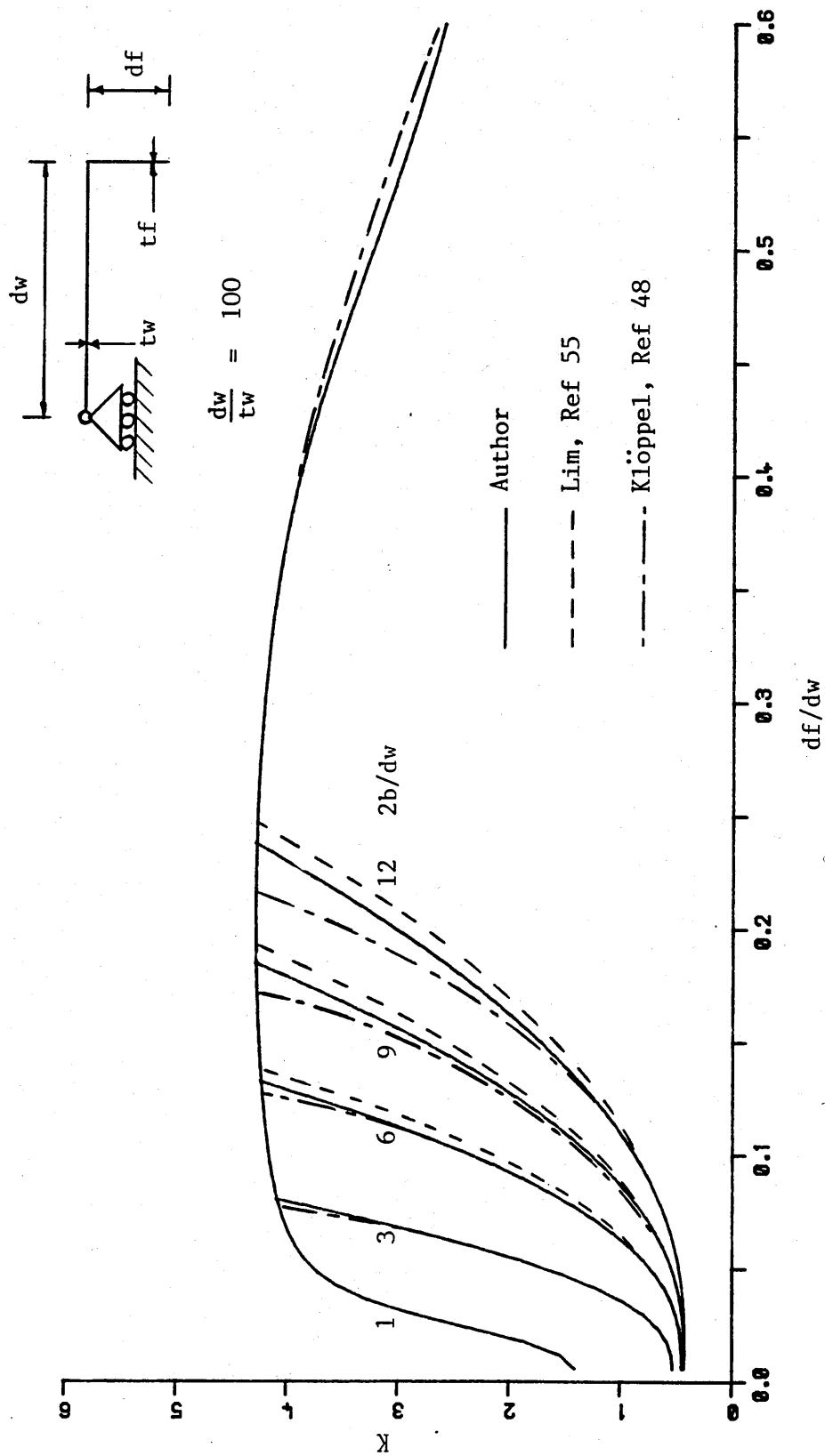
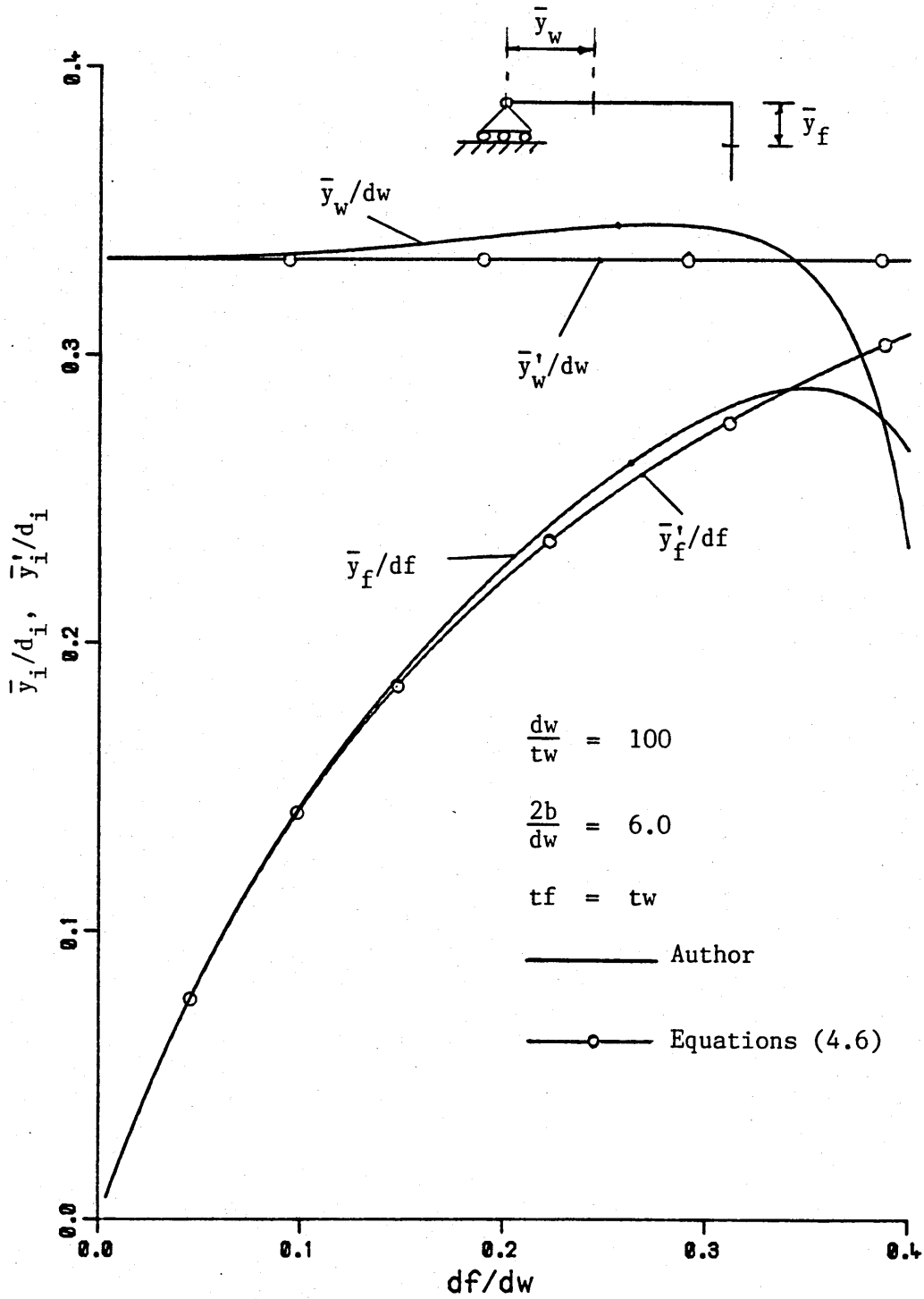


Fig 3.18 Variation of buckling coefficient K with df/dw ratio

Fig 3.19 Variation of neutral axis position with df/dw

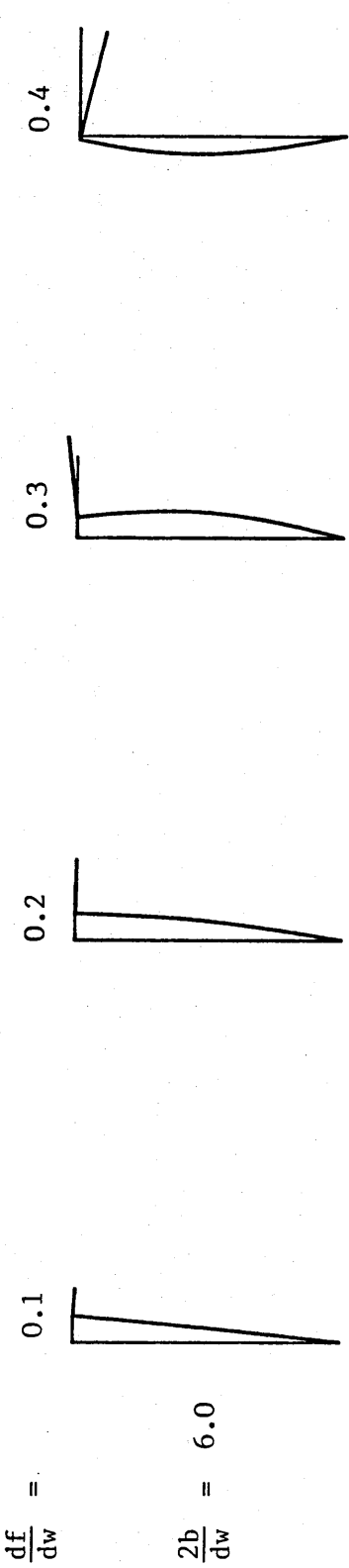


Fig 3.19a Buckled modes

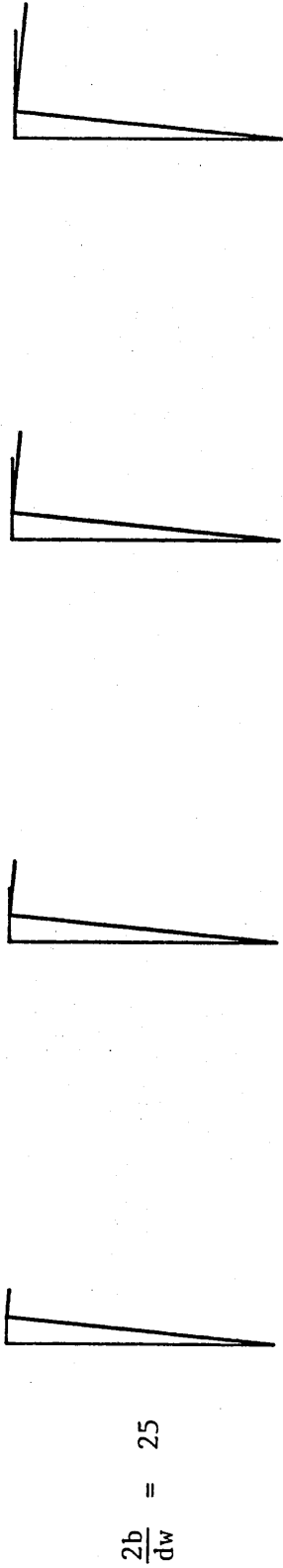


Fig 3.19b Buckled modes

$$\frac{dw}{tw} = 100, \quad tf = tw$$

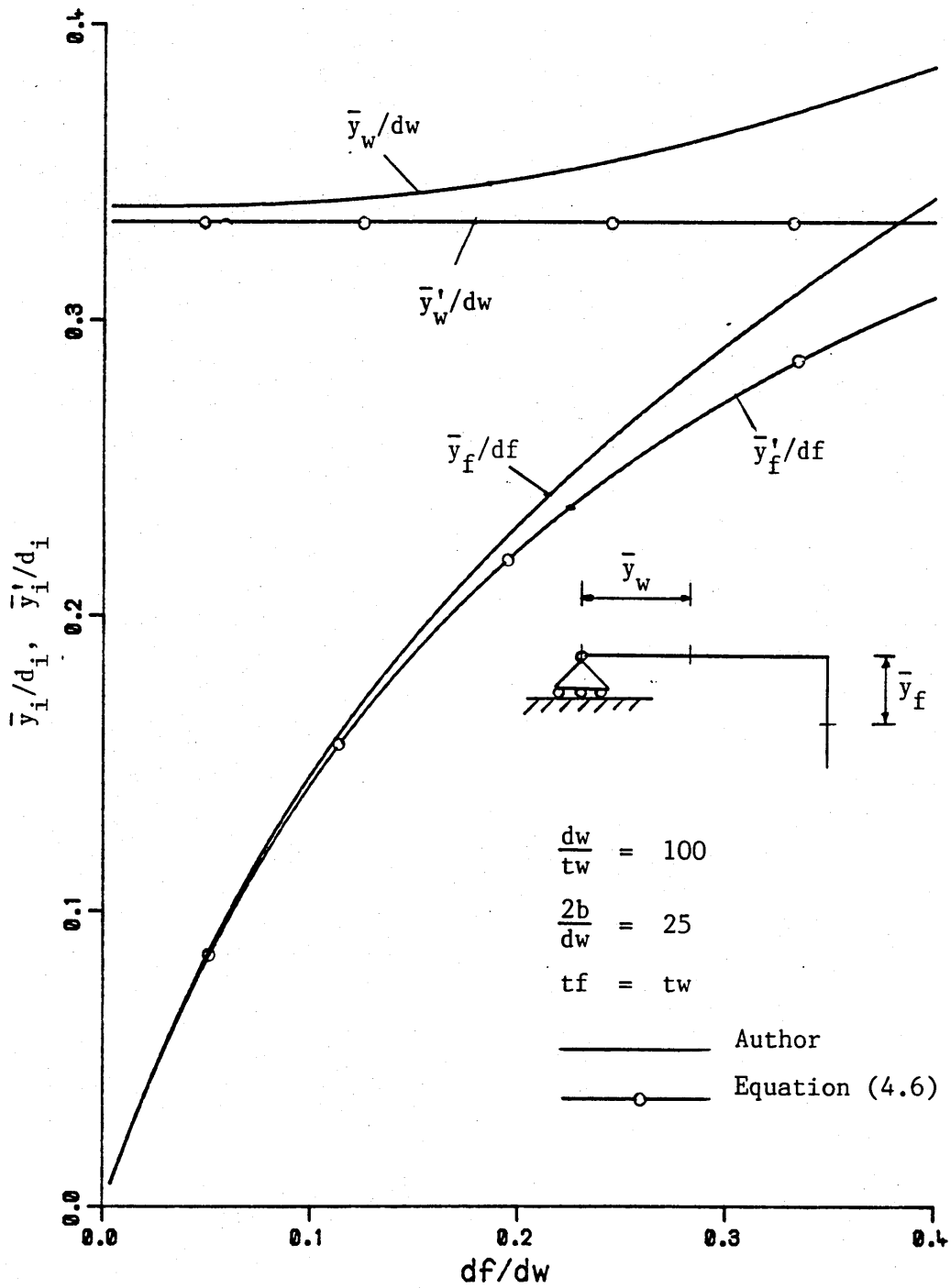
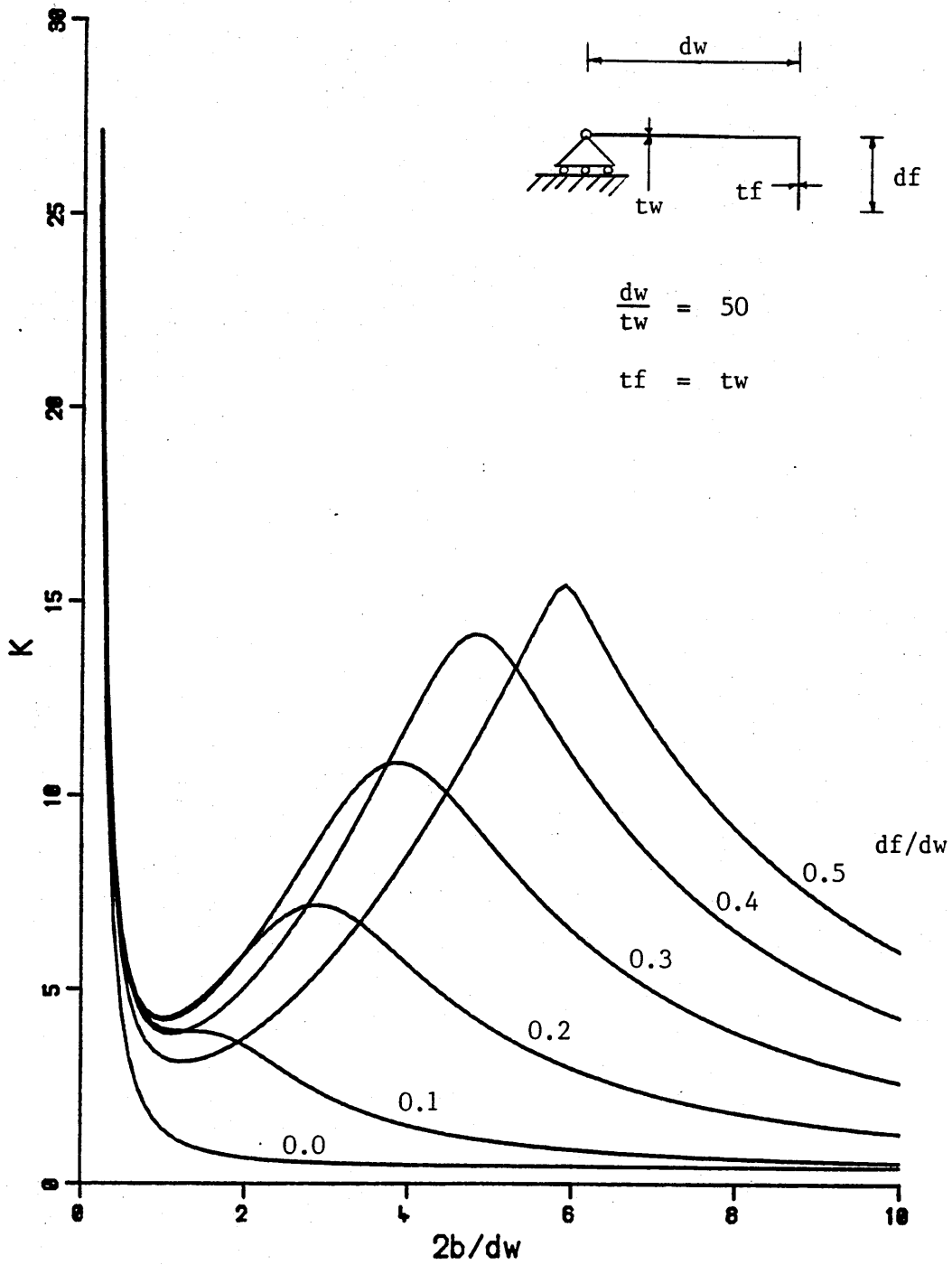


Fig 3.20 Variation of neutral axis position with df/dw

Fig 3.21 Variation of K with $2b/dw$

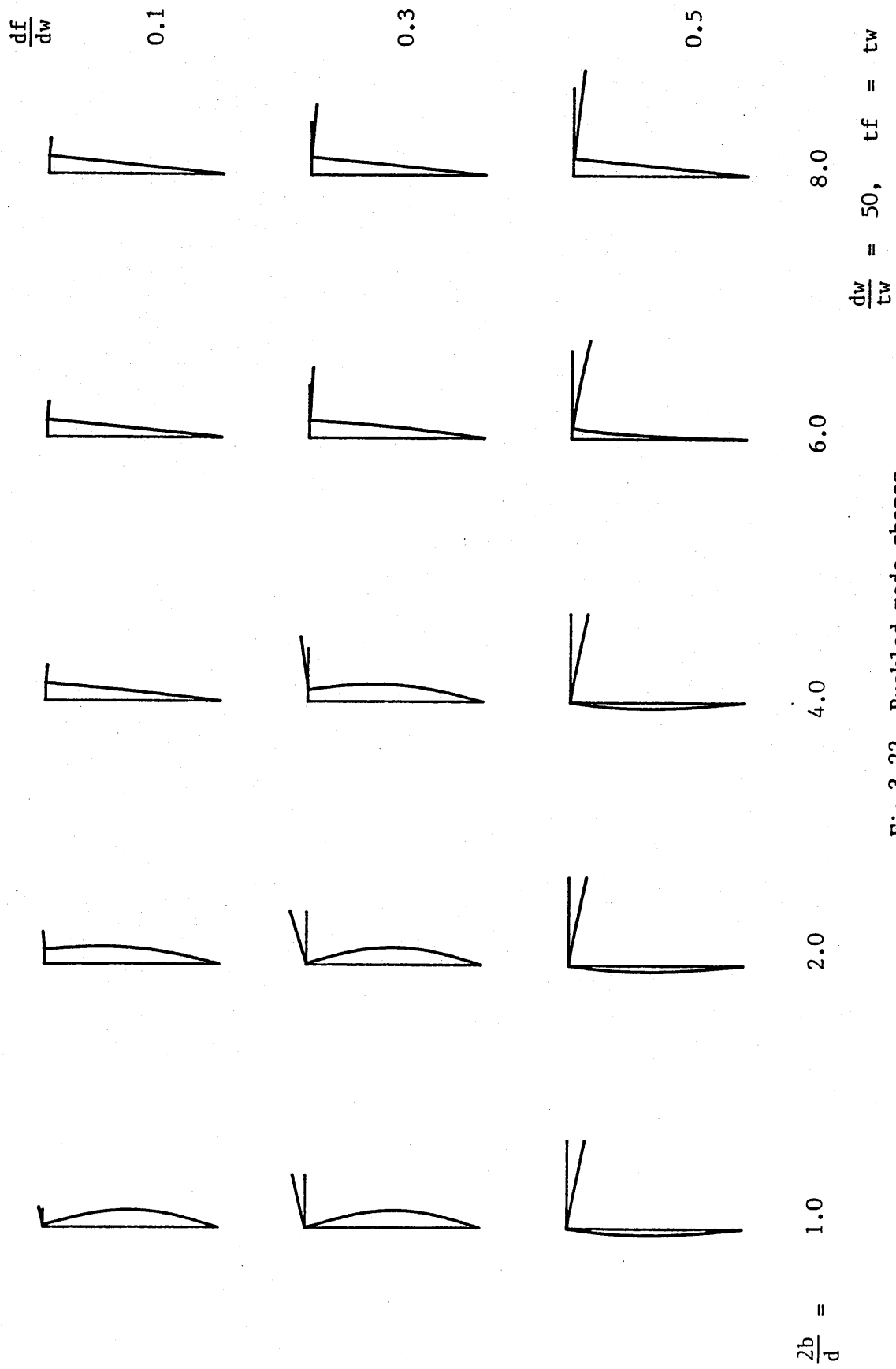
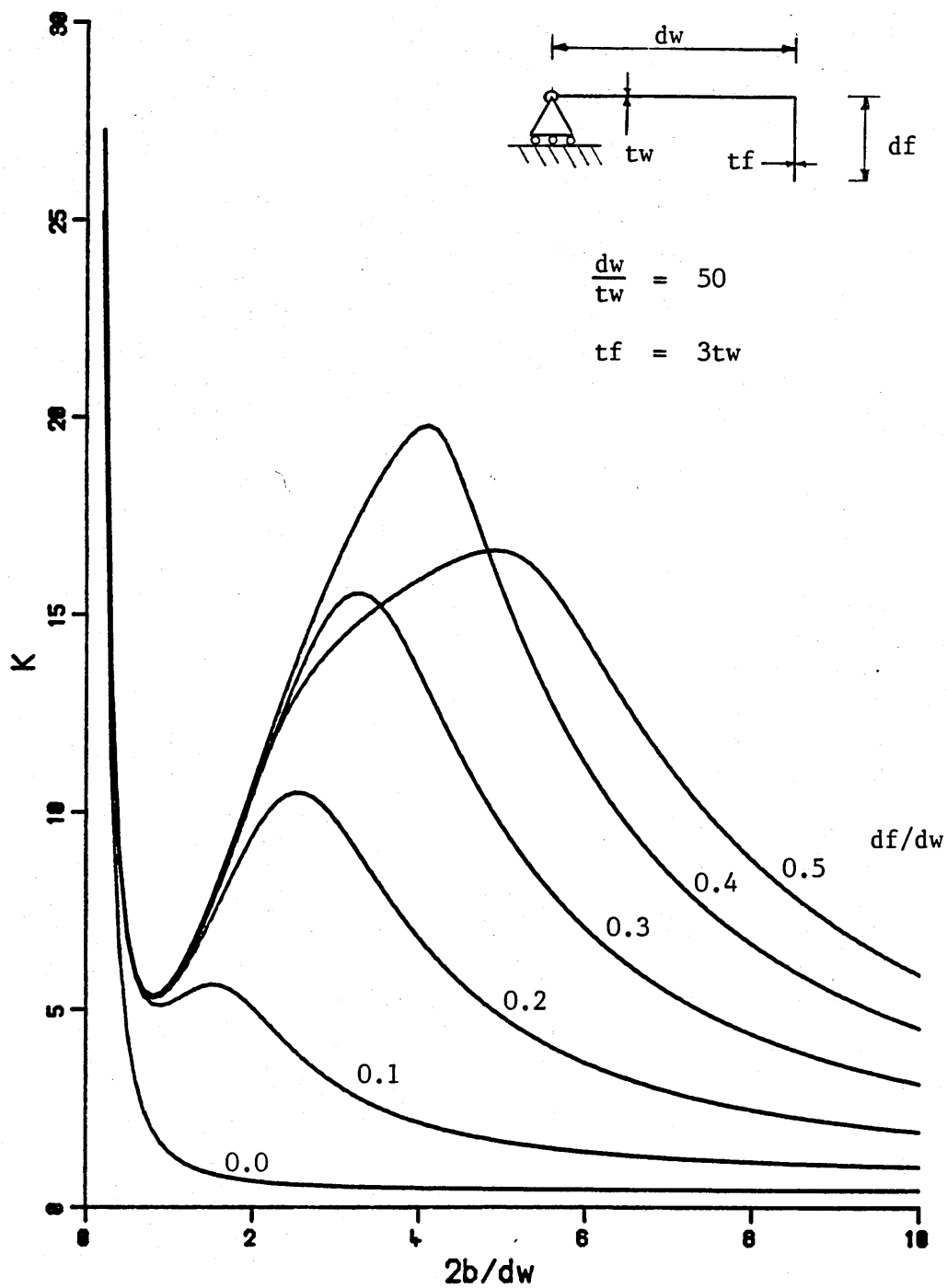
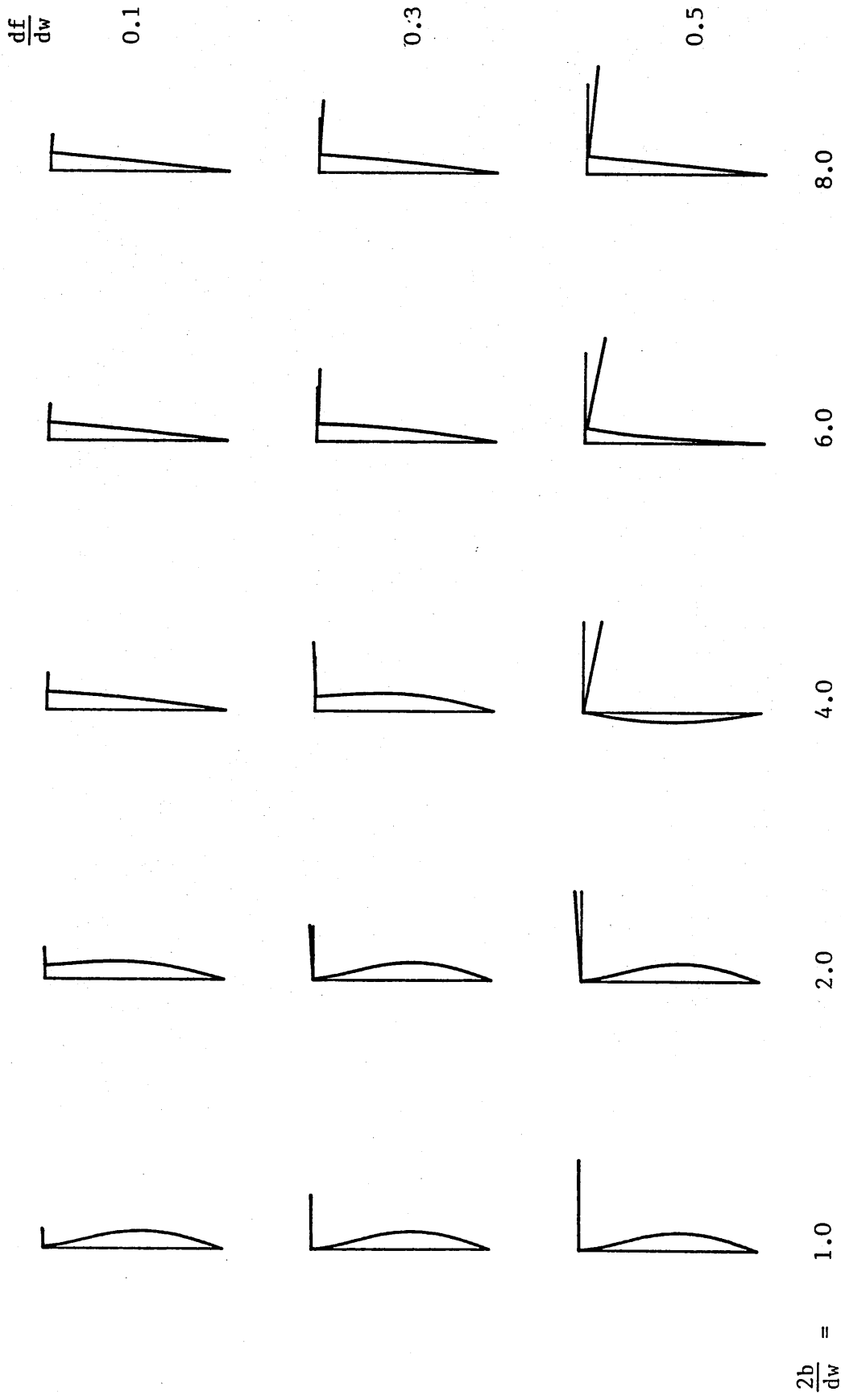


