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Orbit Determination Under Uncertainty: Differential Algebra
Applications for Apophis Precovery

Faculty of Engineering and Applied Sciences
Astronautics and Space Engineering

MSc
Academic Year: 2024–2025

Supervisor: Dr Marta Cecceroni
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requirements for the degree of MSc.

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Abstract

Rapid and reliable determination of Near-Earth Object (NEO) orbits is essential for Planetary Defence planning. Precovery, the association of new discoveries with archived observations, can immediately extend an observation arc and reduce impact-risk uncertainty. However NEO observations are typically short arcs, where measurement uncertainty dominates and a large set of possible orbits are defined. Traditionally, orbit uncertainty is captured by a sample-based Monte Carlo approach. This forces computationally intensive, and often slow, uncertainty propagation. Ultimately, this delays precovery and subsequent Planetary Defence planning.

This thesis investigates the feasibility of the established Differential Algebra (DA), and Automatic Domain Splitting (ADS), algorithms to accelerate the precovery pipeline. DA utilises high order Taylor expansions to efficiently map the uncertainty domain to the orbit solution. While ADS adaptively partitions the uncertainty domain so a lower order Taylor expansions remain valid within each sub-domain. Combined, these techniques have been thoroughly investigated within spacecraft Orbit Determination (DAIOD+ADS) and propagation. However, these developments are absent within the NEO precovery pipeline.

The DA and ADS initial orbit determination (DAIOD+ADS) and propagation algorithms were implemented on (99942) Apophis Ephemeris data to simulate short-arc discovery observations which spanned from one hour to one day. The pipeline's sky-patch coverage, number of false candidates, and computational performance were benchmarked

against classical Monte Carlo approaches.

Results show that DA+ADS based Initial Orbit Determination reduce the computational cost of precovery by orders of magnitude. Furthermore, DA+ADS achieves smaller initial sky-patch coverage therefore a reduction of false candidates. However, the accuracy of the solution deteriorates in DA propagation if expansion orders based on wider literature were used. Moreover, the study highlights the failure of the DAIOD+ADS algorithm to outperform traditional approaches for arcs less than 4 hours, this was attributed to the Gauss defined initial solution which the DAIOD+ADS algorithm relies on. Applied to Apophis, the integrated DA+ADS pipeline recovered historical detections up to 3 months prior to the discovery observation. It is faster than traditional methods, reducing the time to obtain a constrained impact corridor. These findings highlight the potential of DA-enabled precovery to deliver earlier impact-risk assessments. It's suggested an investigation of expansion order, ADS split tolerance and the computational consequence would enable an operations-ready precovery pipeline that would outperform traditional sample based techniques.

Keywords:

Precovery, Uncertainty Propagation, Short Arc Observations, Automatic Domain Splitting, Planetary Defence, Non-linear Dynamics, Near Earth Objects.

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Contents

Academic Integrity Declaration	i
Abstract	ii
Acknowledgements	iv
Contents	v
List of Figures	viii
List of Tables	xi
List of Abbreviations	xii
1 Introduction	1
1.1 Planetary Defence	2
1.1.1 Observations, Discovery and Uncertainty	2
1.1.2 The Relationship between Discovery, Assessment and Mitigation	5
1.2 The Importance of Pre-Discovery Observations	6
1.2.1 Gap within Preccovery from Initial Arc Extension	7
1.3 Proposed Solution to Preccovery: Differential Algebra	8
1.4 Case Study: Application of Preccovery for Apophis	9
1.5 Aims and Objectives	9
2 Literature Review	12
2.1 Differential Algebra Fundamentals	12
2.1.1 Principles of DA	12
2.1.2 Truncated Polynomial Series Algebra	13
2.1.3 Expansion of the solution of Parametric Implicit Equations	16
2.1.4 Differential Algebra effect on Preccovery Operations	18
2.2 Automatic Domain Splitting	19
2.2.1 Single-variable examples	22
2.2.2 ODE propagation	22
2.2.3 ADS Application on Preccovery	24
2.3 Orbit Determination	26
2.3.1 Classical Initial Orbit Determination	27
2.3.1.1 Lambert's Problem	32

