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THE PRESSURE FIELD ASSOCIATED WITH A
SPHERICAL DIFFUSION FLAME

by

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Summary

The pressure field associated with a spherically symmetric diffusion flame is studied. The flame itself is described using the Burke-Schumann infinitely-thin-flame-sheet model and the associated pressure field is analysed using the method of inner and outer expansions. Rapid pressure variations are found to be present in the regions close to the flame sheet.

LIST OF SYMBOLS

C_α	specific heat at constant pressure of species α
c_α	mass fraction of species α
$\mathcal{D}$	diffusion coefficient
h_α	enthalpy per unit mass of species α
M	Mach number
$4\pi\dot{m}$	constant mass flux injected into region $r \geq r_i$
p	pressure ($\bar{p} \equiv p-1$)
R	universal gas constant
r	radial co-ordinate
T	temperature
δ	'equilibrium-constant' parameter
ε	inverse Damköhler number
ν_α	stoichiometric integer of species
ρ	mixture density

1. Introduction

The work reported here records an attempt to increase the understanding of the properties of the flow fields associated with diffusion flames. In this context a diffusion flame is considered to be a region of intense chemical reaction caused by the diffusive mixing of two initially separated fluids (e.g. a Bunsen flame with the air-hole of the Bunsen burner closed), in general a fuel (F) and an oxidant (X). The analysis here is restricted to the consideration of laminar flows and situations in which all of the reactants are in the gaseous phase.

The simplest theoretical description of a diffusion flame is that proposed by Burke and Schumann (1928). Since certain properties of the flow fields predicted using this model are of importance in the calculations to follow, it is instructive to take a close look at the Burke-Schumann model and its implications. To achieve this end Clarke's (1972) analysis of the Burke-Schumann model viewed in terms of singular perturbation theory is utilised.

2. The Burke-Schumann Model of a Diffusion Flame

In the Burke-Schumann model of a diffusion flame the flame is considered to be a surface into which the reactants

diffuse and are instantly consumed by reaction to form the product species. It will now be shown that this model can be viewed as the first approximation to the full solution of the species field problem in the limit of the reaction proceeding irreversibly at a very rapid rate.

In equation (1) the species continuity equations are given in dimensionless form for the fuel (F) and oxidant (X) gases which are presumed to be participating in the elementary reversible reaction whose stoichiometric reaction is given in (2).

$$\begin{aligned} e \, dv \left\{ \rho \frac{\gamma_\alpha}{v_\alpha} (\underline{u} + \underline{u}_\alpha) \right\} &= - \left(\frac{\rho}{T_F} \right) \left\{ C_F^{\nu_F} C_X^{\nu_X} - (1 - \alpha - C_F)^{\nu_P} \delta \right\} \\ &= \left(\frac{\rho}{T_F} \right) R \quad \alpha = X, F \end{aligned} \quad (1)$$



where

$$\gamma_\alpha = C_\alpha / W_\alpha, \quad E = T_{\infty}' U' / L', \quad \delta = T_F' / T_r' \quad (3)$$

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The primed quantities are all dimensional whereas all other variables are dimensionless. Equation (1) has been obtained from its dimensional form by introducing the characteristic dimensional velocity U' , length L' , density ρ' and chemical time τ'_c . u is the dimensionless mass-average velocity and u_α is the dimensionless diffusion velocity for a particular chemical species α . The quantities ν_α are stoichiometric integers and X, F and P represent the oxidant, fuel and product species respectively. τ'_f is the chemical time typifying the forward chemical reaction rate and τ'_r typifies the reverse reaction. The chemical time is simply related to the reciprocal of the more familiar specific rate constant.

C_α and W_α are respectively the mass fraction and the molecular weight of species α . R is the dimensionless rate of progress of the reaction. In terms of this analysis the most important feature of equation (1) is the appearance of the two dimensionless parameters ϵ and δ . From equation (3) and the relationship between the chemical time and specific rate constant it is clear that the parameter δ is closely related to the equilibrium constant for the mass fractional composition. As such it must be a very small number indeed if species X and F are to form a combustible pair.

The second parameter ϵ is the ratio of a typical chemical time (in this case a time typical of the forward, burning reaction) to some other time L'/U' which typifies some other important feature of the system. For a diffusion flame it is appropriate to assume that L'/U' is a time typical of a diffusion process so that U' measures a diffusion velocity and L' measures a diffusion-layer thickness. If we now assume that diffusion is a much slower process than reaction then we are studying a diffusion-dominated process and such a process is defined by the requirement that $\epsilon \ll 1$.

Since both δ and ϵ are very small for a diffusion-dominated combustion process such as we assume holds in a diffusion flame a plausible first approximation to the solution of the species equations may well be obtained by taking the limiting (zero) values of the small parameters. This limit of vanishing ϵ and δ is taken with ζ fixed, when, by implication, all of the dependent variables and their derivatives will also be fixed. The result of applying this limiting process to equation (1) is

$$0 = C_F^{(0)} v_F - C_X^{(0)} v_X \quad (4)$$

where $C_\alpha^{(0)}$ is the first approximation to C_α . The very

simple equation (4) has an equally elementary non-trivial solution given below;

$$C_X^{(0)} = 0, C_F^{(0)} \neq 0 \quad \text{in } \mathcal{D}_F \quad (5a)$$

$$C_X^{(0)} \neq 0, C_F^{(0)} = 0 \quad \text{in } \mathcal{D}_X \quad (5b)$$

The domains $\mathcal{D}_F$ and $\mathcal{D}_X$ must not overlap but they will be separated by a surface S , which will later be shown to be an infinitely thin flame sheet. It is now useful to define a co-ordinate n which is both normal to S and has the value zero on S . For physical reasons continuity of mass fractions is required throughout the union of $\mathcal{D}_X$ and $\mathcal{D}_F$. The additions of this physical requirement at the surface S to the solutions given in equation (5) produces two important results. Firstly there are boundary conditions on both $C_X^{(0)}$ and $C_F^{(0)}$ as $n \rightarrow 0$, which say that both mass fractions vanish on S . Secondly, although $C_X^{(0)}$ is continuous its derivative with respect to n is discontinuous; the derivative is zero as $n \rightarrow 0$ in $\mathcal{D}_F$ and it is non-zero as $n \rightarrow 0$ in $\mathcal{D}_X$. The inference is that the second derivative of $C_X^{(0)}$ with respect to n is unbounded on S .

This lack of boundedness is extremely important in the analysis of the species continuity equations, since those equations involve terms of the type ϵ times $\text{div } \rho c_\alpha u_\alpha$. If Fick's Law of Mass Diffusion is used to model the diffusive mass flux of species α then $c_\alpha u_\alpha$ is proportional to the gradient of c_α and thus it follows that there must be a term appearing in the species continuity equations which is proportional to ϵ times the second derivative of c_α with respect to n . Thus the limiting parameter approximation is attempting to model this term as 'zero times infinity'. The indeterminacy of such a product means that the use of the limit of zero ϵ and zero δ with ζ fixed if ζ is on the surface S must not be allowed, and thus the approximation is not uniformly valid throughout the region. This fact signifies that the small parameter approximation applied to the problem at hand involves the use of singular, rather than regular, perturbation techniques.

If the surface S is taken to be a flame sheet then the nature of the solutions for $c_\alpha^{(1)}$ makes it plain that at the level of approximation there will be no reaction above or below the flame sheet but that the reaction must be confined to within the flame sheet itself. In this sense it is clear

that this is indeed an outline of the Burke-Schumann model of the diffusion flame. Moreover Clarke (1972) has shown by further analysis that the reactants diffuse into the flame sheet at molar rates which are in stoichiometric proportion, this being a further condition in the Burke-Schumann model.

3. Motivation

The prime objective in this work is to increase the understanding of the flow fields associated with diffusion flames. However the motivation for the work stemmed from something much more specific - the discovery that a widely accepted statement concerning discontinuous variables at the flame sheet was in fact false. Throughout this work all flows considered will be assumed to be steady. It has been shown quite generally that in the Burke-Schumann flame sheet model the normal gradients of the mass fractions C_α of the reacting species are discontinuous across the flame sheet. Then, by reference to the energy equation, it can be shown that the discontinuities in the normal gradients of the C_α imply that there is a discontinuity

in the normal temperature gradient across the sheet. These facts concerning the behaviour of the temperature and mass fraction gradients in the Burke-Schumann flame sheet model are widely known and accepted.

Unfortunately the argument then follows that since $\rho T = 1$, an approximation frequently used in combustion problems, a discontinuity in the normal gradient of temperature across the sheet must imply a similar discontinuity in the normal gradient of the density across the sheet. It is with this latter statement that we wish to take issue.

Clarke (1967) assumes that all flow variables are continuous across the flame sheet and uses this assumption to show that not only the velocity but also the velocity derivatives must be continuous on crossing the flame sheet. It is then possible to show quite generally that continuity of velocity derivatives at the sheet implies the continuity of density gradient across the flame sheet. This result is not consistent with the discontinuity of the temperature gradient across the sheet and the use of $\rho T = 1$, and poses the question of how these facts can be allied. The answer to this question comes from the fact that $\rho T = 1$ is a simplification of the more realistic thermal equation of state.

$$p = \frac{\rho RT}{C_c} \sum \frac{C_{\alpha}}{W_{\alpha}} \quad (6)$$

and examination of this equation makes it clear that discontinuities in the normal gradients of the temperature and mass fractions can be compensated by a discontinuity in the normal gradient of the pressure across the sheet. In the remainder of this work the major effort will be afforded to the analysis of the pressure field associated with a diffusion flame with particular reference to its behaviour in the vicinity of the flame sheet.

4. Basic Equations

The choice of the spherical flame configuration for this work was made for two reasons. Firstly, for the case of steady flow, the equations upon which the analysis is based contain only one independent variable, the radial co-ordinate r and secondly, Clarke and Moss (1970) have partially solved the steady flow problem for a spherical $H_2 - O_2$ flame. Using the species, mass and energy equations Clarke and Moss obtained solutions for the temperature and mass fractions of the reactants.

Unfortunately, these solutions were given in terms of a newly defined independent variable y which was related in a potentially complicated way with the radial co-ordinate r . Nevertheless the overall simplifications obtained by utilising the spherical configuration out-weighed significantly any complications introduced by the use of the new variable y . To obtain information about the pressure, velocity and density fields associated with the spherical diffusion flame it is necessary to augment the basic equations of Clarke and Moss with the momentum equation and the thermal equation of state.

The basic equations for steady flow in the spherically symmetric configuration are

$$\text{Mass} \quad \dot{m} = \rho r^2 u \quad (7)$$

$$\text{Species} \quad \epsilon r^{-2} \frac{d}{dr} \left\{ r^2 \left[\rho u c_\alpha + \rho T_\alpha \right] \right\} = K_\alpha ; \alpha = X, F, P, (D) \quad (8)$$

$$\text{Energy} \quad r^{-2} \frac{d}{dr} \left\{ r^2 \rho u \left[h + O(M^2) \right] + r^2 q \right\} = 0 \quad (9)$$

Momentum

$$\rho u \frac{du}{dr} = - \frac{d}{dr} \left\{ \frac{p}{M^2} + \frac{2}{3} \rho \mathcal{D} Sc \frac{1}{r^2} \frac{d}{dr} (ur^2) - 2 \rho \mathcal{D} Sc \frac{du}{dr} \right\} + \frac{2}{r} \left\{ 2 \rho \mathcal{D} Sc \left(\frac{du}{dr} - \frac{u}{r} \right) \right\} \quad (10)$$

Thermal

$$p = \frac{\rho R T}{C_c} \sum_{\alpha} \frac{C_{\alpha}}{W_{\alpha}} \quad (6)$$

All of the variables appearing in the above equations are dimensionless and are defined with reference to a set of characteristic dimensional quantities including a density ρ_c , velocity u_c , background pressure p_c , diffusion coefficient $\mathcal{D}_c$, chemical time τ_c , specific heat C_c and temperature T_c . From these characteristic variables a typical diffusion length $\mathcal{D}_c / u_c$ and a typical time $\mathcal{D}_c / u_c^2$ can be manufactured. The parameter $\mathcal{E}$ is then the ratio of two times

$$\mathcal{E} = \tau_c u_c^2 / \mathcal{D}_c$$

u_c is characteristic of the diffusion velocities which exist in the system but further it is assumed that u_c is characteristic of all velocities in the system, including flow velocities. When this assumption is made it can be shown that the kinetic energy and viscous dissipation terms in the energy equation are $O(M^2)$. The parameter M^2 is equal to $\rho_c u_c^2 / p_c$, and is proportional to the square of the diffusive Mach number and thus is small enough to

be negligible compared with unity. The form of the energy equation given above is only correct if the dimensional enthalpy is given by $C_c T_c h$ and the dimensional energy flux by $\rho_c u_c C_c T_c q$. $4\pi \dot{m}$ is the constant dimensionless mass flux injected into the region $r \geq r_i$ through the porous fuel sphere whose surface is at r_i , K_α is the dimensionless chemical source term for species α ($\rho_c K_\alpha / \tau_c$ is the nett mass rate of production of α per unit volume) and J_α is given by

$$J_\alpha = C_\alpha u_\alpha \quad (11)$$

q is given by

$$q = -\frac{\lambda}{Le} \frac{dT}{dr} + \rho \sum_{\alpha} J_\alpha h_\alpha \quad (12)$$

where $\lambda_c \lambda$ is the thermal conductivity of the mixture,

Le is the Lewis number

$$Le = \rho_c C_c D_c / \lambda_c \quad (13)$$

and h_α is the dimensionless specific enthalpy which is assumed to be of the form

$$h_{\alpha} = C_{\alpha}T + Q_{\alpha} \quad h = \sum_{\alpha} h_{\alpha} C_{\alpha} \quad (14)$$

where $C_{\alpha}T + Q_{\alpha}$ is the energy of formation of species α and $C_{\alpha}C_{\alpha}$ is its specific heat at constant pressure.

Sc is the Schmidt number. The J_{α} are not all independent

$$\sum_{\alpha} J_{\alpha} = 0 \quad (15)$$

and the mass fractions by definition sum to one.

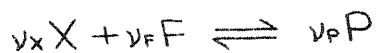
$$\sum_{\alpha} C_{\alpha} = 1 \quad (16)$$

It is assumed that Fick's Law gives a satisfactory description of the diffusion processes taking place and further assumptions are made concerning the diffusion coefficients. In fact the simple fuel, oxidant and product system is treated in situations where there is also a diluent species (denoted by D) present.

5. The Temperature and Mass Fraction Fields

As stated earlier, to obtain solutions for the temperature and mass fraction fields the method of Clarke and Moss (1970) is utilised. Clarke and Moss used a moderately realistic kinetic scheme to describe the H_2-O_2 reaction but here a much cruder reaction scheme is utilised since the precise behaviour of the mass fraction fields is of secondary interest. Moreover no diluent species was included in the Clarke and Moss analysis and thus it seems worthwhile to include the detailed calculations of the temperature and mass fractions for the precise reaction scheme considered in this work.

The reaction considered here is that given in equation (2)



and the presence of a diluent species D is allowed.