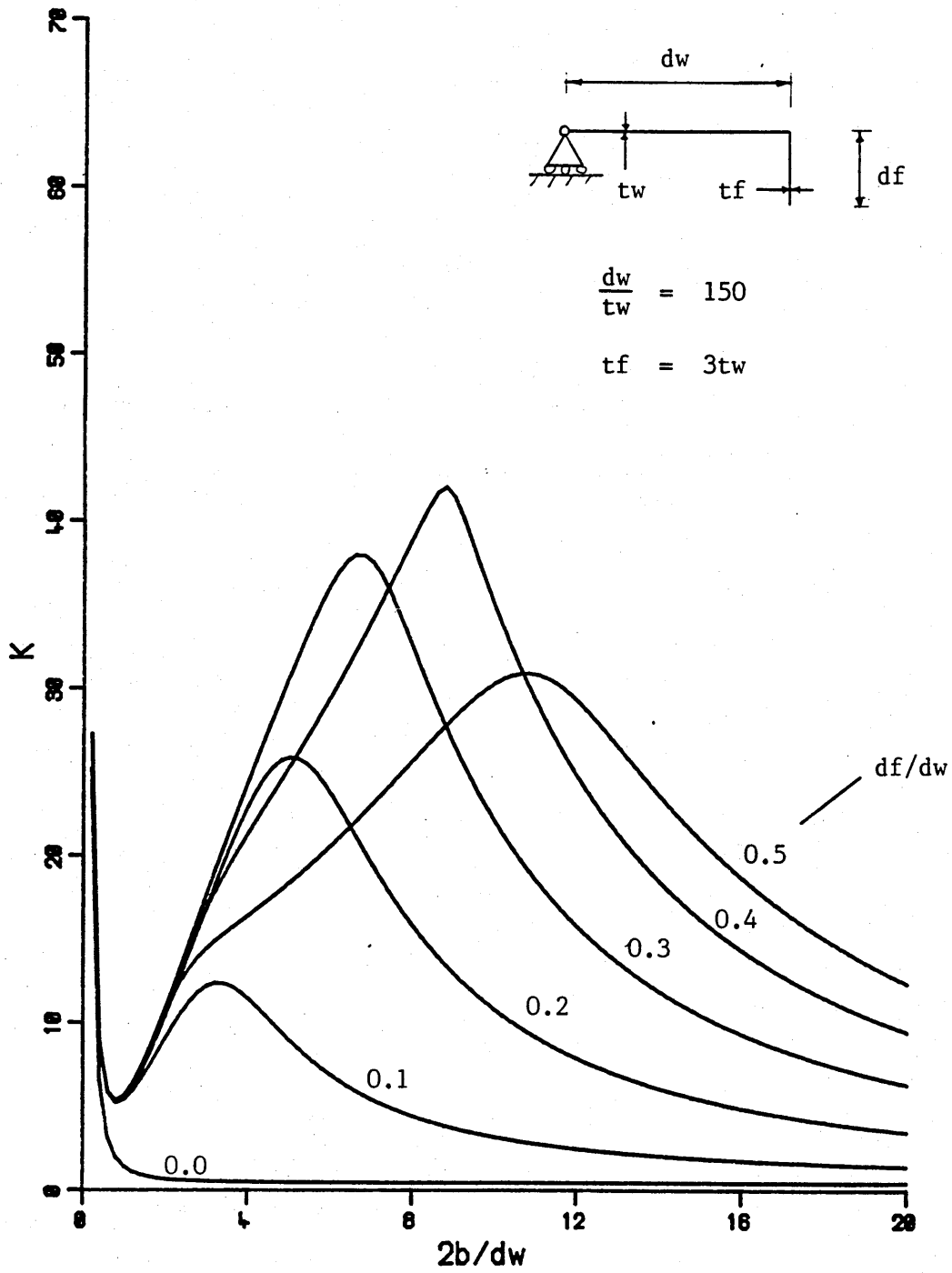
Fig 3.22 Buckled mode shapes

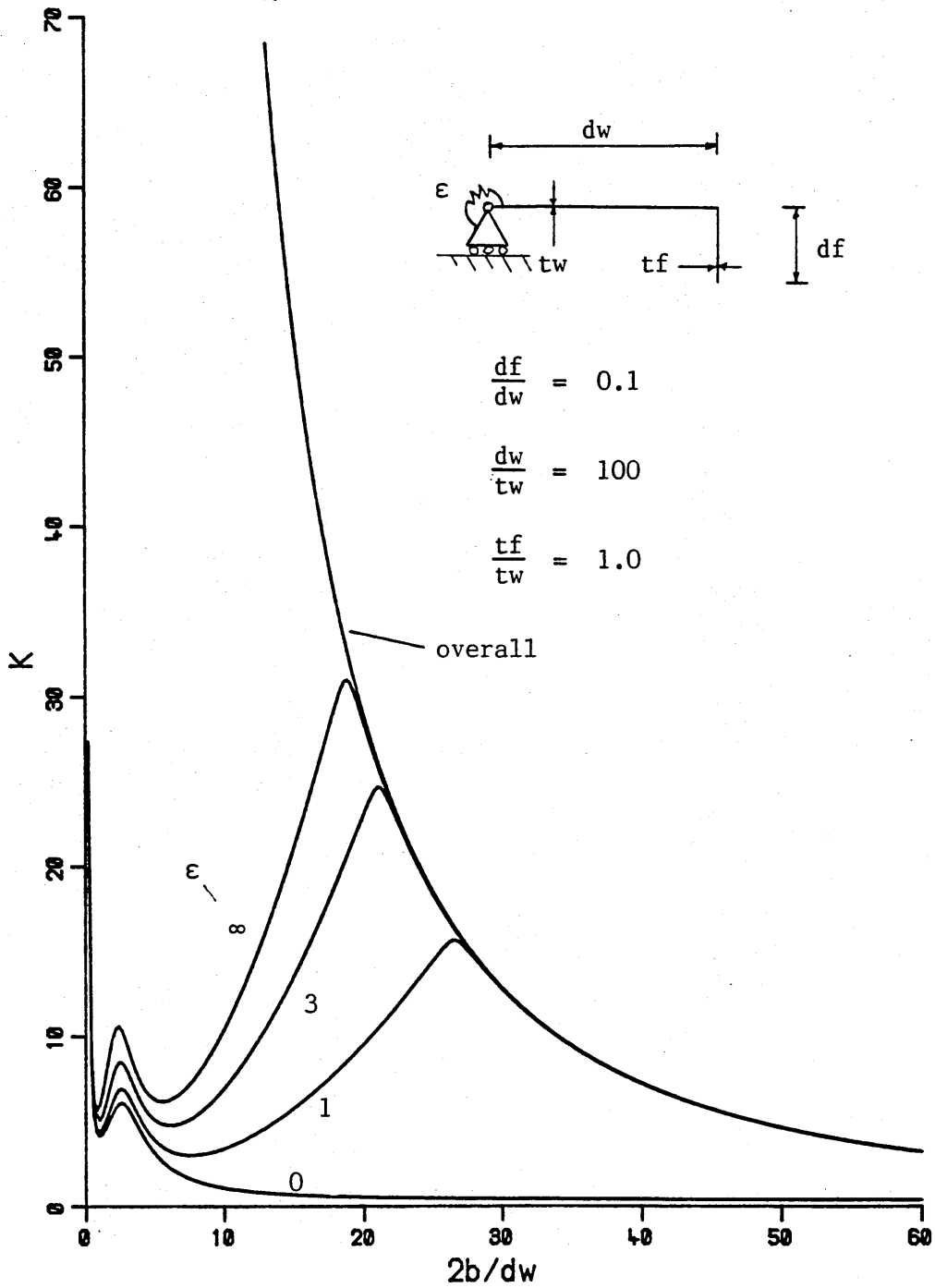
Fig 3.23 Variation of K with $2b/dw$

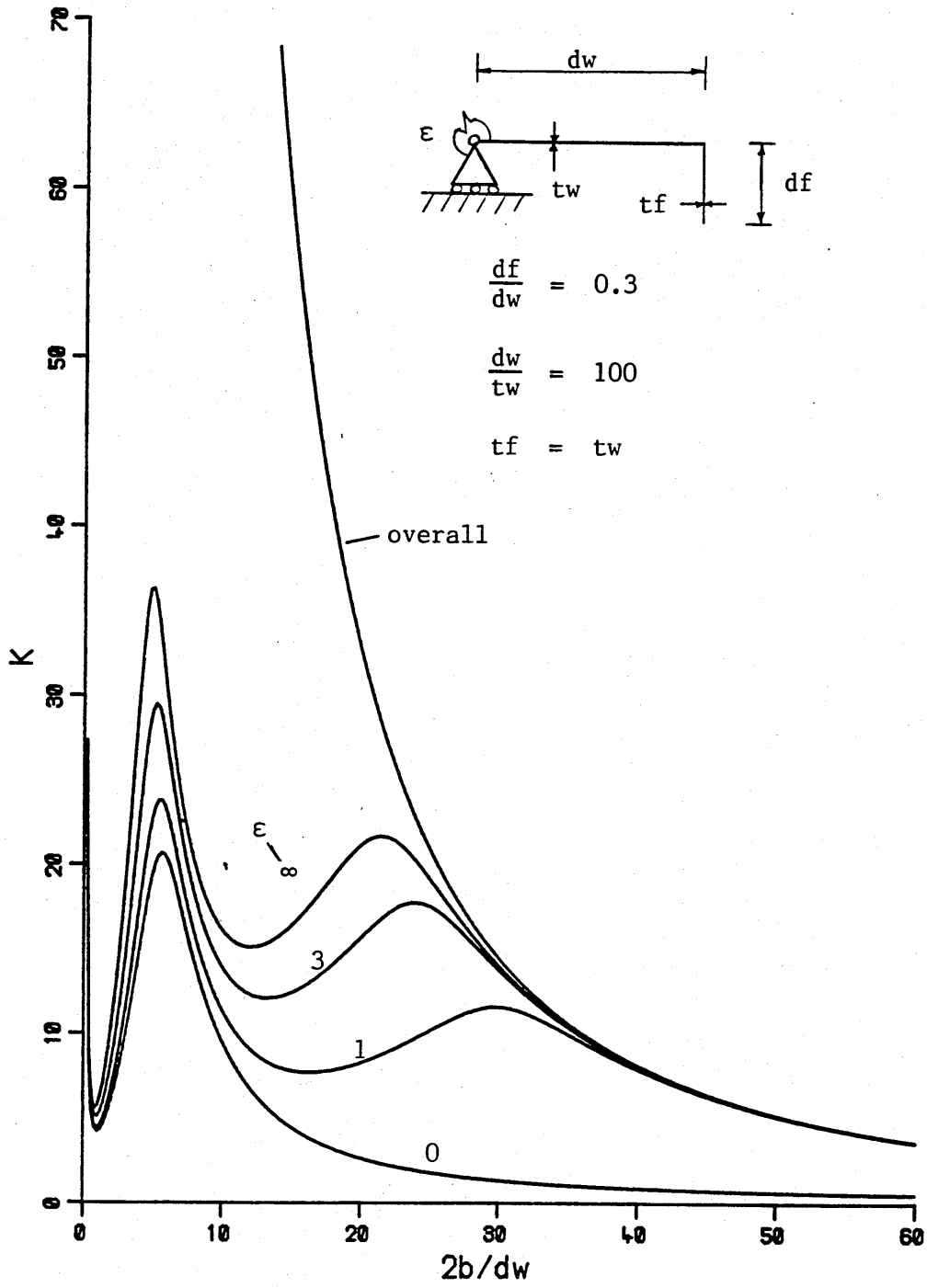


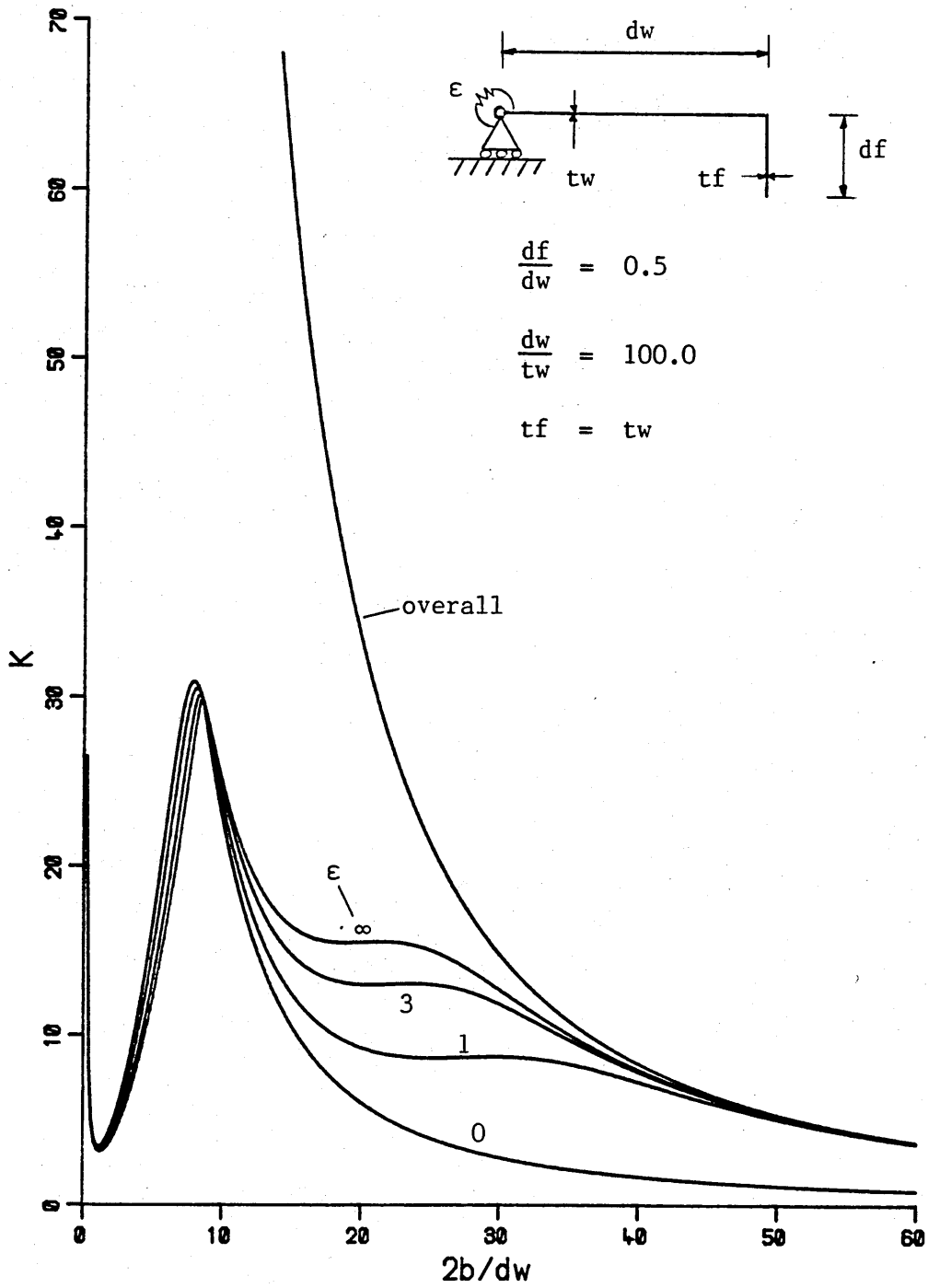
$$\frac{dw}{tw} = 50, \quad tf = 3tw$$

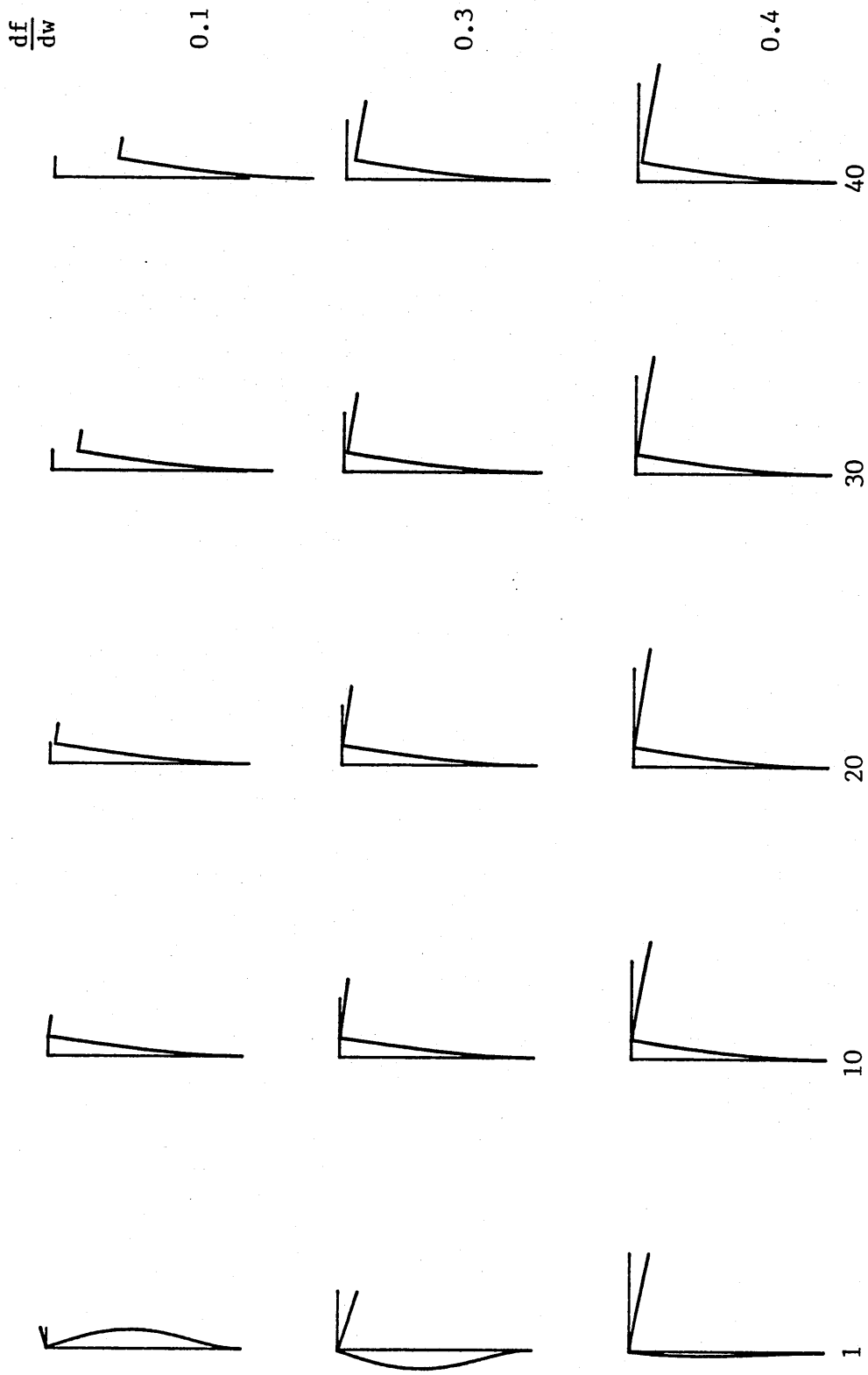
Fig 3.24 Buckled mode shapes

Fig 3.25 Variation of K with $2b/dw$

Fig 3.26 Variation of K with $2b/dw$

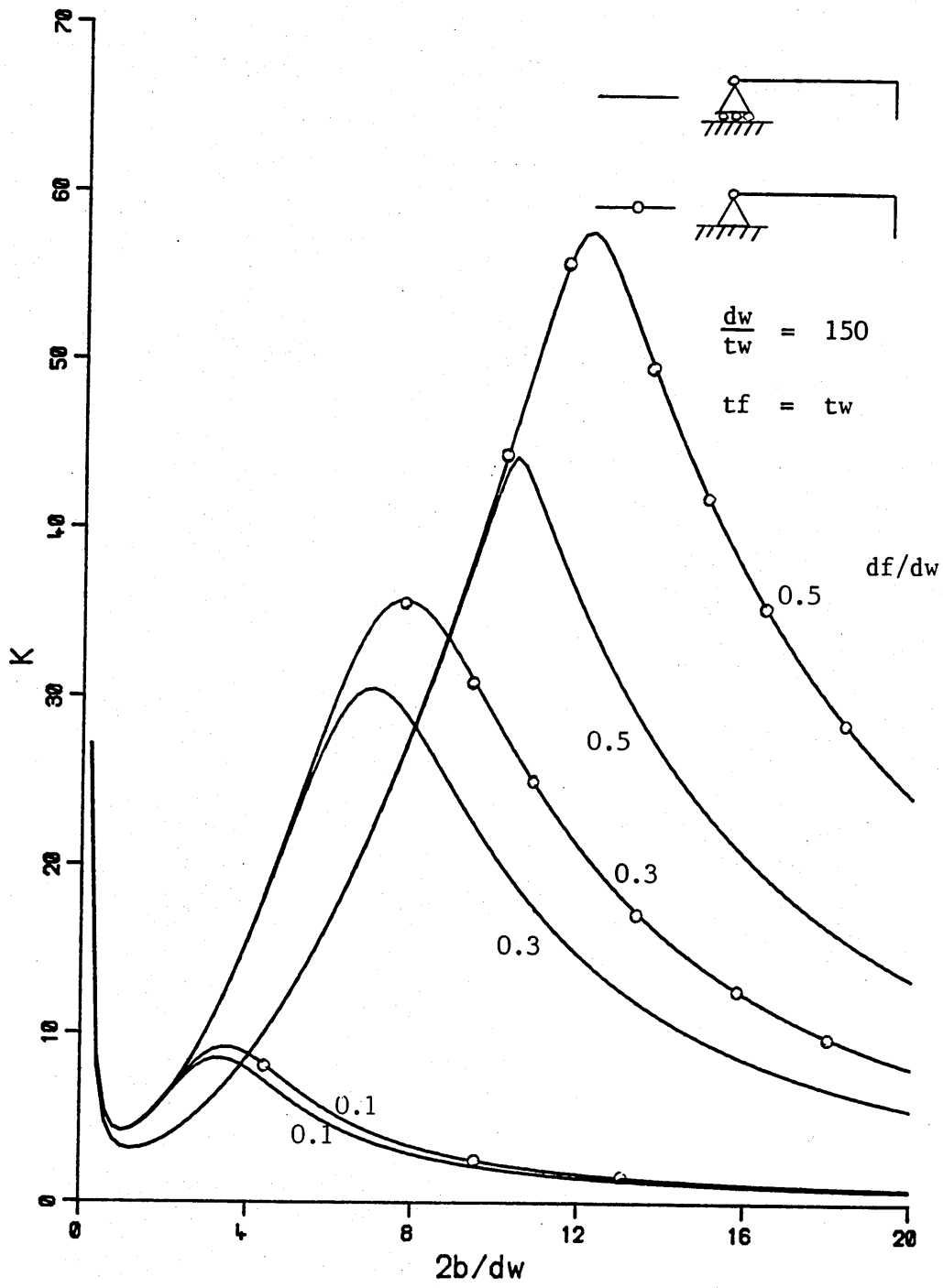
Fig 3.27 Variation of K with $2b/dw$

Fig 3.28 Variation of K with $2b/dw$



$$\frac{dw}{tw} = 100, \quad \frac{tf}{tw} = 1.0, \quad \epsilon = \infty$$

Fig 3.29 Buckled mode shapes

Fig 3.30 Variation of K with $2b/dw$

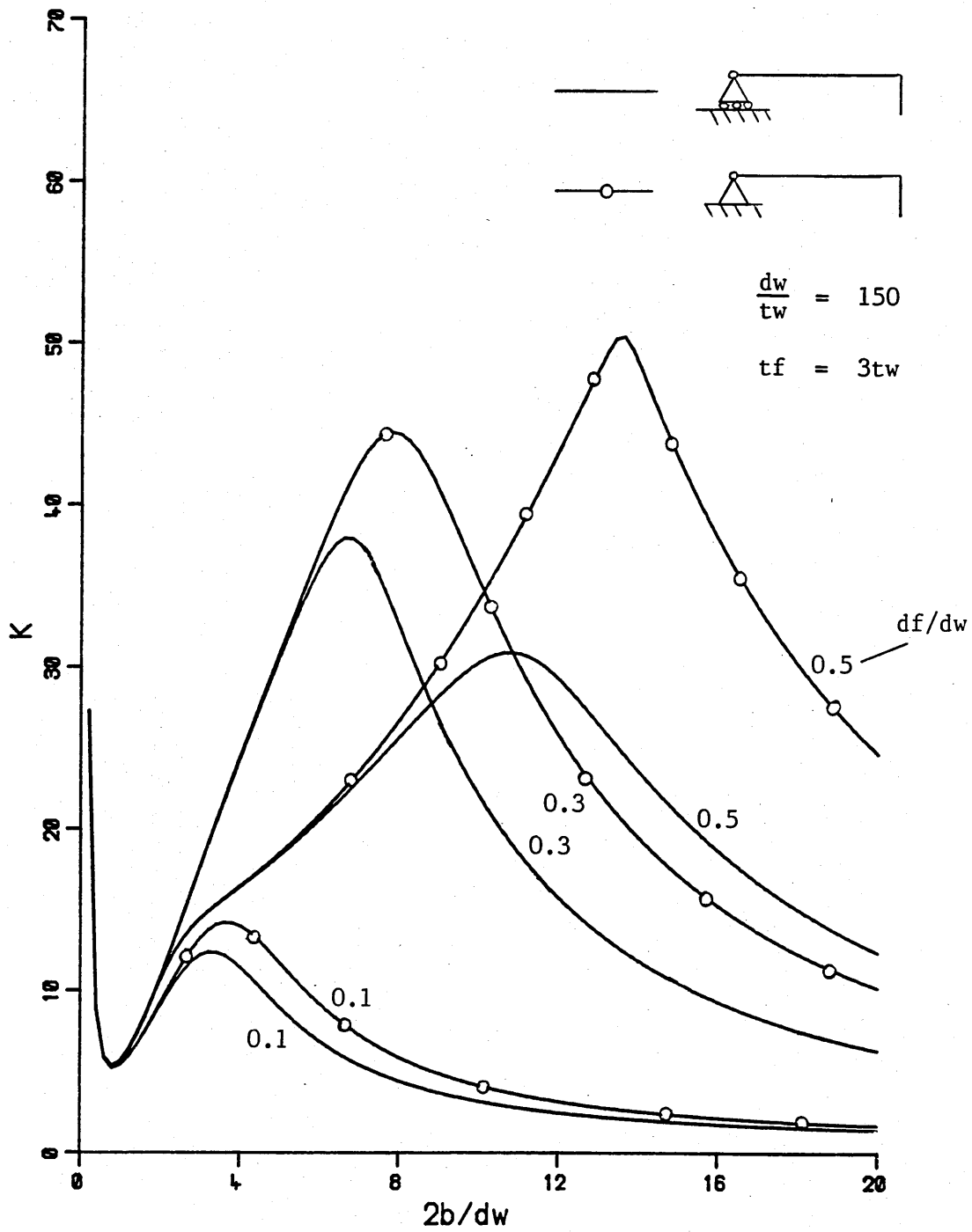


Fig 3.31 Variation of buckling coefficient K with web aspect ratio $2b/dw$

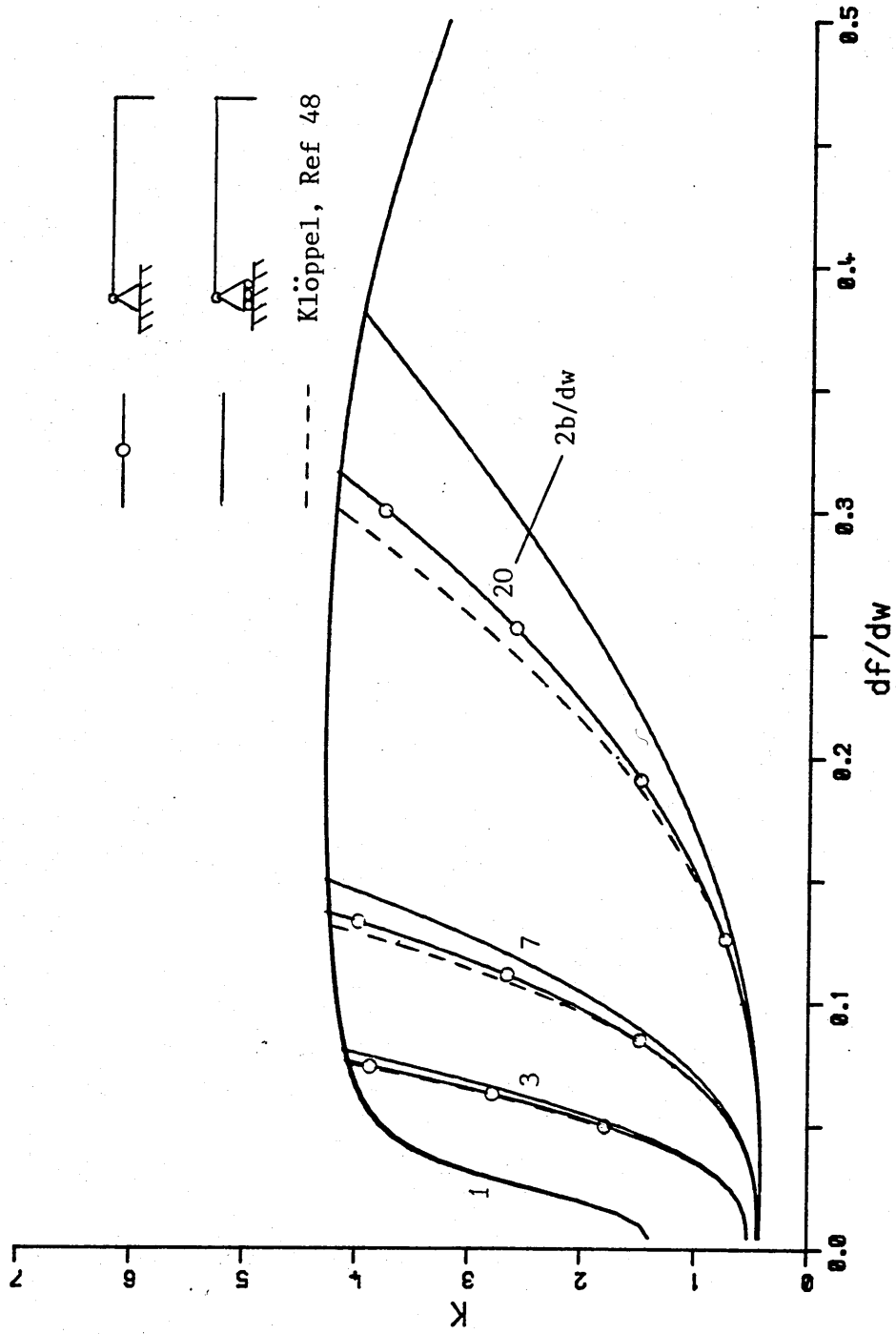


Fig 3.32 Variation of K with df/dw $\frac{dw}{tw} = 100, \quad t_f = t_w$

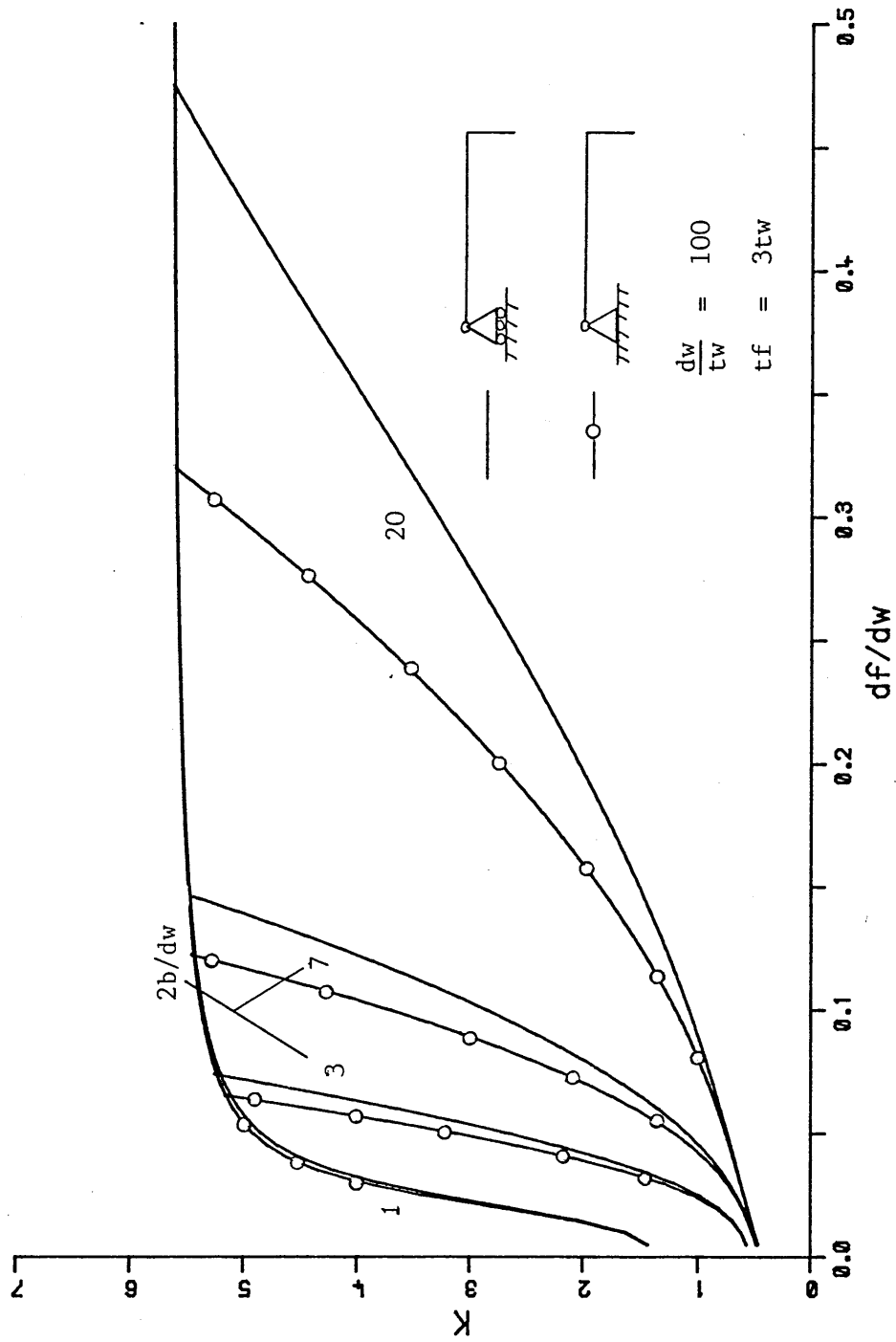


Fig 3.33 Variation of K with df/dw

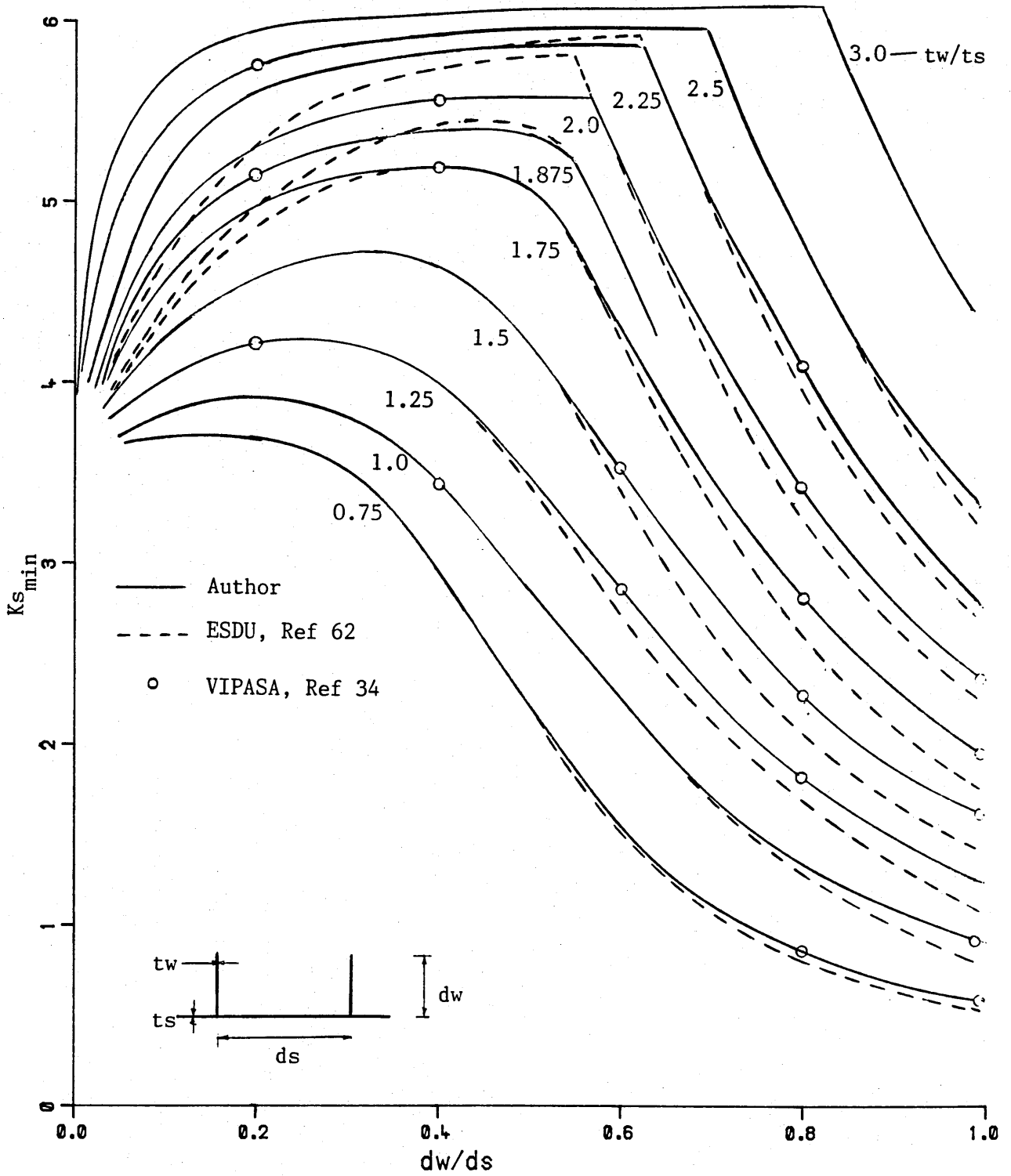
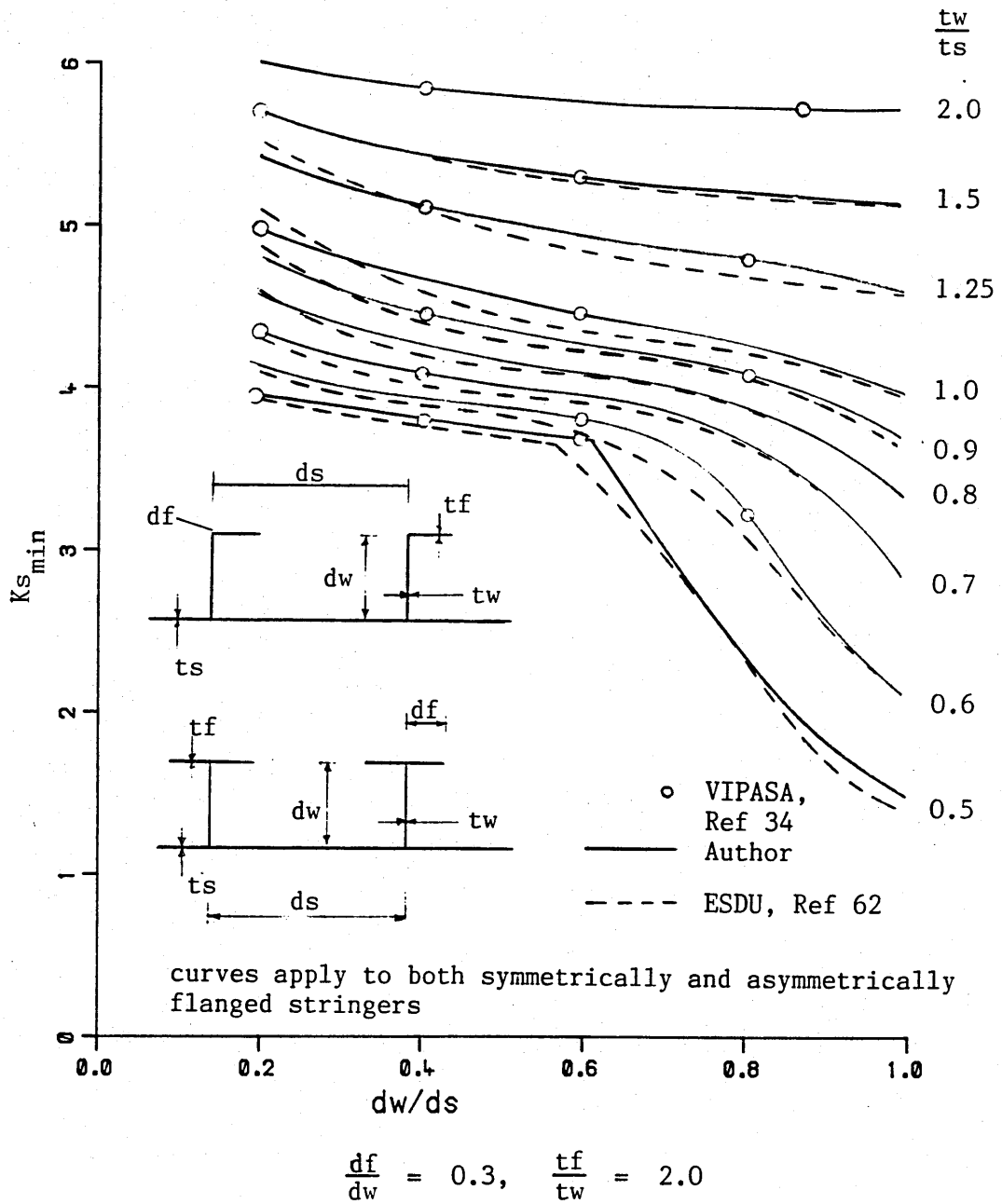


Fig 3.34 Variation of $K_{s_{min}}$ with dw/ds

Fig 3.35 Variation of $K_{s_{min}}$ with dw/ds

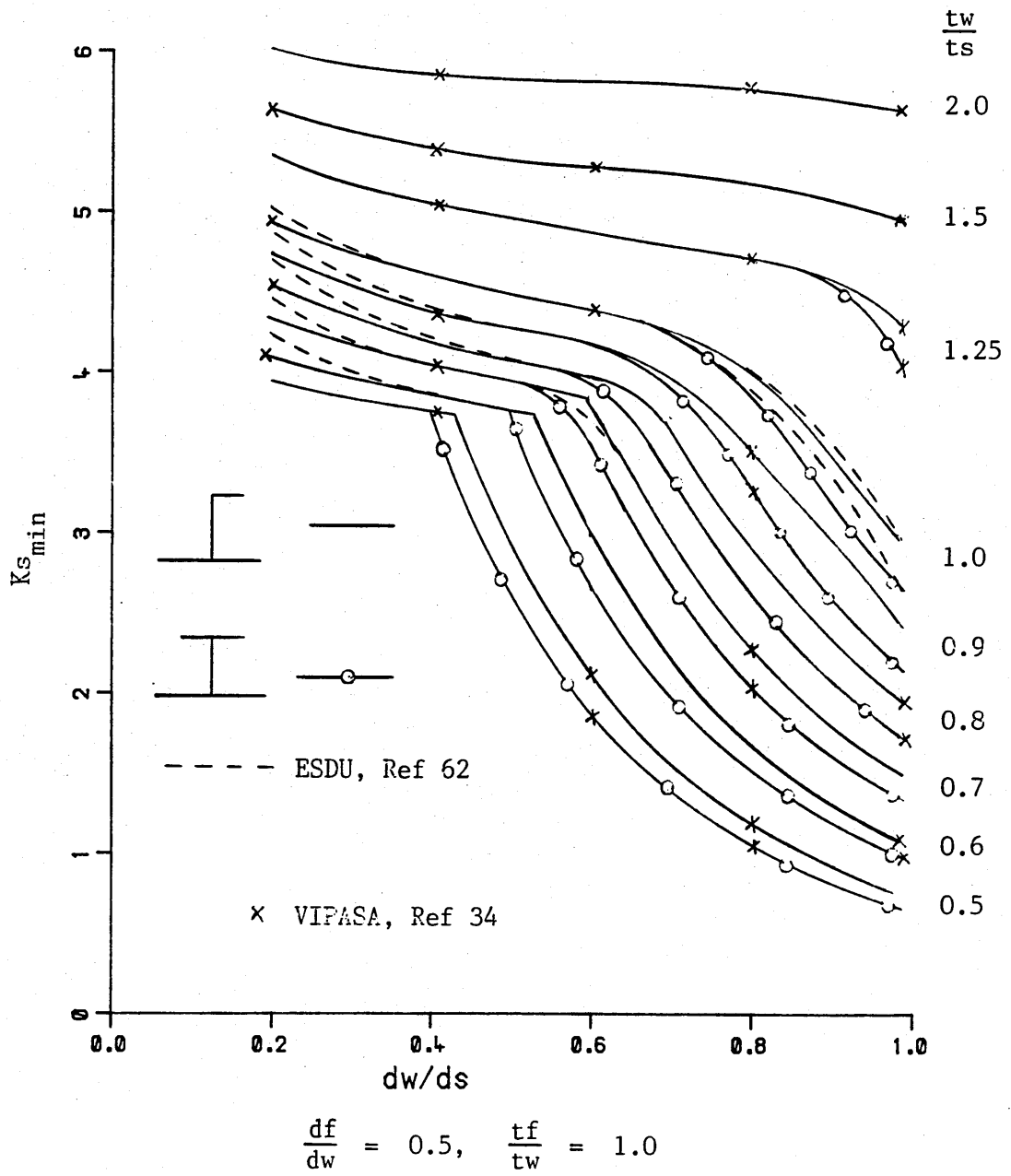


Fig 3.36 Variation of K_{min} with dw/ds

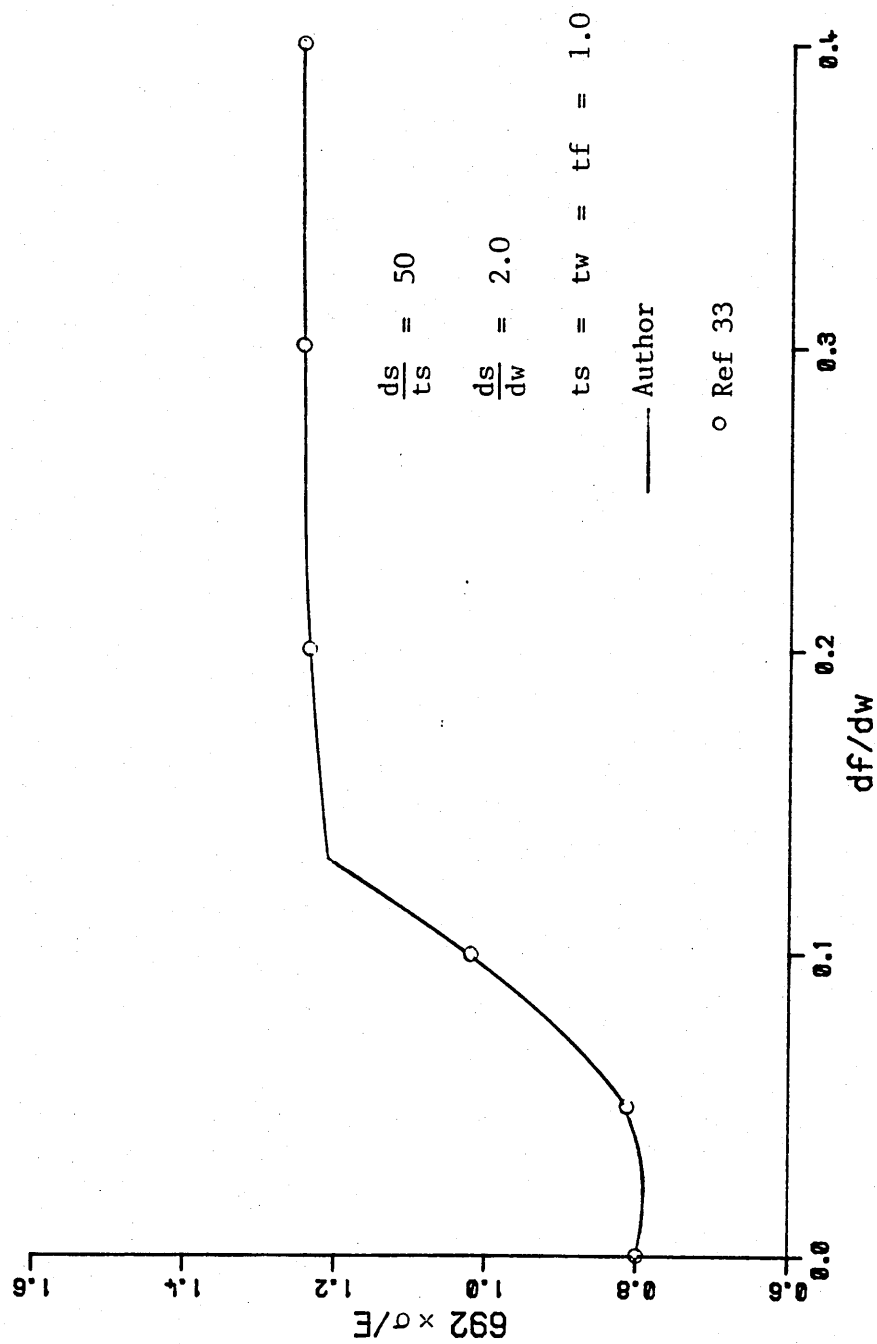


Fig 3.37 Variation of critical stress factor with df/dw

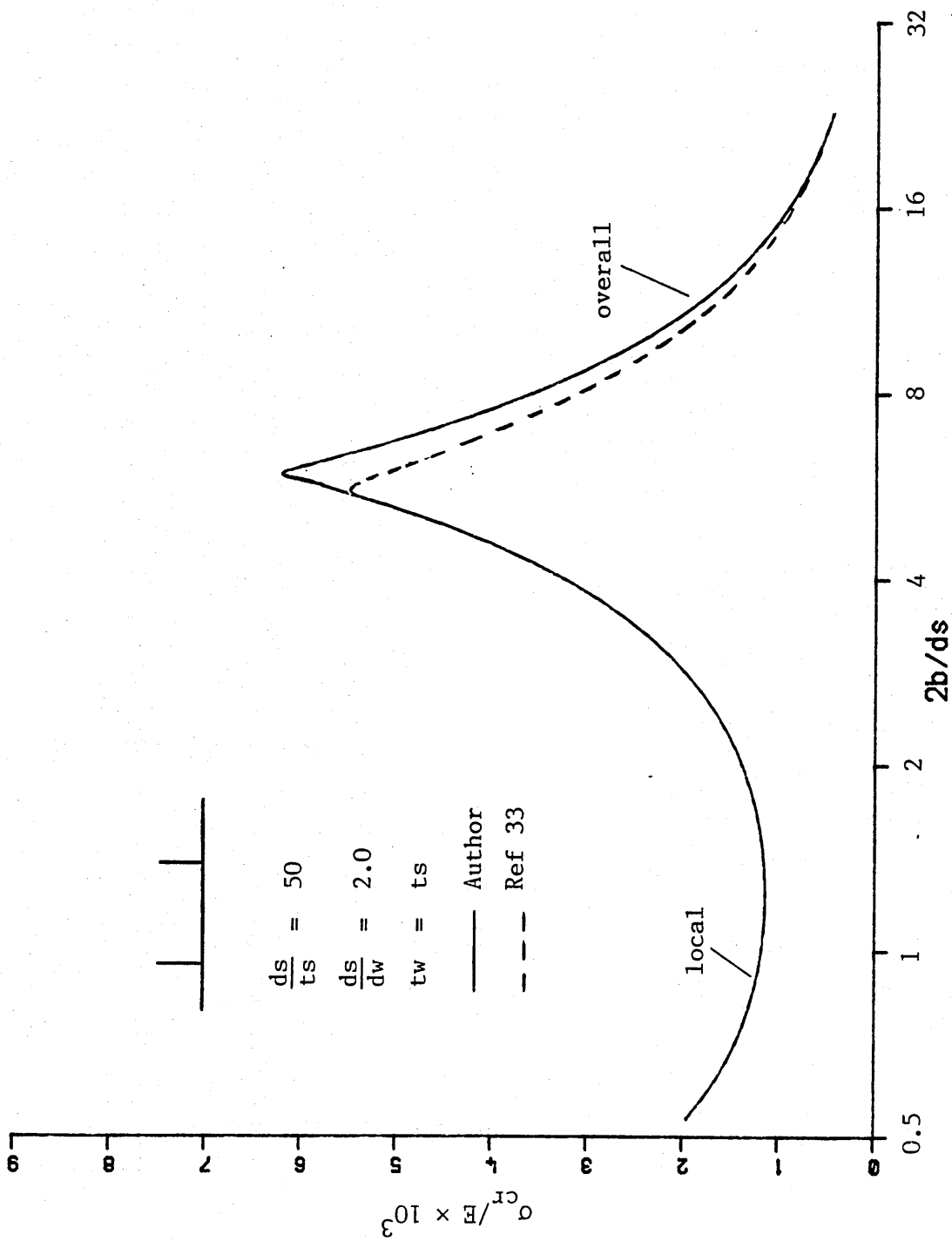


Fig 3.38 Variation of a critical stress factor with $2b/ds$ for an infinite panel and a finite panel

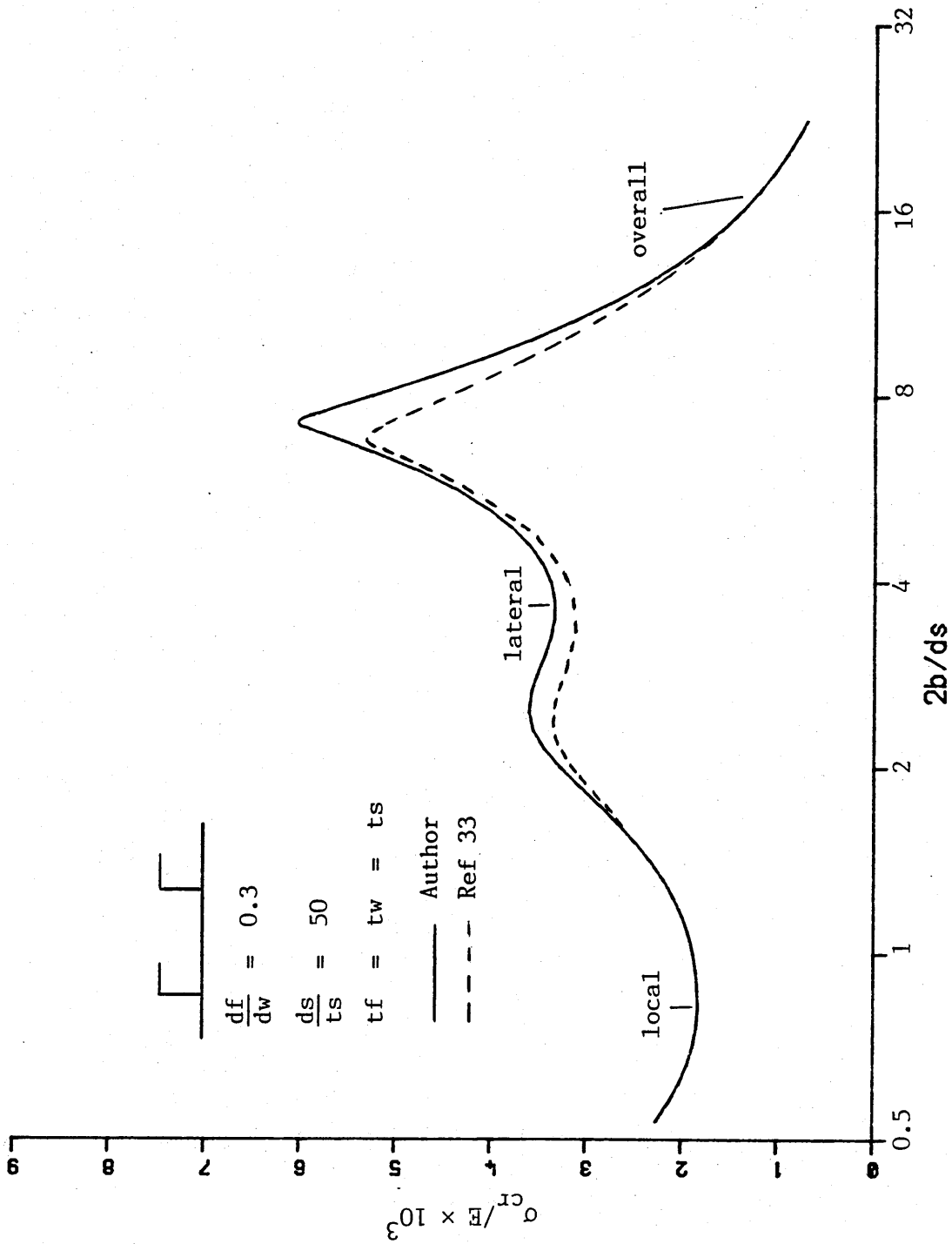


Fig 3.39 Variation of critical stress factor with $2b/ds$ for an infinite panel and a finite panel

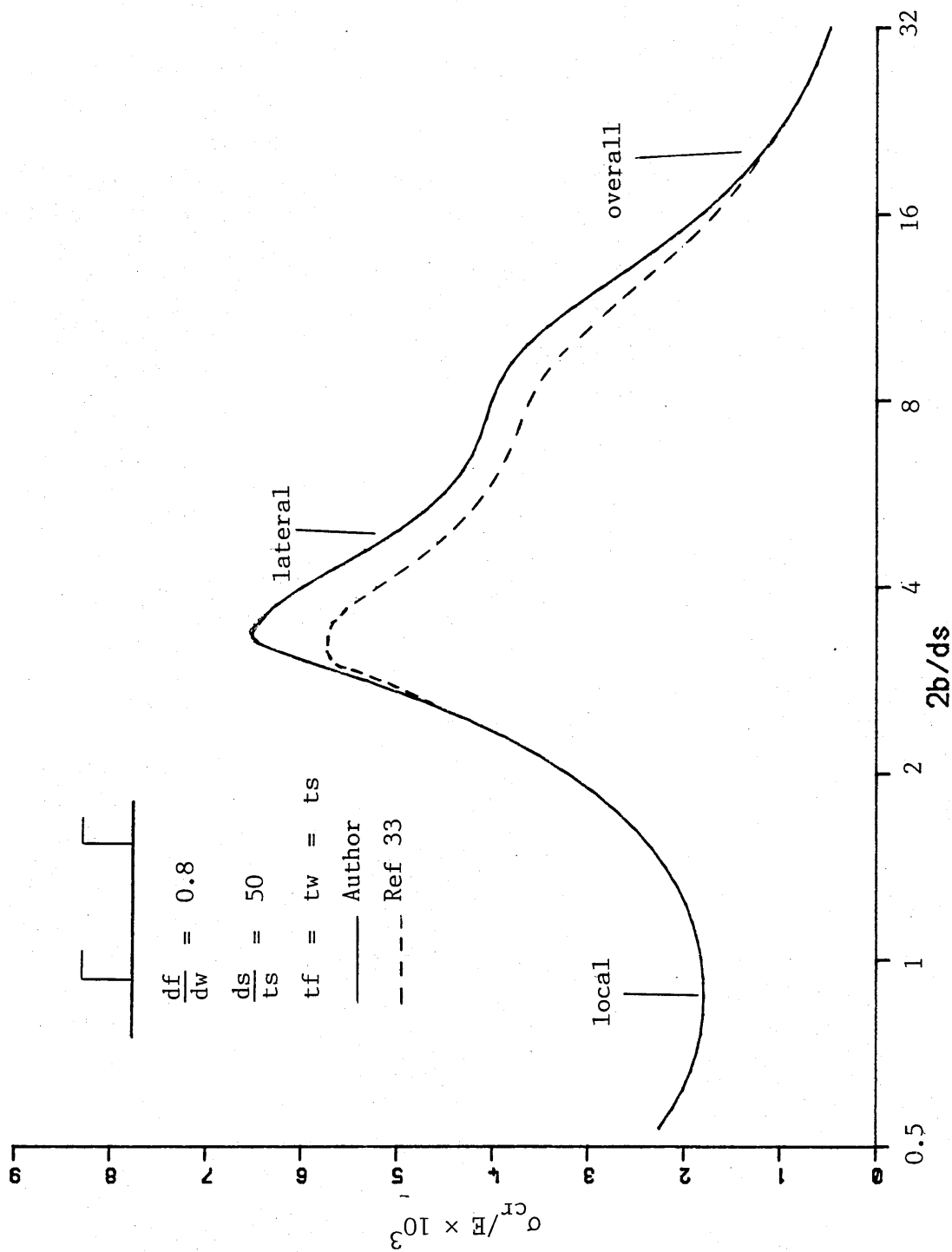


Fig 3.40 Variation of critical stress factor with $2b/ds$ for an infinite panel and a finite panel

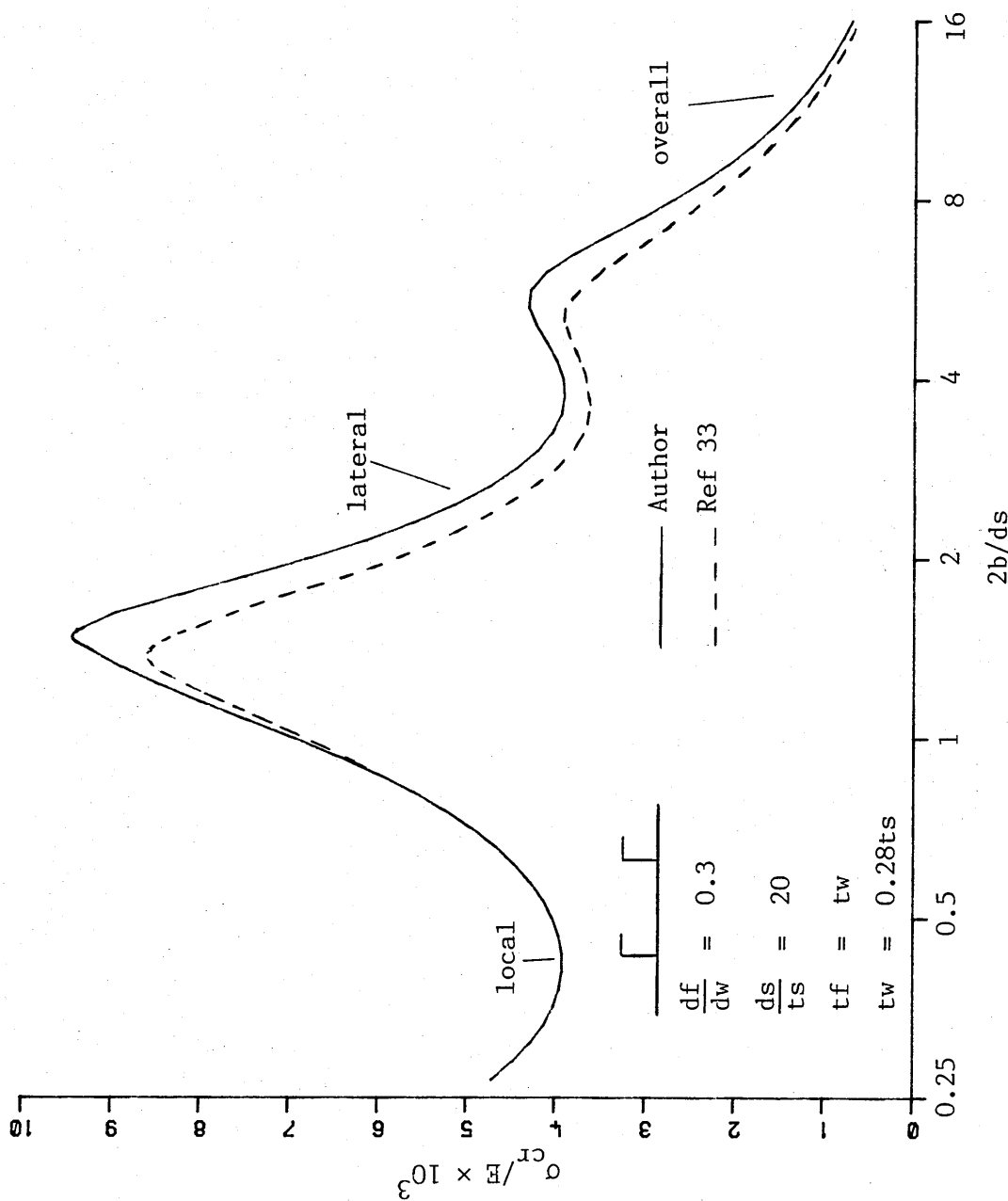


Fig 3.41 Variation of critical stress factor with $2b/ds$ for an infinite panel and a finite panel

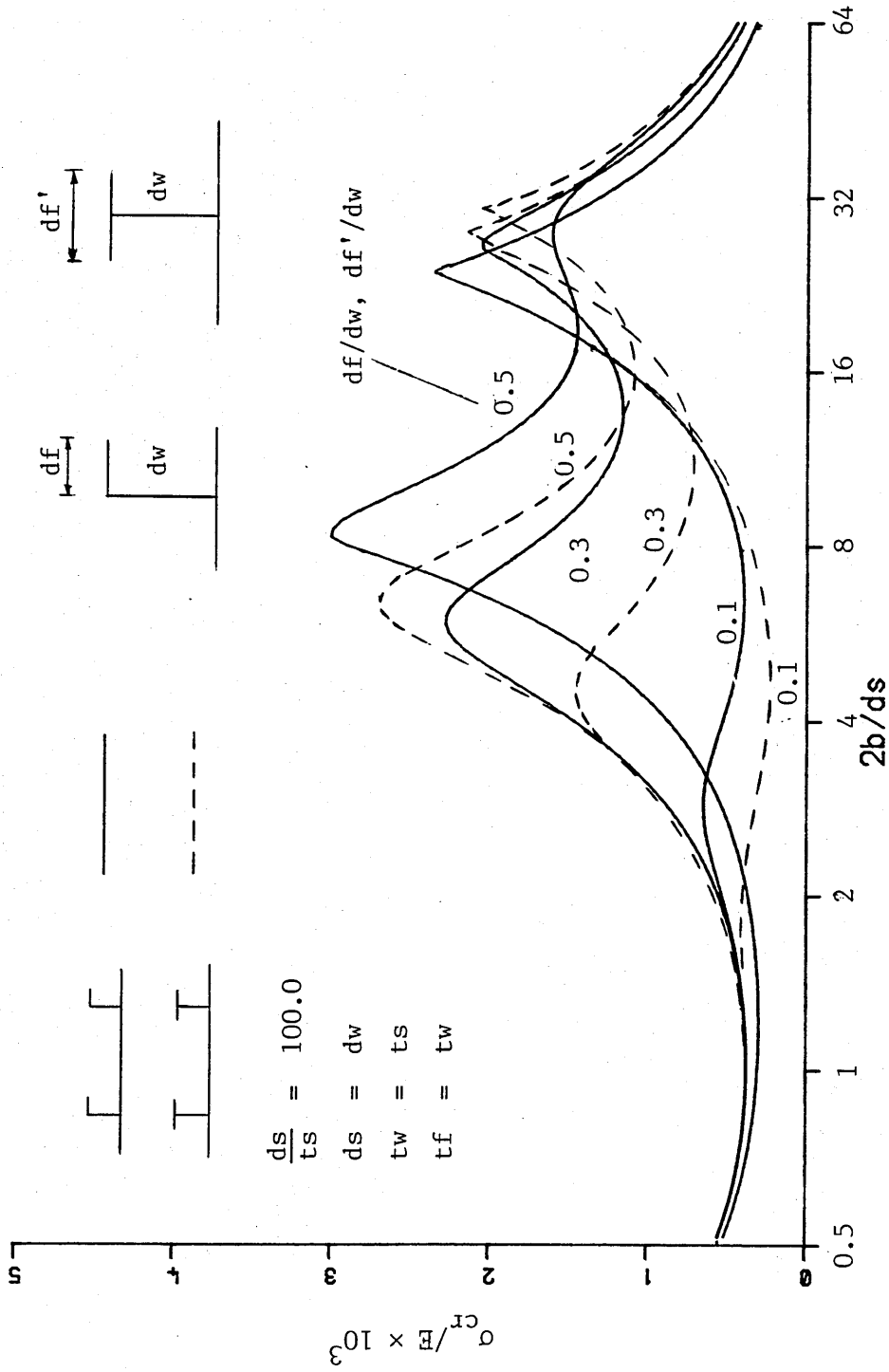


Fig 3.42 Variation of critical stress factor with $2b/ds$

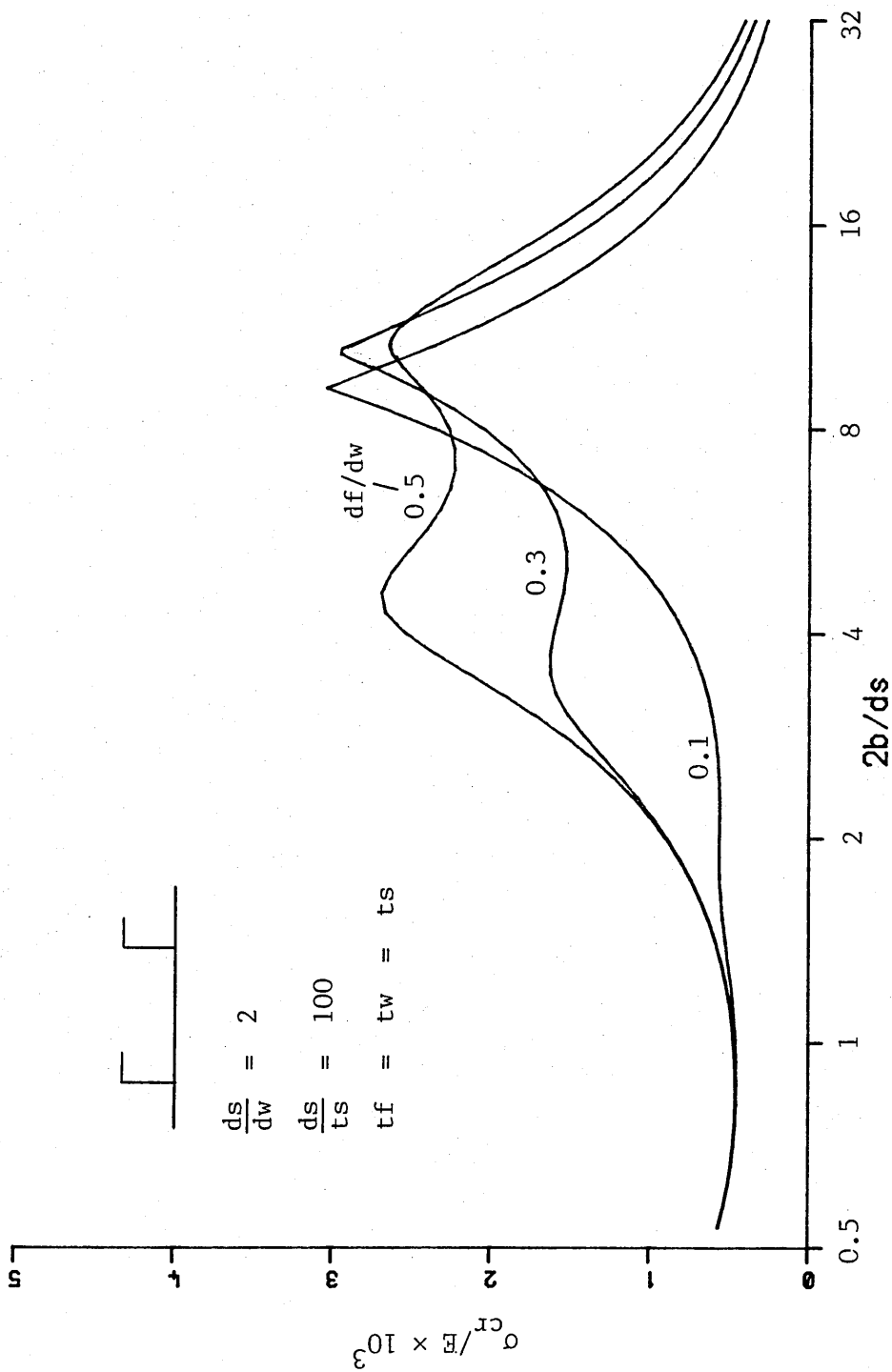
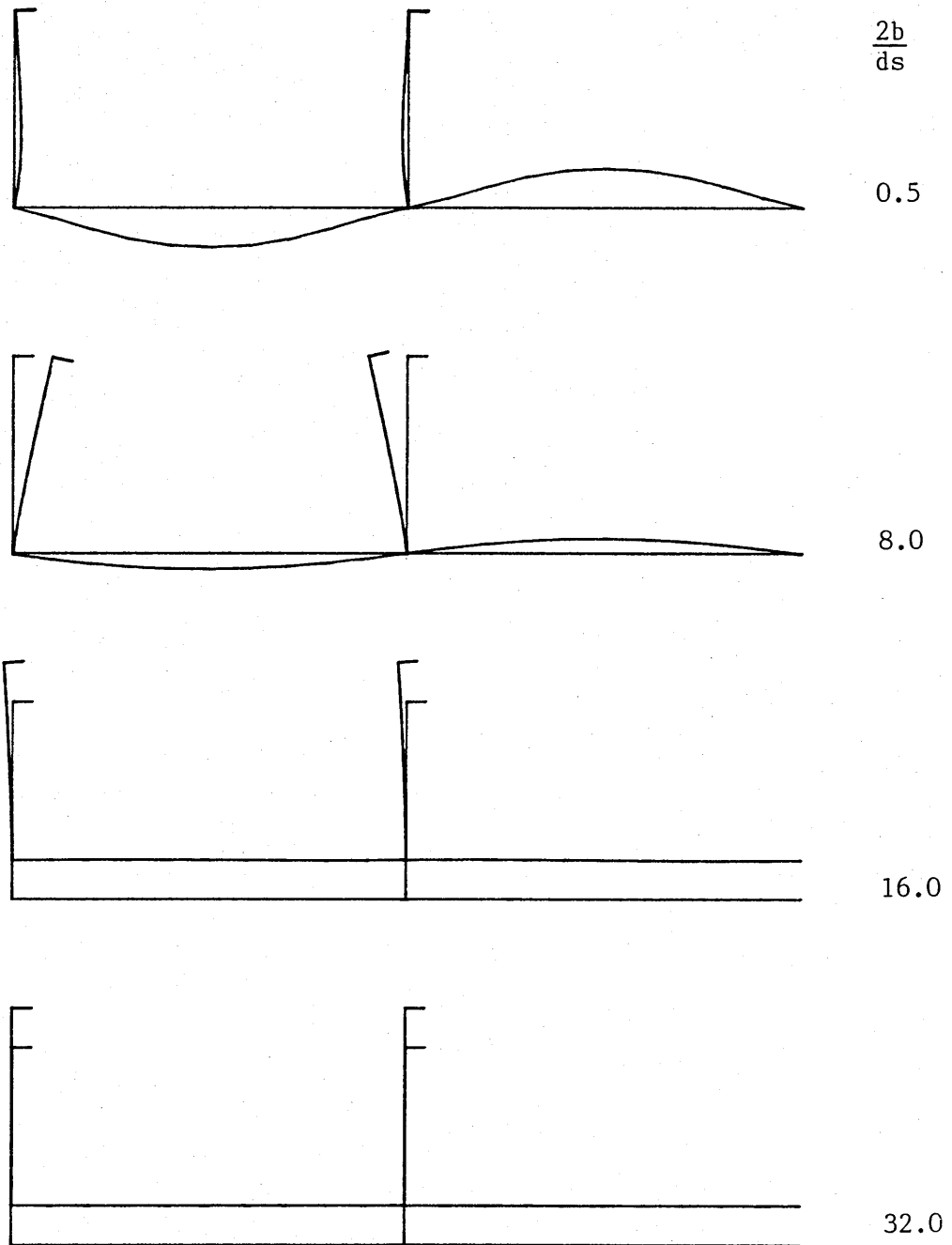


Fig 3.43a Variation of critical stress factor with $2b/ds$



$$\frac{dw}{ds} = 2, \quad \frac{ds}{ts} = 100, \quad tf = tw = ts, \quad \frac{df}{dw} = 0.1$$

Fig 3.43b Buckled mode shapes at various values of half wavelength

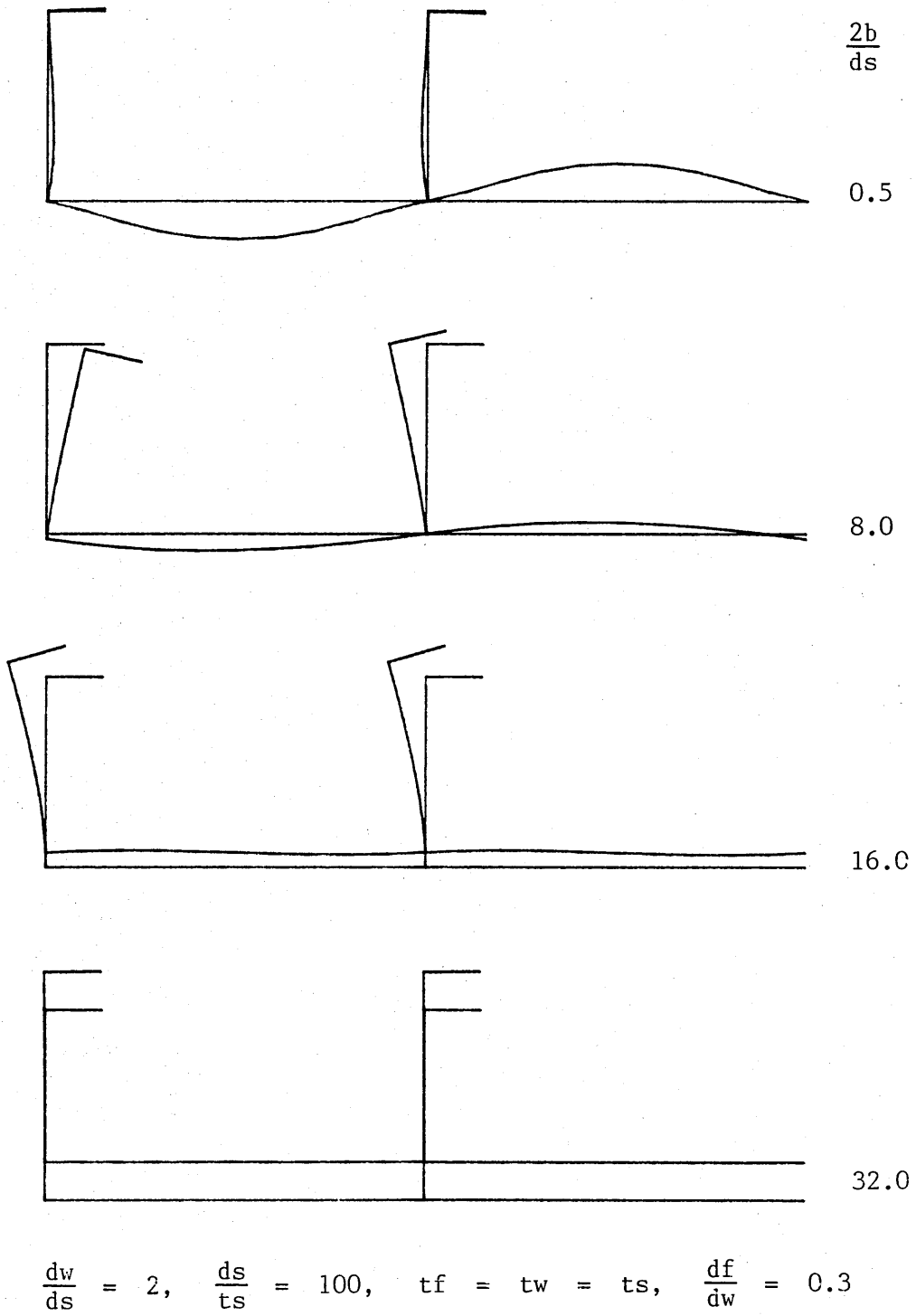


Fig 3.43c Buckled mode shapes at various values of half wavelength

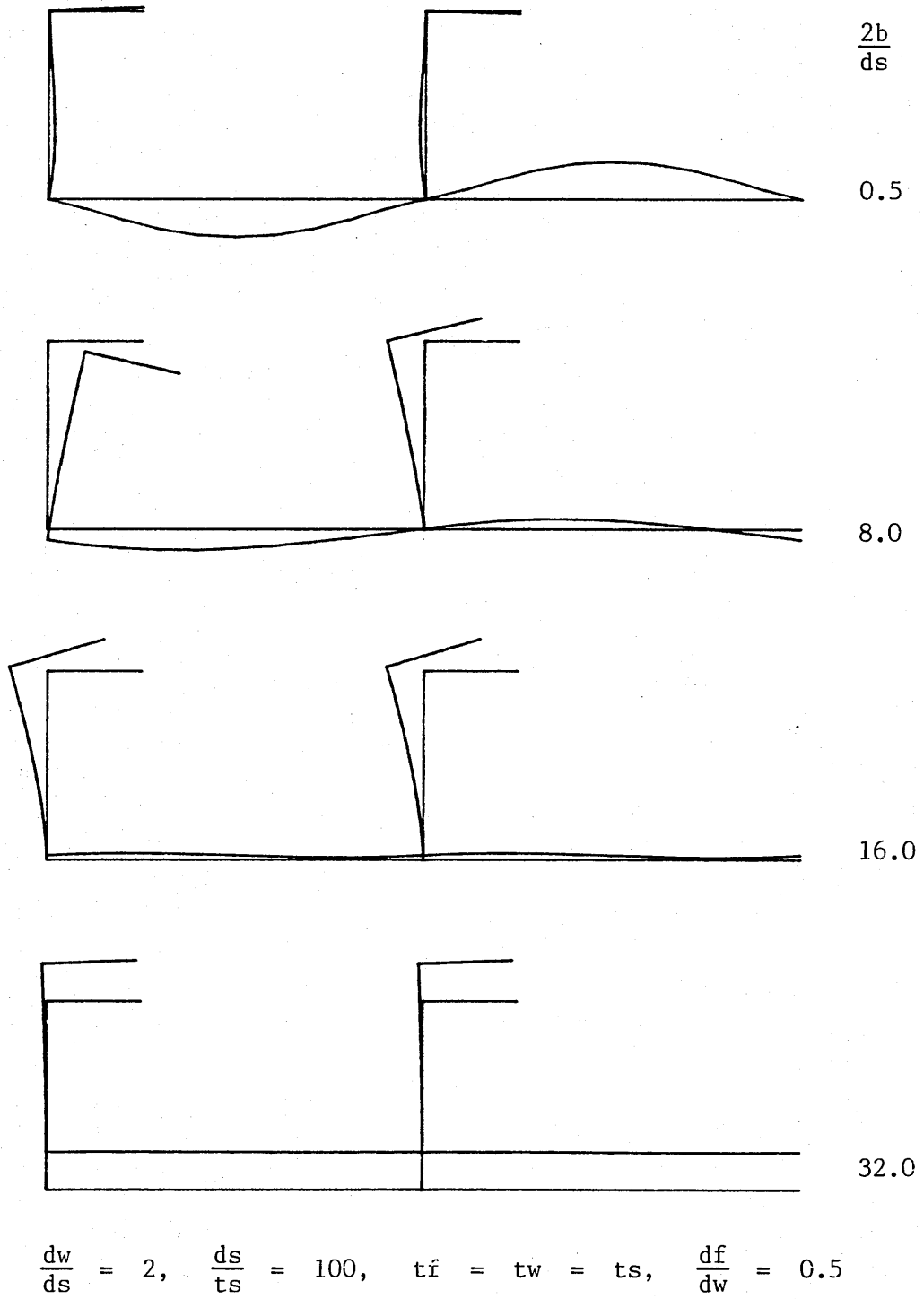


Fig 3.43d Buckled mode shapes at various values of half wavelength

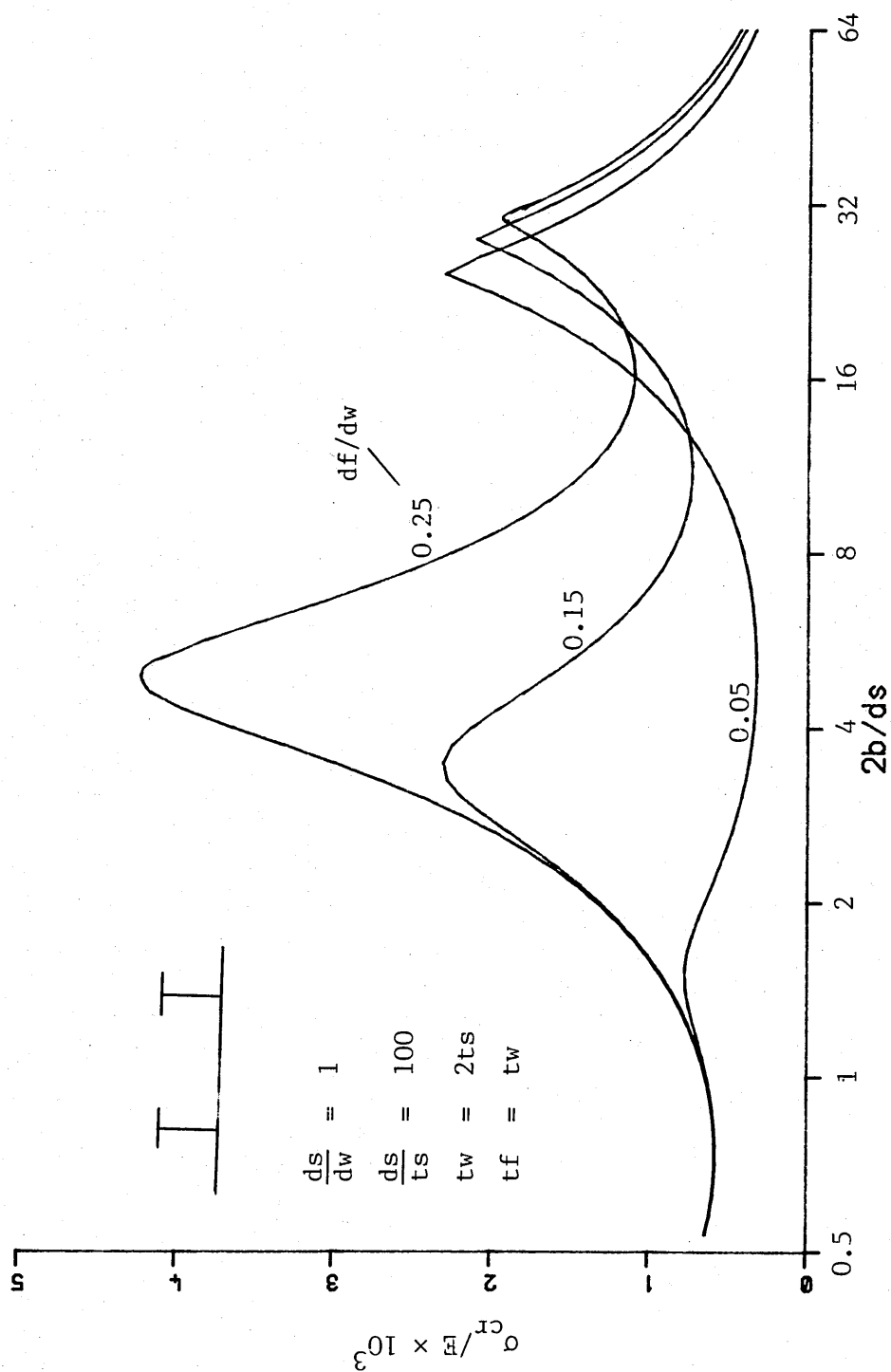
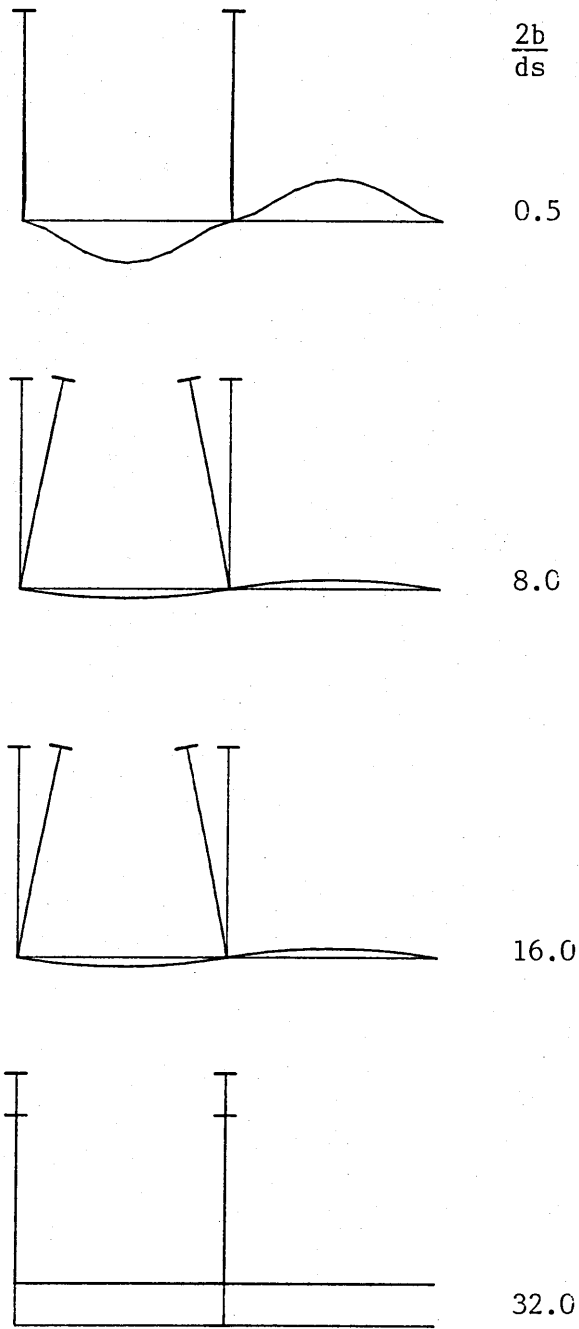
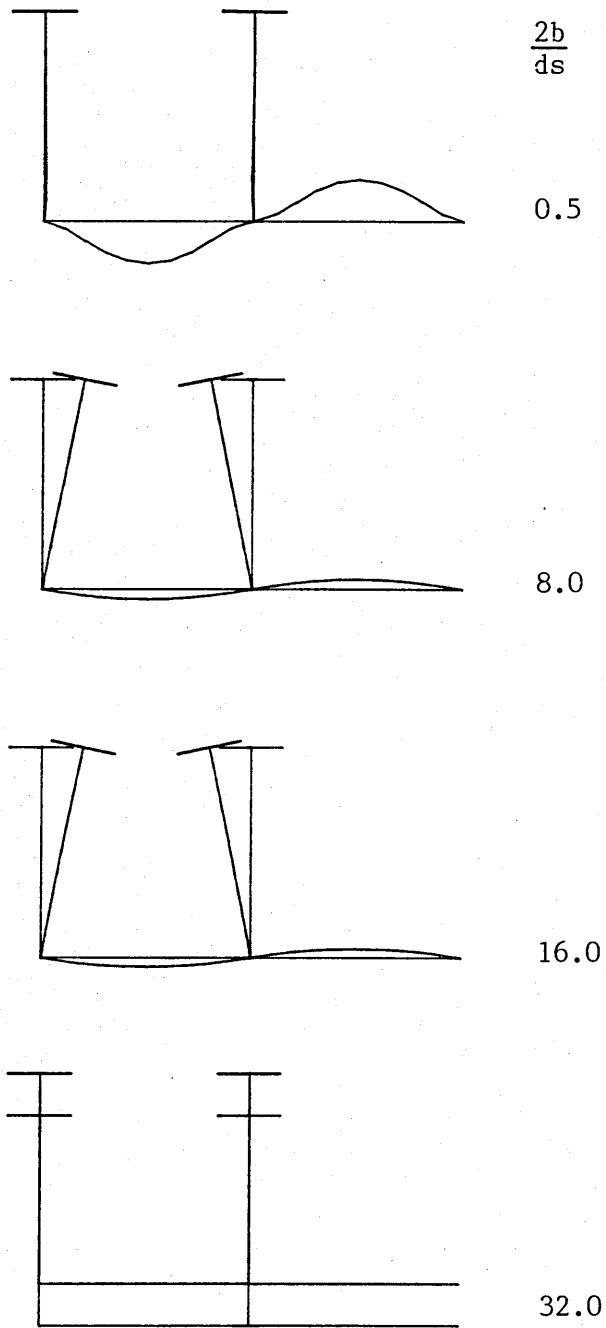