2.3.2	Admissible Region	32
2.3.3	Differential Algebra based Initial Orbit Determination	35
2.3.4	DAIOD for Preccovery	43
2.4	Orbit Propagation	44
2.4.1	Monte Carlo Propagation	45
2.4.2	Linearised Propagation	47
2.4.3	Non-Linear Propagation	48
2.5	Preccovery Data Association	50
2.5.1	CADC, ESO and SSOIS	51
2.6	Apophis	53
2.7	Research Gap and Motivation	54
3	Methodology	55
3.1	Figures of Merit	57
3.2	Input vector	57
3.3	Orbit Determination	58
3.3.1	Assumptions	58
3.3.2	Virtual Observations, Observer Position and Orbit dynamics . . .	60
3.3.3	DAIOD	60
3.3.4	DAIOD+ADS	64
3.4	Orbit Propagation	66
3.4.1	Assumptions	67
3.4.2	DA-based	67
3.4.3	ADS-based	68
3.4.4	ADS Propagation of multiple domains	69
3.5	Differential Algebra Image Search Preparation	69
3.5.1	Heliocentric to Geocentric conversion	70
3.5.2	Coordinate Transform	70
3.5.3	DA/ADS Evaluation	70
3.6	Fixed Region Query	71
3.6.1	Database Query Strategy	74
3.7	Validation	75
3.7.1	Orbit determination	75
3.7.2	Orbit Propagation	77
3.7.3	Fixed Region Query	77
4	Validation Results	79
4.1	Orbit Determination	79
4.2	Orbit Propagation	81
4.3	Image Query	82
5	Results and Discussion	83
5.1	Differential Algebra Performance	83
5.1.1	Assessment of the Taylor Series Construction	83
5.1.2	Coverage of the Celestial Sphere	87
5.2	Computational Performance	94
5.2.1	Differential Algebra comparison	95

5.2.2	Computational time scaling	97
5.3	Precovery Performance	99
6	Limitations and Future Work	101
6.1	Computational Limitations	101
6.2	Methodology	102
6.3	Time Constraint	103
7	Conclusion	105
8	Appendix	110
8.1	Appendix A: CURES	110
8.2	Appendix B: Example of DA as Taylor polynomials	112

List of Figures

1.1	NEO Discovery and Risk Assessment pipeline.	2
1.2	Observation coordinates: Right Ascension, α , Declination, δ of observation N at time t . Measurement Precision, σ , defines initial uncertainty. For Optical systems, range, ρ , is defined through Initial Orbit Determination Algorithms. Source: (Pirovano, 2020, p. 1)	3
1.3	A planar view of the effect of observation arc length/ measurement precision on orbit uncertainty. The black dots are the observations. The red circles represent the measurement uncertainty. The light green region show the orbit uncertainty as a function of arc length. Source: (Pirovano, 2020, p. 1)	4
1.4	Evolution of Asteroid 2024 YR4 impact corridor. The red dots are the individual Virtual Asteroids. (Gif only works in pdf viewer) Source: (Agency, 2025a, p. 1)	5
1.5	Evolution of Probability of Impact for Apophis (above) and YR4 2024 (Below). On the 27th December 65 observations over 192 days led to 2.7% Impact Probability (Valvano et al., 2024, p. 1). On 28th December Probability of Impact eliminated via Preccovery.	10
2.1	Analogy between floating point representation of real numbers in the computer environment, and the Taylor series representation of functions in a computer environment. Source: (Armellin et al., 2012, p. 1)	13
2.2	The mechanics of ADS in propagation (Wittig et al., 2015, p. 1)	20
2.3	Estimation of truncation error of the 9 th order Taylor expansion of $\sqrt{1 + \frac{x}{2}}$. S_i of each order is shown by the hatched bars and the fitted function is shown as a line. The true 10 th order S_i is shown by the white bar. The difference between the true and predicted S_{10} is 2×10^{-6} (Wittig et al., 2015, p. 1)	21
2.4	ADS logic: The initial domain is copied in a list, for each element of the list checks are performed to detect whether the truncation error is above the threshold and/or the maximum number of splits have been reached. If necessary the domain considered is split and inserted in the list. This list is checked again for further possible splits. Once the entire list is checked, the routine is ended (Pirovano, 2020, p. 1).	23
2.5	Polynomial approximate of the sine function (Pirovano, 2020, p. 1).	24
2.6	Polynomial approximation of the inverse-square function (Pirovano, 2020, p. 1).	25