In the specific case where there is no diluent present it is useful to model the behaviour of the diffusion coefficients such that the ratio of the diffusion coefficient for fuel through product to that of the diffusion coefficient for oxidant through product is constant for all temperatures. This ratio then serves as a useful parameter which can be varied to produce better numerical agreement with experimental results from the theoretical model. The particular model of the diffusion coefficient as a function of temperature is then chosen to produce good agreement

between theory and experiment. However the presence of a diluent species complicates the rather simple model of the diffusion processes given by Fick's Law applied to a Burke-Schumann model of a diffusion flame since there are now non-negligible mass fractions of three species on each side of the flame sheet rather than two when the diluent species is absent. Then, for simplicity, the gross approximation that all diffusion coefficients appearing in the description of the system are equal is made. Again, even with such a gross approximation, the qualitative features of the mass fractions should be given reasonably accurately and for the purposes of these calculations such qualitative accuracy is adequate. At a later stage in the calculation it is necessary to consider what model to use to describe the temperature dependence of the diffusion coefficients but for now it is merely assumed that all diffusion coefficients are equal. Then, applying this assumption and using Fick's Law the diffusive fluxes J_x and J_F are given by

$$J_x^{(u)} = -\Phi^{(u)} \frac{dc_x^{(u)}}{dr} \quad (17a)$$

$$J_F^{(u)} = -\Phi^{(u)} \frac{dc_F^{(u)}}{dr} \quad (17b)$$

where the superscripts '1' refer to the first outer solutions to the basic equations considered for the Burke-Schumann model of the diffusion flame in the limit of an infinitely fast irreversible reaction as described in the section on the Burke-Schumann model.

In general the species equation can be written in the form

$$\varepsilon (\rho r^2)^{-1} \frac{d}{dr} \left\{ m \gamma_\alpha + \rho r^2 J_\alpha' \right\} = (\nu_\alpha'' - \nu_\alpha') R \quad (18)$$

where $J_\alpha' = J_\alpha / W_\alpha$, $R = \tau_f^{-1} \left\{ C_\alpha^{\nu_\alpha} C_F^{\nu_F} - \delta C_P^{\nu_P} \right\}$ (19)

and the quantities ν_α' , ν_α'' are, respectively, the left and right hand side stoichiometric integers for species

α . Equation (18) can now be used to write down the species equations for all of the species present in the system. Of particular interest is the species equation for the diluent:

$$\varepsilon (\rho r^2)^{-1} \frac{d}{dr} \left\{ m \gamma_D + \rho r^2 J_D' \right\} = 0 \quad (20)$$

in which the right hand side is zero since no diluent species is created or destroyed in the reaction ($K_D=0$). The simplest solution of this equation is $C_D = \text{constant}$ over the whole system ($C_D = \text{constant} \Rightarrow J_D=0$) and purely for simplicity this solution will be used throughout the remainder of this work. The equations obtained from equation (18) for the fuel and oxidant species can be combined to give

$$\frac{d}{dr} \left\{ \dot{m} (\nu_F \gamma_X - \nu_X \gamma_F) + \rho r^2 (\nu_F J_X' - \nu_X J_F') \right\} = 0 \quad (21)$$

which represents the indestructability of atoms in the system and is referred to by Clarke (1972) as an "atom conservation" equation. Equation (21) is now integrated with respect to the variable r and the boundary condition that only fuel and the fixed mass fraction of diluent flow out of the fuel sphere (i.e. at $r=r_c$) is applied giving

$$\dot{m} (\nu_F \gamma_X - \nu_X \gamma_F) + \rho r^2 (\nu_F J_X' - \nu_X J_F') = - \frac{\nu_X}{W_F} \dot{m} (1 - C_D) \quad (22)$$

(N.B. $\dot{m}_\alpha + \rho r^2 J_\alpha$ is the total mass flux of any species at a given value of r). Now as we saw in an earlier section the Burke-Schumann model of the diffusion flame

has both ε and δ approaching limiting values of zero in the species equation producing the limiting form of the equation

$$C_F^{yy} C_X^{yy} = 0 \quad \text{everywhere} \quad (23)$$

The solution of this equation for the experimental configuration considered here is

$$\begin{aligned} C_F &= O(1) & r_s > r \geq r_i & & C_X &= 0 & r_s > r \geq r_i \\ &= 0 & r > r_s & & &= O(1) & r > r_s \end{aligned} \quad (24)$$

Then in the terminology of inner and outer expansions the first outer solution (that is the solution valid everywhere except inside the inner region, which in this instance means within the flame sheet) for the mass fractions is

$$\begin{aligned} r_s > r \geq r_i & ; & C_F &= C_F^{(0)} , & C_X &= 0 \\ r > r_s & ; & C_F &= 0 , & C_X &= C_X^{(0)} \end{aligned} \quad (25)$$

Then substituting equations (17) and (25) into equation (22) the following equations are obtained:

$$-\dot{m} C_F^{(0)} + \rho^{(0)} r^2 D^{(0)} \frac{dC_F^{(0)}}{dr} = -\dot{m} (1 - C_D) \quad r_s > r \geq r_i \quad (26a)$$

$$\dot{m} C_x^{(0)} - \rho^{(0)} r^2 \Phi^{(0)} \frac{dC_x^{(0)}}{dr} = - \left(\frac{\nu_x}{\nu_F} \right) \frac{W_x}{W_F} \dot{m} (1 - C_D), \quad r > r_s \quad (26b)$$

To simplify solution of equations (26a and b) the variable y is introduced

$$y = \int_{r_s}^r \left\{ \frac{\dot{m}}{\rho^{(0)}(\tilde{r}) \Phi^{(0)}(\tilde{r}) \tilde{r}^2} \right\} d\tilde{r} \quad (27)$$

Then equation (26) can be written as

$$-C_F^{(0)} + \frac{dC_F^{(0)}}{dy} = -(1 - C_D) \quad 0 > y \geq y_i \quad (28a)$$

$$C_x^{(0)} - \frac{dC_x^{(0)}}{dy} = - \frac{\nu_x W_x}{\nu_F W_F} (1 - C_D) \quad y_i \geq y > 0 \quad (28b)$$

Equations (28) must be solved with their associated boundary conditions. As shown earlier one condition which must be imposed in the Burke-Schumann flame sheet is that the mass fractions of fuel and oxidant must drop to zero at the flame sheet

$$C_F^{(0)} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow r_s - \quad (y \longrightarrow 0 -) \quad (29a)$$

$$C_x^{(0)} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow r_s + \quad (y \longrightarrow 0 +) \quad (29b)$$

The solutions of equations (28) with boundary conditions (29) applied are

$$C_F^{(1)} = (1 - c_D)(1 - \exp y) \quad , \quad C_P^{(1)} = (1 - c_D) \exp y \quad ; \quad 0 > y \geq y_i \quad (30a)$$

$$C_X^{(1)} = (1 - c_D)A(\exp y - 1) \quad , \quad C_P^{(1)} = (1 - c_D) \left[1 - A(\exp y - 1) \right] \quad ; \quad y_\infty \geq y > 0 \quad (30b)$$

$$A = \nu_X W_X / \nu_F W_F$$

where the solutions for $C_P^{(1)}$ are obtained by reference to equation (16). However there is another boundary condition to be imposed in the oxidant region. As $r \rightarrow \infty$ it is anticipated that $C_P^{(1)} \rightarrow 0$ and this will only be true if

$$1 + A = A \exp y_\infty$$

Then

$$y_\infty = \int_{r_s}^{\infty} \left\{ \frac{\dot{m}}{Q^{(1)} Q^{(1)} r^2} \right\} dr = \ln \left[1 + \frac{1}{A} \right] \quad (31)$$

which means that y_∞ is fixed in terms of known constants. The co-ordinate transformation given in equation (27) can be inverted giving

$$\dot{m} \left(\frac{1}{r_s} - \frac{1}{r_i} \right) = \int_0^y \rho^{(n)}(\tilde{y}) \Phi^{(n)}(\tilde{y}) d\tilde{y} \quad (32)$$

and then

$$\frac{\dot{m}}{r_s} = \int_0^{y_\infty} \rho^{(n)}(\tilde{y}) \Phi^{(n)}(\tilde{y}) d\tilde{y} \quad (33)$$

It is now possible to calculate the temperature as a function of the variable y using the known expressions for the mass fractions C_α given in equations (30a and b). Equation (9) is integrated with respect to the variable r and then equations (7) and (12) are substituted into the resulting expression giving

$$\dot{m} h_\infty - \frac{\lambda}{k_e} r^2 \frac{dT}{dr} + \rho r^2 \sum_\alpha J_\alpha h_\alpha = \dot{m} H \quad (34)$$

where H is a constant to be determined later in the calculation. It is now assumed that the temperature can be written as

$$T = T^{(n)}(r) + o(1)$$

where it is anticipated that $T^{(n)}$ will be a continuous

function of r with a discontinuous derivative with respect to r at $r = r_s$. Expressions for h and h_x from equation (14) are substituted into equation (34) giving

$$\dot{m} \sum_{\alpha} c_{\alpha} (C_{\alpha} T + Q_{\alpha}) - \frac{\lambda}{Le} r^2 \frac{dT}{dr} + \rho r^2 \sum_{\alpha} J_{\alpha} (C_{\alpha} T + Q_{\alpha}) = \dot{m} H \quad (35)$$

Expressions for the mass fractions given in equations (30a and b) are now introduced, and after some rearrangement the following equations are obtained:

$$\frac{C_F}{Le} \frac{dT^{(u)}}{dy} - \left\{ (1-c_D)C_F + c_D C_D \right\} T^{(u)} = -H \quad 0 > y \geq y_1 \quad (36a)$$

$$\frac{C_F}{Le} \frac{dT^{(u)}}{dy} - \left\{ (A+1)(1-c_D)C_F + c_D C_D - A(1-c_D)C_X \right\} T^{(u)} = (A+1)(1-c_D)Q_F - H$$

$$y_{\infty} \geq y > 0 \quad (36b)$$

where

$$Le = \rho^{(u)} C_F D^{(u)} Le / \lambda^{(u)} \quad (36c)$$

and where Le is assumed constant and where it is also assumed that

$$Q_F = Q_X = Q_D = 0, \quad C_F = 1$$

Now for simplicity put

$$\alpha = (1 - c_D) + c_D C_D, \quad \alpha^* = (1 + A)(1 - c_D)C_P + c_D C_D - A(1 - c_D)C_X \quad (37)$$

The solutions of equations (36a and b) for $T^{(u)}$ are then

$$T^{(u)} = \frac{H}{\alpha} + \left(T_i - \frac{H}{\alpha} \right) \exp \alpha k e (y - y_i) \quad 0 > y \geq y_i \quad (38a)$$

$$T^{(u)} = \left[\frac{H - (A+1)(1 - c_D)Q_P}{\alpha^*} \right] + \left\{ T_\infty - \left[\frac{H - (A+1)(1 - c_D)Q_P}{\alpha^*} \right] \right\} \exp \alpha^* k e (y - y_\infty) \quad (38b)$$

$$y_\infty \geq y > 0$$

where T_i is the temperature at the fuel sphere and T_∞ is the temperature as $r \rightarrow \infty$. It was stated previously that the constant H would be determined at a later stage in the calculation and that stage has now been reached. The determination is effected by noting that the temperature must be continuous across the Burke-Schumann flame sheet and thus $T^{(u)}(r_s^-)$ given by equation (38a) must equal $T^{(u)}(r_s^+)$ given by equation (38b). Then algebraic

rearrangement gives

$$H = \frac{T_{\infty} \exp - \alpha^* k y_0 - \frac{1}{\alpha^*} (A+1)(1-C_0) Q_F (1 - \exp - \alpha^* k y_0) - T_i \exp - \alpha k y_i}{\left\{ \frac{1}{\alpha} (1 - \exp - \alpha k y_i) - \frac{1}{\alpha^*} (1 - \exp - \alpha^* k y_0) \right\}} \quad (39)$$

Unfortunately this does not completely specify H since

y_i is not known unless the transformation connecting r and y is specified completely. To do this a model for

$\rho^{(n)} D^{(n)}$ as a function of either r or y must be forwarded and this is a subject which will be discussed at greater length later. However when such a model for

$\rho^{(n)} D^{(n)}$ is provided all of the mass fractions and the temperature are known throughout the region of interest to first order.

At this point in the calculation the mass fractions and temperature are assumed to be known sufficiently accurately and interest shifts to the calculation of the pressure, velocity and density fields. To study these fields it is necessary to examine the mass and momentum equations and the thermal equation of state. However before undertaking this calculation it is useful to demonstrate some of the aspects of the continuity and discontinuity of various quantities across the flame sheet

with particular reference to the spherical diffusion flame.

6. The Discontinuous Jumps in Gradients Across the Flame Sheet

Initially it is assumed that all flow variables are continuous across the flame sheet. The difference between the values of any quantity on the two sides of the flame sheet will be denoted by enclosing it in square brackets; for example

$$[f]_{-}^{+} = f(r_{s+}) - f(r_{s-})$$

The momentum equation is integrated with respect to the variable r from r_{s-} to r_{s+} . Then

$$\int_{r_{s-}}^{r_{s+}} \rho u \frac{du}{dr} dr = - \left[\frac{p}{M^2} + \frac{2}{3} \rho D Sc \frac{1}{r^2} \frac{d(ur^2)}{dr} - 2 \rho D Sc \frac{du}{dr} \right]_{-}^{+} \quad (40)$$

$$+ \int_{r_{s-}}^{r_{s+}} 4 \rho D Sc \frac{d}{dr} \left(\frac{u}{r} \right) dr$$

giving

$$\rho_s u_s [u]_{-}^{+} = - \frac{[p]_{-}^{+}}{M^2} + \frac{4}{3} \rho_s D_s Sc \left[\frac{du}{dr} \right]_{-}^{+} + 4 \rho_s D_s Sc \left[\frac{u}{r} \right]_{-}^{+} \quad (41)$$

where the quantities subscripted 'S' denote the values of the quantities at the flame sheet and where it has been assumed that the Schmidt number is a constant throughout the system considered and that the diffusion coefficient is a function of the flow variables but not of their derivatives. Since all flow variables are assumed to be continuous

$$[u]_{-}^{+} = [p]_{-}^{+} = [u/r]_{-}^{+} = 0$$

and thus

$$\left[\frac{du}{dr} \right]_{-}^{+} = 0 \quad (42)$$

The continuity of $\frac{du}{dr}$ across the flame sheet when considered in conjunction with the mass conservation equation, equation (7), requires that the density derivative $\frac{dp}{dr}$ must also be continuous and this leads to the necessity of the discontinuity of the pressure derivative $\frac{dp}{dr}$. By combining the mass conservation equation and the thermal equation of state the density variable can be eliminated giving

$$p u r^2 = \frac{\dot{m} R T}{C_c} \sum_{\alpha} \frac{C_{\alpha}}{W_{\alpha}} \quad (43)$$

Then

$$u_s r_s^2 \left[\frac{dp}{dr} \right]_-^+ = \frac{\dot{m} R}{C_c} \left\{ \sum_{\alpha} \frac{C_{\alpha s}}{W_{\alpha}} \left[\frac{dT}{dr} \right]_-^+ + T_s \sum_{\alpha} \frac{1}{W_{\alpha}} \left[\frac{dc_{\alpha}}{dr} \right]_-^+ \right\} \quad (44)$$

Equation (44), which relates the discontinuities in the gradients of pressure, temperature and mass fraction across the flame sheet, will be used further at a later stage in the calculation but at the moment it is only used to provide a rough guide to the magnitude of the pressure gradient discontinuity across the flame sheet. Using the expressions for $T^{(1)}$ and $C_{\alpha}^{(1)}$ in equations (38) and (30) it is easy to show that both $\left[\frac{dT^{(1)}}{dr} \right]_-^+$ and $\left[\frac{dc_{\alpha}^{(1)}}{dr} \right]_-^+$ are $O(1)$ and consequently $\left[\frac{dp}{dr} \right]_-^+$ will be $O(1)$ to lowest order. At this point it must be stressed that the superscript '1' refers to the first approximation to the temperature and mass fraction fields outside the flame sheet. These solutions were obtained in the limit of the ϵ and δ approaching zero but also it was assumed that the Mach number M was approaching its zero limiting value. In this calculation as a whole the parameters ϵ and δ retain their limiting values, but the Mach number is considered as a non-zero small parameter. In this way the quantities $T^{(1)}$ and $C_{\alpha}^{(1)}$ are seen to be also the first approximations in

a Mach number expansion outside the flame sheet.