Fig 3.44a Variation of critical stress factor with $2b/ds$



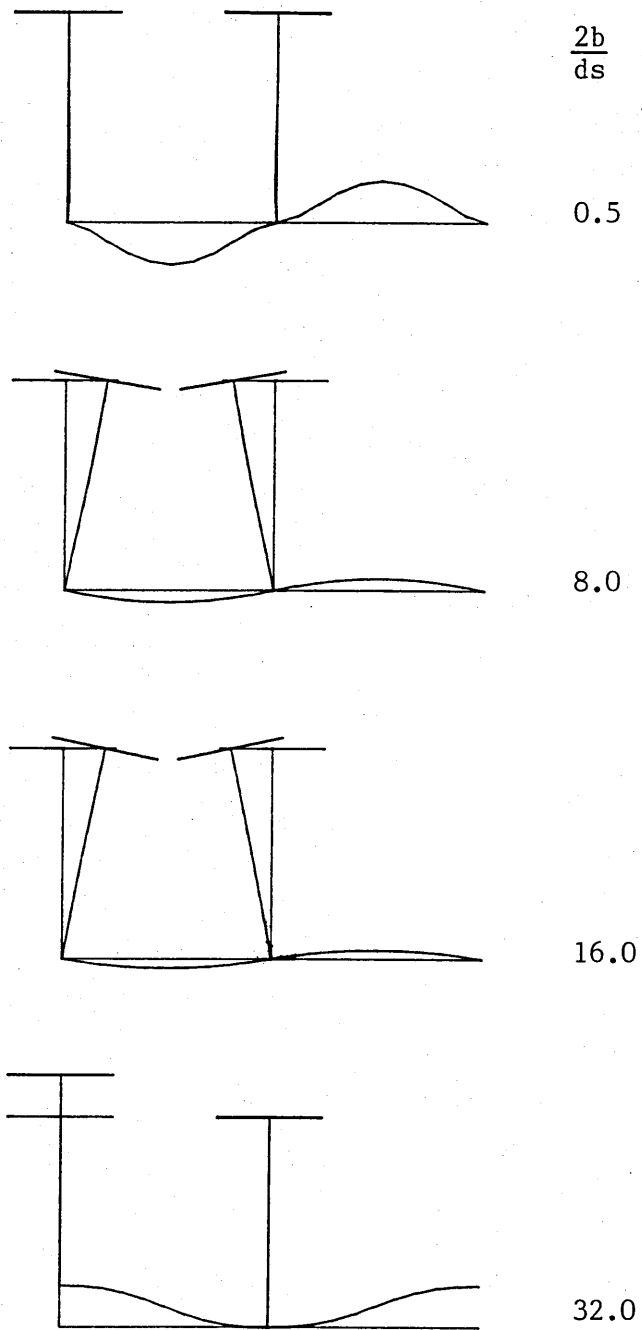
$$\frac{df}{dw} = 0.05, \quad \frac{ds}{dw} = 1, \quad \frac{ds}{ts} = 100, \quad tw = 2ts, \quad tf = tw$$

Fig 3.44b Buckled modes at various values of half wavelength



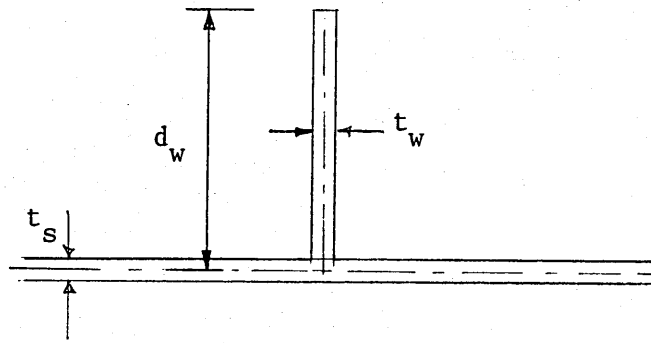
$$\frac{df}{dw} = 0.15, \quad \frac{ds}{dw} = 1, \quad \frac{ds}{ts} = 100, \quad tw = 2ts, \quad tf = tw$$

Fig 3.44c Buckled modes at various values of half wavelength

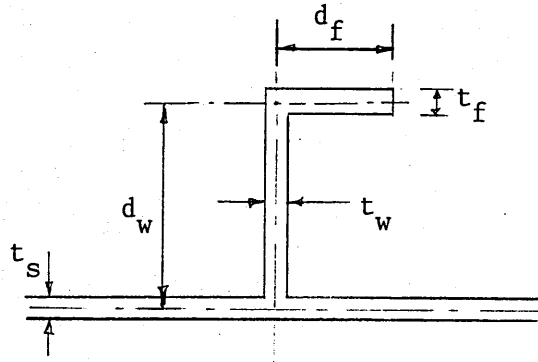


$$\frac{df}{dw} = 0.25, \quad \frac{ds}{dw} = 1, \quad \frac{ds}{ts} = 100, \quad tw = 2ts, \quad tf = tw$$

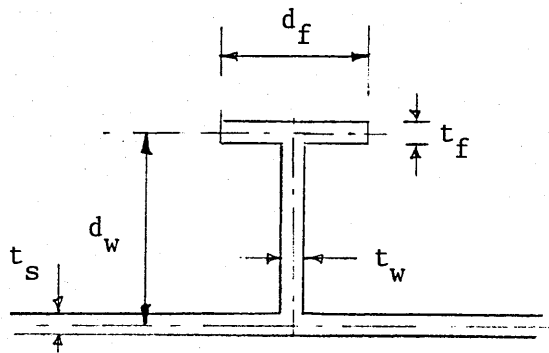
Fig 3.44d Buckled modes at various values of half wavelength



I-section stiffener



L-section stiffener



T-section stiffener

Fig 3.45 Cross-section details

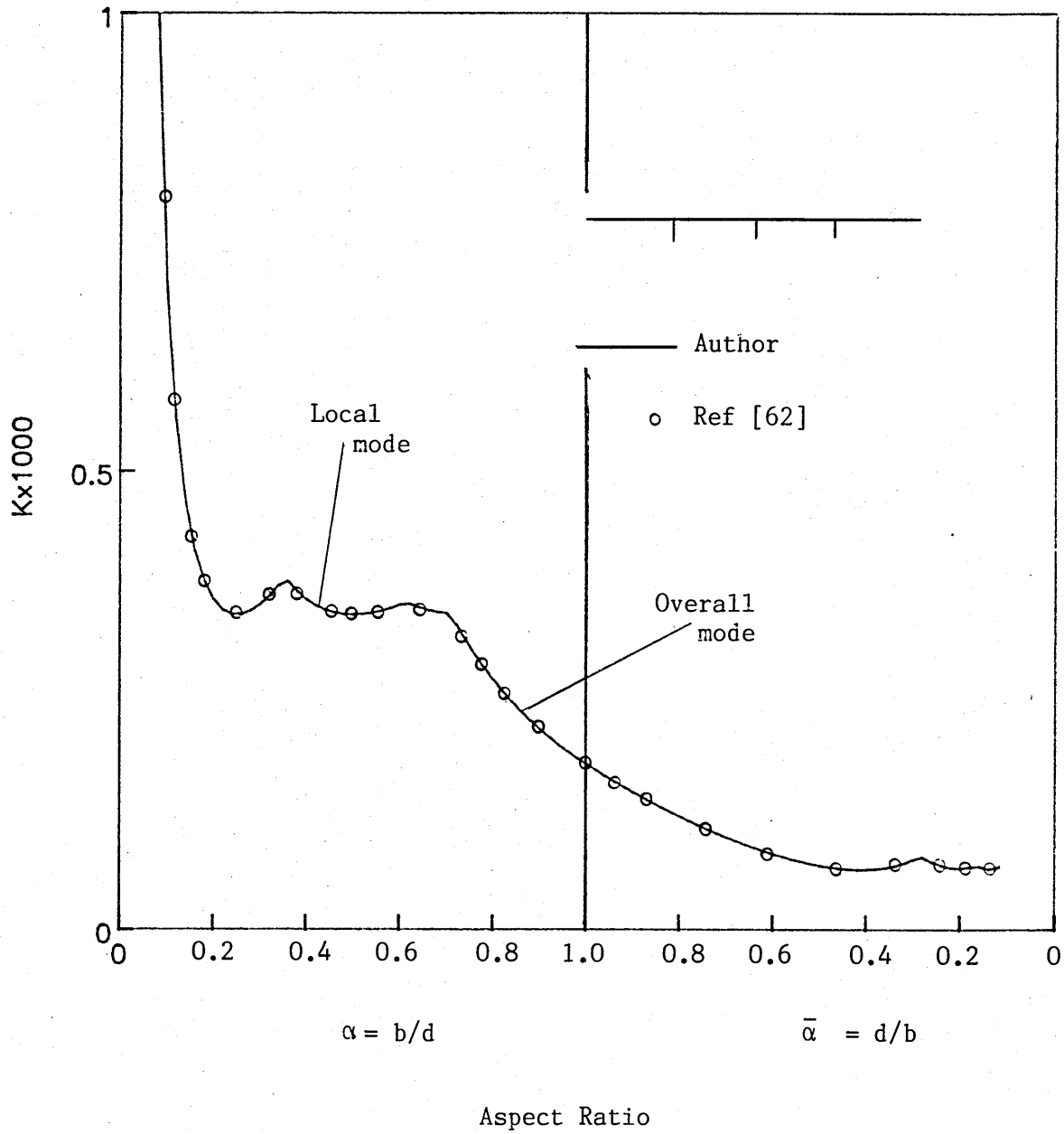


Fig 3.46 Variation of buckling coefficient K with aspect ratio for panel II

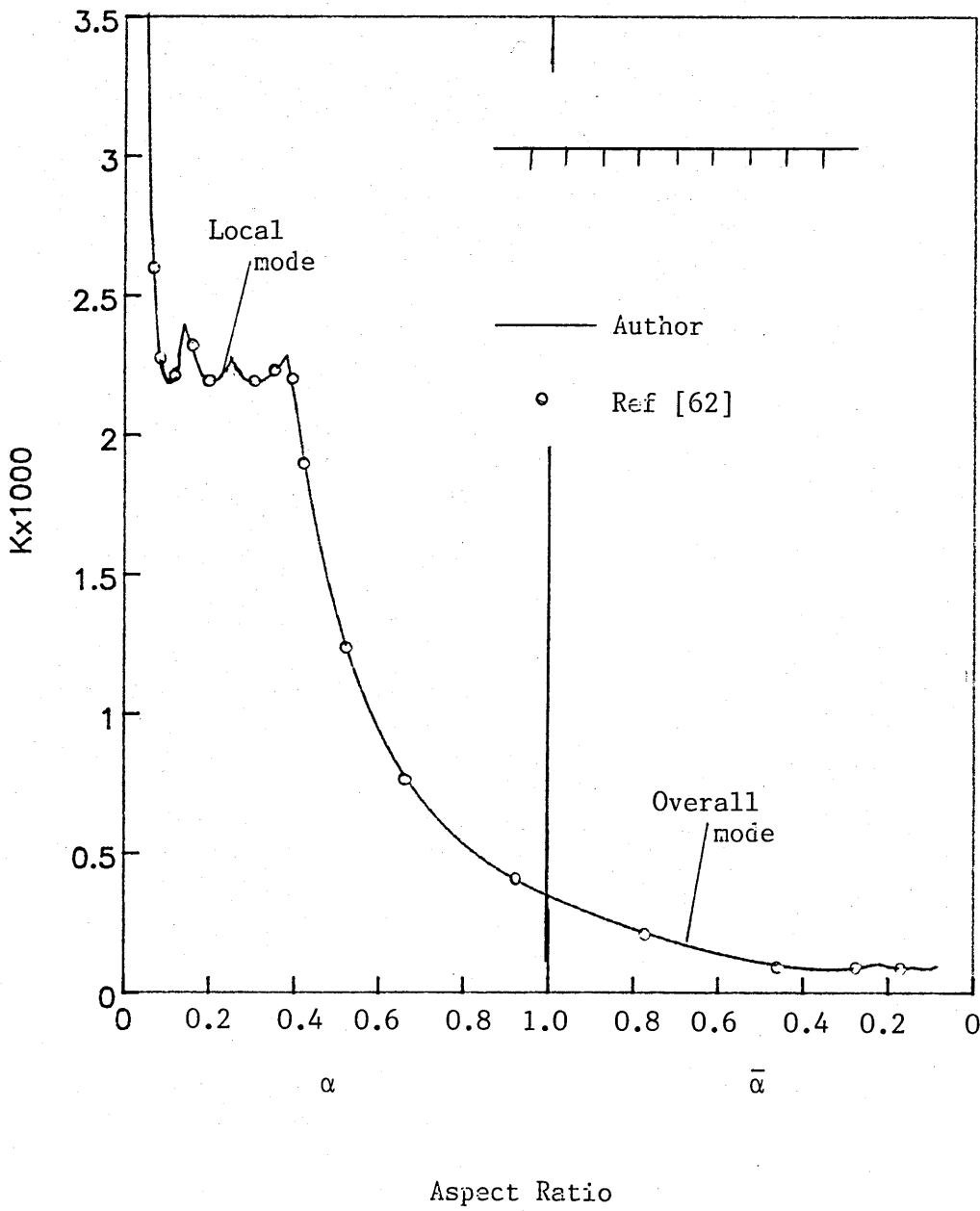


Fig 3.47 Variation of buckling coefficient with aspect ratio for panel 12

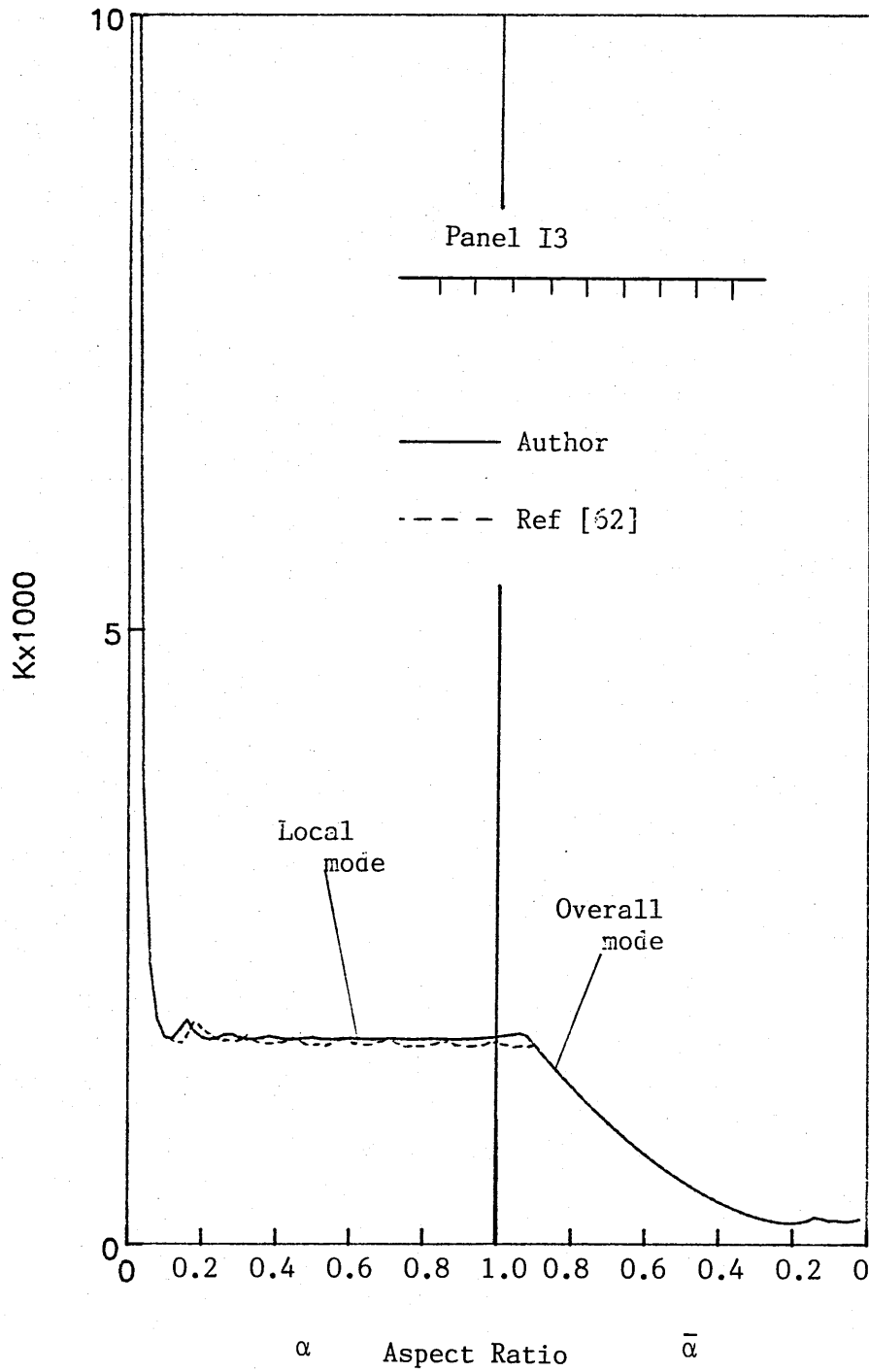


Fig 3.48 Variation of buckling coefficient K with the aspect ratio α , $\bar{\alpha}$

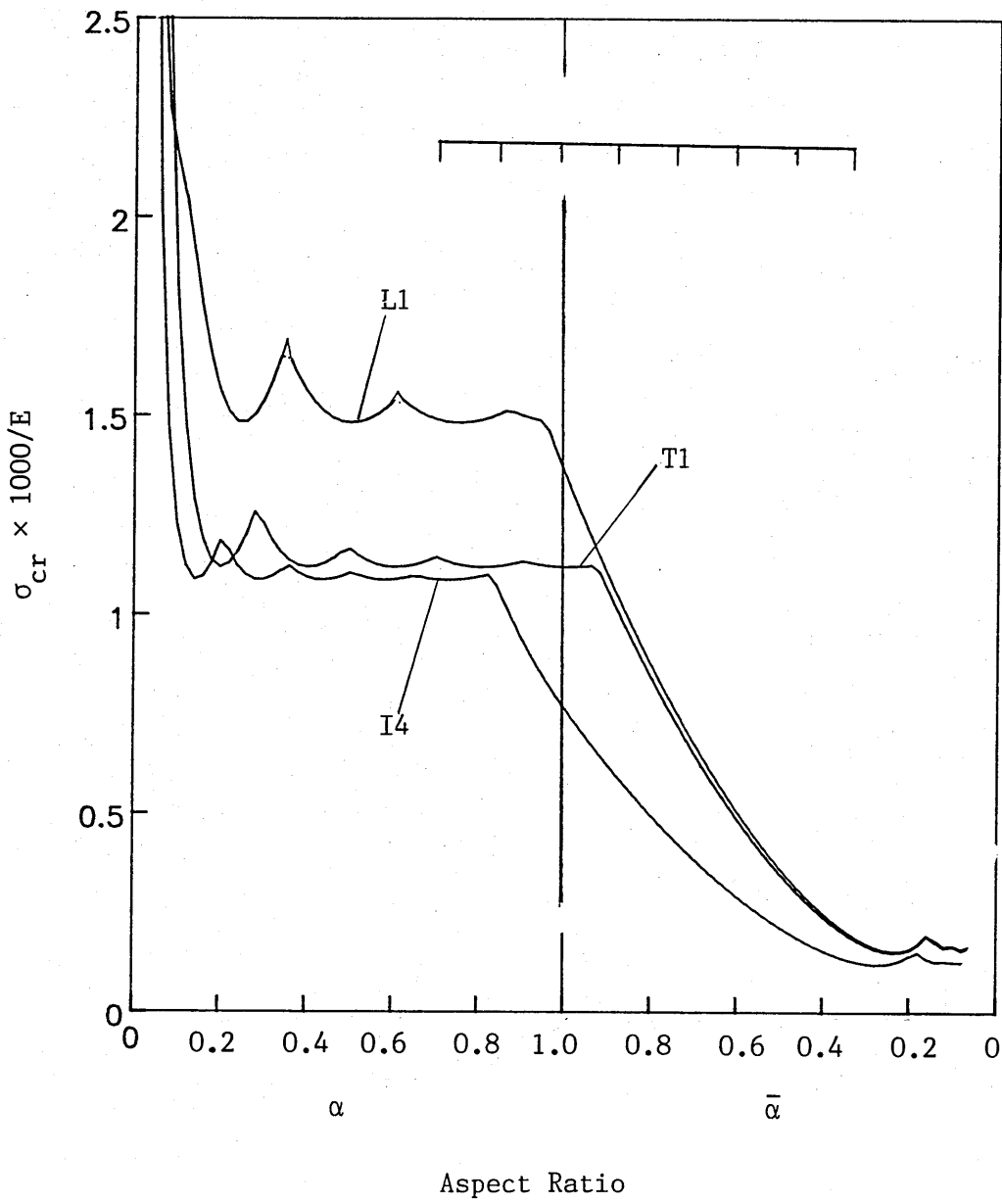


Fig 3.49 Variation of buckling stress with aspect ratio for various panels

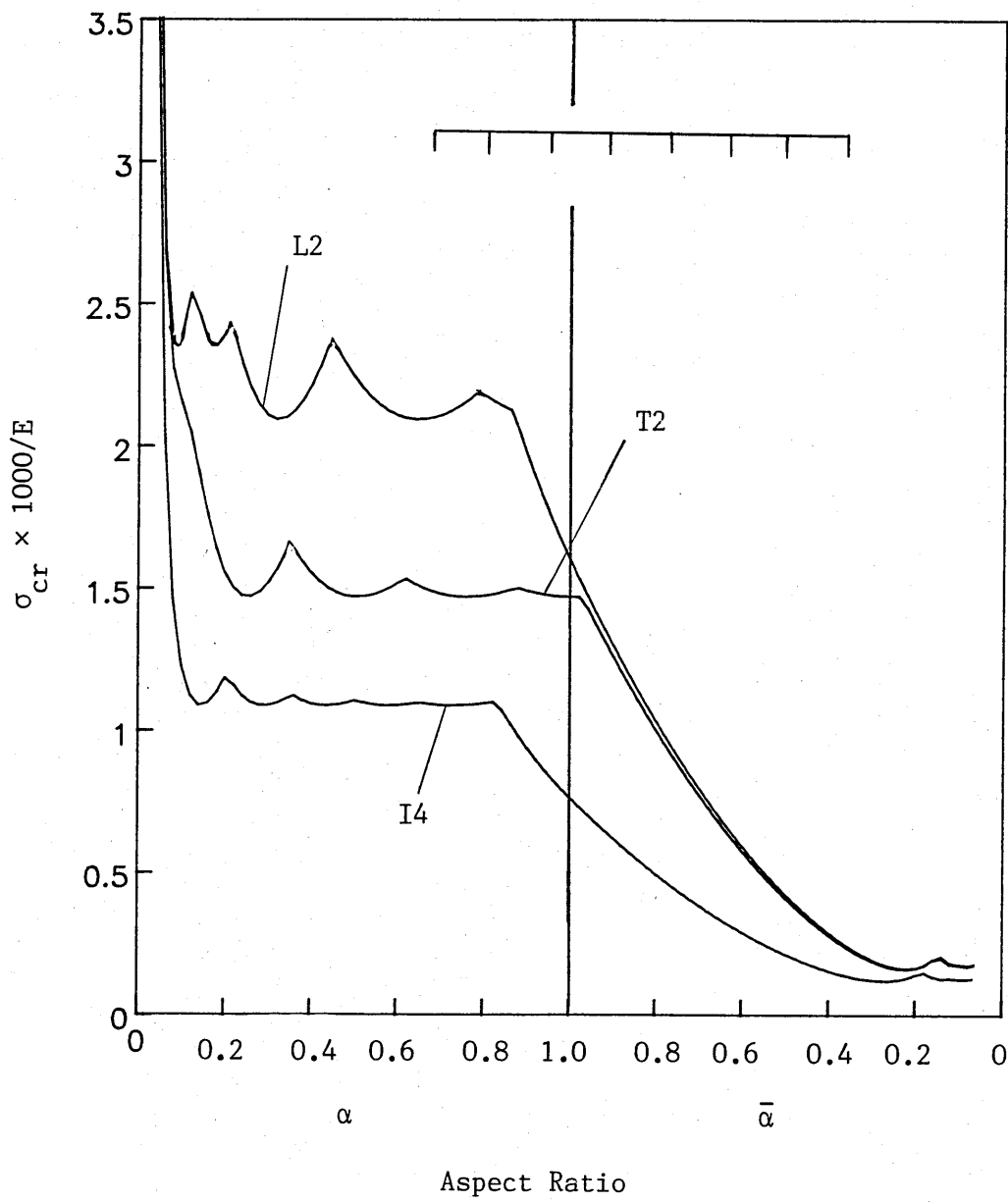


Fig 3.50 Variation of critical stress with the aspect ratio for various panels

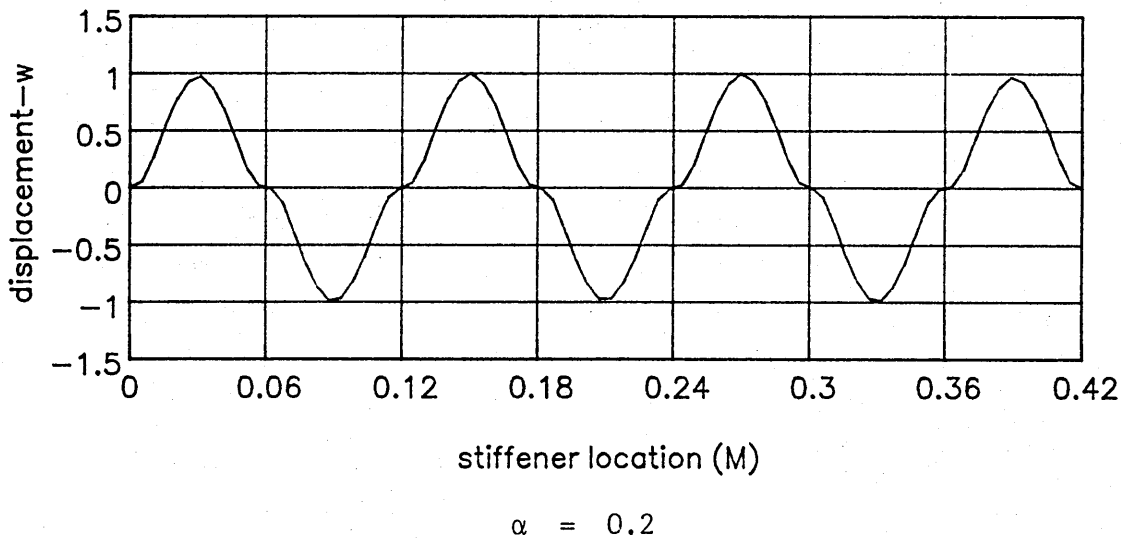


Fig 3.51a Buckled mode for panel L2

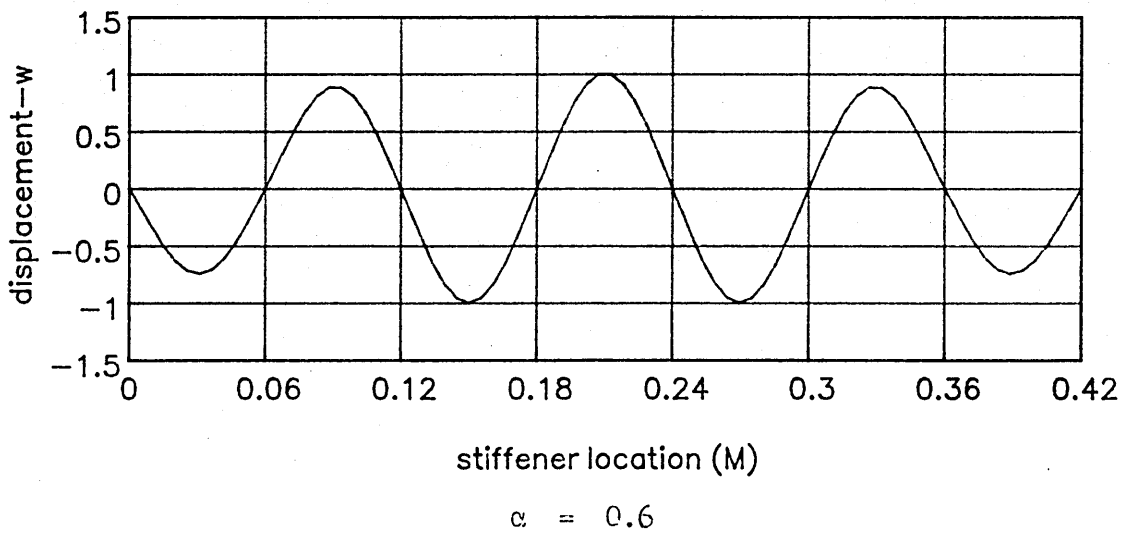


Fig 3.51b Buckled mode for panel L2

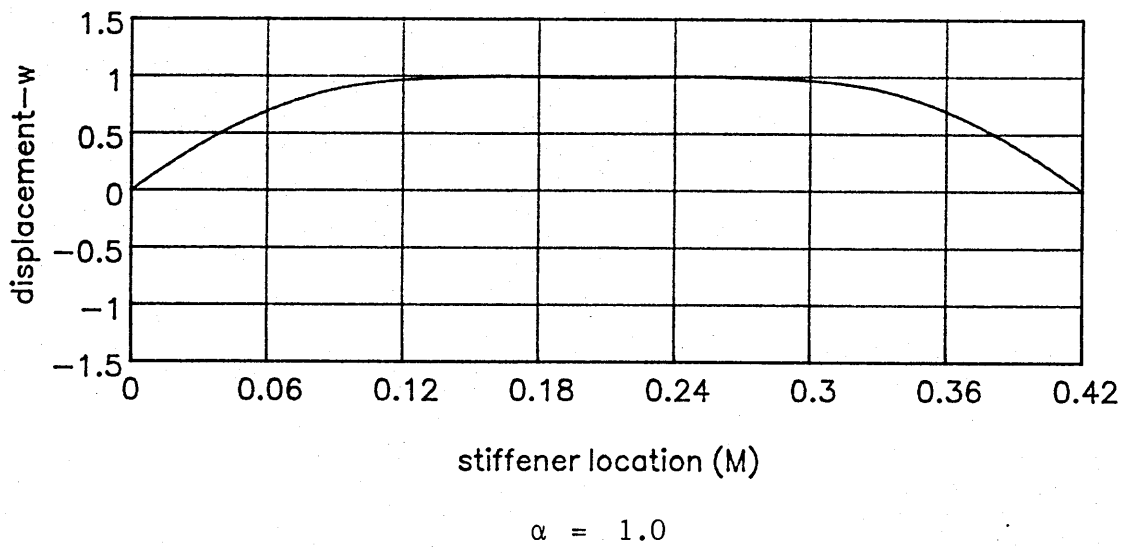


Fig 3.51c Buckled mode for panel L2

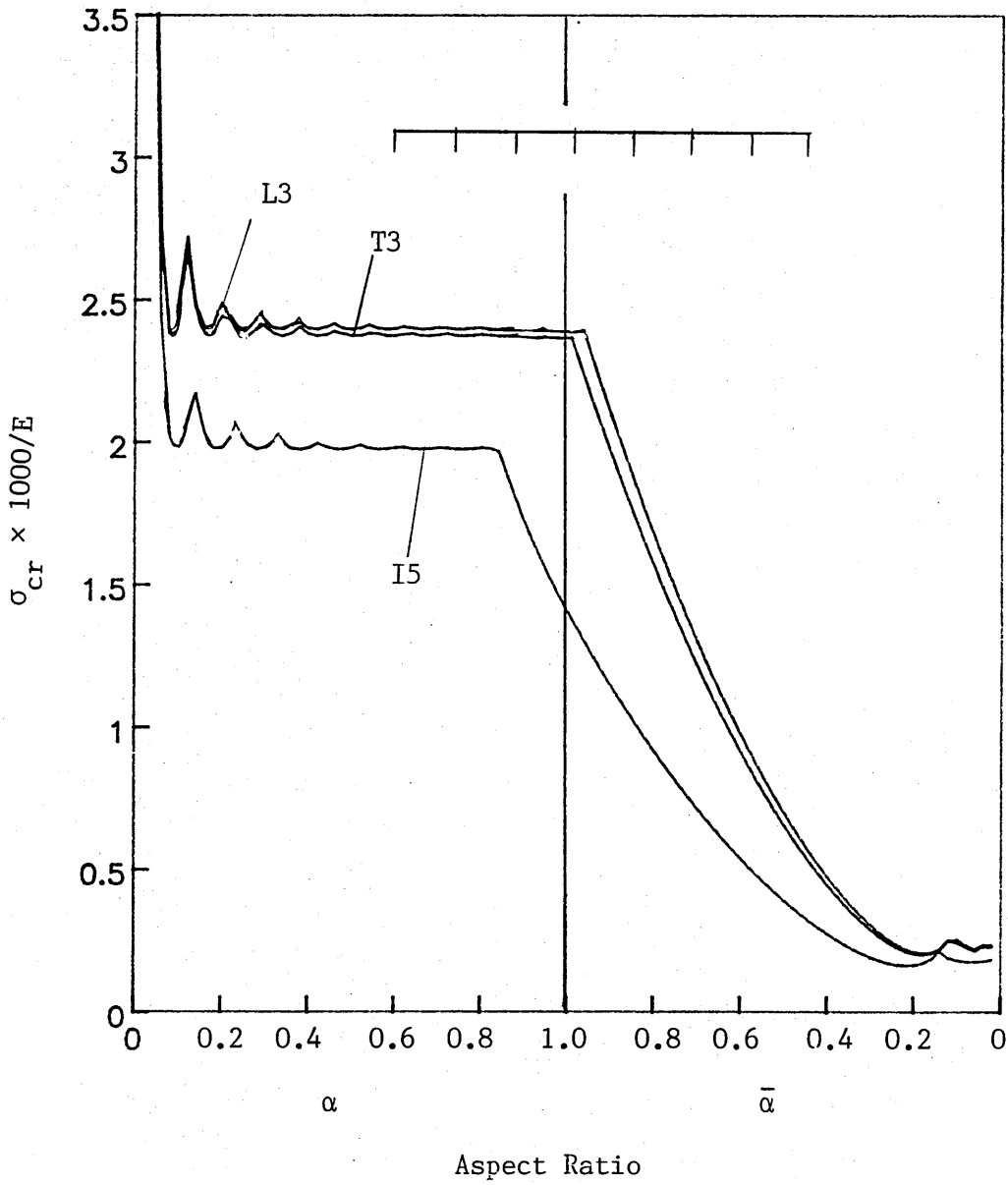


Fig 3.52 Variation of critical stress with the aspect ratio for various panels

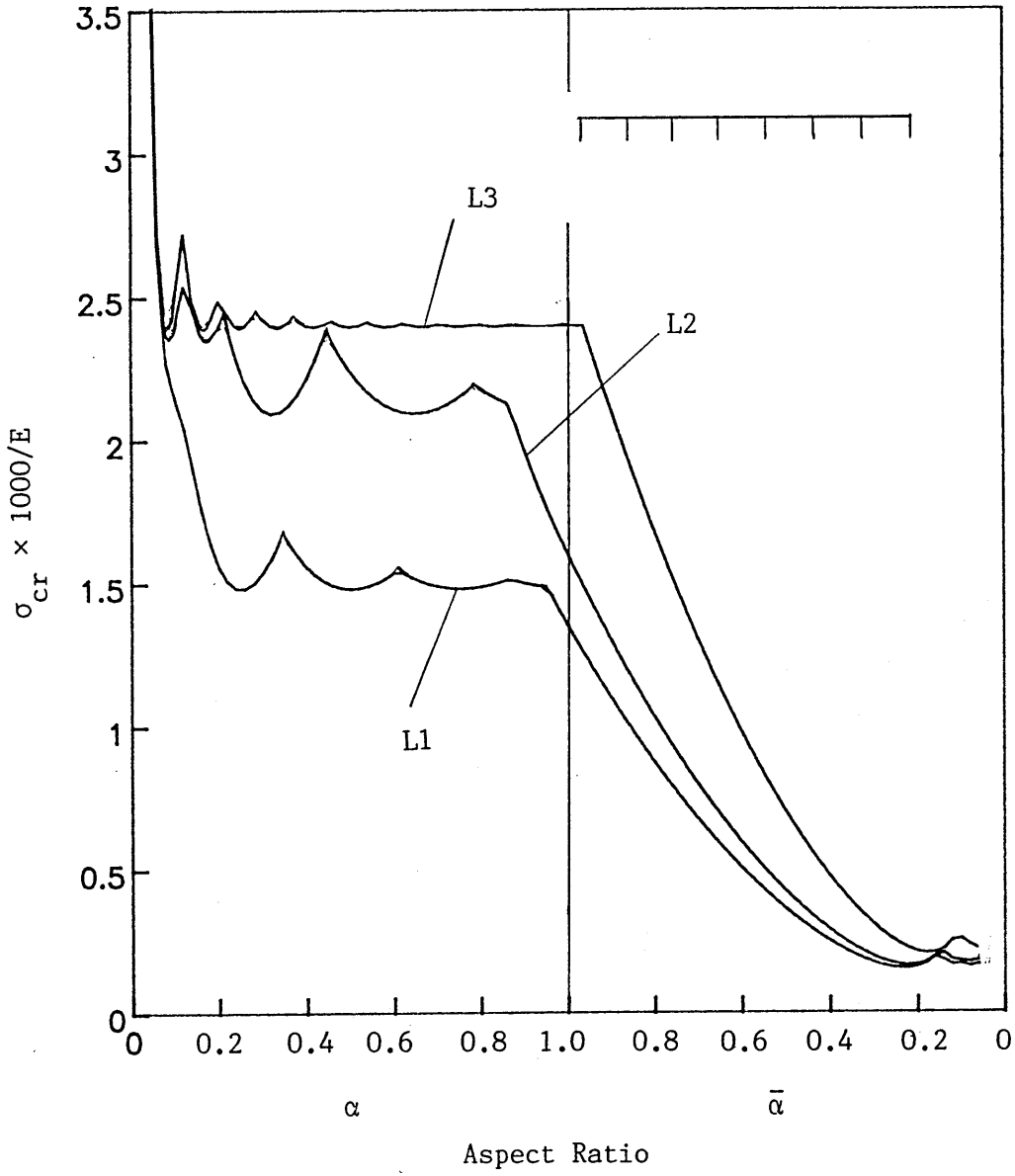


Fig 3.53 Variation of critical stress with the aspect ratio

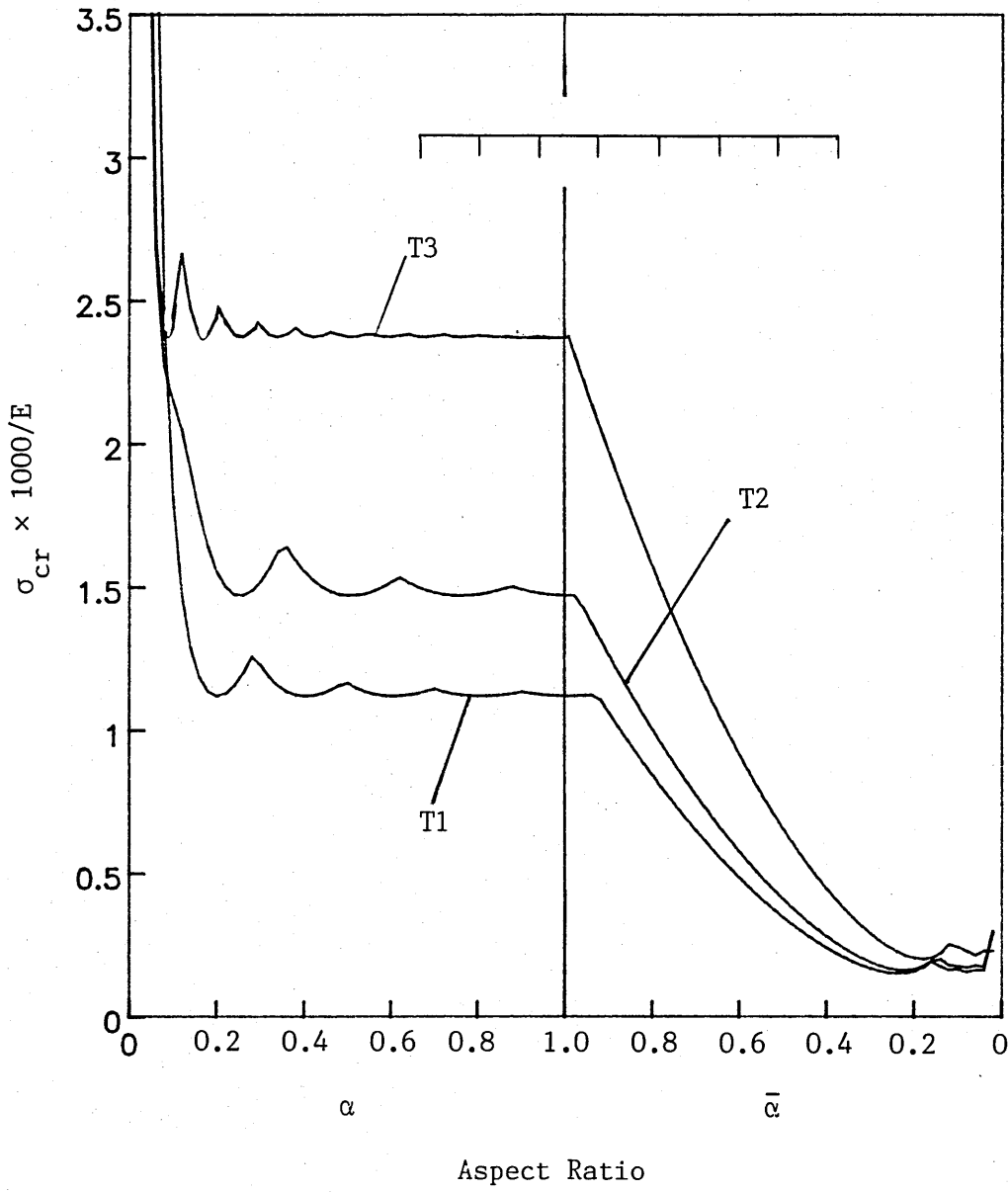


Fig 3.59 Variation of critical stress with aspect ratio

4 NONLINEAR ANALYSIS

4.1 Introduction

In linear mechanics of deformable bodies, it is assumed that displacements are proportional to applied loads. The concept of buckling is defined as a disproportionate increase in displacements resulting from a small increase in load. Buckling therefore considered as a subtopic of nonlinear rather than linear mechanics. Nonlinearity in mechanics of deformable bodies can result from either physical or geometrical conditions or a combination of both. Physical nonlinearity results from nonlinear stress-strain relations and is also known as material nonlinearity. Geometrical nonlinearity results from the influence of rotations of structural elements on the behaviour of structure. The work presented in this chapter assumes stresses and strains obey Hook's law and consequently, the nonlinearity is purely geometrical.

In general the stability behaviour of structures can be characterized by two fundamentally different types of loss of stability. These two types of singular behaviour are related to the concepts of limit point and bifurcation point as depicted by the load displacement curves in Fig (4.1). The elastic behaviour of structures which is associated with the loss of stability at a limit point is also known as snap buckling. This type of behaviour is typical for a shallow conical shell where the load p refers to the axial compression and displacement u refers to the axial displacement of one end of the shell relative to the other. In general the non-linear snapping condition is not of great practical importance, since the snapping condition is reached only after a very considerable deformation, and such large deformations are not characteristic of practical structures at their working load [63]. If the loss of stability occurs in a bifurcation point then, the behaviour of the structure is called buckling. This type of behaviour is characterized by the existence of a primary (or fundamental) equilibrium path which is intersected by a secondary (or adjacent) path. The point of intersection is called the bifurcation point. The loss of stability of a structure through bifurcation can be characterized in three distinct manners as shown in Fig (4.2). In general the state of bifurcation is unsymmetric as

shown in Fig (4.2a), ie the secondary path is asymmetric about the load axis. This type of behaviour is typical, for example, for a two bar frame with a load p applied at the junction, there the displacement u represents the rotation of the junction [64]. For many practical problems the bifurcation is stable-symmetric as shown in Fig (4.2b). Here the secondary path is symmetric about the p axis and the structure can continue to sustain further load after buckling has occurred. This type of behaviour is typical for structures such as columns and plates loaded axially by a load p , u being the lateral displacement of the structure.

Fig (4.2c) depicts unstable-symmetric bifurcation behaviour where the structure cannot sustain further increase in load after buckling has occurred. This type of behaviour is typical for structures such as arches and shells.

In reality, structures are seldomly free from imperfections.

In the presence of imperfections, each of the three categories of bifurcation mentioned above gives a distinct style of behaviour. In general, the imperfect structures approach asymptotically the behaviour of the perfect structures at bifurcation as shown in Fig (4.2). For the case where bifurcation is asymmetric, two possible forms of behaviour occur, either the structure reaches a peak load below the critical load for a perfect structure and is unable to sustain any further increase in load, or a zone of instability is reached near the same critical load and the structure can continue to sustain further increases in load.

For the stable-symmetric case the imperfect structure does not reach a peak load and can still sustain loads above the critical load for the perfect structure. For the unstable-symmetric case a peak load for the imperfect structure exists below the critical load of the perfect structure. In general, imperfect structures which exhibit peak load type of behaviour are sensitive to initial imperfections. For the asymmetric case the reduction of the peak load is proportional to the square root of the imperfection ϵ . For the symmetric unstable case the reduction of the peak load is proportional to the power $\frac{2}{3}$ of the imperfection ϵ [64] [65]. In reality, structures may fail through yeilding, (Fig 4.3) shows a typical

structure exhibiting symmetric stable behaviour such as a plate or a column. $abcd$ represents the elastic loading curve for a structure with an initial imperfection. ecg is the plastic unloading curve which crosses the elastic loading curve at c . As the structure is loaded, yielding occurs at point b and the structure eventually fails at f .

The work presented in this chapter assumes the failure of stiffened plates occurs when the maximum edge stress reaches yielding. This criterion has been shown in the past to give favourable results when compared with experiments [50, 49].

The post-buckling analysis of plates requires the use of non-linear large-deflection theory, the resulting mathematical difficulties may be considerable. Additional difficulties are also encountered in the choice of a suitable displacement function to describe accurately the post-buckled form of the plate. That is, after reaching buckling, the shape of the buckled plate may change as the load is further increased. These changes are usually progressive. For instance, the first form of buckling keeps developing until the energy stored in the plate is sufficient to carry it into the second buckled form and so on until failure occurs.

The governing von Karman's large-deflection equations for the investigation of the post buckling behaviour of plates have been outlined in section 2.5. Since these differential equations are nonlinear, exact solutions are seldomly obtained. Therefore, the most common techniques used for the post-buckling analysis of plated structures are based on energy methods. The strain energy method requires the choice of both out-of-plane as well as in-plane displacement fields. Whilst, the out-of-plane displacements may be easily perceived, the in-plane displacements can be much more difficult to envisage.

An alternative method of analysis is the combination of variational methods coupled with the direct solution to one of the governing differential equations. The most common of these is the Semi-Energy method [8], which requires the solution of von Karman's equation of compatibility in conjunction with the vanishing of the first variation of the total potential energy. The edge displacements

of the plate are usually introduced into the analysis through the satisfaction of displacement boundary conditions, and not as an independent displacement variable, as is the strain energy method. In essence, the method obviates the necessity of prescribing complicated in-plane displacement fields.

The work presented in this chapter makes use of the semi-energy approach for the geometric nonlinear analysis of plates and stiffened panels, the loaded edge displacements being expressed by simple displacement functions with independent displacement variables.

The post-buckling behaviour of plates has received a considerable amount of attention since plates can carry loads beyond the critical load. The use of this additional strength is of great practical importance in the design of aerospace, naval and civil engineering structures since considerable weight savings can be achieved. A comprehensive literature review on the subject can be found in the text book by Chia [29]. In comparison the post buckling of integrally stiffened panels has received far less attention. Most of the published literature concentrates on interaction between the local and overall buckling modes and the sensitivity of panels to geometrical imperfections when the local and the overall buckling loads are coincident [35-37, 66-69]. In the civil engineering field, the collapse of a few large box girder bridges during erection, precipitated a large effort on the various design criterion for these structures. Horne and Narayanan et al [39, 70-75] carried out a comprehensive experimental investigation and proposed approximate analytical models for the prediction of the collapse load of welded stiffened panels under uniform axial compression.

In all the above investigations, the overall buckling load is assumed to be equal to the Euler load for a pin-ended strut with the cross-section equal to the repeated panel cross-section. This model has been assumed to give quite acceptable results for the case where the panel length to the stiffener pitch ratio is between 8 and 32 [38].

In general, stiffened panels are supported laterally by a system of cross-frames. The panels can span several bays. A consequence of this layout is that lateral support is provided on all boundaries of the panel.

In the present work, panels of length to stiffener pitch ratios of below eight are considered. Such panels have a likely use in the aerospace industry. The panels are assumed to be laterally supported along all four edges and integrally stiffened with stiffeners of non-deformable cross-section.

The post-buckling behaviour of stiffened panels involves interaction between the local and overall modes so that buckling in one mode influences the other. The influence of local buckling results from the fact that plate elements which act as load carrying members lose part of their axial stiffness under local buckling. This causes a drop in the axial stiffness and bending stiffness of the panel as a whole. With this decrease in stiffness of the panel section there is a shift in the effective neutral axis away from the buckled skin resulting in loading eccentricity. The combination of a loss in bending stiffness and loading eccentricity leads to overall buckling. For stiffened panel models based upon a broad Euler column assumptions, this interaction can lead to unstable buckling as well as being very sensitive to imperfections [35].

4.2 Energy Equations for the Nonlinear Analysis of Plates

An initially imperfect plate of thickness t , Young's modulus E and Poisson's ratio ν is shown in Fig (4.4). x, y, z represent the axis, u, v, w the corresponding displacements and w_0 the initial imperfection. Equation (2.14) provides a basis for the Newton-Rapson procedure for the solution of nonlinear equilibrium equations and is restated here for convenience:

$$\int \sigma'_i \delta \epsilon_i + \sigma_i \delta \epsilon'_i \, d \text{ vol} - p'_i \delta r_i = p_i \delta r_i - \int \sigma_i \delta \epsilon_i \, d \text{ vol} \quad (4.1)$$

For plate problems, equation (4.1) may be written as:

$$\int \sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_{xy} \delta \epsilon_{xy} + \sigma_x \delta \epsilon_{xy} \, d \text{ vol} - p'_i \delta r_i = p_i \delta r_i - \int \sigma_z \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_{xy} \delta \epsilon_{xy} \, d \text{ vol} \quad (4.2)$$

where $\sigma_x, \sigma_y, \sigma_{xy}$ represent the total internal stresses and $\delta \epsilon_x, \delta \epsilon_y, \delta \epsilon_{xy}$ are the first variations of the corresponding strains. p_i represents the externally applied loads and δr_i the first variation of the corresponding displacements. For most problems r_i can be expressed as a linear function of the displacement variables.

To investigate the post-buckling behaviour of plates, the total strains $\epsilon_x, \epsilon_y, \epsilon_{xy}$ can be assumed to be composed of strains which refer to zero displacement of plate edges $\epsilon_{bx}, \epsilon_{by}, \epsilon_{bxy}$ and $\epsilon_{mx}, \epsilon_{my}, \epsilon_{mxy}$ and those which are induced by displacement of edges $\bar{\epsilon}_{mx}, \bar{\epsilon}_{my}, \bar{\epsilon}_{mxy}$. The subscripts b and m denote bending and membrane strains.

The total strains may then be written as:

$$\begin{aligned} \epsilon_x &= \epsilon_{bx} + \bar{\epsilon}_{mx} + \epsilon_{mx} \\ \epsilon_y &= \epsilon_{by} + \bar{\epsilon}_{my} + \epsilon_{my} \\ \epsilon_{xy} &= \epsilon_{bxy} + \bar{\epsilon}_{mxy} + \epsilon_{mxy} \end{aligned} \quad (4.3)$$

Similarly, the total stresses $\sigma_x, \sigma_y, \sigma_{xy}$ may be written as:

$$\begin{aligned}
\sigma_x &= \sigma_{bx} + \bar{\sigma}_{mx} + \sigma_{mx} \\
\sigma_y &= \sigma_{by} + \bar{\sigma}_{my} + \sigma_{my} \\
\sigma_{xy} &= \sigma_{bxy} + \bar{\sigma}_{mxy} + \sigma_{mxy}
\end{aligned} \tag{4.4}$$

where σ_b , $\bar{\sigma}_m$, σ_m are the total internal stresses corresponding to strains ϵ_b , $\bar{\epsilon}_m$, ϵ_m .

Utilizing the relationship between stresses and strains given by equations (2.94) to (2.96) and (2.99), the total strains and stresses may be written as:

$$\begin{aligned}
\epsilon_x &= \epsilon_{bx} + \bar{\epsilon}_{mx} + \frac{1}{E} (\sigma_{mx} - \nu \sigma_{my}) \\
\epsilon_y &= \epsilon_{by} + \bar{\epsilon}_{my} + \frac{1}{E} (\sigma_{my} - \nu \sigma_{mx}) \\
\epsilon_{xt} &= \epsilon_{bxy} + \bar{\epsilon}_{mxy} + \frac{1}{G} \sigma_{mxy}
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\sigma_x &= C(\epsilon_{bx} + \nu \epsilon_{by}) + C(\bar{\epsilon}_{mx} + \nu \bar{\epsilon}_{my}) + \sigma_{mx} \\
\sigma_y &= C(\epsilon_{by} + \nu \epsilon_{bx}) + C(\bar{\epsilon}_{my} + \nu \bar{\epsilon}_{mx}) + \sigma_{my} \\
\sigma_{xy} &= G\epsilon_{bxy} + G\bar{\epsilon}_{mxy} + \sigma_{mxy}
\end{aligned} \tag{4.6}$$

where

$$C = \frac{E}{(1-\nu^2)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)}$$

Referring to equations (4.5) and (4.6), σ' , ϵ' , $\delta\epsilon$ may be expressed as:

$$\begin{aligned}
\sigma'_x &= C(\epsilon'_{bx} + \nu \epsilon'_{by}) + C(\bar{\epsilon}'_{mx} + \nu \bar{\epsilon}'_{my}) + \sigma'_{mx} \\
\sigma'_y &= C(\epsilon'_{by} + \nu \epsilon'_{bx}) + C(\bar{\epsilon}'_{my} + \nu \bar{\epsilon}'_{mx}) + \sigma'_{my} \\
\sigma'_{xy} &= G\epsilon'_{bxy} + G\bar{\epsilon}'_{mxy} + \sigma'_{mxy}
\end{aligned}$$

$$\epsilon'_x = \epsilon'_{bx} + \bar{\epsilon}'_{mx} + \frac{1}{E} (\sigma'_{mx} - \nu \sigma'_{my})$$

$$\epsilon'_y = \epsilon'_{by} + \bar{\epsilon}'_{my} + \frac{1}{E} (\sigma'_{my} - \nu \sigma'_{mx})$$

$$\epsilon'_{xy} = \epsilon'_{bxy} + \bar{\epsilon}'_{mxy} + \frac{1}{G} \sigma'_{mxy}$$

$$\delta \epsilon_x = \delta \epsilon_{bx} + \delta \bar{\epsilon}_{mx} + \frac{1}{E} (\delta \sigma_{mx} - \nu \delta \sigma_{mx})$$

$$\delta \epsilon_y = \delta \epsilon_{by} + \delta \bar{\epsilon}_{my} + \frac{1}{E} (\delta \sigma_{my} - \nu \delta \sigma_{mx})$$

$$\delta \epsilon_{xy} = \delta \epsilon_{bxy} + \delta \bar{\epsilon}_{mxy} + \frac{1}{G} \delta \sigma_{mxy}$$

$$\delta \epsilon'_x = \delta \epsilon'_{bx} + \delta \bar{\epsilon}'_{mx} + \frac{1}{E} (\delta \sigma'_{mx} - \nu \delta \sigma'_{my})$$

$$\delta \epsilon'_y = \delta \epsilon'_{by} + \delta \bar{\epsilon}'_{my} + \frac{1}{E} (\delta \sigma'_{my} - \nu \delta \sigma'_{mx})$$

$$\delta \epsilon'_{xy} = \delta \epsilon'_{bxy} + \delta \bar{\epsilon}'_{mxy} + \frac{1}{G} \delta \sigma'_{mxy}$$

(4.7)

The strains ϵ_b , $\bar{\epsilon}_m$ can be expressed in terms of displacements u, v, w and the initial imperfections w_0 as:

$$\epsilon_{bx} = -z(w_{xx} - w_{0xx})$$

$$\epsilon_{by} = -z(w_{yy} - w_{0yy})$$

$$\epsilon_{bxy} = -2z(w_{xy} - w_{0xy})$$

$$\bar{\epsilon}_{mx} = u_x$$

$$\bar{\epsilon}_{my} = v_y$$

$$\bar{\epsilon}_{mxy} = u_y + v_x$$

(4.8)

In equations (4.7) $\delta \epsilon_b$, $\delta \bar{\epsilon}_m$, $\delta \epsilon'_b$, $\delta \bar{\epsilon}'_m$ and ϵ'_b , ϵ'_m can then be expressed as:

$$\delta \epsilon_{bx} = -z \delta w_{xx}, \quad \delta \epsilon_{by} = -z \delta w_{yy}, \quad \delta \epsilon_{bxy} = -2z \delta w_{xy}$$

$$\delta \bar{\epsilon}_m = \delta u_x, \quad \delta \bar{\epsilon}_{my} = \delta v_y, \quad \delta \bar{\epsilon}_{mxy} = \delta u_y + \delta v_x$$

(4.9)

$$\begin{aligned}
\delta \epsilon'_{bx} &= \delta \bar{\epsilon}'_{mx} = 0 \\
\delta \epsilon'_{by} &= \delta \bar{\epsilon}'_{my} = 0 \\
\delta \epsilon'_{bxy} &= \delta \bar{\epsilon}'_{mxy} = 0
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\epsilon'_{bx} &= -z \Delta w_{xx} \\
\epsilon'_{by} &= -z \Delta w_{yy} \\
\epsilon'_{bxy} &= -2z \Delta w_{xy}
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\bar{\epsilon}'_{mx} &= \Delta u_x \\
\bar{\epsilon}'_{my} &= \Delta v_y \\
\bar{\epsilon}'_{mxy} &= \Delta u_y + \Delta v_x
\end{aligned} \tag{4.12}$$

The membrane stresses σ_m can be expressed in terms of an Airy stress function ϕ defined in section 2.5 by equations (2.88), which are restated here as:

$$\sigma_{mx} = \phi_{yy}, \quad \sigma_{my} = \phi_{xx}, \quad \sigma_{mxy} = -\phi_{xy} \tag{4.13}$$

For plate problems, the stress function ϕ can be expressed as a second order function in the out-of-plane displacement variables w (to be shown later). In equations (4.7), $\delta \sigma_m$, $\delta \sigma'_m$ and σ'_m can then be expressed as:

$$\begin{aligned}
\delta \sigma_{mx} &= \delta \phi_{yy}, \quad \delta \sigma_{my} = \delta \phi_{xx}, \quad \delta \sigma_{mxy} = -\delta \phi_{xy} \\
\delta \sigma'_{mx} &= \delta \phi'_{yy}, \quad \delta \sigma'_{my} = \delta \phi'_{xx}, \quad \delta \sigma'_{mxy} = -\delta \phi'_{xy} \\
\sigma'_m &= \phi'_{yy}, \quad \sigma'_{my} = \phi'_{xx}, \quad \sigma'_{mxy} = -\phi'_{xy}
\end{aligned} \tag{4.14}$$

Also p'_i can be expressed as:

$$p'_i = \Delta p_i \tag{4.15}$$

In equation (4.2), the integration over the volume $d \text{ vol}$ may be replaced by $dx dy dz$. The following integrals then apply:

$$\int_{-t/2}^{t/2} dz = t, \quad \int_{-t/2}^{t/2} z dz = 0, \quad \int_{-t/2}^{t/2} z^2 dt = \frac{t^3}{12} \quad (4.16)$$

Substituting equations (4.7) and (4.2) and making use of equations (4.8) to (4.16), equation (4.2) can then be written as:

$$\begin{aligned} & \iint_D \{ \delta w_{xx} \Delta w_{xx} + \nu (\delta w_{xx} \Delta w_{yy} + \delta w_{yy} \Delta w_{xx}) + \delta w_{yy} \Delta w_{yy} + 2(1-\nu) \delta w_{xy} \Delta w_{xy} \} \\ & + Ct \{ \delta u_x \Delta u_x + \nu (\delta u_x \Delta v_y + \delta v_y \Delta u_x) + \delta v_y \Delta v_y \\ & + \frac{1}{2} (1-\nu) (\delta u_y + \delta v_x) (\Delta u_y + \Delta v_x) \} \\ & + t \{ \delta \phi_{yy} \Delta u_x + \delta \phi_{xx} \Delta v_y - \delta \phi_{xy} (\Delta u_y + \Delta v_x) + \\ & + \delta \phi'_{yy} u_x + \delta \phi'_{xx} v_y - \delta \phi'_{xy} (u_y + v_x) \\ & + \delta u_x \phi'_{yy} + \delta v_y \phi'_{xx} - (\delta u_y + \delta v_x) \phi'_{xy} \} \\ & + \frac{t}{E} \{ \delta \phi_{yy} \phi'_{yy} - \nu (\delta \phi_{xx} \phi'_{yy} + \delta \phi_{yy} \phi'_{xx}) + \delta \phi_{xx} \phi'_{xx} \\ & + 2(1+\nu) \delta \phi_{xy} \phi'_{xy} \\ & + \delta \phi'_{yy} \phi_{yy} - \nu (\delta \phi'_{xx} \phi_{yy} + \delta \phi'_{yy} \phi_{xx}) + \delta \phi'_{xx} \phi_{xx} \\ & + 2(1+\nu) \delta \phi'_{xy} \phi_{xy} \} \quad dx dy - \Delta p_i \delta r_i = \\ & = \iint_D \{ [(w_{xx} - w_{0xx}) + \nu (w_{yy} - w_{0yy})] \delta w_{xx} \\ & + [(w_{yy} - w_{0yy}) + \nu (w_{xx} - w_{0xx})] \delta w_{yy} + 2(1-\nu) (w_{xy} - w_{0xy}) \delta w_{xy} \} \\ & + Ct \{ (u_x + \nu v_y) \delta u_x + (v_y + \nu u_x) \delta v_y + \frac{1}{2} (1-\nu) (u_y + v_x) (\delta u_y + \delta v_x) \} \\ & + t \{ u_x \delta \phi_{yy} + v_y \delta \phi_{xx} - (u_y + v_x) \delta \phi_{xy} \} \\ & + t \{ \phi_{yy} \delta u_x + \phi_{xx} \delta v_y - \phi_{xy} (\delta u_y + \delta v_x) \} \end{aligned}$$

... [continued]

$$\begin{aligned}
& + \frac{t}{E} \{ (\phi_{yy} - \nu \phi_{xx}) \delta \phi_{yy} + (\phi_{xx} - \nu \phi_{yy}) \delta \phi_{xx} \\
& + 2(1-\nu) \phi_{xy} \delta \phi_{xy} \} \quad dx dy - p_i \delta r_i
\end{aligned} \tag{4.17}$$

where

$$C = \frac{E}{(1-\nu^2)}, \quad D = C \frac{t^3}{12}$$

Equation (4.17) may be expressed in matrix form as:

$$\begin{aligned}
& \iint [\delta Q_1]^T [D] [\Delta Q_1] + [\delta Q_2]^T [M_1] [\phi'] + [\delta \phi]^T [M_1]^T [\Delta Q_2] \\
& + [\delta \phi]^T [M_2] [\phi'] + [\delta \phi']^T [M_2] [\phi] \quad dx dy \\
& - \Delta p_i \delta r_i = p_i \delta r_i + \iint [Q_1]^T [D_1] [\delta Q_1] \\
& + [\delta Q_2]^T [M_1] [\phi] + [\delta \phi]^T [M_1]^T [Q_1] \\
& + [\delta \phi] [M_2] [\phi] \quad dx dy
\end{aligned} \tag{4.18}$$

where

$$[\delta Q_1]^T = [\delta u_x, \delta u_y, \delta v_x, \delta v_y, \delta w_{xx}, \delta w_{yy}, 2\delta w_{xy}]$$

$$[\Delta Q_1]^T = [\Delta u_x, \Delta u_y, \Delta v_x, \Delta v_y, \Delta w_{xx}, \Delta w_{yy}, 2\Delta w_{xy}]$$

$$[\delta Q_2]^T = [\delta u_x, \delta v_y, \delta u_y, \delta v_x]$$

$$[\Delta Q_2] = [\Delta u_x, \Delta v_y, \Delta u_y, \Delta v_x]$$

$$[\phi] = [\phi_{yy}, \phi_{xx}, \phi_{xy}]$$

$$[\delta \phi] = [\delta \phi_{yy}, \delta \phi_{xx}, \delta \phi_{xy}]$$

$$[\phi'] = [\phi'_{yy}, \phi'_{xx}, \phi'_{xy}]$$

$$[\delta \phi'] = [\delta \phi'_{yy}, \delta \phi'_{xx}, \delta \phi'_{xy}]$$

$$\begin{aligned}
 [M_1]^T &= \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & -t & -t \end{bmatrix} \\
 [M_2] &= \frac{t}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}
 \end{aligned} \tag{4.19}$$

[D] is defined in equations (2.114).

Equation (4.18) provides a basis for the Newton-Raphson procedure to the solution of nonlinear plate problems. Before proceeding to outline the solution procedure, it is necessary to define the form of the stress function ϕ , and therefore $\delta\phi$, $\delta\phi'$, ϕ' in equation (4.18).

4.2.1 Stress Function and Boundary Conditions

For a plate with an initial out-of-plane imperfection w_0 shown in Fig (4.4), the nonlinear expressions for the membrane strains in terms of displacements u, v, w in the x, y, z directions respectively are given by:

$$\epsilon_x = u_x + \frac{1}{2} (w_x^2 - w_{0x}^2) \quad (a)$$

$$\epsilon_y = v_y + \frac{1}{2} (w_y^2 - w_{0y}^2) \quad (b)$$

$$\epsilon_{xy} = u_y + v_x + (w_x w_y - w_{0x} w_{0y}) \quad (c)$$

(4.20)

Using equations (4.20) and the procedure outlined in section 2.5 for the derivation of Von Karman's equilibrium and compatibility equations, Von Karman's equation of compatibility can be written as:

$$\nabla^4 \phi = E \left\{ (w_{xy}^2 - w_{xx} w_{yy}) - (w_{0xy}^2 - w_{0xx} w_{0yy}) \right\} \quad (4.21)$$

If the out-of-plane displacement w and the initial imperfection w_0 are defined by:

$$w = \sum_i^r \sum_j^s W_{ij} Y_i \sin \frac{j\pi x}{2b}$$

$$w_0 = \sum_i^r \sum_j^s W_{0ij} Y_i \sin \frac{j\pi x}{2b} \quad (4.22)$$

then by substituting equations (4.22) into equation (4.21) it can be shown that:

(i) for $j \neq \ell$

$$\begin{aligned} \nabla^4 \phi = \frac{\gamma}{2} \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \bar{w}_{ijkl} \left[\left\{ 2(j\ell)Y_{iy}K_{ky} + (j^2Y_iY_{kyy} \right. \right. \\ \left. \left. + \ell^2Y_{iyy}K_k) \right\} \cos \frac{(j-\ell)\pi x}{2b} \right. \\ \left. + \left\{ 2(j\ell)Y_{iy}Y_{ky} - (j^2Y_iY_{kyy} \right. \right. \\ \left. \left. + \ell^2Y_{iyy}Y_k) \right\} \cos \frac{(j+\ell)\pi x}{2b} \right] \end{aligned} \quad (4.23)$$

(ii) for $j = \ell$

$$\begin{aligned} \nabla^4 \phi = \frac{\gamma}{2} \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \bar{w}_{ijkl} (j^2) \left[\left\{ 2Y_{iy}Y_{ky} + (Y_iY_{kyy} + Y_{iyy}Y_k) \right\} \right. \\ \left. + \left\{ 2Y_{iy}Y_{ky} - (Y_iY_{kyy} \right. \right. \\ \left. \left. + Y_{iyy}Y_k) \right\} \cos \frac{2j\pi x}{2b} \right] \end{aligned} \quad (4.24)$$

where

$$\gamma = \frac{E}{2} \left(\frac{\pi}{2b} \right)^2$$

$$\bar{w}_{ijkl} = w_{ij}w_{kl} - w_{Oij}w_{Ok\ell} \quad (4.25)$$

and the subscript y relating to Y denotes differentiation with respect to y .

Consider first the case where $j \neq \ell$, the stress function ϕ can be assumed to be of the form:

$$\phi = \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \bar{w}_{ijkl} \left\{ \phi_1 \cos \frac{(j-\ell)\pi x}{2b} + \phi_2 \cos \frac{(j+\ell)\pi x}{2b} \right\} \quad (4.26)$$

Substituting equation (4.26) into equation (4.23) and separating the final equation into its component parts yields the following two equations:

$$\phi_{1yyyy} + \alpha_{1j\ell}\phi_{1yy} + \alpha_{2j\ell}\phi_1 = \{2(j\ell)Y_{iy}Y_{ky} + (j^2Y_iY_{kyy} + \ell^2Y_{iyy}Y_k)\} \quad (4.27)$$

$$\phi_{2yyyy} + \rho_{1j\ell}\phi_{1yy} + \rho_{2j\ell}\phi_1 = \{2(j\ell)Y_{iy}Y_{ky} - (j^2Y_iY_{kyy} + \ell^2Y_{iyy}Y_k)\} \quad (4.28)$$

where

$$\begin{aligned} \alpha_{1j\ell} &= -2\left[\frac{(j-\ell)\pi}{2b}\right]^2 \\ \alpha_{2j\ell} &= \left[\frac{(j-\ell)\pi}{2b}\right]^4 \\ \rho_{1j\ell} &= -2\left[\frac{(j+\ell)\pi}{2b}\right]^2 \\ \rho_{2j\ell} &= \left[\frac{(j+\ell)\pi}{2b}\right]^4 \end{aligned} \quad (4.29)$$

Assuming that the plate is simply supported along edges parallel to the x axis, then Y_i can be assumed to be of the form:

$$Y_i = \sin \frac{i\pi y}{2d} \quad (4.30)$$

Substituting equation (4.30) into equations (4.27) and (4.28) gives:

$$\begin{aligned} \phi_{1yyyy} + \alpha_{1j\ell}\phi_{1yy} + \alpha_{2j\ell}\phi_1 &= \frac{\gamma}{4}\left(\frac{\pi}{2d}\right)^2 \left\{ \theta_{1ijkl} \cos \frac{(i-k)\pi y}{2d} \right. \\ &\quad \left. + \theta_{2ijkl} \cos \frac{(i+k)\pi y}{2d} \right\} \end{aligned} \quad (4.31)$$

$$\begin{aligned} \phi_{2yyyy} + \rho_{1j\ell}\phi_{1yy} + \rho_{2j\ell}\phi_1 &= \frac{\gamma}{4}\left(\frac{\pi}{2d}\right)^2 \left\{ \theta_{2ijkl} \cos \frac{(i-k)\pi x}{2b} \right. \\ &\quad \left. + \theta_{1ijkl} \cos \frac{(i+k)\pi y}{2d} \right\} \end{aligned} \quad (4.32)$$

where

$$\begin{aligned} \theta_{1ijkl} &= 2ijkl - (j^2k^2 + i^2\ell^2) \\ \theta_{2ijkl} &= 2ijkl + (j^2k^2 + i^2\ell^2) \end{aligned} \quad (4.33)$$

The general solution of equations (4.31) and (4.32) can be expressed as the sum of the particular integral and the complementary function. Consider first equation (4.31), the particular integral ϕ_{1PI} can be taken as:

$$\phi_{1PI} = D_{1ijkl} \cos \frac{(i-k)\pi y}{2d} + D_{2ijkl} \cos \frac{(i+k)\pi y}{2d} \quad (4.34)$$

Substituting equation (4.34) into equation (4.31) and equating like terms in y gives:

$$\begin{aligned} D_{1ijkl} &= \frac{\gamma \left(\frac{\pi}{2d} \right)^2 \theta_{1ijkl}}{4 \left(\mu_{ik}^4 - \alpha_{1jl} \mu_{ik}^2 + \alpha_{2jl} \right)} \\ D_{2ijkl} &= \frac{\gamma \left(\frac{\pi}{2d} \right)^2 \theta_{2ijkl}}{4 \left(\eta_{ik}^4 - \alpha_{1jl} \eta_{ik}^2 + \alpha_{2jl} \right)} \end{aligned} \quad (4.35)$$

where

$$\begin{aligned} \mu_{ik} &= \frac{(i-k)\pi}{2d} \\ \eta_{ik} &= \frac{(i+k)\pi}{2d} \end{aligned} \quad (4.36)$$

The complementary function ϕ_{1CF} for equation (4.31) is of the form:

$$\begin{aligned} \phi_{1CF} &= t_{11ijkl} \cosh (\beta_{1jl} y) + t_{12ijkl} \sinh (\beta_{1jl} y) \\ &\quad + t_{13ijkl} y \cosh (\beta_{1jl} y) \\ &\quad + t_{14ijkl} y \sinh (\beta_{1jl} y) \end{aligned} \quad (4.37a)$$

where

$$\beta_{1jl} = \frac{(j-l)\pi}{2b} \quad (4.37b)$$

The general solution of equation (4.31) is then given by:

$$\begin{aligned}
\phi_1 = & D_{1ijkl} \cos \frac{(i-k)\pi y}{2d} + D_{2ijkl} \cos \frac{(i+k)\pi y}{2d} + t_{11ijkl} \cosh (\beta_{1jl} y) \\
& + t_{12ijkl} \sinh (\beta_{1jl} y) \\
& + t_{13ijkl} y \cosh (\beta_{1jl} y) \\
& + t_{14ijkl} y \sinh (\beta_{1jl} y)
\end{aligned} \tag{4.38}$$

Similarly, the general solution of equation (4.38) is given by:

$$\begin{aligned}
\phi_2 = & E_{1ijkl} \cos \frac{(i-k)\pi y}{2d} + E_{2ijkl} \cos \frac{(i+k)\pi y}{2d} + t_{21ijkl} \cosh (\beta_{2jl} y) \\
& + t_{22ijkl} \sinh (\beta_{2jl} y) \\
& + t_{23ijkl} y \cosh (\beta_{2jl} y) \\
& + t_{24ijkl} y \sinh (\beta_{2jl} y)
\end{aligned} \tag{4.39a}$$

where

$$\beta_{2jl} = \frac{(j+l)\pi}{2b} \tag{4.39b}$$

where

$$\begin{aligned}
E_{1ijkl} &= \frac{\gamma \left(\frac{\pi}{2d} \right)^2 \theta_{2ijkl}}{4 \left(\mu_{ik}^4 - \rho_{1jl} \mu_{ik}^2 + \rho_{2jl} \right)} \\
E_{2ijkl} &= \frac{\gamma \left(\frac{\pi}{2d} \right)^2 \theta_{1ijkl}}{4 \left(\eta_{ik}^4 - \rho_{1jl} \eta_{ik}^2 + \rho_{2jl} \right)}
\end{aligned} \tag{4.40}$$

The constants $t_{11}-t_{14}$, $t_{21}-t_{24}$ in equations (4.38) and (4.39) are used to satisfy boundary conditions on the plate unloaded edges.

Consider now the solution to equation (4.24) for the case where $j = l$, the stress function ϕ can be represented by the expression:

$$\phi = \sum_i^r \sum_j^s \sum_k^r \sum_l^s \bar{w}_{ijkl} \left\{ \phi_1 + \phi_2 \cos \frac{2j\pi x}{2b} \right\} \tag{4.41}$$

Substituting equation (4.41) into equation (4.24) and separating the final equation into its component parts gives:

$$\phi_{1yyyy} = \frac{\gamma}{2} (j^2) \{ 2Y'_{iy} Y'_{ky} + Y_i Y_{ky} + Y_{iyy} Y_k \} \quad (4.42)$$

$$\phi_{2yyyy} + \rho_{1j\ell} \phi_{2yy} + \rho_{2j\ell} \phi_2 = \{ 2(j\ell) Y_{iy} Y_{ky} - (j^2 Y_i Y_{ky} + \ell^2 Y_{iyy} Y_k) \} \quad (4.43)$$

Integrating equation (4.42) twice by parts gives:

$$\phi_{1yy} = \frac{\gamma}{2} (j^2) \{ Y_i Y_k \} + C_{1y} + C_2 \quad (4.44)$$

The constants of integration C_1 and C_2 are used to satisfy displacement boundary conditions regarding the in-plane movement of the loaded edges of the plate. The complete expression for ϕ_1 is given by:

$$\phi_1 = \frac{\gamma}{2} (j^2) \iint \{ Y_i Y_k + C_{1y} + C_2 \} dy dy \quad (4.45)$$

The general solution of equation (4.43) has already been derived and is given by equation (4.39) assuming $j = \ell$.

The general expressions defining the stress function ϕ have now been derived. To evaluate constants $t_{11}-t_{14}$, $t_{21}-t_{24}$ in equations (4.38) and (4.39), the boundary conditions for stress free edges are used, these being:

$$\left. \begin{aligned} \sigma_y &= \phi_{xx} = 0 \\ \sigma_{xy} &= -\phi_{xy} = 0 \end{aligned} \right\} \text{ at } y = 0 \text{ and } y = 2d \quad (4.46)$$

Substituting equation (4.26) for $j \neq \ell$ into equation (4.46) yields:

$$\left. \begin{aligned} \phi_1 &= \phi_2 = 0 \\ \phi_{1y} &= \phi_{2y} = 0 \end{aligned} \right\} \text{ at } y = 0 \text{ and } y = 2d \quad (4.47)$$

Substituting equation (4.41) for $j = \ell$ into equation (4.46) gives:

$$\left. \begin{aligned} \phi_2 &= 0 \\ \phi_{2y} &= 0 \end{aligned} \right\} \text{ at } y = 0 \text{ and } y = 2d \quad (4.48)$$

Substituting for ϕ_1 and ϕ_2 from equations (4.38) and (4.39) into equations (4.47) and solving for $t_{11}-t_{14}$, $t_{21}-t_{24}$ gives:

$$\begin{aligned}
t_{11ijkl} &= -D_{1ijkl} - D_{2ijkl} \\
t_{12ijkl} &= -(G_{1ijkl}G_{6jl} - G_{4ijkl}G_{3jl}) / (G_{2jl}G_{6jl} - G_{5jl}G_{6jl}) \\
t_{13ijkl} &= -\beta_{1jl} t_{12ijkl} \\
t_{14ijkl} &= -(G_{4ijkl} - G_{5jl} t_{12ijkl})
\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}
G_{1ijkl} &= D_{1ijkl} \cos (i-k)\pi + D_{2ijkl} \cos (i+k)\pi + t_{11ijkl} \cosh (\beta_{1jl} 2d) \\
G_{2jl} &= \sinh (\beta_{2jl} 2d) \\
G_{4ijkl} &= -(D_{1ijkl} + D_{2ijkl}) \beta_{1jl} \sinh (\beta_{1jl} 2d) \\
G_{5jl} &= -\beta_{1jl}^2 2d \sinh (\beta_{1jl} 2d) \\
G_{6jl} &= \sinh (\beta_{1jl} 2d) + \beta_{1jl} 2d \cosh (\beta_{1jl} 2d)
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
t_{21ijkl} &= -E_{1ijkl} - E_{2ijkl} \\
t_{22ijkl} &= -(H_{ijkl}H_{6jl} - H_{4ijkl}) / (H_{2jl}H_{6jl} - H_{5jl}H_{6jl}) \\
t_{23ijkl} &= -\beta_{2jl} t_{12ijkl} \\
t_{24ijkl} &= -(H_{4ijkl} - H_{5jl} t_{22ijkl})
\end{aligned} \tag{4.51}$$

where

$$\begin{aligned}
H_{1ijkl} &= E_{1ijkl} \cos (n-q)\pi + E_2 \cos (n+q)\pi + t_{21ijkl} \cosh (\beta_{2jl} 2d) \\
H_{2jl} &= \sinh (\beta_{2jl} 2d) - \beta_{2jl} 2d \cosh (\beta_{2jl} 2d) \\
H_{3jl} &= 2d \sinh (\beta_{1jl} 2d) \\
H_{4ijkl} &= -(E_{1ijkl} + E_{2ijkl}) \beta_{2jl} \sinh (\beta_{2jl} 2d) \\
H_{5jl} &= -\beta_{2jl}^2 2d \sinh (\beta_{2jl} 2d)
\end{aligned}$$

$$H_{6j\ell} = \sinh(\beta_{2j\ell} 2d) + \beta_{2j\ell} 2d \cosh(\beta_{2j\ell} 2d) \quad (4.52)$$

To solve for constants of integrations C_1 and C_2 in equation (4.45), equation (4.20a) is utilized in the following form:

$$u = \int \epsilon_x - \frac{1}{2} (w_x^2 - w_{0x}^2) dx \quad (4.53)$$

Expressing the strain ϵ_x in terms of stresses gives:

$$u = \int \frac{1}{E} (\phi_{yy} - \nu \phi_{xx}) - \frac{1}{2} (w_x^2 - w_{0x}^2) dx \quad (4.54)$$

Substituting equations (4.22), (4.41) into (4.54) and using equations (4.39), (4.45) and integrating, it can be shown that the displacements at the plate loaded edges $x = 0$ and $x = 2b$ are given by:

$$\begin{aligned} u|_{x=0} &= 0 \\ u|_{x=2b} &= (C_1 + C_{2y})2b \end{aligned} \quad (4.55)$$

The work presented here assumes that the strains associated with the stress function ϕ are purely extensional along the axial direction and therefore:

$$u = 0 \quad \text{at } x = 0 \quad \text{and } x = 2b \quad (4.56)$$

This condition is satisfied if:

$$C_1 = C_2 = 0 \quad (4.57)$$

Using equations (4.30) and (4.57) equation (4.45) can then be expressed as:

$$\phi_1 = \frac{\gamma}{4} (j)^2 \iint \cos \frac{(i-k)\pi y}{2d} - \cos \frac{(i+k)\pi y}{2d} dy dy \quad (4.58)$$

The stress function ϕ given by equations (4.26) and (4.41) have now been completely defined for the conditions $j \neq \ell$ and $j = \ell$.

ϕ can now be expressed in shorthand notation as:

$$\phi = \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \bar{w}_{ijkl} \xi_{ijkl} \quad (4.59a)$$

where for $j \neq \ell$

$$\xi_{ijkl} = \phi_1 \cos \frac{(j-\ell)\pi x}{2b} + \phi_2 \cos \frac{(j+\ell)\pi x}{2b} \quad (4.59b)$$

and for $j = \ell$

$$\xi_{ijkl} = \phi_1 + \phi_2 \cos \frac{2j\pi x}{2b} \quad (4.59c)$$

Having defined the general form of ϕ , then using equations (4.25), $\delta\phi$, $\delta\phi'$, ϕ' in equation (4.18) are of the form:

$$\begin{aligned} \delta\phi &= \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \delta w_{ij} w_{kl} (\xi_{ijkl} + \xi_{klij}) \\ \delta\phi' &= \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \delta w_{ij} \Delta w_{kl} (\xi_{ijkl} + \xi_{klij}) \\ \phi' &= \sum_i^r \sum_j^s \sum_k^r \sum_\ell^s \Delta w_{ij} w_{kl} (\xi_{ijkl} + \xi_{klij}) \end{aligned} \quad (4.60)$$

4.2.2 Application to Plates

Consider an initially imperfect plate under end compression along two opposite edges as shown in Fig (4.4). All edges are assumed simply supported. Two types of in-plane boundary conditions are considered. The first case is where the unloaded edges are stress free and therefore free to distort in the plane of the plate. The second case is for unloaded edges which are maintained straight but allowed to move bodily in the plane of the plate.

Case (a)

If the unloaded edges of the plate are allowed free to wave in the plane of the plate, the stress function ϕ must comply to stress free conditions along these edges - this case has already been considered during the derivation of the stress function in the previous section - see equations (4.46) to (4.52).

The in-plane displacements u and v which allow uniform displacement of the loaded edges and stress free conditions along the unloaded edges can be taken as:

$$\begin{aligned} u &= u_L(b-x) \\ v &= -u_L v(d-y) \end{aligned} \quad (4.61)$$

The out-of-plane displacement and the form of the initial imperfection are given by equations (4.22), (4.30) and restated here for completeness:

$$\begin{aligned} w &= \sum_i^r \sum_j^s w_{ij} \sin \frac{2\pi y}{2d} \sin \frac{j\pi x}{2b} \\ w_0 &= \sum_i^r \sum_j^s w_{0ij} \sin \frac{2\pi y}{2d} \sin \frac{j\pi x}{2b} \end{aligned} \quad (4.62)$$

Substituting equations (4.59), (4.61), (4.62) into equation (4.18) yields:

$$\{\delta\alpha\}^T [K] \{\Delta\alpha\} + \{\delta\alpha\}^T [K_g] \{\Delta\alpha\} - \{\delta\alpha\}^T \{\Delta P\} = \{\delta\alpha\}^T \{P\} - \{\delta\alpha\}^T [K_e] \quad (4.63)$$

where

$$\begin{aligned}\{\delta\alpha\}^T &= \{\delta u_L, \delta w_{11}, \delta w_{12}, \dots, \delta w_{ij}, \dots, \delta w_{k\ell}, \dots, \delta w_{mn}, \dots, \delta w_{pq}, \dots\} \\ \{\Delta\alpha\}^T &= \{\Delta u_L, \Delta w_{11}, \Delta w_{12}, \dots, \Delta w_{ij}, \dots, \Delta w_{k\ell}, \dots, \Delta w_{mn}, \dots, \Delta w_{pq}, \dots\}\end{aligned}\quad (4.64)$$

The general terms in $[K]$, $[K_g]$, $[K_e]$, $\{\Delta P\}$ and $\{P\}$ associated with displacement variables u , w_{ij} , $w_{k\ell}$, w_{mn} , w_{pq} are:

$$[K] = D \begin{bmatrix} K_{uu} & K_{uwij} \\ K_{wuk\ell} & K_{wwijk\ell} \end{bmatrix} \quad (4.65)$$

where

$$\begin{aligned}K_{uu} &= Et \int_0^{2d} \int_0^{2b} 1.0 \, dx dy \\ K_{uwij} &= K_{wuk\ell} = 0 \\ K_{wwijk\ell} &= D \left\{ i^2 k^2 \left(\frac{\pi}{2d} \right)^2 + v(i^2 \ell^2 + j^2 k^2) \left(\frac{\pi}{2b} \right)^2 \left(\frac{\pi}{2d} \right)^2 \right. \\ &\quad \left. + j^2 \ell^2 \left(\frac{\pi}{2b} \right)^2 \right\} \int_0^{2d} \int_0^{2b} \sin \frac{i\pi y}{2d} \sin \frac{k\pi y}{2d} \sin \frac{j\pi x}{2b} \\ &\quad \sin \frac{\ell\pi x}{2b} \, dx dy \quad (4.66)\end{aligned}$$

$$[K_g] = \begin{bmatrix} A_{uu} & & & & \\ & A_{uw} & & & \\ & & A_{wwijk\ell} & A_{wwijmn} & A_{wwijpq} \\ A_{wu} & A_{wwk\ell ij} & & A_{wwk\ell mn} & A_{wwk\ell pq} \\ & & & & A_{wwmnpq} \\ & & & A_{wwpqmn} & \end{bmatrix} \quad (4.67)$$

where

$$\begin{aligned}
 A_{uu} &= 0 \\
 A_{uw} &= A_{wu} = -W_{ij}\Gamma_{ijkl} - W_{kl}\Gamma_{ijkl} \\
 A_{wwijmn} &= W_{kl}W_{pq}\Lambda_{ijklmnpq} \\
 A_{wwijpq} &= W_{kl}W_{pq}\Lambda_{ijklmnpq} \\
 A_{wwklmn} &= W_{ij}W_{pq}\Lambda_{ijklmnpq} \\
 A_{wwklpq} &= W_{ij}W_{mn}\Lambda_{ijklmnpq} \\
 A_{wwmnpq} &= A_{wwpqmn} = (W_{ij}W_{kl} - W_{Oij}W_{Ok\ell})\Lambda_{ijklmnpq} \\
 A_{wwijkl} &= A_{wwkl ij} = -U_L\Gamma_{ijkl}
 \end{aligned} \tag{4.68}$$

in which

$$\begin{aligned}
 \Gamma_{ijkl} &= t \int_0^{2d} \int_0^{2b} \{ \xi_{ijkl yy} + \nu \xi_{ijkl xx} \} \, dx dy \\
 \Lambda_{ijklmnpq} &= \frac{t}{E} \int_0^{2d} \int_0^{2b} \{ \xi_{ijkl yy} \xi_{mnpq yy} - \nu (\xi_{ijkl yy} \xi_{mnpq xx} \\
 &\quad + \xi_{ijkl xx} \xi_{mnpq yy}) \\
 &\quad + \xi_{ijkl xx} \xi_{mnpq xx} \\
 &\quad + 2(1+\nu) \xi_{ijkl xy} \xi_{mnpq xy} \} \, dx dy
 \end{aligned} \tag{4.69}$$

$$[K_e] = [K]\{\alpha\} + [K_{ne}] \tag{4.70}$$

where

$$[K_{ne}] = \begin{bmatrix} C_u \\ C_{ij} \\ C_{kl} \end{bmatrix} \tag{4.71}$$

in which

$$\begin{aligned}
 C_u &= -(W_{ij}W_{kl} - W_{Oij}W_{Ok\ell})\Gamma_{ijk\ell} \\
 C_{ij} &= (-U_L W_{kl})\Gamma_{ijk\ell} + W_{kl}(W_{mn}W_{pq} - W_{Omn}W_{Opq})\Lambda_{ijk\ell mnpq} \\
 C_{kl} &= (-U_L W_{ij})\Gamma_{ijk\ell} + W_{ij}(W_{mn}W_{pq} - W_{Omn}W_{Opq})\Lambda_{ijk\ell mnpq} \quad (4.72)
 \end{aligned}$$

$$\{\Delta P\} = \begin{pmatrix} E_u \\ E_{wij} \end{pmatrix}$$

where

$$\begin{aligned}
 E_u &= \Delta P 2b \\
 E_{wij} &= 0 \quad (4.73)
 \end{aligned}$$

and

$$\begin{aligned}
 \{P\} &= \begin{pmatrix} F_u \\ F_{wij} \end{pmatrix} \\
 F_u &= P 2b \\
 F_{wij} &= 0 \quad (4.74)
 \end{aligned}$$

The general terms in equation (4.63) have now been completely defined. The following orthogonality relationships are then useful in the integration of equations (4.66) and (4.69).

$$\begin{aligned}
 \int_0^{\ell} \sin \frac{r\pi x}{\ell} \sin \frac{s\pi x}{\ell} dx &= \begin{cases} \frac{\ell}{2} & \text{for } r=s \\ 0 & \text{for } r \neq s \end{cases} \\
 \int_0^{\ell} \cos \frac{r\pi x}{\ell} \cos \frac{s\pi x}{\ell} dx &= \begin{cases} \frac{\ell}{2} & \text{for } r=s \\ 0 & \text{for } r \neq s \end{cases} \quad (4.75)
 \end{aligned}$$

Since equation (4.63) is valid for all admissible variations $\{\delta\alpha\}$ then equation (4.63) may be written as:

$$[K_{inc}]\{\Delta\alpha\} - \{\Delta P\} = \{P\} - [K_e] \quad (4.76)$$

where $[K_{inc}]$ is the incremental stiffness matrix given by:

$$[K_{inc}] = [K] + [K_g] \quad (4.77)$$

Equation (4.76) represents a set of linear simultaneous equation in the unknown $\{\Delta\alpha\}$ and $\{\Delta P\}$. The right hand side of equation (4.76) represents the equilibrium equations and vanishes if equilibrium is satisfied. Equation (4.76) is solved through Newton's method which is discussed in the next section.

Case (b)

If the unloaded edges of the plate are kept straight but allowed to move in the plane of the plate, then the constants t_{11} - t_{14} , t_{21} - t_{24} in equations (4.38) and (4.39) must be such that the stress function ϕ complies to the following boundary conditions along these edges:

$$(i) \quad \sigma_{xy} = -\phi_{xy} = 0 \quad \text{along } y=0 \text{ and } y=2d$$

$$(ii) \quad v = \text{constant in } x \text{ along } y=0 \text{ and } y=2d$$

To solve condition (ii), equation (4.20b) is utilized in the following form:

$$v = \int \epsilon_y - \frac{1}{2} (w_y^2 - w_{0y}^2) dy \quad (4.78)$$

Expressing the strain ϵ_y in terms of the stress function ϕ , equation (4.78) can be written as:

$$v = \int \frac{1}{E} (\phi_{xx} - \nu \phi_{yy}) - \frac{1}{2} (w_y^2 - w_{0y}^2) dy \quad (4.79)$$

Using equations (4.59) representing ϕ , it can be shown that conditions (i) and (ii) are satisfied if:

$$t_{11ijkl} = \frac{-Et_{14ijkl} S_{1jl}}{\left[(j-l)^2 \left(\frac{\pi}{2b} \right)^2 \frac{1}{\beta_{1jl}} + \nu \beta_{1jl} \beta_{1jl} \sinh(\beta_{1jl} 2d) \right]}$$

$$\begin{aligned}
 t_{12ijkl} &= 0 \\
 t_{13ijkl} &= 0 \\
 t_{14ijkl} &= \frac{S_{2ijkl}}{S_{3jl} - S_{1jl}}
 \end{aligned} \tag{4.80}$$

where

$$S_{1jl} = \frac{1}{E} \left((j-l)^2 \left(\frac{\pi}{2b} \right)^2 \frac{1}{\beta_{1jl}} + \nu \beta_{1jl} \right) \left(\sinh (\beta_{1jl} 2d) + 2d \beta_{1jl} \cosh (\beta_{1jl} 2d) \right)$$

If $i=k$

$$S_{2ijkl} = -\beta_{1jl} \left(\frac{1}{8} i k \left(\frac{\pi}{2d} \right)^2 2d + \frac{1}{E} (j-l)^2 \left(\frac{\pi}{2b} \right)^2 D_{12jkl} 2d \right)$$

If $i \neq k$

$$S_{2ijkl} = 0$$

$$\begin{aligned}
 S_{3jl} = \frac{1}{E} \beta_{1jl} \left[\left((j-l)^2 \left(\frac{\pi}{2b} \right)^2 \frac{1}{\beta_{1jl}} + \nu \beta_{1jl} \right) 2d \cosh (\beta_{1jl} 2d) \right. \\
 \left. - \left((j-l)^2 \left(\frac{\pi}{2b} \right)^2 \frac{1}{\beta_{1jl}^2} + \nu \right) \sinh (\beta_{1jl} 2d) \right]
 \end{aligned} \tag{4.81}$$

Similarly

$$\begin{aligned}
 t_{21ijkl} &= \frac{-E t_{24ijkl} T_{ijl}}{\left[(j+l)^2 \left(\frac{\pi}{2b} \right)^2 \frac{1}{\beta_{2jl}} + \nu \beta_{2jl} \beta_{2jl} \sinh (\beta_{2jl} 2d) \right]} \\
 t_{22ijkl} &= 0 \\
 t_{23ijkl} &= 0 \\
 t_{24ijkl} &= \frac{T_{2ijkl}}{T_{3jl} - T_{1jl}}
 \end{aligned} \tag{4.82}$$

where

$$T_{1j\ell} = \frac{1}{E} (j+\ell)^2 \left(\frac{\pi}{2b}\right)^2 \frac{1}{\beta_{2j\ell}} + v\beta_{2j\ell} \sinh (\beta_{2j\ell} 2d) + 2d\beta_{2j\ell} \cosh (\beta_{2j\ell} 2d)$$

If $i=k$

$$T_{2ijk\ell} = -\beta_{2j\ell} \left(-\frac{1}{8} ik \left(\frac{\pi}{2d}\right)^2 2d + \frac{1}{E} (j+\ell)^2 \left(\frac{\pi}{2b}\right)^2 E_{ijk\ell} 2d \right)$$

If $i \neq k$

$$T_{2ijk\ell} = 0$$

$$T_{3j\ell} = \frac{1}{E} \beta_{2j\ell} \left[\left((j+\ell)^2 \left(\frac{\pi}{2b}\right)^2 \frac{1}{\beta_{2j\ell}} + v\beta_{2j\ell} \right) 2d \cosh (\beta_{2j\ell} 2d) - \left((j+\ell)^2 \left(\frac{\pi}{2b}\right)^2 \frac{1}{\beta_{2j\ell}^2} + v \right) \sinh (\beta_{2j\ell} 2d) \right] \quad (4.83)$$

The in-plane displacements u and v which allow uniform displacement of the loaded edges and the unloaded edges are given by equations (4.61). Substituting equations (4.59), (4.61), (4.62) into equation (4.17) and following the procedure outlined for Case (a), the final equation can be expressed in the form of equation (4.76) which can be solved through Newton's method discussed in the next section.

4.2.3 Basic Nonlinear Solution Techniques

The solution of geometrically nonlinear problems is usually attempted by one of three basic techniques [76] [77]:

- (1) incremental procedures, (2) iterative or Newton methods,
- (3) incremental-iterative or mixed procedures.

In this section the fundamentals of each of these techniques is presented for a nonlinear system whose equilibrium equations are given by:

$$[K_e]\{\alpha\} - \{P\} = 0 \quad (4.84)$$

where the nonlinearity occurs in the stiffness matrix $[K_e]$ which is a function of displacement variables, and $\{P\}$, $\{\alpha\}$ are load and displacement vectors respectively.

Applying a Taylor series expansion to equation (4.84) and neglecting terms containing second and higher order terms in increments $\Delta\alpha$ and ΔP yields:

$$[K_{inc}]\{\Delta\alpha\} - \{\Delta P\} = \{P\} - [K]\{\alpha\} \quad (4.85)$$

where $[K_{inc}]$ is the symmetric incremental stiffness matrix and $\{\Delta\alpha\}$, $\{\Delta P\}$ are incremental displacement and load vectors respectively.

(1) Incremental Procedures

The solution resulting from a pure incremental procedure does not necessarily satisfy equilibrium. The analysis is based upon the solution to the left hand side of equation (4.85) with the right hand side, representing equilibrium, set to zero, ie:

$$[K_{inc}]\{\Delta\alpha\} - \{\Delta P\} = 0 \quad (4.86)$$

Equation (4.86) represents a set of linear simultaneous equations in the unknowns $\Delta\alpha$ and ΔP . Either displacements $\Delta\alpha$ or load ΔP may be incremented and equation (4.86) solved for the unknown increments. Hence after the $i-1$ th increment, equation (4.86) may be written as:

$$K_{inc(i-1)} \{\Delta\alpha_i\} - \{\Delta P_i\} = 0 \quad (4.87)$$

The solution of equation (4.87) yields the total loads $\{P_i\}$ and displacements $\{\alpha_i\}$ given by:

$$\begin{aligned}\{P_i\} &= \{P_{i-1}\} + \{\Delta P_i\} \\ \{\alpha_i\} &= \{\alpha_{i-1}\} + \{\Delta \alpha_i\}\end{aligned}\quad (4.88)$$

The incremental procedure is schematically illustrated in Fig (4.5) for a two-degree-of-freedom system in P_i and α_i . Since equilibrium is not satisfied it can be seen that the solution resulting from this approach diverges from the exact solution. The accuracy of the incremental procedure may be improved by considering smaller increments, however, since a new incremental stiffness matrix has to be computed for each step, accuracy is purchased at the cost of additional computational effort.

(2) Iterative Procedures

Iterative or Newton iterative procedures involve the solution to equation (4.84) for the case where the structure is either fully loaded or displaced in each iteration. If the structure is loaded to a prescribed value $\{P\}$, equation (4.8) for the i th iteration may be written as:

$$[K_{inc}^{i-1}]\{\Delta \alpha^i\} = \{P\} - [K_e^{i-1}]\{\alpha^{i-1}\} \quad (4.89)$$

Equation (4.89) is solved for $\{\Delta \alpha^i\}$ and $[K_{inc}^{i-1}]$, $\{\Delta \alpha^i\}$ and $[K_{inc}^{i-1}]$, $[K_e^{i-1}]$, $\{\alpha^{i-1}\}$ are updated. The displacements after the i th iteration are given by:

$$\{\alpha^i\} = \{\alpha^{i-1}\} + \{\Delta \alpha^i\} \quad (4.90)$$

The cycle is repeated until the right hand side of equation (4.89) representing the equilibrium equations vanishes or reaches a prescribed error bound. In equation (4.89), the incremental stiffness matrix has to be updated after each iteration which results in extra computation. A modified iterative procedure may be employed where the incremental stiffness matrix is kept constant at its initial value. The two methods are schematically illustrated in Figs (4.6). Obviously the modified procedure requires greater number of iterations, however there is a saving on computation since it is not necessary to invert

a new stiffness at each cycle. The iterative procedures described here analogous to the various iterative schemes used for the numerical solutions of nonlinear equations, such as the Newton and Newton-Raphson procedures.

(3) Incremental - Iterative Procedures

The incremental-iterative or mixed procedures are in essence a combination of the incremental and iterative schemes. One such method is shown in Fig (4.7). Here the load is applied incrementally and iterations are carried out until equilibrium is satisfied. The load is further incremented followed by further iterations and so on.

Either the loads or the displacements may be incremented. Assuming that the load is incremented by $\{\Delta P\}$ and the structure is in equilibrium after the $(i-1)$ th iteration, equation (4.85) may be written as:

$$\left[K_{inc(i-1)} \right] \{\Delta \alpha_i\} - \{\Delta P\} = 0 \quad (4.91)$$

Equation (4.91) is then solved for the unknown displacements $\{\Delta \alpha_i\}$, the loads and displacements are then given by:

$$\begin{aligned} \{P_i\} &= \{P_{i-1}\} + \{\Delta P\} \\ \{\alpha_i\} &= \{\alpha_{i-1}\} + \{\Delta \alpha_i\} \end{aligned} \quad (4.92)$$

Holding the load constant at $\{P\} = \{P_i\}$, then for the system in its new configuration, equation (4.85) can be written as:

$$\left[K_{inc_i} \right] \{\Delta \alpha_i\} = \{P_i\} - [K_{ei}] \{\alpha_i\} \quad (4.93)$$

The iterative procedure is now used to solve for the correct values of displacements which satisfy equilibrium, ie resulting in the vanishing of the right hand side of equation (4.93). For the j th iteration, equation (4.93) may be written as:

$$\left[K_{inc_i}^{(j-1)} \right] \{\Delta \alpha_i^j\} = \{P_i\} - \left[K_{ei}^{(j-1)} \right] \{\alpha_i^{(j-1)}\} \quad (4.94)$$

As before, it is not necessary to update the incremental stiffness matrix during the iterative procedure and can be kept constant at the incremental value.

Comparison of Various Techniques

The principal advantage of the pure incremental approach is its relative simplicity, however its behaviour of predicting solutions which diverge from the true solution makes the approach unfavourable. The iterative procedure is easier to use, however, its limitation is that it provides no information concerning behaviour at intermediate loads. The mixed method combines the advantages of both the incremental and the iterative procedures, the additional computational effort is justified by the fact that the iterative part of the procedure permits one to assess the quality of equilibrium at each incremental stage. However it is well known that the incremental-iterative procedure fails to predict the equilibrium states in the vicinity of a limit point if load is incremented. Variants of Newton's methods have been proposed [78] [79] to get around this problem by solving the incremental equations in conjunction to either a normal to the tangent [78] or on arc subtended radially from the tangent [79].

An alternative approach to get around this problem is to increment displacements rather than loads [80] [81].

In the work that follows the incremental-iterative approach is used where the in-plane displacements are incremented.

4.3 Nonlinear Analysis of Stiffened Plates

4.3.1 Energy Equations for Stiffeners

The energy method described in section 4.2 for the nonlinear analysis of plate problems can be extended to analyse the nonlinear behaviour of stiffened panels, by taking into account the energy contribution of the stiffeners.

Consider a panel of length $2b$, width $2d$ shown in Fig (3.5). The panel is assumed to be simply supported along all four edges and stiffened by stiffeners of rigid cross-section which are allowed to warp.

The nonlinear expressions for strains occurring in a thin walled bars of open cross-section are given in section 3.3.2. If small initial imperfections defined by v_{0s} , w_{0s} , θ_s are present and u_s , v_s , w_s and θ_s are the total displacements, the nonlinear expressions for ϵ_x and ϵ_{xy} given by equations (3.127) and (3.129) can be written as:

$$\begin{aligned}
 \epsilon_x &= -y\{(v_{sxx} - v_{0sxx}) + (w_{sxx}\theta_s - w_{0sxx}\theta_{0s})\} \\
 &\quad -z\{(w_{sxx} - w_{0sxx}) - (v_{sxx}\theta_s - v_{0sxx}\theta_{0s})\} \\
 &\quad -u_{sx} + \frac{1}{2}(v_{sx}^2 - v_{0sx}^2) + \frac{1}{2}(w_{sx}^2 - w_{0sx}^2) \\
 &\quad + \frac{1}{2}\rho^2(\theta_{sx}^2 - \theta_{0sx}^2) \\
 \epsilon_{xy} &= 2\xi(\theta_{sx} - \theta_{0sx}) - (w_{sxx}v_{sx} - w_{0sxx}v_{0sx}) \\
 &\quad + (v_{sxx}w_{sx} - v_{0sxx}w_{0sx}) \quad (4.95)
 \end{aligned}$$

Equation (2.14) provides a basis for Newton-Raphson procedure to the solution of nonlinear equilibrium equations. For a thin walled bar of open cross-section, equation (2.14) can be written as:

$$\begin{aligned}
 \int \sigma'_x \delta \epsilon_x + \sigma'_{xy} \delta \epsilon_{xy} + \sigma_x \delta \epsilon'_x + \sigma_{xy} \delta \epsilon'_{xy} \quad d \text{ vol} \\
 - P'_i \delta r_i = P_i \delta r_i - \int \sigma_x \delta \epsilon_x + \sigma_{xy} \delta \epsilon_{xy} \quad d \text{ vol} \quad (4.96)
 \end{aligned}$$

Expressing stresses in terms of strains through equations (3.131) and rearranging equation (4.96) can be written as:

$$\int E\{\epsilon'_x \delta \epsilon_x + \epsilon_x \delta \epsilon'_x\} + G\{\epsilon'_{xy} \delta \epsilon_{xy} + \epsilon_{xy} \delta \epsilon'_{xy}\} d \text{ vol} - P'_i \delta r_i = P_i \delta r_i - \int E\{\epsilon_x \delta \epsilon_x\} + G\{\epsilon_{xy} \delta \epsilon_{xy}\} d \text{ vol} \quad (4.97)$$

In which $\delta \epsilon$, $\delta \epsilon'$ and ϵ' are of the following form:

$$\begin{aligned} \delta \epsilon_x = & -y\{\delta v_{sxx} + \delta w_{sxx} \theta_s + w_{sxx} \delta \theta_s\} \\ & -z\{\delta w_{sxx} - \delta v_{sxx} \theta_s - v_{sxx} \delta \theta_s \\ & + \delta u_x + v_{sx} \delta v_{sx} + w_{sx} \delta w_{sx} + \rho^2 \theta_{sx} \delta \theta_{sx}\} \end{aligned} \quad (4.98)$$

$$\delta \epsilon_{xy} = 2\xi\{\delta \theta_{sx} - \delta w_{sxx} v_{sx} - w_{sxx} \delta v_{sx} + \delta v_{sxx} w_{sx} + v_{sxx} \delta w_{sx}\} \quad (4.99)$$

$$\begin{aligned} \delta \epsilon'_x = & -y\{\delta w_{sxx} \Delta \theta_s + \Delta w_{sxx} \delta \theta_s\} \\ & +z\{\delta v_{sxx} \Delta \theta_s + \Delta v_{sxx} \delta \theta_s\} \\ & + \Delta v_{sx} \delta v_{sx} + \Delta w_{sx} \delta w_{sx} + \rho^2 \Delta \theta_{sx} \delta \theta_{sx} \end{aligned} \quad (4.100)$$

$$\delta \epsilon'_{xy} = -2\xi\{\delta w_{xx} \Delta v_{sx} + \Delta w_{sxx} \delta v_{sx} - \delta v_{sxx} \Delta w_{sx} - \Delta v_{sxx} \delta w_{sx}\} \quad (4.101)$$

$$\begin{aligned} \epsilon'_x = & -y\{\Delta v_{sxx} + \Delta w_{sxx} \theta_s + w_{sxx} \Delta \theta_s\} \\ & -z\{\Delta w_{sxx} - \Delta v_{sxx} \theta_s - v_{sxx} \Delta \theta_s \\ & + \Delta u_{sx} + v_{sx} \Delta v_{sx} + w_{sx} \Delta w_{sx} + \rho^2 \theta_{sx} \Delta \theta_{sx}\} \end{aligned} \quad (4.102)$$

$$\epsilon'_{xy} = 2\xi\{\Delta \theta_{sx} - \Delta w_{sxx} v_{sx} - w_{sxx} \Delta v_{sx} + \Delta v_{sxx} w_{sx} + v_{sxx} \Delta w_{sx}\} \quad (4.103)$$

In equation (4.97), the limit of integration over the volume $d \text{ vol}$ can be replaced by $dA dx$, where A represents the area of cross-section of the stiffener.

Equation (4.97) is now completely defined by substituting for ϵ_x , ϵ_{xy} , $\delta\epsilon_x$, $\delta\epsilon_{xy}$, ϵ'_x , ϵ'_{xy} , $\delta\epsilon'_x$ and $\delta\epsilon'_{xy}$ from equations (4.98) to (4.103). Equation (4.97) can then be expressed in matrix form as:

$$\int_0^{2b} \{\delta S_1\}^T [K_1] \{\Delta S_1\} + \{\delta S_2\}^T [K_2] \{\Delta S_2\} - \{\delta r\}^T \{\Delta P\} = \{\delta r\} \{P\} \\ - \int_0^{2b} \{\delta S_1\}^T [E_1] - \{\delta S_1\}^T [E_2] dx \quad (4.104)$$

in which

$$\begin{aligned} \{\delta S_1\}^T &= \{\delta v_{sxx}, \delta w_{sxx}, \delta\theta_s, \delta\theta_{sx}, \delta v_{sx}, \delta w_{sx}\} \\ \{\Delta S_1\}^T &= \{\Delta v_{sxx}, \Delta w_{sxx}, \Delta\theta_s, \Delta\theta_{sx}, \Delta v_{sx}, \Delta w_{sx}\} \\ \{\delta S_2\}^T &= \{\delta u_{sx}, \delta v_{sx}, \delta w_{sx}, \delta\theta_{sx}\} \\ \{\Delta S_2\}^T &= \{\Delta u_{sx}, \Delta v_{sx}, \Delta w_{sx}, \Delta\theta_{sx}\} \end{aligned} \quad (4.105)$$

The matrices $[K_1]$, $[E_1]$ represent the contribution of bending and St Venant torsion strains, and the matrices $[K_2]$, $[E_2]$ represent the contribution of 'membrane' strains. The elements of matrices $[K_1]$, $[K_2]$, $[E_1]$ and $[E_2]$ are given in Appendix 4 where the following integrals are used:

$$\begin{aligned} \int y dA &= \int z dA = 0 \\ \int y^2 dA &= I_z \quad \int z^2 dA = I_y \\ \int 4\xi^2 dA &= J \\ \int \rho^2 dA &= I_0 \quad \int \rho^4 dA = I_\rho \\ \int dA &= A \end{aligned} \quad (4.106)$$

4.3.2 Displacement Compatibility Conditions

Consider a stiffener placed at a distance $y = s$ as shown in Fig (3.5). The co-ordinate system for a typical stiffener and skin are shown in Fig (3.6).

The stiffener displacements u_s , v_s , w_s and θ_s can be related to the skin displacements u , v and w through the following compatibility equations:

$$\begin{aligned} u_s &= (u - h_{cz} w_x - h_{cy} v_x)_{y=s} \\ v_s &= (v - h_s w_y)_{y=s} \\ w_s &= (w)_{y=s} \\ \theta_s &= (w_y)_{y=s} \end{aligned} \quad (4.107)$$

where h_{cy} and h_{cz} represent the distance l_0 the centroid from the y and z axis respectively, and h_s represents the distance to the shear centre as depicted in Fig (3.6).

Then using equations (4.107), equations (4.105) can be expressed as:

$$\begin{aligned} \{\delta S_1\} &= [T_1] \{\delta S_{p1}\} \\ \{\Delta S_1\} &= [T_1] \{\Delta S_{p1}\} \\ \{\delta S_2\} &= [T_2] \{\delta S_{p2}\} \\ \{\Delta S_2\} &= [T_2] \{\Delta S_{p2}\} \end{aligned} \quad (4.108)$$

where $\{S_{p1}\}$ and $\{S_{p2}\}$ are vectors representing the skin displacements and are of the following form:

$$\begin{aligned} \{\delta S_{p1}\} &= \{\delta v_{xx}, \delta w_{yxx}, \delta w_{xx}, \delta w_y, \delta w_{yx}, \delta v_x, \delta w_x\}_{y=s} \\ \{\Delta S_{p1}\} &= \{\Delta v_{xx}, \Delta w_{yxx}, \Delta w_{xx}, \Delta w_y, \Delta w_{yx}, \Delta v_x, \Delta w_x\}_{y=s} \\ \{\delta S_{p2}\} &= \{\delta u_x, \delta w_{xx}, \delta v_{xx}, \delta v_x, \delta w_{yx}, \delta w_x\}_{y=s} \\ \{\Delta S_{p2}\} &= \{\Delta u_x, \Delta w_{xx}, \Delta v_{xx}, \Delta v_x, \Delta w_{yx}, \Delta w_x\}_{y=s} \end{aligned} \quad (4.109)$$

and $[T_1]$, $[T_2]$ are transformation matrices given by:

$$\begin{aligned}
 [T_1] &= \begin{bmatrix} 1 & -h_s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -h_s & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 [T_2] &= \begin{bmatrix} 1 & -h_{cz} & -h_{cy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -h_s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.110)
 \end{aligned}$$

Using equations (4.108), equation (4.104) can be written as:

$$\begin{aligned}
 \int_0^{2b} \{ \delta S_{p1} \}^T [T_1]^T [K_1] [T_1] \{ \Delta S_{p1} \} + \{ \delta S_{p2} \}^T [T_2]^T [K_2] [T_2] \{ \Delta S_{p2} \} \quad dx \\
 - \{ \delta r \}^T \{ \Delta P \} = \{ \delta r \}^T \{ P \} \\
 - \int_0^{2b} \{ \delta S_{p1} \}^T [T_1]^T [E_1] \\
 - \{ \delta S_{p2} \} [T_2]^T [E_2] \quad dx \quad (4.111)
 \end{aligned}$$

Equation (4.111) is now expressed completely in terms of skin displacements. Then following the procedure outlined in section (4.4), equation (4.111) can be expressed as:

$$\left[K_{inc_s} \right] \{ \Delta \alpha \} - \{ \Delta P_s \} = \{ P_s \} - \left[K_{e_s} \right] \quad (4.112)$$

where the subscript s refers to terms relating to a stiffener.

If n stiffeners are present, then equation (4.112) can be written as:

$$\left[K_{inc_s} \right]_t \{ \Delta \alpha \} - \{ \Delta P_s \}_t = \{ P_s \}_t - \left[K_{e_s} \right]_t \quad (4.113)$$

where

$$\begin{aligned}
 \left[K_{inc_s} \right]_t &= \sum_{i=1,n}^n \left[K_{inc_s} \right]_i \\
 \left[K_{e_s} \right]_t &= \sum_{i=1,n}^n \left[K_{e_s} \right]_i \\
 \{ \Delta P_s \}_t &= \sum_{i=1,n}^n \{ \Delta P_s \}_i \\
 \{ P_s \}_t &= \sum_{i=1,n}^n \{ P_s \}_i
 \end{aligned} \tag{4.114}$$

The nonlinear equations for stiffeners have now been completely defined in terms of plate displacements. It now remains to define the plate displacement fields which are discussed in the next section.

4.3.3 Application to Stiffened Panels

Consider a stiffened panel of length $2b$, width $2d$ depicted in Fig (3.5). The panel is assumed to be simply supported along all four edges and stiffened by stiffeners of rigid cross-section which are allowed to warp.