7. The Pressure Field

The basic equations involved in the study of the pressure field associated with a spherical diffusion flame are

$$\frac{dp}{dr} = M^2 \left\{ \frac{d}{dr} \left[\frac{4}{3} \rho D Sc \left(\frac{du}{dr} - \frac{u}{r} \right) \right] + 4 \rho D Sc \frac{d}{dr} \left(\frac{u}{r} \right) - \frac{\dot{m}}{r^2} \frac{du}{dr} \right\} \quad (45)$$

$$p r^2 = \frac{\dot{m} R T}{C_c} \sum_{\alpha} \frac{C_{\alpha}}{W_{\alpha}} \quad (43)$$

where equation (45) is obtained by rearrangement of equation (10) and use of the mass conservation equation. The temperature and mass fraction fields are known to $O(1)$ and thus

$$p u = f(r) + o(1) \quad (46)$$

where $f(r)$ is $ord(1)$ and is given by

$$f(r) = \frac{\dot{m}RT^{(1)}}{C_0 r^2} \sum_{\alpha} \frac{C_{\alpha}^{(1)}}{W_{\alpha}} \quad (47)$$

A single differential equation for the pressure can be obtained by eliminating the velocity u between equations (45) and (46) and this procedure will be carried out after summarising the discussion of possible models for the diffusion coefficient Φ given in the Appendix. Up to this point in the calculation the only assumptions made about the diffusion coefficients have been a) that they are equal and b) that they are functions of the flow variables themselves rather than of their first or higher derivatives.

In choosing a model of the diffusion coefficient to use in this calculation two criteria are employed, simplicity and the degree of realism. Starting from these bases the obvious choice is

$$\Phi \propto \frac{T}{\rho} \quad (48)$$

and this indeed is the model used in the main calculation. However, the calculations themselves become much simpler

if it is assumed that ϕ_D is a constant and although this model is not particularly realistic its use should produce results of qualitative significance at least, and for this reason supplementary calculations are carried out using this very simple model.

Whilst discussing the degree of realism of the models used to describe the diffusion coefficient it is important to consider the modelling of the diffusion coefficient in its true perspective with reference to the whole calculation. In describing the diffusion process it has been assumed that multicomponent diffusion processes are unimportant and that diffusive processes are adequately described by Fick's Law and moreover that all diffusion coefficients are equal. These are such swingeing approximations that any attempt at quantitatively realistic results has already vanished far from view and consequently the detailed modelling of the diffusion coefficient as a function of temperature is not of great importance, and it is therefore not unreasonable to place convenience and simplicity to the forefront when deciding criteria for choosing the models to be used.

Equation (48) gives the product ϕ_D to be proportional to the temperature. However as a result

of the process of non-dimensionalising the variables in the calculation the coefficient of proportionality is unity and thus the relationship becomes

$$\rho D = T \quad (49)$$

This equation is now substituted into equation (45) giving

$$\frac{dp}{dr} = M^2 \left\{ \frac{d}{dr} \left[\frac{4}{3} Sc T \left(\frac{du}{dr} - \frac{u}{r} \right) \right] + 4 Sc T \frac{d}{dr} \left(\frac{u}{r} \right) - \frac{\dot{m}}{r^2} \frac{du}{dr} \right\} \quad (50)$$

The velocity u is now eliminated between equations (46) and (50) and the new variable $\bar{p}$ is introduced giving

$$\begin{aligned} (\bar{p}+1)^3 \frac{d\bar{p}}{dr} = M^2 \left\{ \frac{4}{3} Sc T \left[-f(\bar{p}+1)\bar{p}_{rr} + (\bar{p}+1)^2 f_{rr} - 2(\bar{p}+1)f_r \bar{p}_r \right. \right. \\ \left. \left. + 2f\bar{p}_r^2 + \frac{2(\bar{p}+1)^2}{r} f_r - \frac{2(\bar{p}+1)f\bar{p}_r}{r} - 2(\bar{p}+1)^2 \frac{f}{r^2} \right] \right. \\ \left. - \frac{\dot{m}}{r^2} \left[(\bar{p}+1)^2 f_r - (\bar{p}+1)f\bar{p}_r \right] \right\} + o(M^2) \quad (51) \end{aligned}$$

$$\bar{p} \equiv p - 1 \quad (52)$$

where f_r denotes the first derivative of f with respect to r , f_{rr} the second derivative with respect to r etc. It is clear from the appearance of the second derivative of $\bar{p}$ with respect to r that equation (51) is a second

order ordinary differential equation for $\bar{p}$ and consequently two boundary conditions must be given for $\bar{p}$ to be specified completely. Moreover the term containing $\bar{p}_{rr}$ is multiplied by M^2 which in this calculation is a very small parameter indeed and the appearance of the small parameter multiplying the highest derivative in the equation signals the fact that the problem to be solved is a singular perturbation problem. This means that the presence of a 'boundary layer', which in this context is a region of rapid pressure variation, is anticipated, although it is not possible to say at this time where this region of rapid pressure variation is situated.

Initially the pressure field in the region where $r > r_s$ is examined. In this region, henceforth called region B, it is assumed that the solution of equation (51) can be usefully approximated by a series expansion:-

$$\bar{p}_B(r) = \delta_B^{(1)}(M) \bar{p}_B^{(1)}(r) + \delta_B^{(2)}(M) \bar{p}_B^{(2)}(r) + \dots \quad (53)$$

where the δ 's are scale factors depending solely on

the small parameter appearing in equation (51), the Mach number M , and the subscripts B denote variables appropriate to region B . At this point the only further information available concerning the scale factors is that

$$\lim_{M \rightarrow 0} \frac{\delta_B^{(n)}(M)}{\delta_B^{(n-1)}(M)} = 0 \quad ; \quad \delta_B^{(n)}(M) = o(\delta_B^{(n-1)}(M)) \quad (54)$$

Since M is **very** small indeed it is hoped that a very small number of terms will provide a very good approximation to the full solution in the regions of interest.

The only further information available concerning the pressure field in region B is related to its value as $r \rightarrow \infty$. It is convenient to choose the characteristic thermodynamic variables in such a way that the dimensionless pressure is unity as $r \rightarrow \infty$. This information proves to be vital in the discussion leading to the discovery of the precise form of $\delta_B^{(n)}(M)$. The method used to find $\delta_B^{(n)}(M)$ is essentially a systematic trial and error technique and this type of technique is used frequently in the following work. As a first trial it is assumed that

$$1/\delta_B^{(n)}(M) = o(1)$$

The technique of trying various magnitudes until finding one for which the first term is non-zero is equivalent to substitution of $\sum_i \delta_B^{(i)} p_B^{(i)}$ into the differential equation and finding the first non-zero $\delta_B^{(i)}$.

From the solutions obtained earlier for the temperature and mass fraction fields, and the definition of $f(r)$ given in equation (47) the following magnitudes can be assigned

$$f = O(1), f_r = O(1), f_{rr} = O(1)$$

These magnitudes given here are completely independent of the choice of any of the δ 's and will be used henceforth in the calculations along with the specific choices made for the δ 's to fix the magnitudes of the various terms in equation (51). For the particular magnitude (or set of possible magnitudes) under consideration at present, substitution of equation (53) into equation (51) produces the following expression when only terms of the highest orders present are retained.

$$\delta_B^{(4)}(M)^4 p_B^{(4)}(r)^3 \frac{dp_B^{(4)}(r)}{dr} = 0$$

This equation has two possible solutions:-

$$p_B^{(4)} = 0 \quad \text{or} \quad p_B^{(4)} = \text{constant}$$

The first of these is not a significant first solution and application of the condition on p as $r \rightarrow \infty$ suggests that the constant must also be zero. Thus, it is seen that this set of possible values for $\delta_B^{(4)}(M)$ produce no significant first solutions for $\bar{p}_B$.

The next trial solution for $\delta_B^{(4)}(M)$ is

$$\delta_B^{(4)}(M) = O(1)$$

Substitution of equation (53) into equation (51) and retention of the leading terms gives

$$(\delta_B^{(4)}(M) p_B^{(4)}(r) + 1)^3 \delta_B^{(4)}(M) \frac{dp_B^{(4)}}{dr} = 0$$

which again only has solutions either of the form $p_8^{(0)}(r) = \text{constant}$ which are inconsistent with the boundary condition as $r \rightarrow \infty$ unless the constant is zero, in which case the solutions are not significant anyway or of such a form that the pressure itself is zero everywhere and this too is clearly not a physically significant solution.

The next possible set of magnitudes is given by:

$$M^2 = O(\delta_B^{(0)}(M)), \delta_B^{(0)}(M) = O(1)$$

Substitution of equation (53) into equation (51) and retention of leading terms gives

$$\delta_B^{(0)}(M) \frac{dp_8^{(0)}(r)}{dr} = 0$$

The only possible solutions of this equation are $p_8^{(0)}(r) = \text{constant}$ and again consideration of the boundary condition on p as $r \rightarrow \infty$ shows that the constant must be zero meaning again no significant solution is obtained.

The next choice is

$$\delta_B^{(0)}(M) = O(M^2)$$

For convenience $\delta_B^{(0)}(M)$ is put equal to M^2 .

The resulting expression of the leading terms obtained

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by substituting equation (53) into equation (51) can be written in the form

$$\frac{dp_B^{(u)}(r)}{dr} = \frac{d}{dr} \left[T^{(u)} \frac{4Sc}{3} \left(f_r - \frac{f}{r} \right) \right] + 4Sc T^{(u)} \frac{d}{dr} \left(\frac{f}{r} \right) - \frac{\dot{m} f_r}{r^2} \quad (55)$$

Then it is clear that the first significant term in the expression for $\bar{p}$ is produced when $\delta_B^{(u)} = M^2$, a result in agreement with expectations based upon examination of the thermal equation of state. Moreover

$$\begin{aligned} p_B^{(u)}(r) &= T^{(u)} \frac{4Sc}{3} \left(f_r - \frac{f}{r} \right) + 4Sc T^{(u)} \frac{f}{r} - 4Sc \int \frac{f}{r} \frac{dT^{(u)}}{dr} dr - \int \frac{\dot{m} f_r}{r^2} dr + \text{const.} \\ &= T^{(u)} \frac{4Sc}{3} \left(f_r + \frac{2f}{r} \right) - 4Sc \int \frac{f}{r} \frac{dT^{(u)}}{dr} dr - \int \frac{\dot{m} f_r}{r^2} dr + \text{const.} \end{aligned} \quad (56)$$

where $T^{(u)}$, f , f_r etc. are functions appropriate to region B.

It is immediately clear that the value of the pressure at the flame sheet given by this solution will be of the form

$$p_B(r_s+) = 1 + M^2 p_B^{(u)}(r_s+) \quad (57)$$

and then using the condition of continuity of the pressure field across the flame sheet gives

$$p_A(r_s-) = p_B(r_s+) = 1 + M^2 p_B^{(0)}(r_s+) \quad (58)$$

This expression can now be used as a boundary condition on the pressure in region A and allows the magnitude of the scale parameter $\delta_A^{(0)}(M)$ to be fixed. Following arguments parallel to those used for region B it is possible to find a solution to equation (57) for region A of the form

$$p_A(r) = 1 + M^2 p_A^{(0)}(r) \quad (59)$$

However if $\left[\frac{dp}{dr} \right]_-^+$ is now calculated from equation (57) and its equivalent for region A it is found to be $O(M^2)$ which is not in agreement with the result shown earlier in this work. Thus it is seen that solutions of the form given in equations (57) and (59) are not uniformly valid throughout the domain of interest since it is clear that the pressure gradients must be very much greater in the vicinity of the sheet than elsewhere.

Thus it can be seen that the region of rapid pressure variations signalled by the presence of the product

$M^2 \bar{p}_{rr}$ in equation (51) is situated in the immediate vicinity of the flame sheet. It is in tackling precisely such singular perturbation problems as this that the method of inner and outer expansions is most useful and this method is employed here. There is no reason to suppose that the solutions given in equations (57) and (59) are invalid far from the flame sheet but clearly a different solution is required in the regions near the flame sheet, the inner regions.

For the remainder of this work both regions A and B are subdivided into inner and outer regions. The inner regions are those regions nearest the flame sheet and are characterised by a rapid variation of pressure whereas the outer regions are further from the flame sheet and are characterised by slow pressure variations.

The solution for $\bar{p}_a$ obtained by substituting equation (56) into equation (53) is regarded as the first outer approximation to the pressure field in region

B. The next step in the calculation is to examine the first inner approximation in region B. However before proceeding with the calculation it is necessary

to characterise the inner regions with a new spatial variable in an equivalent way to the spatial characterisation of the outer regions by the dimensionless radial coordinate σ . The new characteristic variable R is given by

$$r - r_s = \sigma(M) R, \quad \sigma(M) = o(1) \quad (60)$$

where $\sigma(M)$ is a scaling function dependent solely on the Mach number M . This equation is used along with equation (51) to give the differential equation relating $\bar{p}$ and R , that is the equation appropriate to the inner regions

$$\begin{aligned} (\bar{p}+1)^3 \bar{p}_R = M^2 \left\{ \frac{4Sc}{3} \frac{T}{\sigma} \left[-f(\bar{p}+1) \bar{p}_{RR} + (\bar{p}+1)^2 f_{RR} - 2(\bar{p}+1) f_R \bar{p}_R + 2f \bar{p}_R^2 \right] \right. \\ + \frac{4Sc}{3} T \left[\frac{2(\bar{p}+1)^2 f_R}{r_s + \sigma R} - \frac{2(\bar{p}+1) f \bar{p}_R}{r_s + \sigma R} - \frac{2(\bar{p}+1)^2 f \sigma}{(r_s + \sigma R)^2} \right] \\ - \frac{\dot{m}}{(r_s + \sigma R)^2} \left[(\bar{p}+1)^2 f_R - (\bar{p}+1) f \bar{p}_R \right] \\ \left. + T_R \frac{4Sc}{3} \left[\frac{(\bar{p}+1)^2 f_R}{\sigma} - \frac{(\bar{p}+1) f \bar{p}_R}{\sigma} - \frac{(\bar{p}+1)^2 f}{r_s + \sigma R} \right] \right\} \\ + o(M^2) \quad (61) \end{aligned}$$

The solution of this equation in region B is obtained by writing $\bar{p}$ in the form

$$\bar{p}_B = \Delta_B^{(1)}(M) P_B^{(1)}(R) + \Delta_B^{(2)}(M) P_B^{(2)}(R) + \dots \quad (62)$$

where

$$\lim_{M \rightarrow 0} \Delta_B^{(2)}(M) / \Delta_B^{(1)}(M) = 0$$

and substituting equation (62) into equation (61).

At each stage in the calculation only the terms with the largest magnitudes are retained and these form the equations from which the forms for $P_B^{(n)}$ are calculated.

The functions $P_B^{(1)}, P_B^{(2)}, \dots$ are all ord (1) functions of the characteristic variable R .

It has been shown previously in this work that

$\left[\frac{dp}{dr} \right]^+ = O(1)$. This suggests that $\frac{dp}{dr}$ is itself $O(1)$ in the vicinity of the flame sheet, although it could be of greater magnitude. Thus $\frac{dp}{dr} = O(\sigma)$ or larger and consequently interest centres on cases where $\Delta_B^{(1)}(M)$ is not smaller in magnitude than σ .

In the inner regions

It has been stated previously that $\sigma = o(1)$ but to pursue the calculations further tighter restrictions must be placed on the magnitude of σ .

$$\underline{I \quad \sigma = o(1), \quad M^2 = o(\sigma)}$$

(i) If $1/\Delta_B''(M)$ is $o(1)$ then retention of the largest terms in the expression obtained by substituting equation (62) into equation (61) gives

$$\Delta_B'' (\Delta_B'' P_B''(R))^3 \frac{dP_B''(R)}{dR} = 0$$

The only non-zero solution of this equation is $P_B''(R) = \text{constant}$. To determine the value of this constant the method of order matching is utilised.