The out-of-plane displacement and the form of the initial imperfection are assumed to be of the form:

$$\begin{aligned}
 w &= \sum_i^r \sum_j^s w_{ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b} \\
 w_0 &= \sum_i^r \sum_j^s w_{0ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b}
 \end{aligned} \tag{4.115}$$

The in-plane displacement u, v which allow uniform displacement of the loaded edges are taken as:

$$\begin{aligned}
 u &= U_L(b-x) + \sum_r^\ell U_r \cos \frac{r\pi x}{2b} \\
 v &= -U_L v(d-y) - \sum_r^\ell U_r v\left(\frac{r\pi}{2b}\right) \sin \frac{r\pi x}{2b} (d-y)
 \end{aligned} \tag{4.116}$$

It should be noted that the above in-plane displacements satisfy the conditions of zero normal stress along the unloaded edges. The condition of zero shear stress along the unloaded edges is approximately satisfied in the present solution by considering a large number of terms in u and v .

Using equations (4.115), (4.116) and following the procedure outlined in section 4.4. The Newton-Raphson equations for the skin can be written in the form of equation (4.76) which is rewritten here for as:

$$[K_{inc}]\{\Delta\alpha\} - \{\Delta P\} = \{P\} - [K_e] \tag{4.117}$$

The Newton-Rapson equations for the complete panel, ie the stiffener-skin combination are given by the sum of equations (4.113) and (4.117) and can be written as:

$$[K_{inc}]_t \{\Delta \alpha\} - \{\Delta P\}_t = \{P\}_t - [K_e]_t \quad (4.118)$$

in which

$$\begin{aligned} [K_{inc}]_t &= [K_{inc}] + K_{inc_s t} \\ [K_e]_t &= [K_e] + K_{e_s t} \\ \{\Delta P\}_t &= \{\Delta P\} + \{\Delta P_s\}_t \\ \{P\}_t &= \{P\} + \{P_s\}_t \end{aligned} \quad (4.119)$$

In equation (4.119), the general terms in $\{\Delta P\}$, $\{\Delta P_s\}$, $\{P\}$ and $\{P_s\}$ relating to displacement variables U_L , U_r , W_{ij} are given by:

$$\begin{aligned} \{\Delta P\} &= \Delta P \frac{A_p}{A_t} \left\{ \begin{array}{c} 2b \\ \vdots \\ (1 + \cos \pi r) \\ \vdots \\ 0 \end{array} \right\} \\ \{P\} &= P \frac{A_p}{A_t} \left\{ \begin{array}{c} 2b \\ \vdots \\ (1 + \cos \pi r) \\ \vdots \\ 0 \end{array} \right\} \\ \{\Delta P_s\} &= P \frac{A}{A_t} \left\{ \begin{array}{c} 2b \\ \vdots \\ (1 + \cos \pi r) \\ \vdots \\ h_{cz} \frac{d\pi}{2b} \sin \frac{i\pi s}{2d} (1 + \cos j\pi) \end{array} \right\} \end{aligned}$$

$$\{P_s\} = P \frac{A}{A_T} \left\{ \begin{array}{l} 2b \\ \vdots \\ (1 + \cos \pi r) \\ \vdots \\ h_{cz} \frac{d\pi}{2b} \sin \frac{i\pi s}{2d} (1 + \cos j\pi) \end{array} \right\} \quad (4.120)$$

in which A_p represents the cross-sectional area of the skin plate and A represents the cross-sectional area of the stiffener. A_t represents the total cross-sectional area of the panel.

Equation (4.118) can now be solved for the nonlinear analysis of stiffened panels using the incremental iterative approach described in section (4.5).

4.4 Results

4.4.1 Nonlinear Analysis of Square Plates

The theoretical basis for the nonlinear analysis of plates and basic solution techniques has been discussed earlier in section 4.2.

This section considers the results for the post-buckling behaviour of square plates. The effects of geometrical imperfections are included. The results are compared with those obtained by other authors.

The plates are assumed to be square, simply supported about the plate boundaries and loaded uniaxially. Two cases of in-plane displacement boundary conditions are considered regarding the unloaded edges of the plate, these being:

Case 1: The unloaded edges are assumed stress free and therefore free to deform in the plane of the plate.

Case 2: The unloaded edges are constrained to remain straight but free to move in the plane of the plate.

These two cases are shown schematically in Figs 4.8 to 4.10. The stress distribution in the post-buckling range for these two cases is shown in Figs 4.9, where σ_e represents the axial stress along the edge of the plate. Fig 4.10 shows the variation of edge stress along the length of the plate. It can be shown that for case 1, the edge stress σ_e is the greatest at the cross-section through the crest of the buckle. For case 2, σ_e is the greatest at the cross-section through the wave inflection [82].

Fig 4.11 shows the variation of a non-dimensional load factor λ with a non-dimensional edge displacement factor u^* . For case 1, where:

$$\lambda = 4pd^2/\pi^2Et$$

$$u^* = 2ub/t^2$$

in which E is the Young's Modulus, p is the average compressive stress in the x direction, u is edge shortening in the x direction and is given by:

$$u = u \Big|_{x=0} + u \Big|_{x=2b}$$

b , d and t are plate dimensions.

The results obtained for a perfect plate and for a plate with an imperfection of w_m/t where:

$$w_m = w - w_0$$

The results can be seen to be in good agreement with those obtained by Yamaki [17], and furthermore highlight the stable post-buckling behaviour of plates with simple support boundary conditions.

Similarly, Figs 4.13 and 4.14 show the variation of λ with u^* and w_m/t respectively for case 2. The results are again compared with existing results [17] and can be seen to be in excellent agreement. Looking at Figs 4.5 and 4.14 it can be seen that out of plane deflection for case 2 is always smaller than the corresponding one for case 1. This can be attributed to the fact that for case 2, tensile stresses act along the central position of the transverse edges to constrain them straight.

Figs 4.15 shows the variation of the effective width ratio R with the non-dimensional load factor λ . The effective width ratio R is defined here as the ratio of the actual load carried by the plate to the load the plate would have carried if the stresses had been uniform and equal to the Young's modulus times the average strain, where:

$$R = \frac{P}{E(u^*/2b)}$$

The results show that the effect of the initial imperfection is the greatest in the vicinity of the critical load for both cases considered. It can also be seen that the present results are in good agreement with the results obtained by Yamaki [17]. Fig 4.16 shows the variation of the post-buckling stiffness ratio E^*/E with the edge displacement ratio u_{cr}/u for case 1, where E is the Young's Modulus of elasticity and E^* is the effective modulus of elasticity in the post-buckling region and is given by:

$$E^* = \Delta p / \Delta u$$

and u is the edge displacement, u_{cr} is the edge displacement at critical load. The assumed displacement field and the initial imperfections are of the form of a double series given by:

$$w = \sum_{i=1,3,5..}^n \sum_{j=1,3,5..}^m W_{ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b}$$

$$w_0 = \sum_{i=1,3,5..}^n \sum_{j=1,3,5..}^m W_{0ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b}$$

The results are shown for a single term using $n = 1$ and multiple terms allowing the change in buckled form using $n = 11$. It can be seen that if a single term is used the post-buckling stiffness ratio has a constant value of 0.408. If multiple terms are used, the post-buckling stiffness ratio drops below 0.408 and gradually decreases with increasing value of u/u_{cr} . The results also show that the post-buckling stiffness of an imperfect plate, asymptotically approaches that for a perfect plate. The results for the multi-term perfect plate are compared against the results obtained by Rhodes and Harvey [25] showing good agreement.

Similarly, Fig 4.17 shows the variation of E^*/E with u/u_{cr} for case 2. The assumed displacement field and form of the initial imperfection are as those for case 1. It can be seen that if a single term is used ($n=1$) to describe the out of plane displacement, the post-buckling stiffness ratio has a constant value of 0.5. If multiple terms are used ($n=11$), then the post-buckling gradually decreases below 0.5 with increasing value of u/u_{cr} .

Comparing Figs 4.16 and 4.17 it can be seen that the post-buckling stiffness for case 2 is greater than that for case 1.

4.4.2 Nonlinear Analysis of Stiffened Panels

The theoretical basis for the nonlinear analysis of stiffened plates has been discussed in section 4.3. This section considers the results for the geometric nonlinear behaviour of stiffened panels under uniaxial compression. Comparison with existing literature is included wherever possible.

(i) Statement of problem

Consider the stiffened panel depicted in Fig 4.18. The panel is assumed to be simply supported along all four edges so that the out-of-plane displacement w can be represented by:

$$w = \sum_{i=1,2,\dots}^m \sum_{j=1,2,\dots}^n W_{ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b} \quad (4.121a)$$

The initial imperfection w_0 is given by:

$$w_0 = \sum_{i=1,2,\dots}^m \sum_{j=1,2,\dots}^n W_{0ij} \sin \frac{i\pi y}{2d} \sin \frac{j\pi x}{2b} \quad (4.121b)$$

The imperfection parameters ξ_0 and ξ_L corresponding to the overall and the local modes respectively are defined by the ratio of maximum skin displacement to the skin thickness. The loaded edges of the skin are assumed to displace uniformly in the plane of the skin and the unloaded edges are assumed stress free.

Bending due to the overall mode of buckling is assumed to occur about the centroidal axis of the panel cross-section so that the axial displacement $\bar{u}$ of the panel loaded edges along the centroidal axis is given by:

$$\begin{aligned} \bar{u} \Big|_{x=0} &= \left(u - h_c^0 (w_x - w_{0x}) \right)_{x=0} \\ \bar{u} \Big|_{x=2b} &= \left(u - h_c^0 (w_x - w_{0x}) \right)_{x=2b} \\ &\dots (4.122) \end{aligned}$$

in which h_c^0 represents the distance to the panel cross-section centroidal axis from the skin plane as shown in Fig 4.18.

The first term in equations (4.122) represents the uniform displacement of the panel loaded edges due to an axial displacement u . The second term in equations (4.122) represents the net axial displacement along the centroidal axis due to the rotation along the panel edges resulting from the out-of-plane displacement w of the skin.

From equations (4.121), it can be seen that w_x and w_{0x} are not constant in y and, therefore, $\bar{u}$ in equations (4.122) is also not constant in y but differs at various stiffener location points. However, the axial displacement of the skin loaded edges is uniform in y and therefore in the following analysis, the axial displacement is measured along the skin line and not about the centroidal axis of the panel cross-section. Fig 4.19 shows that the overall buckling may occur in two distinct ways. The first is where the overall buckling occurs in the positive direction (ie this is where the panel bows outwards so that the skin displacement is in the positive direction) and the additional strains in the skin due to curvature in the axial direction are compressive and therefore, additive to those due to overall compression. The second type is where the overall buckling occurs in the negative direction (ie where the skin displacement is in the negative direction) and the additional strains in the skin due to curvature are now tensile along the axial direction.

(ii) Overall buckling

Consider now a square panel stiffened by six stiffeners of rectangular cross-section and of the following geometry (see Figs 3.5 and 3.45)):

$$\begin{aligned} N &= 6 \\ 2b &= 2d = 1.0M \\ t_w &= 0.001M \\ d_w &= 0.015M \\ t_s &= 0.0075M \end{aligned}$$

Figs 4.21 and 4.22 show the variation of the applied load with the maximum out-of-plane displacement of the skin and the end shortening respectively for the above panel. For comparative purposes only a

single term solution to the problem is considered here so that $m = n = 1$ in equation (4.21). The results obtained are compared against those obtained by Steen [83] who assumes that the panel is simply supported along the skin line at the loaded edges whilst the present analysis assumes that the panel is effectively simply supported along the neutral axis of the stiffener-skin combination. In both analysis it is assumed that the loaded edges of the skin displace uniformly, and also, the end shortening is measured about the skin line. It can be seen that the two assumptions lead to considerably different results. Steen's [83] results show that the bifurcation point has a high degree of asymmetry whilst the present analysis shows a lower degree of asymmetry. It should also be noted that Steen [83] effectively assumes the strain expression for a perfect stiffener to be of the form:

$$\epsilon_x = u_{sx} + \frac{1}{2} w_{sx}^2 - zw_{sxx} \quad (4.123)$$

The present analysis utilizes a fuller expression given by equation (3.127). It is unclear what degree of approximation is made when using Steen's [83] expressions for strains. However it is quite possible that its neglect of torsional terms will have some effect on the results since the overall mode involves a certain amount of torsional deformation of the stiffeners.

(iii) Local-overall buckling

Fig 4.23 shows the variation of the buckling stress with the aspect ratio for a panel stiffened by a single centrally located stiffener of rectangular cross section. The cross sectional geometry of the panel is given by the following parameters (see Figs 3.5 and 3.45):

$$\begin{aligned} N &= 1 \\ d &= 0.5M \\ t_w &= 0.001M \\ d_w &= 0.0128M \\ t_s &= 0.001M \end{aligned}$$

Consider now the case where the aspect ratio of the panel is equal to 1 (ie $\alpha = \bar{\alpha} = 1$).

Fig 4.23 shows that such a panel will buckle locally with two half waves in the longitudinal direction. The ratio of the local buckling load p_L with the overall buckling load p_0 is 0.86. The variation of the applied load p with the edge displacement u is shown in Fig (4.24). It can be seen that the post buckling behaviour of a perfect panel ($\xi_L = \xi_0 = 0$) comprises of two bifurcation points. The initial bifurcation point represents the local buckling mode which occurs over two half waves in the longitudinal direction. The secondary bifurcation point represents the overall mode which occurs over a single half wave in the longitudinal direction. Fig 4.17 also shows that the type of initial bifurcation can have important effects on the post buckling behaviour. It can be seen that if an imperfection which is similar to the local buckling mode ($\xi_L = 0.1$) is present, then the nonlinear path eventually converges asymptotically to the unstable path representing the local buckling mode. However, if an imperfection which is similar to the overall buckling mode ($\xi_0 = 0.1$) is present, then the nonlinear path converges asymptotically to the stable path representing the overall buckling mode. The stability of the post buckling paths is checked by examining the eigenvalues of the second variation of the total potential energy which is given by $\frac{1}{2} \times$ tangent stiffness matrix. It can be shown [84] that the presence of a bifurcation point is represented by the vanishing of an eigenvalue and the presence of an unstable path is represented by a negative eigenvalue.

The ratio of the post-buckling stiffness to the pre-buckling stiffness is often illustrated by using an effective modulus of elasticity E^* which represents the slope of the curves shown in Fig 4.24. The ratio of the post-buckling to the pre-buckling stiffness is therefore given by E^*/E where E is the modulus of elasticity.

Fig 4.25 shows the variation of E^*/E with the edge displacement. It can be seen that there is a distinct drop in the stiffness from the post-buckling behaviour resulting from the initial bifurcation point representing local buckling to the secondary bifurcation point representing overall buckling. It can also be seen that if the

initial imperfection is similar to the local buckling mode ($\xi_L = 0.1$) then the stiffness of the panel is higher than when the imperfection is similar to the overall buckling mode ($\xi_0 = 0.1$).

Fig 4.26 shows the variation of the out-of-plane skin displacement at $x = d/2$ for a perfect panel. It can be seen that immediately after buckling for $u/ucr = 1.1$, the deflection mode is predominantly local which occurs over two half-waves along the longitudinal direction. The small movement of the skin-stiffener junction indicates the presence of an overall mode - this results from the fact that when local buckling occurs as the primary buckling, the effective rigidity of the plate reduces and the center of effective area moves out of the center of cross-section. This shift becomes the eccentricity of the loading and produces an out-of-plane bending moment under the action of axial compression to deflect the stiffened panel in the overall mode.

Fig 4.26 also shows that for values of u/ucr of 2, 3 and 4 the deflected form plate is predominantly of the overall type which occurs over a single half-wave in the longitudinal direction.

Consider now a panel of the following cross-sectional geometry (see Figs 3.5 and 3.45):

$$\begin{aligned} N &= 8 \\ d'_s &= 113.6t_s \\ d'_w &= \frac{d'_s}{10} \\ t_w &= d'_s/100 \\ 2b &= 4.16d'_s \\ 2d &= 7d'_s \end{aligned}$$

where d'_s is the stiffener pitch. The panel is composed of a slender plate stiffened by stocky ($t_w/t_s \approx 4.5$) stiffeners of rectangular cross-section. This panel problem has been studied by Tvergaard [35] who assumed that the panel was infinitely wide and that the overall mode was essentially that of a broad Euler column where the flexural torsional mode of the stiffeners does not occur. In the present analysis, the panel is assumed to be of a finite width $2d$ (composed of seven bays) with all four edges simply supported. These boundary

conditions result in the torsional flexural deformation of the stiffeners during overall buckling. Both the Tvergaard panel and the panel considered here are designed for near-coincident buckling so that $p_0/p_L = 1.036$ where p_L and p_0 represent the local and overall buckling loads respectively. It can be seen from the panel geometry that the length of the panel required to obtain the near-coincident buckling ratio $p_0/p_L = 1.036$ is given by $2b = 4.16 d'_s$. However, for the Tvergaard panel [35] the length is given by $2b = 4d'_s$. This difference in the lengths can be attributed to the differences in the boundary conditions along the transverse edges of the panel between the two models.

In the present analysis, eight terms were used to describe the out-of-plane displacement of the skin in the y direction, there being $i = 1, 13, 15, 27, 29, 41, 43, 57$ (see equations (4.121)). The local buckling of the panel was shown to occur over six half-waves in the longitudinal direction - this is the same for the Tvergaard panel. The load deflection characteristics of this panel are shown in Fig 4.27 in which the edge displacement ratio u/u_0 is plotted against the load ratio p/p_0 , where p_0 is the critical load for the overall mode and u_0 is the corresponding edge displacement. Four cases are illustrated; Case (i) - this is where only an overall type of deformation is considered so that $j = 1$ in equation (4.121); Case (ii) - this is where both local and overall degrees of freedom are included in the analysis so that $j = 1, 6$ in equation (4.121); Cases (iii) and (iv) are the same as case (ii) but where an overall type of imperfection $\xi_0 = 0.005$ and $\xi_0 = 0.05$ are included respectively as well as a local type of imperfection $\xi_L = 0.001$. From examination of cases (i) and (ii), it can be seen that the introduction of local degrees of freedom has the effect of reducing the load p in post buckling region. From examination of cases (ii) to (iv), it can be seen that the effect of introducing imperfection results in a further reduction of load p - this is especially true in the vicinity of the critical load where the reduction is the greatest. Also, from examination of cases (ii) to (iv) it can be seen that the nonlinear behaviour of all these cases is very strongly

governed by the overall type of mode since all nonlinear paths converge asymptotically to the overall-nonlinear path of case (i).

Fig 4.28 shows the variation of the load ratio p/p_L with the relative overall displacement ratio $(w - w_0)/t_s$ of the panel at midspan. Three cases are illustrated for various levels of overall imperfection ξ_0 , these being $\xi_0 = 2b/4000$, $2b/2000$. The local imperfection ξ_L is assumed to be 0.05 for all three cases.

The results obtained are compared against those obtained by Ali and Sridharan [85] who utilize the model of an infinitely wide panel with an Euler type of an overall mode. It can be seen that the present results and those of Ref [85] show completely different trends. Ref [85] shows that the behaviour of panel resembles that of a solid Euler column with a reduced stiffness. In the present analysis the transverse edges of the panel are assumed simply supported so that the nonlinear behaviour resembles the post-buckling behaviour of a simply supported plate (which is stable) with a reduced stiffness.

Although the two sets of results are quite different, however, for case 1 where $\xi_0 = 0$, both sets of results show that there is interaction between the local and overall modes from the very outset of the loading history.

Fig 4.29 shows further results for the variation of load ratio p/p_0 with the relative maximum overall deflection $(w - w_0)/t_s$ for values of overall imperfections ξ_0 of 0.05 and 0.005. A constant local imperfection $\xi_L = 0.001$ is assumed for both cases. The results obtained are compared against those obtained by Sridharan and Peng [86] for the Tvergaard panel. It can again be seen that the two sets of results show different trends. The results obtained by Ref [86] show the existence of a maximum load above which the load p cannot rise. The present analysis shows that there is no maximum load. These differences in results can again be attributed to the different displacement boundary conditions along the transverse edges of panels for the two models.

Consider next the range of panel models tested by Thompson et al [37]. The panels were constructed from an epoxy plastic (which remains elastic up to a large range) and were stiffened by stiffeners

of rectangular cross section. The ends of the panel assembly at which the load was applied were assumed to be simply supported while the transverse edges were free. Therefore the overall buckling mode was essentially that of a broad Euler column. During the tests imperfection in the overall and local mode were varied - the former was varied continuously over a wide range. Tests were carried out on a family panels with critical load ratio p_L/p_0 of 0.64, 1.02 and 1.5 - where p_L and p_0 refer to the local and overall buckling critical loads respectively. The results presented indicated the existence of a maximum value or an asymptotic maximum value in the load deflection characteristics. The panels tested were of the following cross-sectional geometry [see Figs 3.5 and 3.45]:

$$\begin{aligned} N &= 9 \\ d &= 0.23M \\ t_s &= 0.00075M \\ d_w &= 0.009625M \\ t_w &= 0.0049M \end{aligned}$$

In the present analysis, panels of the same cross-sectional geometry and critical load ratio p_L/p_0 are considered. The transverse edges of the panel are however, considered to be simply supported rather than free as in the above investigation by Thompson et al [37]. Fig 4.30 shows the variation of load with edge displacement for a panel with a critical load ratio of 0.64. In this case the panel is of a relatively short length so that the initial buckling mode is of a local type. Fig 4.30 shows that after the onset of local buckling the subsequent post buckling behaviour of the panel involves interaction between both the local and overall modes. This is evident since the nonlinear post buckling path of the panel tends towards the nonlinear path involving the overall mode only. This behaviour results from the fact that under local buckling, plate elements between the stringers which act as load carrying members, lose part of their effective axial stiffness and also cause a drop in the axial and bending stiffness of the panel as a whole. This loss in stiffness results in a shift in the effective neutral axis away from the skin which leads to loading eccentricity which in turn promotes overall buckling.

Consider next the case where both local and imperfections are present simultaneously, both positive and negative imperfections are considered where $\xi_L = \pm 0.01$ and $\xi_0 = \pm 0.5$. For the case where positive imperfections are present, the post-buckling behaviour is such that the overall buckling mode occurs in the positive direction (ie where the skin is on the concave side) so that the strains produced in the skin due to the overall mode are compressive and therefore promote local buckling. Here the interaction between the overall and local buckling is especially strong so that buckling in one mode positively encourages buckling in the other. However, when negative imperfections are present, the resulting overall mode is in the negative direction, so that the additional strains present in the skin are tensile. This tends to reduce local buckling in the skin and the overall stiffness of the panel is higher than if imperfections are positive.

An important observation of the results is that there is no maximum or an asymptotic maximum in the load deflection characteristics for the panel.

The interaction between the local and overall modes of buckling is depicted in Figs 4.31 and 4.32. These figures show the development of the overall deflection for the case where the initial buckling mode is of an overall type. Each figure shows the variation of the out-of-plane skin displacement for various values of edge displacement u .

The first figure shows the displacement across the panel cross-section at midspan. The second figure shows the variation of the out-of-plane skin displacement in the axial direction at $y = 9/20d$.

Fig 4.33 shows the variation of load with edge displacement for a panel with a critical load ratio of 1.05. This is a case where the critical loads in the two modes are nearly coincident and the initial buckling mode is of an overall type. The important feature in Fig 4.33 is that there is no maximum or an asymptotic maximum in the load displacement characteristics. This is contrary to results obtained from panel models where the transverse edges of the panel are assumed to be free. Fig 4.33 also shows that the presence of imperfections

results in a large decrease in the load p in the vicinity of the buckling load.

Fig 4.34 and Fig 4.35 show the variation in the out-of-plane displacement with the edge displacement u . Although the initial buckling mode is of an overall type, it can be seen that there is a rapid development of the local mode as the edge displacement is increased. This suggests that interactive buckling effects are especially important in the case where the critical loads in the two buckling modes are nearly coincident.

Fig 4.36 shows the variation of the load p with the edge displacement u for the case of a panel with a critical load ratio of 1.5. This is the case of a relatively long panel where the theoretical overall buckling load is well below the loads required to initiate local buckling under direct compression. Fig 4.37 shows that the behaviour of the panel is strongly dominated by the Euler mode with relatively little influence from the local buckling mode. This is depicted quite clearly in Figs 4.37 and 4.38 which show the out-of-plane displacement of the skin at various values of edge displacement u . It can be seen that the local mode develops slowly as the edge displacement is increased.

Consider now a panel stiffened by 8 L section stiffeners under axial compression. The panel has the following cross-sectional geometry (see Fig 3.45):

$$\begin{aligned} N &= 8 \\ d &= 0.21M \\ t_s &= 0.001M \\ d_w/d'_s &= 0.25 \\ d_f/d_w &= 0.5 \\ t_w/t_s &= 1.0 \\ t_f/t_w &= 1.0 \end{aligned}$$

The variation in buckling stress with aspect ratio is depicted in Fig 3.50 in which it is referred to as panel L2, (see Table 1 in section 3.4.3). Figs 4.39, 4.40 and 4.41 show the load displacement

characteristics in the presence of imperfections for the above panel with critical load ratio p_L/p_0 of 0.65, 0.99 and 1.5 respectively. Various levels of overall imperfection ξ_0 are considered and a constant local imperfection $\xi_L = 0.02$ is assumed in all cases. The figures show that the post-buckling behaviour is essentially stable for all three cases and the influence of the imperfections is the greatest in the vicinity of the overall bifurcation point. The effect of imperfection in the vicinity of the buckling load is shown in Fig 4.42. The figure shows that for the case where the buckling loads are nearly coincident, ie $p_L/p_0 = 0.99$, the effect of the imperfection is the greatest. This is in line with the conclusion reached by Tvergaard [35] who's analysis was based upon a broad Euler column model.

Consider next a panel with T section stiffeners of the following cross-sectional geometry (see Fig 3.45):

$$\begin{aligned} N &= 8 \\ d &= 0.21M \\ t_s &= 0.001M \\ d_w/d_s &= 0.25 \\ d_f/d_w &= 0.5 \\ t_w/t_s &= 1.0 \\ t_f/t_w &= 1.0 \end{aligned}$$

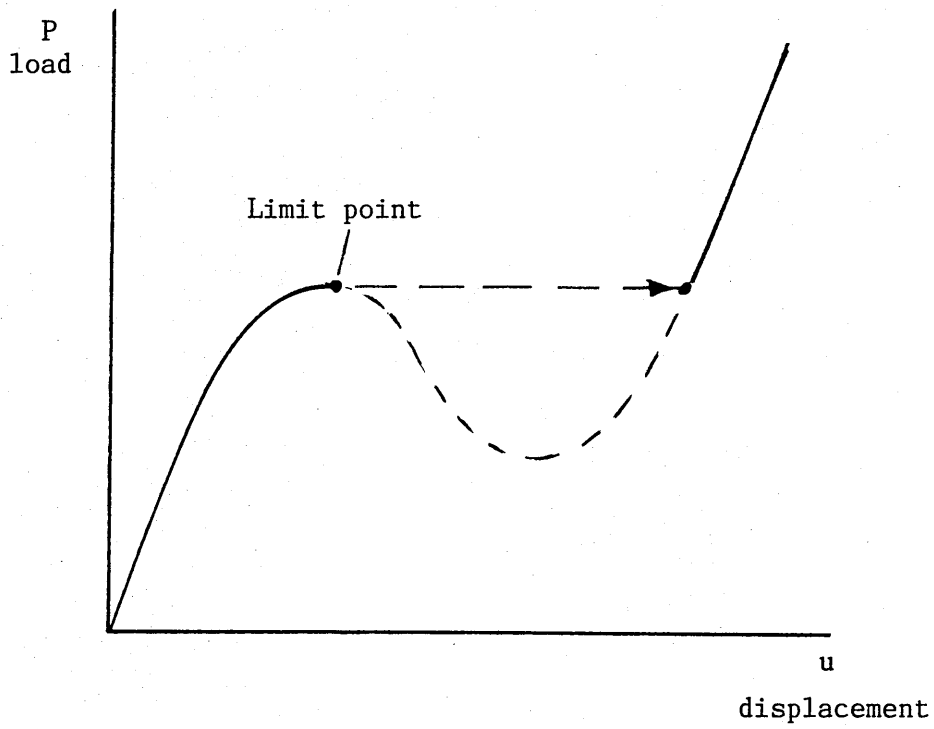
The variation of the buckling stress with the aspect ratio for this panel is shown in Fig 3.50 in which it is referred to as panel T2, (see Table 1 in section 3.43). Figs 4.43, 4.44 and 4.45 show the load displacement characteristics in the presence of imperfections for the above panel cross-section with the critical load ratio p_L/p_0 of 0.65, 0.99 and 1.5 respectively. The results are presented for various levels of overall imperfection ξ_0 and constant local imperfection $\xi_L = 0.02$ for all three cases of critical load ratio. As in the previous case of a panel with L section stiffeners, it can be seen that the effect of imperfections is the greatest in the vicinity of the overall bifurcation point. The sensitivity of the panels to imperfection in the locality of the buckling load is shown in Fig 4.46.

It can be seen that the panel with near coincident buckling load p_L/p_0 has the greatest sensitivity to imperfections.

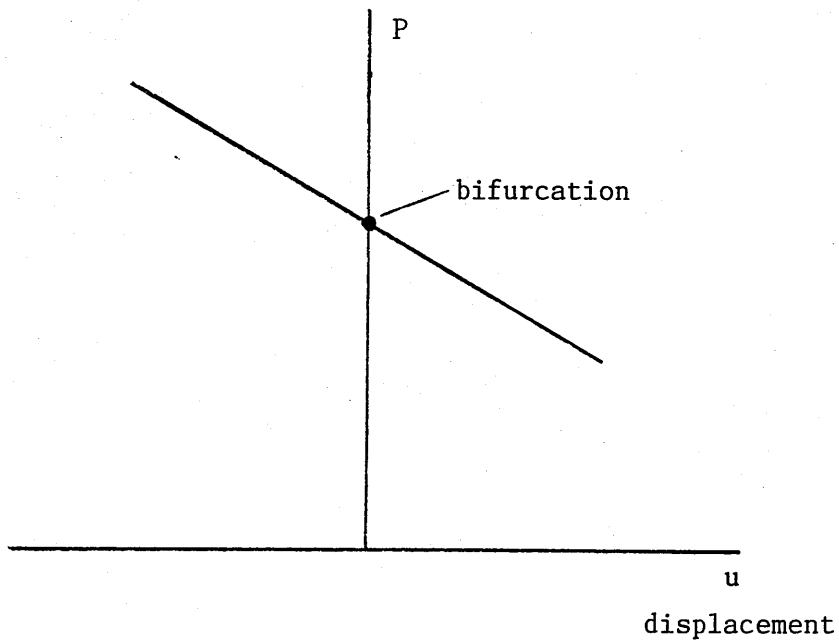
The variation of the out-of-plane displacement across the panel cross-section and in the longitudinal direction for the case of the panel with the critical load ratio of 0.65, overall imperfection $\xi_0 = 0.6$ and local imperfection $\xi_L = 0.02$ is shown in Figs 4.47 and 4.48 respectively. The results are presented for various levels of edge displacement and show that there is strong interaction between the local and overall buckling modes. This is indicated by the development of the local mode as the edge displacement is increased. The variation of the axial stress along the panel cross-section is shown in Fig 4.49.

The results presented so far assume that the behaviour of the panel is purely elastic. However in practice structures often fail through plasticity. This would indeed be the case for the panels considered here where the post-buckling behaviour is stable. Figs 4.50 and 4.51 show the variation of the failure stress σ_f with the aspect ratio in the presence of imperfection for panels L2 and T2 considered above. Failure is assumed to occur when the maximum edge stress in the skin reaches yield. This criterion has been shown to give good results when compared with experiments [49,50].

The flat portion of the curves shown in Figs 4.50 and 4.51 represent failure through local buckling and the curved portion represents the influence of overall buckling. The results also show that the influence of imperfections is the greatest in the region where failure results from the combination of both overall and local buckling.

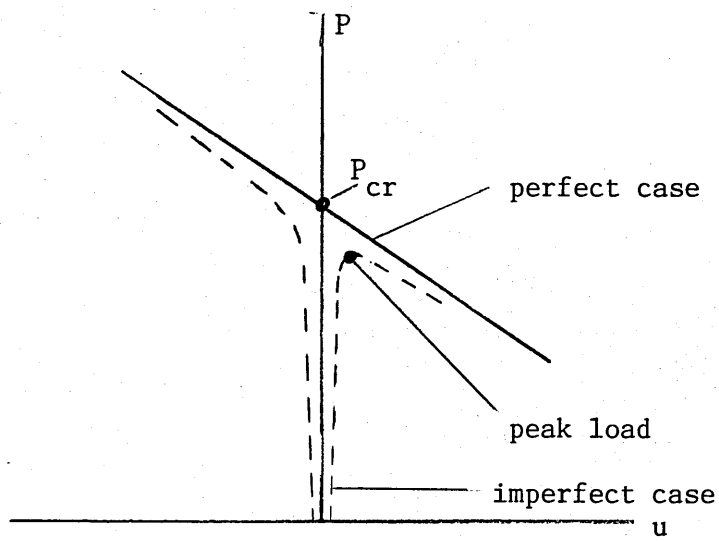


(a) Limit point

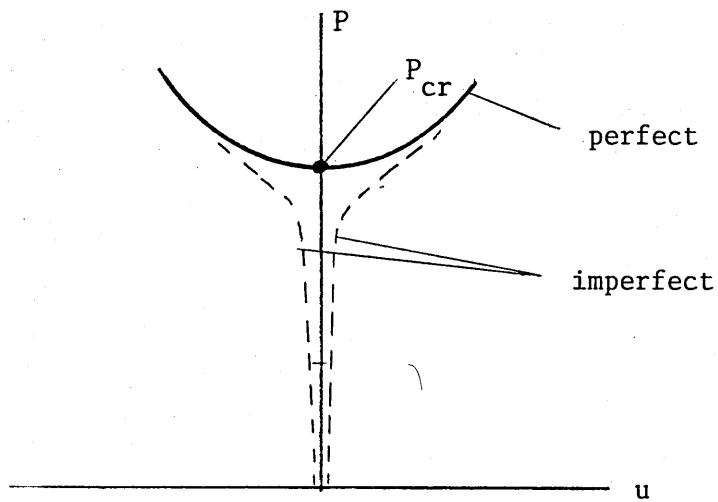


(b) Bifurcation

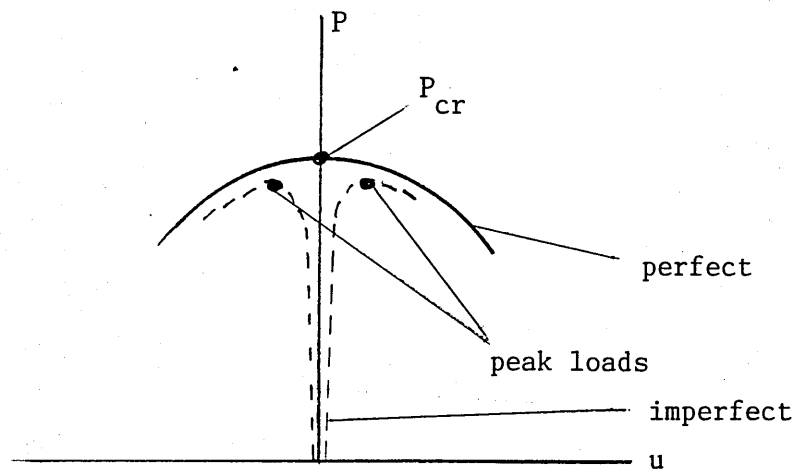
Fig 4.1 Forms of loss of stability by limit point and bifurcation



(a) Unsymmetric



(b) Symmetric stable



(c) Symmetric unstable

Fig 4.2 Asymmetric and symmetric forms of bifurcation

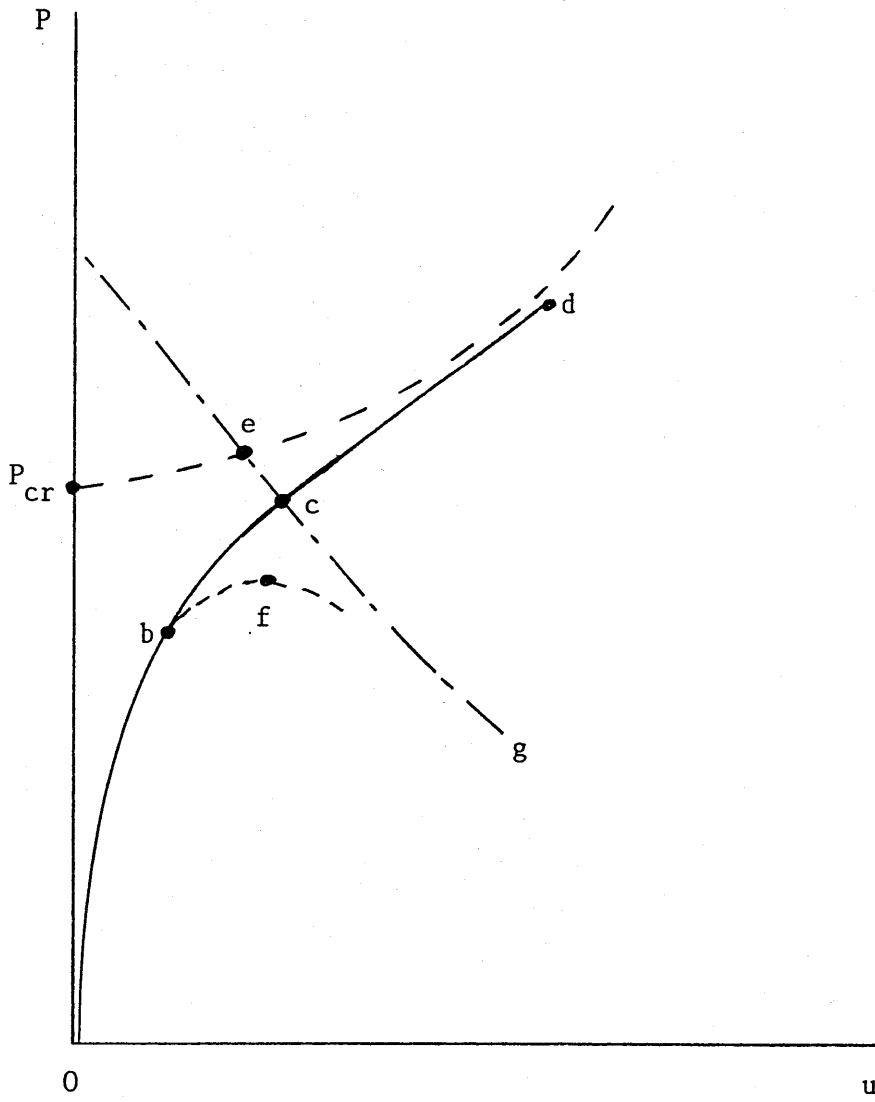


Fig 4.3 Failure of an imperfect structure through yielding

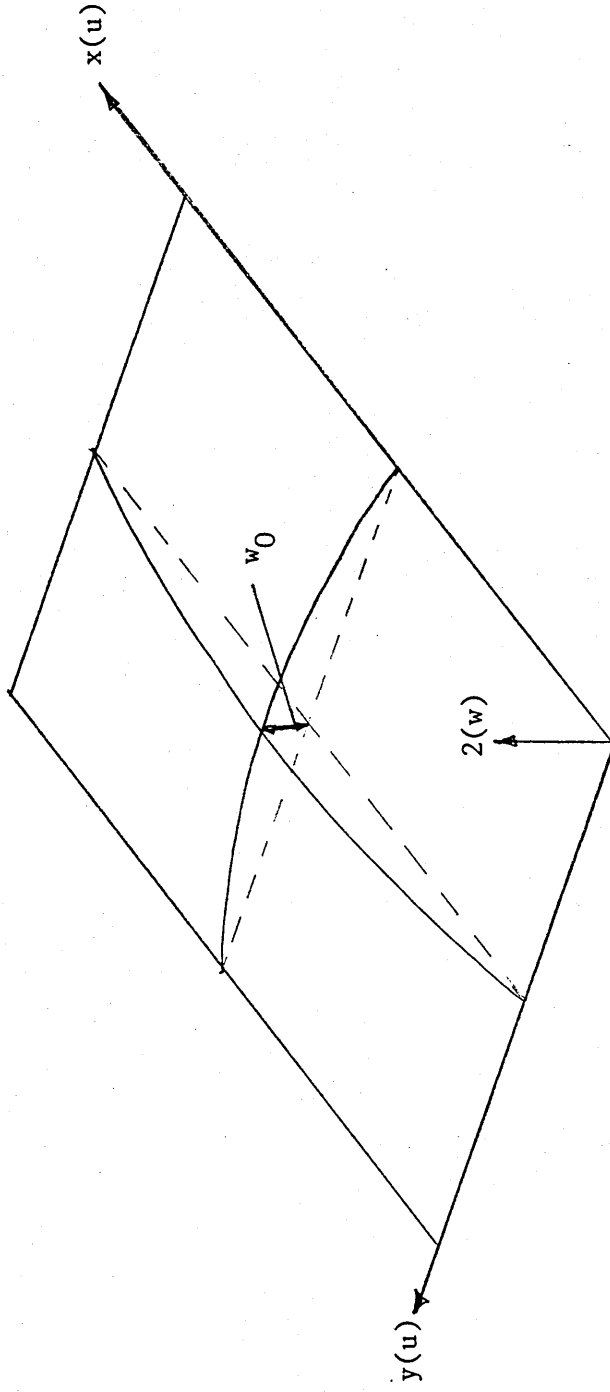


Fig (4.4) Initially imperfect plate

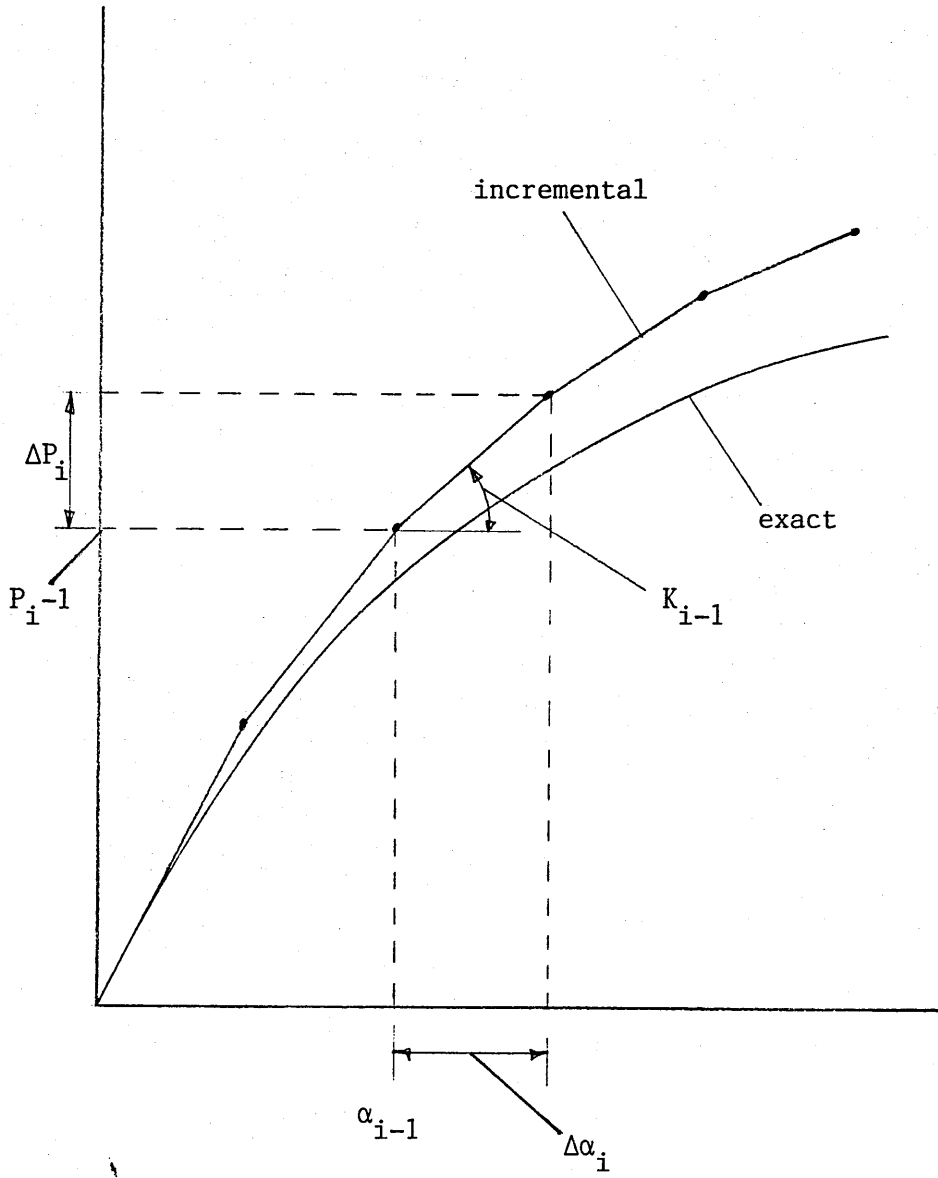
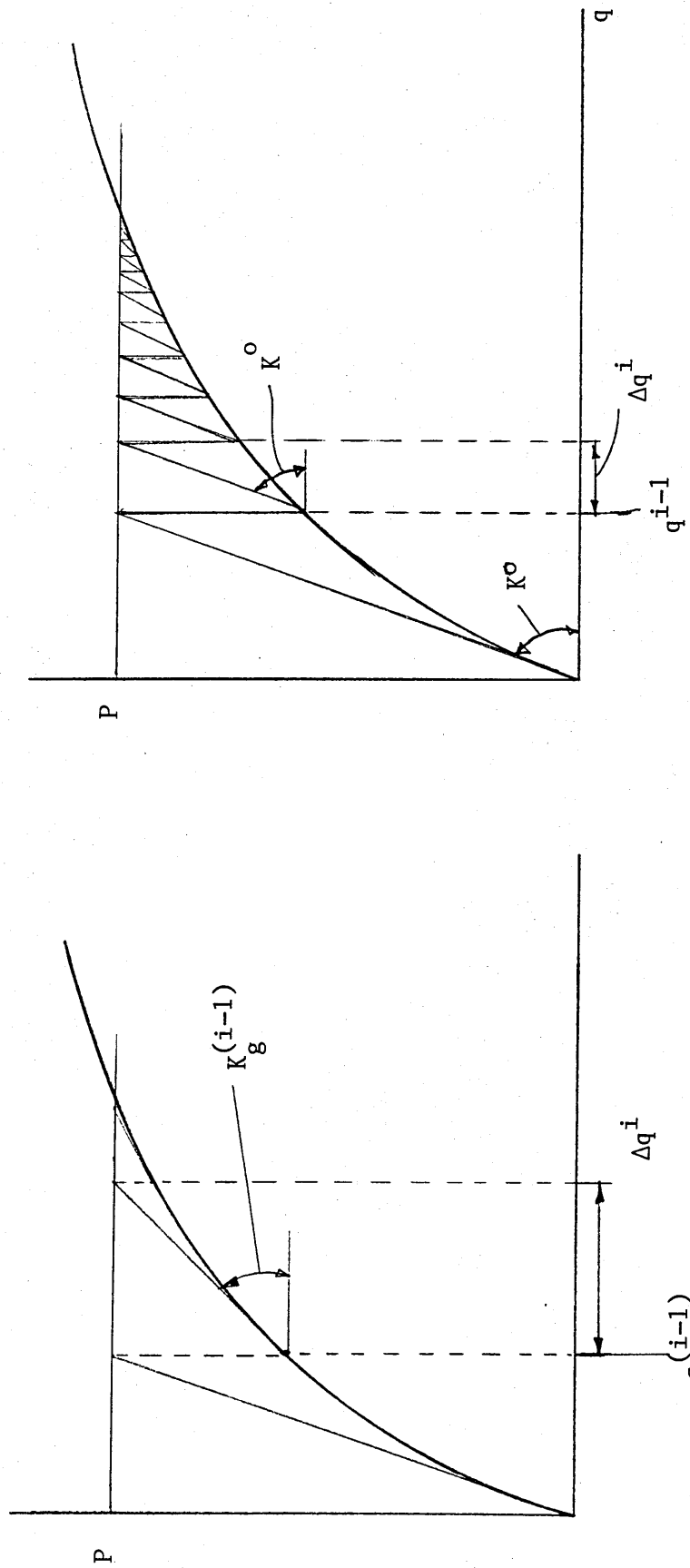


Fig 4.5 Pure incremental analysis



(a) Tangent stiffness procedure (b) Modified procedure

Fig 4.6 Iterative or Newton method

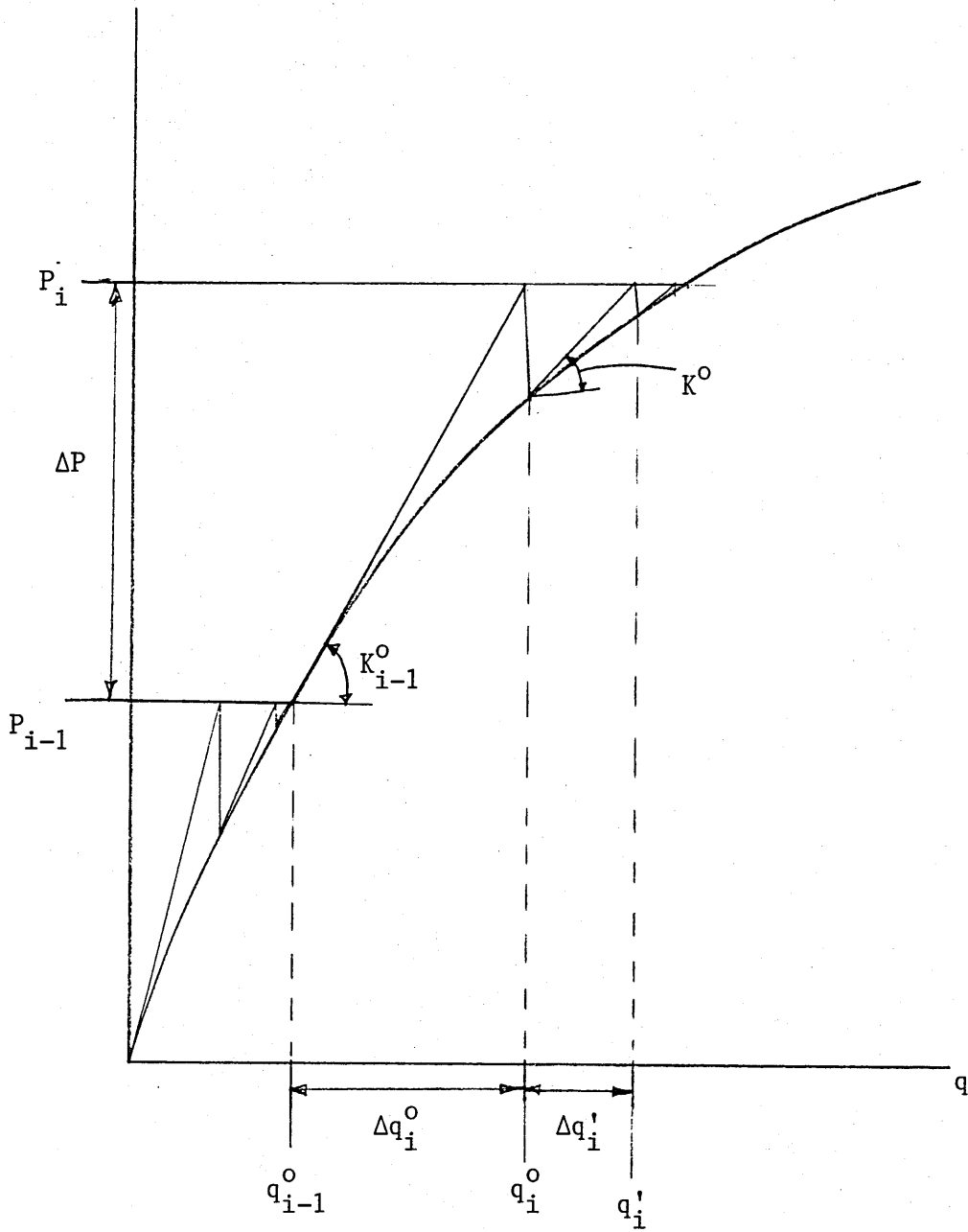
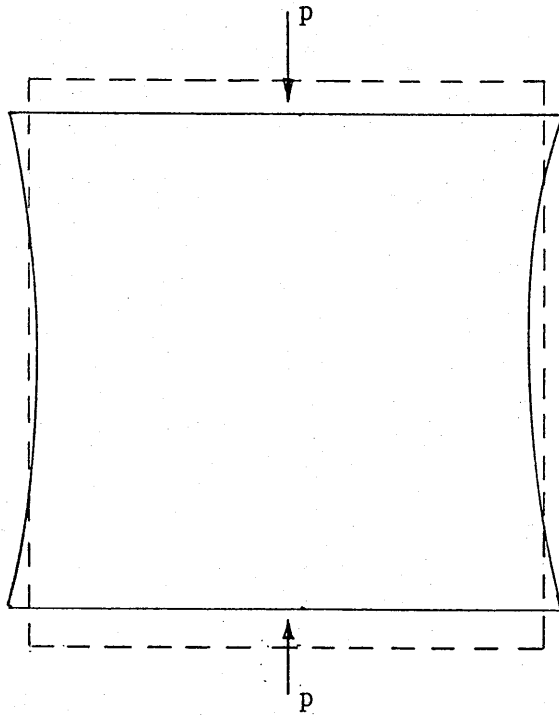
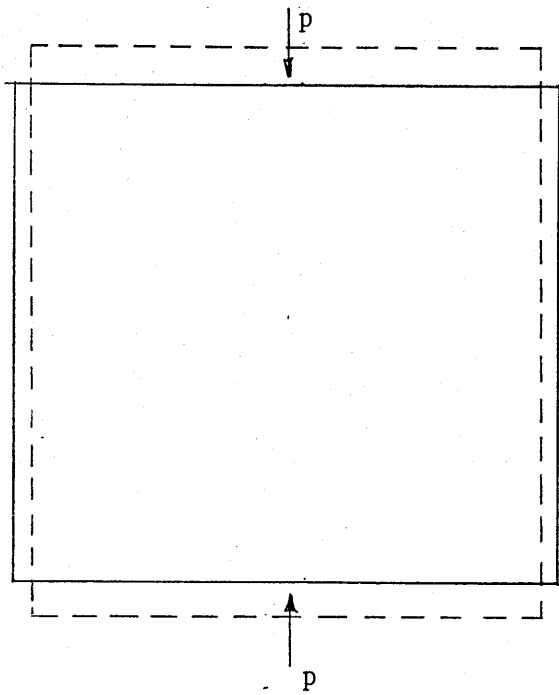