In the present calculation the outer solution is given to $O(\delta_B''(M))$ and the inner solution to $O(\Delta_B''(M))$.

The order matching procedure requires that

(a) the outer solution to $O(\delta_B''(M))$ is written in

inner co-ordinates when the inner limiting process
 (fix $R, M \rightarrow 0$) is applied to find the outer solution
 in inner co-ordinates to $O(\Delta_B^{(0)}(M))$ and

b) the inner solution to $O(\Delta_B^{(0)}(M))$ is written in
 outer co-ordinates when the outer limiting process
 (fix $r-r_s, M \rightarrow 0$) is applied to find the inner solution
 in outer co-ordinates to $O(\delta_B^{(0)}(M))$ and this is then
 rewritten in inner co-ordinates.

The two expressions obtained from a) and b) are
 then equated. This order matching procedure is now
 applied specifically to the problem at hand.

The one-term outer solution has the form

$$\bar{p}_B = M^2 p_B^{(0)}(r) \quad (63)$$

and the one-term inner solution proposed is

$$\bar{p}_B = \Delta_B^{(0)}(M) P_B^{(0)}(R) \quad (64)$$

where $1/\Delta_B^{(0)} = o(1)$ and $P_B^{(0)}(R)$ is a constant,

C say. Since $p_B^{(0)}(r)$ must be zero for $r \rightarrow \infty$
 equation (62) can be rewritten as

$$p_B^{(0)}(r) = S(r) - S(r \rightarrow \infty) \quad (65)$$

where $S(r)$ can be obtained from equation (56).

Order matching is now applied to these solutions.

Consider the one-term outer solution, equation (63), rewritten in inner co-ordinates

$$\bar{p}_B = M^2 \left\{ S(r_s + \sigma R) - S(\infty) \right\}$$

Applying the inner limit this becomes ($\sigma = O(1)$)

$$\bar{p}_B = M^2 \left\{ S(r_s) - S(\infty) \right\}$$

which to $O(\Delta_B^{(0)})$ is

$$\bar{p}_B = 0$$

The one-term inner solution written in outer co-ordinates is

$$\bar{p}_B = \Delta_B^{(0)}(M) C$$

which with the outer limit applied and retention of terms to $O(M^2)$ gives

$$\bar{p}_B = \Delta_B^{(n)}(M) C$$

Then order matching requires the constant C to be zero and thus the choice of $1/\Delta_B^{(n)} = O(1)$ does not produce a significant first inner solution for the magnitudes of σ considered in this section.

(ii) If $\Delta_B^{(n)}(M) = O(1)$ then the leading terms obtained from the substitution of equation (61) into equation (62) are

$$(\Delta_B^{(n)}(M) P_B^{(n)}(R) + 1)^3 \Delta_B^{(n)} \frac{dP_B^{(n)}(R)}{dR} = 0$$

The only possible significant solution of this equation is $P_B^{(n)}(R) = \text{constant}$, but arguments similar to those used in I(i) show that the constant must be zero and that consequently no significant solution is obtained again.

(iii) If $\Delta_B^{(n)}(M) = o(1)$ and $M^2 = o(\Delta_B^{(n)}(M))$ then the equation for $P_B^{(n)}(R)$ is

$$\Delta_B^{(n)}(M) \frac{dP_B^{(n)}(R)}{dR} = 0$$

which again has the solution $P_B^{(n)}(R) = \text{constant}$ and this constant can be shown to be zero using the arguments of I(i).

Since magnitudes of $\Delta_B^{(n)}(M)$ smaller than those of $\sigma(M)$ are not of interest a new magnitude for σ is now examined.

II $\sigma(M) = M^2$

(i) If $1/\Delta_B^{(n)}(M) = O(1)$ then the equation for $P_B^{(n)}$ is

$$(\Delta_B^{(n)}(M) P_B^{(n)}(R))^3 \Delta_B^{(n)}(M) \frac{dP_B^{(n)}}{dR} = 0$$

Again using the arguments developed in I(i) it can be shown that no useful inner solution is obtained for this $\Delta_B^{(n)}$.

(ii) If $\Delta_B^{(n)}(M) = O(1)$ the equation for $P_B^{(n)}$ is

$$(\Delta_B^{(n)} P_B^{(n)} + 1)^3 \Delta_B^{(n)} \frac{dP_B^{(n)}}{dR} = \frac{4}{3} Sc Ts \left\{ 2f_0 \Delta_B^{(n)2} \left(\frac{dP_B^{(n)}}{dR} \right)^2 - f_0 (\Delta_B^{(n)} P_B^{(n)} + 1) \Delta_B^{(n)} \frac{d^2 P_B^{(n)}}{dR^2} \right\}$$

This equation can be rewritten in the form

$$\frac{d}{dR} \Delta_B^{(0)}(M) P_B^{(0)}(R) = T_s f_0 \frac{4Sc}{3} \left\{ \frac{d}{dR} \left[- \frac{1}{(\Delta_B^{(0)} P_B^{(0)} + 1)^2} \Delta_B^{(0)} \frac{dP_B^{(0)}}{dR} \right] \right\}$$

Then writing

$$Y_B = 1 + \Delta_B^{(0)} P_B^{(0)}, \quad \gamma = \frac{4Sc T_s f_0}{3}$$

the equation becomes

$$Y_B = - \frac{\gamma}{Y_B^2} \frac{dY_B}{dR} + \bar{a}, \quad \bar{a} = \text{constant}$$

Then this can be rewritten as

$$\frac{dY_B}{dR} = \frac{\bar{a}}{\gamma} Y_B^2 - \frac{Y_B^3}{\gamma}$$

which is an Abel equation of the first kind. The

solution of this equation is

$$e^{\bar{a}/Y_B} \left[1 - \bar{a}/Y_B \right] = F e^{-\bar{a}R}$$

where F is an arbitrary constant. $[F=0$ produces no significant solution].

Order matching requires consideration of the behaviour of the function Y_B as $R \rightarrow \infty$ and imposes the condition that $Y_B \rightarrow 1$ as $R \rightarrow \infty$ for matching to occur. Three possible cases arise:-

$$\bar{a} < 0 : R \rightarrow \infty \implies e^{-\bar{a}R} \rightarrow \infty \text{ but}$$

this can only be achieved by the L.H.S.

if Y_B takes on smaller and smaller negative

values. However Y_B cannot be negative

and thus no allowable solutions of Y_B

exist for this case.

$$\bar{a} = 0 : Y_B = (R + \text{const})^{-1/2}. \text{ Then as } R \rightarrow \infty, Y_B \rightarrow 0$$

which does not satisfy the matching condition.

$$\bar{a} > 0 : R \rightarrow \infty \implies e^{-\bar{a}R} \rightarrow 0 \implies Y_B \rightarrow \bar{a}$$

Then matching can be achieved with the

first outer solution iff $\bar{a} = 1$.

(ii) If $\Delta_B^{(1)} = o(1)$ and $M^2 = o(\Delta_B^{(1)})$ then the equation for $P_B^{(1)}$ is

$$\Delta_B^{(1)} \frac{dP_B^{(1)}}{dR} = -\frac{4}{3} Sc Ts f_0 \Delta_B^{(1)} \frac{d^2 P_B^{(1)}}{dR^2} \quad (66)$$

The solution of the equation is

$$P_B'''(R) = b_{0B} + b_{1B} \exp -\frac{3R}{4ScT_s f_0} \quad b_{0B}, b_{1B} \text{ constants}$$

Order matching with the one-term outer solution gives no information at all on the coefficient b_{1B} but shows that b_{0B} must equal zero. Then

$$\bar{P}_B \sim \Delta_B'''(M) b_{1B} \exp -\frac{3R}{4Scf_0 T_s} ; \Delta_B''' = o(1), M^2 = o(\Delta_B''') \quad (67)$$

is a possible inner solution.

(iii) If $\Delta_B''' = M^2$ the equation for P_B''' is again

$$\Delta_B''' \frac{dP_B'''}{dR} = -\frac{4ScT_s f_0}{3} \Delta_B''' \frac{d^2 P_B'''}{dR^2} \quad (68)$$

The solution to this equation is

$$P_B'''(R) = a_{0B} + a_{1B} \exp -\frac{3R}{4Scf_0 T_s} ; a_{0B}, a_{1B} \text{ constants} \quad (69)$$

and thus

$$\bar{P}_B \sim M^2 \left(a_{0B} + a_{1B} \exp -\frac{3(r-r_s)}{4M^2 ScT_s f_0} \right)$$

expresses the one term inner solution in outer variables.

When the outer limit is applied this becomes

$$\bar{p}_B \sim M^2 a_{0B} + \text{exponentially small terms}$$

Then this is to $O(M^2)$

$$\bar{p}_B = M^2 a_{0B}$$

and consequently matching produces the result

$$a_{0B} = S(r_s) - S(\infty) \quad (70)$$

since it has already been shown that the inner limit applied to the first outer solution written in inner variables produces to $O(M^2)$

$$\bar{p}_B = M^2 \{ S(r_s) - S(\infty) \}$$

The application of the order matching process furnishes

no information about the constant coefficient a_{1B} .

To progress further in the calculation of the pressure

field it is necessary to focus attention on region A

($r_i < r < r_s$). This region is subdivided into inner

and outer regions in just the same way as was region B

and expressions for $\bar{p}$ in the outer and inner regions

are respectively given by

$$\bar{p}_A = \delta_A^{(0)}(M) p_A^{(0)}(r) + \delta_A^{(2)}(M) p_A^{(2)}(r) + \dots \quad (71)$$

$$\bar{p}_A = \Delta_A^{(0)}(M) p_A^{(0)}(R) + \Delta_A^{(2)}(M) p_A^{(2)}(R) + \dots \quad (72)$$

These expressions are exactly equivalent to those given for $\bar{p}$ in region B, equations (53) and (62). However, there is one major difference between the formulations of the pressure field problem in regions A and B and this is that there is no exact boundary condition known for the outer region in region A. This essentially means that the solutions of the inner and outer problems in region A must be tackled in tandem. Fortunately the absence of a known boundary condition on the pressure in the outer region is compensated for by the knowledge that the pressure at the flame sheet in region B is of one of the forms

$$p_s = 1 + O(1) \quad \text{or} \\ 1 + O(\Delta) \quad \text{where } \Delta = o(1), M^2 = o(\Delta), \text{ or } (73) \\ 1 + O(M^2)$$

and the requirement of continuity of pressure across

the flame sheet means that these can be regarded as the only possible values of the pressure at the flame sheet in the inner region of region A . The detailed methods of finding an outer solution and then trying various inner solutions until order matching can be achieved have been outlined earlier in the discussion of region B . Because of the equivalence of equations (71) and (72) to equations (53) and (62) and the use of the same differential equations for the outer and inner regions, equations (51) and (61) respectively, in the regions A and B exactly the same mathematical arguments are introduced and consequently in the discussion of region A only the framework of the full mathematical treatment will be displayed. Initially it is assumed that $1/\Delta_A^{(u)} = O(1)$. Substitution of equation (72) into equation (61) then leads to the result

$$P_A^{(u)}(R) = C_1 \quad (\text{constant})$$

In order to match this successfully with the one-term outer solution $\delta_A^{(u)}$ must be $ord(\Delta_A^{(u)})$ and then substitution of equation (71) into equation (51) leads

to the result

$$p_A^{(0)}(r) = \text{constant}$$

In particular if $\delta_A^{(0)}(M) = \Delta_A^{(0)}(M)$ matching gives

$$p_A^{(0)}(r) = C_1$$

but although this solution satisfies the matching criteria it does not give continuity of $\bar{p}$ at the sheet unless $C_1 = 0$. Consequently this is not the first significant solution in region A.

If it is now assumed that $\Delta_A^{(0)}(M) = O(1)$ the equation to solve for $p_A^{(0)}(R)$ can be rearranged to produce an Abel equation of the first kind

$$(\Delta_A^{(0)} p_A^{(0)} + 1) = - \frac{4ScTsfo}{3(\Delta_A^{(0)} p_A^{(0)} + 1)^2} \frac{d(\Delta_A^{(0)} p_A^{(0)} + 1)}{dR} + \text{constant}$$

Putting

$$Y_A = 1 + \Delta_A^{(0)} p_A^{(0)} ; \quad \gamma = \frac{4ScTsfo}{3}$$

and the constant equal to $\bar{b}$ the equation becomes

$$Y_A = -\frac{\gamma}{Y_A^2} \frac{dY_A}{dR} + \bar{b}$$

The solution of this equation is

$$e^{\bar{b}/Y_A} [1 - \bar{b}/Y_A] = G e^{-\bar{b}R}$$

Order matching requires consideration of the behaviour of the function Y_A as $R \rightarrow -\infty$. Since $G=0$ produces no significant solution three cases must be considered with $G \neq 0$.

$$\bar{b} < 0 : R \rightarrow -\infty \Rightarrow e^{-\bar{b}R} \rightarrow 0$$

Thus either $Y_A \rightarrow \bar{b}$ or $Y_A \rightarrow 0$ but since $\bar{b}$ is negative in this instance and Y_A must be non-negative the only valid result is $Y_A \rightarrow 0$ as $R \rightarrow -\infty$.

$$\bar{b} = 0 : \text{As } R \rightarrow -\infty \text{ no allowed values of } Y_A \text{ are obtained.}$$

$$\bar{b} > 0 : R \rightarrow -\infty \Rightarrow e^{-\bar{b}R} \rightarrow \infty \text{ Then } Y_A \rightarrow 0$$

Thus as $R \rightarrow -\infty$ the only allowed value of Y_A is zero. However with this result it is impossible to match with any significant outer solution, whatever

the choice of $\delta_A^{(1)}$, and consequently it must be concluded that the choice of $\Delta_A^{(1)} = O(1)$ does not produce the first significant solution for $\bar{p}$ in region A. This in turn means that the solution obtained in region B by choosing $\Delta_B^{(1)} = O(1)$ can now be rejected since the condition of continuity of $\bar{p}$ across the flame sheet is violated and the jump in pressure gradient across the flame sheet is of too great a magnitude.

If it is now assumed that $\Delta_A^{(1)} = o(1)$ and $M^2 = o(\Delta_A^{(1)})$ then $P_A^{(1)}(R)$ is given by

$$P_A^{(1)}(R) = b_{0A} + b_{1A} \exp - \frac{3R}{4Sc\phi T_s}$$

and this can be matched successfully with an outer solution with $\delta_A^{(1)} = \Delta_A^{(1)}$. With such a choice of $\delta_A^{(1)}$

$$\rho_A^{(1)}(r) = C_2 \text{ (constant)}$$

and then order matching gives

$$b_{0A} = C_2, b_{1A} = 0$$

This means however that when this inner solution and the complementary solution appropriate to region B are studied together $\left[\frac{dp}{dr} \right]^+ = O(\Delta)$ where $\Delta = o(1)$ and $M^2 = o(\Delta)$. Now reference to equation (44) gives $\left[\frac{dp}{dr} \right]^+ = O(1)$ and with $\sigma = M^2$ this means that $\left[\frac{dp}{dr} \right]^+ = O(M^2)$. This in turn implies that $\left[\frac{dp}{dr} \right]^+$ to $O(\Delta)$ must be zero and consequently

$$b_{1B} = 0$$

However continuity of $\bar{p}$ across the flame sheet means that

$$b_{0A} = b_{1B}$$

and thus

$$C_2 = b_{0A} = b_{1B} = 0$$

and again a solution which is zero everywhere has been uncovered once the conditions on the pressure and its gradient at the flame sheet have been utilised.