Fig 4.7 Incremental-Iterative procedure

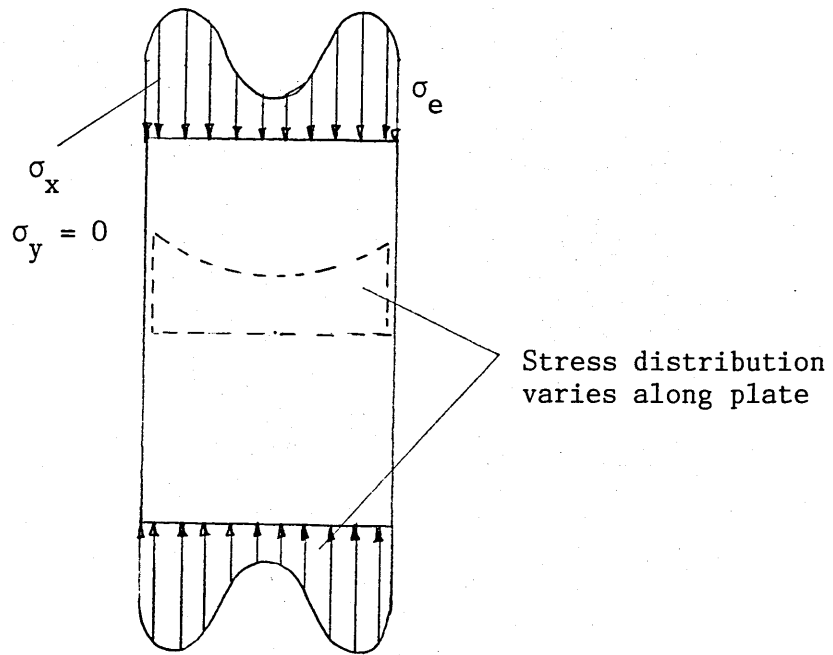


Case 1 - Sides free to wave

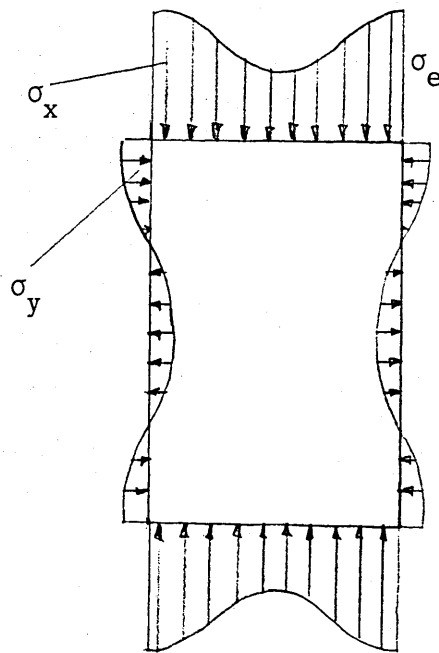


Case 2 - Sides straight; free to move laterally

Fig 4.8 In-plane boundary condition along transverse edges

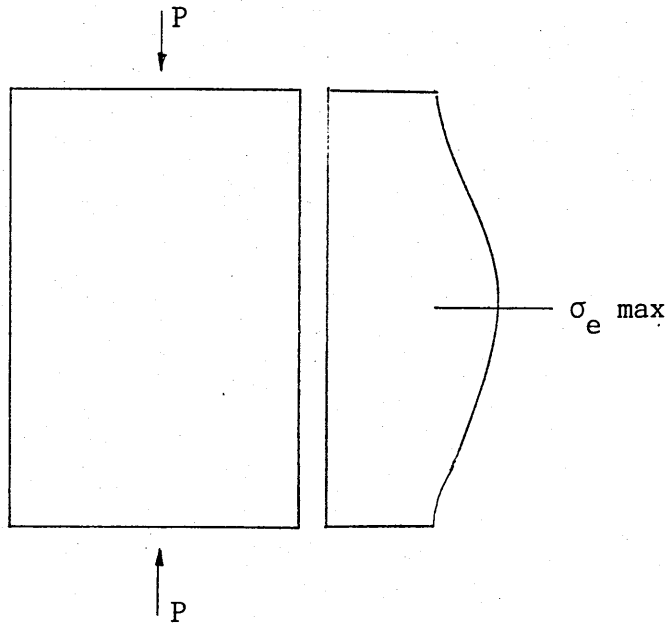


Case 1 - Sides free to wave

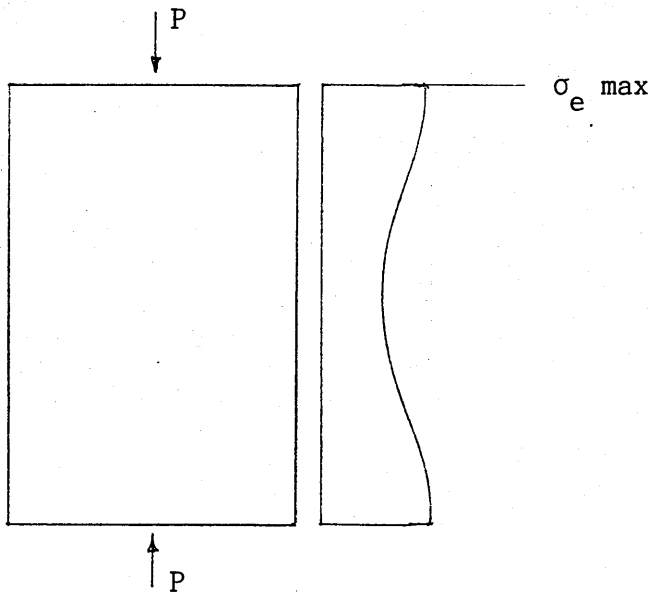


Case 2 - Sides straight; free to move laterally

Fig 4.9 Post-buckled stress distribution



Case 1 - Edges free to wave



Case 2 - Edges straight; free to move laterally

Fig 4.10 Distribution of edge stress σ_e

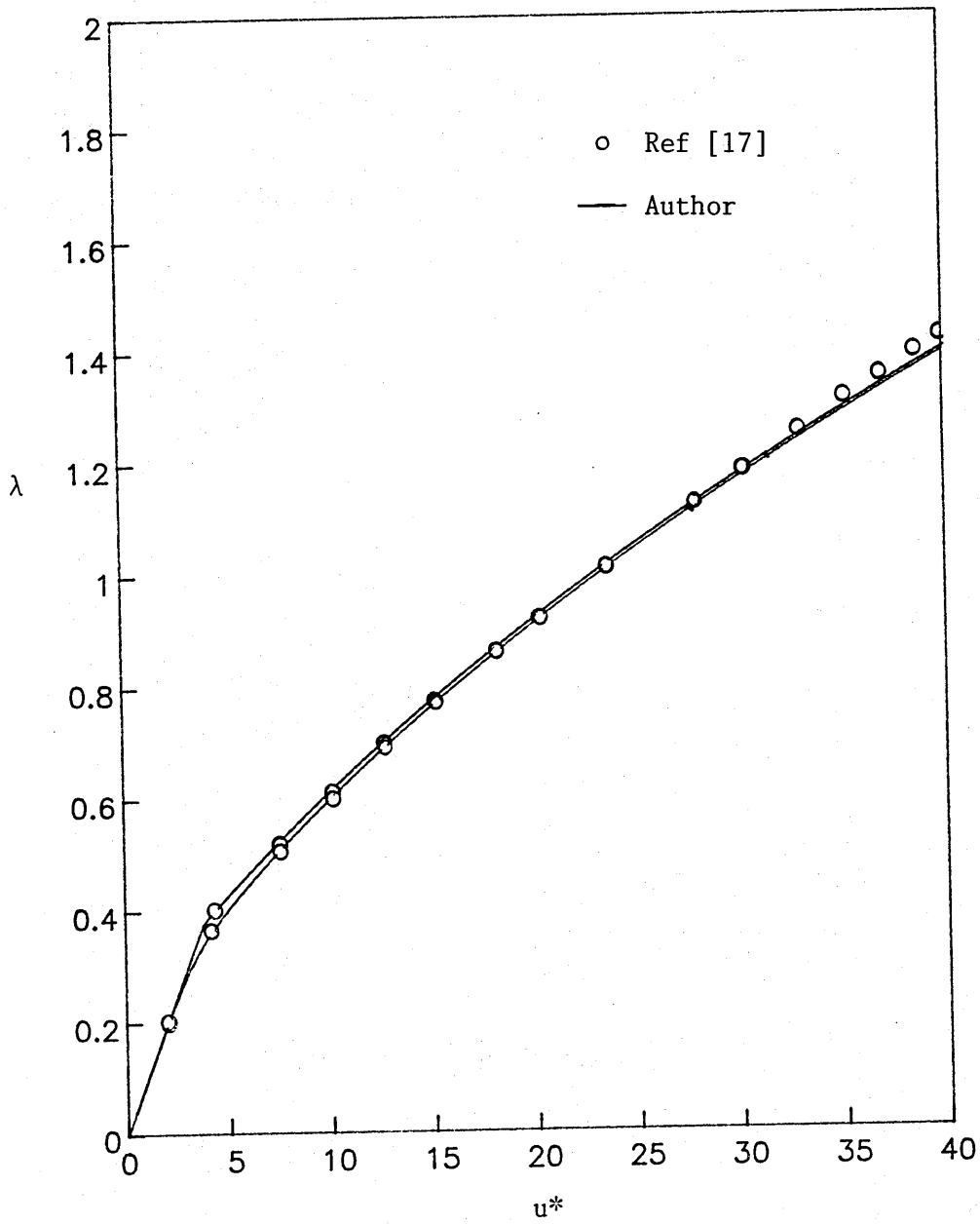


Fig 4.11 Variation of applied load with edge shortening for case 1

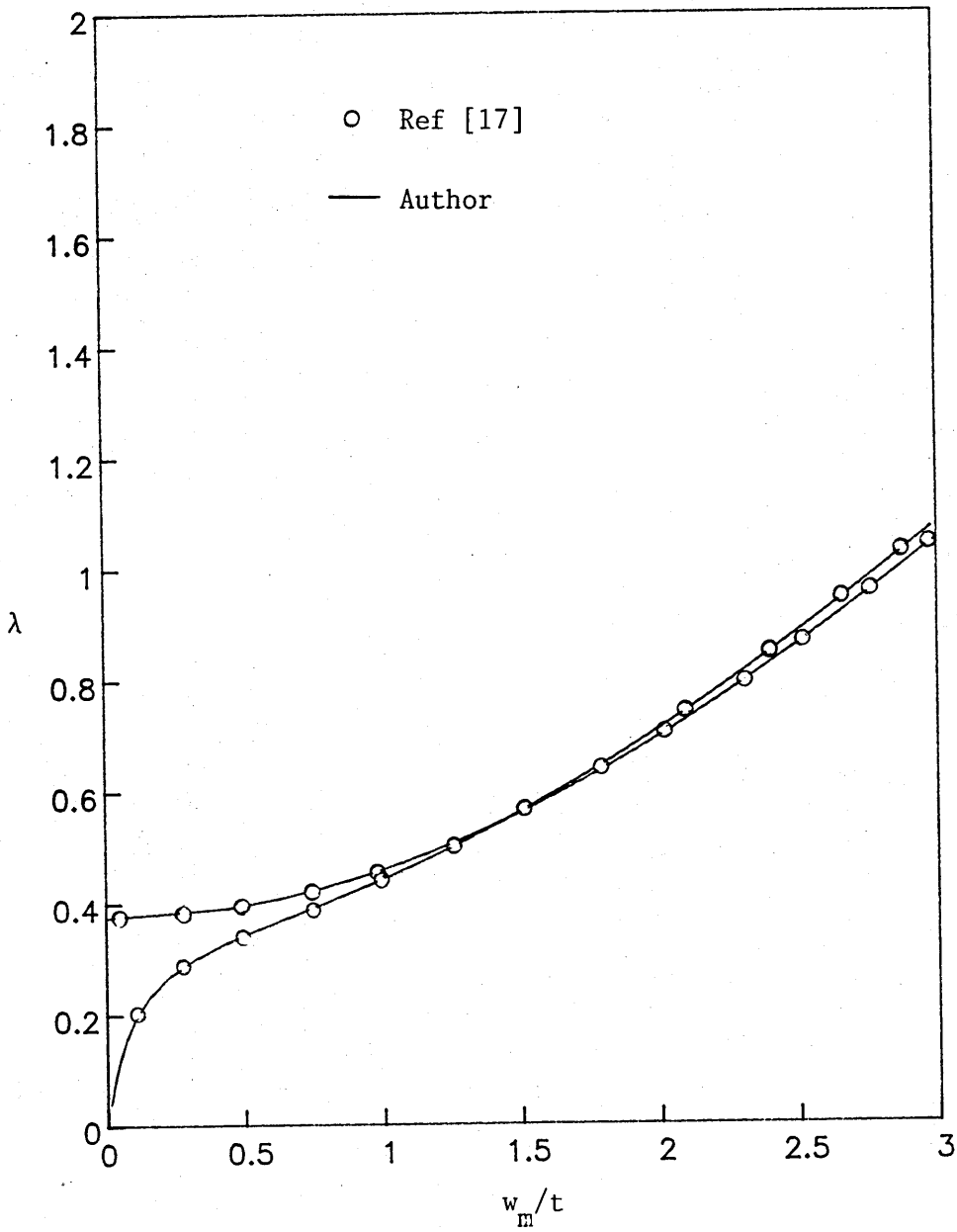


Fig 4.12 Variation of applied load with maximum deflection for case 1

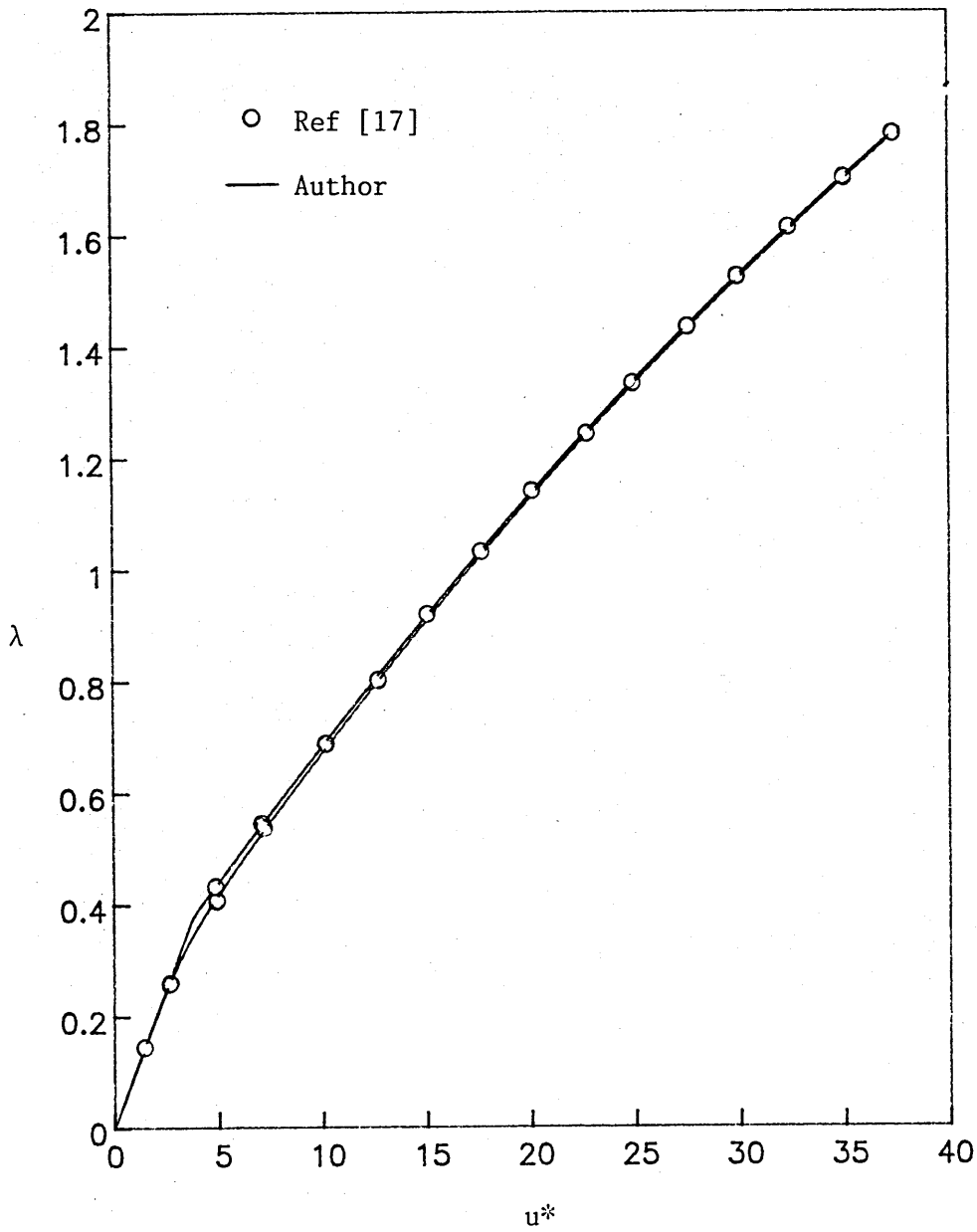


Fig 4.13 Variation of applied load with edge displacement for case 2

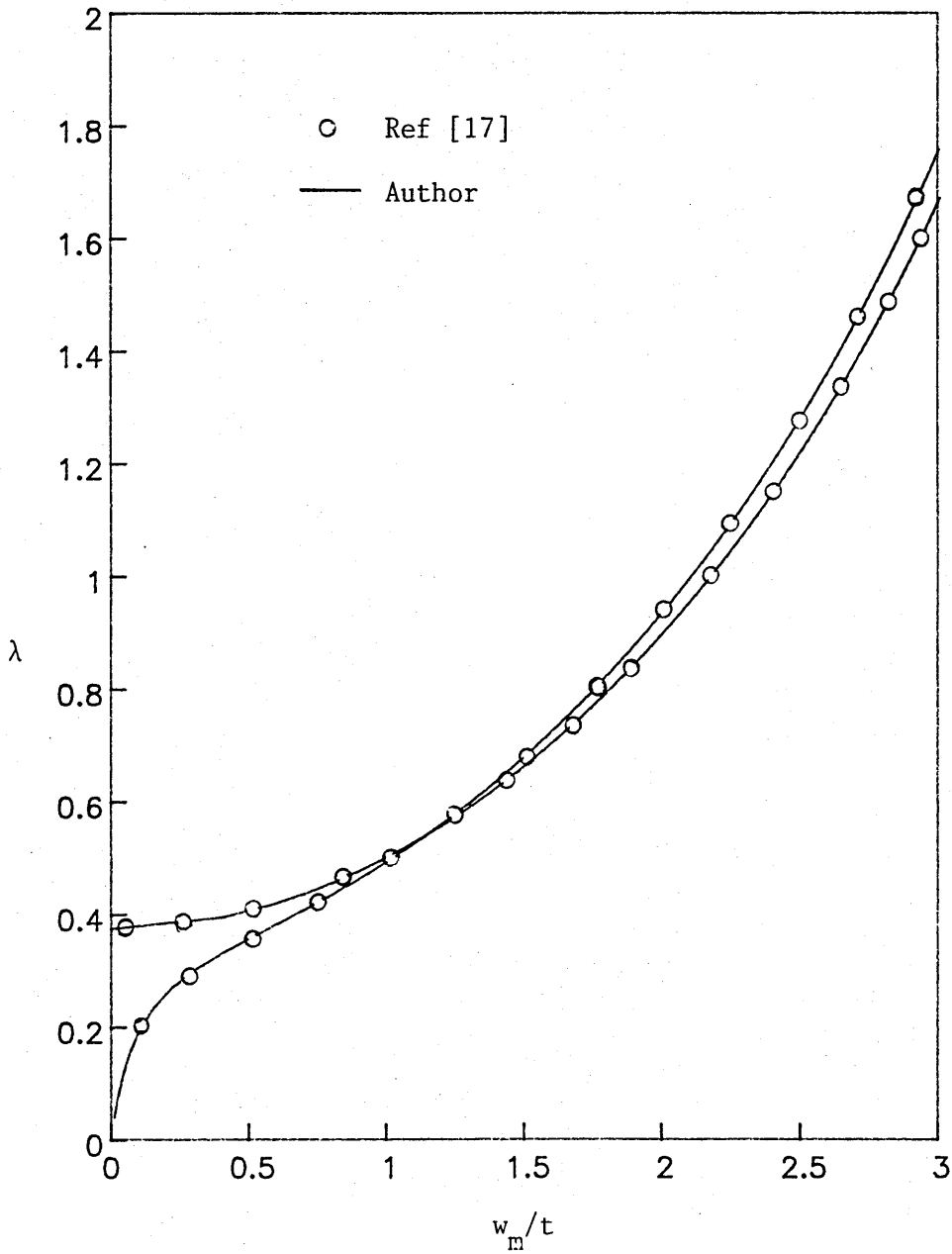


Fig 4.14 Variation of applied load with maximum deflection for case 2

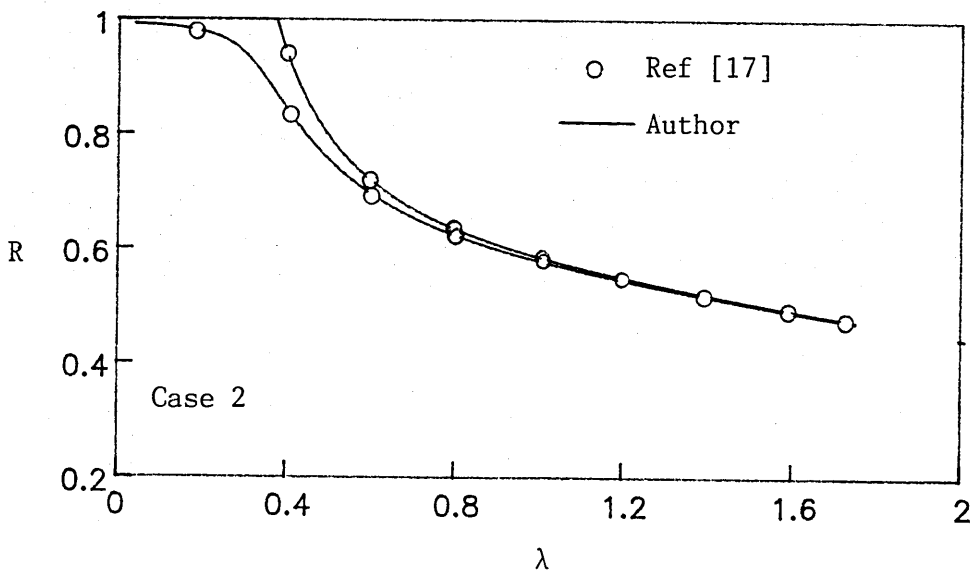
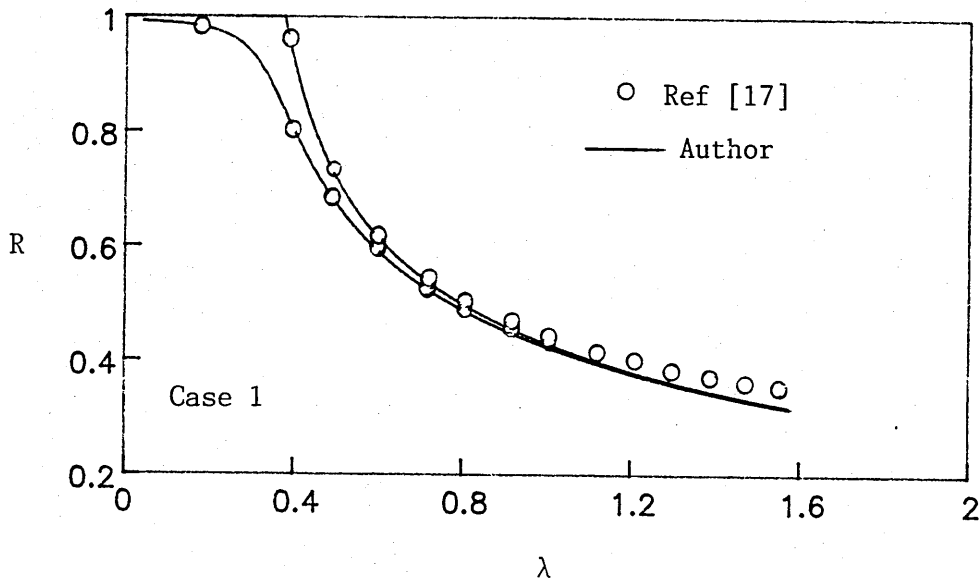


Fig 4.15 Variation of effective width ratio with applied load

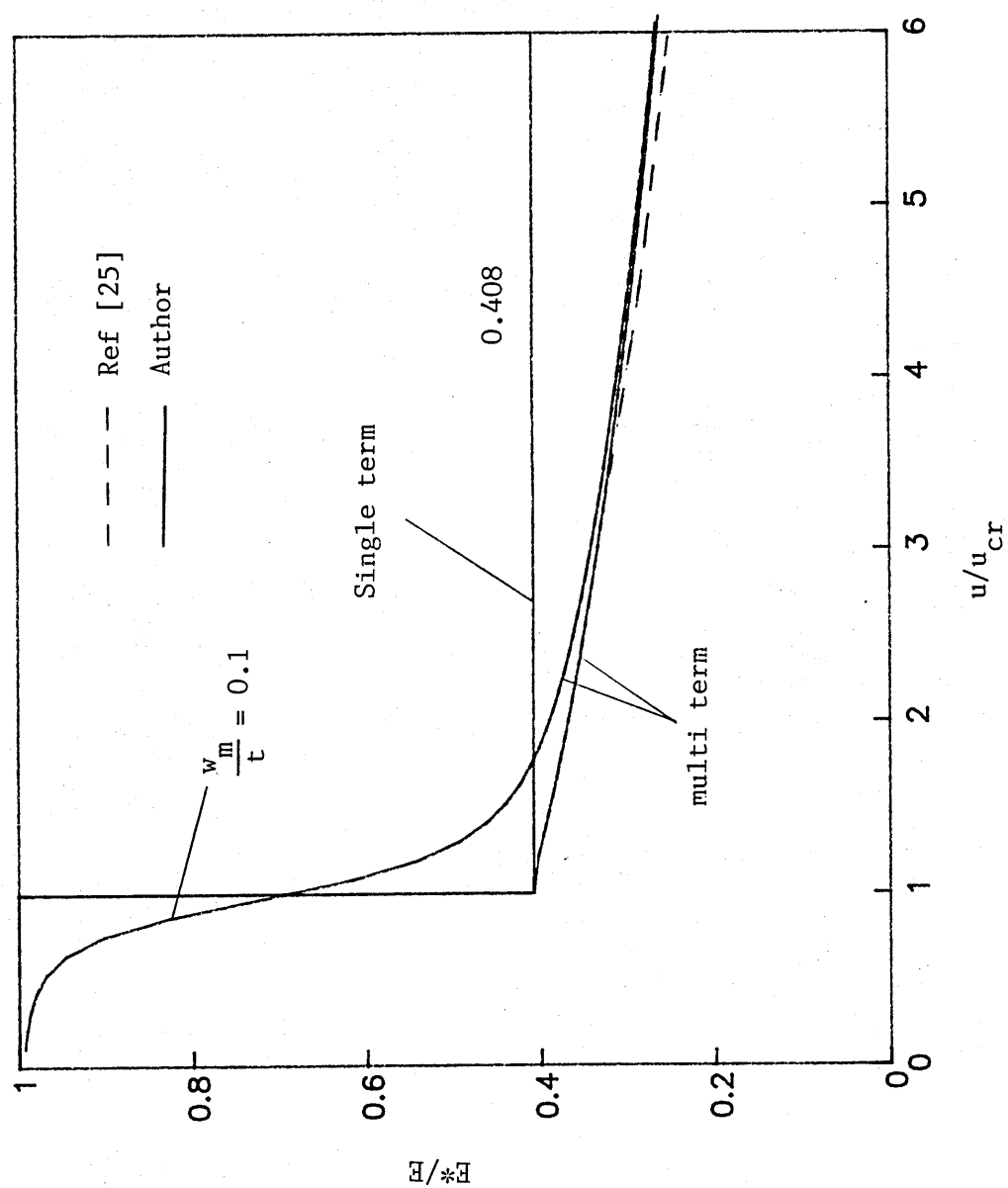


Fig 4.16 Variation of tangent stiffness with edge displacement for case 1

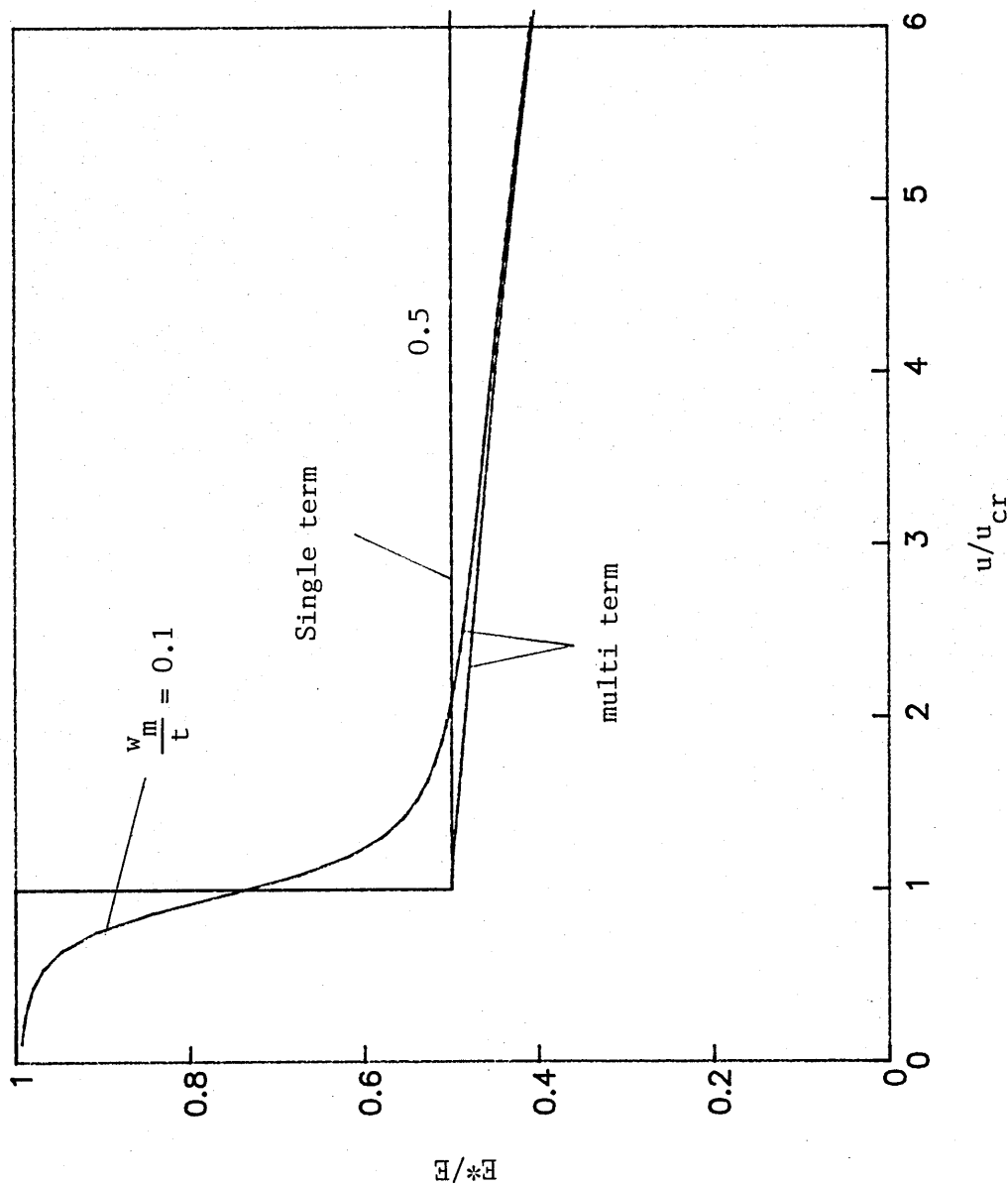


Fig 4.17 Variation of tangent stiffness with edge displacement for case 2

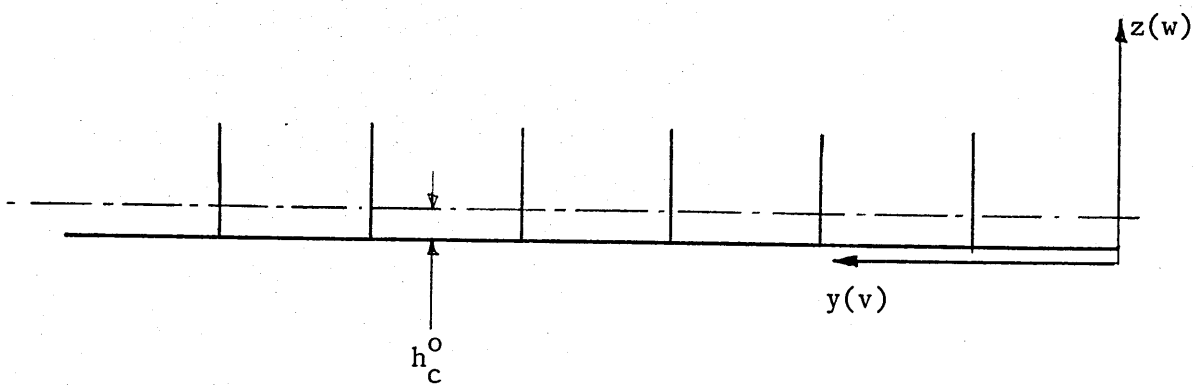
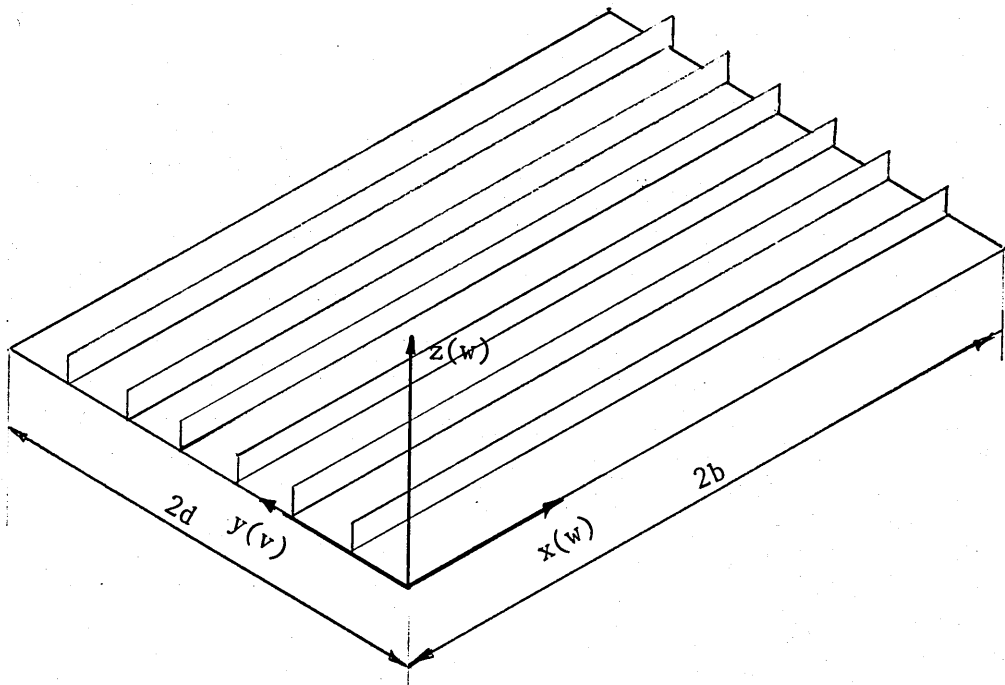
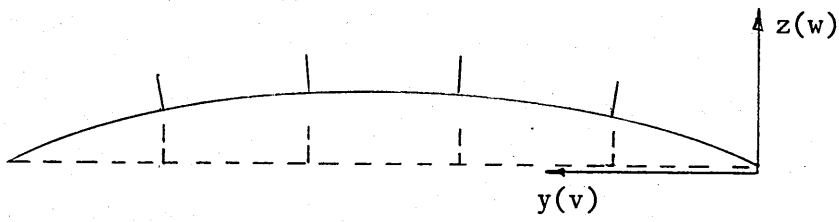
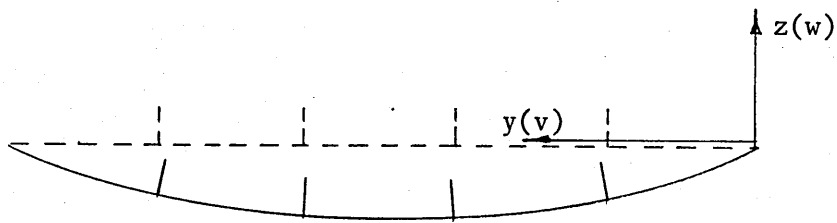


Fig 4.18 Stiffened panel geometry



Overall buckling in the positive direction



Overall buckling in the negative direction

Fig 4.19 Possible overall buckling model

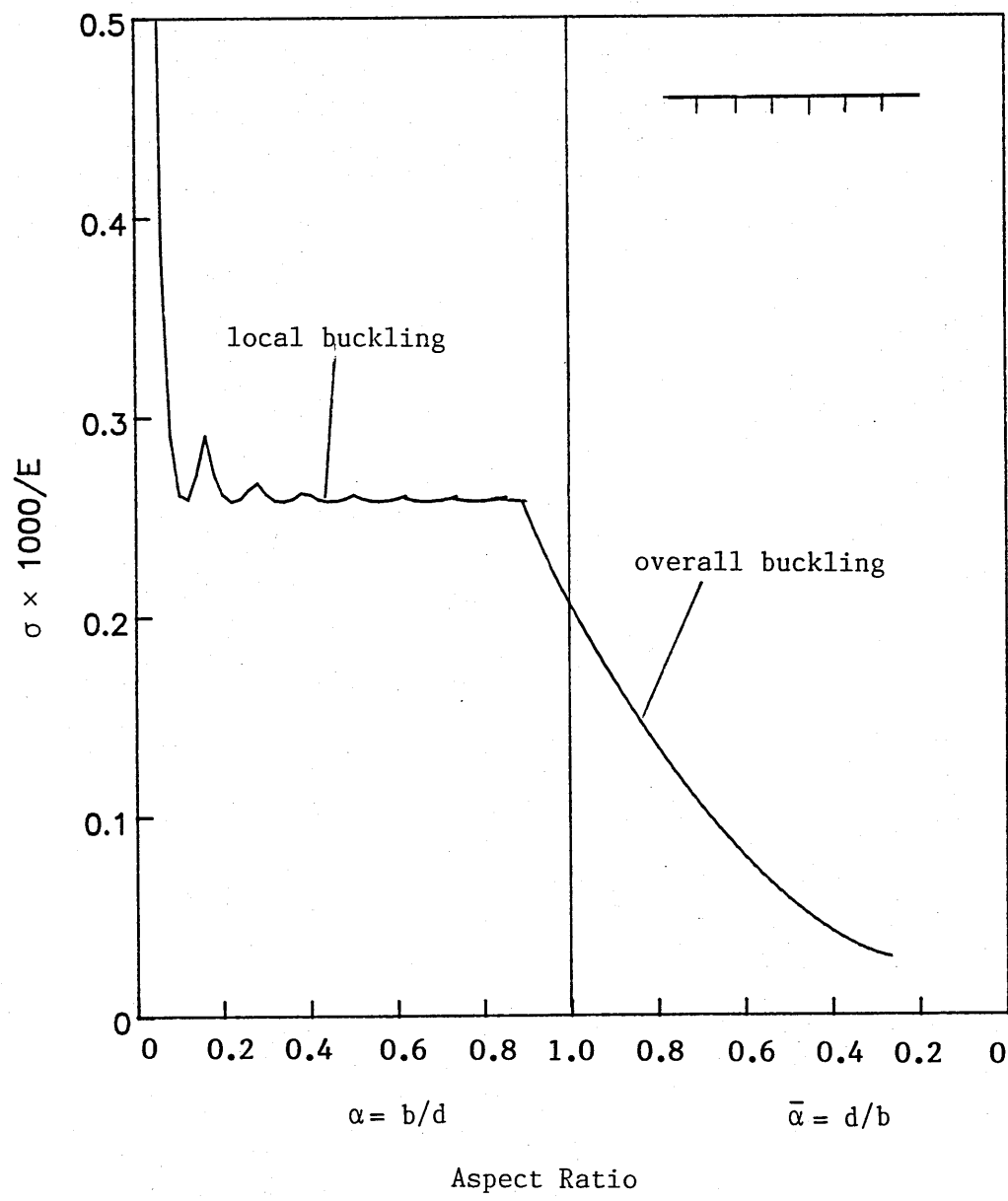


Fig 4.20 Variation of critical stress with Aspect Ratio

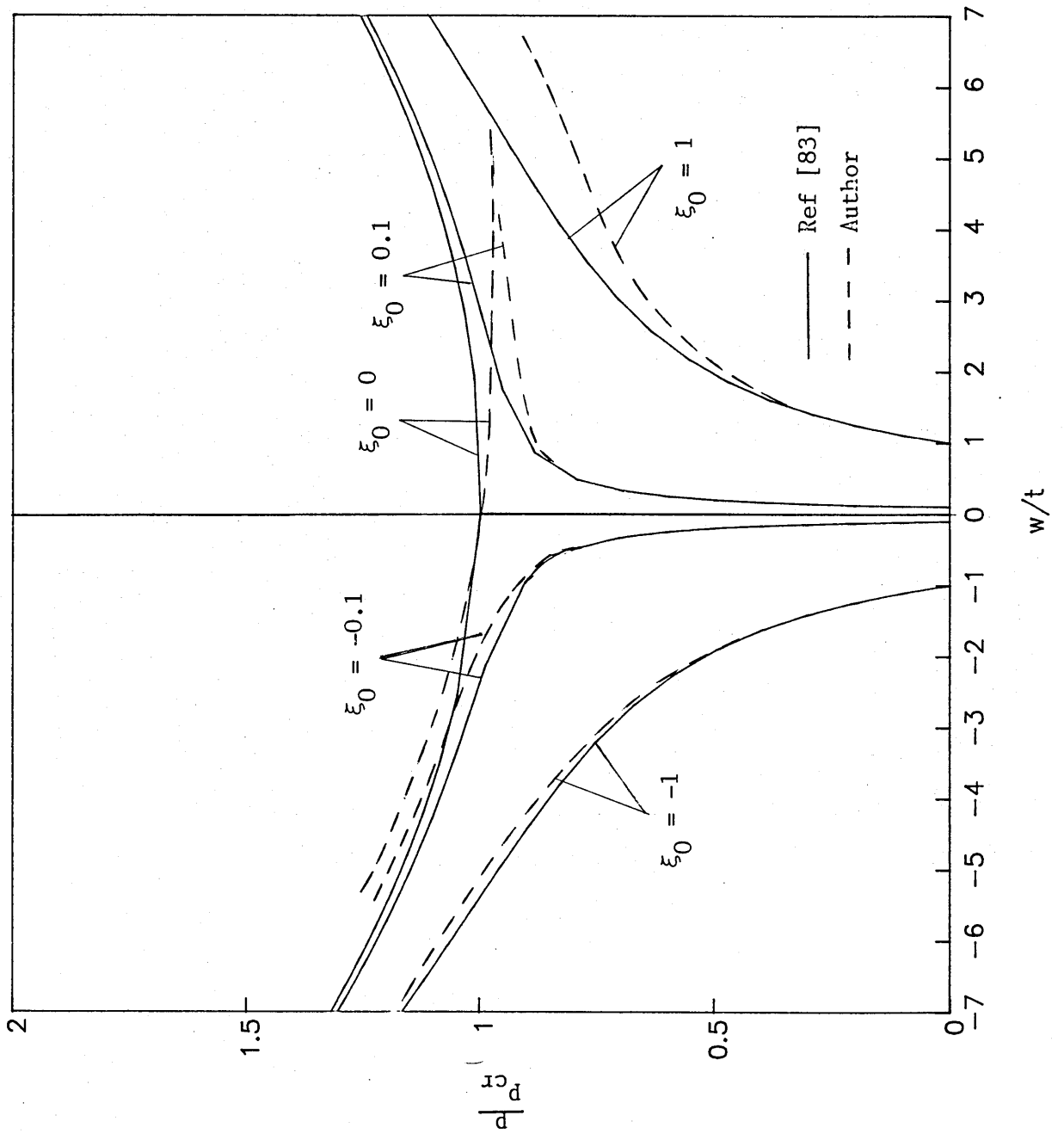


Fig 4.21 Variation of applied load with out-of-plane displacement

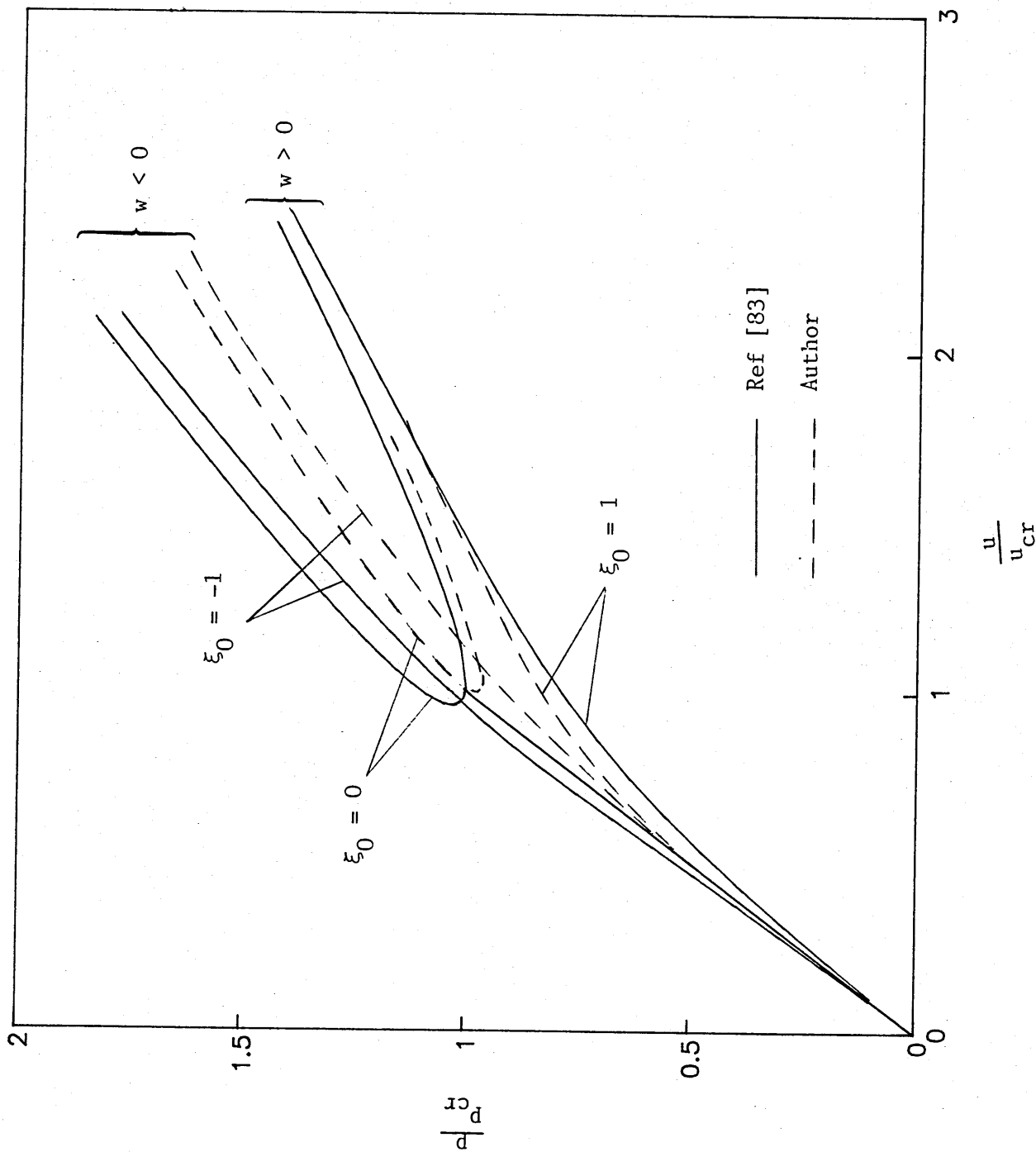


Fig 4.22 Variation of applied load with edge displacement

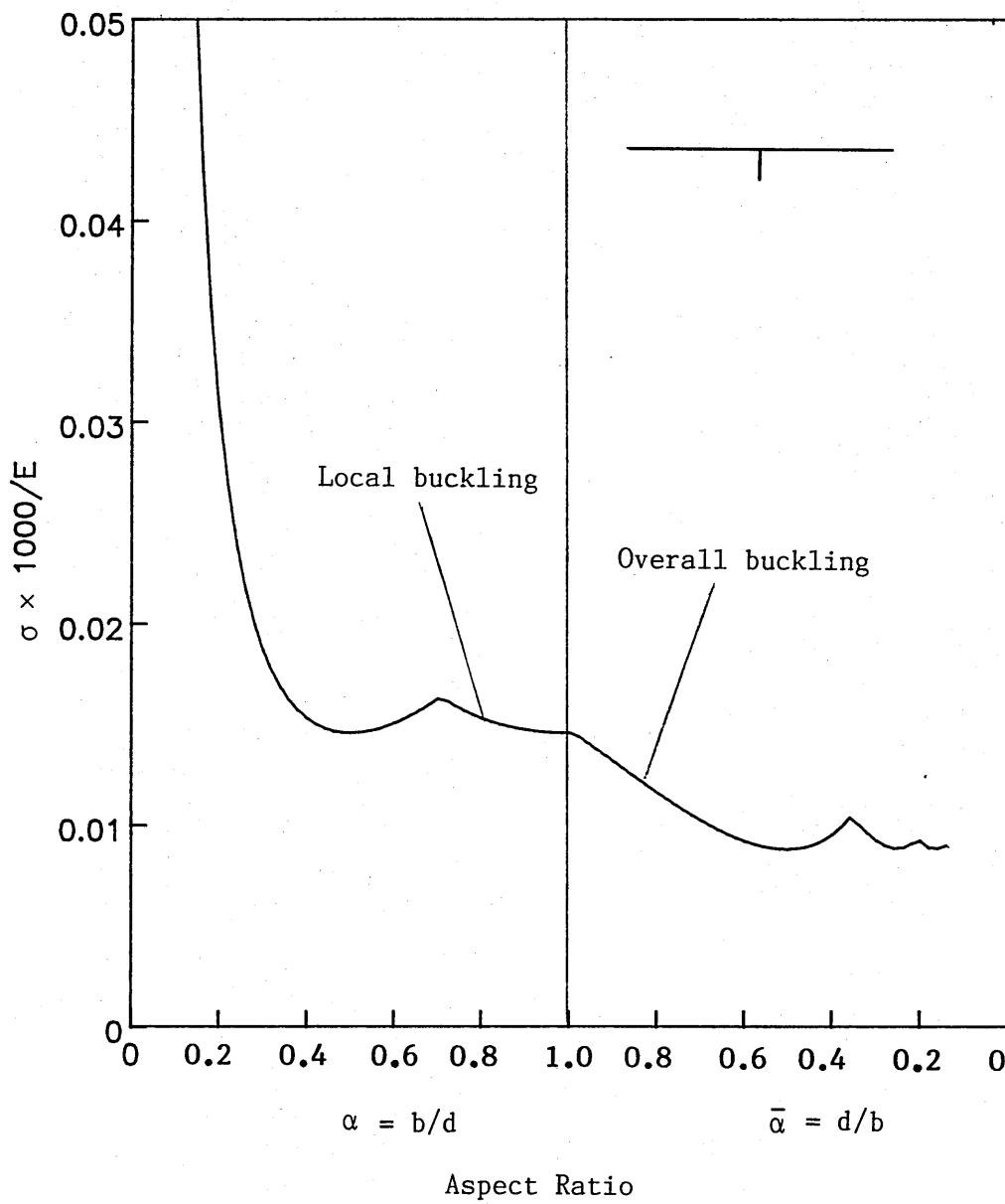


Fig 4.23 Variation of buckling stress with Aspect Ratio

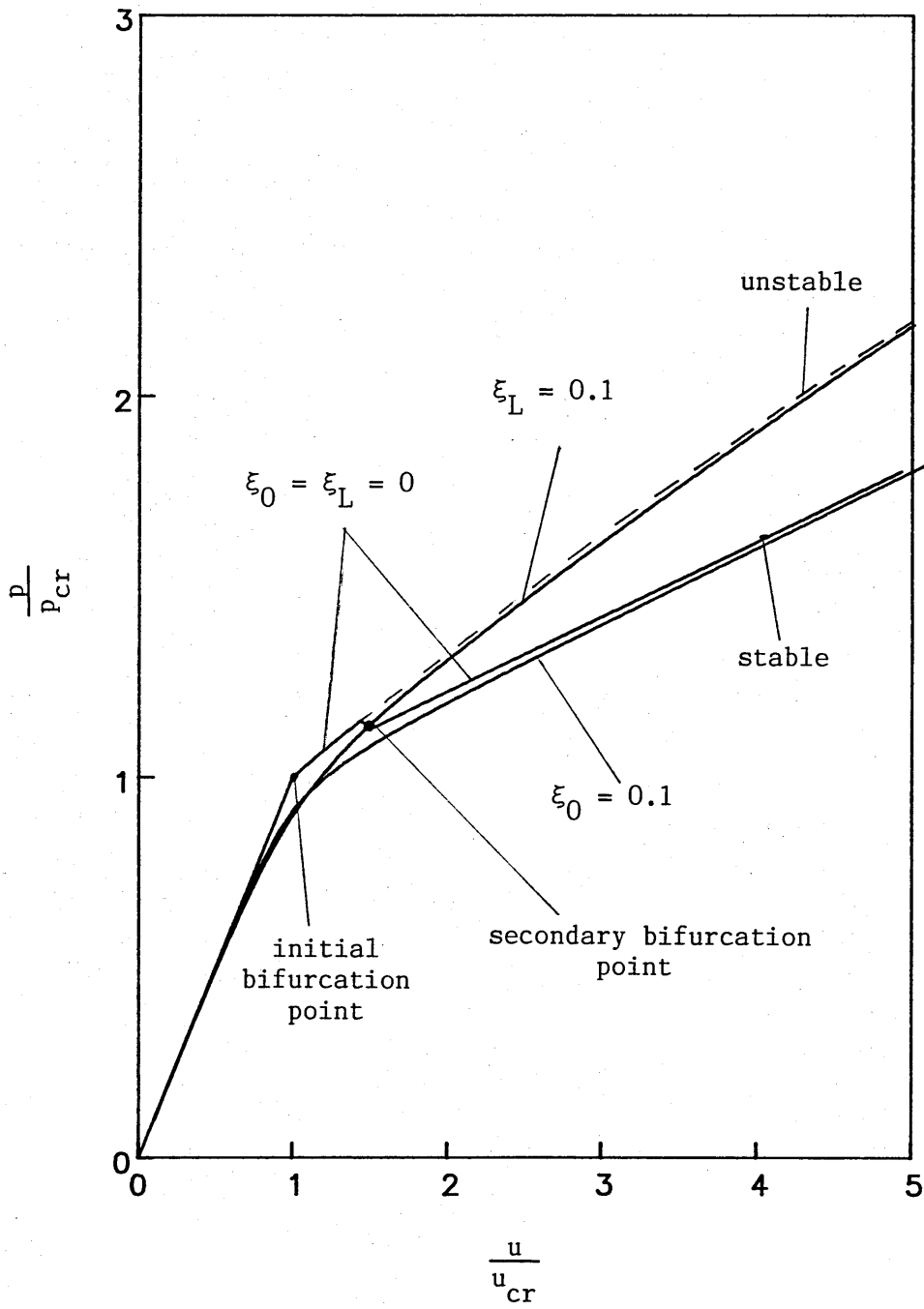


Fig 4.24 Variation of applied load p with edge displacement u

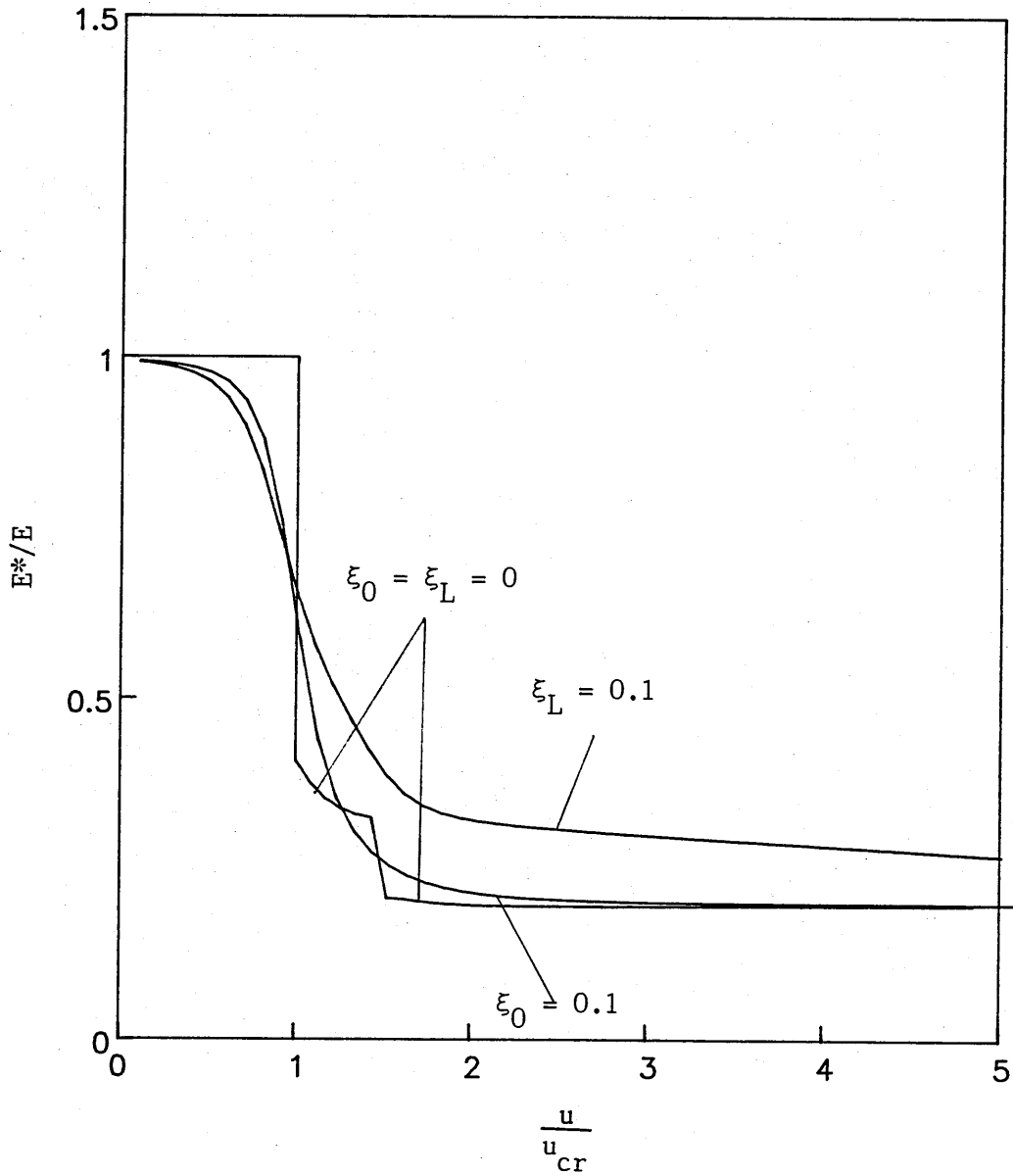


Fig 4.25 Variation of tangent stiffness with edge compression

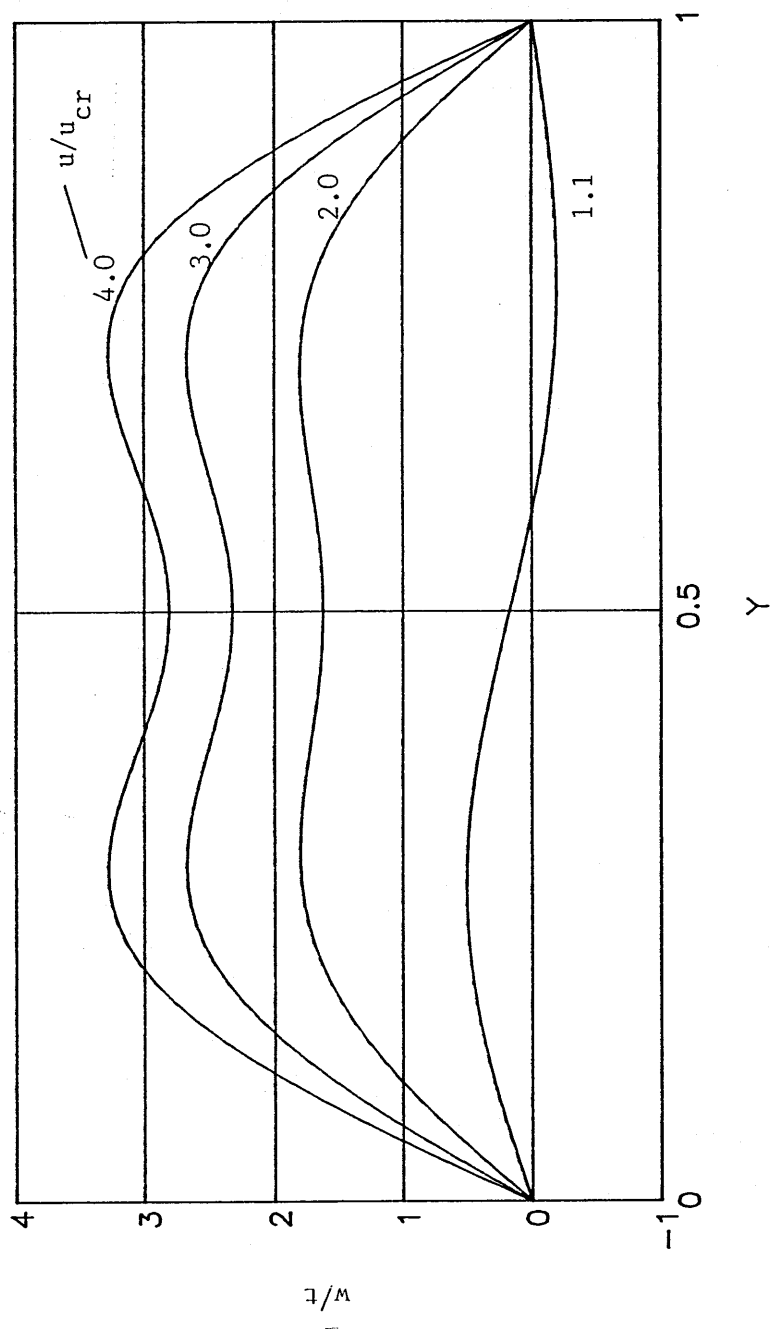


Fig 4.26 Variation of the out-of-plane skin displacement with various values of edge displacement

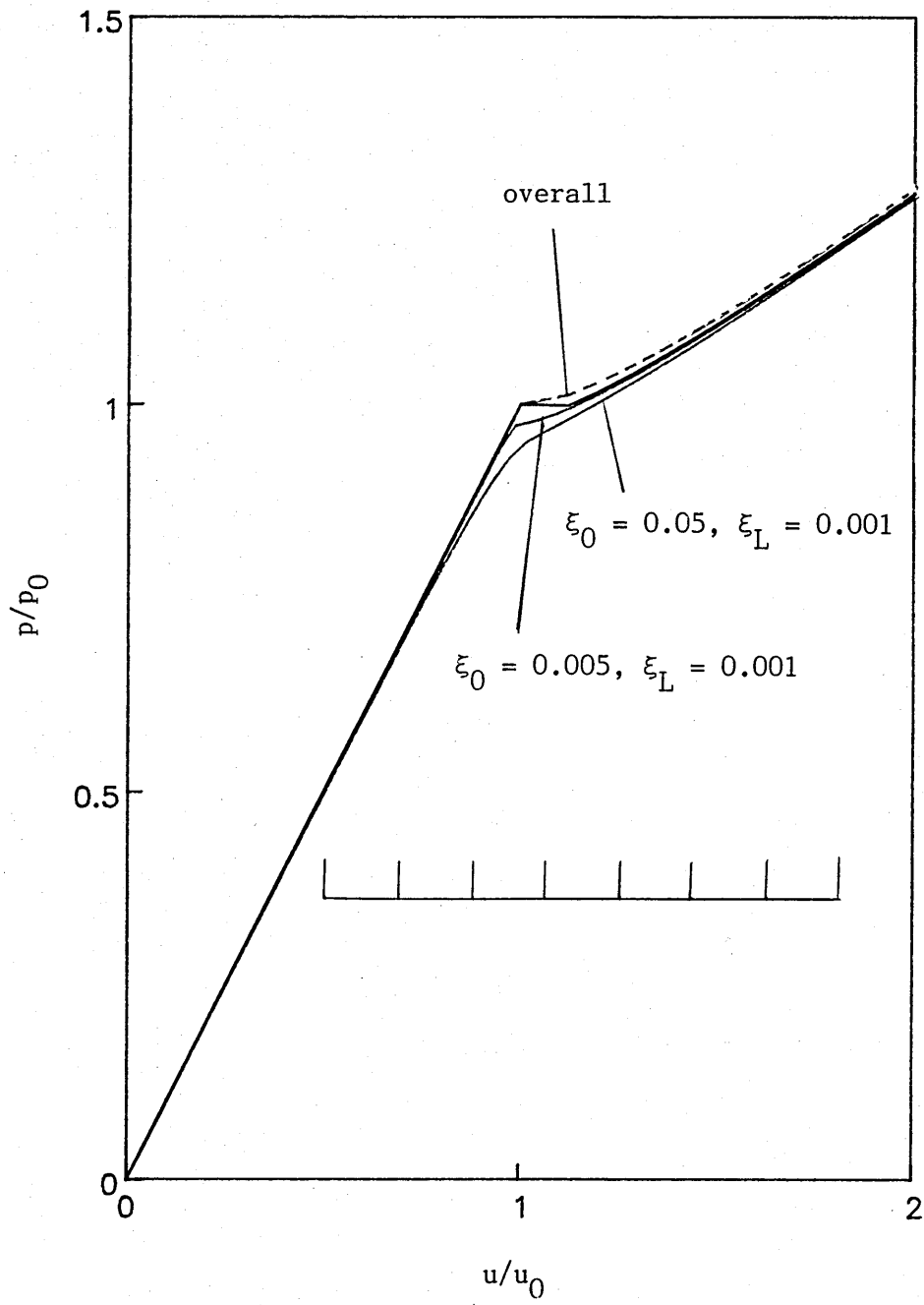


Fig 4.27 Variation of load p with edge displacement u

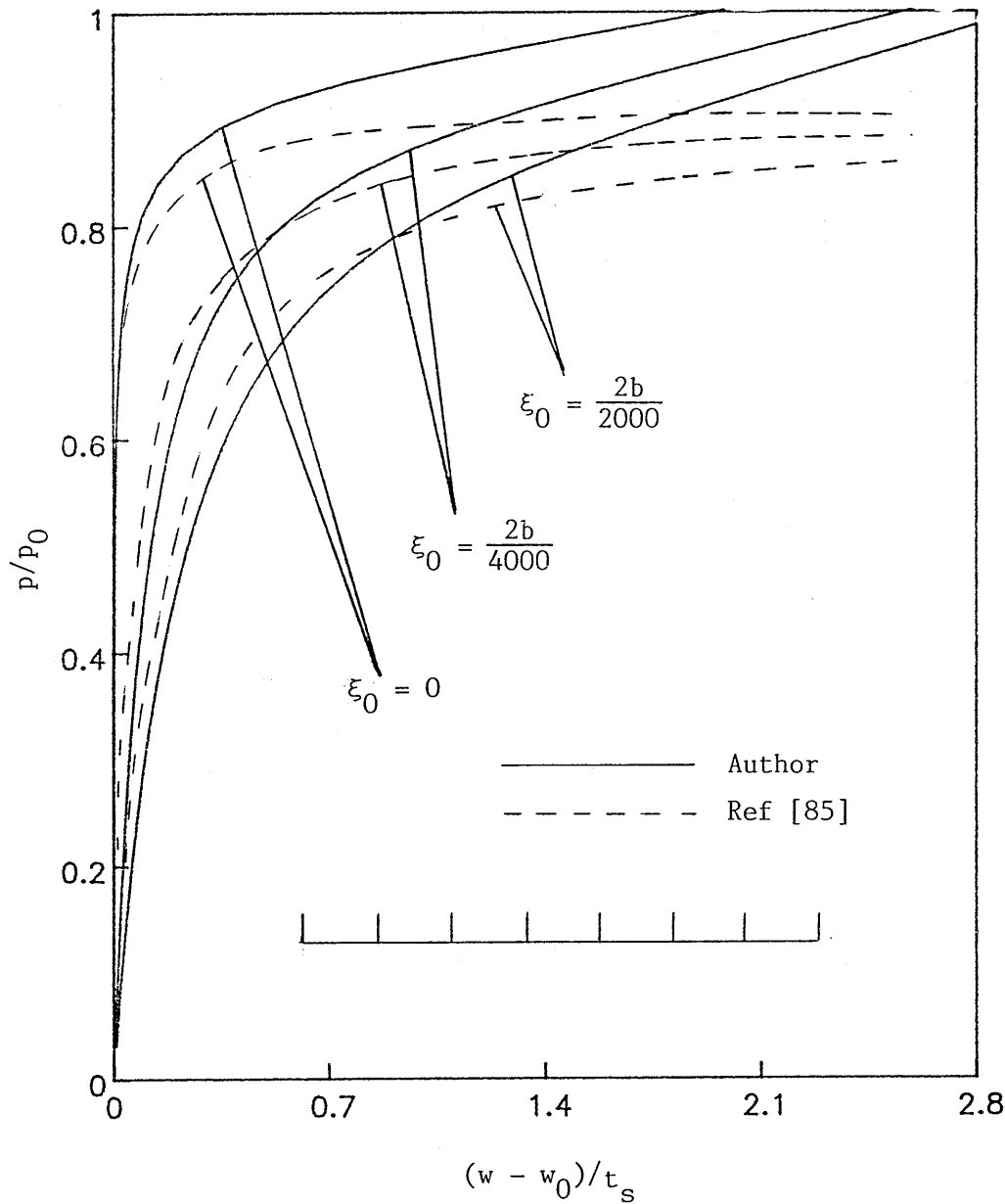


Fig 4.28 Variation of maximum out-of-plane displacement with load

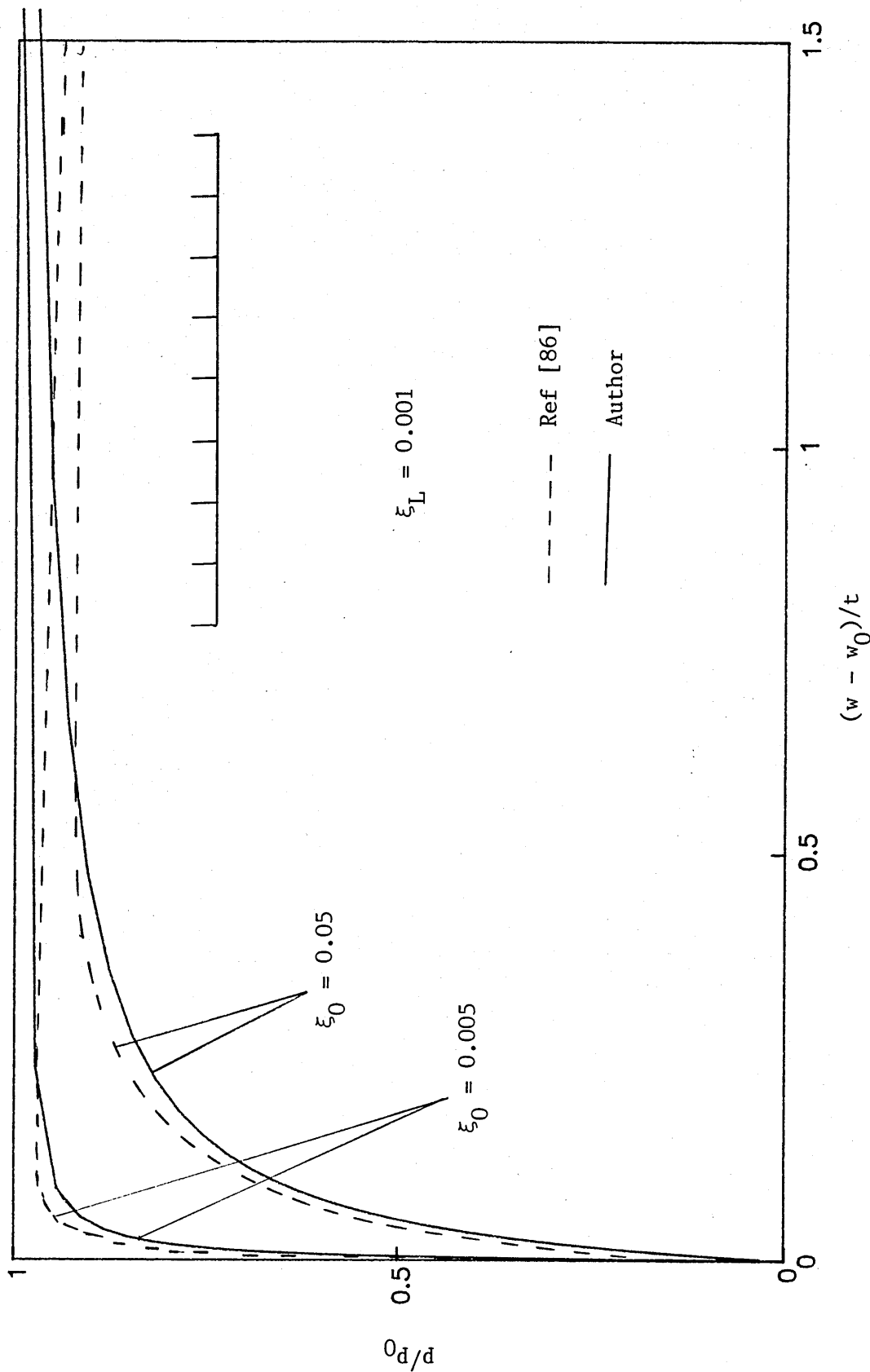


Fig 4.29 Variation of maximum out-of-plane displacement with load

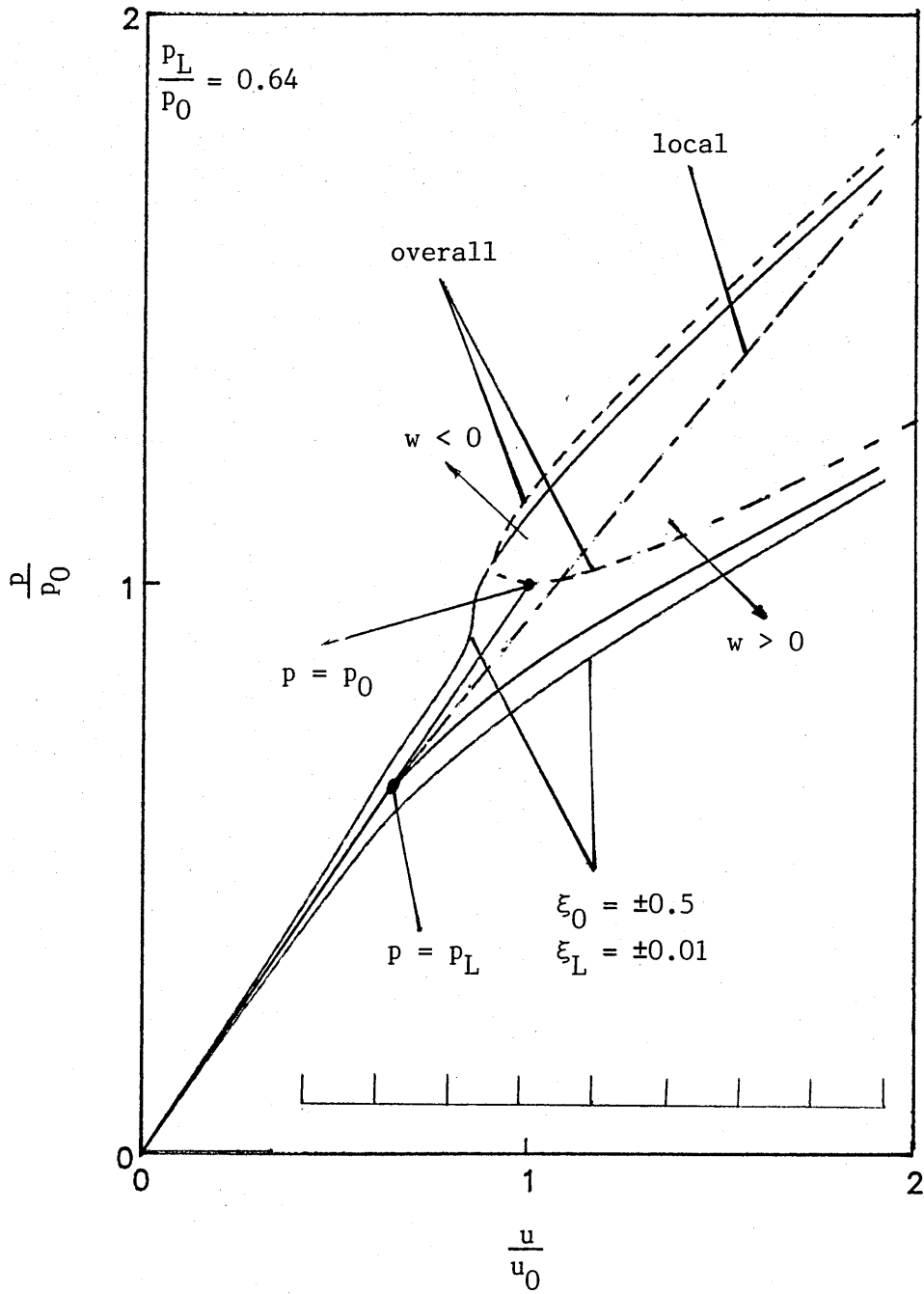


Fig 4.30 Variation of load p with edge displacement u

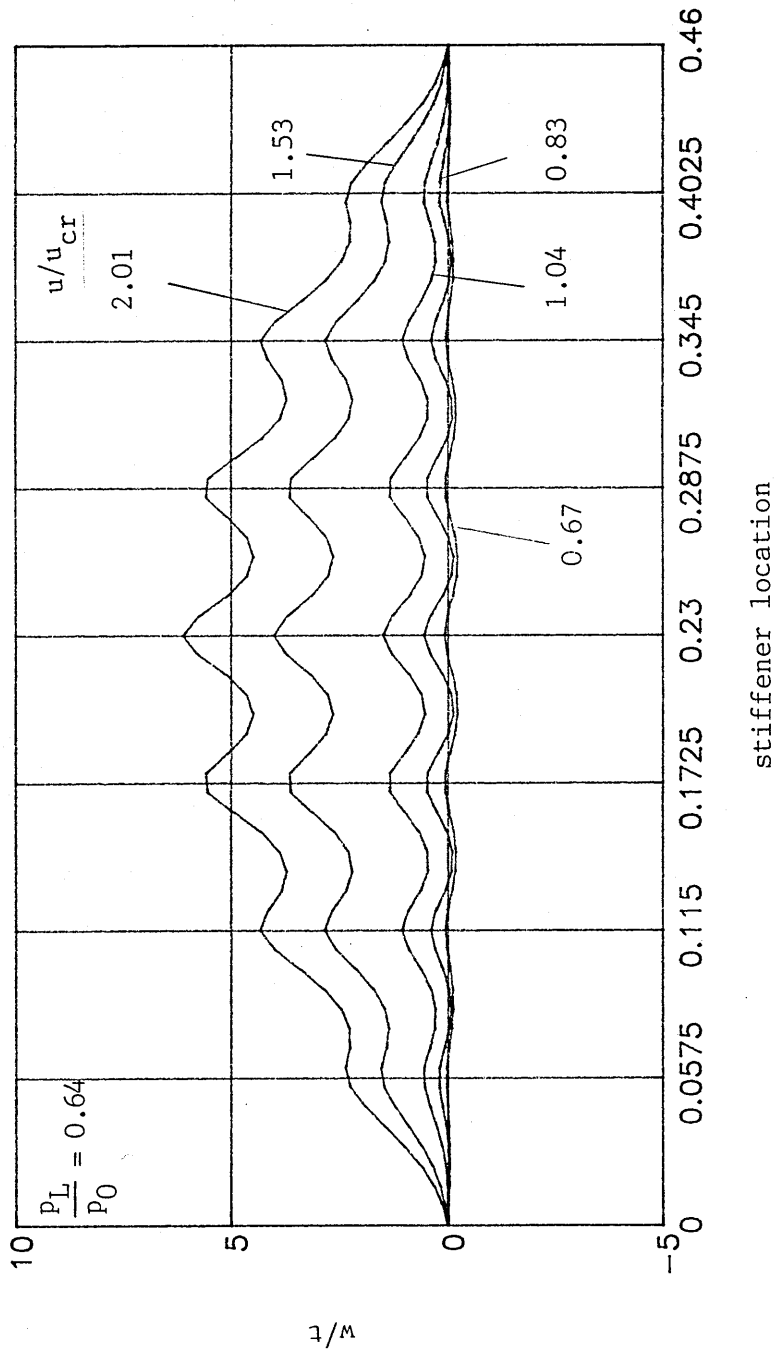


Fig 4.31 Cross-sectional mode shape at $x = b$

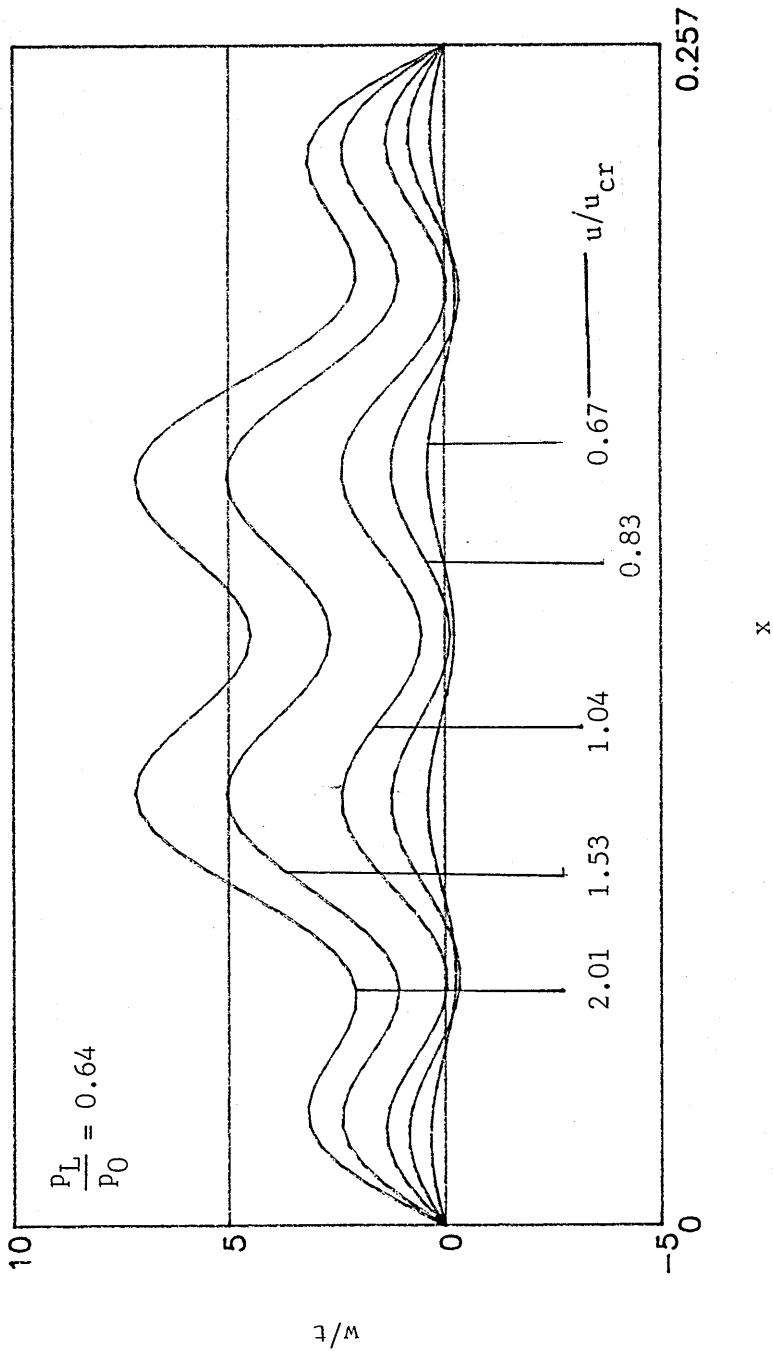


Fig 4.32 Variation of longitudinal mode shape with edge displacement u

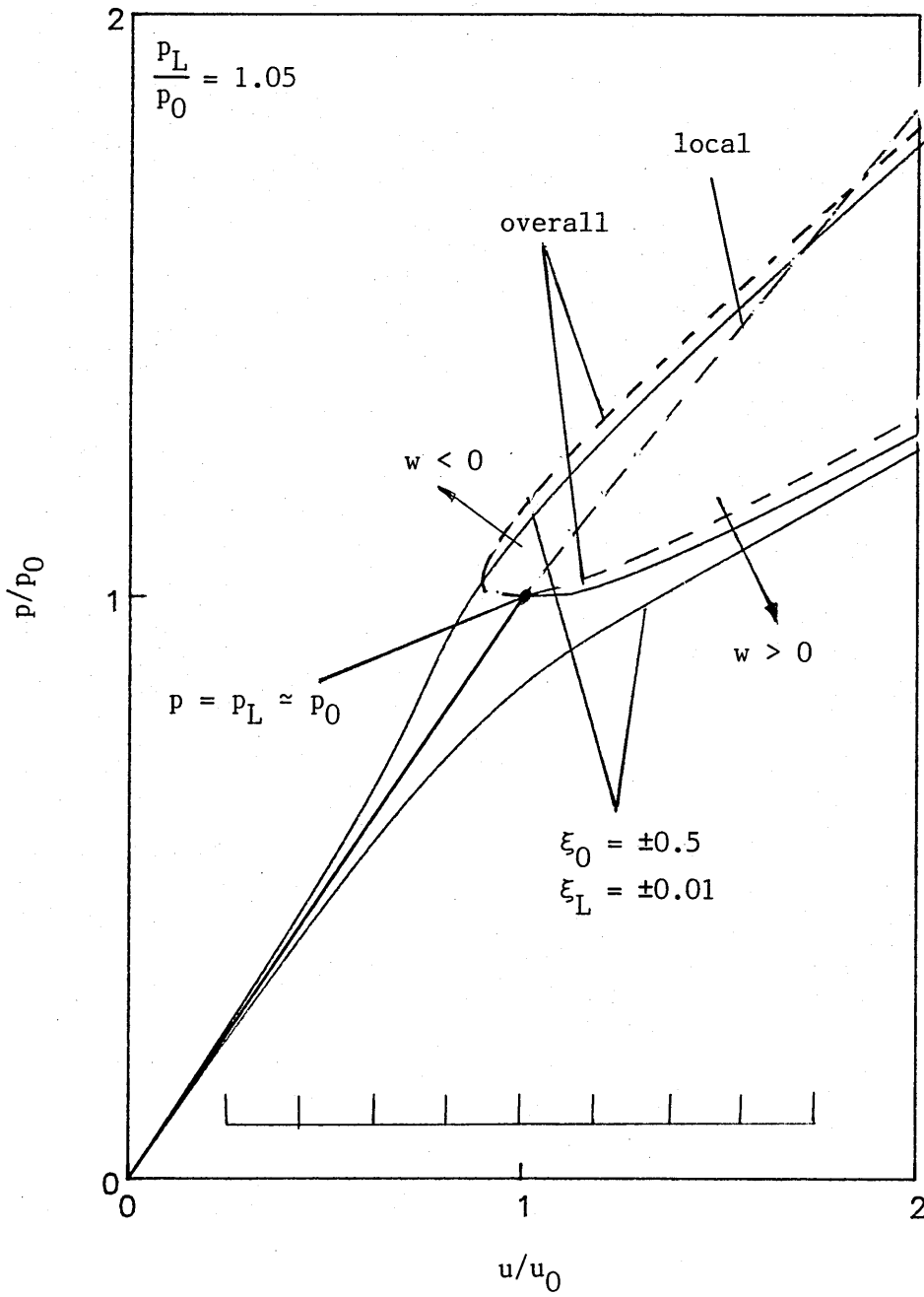


Fig 4.33 Variation of load p with edge displacement u

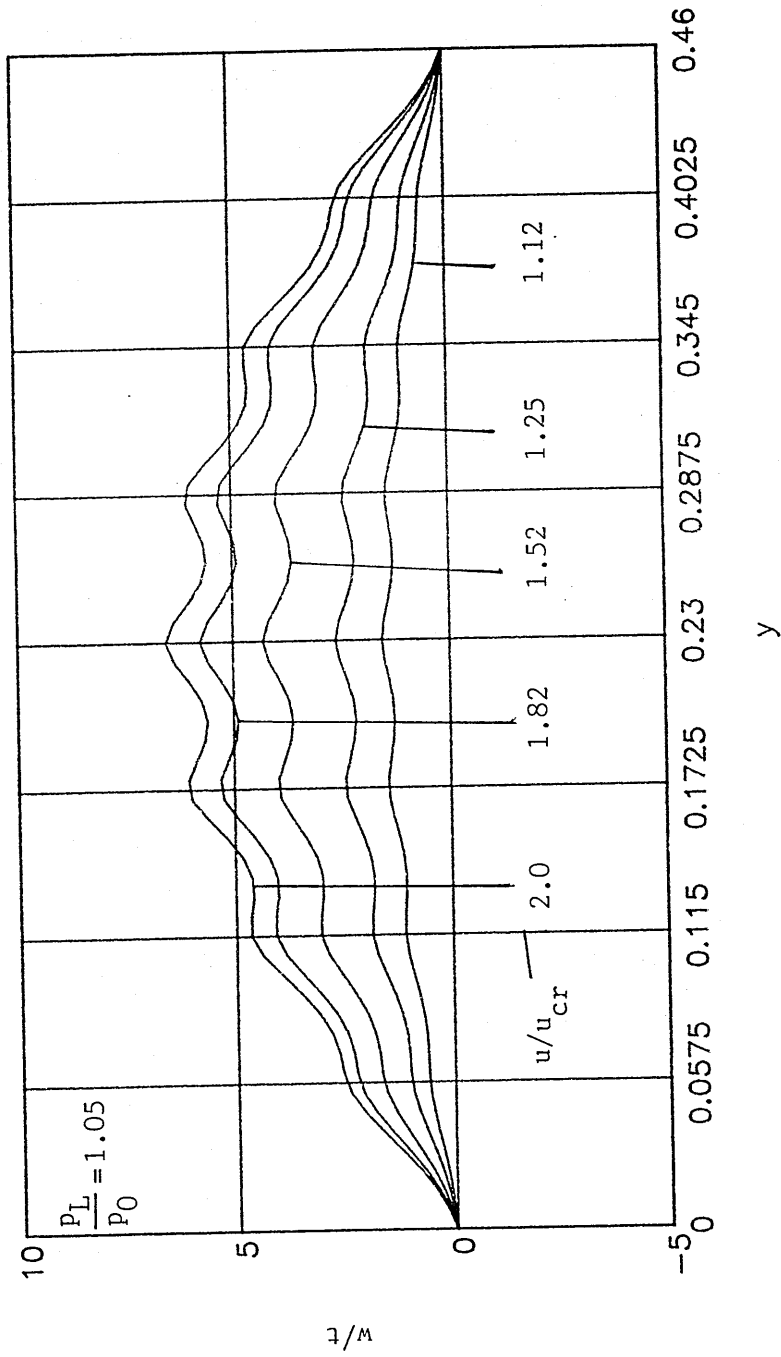


Fig 4.34 Variation of cross-sectional displacement with edge displacement u

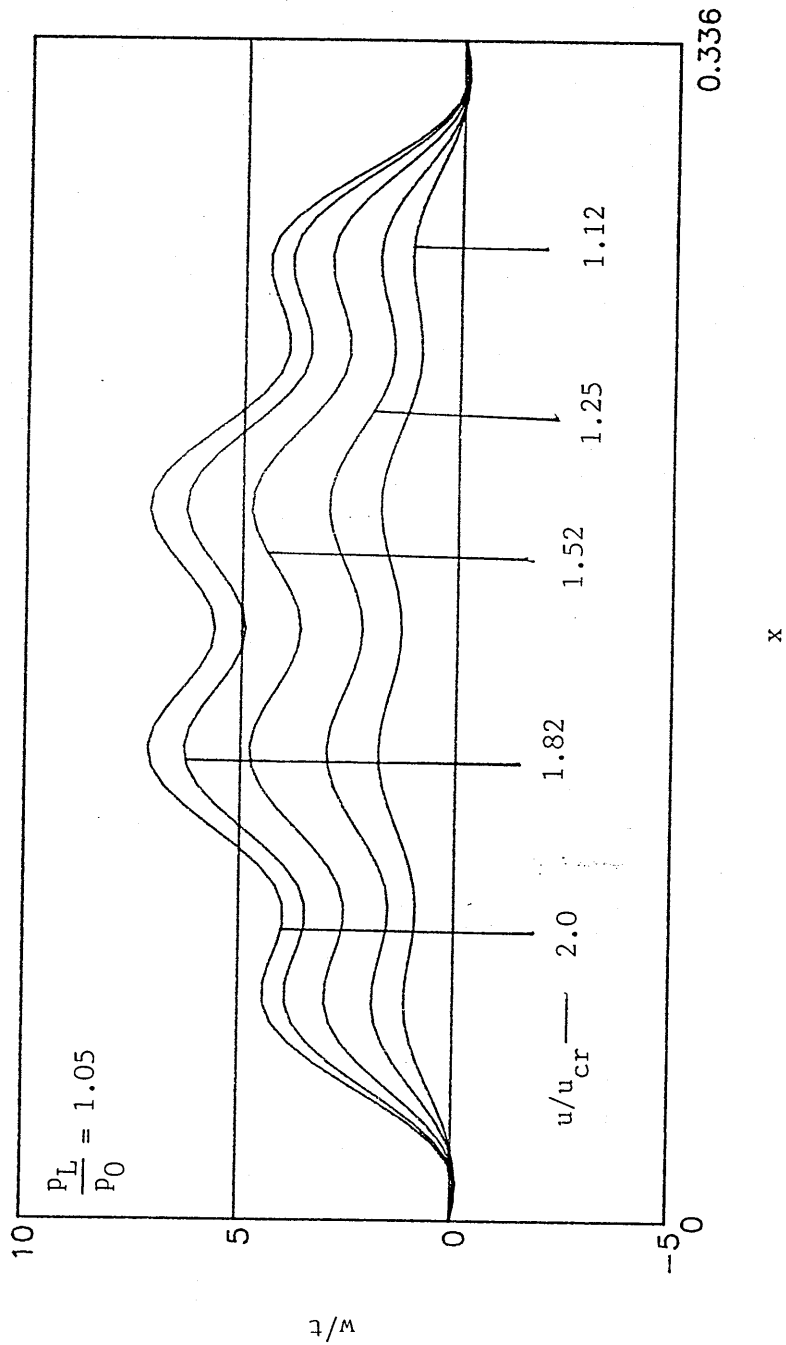


Fig 4.35 Variation of longitudinal displacement with edge displacement u

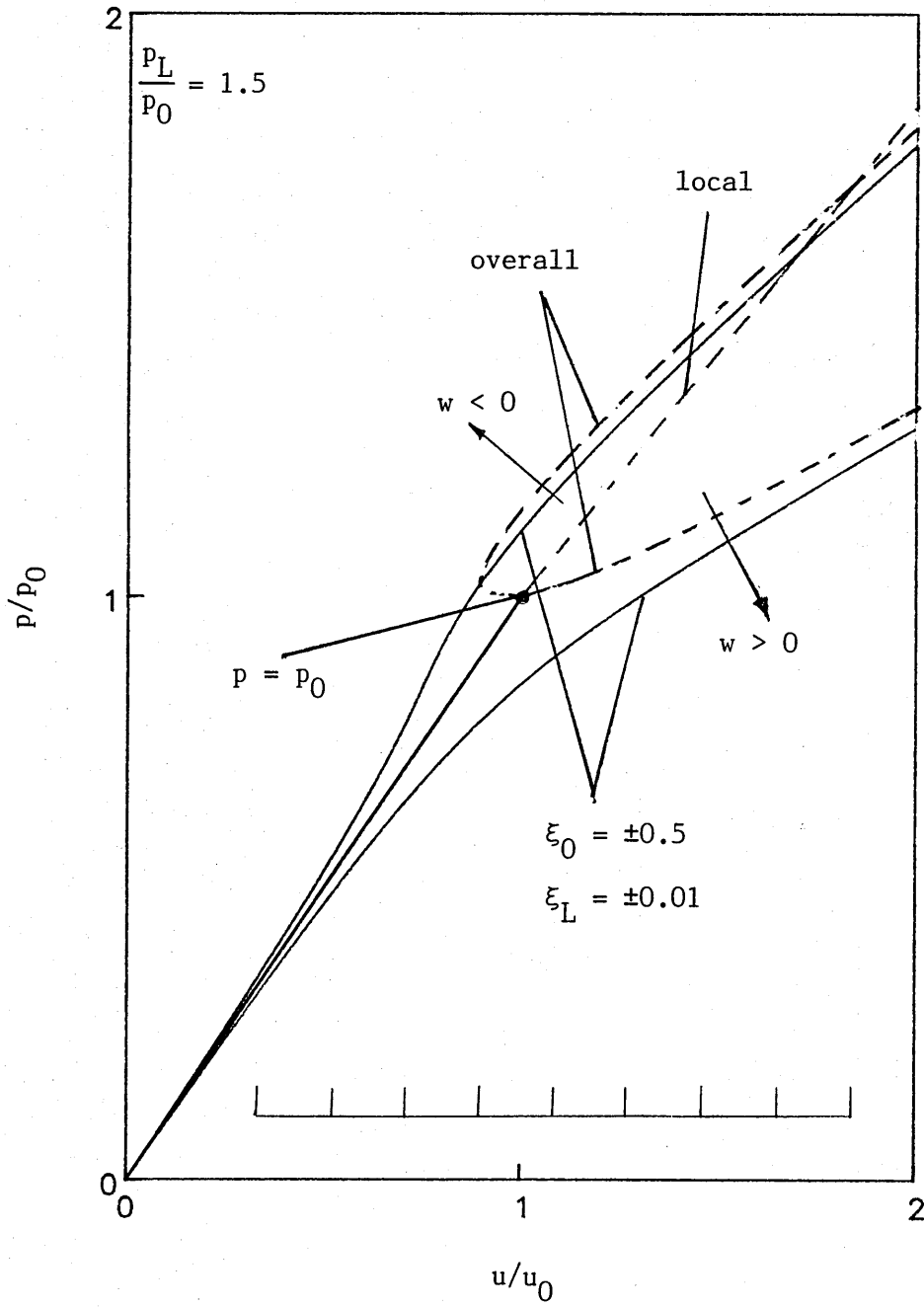


Fig 4.36 Variation of load p with edge displacement u

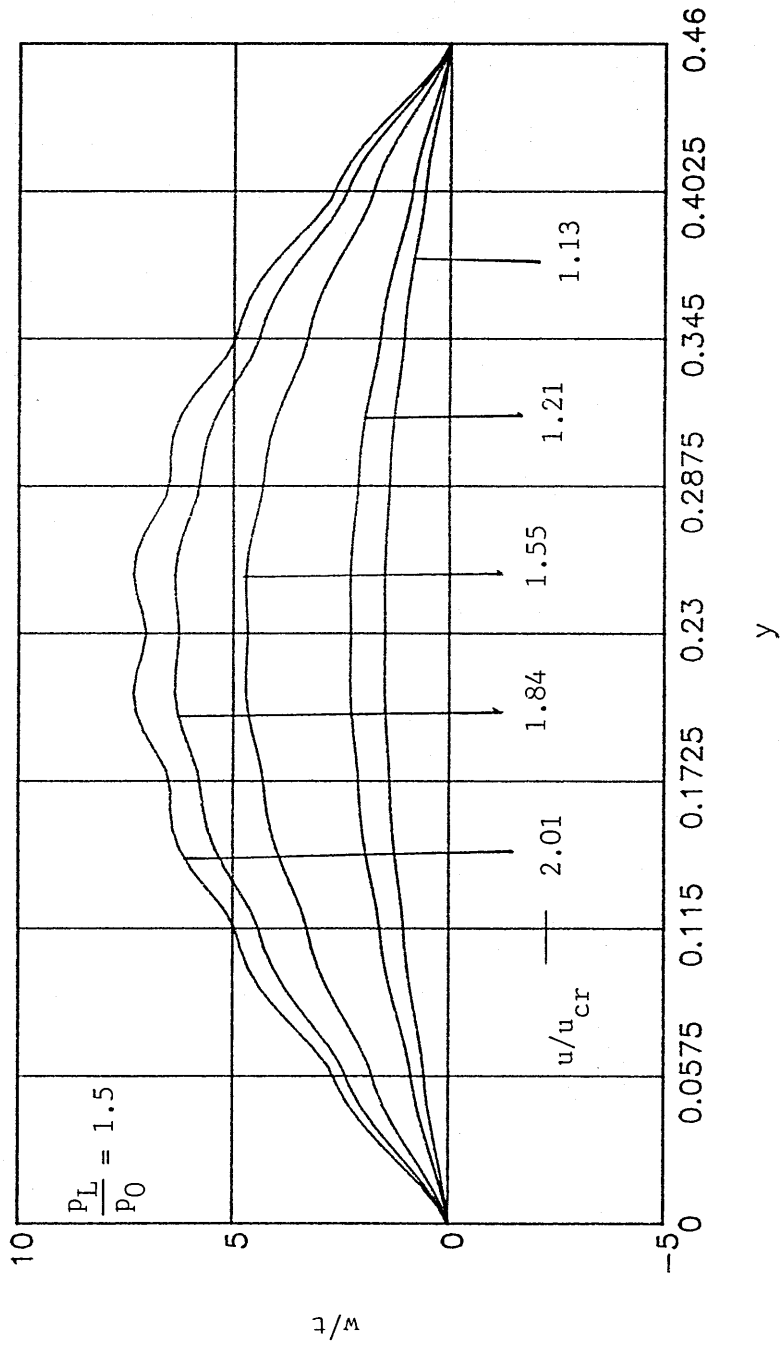


Fig 4.37 Variation of cross-sectional displacement with edge displacement u

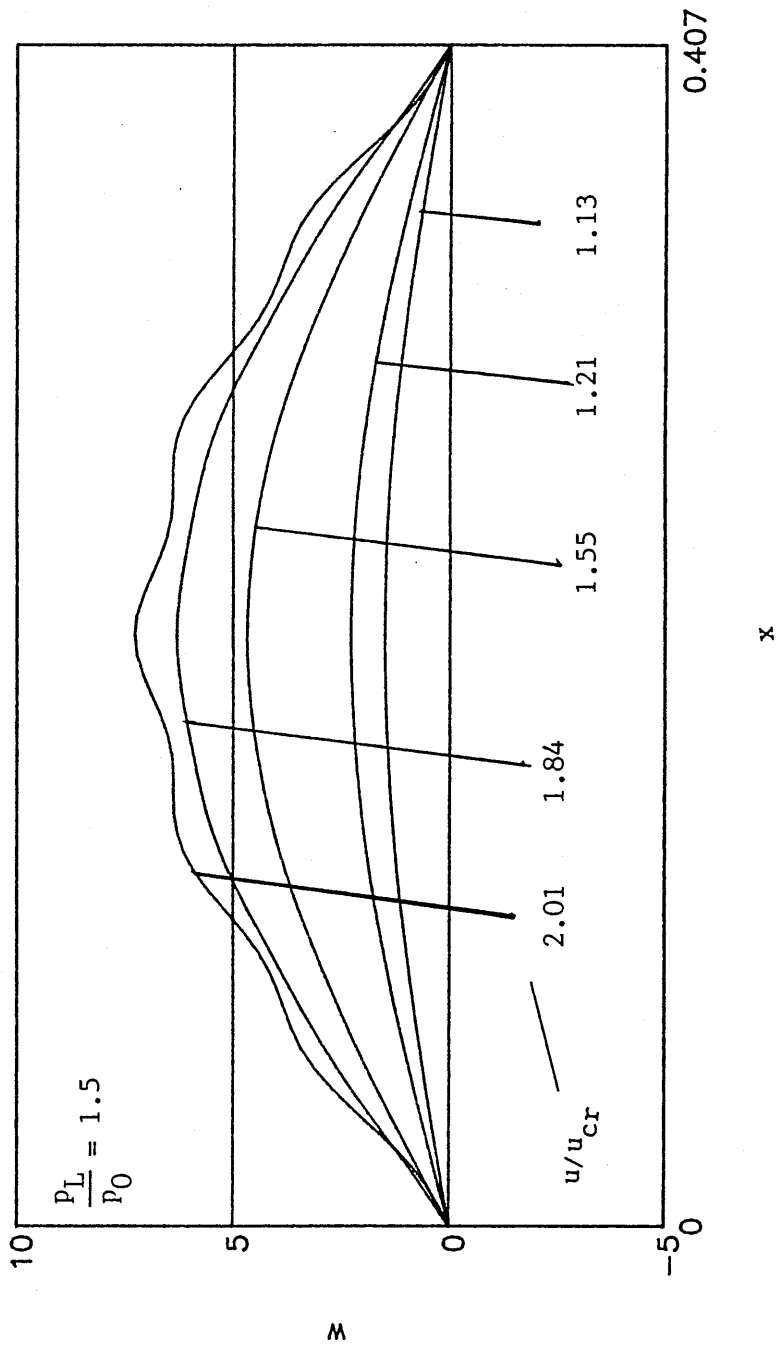


Fig 4.38 Variation of longitudinal out-of-plane displacement with edge displacement u

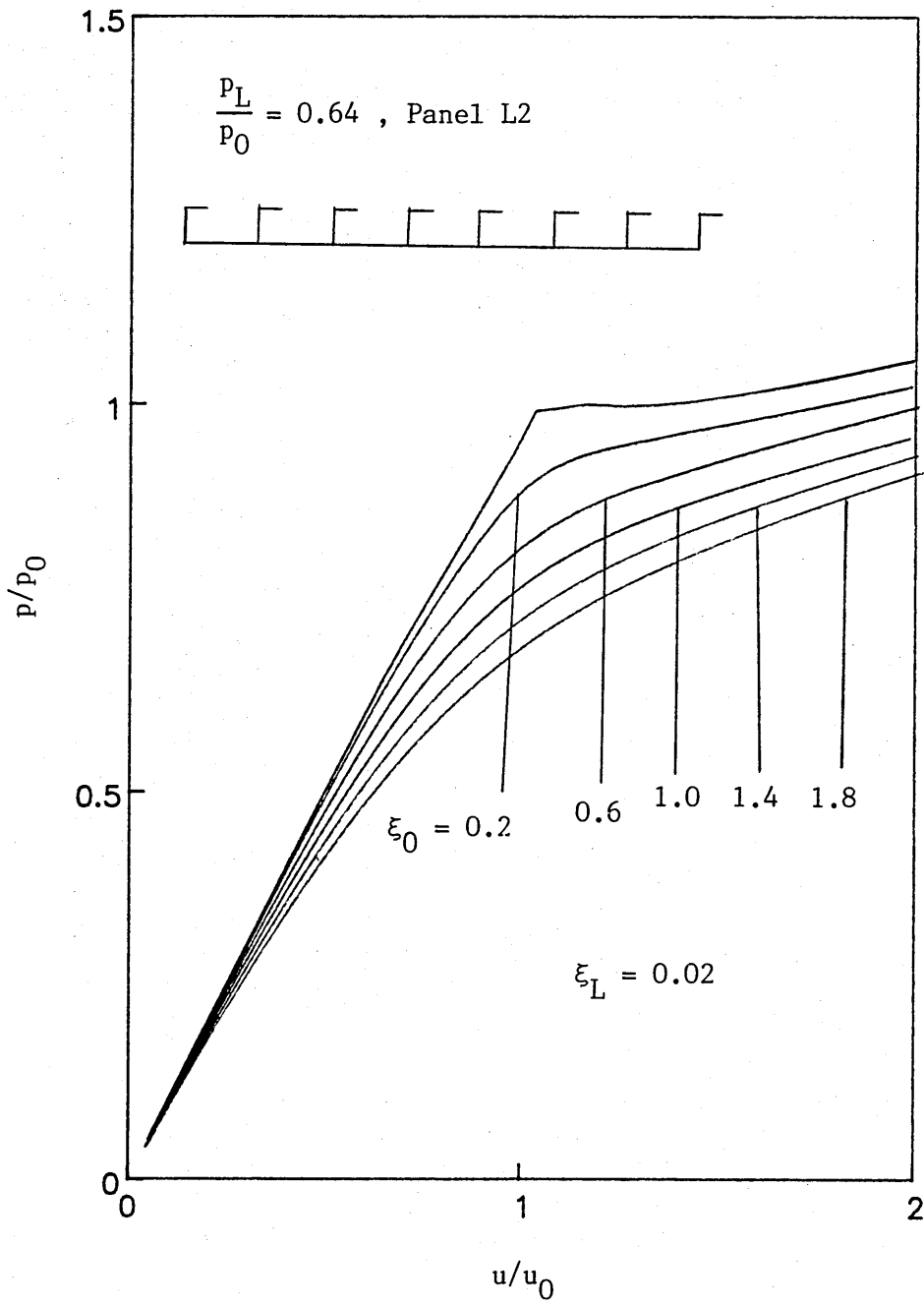


Fig 4.39 Variation of average load with edge displacement

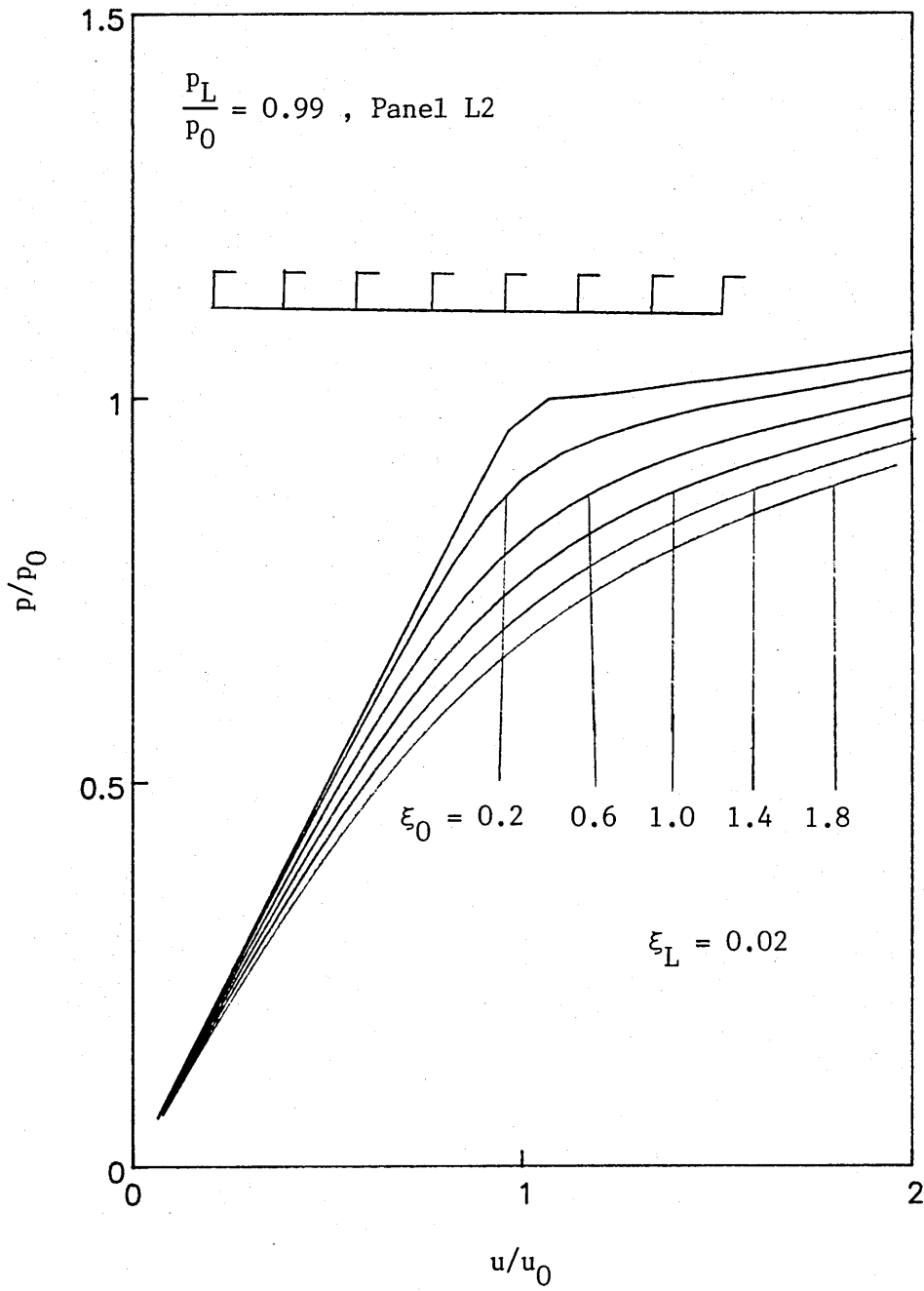


Fig 4.40 Variation of average load with edge displacement

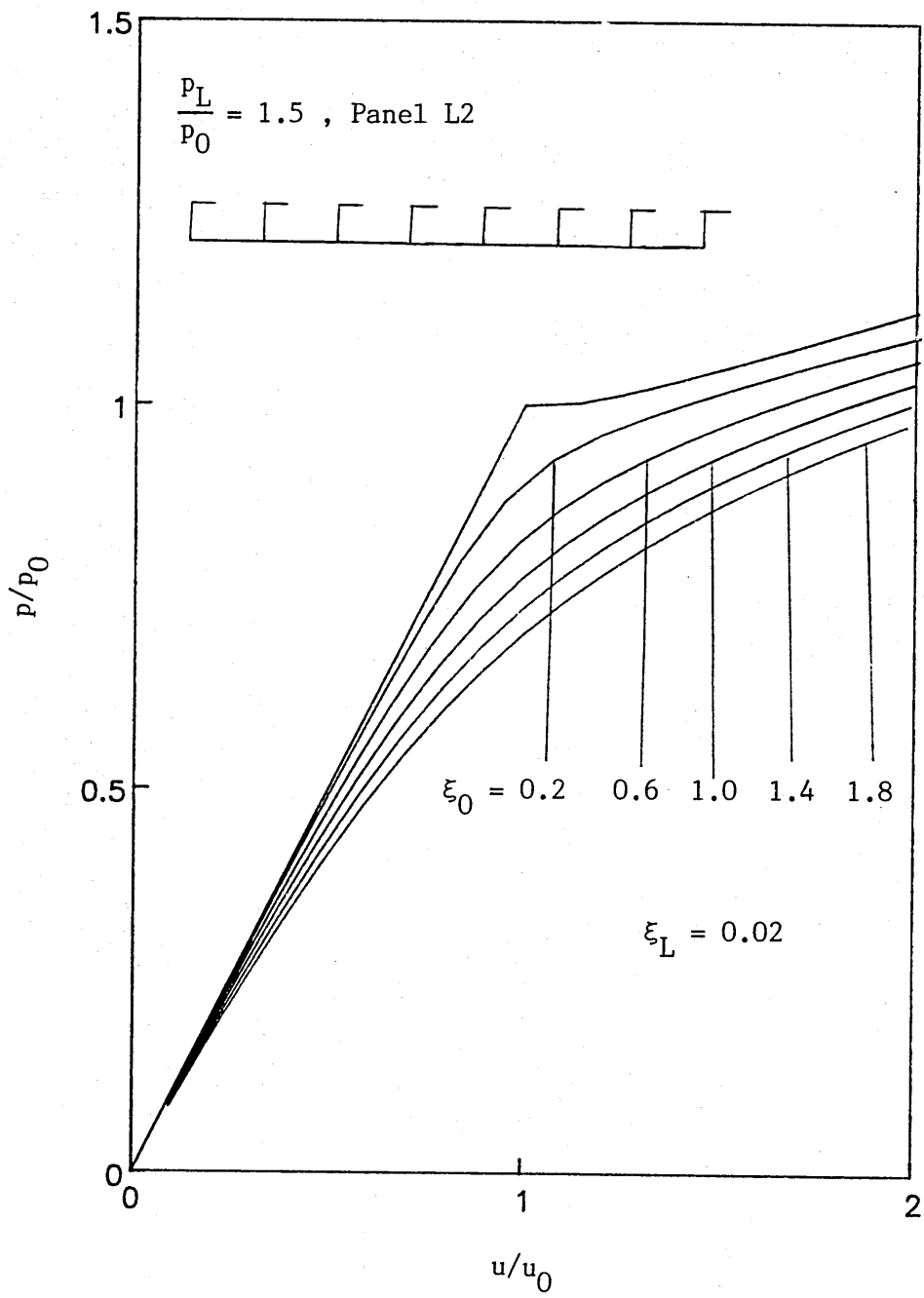


Fig 4.41 Variation of average load with edge displacement

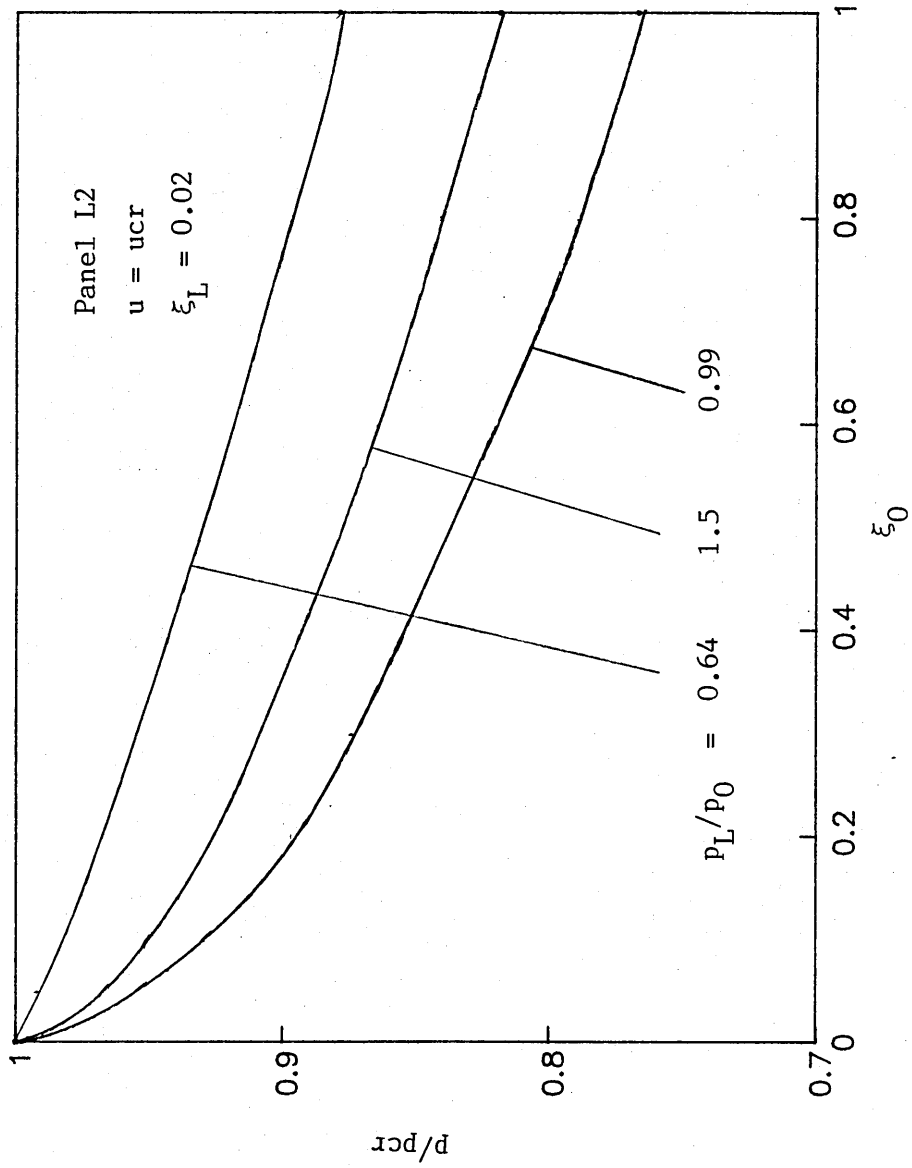


Fig 4.42 Variation of average load with imperfection at critical edge displacement