To produce a solution for $\bar{p}_A$ which is not everywhere zero it is required that

$$S_A^{(1)}(M) = M^2 \quad (74)$$

a result which supports the aforestated belief that

$p = 1 + M^2 \alpha(r)$ is a well-defined outer solution, failing only in the vicinity of the flame sheet.

The calculation now follows the lines laid out in section II (iii) with the one-term outer solution now given by

$$p_A^{(1)}(r) = t(r) + C_A \quad (\text{constant}) \quad (75)$$

where C_A is determined later in the calculation, and

if $\Delta_A^{(1)} = M^2$

$$p_A^{(1)}(R) = a_{0A} + a_{1A} \exp - \frac{3R}{4Sc_0 T_0} \quad (76)$$

Since it is region A under consideration $r < r_s$ ($R < 0$) and order matching of the one-term inner and outer solutions now requires

$$a_{1A} = 0 \quad (77)$$

$$a_{0A} = t(r_s) + C_A \quad (78)$$

The constants a_{1B} and C_A are so far undetermined

and to remedy this solution it is necessary to utilise all of the known information concerning the pressure and its first derivative with respect to the radial co-ordinate at and in the vicinity of the flame sheet.

Since the pressure must be continuous across the flame sheet the values of the pressure at the flame sheet given by the inner solutions in regions A and B must be equal. Thus

$$Q_{OB} + Q_{IB} = Q_{OA} \quad (79)$$

Reference to equation (44) gives

$$\left[\frac{dp}{dr} \right]_{-}^{+} = \frac{\dot{m} R}{C_c u_s r_s^2} \left\{ \sum_{\alpha} \frac{c_{\alpha}}{W_{\alpha}} \left[\frac{dT}{dr} \right]_{-}^{+} + T \sum_{\alpha} \frac{1}{W_{\alpha}} \left[\frac{dc_{\alpha}}{dr} \right]_{-}^{+} \right\} \quad (80)$$

Now the quantity f_0 is defined in such a way that

$$(pu)_s = f_0 + o(1) \quad (81)$$

and

$$p_s = 1 + M^2 (Q_{OB} + Q_{IB}) \quad (82)$$

Thus to $O(1)$ the velocity at the sheet is determined

$$u_s = f_0 + o(1) \quad (83)$$

and then since $T^{(1)}$ and $c_\alpha^{(1)}$ are the $O(1)$ solutions for the temperature and mass fraction fields,

$$\left[\frac{dp}{dr} \right]_-^+ = \frac{mR}{C_c f_0 r_s^2} \left\{ \sum_\alpha \frac{c_\alpha^{(1)}}{W_\alpha} \left[\frac{dT^{(1)}}{dr} \right]_-^+ + T^{(1)} \sum_\alpha \frac{1}{W_\alpha} \left[\frac{dc_\alpha^{(1)}}{dr} \right]_-^+ \right\} + o(1) \quad (84)$$

Thus the pressure gradient discontinuity is known to

$O(1)$ and equation (84) is rewritten as

$$\left[\frac{dp}{dr} \right]_-^+ = \Gamma + o(1) \quad (85)$$

where Γ is given by equation (84). However the discontinuity in pressure gradient can also be calculated from the inner solutions, equations (69) and (76), giving

$$\left[\frac{dp}{dr} \right]_-^+ = -\frac{3M^2 a_{12}}{4Sc f_0 T_s} + o(M^2) \quad (86)$$

and consequently

$$Q_{1B} = -f_0 T_s \Gamma \frac{4Sc}{3} \quad (87)$$

Equation (87) completely specifies Q_{1B} and leaves only C_A to be specified. Eliminating Q_{1B} between equations (79) and (87) produces

$$Q_{0A} = Q_{0B} - f_0 T_s \Gamma \frac{4Sc}{3} \quad (88)$$

which completely determines Q_{0A} and hence with reference to equation (78) determines C_A . At this point it appears that the pressure field problem has been completely solved to $O(M^2)$ in both regions A and B . However there is one aspect of the solution obtained above which might be regarded as unsatisfactory. In calculating the outer solution for the pressure field in region A it was assumed that the pressure at the fuel sphere was not known but subsequently it is seen that the pressure field is completely specified throughout the whole of regions A and B , and in particular, the pressure at the fuel sphere is given. For some problems this situation might be regarded as

completely satisfactory but certainly in others the pressure at the fuel sphere would be a fixed quantity and would be used as another boundary condition on the pressure field and in those circumstances the pressure at the fuel sphere must certainly not be completely specified by the solution obtained so far. In general the solution obtained above can give either a) the correct jump in pressure gradient across the sheet and an incorrect pressure at the fuel sphere or b) the correct pressure at the sphere and an incorrect pressure gradient jump across the sheet provided that the pressure at the sphere is $1 + O(M^2)$.

In this next part of the calculation a solution is found which is correct both in its value at the fuel sphere and in its pressure gradient jump across the sheet. The simplest procedure is to take the solution obtained above when the pressure gradient jump is given correctly and the solution is deficient in the vicinity of the fuel sphere. To counter this deficiency another region of rapid pressure variation is postulated to be present in close proximity to the fuel sphere.

This new inner region is characterised by the variable $\bar{R}$ where

$$r - r_i = \bar{\sigma}(M) \bar{R} ; \bar{\sigma}(M) = o(1) \quad (89)$$

The equation relating the pressure and the variable $\bar{R}$ is obtained from equation (51)

$$\begin{aligned} (\bar{p}+1)^3 \bar{p}_{\bar{R}} = M^2 \left\{ \frac{4Sc}{3} \frac{T}{\bar{\sigma}} \left[(\bar{p}+1)^2 f_{\bar{R}\bar{R}} - 2(\bar{p}+1) f_{\bar{R}} \bar{p}_{\bar{R}} + 2f_{\bar{R}}^2 - f(\bar{p}+1) \bar{p}_{\bar{R}\bar{R}} \right] \right. \\ \left. + \frac{4ScT}{3} \left[\frac{2(\bar{p}+1)^2 f_{\bar{R}}}{r_i + \bar{\sigma} \bar{R}} - \frac{2(\bar{p}+1) f \bar{p}_{\bar{R}}}{r_i + \bar{\sigma} \bar{R}} - \frac{2(\bar{p}+1)^2 f \bar{\sigma}}{(r_i + \bar{\sigma} \bar{R})^2} \right] \right. \\ \left. + T_{\bar{R}} \frac{4Sc}{3} \left[\frac{1}{\bar{\sigma}} \left\{ (\bar{p}+1)^2 f_{\bar{R}} - (\bar{p}+1) f \bar{p}_{\bar{R}} \right\} - \frac{(\bar{p}+1)^2 f}{r_i + \bar{\sigma} \bar{R}} \right] \right. \\ \left. - \frac{m}{(r_i + \bar{\sigma} \bar{R})^2} \left[(\bar{p}+1)^2 f_{\bar{R}} - (\bar{p}+1) f \bar{p}_{\bar{R}} \right] \right\} + o(M^2) \quad (90) \end{aligned}$$

where

$$f_{\bar{R}} = O(\bar{\sigma}), f_{\bar{R}\bar{R}} = O(\bar{\sigma}^2)$$

At this stage of the calculation no precise information is known about $\bar{\sigma}$ or $\bar{\Delta}^{(u)}$ where

$$\bar{p} \sim \bar{\Delta}^{(u)} \bar{p}^{(u)}(R) \quad (91)$$

in the inner region characterised by the variable $\bar{R}$.

However it is assumed in all the subsequent work that

$\bar{\sigma} = o(1)$ and moreover it is known that in region

A the outer solution has the form

$$\bar{p} \sim M^2 p_A''(r) \quad (92)$$

$$\text{III} \quad \underline{M^2 = o(\bar{\sigma})}$$

$$(i) \quad 1/\bar{\Delta}^{(0)} = o(1)$$

Substituting equation (91) into equation (90) and retention of the highest-order terms leads to the following equation for $\bar{p}^{(0)}$.

$$\bar{\Delta}^{(0)3} \bar{p}^{(0)3} \bar{\Delta}^{(0)} \frac{d\bar{p}^{(0)}}{d\bar{R}} = 0$$

The only useful solution of this equation is $\bar{p}^{(0)}(\bar{R}) = \text{constant}$, but matching this with the expression for $\bar{p}$ in equation (92) leads to the conclusion that the constant must be zero, that is this solution for

$\bar{p}^{(0)}(\bar{r})$ is zero everywhere and hence is not significant.

$$(ii) \quad \bar{\Delta}^{(0)} = O(1)$$

This choice of $\bar{\Delta}^{(0)}$ again leads to $\bar{p}^{(0)}(\bar{r})$ equal to zero everywhere in the inner region and is consequently not significant

$$(iii) \quad \bar{\Delta}^{(0)} = o(1) ; M^2 = o(\bar{\Delta}^{(0)})$$

This set of choices of $\bar{\Delta}^{(0)}$ again produces no significant solution since $\bar{p}^{(0)}(\bar{r})$ is everywhere zero in the inner region.

$$(iv) \quad \bar{\Delta}^{(0)} = O(M^2)$$

The equation for $\bar{p}^{(0)}(\bar{r})$ is

$$\bar{\Delta}^{(0)}(M) \frac{d\bar{p}^{(0)}(\bar{r})}{d\bar{r}} = 0$$

which has the simple solution

$$\bar{p}^{(u)}(\bar{r}) = \text{constant}$$

Order matching this solution with the solution for $\bar{p}_A$ given in equation (92) produces the result that the constant is equal to $p_A^{(u)}(r=r_i)$. This means that again there is no degree of arbitrariness in the solution with which one can obtain agreement with the boundary condition on the pressure at the fuel sphere.

$$(v) \quad \bar{\Delta}^{(u)} = O(M^2)$$

This set of choices of $\bar{\Delta}^{(u)}$ produces solutions for the inner problem which cannot be matched with the one-term outer solution.

$$\text{IV} \quad \underline{\bar{\sigma}} = M^2$$

$$(i) \quad 1/\bar{\Delta}^{(u)}(M) = O(1)$$

This set of choices for $\bar{\Delta}^{(u)}(M)$ produce the solution $\bar{p}^{(u)}(\bar{r}) = \text{constant}$ which will then only

match the one-term outer solution if the constant is zero.

$$(ii) \quad \bar{\Delta}^{(1)} = O(1)$$

The equation for $\bar{P}^{(1)}(\bar{R})$ is

$$(\bar{\Delta}^{(1)} \bar{P}^{(1)} + 1)^3 \bar{\Delta}^{(1)} \frac{d\bar{P}^{(1)}}{d\bar{R}} = \frac{4ScT_i}{3} \left\{ 2F_0 \bar{\Delta}^{(1)2} \left(\frac{d\bar{P}^{(1)}}{d\bar{R}} \right)^2 - F_0 (\bar{\Delta}^{(1)} \bar{P}^{(1)} + 1) \bar{\Delta}^{(1)} \frac{d^2 \bar{P}^{(1)}}{d\bar{R}^2} \right\}$$

where T_i is the temperature at the surface of the fuel sphere and $F_0 \equiv f(r_i)$. This equation can be rewritten in the form

$$Y = - \frac{4ScT_i F_0}{3 Y^2} \frac{dY}{d\bar{R}} + \text{constant}$$

where

$$Y = 1 + \bar{\Delta}^{(1)} \bar{P}^{(1)}(\bar{R})$$

It can now be readily shown (see Section II (ii)) that this inner solution will only match with the one-term outer solution if the constant is equal to unity.

Thus provided that

$$p_i + \frac{4ScT_iF_0}{3p_i^2} \left(\frac{dp}{d\bar{R}} \right)_{\bar{R}=0} = 1 + o(1)$$

a solution to the inner problem giving

$$p_i = 1 + O(1)$$

is allowable. The solution is then given by

$$e^{1/4} [1 - 1/4] = H e^{-\bar{R}} ; p = \gamma + o(\bar{\Delta}^{(n)})$$

where H is an arbitrary constant

$$(iii) \quad \bar{\Delta}^{(n)}(M) = o(1) , M^2 = o(\bar{\Delta}^{(n)})$$

In this case the equation for $\bar{P}^{(n)}$ has the form

$$\bar{\Delta}^{(n)}(M) \frac{d\bar{P}^{(n)}(\bar{R})}{d\bar{R}} = -T_i \frac{4}{3} Sc F_0 \bar{\Delta}^{(n)}(M) \frac{d^2\bar{P}^{(n)}(\bar{R})}{d\bar{R}^2}$$

and then

$$\bar{P}^{(n)}(\bar{R}) = c_1 + c_2 \exp -3\bar{R}/4ScT_iF_0 \quad (93)$$

Matching with the first order solution in region A requires that the constant C_1 is zero but no information is given concerning C_2 . Thus a significant solution has been obtained if the pressure at the fuel sphere has a magnitude $O(\Delta)$ where $\Delta = o(1)$ and $M^2 = o(\Delta)$ and under these circumstances the coefficient C_2 is fixed by the pressure at the surface of the fuel sphere.

$$(iv) \quad \bar{\Delta}^{(1)}(M) = M^2$$

The equation for $\bar{p}^{(1)}(\bar{r})$ is

$$\frac{d\bar{p}^{(1)}(\bar{r})}{d\bar{r}} = -T_i F_0 \frac{4}{3} Sc \frac{d^2 \bar{p}^{(1)}(\bar{r})}{d\bar{r}^2}$$

and then

$$\bar{p}^{(1)}(\bar{r}) = \bar{a}_0 + \bar{a}_1 \exp - 3\bar{r} / 4ScT_i F_0 \quad (94)$$

Order matching with the one-term outer solution in region

A fixes $\bar{a}_0$ but leaves $\bar{a}_1$ undetermined. This indeterminacy allows the solution to give the correct value of the pressure at the surface of the fuel sphere

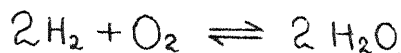
by an appropriate choice of $\bar{\alpha}_i$ provided that p_i has the form $1 + O(M^2)$.

8. Numerical Results

The calculations presented so far in this work have been concerned with a general reaction between fuel, oxidant and product species with a non-reacting diluent species also present. To obtain some numerical results for the pressure field associated with a diffusion flame it is necessary to refer to a specific reaction of the type described by equation (2). Melvin et al (1971) have carried out experiments on the hydrogen-oxygen diffusion flame in a spherically symmetric configuration. In their experiments the temperature and mass fraction fields associated with diffusion flames, obtained by burning hydrogen diluted with nitrogen in air, were examined. Their results for the temperature field associated with a spherical hydrogen-oxygen diffusion flame are now used to obtain reasonable values for some of the parameters appearing in the expressions for the temperature field in regions B and A , equations

(38a) and (38b). For the sake of simplicity it was assumed in the main calculation that the mass fraction of the diluent is constant throughout the region of interest. Fortunately this is quite a good description of one of the mixtures studied by Melvin et al. The mixture in question is one in which the fuel mixture consisted of 25% by mass hydrogen and 75% by mass nitrogen and the oxidant mixture was air. (In dry air the mass fraction of oxygen is about 0.232 and the mass fraction of nitrogen and argon is about 0.768). Thus for making the numerical calculations it is assumed that the mass fraction of diluent is 0.75 throughout the region of interest, and that the fuel is hydrogen and the oxidant oxygen.