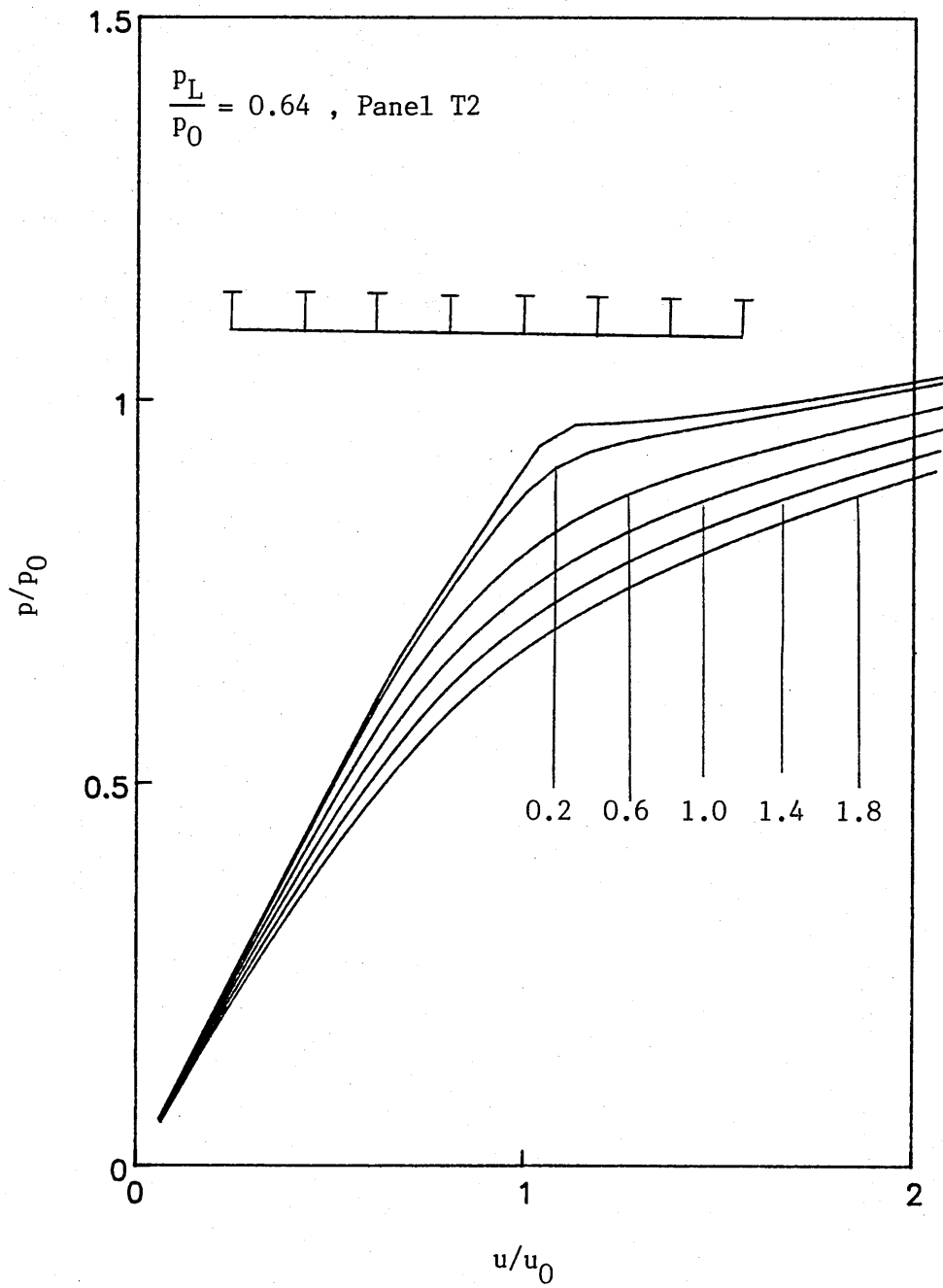


Fig 4.43 Variation of average load with edge displacement

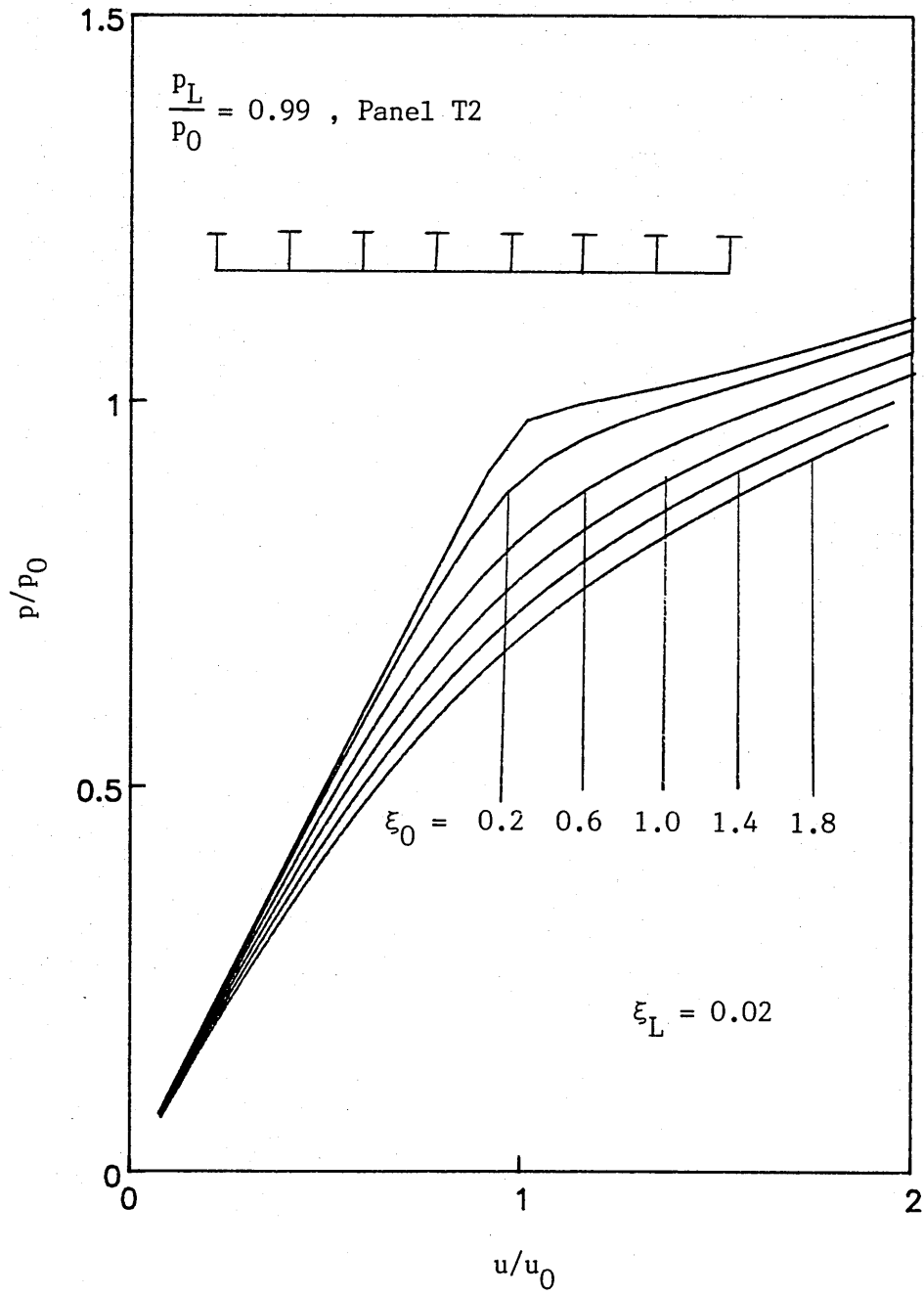


Fig 4.44 Variation of average load with edge displacement

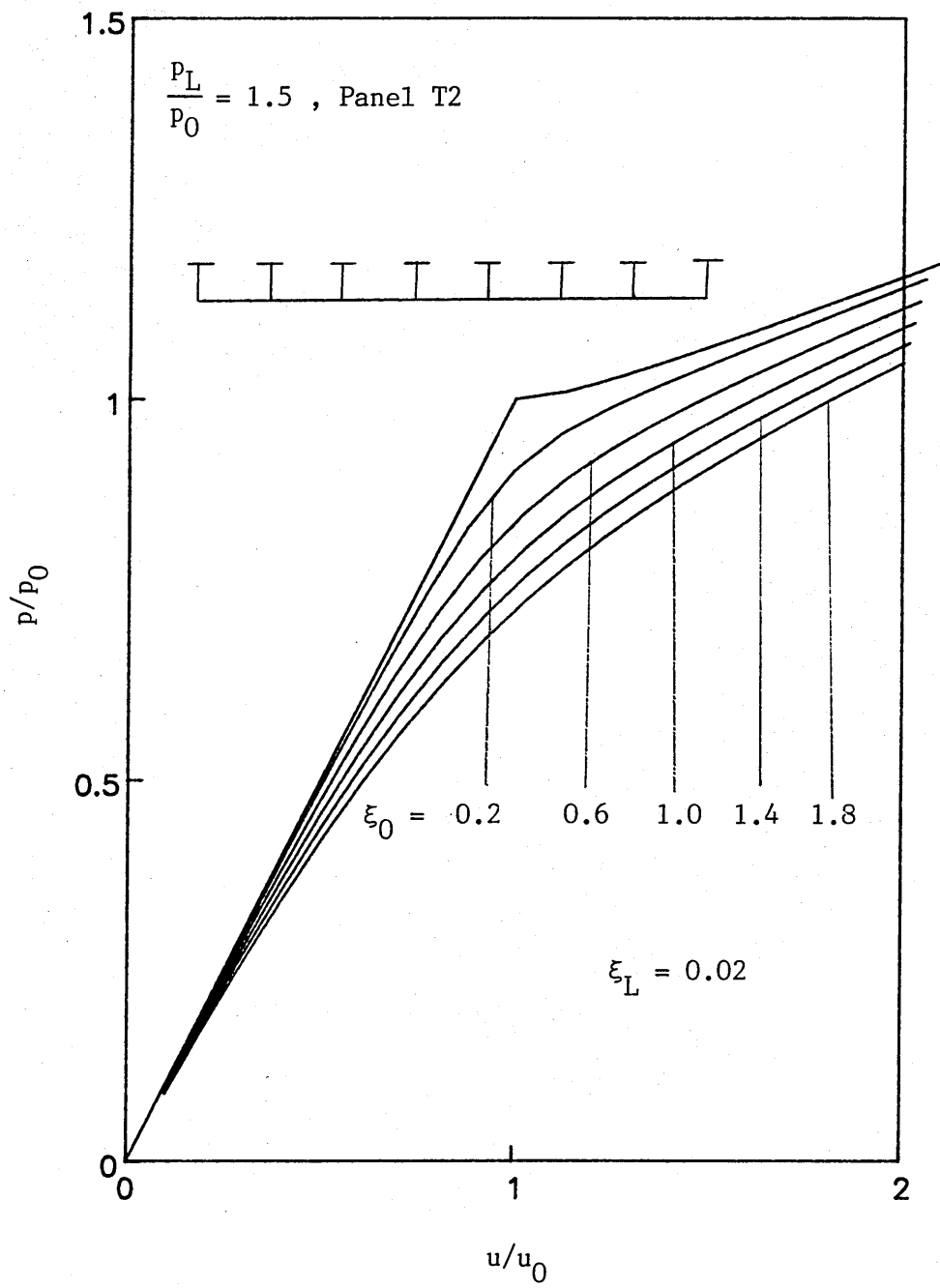


Fig 4.45 Variation of average load with edge displacement

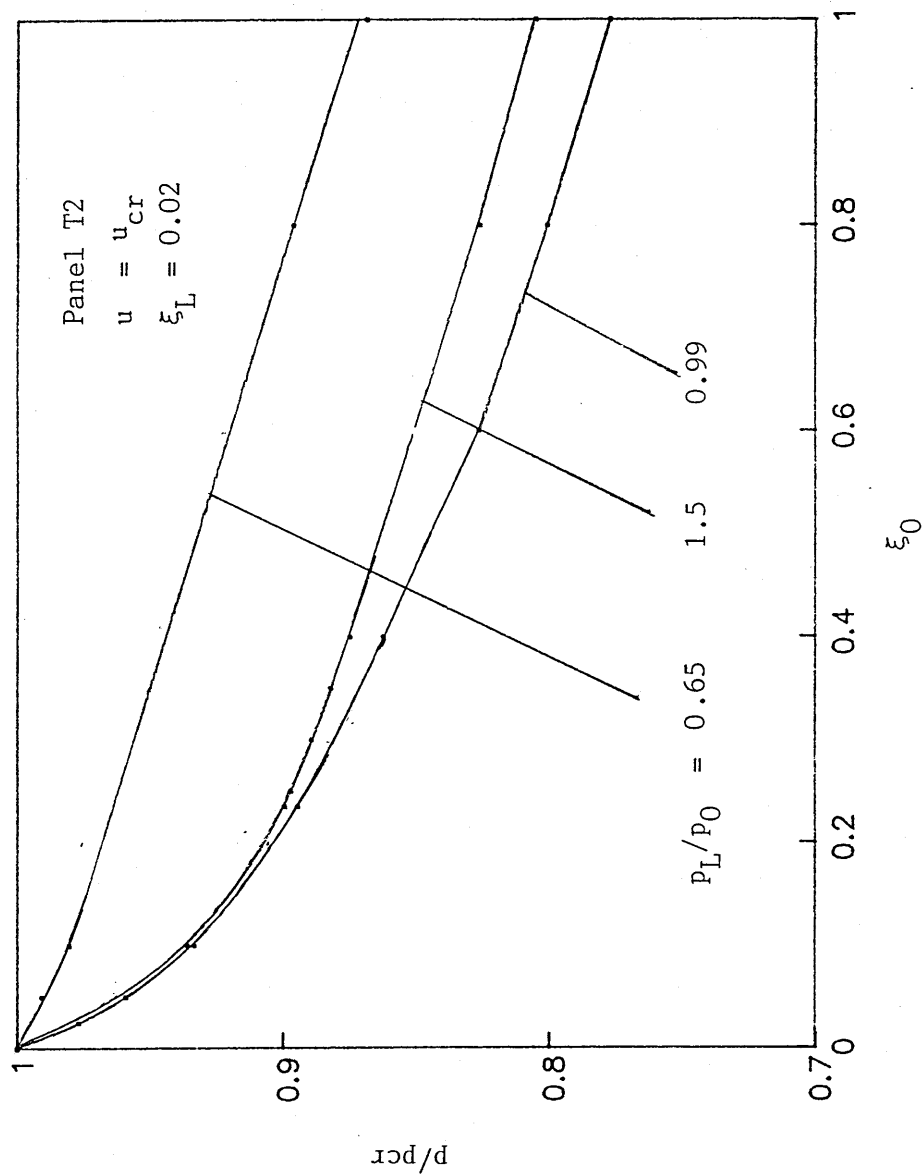


Fig 4.46 Variation of load with overall imperfection at critical load

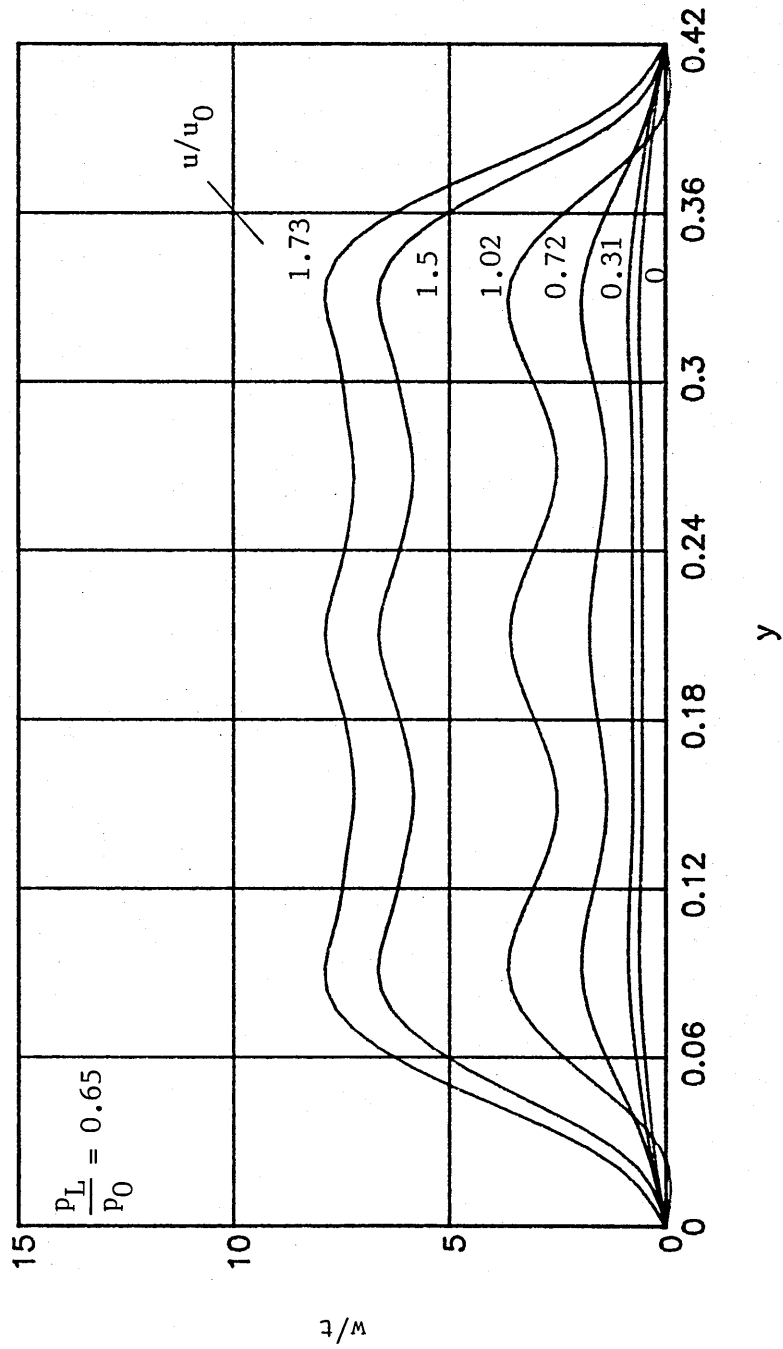


Fig 4.47 Out-of-plane displacement at mid span for Panel T2 $x = b$, $\xi_0 = 0.6$, $\xi_L = 0.002$

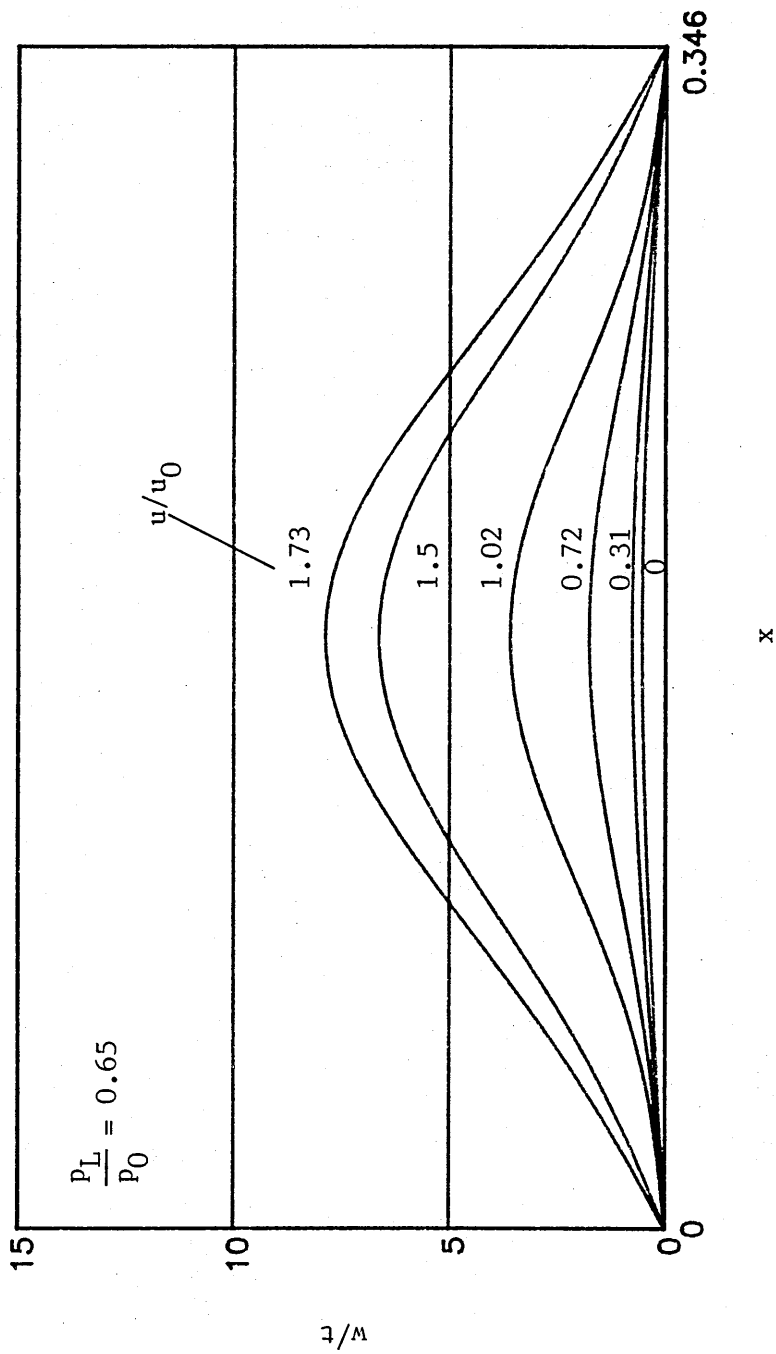


Fig 4.48 Variation of out-of-plane displacement for panel T2 $y = d$, $\xi_0 = 0.6$, $\xi_L = 0.02$

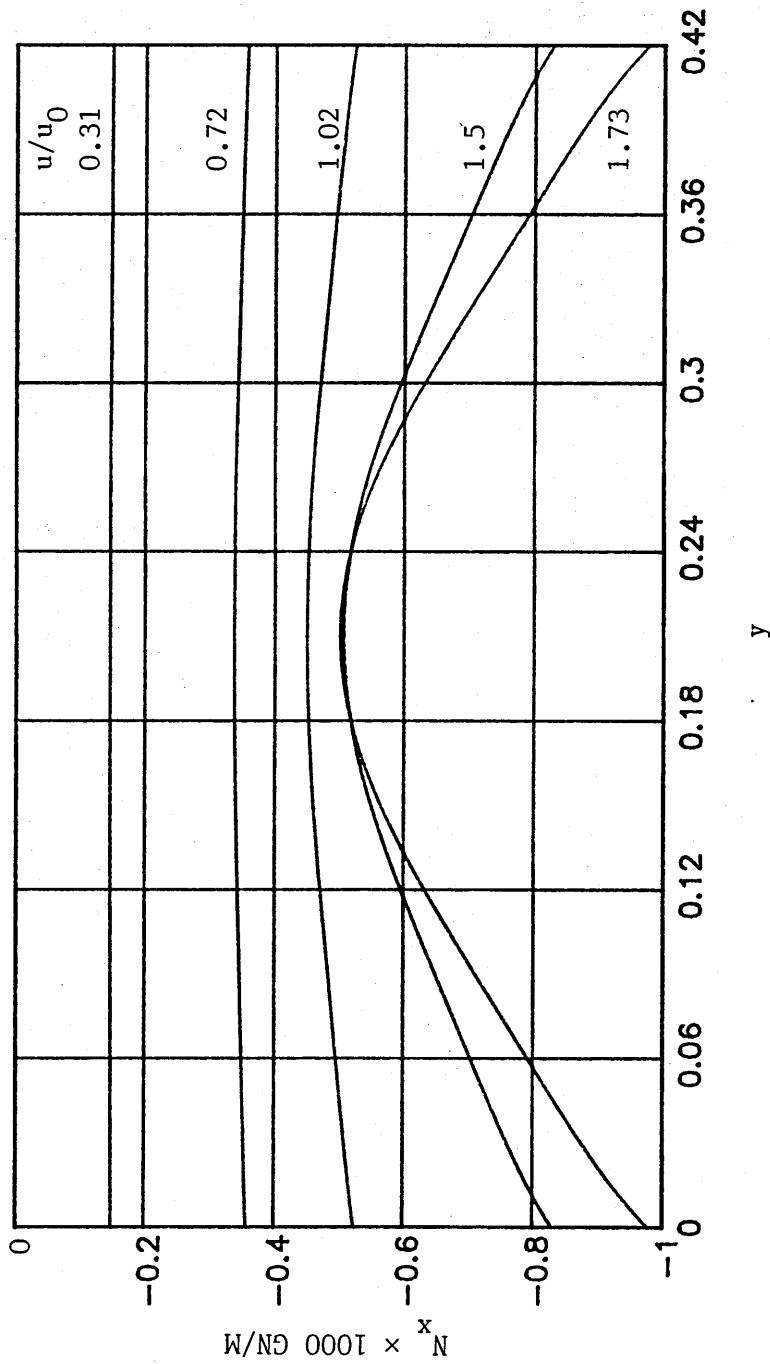


Fig 4.49 Variation of axial stress at mid span for panel T2

$$P_L/P_0 = 0.65, \quad x = b, \quad \xi_0 = 0.6, \quad \xi_L = 0.02$$

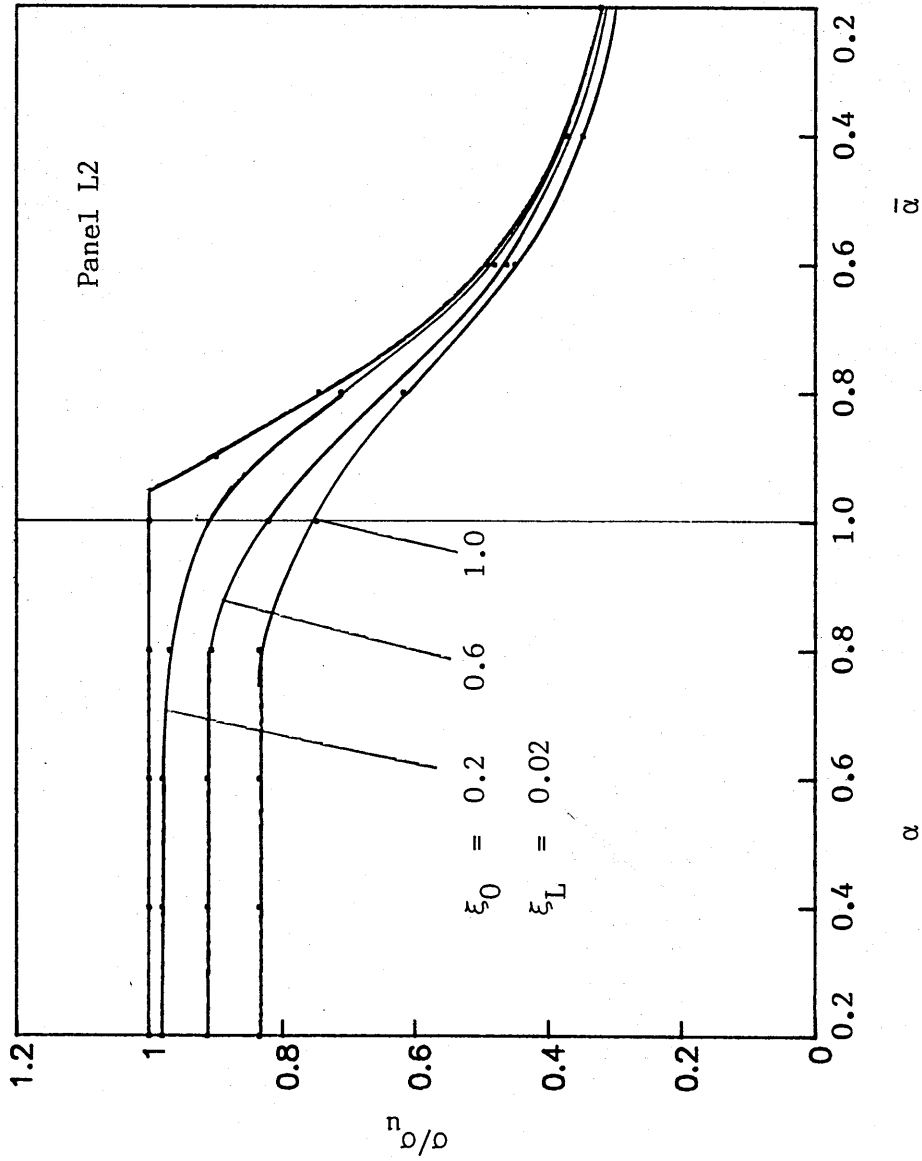


Fig 4.50 Variation of failure stress with Aspect Ratio α , $\bar{\alpha}$ for Panel L2

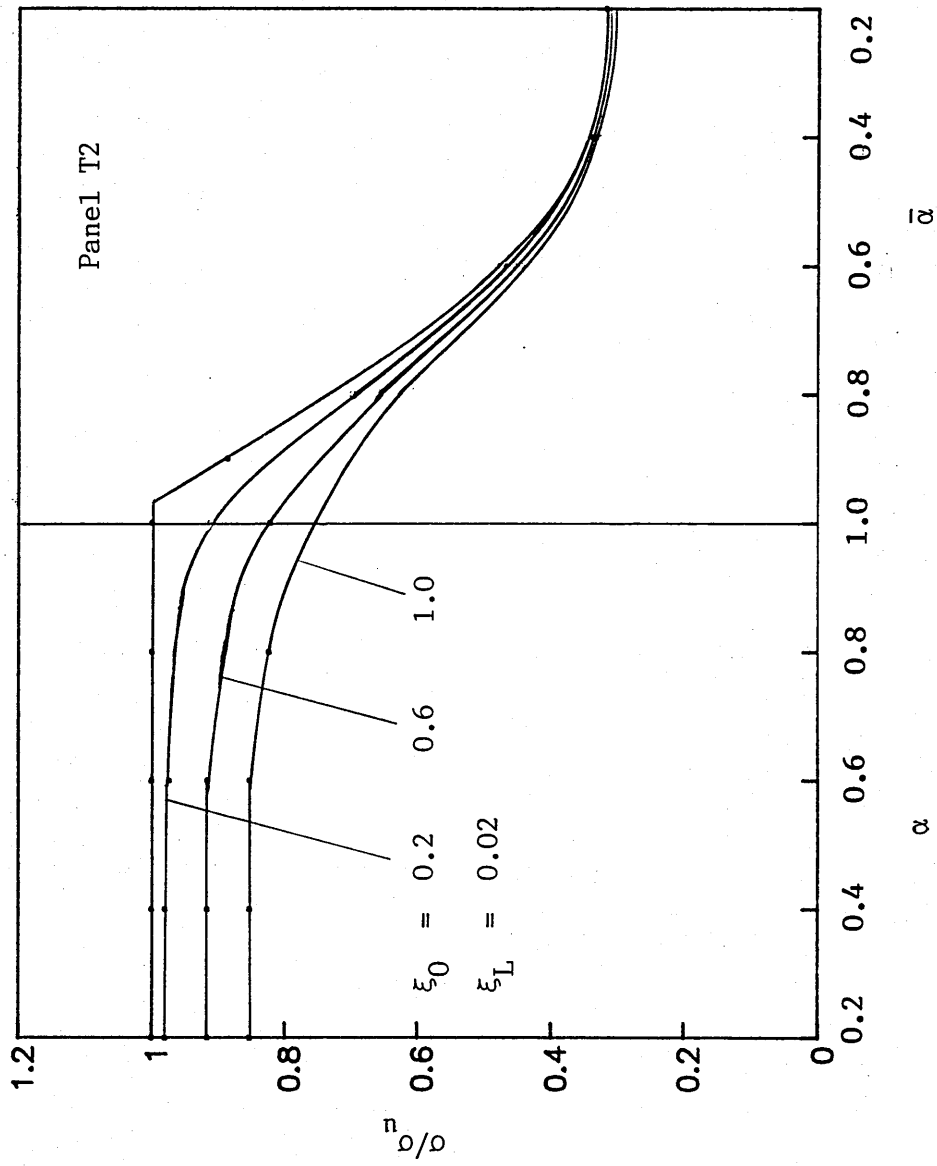


Fig 4.51 Variation of failure load with aspect ratio for panel T2

5 SUMMARY AND CONCLUSIONS

5.1 General Summary

An upper bound energy approach based upon variational principals was used to analyse the buckling behaviour of asymmetrically edge stiffened plates when subjected to end compression. The results obtained were used to point out deficiencies in existing models used to study the buckling behaviour of asymmetrically edge stiffened plates. The method was then extended to predict the buckling of infinitely wide integrally stiffened panels under uniaxial loading. The results obtained from this analysis were used to highlight the differences in the behavior of infinitely wide panels and panels of finite size by comparing with existing results.

An energy model based upon variational principals was then developed to study the buckling behaviour of simply supported panels of finite size, stiffened by stiffeners of non-deformable cross-section. The results obtained compared favourably with existing results. The model was then further developed to analyse the post-buckling behaviour of panels. The subsequent results were used to highlight the differences between existing panel models in which the overall mode is assumed to be that of a broad Euler column so that the transverse edges of the panel are free and that considered in this work where the transverse edges of the panel are assumed to be simply supported. It was shown that the two models lead to considerably different types of post-buckling behaviour.

The analytical model developed here limits itself to the elastic range only. Under these conditions the model is capable of predicting the complicated behaviour of panels and is restricted only by the accuracy in defining the buckled mode of the panel. The model is capable of predicting the behaviour of panels under the influence of imperfections.

5.2 Suggestions For Future Research

The work presented in this thesis highlights the fact that future research on the post-buckling behaviour of light gauge stiffened panels is essential. In particular, attention needs to be directed at the study of various boundary conditions along the panel edges which may exist

in practice as well as the study of the post-buckling behaviour of panels stiffened by stiffeners of deformable cross-section. A finite strip/element type of an approach is recommended for these problems. The effects of shear lag which is neglected in the present analysis should also be included in any future research. Laboratory validation of any theoretical approach is recommended wherever possible.

The above research should be carried out with the desirable aim in producing simple formula or charts to be used during the structural design phase.

5.3 Conclusions

(i) Edge Stiffened Plates

- 1 For asymmetrically edge stiffened plates, the inclusion of both lateral and flexural modes in any analysis is essential (unless one of the modes is physically suppressed) as the two modes are highly coupled.
- 2 Local buckling for edge stiffened plates with small flanges is generally induced by the web. Local buckling for edge stiffened plates with large flanges is induced by the flange at a lower load than for web initiated buckling. The optimum size of the flange (ie the largest) for which local buckling is web induced decreases with increasing thickness of the plate elements comprising the section provided the web and the flange are of equal thickness.
- 3 Interaction between the local and lateral modes which reduces the buckling load was seen not to affect the buckling behaviour of the cross-sectional geometries of edge stiffened plates considered here.

(ii) Stiffened Panels

- 1 The results presented in section 3.4.2 verify the conclusion reached by Williams et al [33] that a model of an infinitely wide panel can be justifiably used to predict the local and overall buckling loads for panels of finite width provided the panel is stiffened by a large number of stiffeners. Torsional buckling loads are over-predicted by the wide strut model.

However the overprediction is not as great as suggested by Williams et al [33] who assume the stiffeners to be of non-deformable cross-section.

- 2 For panels stiffened by thick stiffeners, L section stiffeners are more effective than T section stiffeners in resisting local buckling of the skin. This results from the higher torsional stiffness provided by L section stiffeners when compared with T section stiffeners of equal cross-sectional area and thickness.
- 3 The results presented in section 4.4.2 for the nonlinear analysis of stiffened panels indicate that the post-buckling behaviour of stiffened panels where the transverse edges are simply supported is essentially stable. This is contrary to the post-buckling behaviour of panels modelled as wide Euler struts.
- 4 The results presented in section 4.4.2 concerning the post-buckling behaviour of stiffened panels in the presence of imperfection verify the conclusion that panel designs with near coincident buckling loads are sensitive to imperfections [35].

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Appendix 1EDGE STIFFENED PLATES - DISPLACEMENT FIELDS

Edge stiffened plate - Lateral buckling mode

The lateral buckling mode shapes for various elements of an edge stiffened plate are given by equations 3.28 and 3.27. On satisfaction of the boundary conditions given by equations 3.28 and 3.29, equations 3.28 and 3.29 can be expressed in the form of equations 3.30 and 3.31. The constants F_n and G_n appearing in these equations are given by the following expressions:

$$F_{n1} = 1$$

$$F_{n2} = F_1$$

$$F_{n3} = F_{26X}$$

$$F_{n4} = F_{27X}$$

and

$$G_{n1} = F_{28X}$$

$$G_{n2} = F_{29X}$$

$$G_{n3} = F_{30X}$$

in which the constants F 's appearing on the right hand side of these expressions are given by the following equations:

$$F_1 = -\frac{1}{2} R_{an}$$

$$F_2 = (1/dw) (1 + 2 F_1)$$

$$F_3 = h_n (1/dw)$$

$$F3A = 3 (1/dw)$$

$$F4 = 2 F1 (1/dw)^2 - \mu (1 + F1)$$

$$F5 = (hn) (hn - 1) (1/dw)^2 - \mu$$

$$F5A = 6 (1/dw)^2 - \mu$$

$$F6 = df F2$$

$$F7 = df F3$$

$$F7A = df F3A$$

$$F8 = \frac{1}{2} (Dw/Df) df^2 (1/nf)$$

$$F9 = F4 F8$$

$$F10 = F5 F8$$

$$F10A = F5A F8$$

$$F11 = - \mu$$

$$F12 = 2 (1/df)^2 - \mu$$

$$F13 = (1/df)^2 (gn) (gn - 1) - \mu$$

$$F14 = \emptyset (1/df)$$

$$F15 = 2 \emptyset (1/df)$$

$$F16 = (1/df)^3 gn (gn - 1) (gn - 2) + \emptyset gn (1/df)$$

$$FY = - Df \emptyset (1/df) nf$$

$$FX1 = - Dw \emptyset (1/dw) (1 + 2 F1)$$

$$FX3 = -Dw (hn (hn - 1) (hn - 2) (1/dw)^3 + hn \varnothing (1/dw))$$

$$FA = FX1/FY$$

$$FB = FX2/FY$$

$$FC = FX3/FY$$

$$F17 = - F14/F16$$

$$F18 = - F15/F16$$

$$F19 = F11 + F13 F17$$

$$F20 = F12 + F13 F18$$

$$F21A = F6 F19 + F9 F20$$

$$F21B = F7 F19 + F10 F20$$

$$F21C = F7A F19 + F10A F20$$

$$F21 = - F21A/F21B$$

$$F21X = - F21C/F21B$$

$$F22 = F6 + F21 F7$$

$$F22X = F7A + F21X F7$$

$$F23 = F9 + F10 F21$$

$$F24X = F17 F22 + F18 F23$$

$$F25X = FB + FC F21X - F22X$$

$$F_{25Y} = F_{22} - F_A - F_C F_{21}$$

$$F_{26X} = F_{25Y}/F_{25X}$$

$$F_{27X} = F_{21} + F_{21X} F_{26X}$$

$$F_{28X} = F_{22} + F_{22X} F_{26X}$$

$$F_{29X} = F_{23} + F_{23X} F_{26X}$$

$$F_{30X} = F_{24} + F_{24X} F_{26X}$$

in which

$$\varnothing = (v - 2) \left(\frac{\pi}{2b} \right)^2$$

$$\mu = v \left(\frac{\pi}{2b} \right)^2$$

Edge stiffened plates - Local Buckling mode

The displacement functions associated with the local buckling mode for the various elements of an edge stiffened plate are given by equations 3.33. On satisfaction of the boundary conditions (given by equations 3.21 - 3.29), equations 3.33 can be expressed in the form of equations 3.34 and 3.35. The constants H_m appearing in these equations are given by the following equations:

$$H_{m1} = 1$$

$$H_{m2} = CA$$

$$H_{m3} = C18X$$

$$H_{m4} = C19X$$

and

$$J_{m1} = C20X$$

$$J_{m2} = C21X$$

$$J_{m3} = C22X$$

$$J_{m4} = C23X$$

where the constants C 's are given by the following expressions:

$$CA = -\frac{1}{2} R_{bm}$$

$$CG = - (1 + CA)$$

$$C1 = (1/dw) (1 + 2 CA + hm CG)$$

$$C1X = (1/dw) (3 - hm)$$

$$C2 = df C1$$

$$C2X = df \ C1X$$

$$C3 = (1/dw)^2 (2 CA + (hm)(hm - 1) CG)$$

$$C3X = (1/dw)^2 (6 - hm(hm - 1))$$

$$C4 = \frac{1}{2} (Dw/Df) df^2 C3 (1/nf)$$

$$C4X = \frac{1}{2} (Dw/Df) df^2 C3X (1/nf)$$

$$C5 = (1/df)^2 2 C4$$

$$C5X = (1/df)^2 2 C4X$$

$$C6 = 6 (1/df)^2$$

$$C7 = (1/df)^2 jm (jm - 1)$$

$$C8 = C5 - \mu (C2 + C4)$$

$$C8X = C5X - \mu (C2X + C4X)$$

$$C9 = C6 - \mu$$

$$C10 = C7 - \mu$$

$$C11 = (1/df) \oslash (C2 + 2 C4)$$

$$C11X = (1/df) \oslash (C2X + 2 C4X)$$

$$C12 = 6 (1/df)^3 + 3 (1/df) \oslash$$

$$C13 = (1/df)^3 (jm)(jm - 1)(jm - 2) + (1/df) \oslash jm$$

$$CXX1 = hm(hm - 1)(hm - 2)(1/dw)^3 CG + \oslash (1/dw) \\ (1 + 2 CA + hm CG)$$

$$CXX2 = (1/dw)^3 (6 - hm(hm - 1)(hm - 2)) + \emptyset (1/dw) (3 - hm)$$

$$CYY1 = 6 (1/df)^3$$

$$CYY2 = \emptyset (1/df)$$

$$CXY1 = nf(Df/Dw) CYY1$$

$$CXY2 = nf(Df/Dw) CYY2$$

$$C14A = C11 - C12 C8/C9$$

$$C14AX = C11X - C12 C8X/C9$$

$$C14B = C13 - C12 C10/C9$$

$$C14 = - C14A/C14B$$

$$C14X = - C14AX/C14B$$

$$C15 = - (C8/C9 + C10 C14/C9)$$

$$C15X = - (C8X/C9 + C10 C14X/C9)$$

$$C16X = C15X CXY1 + C2X CXY2 - CXX2$$

$$C17X = CXX1 - C15 CXY1 - C2 CXY2$$

$$C18X = C17X/C16X$$

$$C19X = CG - C18X$$

$$C20X = C2 + C2X C18X$$

$$C21X = C4 + C4X C18X$$

$$C22X = C15 + C15X C18X$$

$$C_{23X} = C_{14} + C_{14X} C_{18X}$$

in which

$$\varnothing = (v - 2) \left(\frac{\pi}{2b} \right)^2$$

$$\mu = v \left(\frac{\pi}{2b} \right)^2$$

Appendix 2STIFFENED PANEL - DISPLACEMENT FIELDS

Skin displacement field - Lateral Mode:

The displacement field associated with the lateral mode for the skin of a stiffened panel is given by equation 3.64. On satisfaction of the appropriate boundary conditions given by equations 3.57-3.63, equation 3.64 may be expressed in the form of equation 3.67. The variables Kn appearing in this expression are given by the following equations:

$$Kn1 = 1$$

$$Kn2 = H1$$

$$Kn3 = - 6 D_w F_{26X1} (1/dw) \phi/ds - H_{XA1}/H_{20E}$$

$$Kn4 = H18 + H39 H18X + H43 H18y$$

$$Kn5 = H19 + H39 H19X + H43 H19y$$

$$Kn6 = H32 H42 + H40$$

in which the variables H are given by the following expressions:

$$H1 = - \frac{1}{2} R_{2n}$$

$$H2 = - \mu$$

$$H3 = 2 (1/ds)^2 - \mu$$

$$H4 = a_n (a_n - 1) (1/ds)^2 - \mu$$

$$H5 = b_n (b_n - 1) (1/ds)^2 - \mu$$

$$H5X = 6.0 (1/ds)^2 - \mu$$

$$H5Y = c_n (c_n - 1) (1/ds)^2 - \mu$$

$$H6 = 1/ds$$

$$H7 = 2/ds$$

$$H8 = an/ds$$

$$H9 = bn/ds$$

$$H9X = 3.0/ds$$

$$H9Y = cn/ds$$

$$H10 = H2 + (R3n/ds) H6$$

$$H11 = H3 + (R3n/ds) H7$$

$$H12 = H4 + (R3n/ds) H8$$

$$H13 = H5 + (R3n/ds) H9$$

$$H13X = H5X + (R3n/ds) H9X$$

$$H13Y = H5Y + (R3n/ds) H9Y$$

$$H14 = 1 + H1$$

$$H15 = H10 + H1 H11$$

$$H16 = 1 - H12/H13$$

$$H17 = H15/H13 - H14$$

$$H17X = H13X/H13 - 1$$

$$H17Y = H13Y/H13 - 1$$

$$H18 = H17/H16$$

$$H18X = H17X/H16$$

$$H18Y = H17Y/H16$$

$$H19 = - H15/H13 - H18 H12/H13$$

$$H19X = - H13X/H13 - H18X H12/H13$$

$$H19Y = - H13Y/H13 - H18Y H12/H13$$

$$H20A = a_n (a_n - 1) (a_n - 2) H18 (1/ds)^{3.0}$$

$$H21A = b_n (b_n - 1) (b_n - 2) H19 (1/ds)^{3.0}$$

$$H20E = 6.0 (1/ds)^{3.0}$$

$$H21E = a_n (a_n - 1) (a_n - 2) H18X (1/ds)^{3.0}$$

$$H22E = b_n (b_n - 1) (b_n - 2) H19X (1/ds)^{3.0}$$

$$H20F = c_n (c_n - 1) (c_n - 2) (1/ds)^{3.0}$$

$$H21F = a_n (a_n - 1) (a_n - 2) H18Y (1/ds)^{3.0}$$

$$H22F = b_n (b_n - 1) (b_n - 2) H19Y (1/ds)^{3.0}$$

$$HXA1 = \emptyset/ds$$

$$HXA2 = 2 \emptyset H1/ds$$

$$HXA3 = a_n \emptyset H18/ds$$

$$HXA4 = b_n \emptyset H19/ds$$

$$HXE1 = 3.0 \emptyset/ds$$

$$HXE2 = a_n \emptyset H18X/ds$$

$$\text{HXE3} = \text{bn } \emptyset \text{ H19X/ds}$$

$$\text{HXF1} = \text{cn } \emptyset/\text{ds}$$

$$\text{HXF2} = \text{an } \emptyset \text{ H18Y/ds}$$

$$\text{HXF3} = \text{bn } \emptyset \text{ H19Y/ds}$$

$$\text{H25} = \text{H20A} + \text{H21A} + \text{HXA1} + \text{HXA2} + \text{HXA3} + \text{HXA4}$$

$$\text{H26} = \text{H20E} + \text{H21E} + \text{H22E} + \text{HXE1} + \text{HXE2} + \text{HXE3}$$

$$\text{H27} = \text{H20F} + \text{H21F} + \text{H22F} + \text{HXF1} + \text{HXF2} + \text{HXF3}$$

$$\text{H28} = 6.0 \text{ F26X1 } (1/\text{dw})^{3.0} + \emptyset/\text{dw}$$

$$\text{H29} = - \text{Dw H28}/(2 \text{ Ds H20E})$$

$$\text{H30} = - \text{HXA1}/\text{H20E}$$

$$\text{H31} = - \text{Dw } (6.0 \text{ F26X2 } (1/\text{dw})^{3.0} + \emptyset/\text{dw})$$

$$\text{H32} = - \text{H31}/(2 \text{ Ds H27})$$

$$\text{H33} = - \text{H25}/\text{H27}$$

$$\text{H34} = - \text{H26}/\text{H27}$$

$$\text{H35} = \text{dw/ds}$$

$$\text{H36} = \text{dw/ds } (1 + 2 \text{ H1} + \text{an H18} + \text{bn H19})$$

$$\text{H37} = \text{dw/ds } (3.0 + \text{an H18X} + \text{bn H19X})$$

$$\text{H38} = \text{dw/ds } (\text{cn} + \text{an H18Y} + \text{bn H19Y})$$

$$\text{H39} = \text{H29 H35} + \text{H30}$$

$$H40 = H33 + H34 H39$$

$$H41 = H36 + H37 H39$$

$$H42 = (H41 + H38 H40) / (1 - H32 H38)$$

$$H43 = H32 H42 + H40$$

$$H44 = H18 + H39 H18X + H43 H18Y$$

$$H45 = H19 + H39 H19X + H43 H19Y$$

in which

$$\varnothing = (v - 2) \left(\frac{\pi}{2b} \right)^2$$

$$\mu = v \left(\frac{\pi}{2b} \right)^2$$

Skin displacement field - Local mode

The displacement field associated with the local mode of buckling for the skin of stiffened panel is given by equation 3.70. On satisfaction of the appropriate boundary conditions, equation 3.70 can be expressed in the form of equation 3.73. The constants I_m appearing in this equation are given by the following equations:

$$I_{m1} = 1$$

$$I_{m2} = G_1$$

$$I_{m3} = G_{50}$$

$$I_{m4} = G_{55}$$

$$I_{m5} = G_{56}$$

$$I_{m6} = G_{57}$$

$$I_{m7} = G_{54}$$

In which the constants G are given by the following equations:

$$G_A = d_m (0.5)^{(d_m - 1)}$$

$$G_B = e_m (0.5)^{(e_m - 1)}$$

$$G_C = k_m (0.5)^{(k_m - 1)}$$

$$G_D = 3 (0.5)^2$$

$$G_E = l_m (0.5)^{(l_m - 1)}$$

$$G_1 = - 0.5 ds (R_{2m} / (ds))$$

$$G2 = -\mu$$

$$G3 = 2 \left(1/ds \right)^2 - \mu$$

$$G4 = (dm)(dm - 1) \left(1/ds \right)^2 - \mu$$

$$G5 = (em)(em - 1) \left(1/ds \right)^2 - \mu$$

$$G6 = (km)(km - 1) \left(1/ds \right)^2 - \mu$$

$$G6X = 6 \left(1/ds \right)^2 - \mu$$

$$G6Y = lm(lm - 1) \left(1/ds \right)^2 - \mu$$

$$G7 = 1/ds$$

$$G8 = 2/ds$$

$$G9 = dm/ds$$

$$G10 = em/ds$$

$$G11 = km/ds$$

$$G11X = 3/ds$$

$$G11Y = lm/ds$$

$$G12 = G2 + (R3m/ds) G7$$

$$G13 = G3 + (R3m/ds) G8$$

$$G14 = G4 + (R3m/ds) G9$$

$$G15 = G5 + (R3m/ds) G10$$

$$G16 = G6 + (R3m/ds) G11$$

$$G16X = G6X + (R3m/ds) G11X$$

$$G16Y = G6Y + (R3m/ds) G11Y$$

$$G17 = 1 + G1$$

$$G18 = G12 + G1 G13$$

$$G19 = G17 - GA G17$$

$$G20 = GB - GA$$

$$G21 = GC - GA$$

$$G21X = GD - GA$$

$$G21Y = GE - GA$$

$$G22 = G18 - G14 G17$$

$$G23 = G15 - G14$$

$$G24 = G16 - G14$$

$$G24X = G16X - G14$$

$$G24Y = G16Y - G14$$

$$G25 = - G19/G20$$

$$G26 = - G21/G20$$

$$G26X = - G21X/G20$$

$$G26Y = - G21Y/G20$$

$$G27 = - G22 - G23 G25$$

$$G28 = G24 + G23 G26$$

$$G28X = - G24X - G23 G26X$$

$$G28Y = - G24Y - G23 G26Y$$

$$G29 = G27/G28$$

$$G29X = G28X/G28$$

$$G29Y = G28Y/G28$$

$$G30 = G25 + G26 G29$$

$$G30X = G26X + G26 G29X$$

$$G30Y = G26Y + G26 G29Y$$

$$G31 = - G17 - G30 - G29$$

$$G31X = - 1 - G30X - G29X$$

$$G31Y = - 1 - G30Y - G29Y$$

$$G32A = dm (dm - 1) (dm - 2) G31 (1/ds)^3$$

$$G33A = em (em - 1) (em - 2) G30 (1/ds)^3$$

$$G34A = km (km - 1) (km - 2) G29 (1/ds)^3$$

$$G32F = 6 (1/ds)^3$$

$$G33F = dm (dm - 1) (dm - 2) G31X (1/ds)^3$$

$$G34F = em (em - 1) (em - 2) G30X (1/ds)^3$$

$$G35F = km (km - 1) (km - 2) G29X (1/ds)^3$$

$$G32H = lm (lm - 1) (lm - 2) (1/ds)^3$$

$$G33H = dm (dm - 1) (dm - 2) G31Y (1/ds)^3$$

$$G34H = em (em - 1) (em - 2) G30Y (1/ds)^3$$

$$G35H = km (km - 1) (km - 2) G29Y (1/ds)^3$$

$$GXA1 = \emptyset/ds$$

$$GXA2 = 2 \emptyset G1/ds$$

$$GXA3 = \emptyset dm G31/ds$$

$$GXA4 = \emptyset em G30/ds$$

$$GXA5 = \emptyset km G29/ds$$

$$GXF1 = 3 \emptyset/ds$$

$$GXF2 = \emptyset dm G31X/ds$$

$$GXF3 = \emptyset em G30X/ds$$

$$GXF4 = \emptyset km G29X/ds$$

$$GXH1 = \emptyset lm/ds$$

$$GXH2 = \emptyset dm G31Y/ds$$

$$GXH3 = \emptyset em G30Y/ds$$

$$GXH4 = \emptyset km G29Y/ds$$

$$G36 = G32A + G33A + G34A + GXA1 + GXA2 + GXA3 + GXA4 + GXA5$$

$$G37 = G32F + G33F + G34F + G35F + GXF1 + GXF2 + GXF3 + GXF4$$

$$G38 = G32H + G33H + G34H + G35H + GXH1 + GXH2 + GXH3 + GXH4$$

$$G39 = 6 C18X1 (1/dw)^3 + \emptyset/dw$$

$$G40 = - Dw G39 / (2 Ds G32F)$$

$$G41 = - GXA1/G32F$$

$$G42 = - Dw (6 C18X2 (1/dw)^3 + \emptyset/dw)$$

$$G43 = - G42 / (2 Ds G38)$$

$$G44 = - G36/G38$$

$$G45 = - G37/G38$$

$$G46 = dw/ds$$

$$G47 = (dw/ds) (1 + 2 G1 + dm G31 + em G30 + km G29)$$

$$G48 = (dw/ds) (3 + dm G31X + em G30X + km G29X)$$

$$G49 = (dw/ds) (lm + dm G31Y + em G30Y + km G29Y)$$

$$G50 = G40 G46 + G41$$

$$G51 = G44 + G45 G50$$

$$G52 = G47 + G48 G50$$

$$G53 = (G52 + G49 Q14) / (1 - G43 G49)$$

$$G54 = G43 G53 + Q14$$

$$G55 = G31 + G50 G31X + G54 G31Y$$

$$G56 = G30 + G50 G30X + G54 G30Y$$

$$G57 = G29 + G50 \ G29X + G54 \ G29Y$$

in which

$$\mu = v \left(\frac{\pi}{2b} \right)^2$$

$$\varnothing = (v - 2) \left(\frac{\pi}{2b} \right)^2$$

Skin displacement field w_{c4}

The displacement field for the skin associated with the alternating up and down stiffener mode shown in fig(3.4)c is given by equation 3.85. On satisfaction of boundry conditions, equation 3.85 can be expressed in the form of equation 3.86. The constants U_r appearing in these equations are given by the following equations:

$$U_{r1} = 1$$

$$U_{r2} = G_1 + G_2 G_5$$

$$U_{r3} = G_5$$

in which the constants G are given by the following equations:

$$G_1 = -g_r/h_r$$

$$G_2 = -j_r/h_r$$

$$G_3 = g_r(g_{r-1}) + G_1 h_r(h_{r-1})$$

$$G_4 = j_r(j_{r-1}) + G_2 h_r(h_{r-1})$$

$$G_5 = -G_3/G_4$$

Appendix 3

LOCATION OF NEUTRAL AXIS

The following assumptions were used in section 4.1.2 to derive explicit expressions for the location of neutral axis $\bar{y}'_w$, $\bar{y}'_f$ for the web and the flange of an asymmetrically edge stiffened plate.

- (a) The net axial load due to the stress distribution is zero.
- (b) The moment about any axis normal to the plane of the web is zero.

Consider now the stress distribution in Fig 3.13

Applying the first assumption, we get:

$$\frac{N_x}{z} \left\{ \frac{(\bar{y}'_w)^2}{(d_w - \bar{y}'_w)} - (d_w - h) - \bar{y}'_f + \frac{1}{\bar{y}'_f} (d - \bar{y}'_f (d_f - \bar{y}'_f)) \right\} = 0 \quad \dots (A3.1)$$

where N_x is the value of the stress distribution at the web flange junction.

Applying the second assumption and taking moments about the flange web junction, gives:

$$\frac{N_x}{z} \left\{ \frac{(\bar{y}'_w)^2}{(d_w - \bar{y}'_w)} (d_w - \frac{1}{3} \bar{y}'_w) - \frac{1}{3} (d_w - \bar{y}'_w)^2 \right\} = 0 \quad (A3.2)$$

From equation (A3.2) we get:

$$\bar{y}'_w = \frac{1}{3} d_w \quad (A3.3)$$

Substituting this into equation (A3.2) and solving for $\bar{y}'_f$ gives:

$$\bar{y}'_f = 2d_f^2 / (d_w - 4d_f) \quad (A3.4)$$

Appendix 4DEFINITION OF MATRICES

The matrices $[K_1]$, $[K_2]$, $[K_3]$, $[E_1]$ and $[E_2]$ which appear in equation (4.104) are as follows:

The non-zero terms in the symmetric matrix $[K_1]$ are as follows:

$$K_{1,11} = E(I_z + I_y \theta_s^2) + GJw_{sx}^2$$

$$K_{1,12} = E(I_z - I_y) \theta_s - GJw_{sx} v_{sx}$$

$$K_{1,13} = E[I_z w_{sxx} + I_y \{(2v_{sxx} \theta_s - v_{0sxx} \theta_{0s}) - (w_{sxx} - w_{0sxx})\}]$$

$$K_{1,14} = GJw_{sx}$$

$$K_{1,15} = -GJw_{sx} w_{sxx}$$

$$K_{1,16} = GJ[(\theta_{sx} - \theta_{0sx}) - (w_{sxx} v_{sx} - w_{0sxx} v_{0sx}) \\ + (2w_{sx} v_{sxx} - w_{0sx} v_{0sxx})]$$

$$K_{1,22} = E[I_z \theta_s^2 + I_y] + GJv_{sx}^2$$

$$K_{1,23} = E[I_z \{(2w_{sxx} \theta_s - w_{0sxx} \theta_s) + (v_{sxx} - v_{0sxx}) - I_y v_{sxx}\}]$$

$$K_{1,24} = -GJv_{sx}$$

$$K_{1,25} = -GJ[(\theta_{sx} - \theta_{0sx}) - (2v_{sx} w_{sxx} - v_{0sx} w_{0sxx}) \\ + (v_{sxx} w_{sx} - v_{0sxx} w_{0sx})]$$

$$K_{1,26} = -GJv_{sx} v_{sxx}$$

$$K_{1,33} = E[I_z w_{sxx}^2 + I_y v_{sxx}^2]$$

$$K_{1,44} = GJ$$

... (A4.1) (continued)

$$\begin{aligned}
K_{1,45} &= -GJw_{sxx} \\
K_{1,46} &= GJv_{sxx} \\
K_{1,55} &= w_{sxx}^2 \\
K_{1,56} &= -GJw_{sxx}v_{sxx} \\
K_{1,66} &= GJv_{sxx}^2
\end{aligned} \tag{A4.1}$$

The terms in the symmetric matrix $[K_2]$ are as follows:

$$\begin{aligned}
K_{2,11} &= A \\
K_{2,12} &= Av_{sx} \\
K_{2,13} &= Aw_{sx} \\
K_{2,14} &= I_0\theta_{sx} \\
K_{2,22} &= A\{u_{sx} + (\frac{3}{2}v_{sx}^2 - \frac{1}{2}v_{0sx}^2) + \frac{1}{2}(w_{sx}^2 - w_{0sx}^2) \\
&\quad + \frac{1}{2}I_0(\theta_{sx}^2 - \theta_{0sx}^2)\} \\
K_{2,23} &= Av_{sx}w_{sx} \\
K_{2,24} &= I_0v_{sx}\theta_{sx} \\
K_{2,33} &= A\{u_{sx} + \frac{1}{2}(v_{sx}^2 - v_{0sx}^2) + (\frac{3}{2}w_{sx}^2 - w_{0sx}^2) \\
&\quad + \frac{1}{2}I_0(\theta_{sx}^2 - \theta_{0sx}^2)\} \\
K_{2,34} &= I_0w_{sx}\theta_{sx} \\
K_{2,44} &= U_{sx} + \frac{1}{2}(v_{sx}^2 - v_{0sx}^2) + \frac{1}{2}(w_{sx}^2 - w_{0sx}^2) + I_p(\frac{3}{2}\theta_{sx}^2 - \theta_{0sx}^2) \\
&\quad \dots \tag{A4.2}
\end{aligned}$$

The terms in matrix $[E_1]$ are as follows:

$$\begin{aligned}
 E_{1,1} &= E(B_1 - B_2 \theta_s) + GJB_3 w_{sxx} \\
 E_{1,2} &= E(B_1 \theta_s + B_2) - GJB_3 v_{sx} \\
 E_{1,3} &= E(B_1 w_{sxx} - B_2 v_{sxx}) \\
 E_{1,4} &= GJB_3 \\
 E_{1,5} &= -GJB_3 w_{sxx} \\
 E_{1,6} &= GJB_3 v_{sxx}
 \end{aligned} \tag{A4.3}$$

where

$$\begin{aligned}
 B_1 &= I_z \{ (v_{sxx} - v_{0sxx}) + (w_{sxx} \theta_s - w_{0sxx} \theta_{0s}) \} \\
 B_2 &= I_y \{ (w_{sxx} - w_{0sxx}) - (v_{sxx} \theta_s - v_{0sxx} \theta_{0s}) \} \\
 B_3 &= \{ (\theta_{sx} - \theta_{0sx}) - (w_{sxx} v_{sx} - w_{0sxx} v_{0sx}) + (v_{sxx} w_{sx} - v_{0sxx} w_{0sx}) \}
 \end{aligned}$$

The terms in matrix $[E_2]$ are as follows:

$$\begin{aligned}
 E_{2,1} &= E\{AB_1 + I_0 B_2\} \\
 E_{2,2} &= E\{AB_1 + I_0 B_2\} v_{sx} \\
 E_{2,3} &= E\{AB_1 + I_0 B_2\} w_{sx} \\
 E_{2,4} &= E\{I_0 B_1 + I_p B_2\} \theta_{sx}
 \end{aligned} \tag{A4.4}$$

where

$$\begin{aligned}
 B_1 &= u_{sx} + \frac{1}{2} (v_{sx}^2 - v_{0sx}^2) - \frac{1}{2} (w_{sx}^2 - w_{0sx}^2) \\
 B_2 &= \frac{1}{2} (\theta_{sx}^2 - \theta_{0sx}^2)
 \end{aligned} \tag{A4.5}$$