The stoichiometric equation for the one-step hydrogen-oxygen reaction is



The molecular weight of hydrogen is 2 and that of oxygen is 32 and consequently the parameter A defined in equation (30b) has the value 8 for this reaction and

then using equation (31) it can easily be seen that

$$y_{\infty} = 0.118$$

The results of Melvin et al suggest that if the temperature far from the fuel sphere and flame sheet (i.e. T_{∞}') is chosen to be the characteristic temperature then the temperature at the surface of the fuel sphere for the reaction under consideration is

$$T_i = 2.6, \quad (T_{\infty} = 1)$$

A good approximation for the specific heats of the hydrogen nitrogen and oxygen species can be obtained by assuming that they behave like ideal diatomic gases and then since the specific heat of the fuel species is chosen to be the characteristic specific heat and the molar heat capacity is equal for all ideal diatomic gases

$$C_F = 1, \quad C_X = 1/16, \quad C_D = 1/14$$

The product species for the reaction under consideration is H_2O and clearly one should not expect this to behave like a diatomic gas, ideal or otherwise, but surprisingly

the assumption that it too has the same molar heat capacity produces quite a good approximation for its specific heat, but a more realistic value is

$$C_p = 1/6$$

With these values for the specific heats the coefficients α and α^* , defined in equation (37), have the numerical values

$$\alpha = \alpha^* = 0.304$$

As the specific heat of hydrogen is taken to be the characteristic specific heat the quantity C_c/R can be simply evaluated. If $\hat{C}'$ is the molar heat capacity then for an ideal diatomic gas in the temperature ranges under consideration

$$\hat{C}' = 7R/2$$

and then since the molecular weight of hydrogen is 2

$$C_c/R = 7/4$$

Q_p is now the dimensionless heat of formation of water,

and taking the temperature T_{∞}' to be 300°K this is

$$Q_p = \frac{\Delta H_f / W_p}{C_c' T_{\infty}'} = \frac{-241,826 / 18}{\gamma/4 \times 8.31 \times 300} = -3.08$$

where ΔH_f is the dimensionless heat of formation of water and is equal to $-241,826$ joules/mole.

Reference to equations (38a) and (38b) for the temperature fields in regions A and B show that two parameters appear which are as yet undetermined; y_i and Re .

The choice of these parameters is made in such a way that the temperature fields in regions A and B, including the temperature at the flame sheet, are reasonable realistic. The values of the parameters used here are

$$y_i = -0.18, Re = 6$$

With the values of the various quantities as prescribed above the quantity H , defined by equation (39) has the value

$$H = -0.8403$$

and the temperature at the flame sheet is given by

$$T_s = 4.6845$$

The dimensional characteristic diffusion coefficient is chosen to be $D_c = 1 \text{ cm}^2/\text{sec}$ and this is equal to the product of the characteristic length and characteristic velocity. Diffusion velocities are usually a few centimetres per second so a convenient choice for u_c is

$$u_c = 1 \text{ cm/sec.}$$

when the characteristic diffusion length becomes automatically

$$L_c = 1 \text{ cm.}$$

In the experiments carried out by Melvin et al the fuel sphere used had a radius of 1.25 cm and consequently

$$r_i = 1.25$$

The next step in the calculation is the numerical evaluation of the radius of the flame sheet r_s . This involves the use of the transformation relating the variables r and y and the modelling of the quantity $\rho\theta$. The use of the model $\rho\theta = T$ has been justified earlier and with this model employed equation (32) becomes

$$\dot{m} \left(\frac{1}{r_s} - \frac{1}{r} \right) = \int_0^y T^{(n)}(\tilde{y}) d\tilde{y} \quad (95)$$

In particular using equation (38a)

$$\frac{\dot{m}}{r_s} = \int_0^{y_0} T_B^{(1)}(\tilde{y}) d\tilde{y} = \left[H - (A+1)(1-c_D)Q_P \right] \frac{y_0}{a^*} + \left\{ T_s - \frac{[H - (A+1)(1-c_D)Q_P]}{a^*} \right\} \frac{[\exp a^* k y_0 - 1]}{a^* Re} \quad (96)$$

and using equation (38b)

$$\dot{m} \left(\frac{1}{r_s} - \frac{1}{r_i} \right) = \frac{H y_i}{a} + \left(T_i - \frac{H}{a} \right) \frac{1}{a k e} [1 - \exp - a k e y_i] \quad (97)$$

Then inserting the numerical values for the quantities involved, and dividing the second expression by the first an expression for r_s/r_i is obtained

$$1 - r_s/r_i = -1.8773$$

Then, since r_i is known, it is easy to obtain

$$r_s = 3.5966 \quad (98)$$

and the dimensionless mass flux $\dot{m}$ has the value

$$\dot{m} = 1.2362 \quad (99)$$

In region B the expressions for the temperature field

and $\sum C_{\alpha}^{(1)}/W_{\alpha}$ are of the form

$$T^{(1)}(y) = 20.0319 - 15.3474 \exp 1.824 y \quad (100)$$

$$\sum_{\infty} \frac{C_{\alpha}^{(1)}}{W_{\alpha}} = 0.08928 - 0.04861 \exp y \quad (101)$$

and using equations (38b) and (95) and inserting appropriate numerical values for the quantities involved

$$\frac{1}{r} = -6.5285 - 16.2044 y + 6.8065 \exp 1.824 y; 0 < y \leq y_s \quad (102)$$

Now equation (56) gives $p_B^{(1)}(r)$, the first outer solution for $\bar{p}$ in region B,

$$\begin{aligned} p_B^{(1)}(r) &= T^{(1)} \frac{4Sc}{3} \left(fr - \frac{f}{r} \right) + 4Sc T^{(1)} \frac{f}{r} - 4Sc \int \frac{f}{r} \frac{dT^{(1)}}{dr} dr - \int \frac{\dot{m}}{r^2} fr dr + \text{const.} \\ &= T^{(1)} \frac{4Sc}{3} \left(fr + \frac{2f}{r} \right) - 4Sc \int \frac{f}{r} \frac{dT^{(1)}}{dr} dr - \int \frac{\dot{m}}{r^2} fr dr + \text{const.} \\ &= T^{(1)} \frac{4Sc}{3} \left(fr + \frac{2f}{r} \right) - \int_{y_s}^y 4Sc \frac{f}{r} \frac{dT^{(1)}}{dy} dy - \frac{\dot{m}f}{r^2} - 2\dot{m} \int_{y=y_s}^{y=y} \frac{f}{r^2} dr \end{aligned}$$

But when $\rho\Phi = T$

$$\frac{\dot{m}}{r^2} dr = T dy$$

and consequently

$$p_b^{(1)}(r) = \frac{4Sc}{3} T^{(1)} \left(f_r + \frac{2f}{r} \right) - \int_{y_0}^y \frac{4Sc}{r} \frac{f}{dy} \frac{dT^{(1)}}{dy} dy - \frac{\dot{m}f}{r^2} - 2 \int_{y_0}^y \frac{fT}{r} dy$$

The value of the dimensionless Lewis number Le used in these calculations is $Le = 6$. This implies that the Lewis number Le is itself approximately equal to six since the other factors appearing in the definition of Le , equation (36c), should all be approximately equal to unity. For convenience it is assumed that the Lewis number Le is equal to six. Since the Prandtl number is approximately equal to three-quarters for most gases the Schmidt number Sc is taken to be

$$Sc = \frac{Pr}{Le} = \frac{1}{8}$$

for the remainder of this calculation. Thus

$$p_b^{(1)}(r) = \frac{T^{(1)}}{6} \left(f_r + \frac{2f}{r} \right) - \frac{1}{2} \int_{y_0}^y \frac{f}{r} \frac{dT^{(1)}}{dy} dy - \frac{\dot{m}f}{r^2} - 2 \int_{y_0}^y \frac{fT}{r} dy \quad (103)$$

The function f is defined in equation (47) and by using that equation in conjunction with equations (100)

(101) and (102) $p_B^{(1)}$ can be evaluated as a function of y . Since equation (102) gives r as a function of y it is possible to calculate $p_B^{(1)}$ in terms of r . The numerical results for $p_B^{(1)}$ as a function of y are given below

TABLE I	y	$p_B^{(1)} \times 10^3$	r
	0.0000+	-2.495	3.5966+
	0.0300	-0.364	5.747
	0.0600	-0.038	10.75
	0.0900	-0.001	29.41
	0.1180	0.0000	∞

The outer solution in region A is now considered.

Using equations (30a) and (16) the quantity $\sum_{\alpha} \frac{C_{\alpha}^{(1)}}{W_{\alpha}}$ can be written in the form

$$\sum_{\alpha} \frac{C_{\alpha}^{(1)}}{W_{\alpha}} = \frac{C_D}{W_D} + (1 - C_D) \left\{ \frac{1}{W_F} + \left(\frac{1}{W_P} - \frac{1}{W_F} \right) \exp y \right\}$$

$$= 0.15178 - 0.1111 \exp y \quad 0 > y \geq y_i \quad (104)$$

when $C_D = 3/4$, $W_F = 2$, $W_P = 18$ and $W_D = 28$

The temperature field in region A is given by equation (38c) and when the appropriate numerical values are given to the quantities appearing in the equation this becomes

$$T^{(u)}(y) = -2.7642 + 7.4487 \exp 1.824 y \quad 0 > y \geq y_i \quad (105)$$

Now, inserting this result into equation (95) and putting $m = 1.236$ gives

$$\frac{1}{r} = 3.5815 + 2.2360 y - 3.3034 \exp 1.824 y \quad (106)$$

It is clear from examination of equations (65) and (70) that

$$Q_{0B} = p_B^{(u)}(r_s +)$$

and consequently reference to Table I gives

$$Q_{0B} = -2.495 \times 10^{-3} \quad (107)$$

Combining equations (84), (85) and (87) and noting that

$$\text{if } e^{\Phi} = T$$

$$\frac{dy}{dr} = \frac{m}{Tr^2}$$

the coefficient Q_{1B} can be written as

$$Q_{1B} = -\frac{4}{3} Sc f_0 T_s \Gamma = -\frac{4}{3} Sc \frac{\dot{m}}{R_s^4} \frac{\dot{m} R}{C_c} \left[\left(\sum_{\alpha} \frac{C_{\alpha}^{(11)}}{W_{\alpha}} \right)_s \frac{dT^{(11)}}{dy} \right]_s^+ + T_s^{(11)} \sum_{\alpha} \frac{1}{W_{\alpha}} \left. \frac{dC_{\alpha}^{(11)}}{dy} \right]_s^+$$

Now, both $\left. \frac{dT^{(11)}}{dy} \right]_s^+$ and $\sum_{\alpha} \frac{1}{W_{\alpha}} \left. \frac{dC_{\alpha}^{(11)}}{dy} \right]_s^+$ can be easily obtained, from equations (100) and (105) and equations (101) and (104) respectively, giving the result

$$Q_{1B} = 0.118 \times 10^{-3} \quad (108)$$

and then using equations (79), (107) and (108) Q_{0A} is obtained

$$Q_{0A} = -2.377 \times 10^{-3} \quad (109)$$

The function $p_A^{(11)}(r)$ can be expressed in a similar way to that given for $p_B^{(11)}$ in equation (103)

$$p_A^{(11)}(r) - p_A^{(11)}(r_s) = \left\{ \frac{T^{(11)}}{6} \left(fr + \frac{2f}{r} \right) - \frac{1}{2} \int_0^y \frac{f}{r} \frac{dT^{(11)}}{dy} dy - \frac{\dot{m}f}{r^2} - 2 \int_0^y \frac{fT}{r} dy \right\} - \left[\frac{T^{(11)}}{6} \left(fr + \frac{2f}{r} \right) - \frac{\dot{m}f}{r^2} \right]_{r=r_s} \quad (110)$$

Reference to equations (75) and (78) gives

$$Q_{0A} = p_A^{(11)}(r_s) \quad (111)$$

and using this result and equations (104), (105), (106) and (109) the following results can be obtained

TABLE II	y	$\bar{p}_A^{(1)} \times 10^3$	r
	0.000-	-2.377	3.5966-
	-0.040	-2.37)	2.374
	-0.080	-1.082	1.826
	-0.120	+3.156	1.517
	-0.180	+18.717	1.250

The one-term inner solution in region B has the form

$$\bar{p}_B = M^2 \left[a_{0B} + a_{1B} \exp - \frac{3R}{4Sc_0T_s} \right]$$

which in numerical terms is

$$\bar{p}_B = M^2 \left[-2.495 \times 10^{-3} + 0.118 \times 10^{-3} \exp - 123.1 R \right] \quad (112)$$

The one-term inner solution in region A is

$$\bar{p}_A = - M^2 \times 2.377 \times 10^{-3} \quad (113)$$

The numerical calculations carried out above were

all specifically for the model $\rho\theta = T$. When choosing this model it was made clear that its simplicity was one of its greatest advantages. However, this policy could be pursued further producing an even simpler model, $\rho\theta = \text{constant}$. In general it is expected that this simpler model will produce results inferior to those produced using the earlier model but in this instance the simpler model is used merely to test the sensitivity of the final results to the specific model used to describe $\rho\theta$.

Again the results of Melvin et al (1971) for the spherical hydrogen-oxygen flame are utilised and consequently a, a^*, y_o, T_o, T_i and Q_p have the same numerical values as in the previous calculations. Furthermore the quantities r_i and $\dot{m}$ are also given the same numerical values leaving only y, Re and r_s to be specified. These three quantities are connected through the temperature fields and the $r-y$ transformation and it is simply for convenience that the values of y_i and Re are chosen to be the same as before and then a new value for r_s must be calculated.

With $\rho\theta = \text{constant}$ the transformation between

the variables r and y is given by

$$\frac{\dot{m}}{\rho D} \left(\frac{1}{r_s} - \frac{1}{r} \right) = y \quad (114)$$

and consequently

$$1 - \frac{r_s}{r_i} = \frac{y_i}{y_o}$$

With

$$r_i = 1.25, y_i = -0.18, y_o = 0.118; r_s = 3.1568$$

Also

$$\frac{\dot{m}}{\rho D r_s} = y_o$$

and thus since $m = 1.236$

$$\rho D = 3.319$$

With $\rho D = \text{constant}$ equation (45) can be integrated more easily with respect to r giving

$$p = M^2 \left\{ \frac{4}{3} \rho^{\oplus} \text{Sc} \left(\frac{du}{dr} + \frac{2u}{r} \right) - \int \frac{\dot{m}}{r^2} \frac{du}{dr} dr \right\} + \text{constant}$$

and when the function f is introduced and the outer solutions shown to be $\sim 1 + O(M^2)$ this expression can be replaced by

$$p = M^2 \left\{ \frac{4}{3} \rho^{\oplus} \text{Sc} \left[f r + \frac{2f}{r} \right] - \int \frac{\dot{m}}{r^2} f dr \right\} + \text{constant} + o(M^2)$$

The calculation for $\rho^{\oplus} = \text{constant}$ follows the same lines as did the one for $\rho^{\oplus} = T$ and the results are as follows:-

Region B : Outer solution $p = 1 + M^2 p_B^{(1)}(r)$

TABLE III	r	$p_B^{(1)}(r) \times 10^3$
	3.1566+	-3.334
	4.234	-0.966
	6.423	-0.165
	13.30	-0.007
	∞	0.000

Inner solution $p = 1 + 10^{-3} M^2 \left[-3.334 + 0.198 \exp -133.9 R \right] \quad (115)$

Region A ($r_c < r < r_s$)

Outer solution $p = 1 + M^2 p_A^{(1)}(r)$

TABLE IV	r	$p_A^{(1)}(r) \times 10^3$
	3.1566	-3.316
	2.357	-4.658
	1.882	-6.264
	1.565	-7.333
	1.250	-5.890

Inner solution $p = 1 - 3.136 \times 10^{-3} M^2 \quad (116)$

When a comparison is made of the results obtained for the pressure field using the two distinct models for ρ it is readily seen that they are of the same qualitative form except in the outer part of region A. There, it is seen that one model predicts a pressure increase as the fuel sphere is approached whereas the other predicts a pressure decrease. This contradiction suggests that the sign, but not the magnitude, of $p_A^{(1)}$ is sensitive to the model for ρ whereas $p_B^{(1)}$, $p_B^{(1)}$ and $p_A^{(1)}$ are insensitive qualitatively to the modelling of ρ .

In this work the Mach number M has been assumed to be a small parameter, but no effort has been made so far to quantify this smallness. The Mach number is defined, in this calculation, by

$$M^2 = \rho_c u_c^2 / p_c$$

Now the characteristic pressure and density have been chosen to be the appropriate dimensional quantities as $r \rightarrow \infty$ and consequently, using equation (6) rewritten in dimensional form,

$$\frac{p_c}{\rho_c} = R \left[T' \sum_{\alpha} \frac{C_{\alpha}}{W_{\alpha}} \right]_{r \rightarrow \infty}$$

The temperature as $r \rightarrow \infty$ which is also the characteristic temperature has been taken as 300°K . Moreover as $r \rightarrow \infty$ the mass fraction of product species tends to zero so that

$$\sum_{\alpha} \frac{C_{\alpha}}{W_{\alpha}} \rightarrow \frac{C_x(r \rightarrow \infty)}{W_x} + \frac{C_o(r \rightarrow \infty)}{W_o}$$

For the experiment in question

$$C_x(r \rightarrow \infty) = \frac{1}{4}, \quad C_D(r \rightarrow \infty) = \frac{3}{4}$$

and then writing R in ergs/mole/ $^{\circ}\text{K}$, P_c/ρ_c is given in c.g.s. units.

$$\begin{aligned} \frac{P_c}{\rho_c} &= 8.315 \times 10^7 \times 300 \times \left[\frac{1}{4} \times \frac{1}{32} + \frac{3}{4} \times \frac{1}{28} \right] \text{ cm}^2/\text{sec}^2 \\ &= 8.65 \times 10^8 \text{ cm}^2/\text{sec}^2 \end{aligned}$$

Now $u_c = 1 \text{ cm./sec.}$, thus

$$M^2 = 1.16 \times 10^{-9} \quad (117)$$

and since it is this parameter that appears as the scaling factor in both the one-term inner and outer solutions for the use of the small parameter perturbation methods employed in this work is fully justified.

9. Some Comparisons with the work of Rasmussen (1974)

All of the work described above has been concerned with a diffusion flame in a steady-state situation, but some interesting comparisons can nevertheless be made

with Rasmussen's recent study of diffusion flames as time-dependent phenomena. Rasmussen proposes a linearised theory for weak explosions associated with planar, cylindrical and spherical symmetries. A simple combustion model, identical to the one employed in this work and involving only fuel, oxidant and product species, is used. This theory is then applied to the one-dimensional linearised shock-tube problem and the spherical flame associated with a weak spherical explosion. The conclusions of particular interest and relevance to the work concern the discontinuities of gradient of the pressure and other variables across the flame sheet. When discussing the one-dimensional shock-tube problem Rasmussen calculates the discontinuity in the density gradient across the flame sheet and shows that it is stronger than the discontinuity in pressure gradient after long times. However, earlier in this work it was shown that the density gradient is in fact continuous rather than discontinuous across the flame sheet, and this general result is clearly not in agreement with Rasmussen's specific example. The reason for this apparent contradiction lies in Rasmussen's use of a linearised theory as some

brief analysis using the continuity equation demonstrates. The continuity equation for a time-dependent system with one space variable (denoted by x) is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = -\rho \frac{\partial u}{\partial x} - u \frac{\partial \rho}{\partial x}$$

where ρ and u are the non-dimensional density and velocity respectively and where the equation has been written so that the non-linear terms appear on the right-hand side. In the linearised theory

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0$$

and taking derivatives with respect to x

$$\frac{\partial}{\partial t} \left(\frac{\partial \rho}{\partial x} \right) + \frac{\partial^2 u}{\partial x^2} = 0$$

However, $\frac{\partial^2 u}{\partial x^2}$ is discontinuous across the flame sheet for both steady and unsteady systems and consequently when the linearised theory is in use the discontinuity in $\frac{\partial^2 u}{\partial x^2}$ implies a discontinuity in $\frac{\partial \rho}{\partial x}$. In the full non-linear theory ρ, u and $\frac{\partial u}{\partial x}$ are all continuous across the flame sheet and then reference

to the continuity equation itself immediately gives the requirement of continuity of ρ/ρ_0 across the flame sheet.

Another interesting aspect of Rasmussen's work is his examination of the discontinuities of the gradients of pressure and density across the flame sheet for asymptotically large times. Rasmussen is primarily concerned with time-dependent phenomena, but it seems reasonable to anticipate that as $t \rightarrow \infty$ his results should be equivalent to those obtained for a steady-state system. However Rasmussen shows that as $t \rightarrow \infty$ the discontinuity in the pressure gradient tends to zero; whereas in this work it has been shown that in a steady-state situation there is a finite discontinuity in the pressure gradient. In both Rasmussen's study and this work the initial discontinuity uncovered is that in the mass fraction gradients, and from those discontinuities follow the discontinuity in temperature gradient and hence the discontinuity in pressure gradient.

In this work it is assumed that a steady-state situation can be attained in which there is a finite discontinuity in mass fraction gradient which means essentially that infinite sources of fuel and oxidant

must be available, whereas Rasmussen assumes that there are only small amounts of fuel and oxidant available in the system and thus as time progresses these reactants are used up to form product until literally as $t \rightarrow \infty$ the reaction rate $\rightarrow 0$ since there are no supplies of reactants. In this limit the mass fractions of all species are constant on both sides of the flame sheet and consequently their gradients are all zero producing zero discontinuities in the gradients of the mass fractions across the flame sheet. This, in turn, leads to zero discontinuities in the gradients of temperature and pressure.

In practical terms one cannot become involved with infinite time scales or infinite sources of reactants and moreover a truly steady-state system is unattainable. A real experimental situation would be at best a quasi-steady state in which the variables to be measured are time independent with respect to the time the experiment is in progress, and it is for the analysis of such systems, with large but finite sources of reactants available, that the results obtained above apply. Such a situation is obtained by Rasmussen for large, but finite, times and thus once again no real contradiction is found between

Rasmussen's results and the results given above.

10. Conclusions

In this report a study has been made of the pressure field associated with a spherical diffusion flame. Throughout the whole region under consideration with the possible exception of a very small region close to the surface of the fuel sphere the pressure field has the form

$$\rho = 1 + O(M^2) \quad (118)$$

This immediately implies, using equation (43) that

$$u = \frac{\dot{m}RT^{(1)}}{r^2 C_c} \sum_{\alpha} \frac{C_{\alpha}^{(1)}}{W_{\alpha}} + o(1) \quad (119)$$

and using equation (6)

$$\rho = \left[\frac{RT^{(1)}}{C_c} \sum_{\alpha} \frac{C_{\alpha}^{(1)}}{W_{\alpha}} \right]^{-1} + o(1) \quad (120)$$

These results for the velocity and density fields are

valid throughout the whole domain considered unless

$p_i = 1 + O(1)$ when they will not hold near the surface of the fuel sphere.

Returning to the discussion of the pressure field the region of rapid pressure variation close to the surface of the fuel sphere is required to allow for the possibility of a given pressure at the surface of the fuel sphere either of the form

$$p_i = 1 + O(\Delta) ; M^2 = o(\Delta) , \Delta = o(1)$$

or of the form

$$p_i = 1 + O(1) \left[\text{iff } p_i + \frac{4ScT_iF_0}{3p_i^2} \left(\frac{dp}{dr} \right)_{\bar{r}=0} = 1 + o(1) \right]$$

and in this small region the pressure changes rapidly from $1 + O(M^2)$ to either $1 + O(\Delta)$ or $1 + O(1)$.

In the remainder of the system considered pressure changes are gradual, that is $\frac{dp}{dr} = O(M^2)$, excepting regions of thickness $O(M^2)$ on either side of the flame sheet in which gradients of $O(1)$ are possible although pressure changes are still only $O(M^2)$. For the particular experimental situation described by this

work, that of fuel being blown from a porous (fuel) sphere into an oxidant atmosphere, the pressure gradient in the region of rapid pressure variation on the oxidant side of the flame is $O(1)$ whereas in the similar region on the fuel side of the flame $dp/dr \sim O(1)$. These results hold for any spherical diffusion flame based on the one-step chemical reaction given in equation (2) and numerical results have been obtained for the hydrogen-oxygen flame using this simplest chemical description.

The dimensional pressure at infinity is chosen to be the characteristic pressure so that the non-dimensional pressure at infinity is unity. For the hydrogen-oxygen flame the pressure decreases as the flame is approached from the oxidant side until the first region of rapid variation is reached. In this region the pressure rises sharply until the flame sheet is reached and then maintains its sheet value throughout the region of rapid variation on the fuel side of the sheet. The pressure then increases slowly through the fuel region until the region of rapid variation close to the surface of the fuel sphere is reached, when, depending upon the value of P_i there is either a sharp increase or decrease of pressure in this inner region. It must again be stressed that all

changes of pressure, with the possible exception of those in the region of rapid variation close to the fuel sphere, are $O(M^2)$ since the rapid changes of pressure when $dp/dr = O(1)$ all occur in regions of thickness $O(M^2)$ whereas in regions of thickness $O(1)$ dp/dr is $O(M^2)$.

Unfortunately it is not possible to obtain precise knowledge of whether the pressure increases or decreases in the outer regions for the general reactants X , P and F in the spherical diffusion flame, since the one-term outer solutions in both regions A and B contain some positive and some negative terms and it is the local weights of these terms which determine whether the pressure is increasing or decreasing through a particular region. However it is possible to obtain some information about the inner regions for the general one-step reaction described by equation (2). The discontinuity in the pressure gradient across the flame sheet can be simply related to the discontinuities in temperature gradient and mass fraction gradients (see equation (44)) and it is not difficult to calculate these discontinuities by reference to the solutions for the temperature and mass fraction fields given in equations (38) and (30)

respectively. Since dy/dr is everywhere positive the sign of $\frac{dc_\alpha}{dr}$ is the same as that of $\frac{dc_\alpha}{dy}$.

From equation (30)

$$\left. \frac{d}{dy} \sum_{\alpha} \frac{c_{\alpha}^{(1)}}{W_{\alpha}} \right|_{-}^{+} = (1-c_0) \left\{ A \left(\frac{1}{W_x} - \frac{1}{W_F} \right) - \left(\frac{1}{W_F} - \frac{1}{W_F} \right) \right\}$$

and consequently

$$\left. \frac{d}{dy} \sum_{\alpha} \frac{c_{\alpha}^{(1)}}{W_{\alpha}} \right|_{-}^{+} > 0 \quad \text{when} \quad (v_x + v_F) > \frac{1}{W_F} (v_F W_F + v_x W_x) \quad (121)$$

Similarly using equation (38) it is easy to show that

$$\left. \frac{dT}{dy} \right|_{-}^{+} = \text{Re} \left\{ T_s (a^* - a) + (A+1)(1-c_0) Q_F \right\} \quad (122)$$

However from this expression it is not possible to make general statements concerning the sign of $\left. \frac{dT}{dy} \right|_{-}^{+}$ whereas by expressing $\left. \frac{dT}{dy} \right|_{-}^{+}$ in a less specific way and utilising physical arguments a more general statement concerning the sign of $\left. \frac{dT}{dy} \right|_{-}^{+}$ can be made.

The energy equation is given in equation (34) and taking account of the absence of fuel from region B and the absence of oxidant from region A the following equations are found for those regions respectively

$$\dot{m}h - \frac{\lambda}{\rho_e} r^2 \frac{dT}{dr} + \rho r^2 \left[J_X (h_X - h_F) + J_D (h_D - h_F) \right] = \dot{m}H \quad r_S < r < \infty \quad (123a)$$

$$\dot{m}h - \frac{\lambda}{\rho_e} r^2 \frac{dT}{dr} + \rho r^2 \left[J_F (h_F - h_F) + J_D (h_D - h_F) \right] = \dot{m}H \quad r_L < r < r_S \quad (123b)$$

Then

$$\left(\frac{\lambda}{\rho_e} \right) \frac{dT}{dr} \Big|_s^+ = \left\{ (h_X - h_F)_s J_X \Big|_-^+ + (h_D - h_F)_s J_D \Big|_-^+ + (h_F - h_F)_s J_F \Big|_-^+ \right\} \quad (124)$$

In the calculation of the pressure field associated with the spherical diffusion flame it was assumed that the mass fraction of diluent was constant throughout regions A and B, but for the discussions relating to the evaluation of the discontinuities in gradient of temperature and mass fractions it can be shown that the functional form of the diluent mass fraction is unimportant and only its value at the sheet is required. Integration of equation (20), the species equation for the diluent, with respect to the variable r gives the result

$$\dot{m} C_D + \rho r^2 J_D = \text{constant} \quad (125)$$

and then it can be seen that

$$J_D \Big|_-^+ = \frac{\dot{m}}{\rho_8 r_s^2} C_D \Big|_-^+ = 0 \quad (126)$$

since all mass fractions are assumed to be continuous across the flame sheet. Integration of the atom conservation equation, equation (21), produces the result

$$\dot{m} \left(\nu_F \frac{C_X}{W_X} - \nu_X \frac{C_F}{W_F} \right) + \rho r^2 \left(\nu_F \frac{J_X}{W_X} - \nu_X \frac{J_F}{W_F} \right) = \text{const.} \quad (127)$$

when, again using the continuity of the mass fractions across the sheet

$$\nu_F W_F J_X \Big|_-^+ = \nu_X W_X J_F \Big|_-^+ \quad (128)$$

Substituting equations (126) and (128) into equation (124) gives

$$\left(\frac{\lambda}{\rho k e} \right)_s \frac{dT}{dr} \Big|_-^+ = \frac{Q}{\nu_X W_X} J_X \Big|_-^+ \quad (129)$$

where

$$Q = \nu_x W_x (h_x - h_F)_S + \nu_F W_F (h_F - h_P)_S \quad (130)$$

Equation (129) can be rewritten as

$$-Re \left. \frac{dT}{dr} \right|_-^+ = \frac{Q}{\nu_x W_x} \left. \frac{dc_x}{dr} \right|_-^+$$

where Re is defined in equation (36c) and then

$$-Re \left. \frac{dT^{(1)}}{dr} \right|_-^+ = \frac{Q}{\nu_x W_x} \left. \frac{dc_x^{(1)}}{dr} \right|_-^+ \quad (131)$$

Now, if it is assumed that the specific heats of the species present in the system are all identical, although not necessarily constant, and if it is further assumed that $Q_x = Q_F = 0$ then

$$Q = -Q_P (\nu_F W_F + \nu_x W_x)$$

Now, for a combustible reaction satisfying equation (2), and with $Q_x = Q_F = 0$, the heat of formation of the product Q_P must be negative, and thus under

the conditions imposed here Q is positive. Reference to equation (131) then clearly shows that $\left. \frac{dT^{(u)}}{dr} \right|_-^+$ has the opposite sign to that of $\left. \frac{dcx^{(u)}}{dr} \right|_-^+$. For the experimental configuration under consideration in this study, $\left. \frac{dcx^{(u)}}{dr} \right|_-^+$ must be positive and thus $\left. \frac{dT^{(u)}}{dr} \right|_-^+$ must be negative under the set of conditions imposed here. However the conditions which have been imposed are rather strict and a more general argument would be useful.

One usual feature of the temperature field associated with the diffusive mixing and subsequent burning of cold reactants is that the temperature at the flame sheet is at least a local maximum. If this is so then

$\left. \frac{dT}{dr} \right|_{r_s+} < 0$ and $\left. \frac{dT}{dr} \right|_{r_s-} > 0$ and consequently again the result $\left. \frac{dT}{dr} \right|_-^+ < 0$ is produced.

Further, and more importantly more conclusive, support for the assignment of a negative value to the quantity $\left. \frac{dT}{dr} \right|_-^+$ is obtained by appealing to its physical interpretation. Then quantity $-\lambda_s \left. \frac{dT}{dr} \right|_{r_s+}$ is the conductive heat flux out of the flame sheet into the region B ($r > r_s$) and $-\lambda_s \left. \frac{dT}{dr} \right|_{r_s-}$ is the conductive heat flux into the

flame sheet from the region $A (r_c \leq r < r_s)$.

Consequently $-\lambda_s \frac{dT}{dr} \Big|_+^+$ is the net heat flux out of the flame sheet. If there is to be a combustion reaction maintained in the flame sheet there must be a net heat production in the sheet and this heat must be conducted away from the flame sheet if a steady state is to be maintained. Thus the net heat flux out of the flame sheet must be positive and consequently

$$\frac{dT}{dr} \Big|_+^+ < 0 \quad (132)$$

This result is general as it only depends on a naturally **occurring physical property of a combustible system** interpreted in terms of the Burke-Schumann flame sheet model and not on the specific models chosen for the specific heats of the species involved or on the particular model chosen for the mass fraction field of the diluent. The result does not necessarily imply that no heat is conducted into the flame sheet but rather that if heat is conducted into the flame sheet from the fuel side then more heat is conducted out of the sheet into the oxidant side.

The arguments presented above suggest that $\left. \frac{dT}{dr} \right|_-^+$ is negative in general and thus $\left. \frac{dp}{dr} \right|_-^+$ will certainly be negative if $\sum_{\infty} \frac{1}{W_k} \left. \frac{dc_k}{dr} \right|_-^+ \ll 0$, that is

$$\left. \frac{dp}{dr} \right|_-^+ < 0 \quad \text{if} \quad (\nu_F + \nu_X) < \frac{1}{W_P} (\nu_F W_F + \nu_X W_X) \quad (133)$$

However, there will certainly be instances where $\left. \frac{dp}{dr} \right|_-^+ < 0$ even if $\sum_{\infty} \frac{1}{W_k} \left. \frac{dc_k}{dr} \right|_-^+ > 0$ but in order to produce conditions for the occurrence of this circumstance the models for the specific heats of the reactants and the mass fraction field of the diluent must be specified. If the specific heats are assumed to be constant and the mass fraction of diluent is assumed to be constant throughout regions A and B, as in the main calculation, then

$$\begin{aligned} \left. \frac{dp}{dr} \right|_-^+ < 0 \quad \text{iff} \quad \text{Re} \left[\frac{C_D}{W_D} + \frac{(1-C_D)}{W_P} \right] \left[T_s (\alpha^* - \alpha) + (A+1)(1-C_D) Q_P \right] \\ < \frac{T_s (1-C_D)}{\nu_F W_F} \left\{ -(\nu_F + \nu_X) + \frac{1}{W_P} (\nu_X W_X + \nu_F W_F) \right\} \end{aligned} \quad (134)$$

where

$$T_s = \frac{H}{a} + (T_i - \frac{H}{a}) \exp - a \text{Re} y_i$$

and H is given in equation (39).

Reference to equations (85) and (88) shows that the difference in the pressures given by the one-term outer solutions in regions A and B close to the sheet can be simply related to the discontinuity in the pressure gradient across the sheet. This means that the conditions quoted above for the sign of $\left. \frac{dp}{dr} \right|_{\pm}^{+}$ can also be used as conditions on the sign of the pressure difference occurring when the flame sheet and the regions of rapid variation on either side of it are crossed. Then, if one traverses the regions of rapid variation and the flame sheet starting from the oxidant side there is a pressure increase provided that the condition in equation (133) is satisfied in the general case or the condition given in equation (134) is satisfied in the case where the specific heats and diluent mass fraction are all constant. Thus, in general the pressure increases as the flame in the spherical situation considered here is traversed from the oxidant side to the fuel side provided that

$$(\nu_x + \nu_F) < \frac{1}{W_P} (\nu_F W_F + \nu_x W_x)$$

Also, due to the symmetry with respect to fuel and oxidant

evident in the condition, the same condition will hold in general also for oxidant being blown into an atmosphere of fuel provided that the simple one-step chemical description is retained.

Throughout this work great emphasis has been placed on the criterion of simplicity in choosing the chemical and physical models required to specify the mathematical problem completely. The simple overall picture has been used to glean some insight into the qualitative behaviour of the pressure field associated with the spherical diffusion flame but clearly many of the approximations used would have to be greatly improved if quantitatively realistic results were required. Certain improvements can be readily carried out. The diluent mass fraction field was assumed to be constant throughout the region under consideration since this was certainly the simplest solution to the species equation for the diluent species, but it is certainly true that the general solution of that equation is also very straightforward and this would be completely specified by fixing the diluent mass fraction field at ρ_i and as $\rho \rightarrow \infty$. In an experimental situation such mass fractions would

certainly be known and consequently this improvement in the model is easily carried out if Fick's Law and equal diffusion coefficients are used. Moreover the temperature dependence of ρD was again chosen purely for its simplicity and other more complicated temperature dependence could easily be assimilated by resorting to numerical methods to calculate what proved simple analytically here. However, both of these improvements were not deemed necessary here because of the primary interest in the qualitative features of the pressure field. If quantitative features had been the aim both of these improvements would certainly have been necessary but more importantly a much more realistic chemical description would have been essential.

The one-step reaction used in this work certainly represents the overall chemical reaction, but in no sense represents the molecular description of the flame. For the hydrogen-oxygen diffusion flame the set of five reactions used by Clarke and Moss (1970) is much more realistic and would be a useful starting point but once such a reaction scheme has been accepted considerable thought must be given to the subject of multicomponent diffusion. In the work of Clarke and Moss (1970) diluent

is excluded and thus in the outer regions, outside the flame sheet, only two components are present in $O(1)$ quantities at any given time, namely either fuel and product or oxidant and product. In this instance Fick's Law models the two component diffusion well and different diffusion coefficients for fuel through product and oxidant through product are readily admissible. Unfortunately, in most experimental situations a diluent species is present so that even in the outer regions three components have finite mass fractions and the multicomponent diffusion problem is met. A description of such a system can be obtained by considering binary diffusion between three pairs of reactants but such a description is not realistic and finding a good model for three component diffusion is a major problem well beyond the scope of this work. Thus, it is clear that whilst some of the more minor problems associated with the detailed description of the diffusion flame could be removed quite easily, the basic difficulties relating to the underlying chemical description of the flame are of quite a different magnitude and no attempt has been made here to tackle these problems.

In conclusion it is interesting to note very briefly

two possibilities for further work arising from the presence of large pressure gradients in the vicinity of the flame sheet. Carrier et al (1973) have suggested that the study of diffusion flames under finite tangential strains is pertinent to the description of diffusion flames burning in transitional and turbulent shear flows. They examined a severely distorted flame sheet in the presence of a constant pressure field and obtained solutions for the temperature and mass fraction fields. However, when considering the velocity field associated with a severely distorted flame sheet it is possible that the presence of large pressure gradients close to the sheet will itself be important in describing the onset of local turbulence and consequently the development of a model capable of including the effects of both strain and large local pressure gradients would be very useful.

In this work, only the Burke-Schumann infinitely thin flame sheet model of the diffusion flame has been considered and the Mach number M has been taken to be the small parameter involved in the parameter perturbation expansions employed. However Clarke (1975) has shown that the dimensionless parameter ϵ , the ratio of the chemical time to the diffusion time, can be simply related

to the Mach number M ,

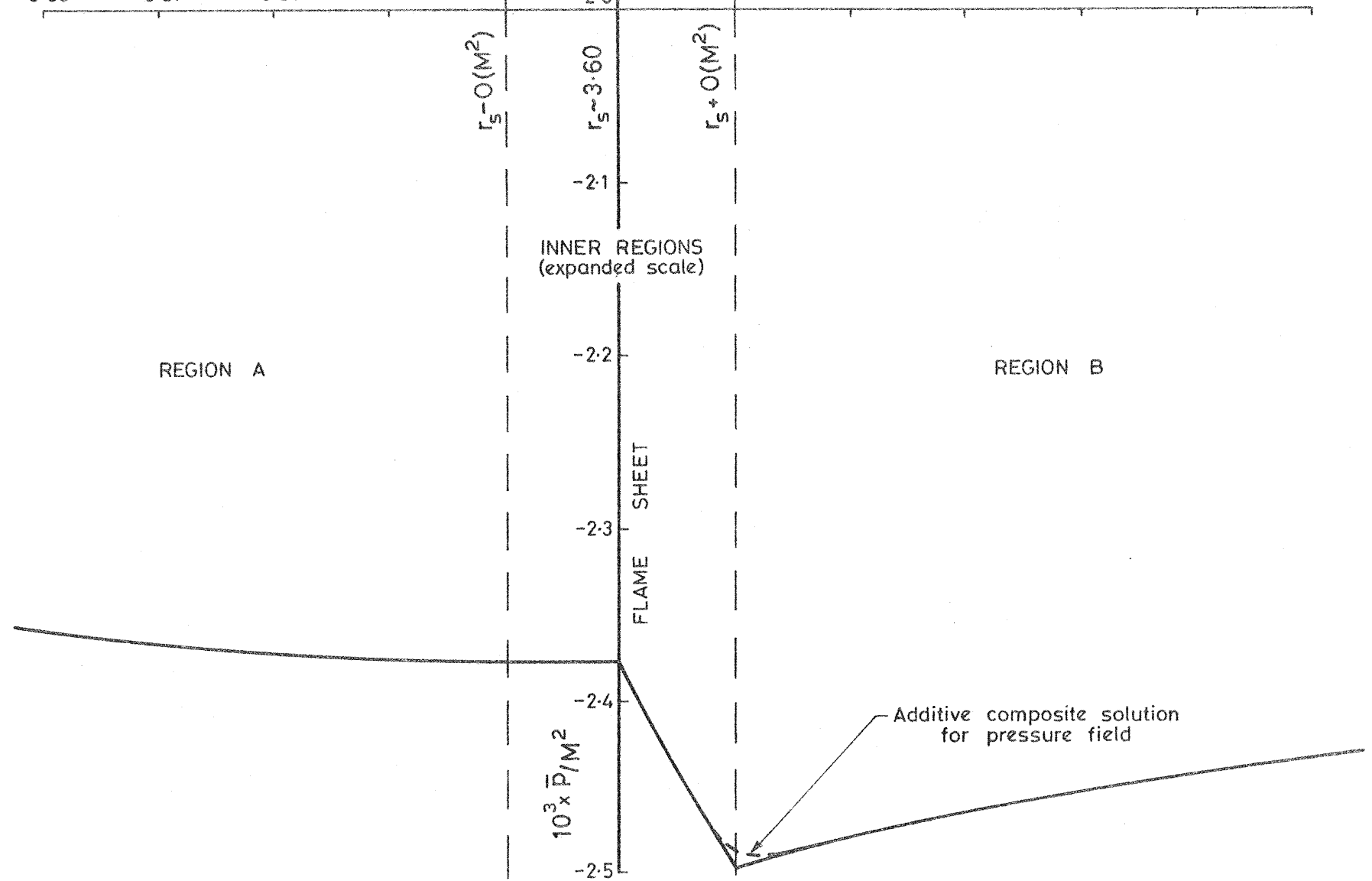
$$\epsilon \sim M^2 e^{\theta}$$

where θ is the dimensionless activation energy.

Thus for $\theta = O(1)$, $\epsilon = O(M^2)$ and for 'reaction-broadened' diffusion flames, where ϵ is treated as the small parameter, Clarke has shown that the thickness of the flame is $O(\epsilon^n)$ where, for example, for the five reaction set describing the hydrogen-oxygen flame (Clarke and Moss 1970) $n = 1/4$. Then the thickness of the flame is $O(M^{1/2})$ and this is considerably larger than the thickness of the regions of rapid pressure variation described in this work. It would thus be of considerable interest to study the regions of rapid pressure variation for a model in which the flame possessed finite thickness to discover whether the smoothing out of the gradient discontinuities achieved by studying a flame of finite thickness leads to major changes in the structure of the pressure field in the vicinity of the flame. It is hoped that further research along these lines will be undertaken in the near future.

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Approximate representation of the pressure field associated with a spherical H_2-O_2 diffusion flame close to the flame sheet ($\rho \mathcal{D} = T$)

APPENDIX A

Modelling of the Diffusion Coefficients

For binary gas mixtures at low pressure the diffusion coefficient is inversely proportional to the pressure, increases with increasing temperature and is almost independent of composition for a given gas-pair (Bird, Stewart and Lightfoot (1960)). The simplest kinetic theory calculations of the diffusion coefficient show that it has the form

$$D \propto T^{3/2} / p$$

However that simple calculation relies upon the molecules behaving like 'hard' spheres in collisions. More realistic temperature dependence can be obtained using a variety of methods to model a more reasonable intermolecular potential. Sutherland (1893 and 1909) introduced a potential which represents rigid spheres of diameter d_{ss} which attract one another according to an inverse power law. This model, which can be viewed as a limiting case of the familiar Lennard-Jones (1924) model, is

both fairly realistic and reasonably easy to handle.

Using Sutherland's model it is assumed that the rigid-sphere diameter d is given by

$$d^2 = d_0^2 (1 + c/T) ; c = \text{constant}$$

when the diffusion coefficient is given by

$$D \propto \frac{T^{3/2}}{\rho(c+T)}$$

Alternatively, use of an intermolecular force law of the form

$$f(r) \propto 1/r^n$$

where $f(r)$ is the intermolecular force and r is the molecular separation, produces a diffusion coefficient of the form

$$D \propto \frac{T^{n+3/2(n-1)}}{\rho} ; \quad \frac{n+3}{2(n-1)} > \frac{1}{2} \quad \text{for } n > 1$$

Usually either one or other of the refinements of the rigid-sphere kinetic theory calculation described here produces a reasonable model for the diffusion coefficient for a particular gas-pair over quite a wide temperature range. Moreover empirical results suggest that

$$D \propto T^{0.85} / \rho$$

is a reasonable model for a wide variety of gas-pairs over wide temperature ranges. In performing a calculation of the type undertaken in this work the detailed modelling of the diffusion coefficient should not be crucially important and therefore appealing to the criterion of simplicity a good model for the purpose of this work is

$$D \propto T / \rho$$

Appendix B

The Effect of Natural Convection on the Problem

In the calculations carried out in this report the effects of natural convection on the flow field have been neglected. The primary reason for this is that the exclusion of natural convection terms brings about an enormous simplification in the basic equations, since the symmetry of the system is then spherical rather than axial. Moreover the experiments of Melvin et al (1971) indicate that the flame sheet outside the lower half of the fuel sphere is indeed approximately spherical suggesting that the spherically symmetric description may well be adequate. However the numerical results presented in this report suggest that for realistic temperatures at the surface of the fuel sphere and at the flame sheet the radius of the flame sheet should be approximately 3.6 cm whereas in practice the radius is close to 1.4 cm, that is the flame sheet is much closer to the surface of the fuel sphere than the spherically symmetric description indicates. It is expected that the effect of natural convection on the position of the flame sheet surrounding

the lower half of the fuel sphere would be to push it up closer to the fuel sphere whereas the flame sheet temperature should not be drastically altered.

Consequently the large flame sheet radius predicted from the spherically symmetric description is consistent with the neglect of the effects of natural convection.

Orloff and de Ris (L. Orloff and J. de Ris , Thirteenth Symposium, Combustion Institute, Pittsburgh, Pa. (1971)) have made some calculations of flow fields associated with flames including natural convection effects but their methods are restricted to processes occurring beneath horizontal surfaces. A natural and interesting extension of the work presented here would be an analysis of the results obtained by Melvin et al for the spherical configuration which includes natural convection in the basic formulation.