2.7	Example of the evolution of ADS splits under two-body dynamics (Wittig et al., 2015, p. 1)	25
2.8	The impact of ADS on Initial Orbit Determination accuracy (Pirovano, 2017, p. 1)	26
2.9	Vectors that define Gauss's Problem (Albini, 2010, p. 1)	28
2.10	Admissible region of a satellite within a Geosynchronous Transfer Orbit .	33
2.11	Admissible Region: Delaney Triangulation	34
2.12	Comparison between AR solution space and the DA-IOD solution space. Note 1) the containment of True Solution within the DA-IOD black coloured box 2) The compared to standard AR (Pirovano, 2020, p. 1)	37
2.13	First step of the DAIOD Algorithm: initial range guess found via Gauss's algorithm, refine solution by imposing $\mathbf{v}_2^- - \mathbf{v}_2^+ = \mathbf{0}$ within Lambert's Algorithm (Pirovano, 2020, p. 1).	39
2.14	Plots of the error distribution for the single Taylor polynomial expansion and Taylor Polynomial expansion which has undergone ADS (Pirovano, 2017, p. 1).	42
2.15	Number of sub-domains with 6-th order expansion. The red line indicates the split threshold (Pirovano, 2017, p. 1).	43
2.16	Success rate (contour) of the DA-IOD algorithm (black-box) depending on Arc length and Uncertainty (Pirovano, 2020, p. 1).	44
2.17	Triangulated admissible region in the plane $(\rho, \dot{\rho})$ for the asteroid 2003 BH84. The units are AU, AU/yr and the coordinates of the asteroid are marked with a circled cross (Gronchi, n.d., p. 1)	52
3.1	High level logic of $(\alpha \pm 3\sigma, \delta \pm 3\sigma)$ region generation	56
3.2	High level logic of observatory database query and Figure of Merit generation	56
3.3	Pre-conditioning for Initial Orbit Determination	60
3.4	Differential Algebra Initial Orbit Determination Logic Diagram	61
3.5	DAIOD+ADS Orbit Determination Logic Diagram	65
3.6	Logic Flow chart of the Query	72
3.7	The Convex Hull of a set of points	73
3.8	The alpha shape of a set of points	73
4.1	2D bounding box representation of the (α, δ) at epoch	80
4.2	Propagated state of an 2005 06 17 23:41:07 epoch state	81
4.3	Comparison of DA and MC methods at initial Orbit determination and through propagation	82
5.1	Range - Range rate Phase space's at observation Epoch	84
5.2	Long Arc Range and Range rate phase space	85
5.3	The Evolution of $\rho, \dot{\rho}$ across 6 months of propagation	86
5.4	Short Observational Arc (4 hours)	88
5.5	Long Observational Arc (24 day)	89
5.6	The epoch $(\rho, \dot{\rho})$ for an 4 hour observation arc	90
5.7	The evolution of the (α, δ) projected on the Celestial Sphere. The 0.18 day Arc results in large uncertainty regions within 3 days	90

5.8	Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 13 days (312 hours)	91
5.9	Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 52.08 days (1249.92 hours)	92
5.10	Evolution of (α, δ) Search Region: Gauss Monte Carlo 13 day (312 hour)	93
5.11	Evolution of (α, δ) Search Region: Gauss Monte Carlo 52.08 day (1249.92 hours)	94
5.12	Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 83 day (1992 hours)	94
5.13	Evolution of (α, δ) Search Region: Gauss Monte Carlo 83 day (1992 hours)	95
5.14	Breakdown of the Mean total computational time across all observation arcs.	96
5.15	Total mean computational time, alpha shape generation removed.	96
5.16	The effect of Arc length on Computational time	98
5.17	Notice the distribution of the Monte Carlo points. The short arc forms a complex shape due to the non-linearities while the longer arc is symmetrical so a less complex alpha shape is required.	99

List of Tables

3.1	Model Inputs	57
3.2	ADS Parameters	65
3.3	IOD and Propagation combination that were tested	66

List of Abbreviations

FEAS	Faculty of Engineering and Applied Sciences
FBAM	Faculty of Business and Management
STEM	Science, Technology, Engineering and Mathematics
DA	Differential Algebra
ADS	Automatic Domain Splitting
DAIOD	Differential Algebra Initial Orbit Determination
NEO	Near Earth Object
DART	Double Asteroid Redirection Test
MCS	Monte Carlo Simulation
LOV	Line of Variation
IOD	Initial Orbit Determination
TPSA	Truncated Polynomial Series Algebra
FP	Floating Point
DACE	Differential Algebra Computational Toolbox
MC	Monte Carlo
AR	Admissible Region
VA	Virtual Asteroid
PDF	Probability Density Function
NASA	National Aeronautic and Space Administration
STM	State Transition Matrix
SSOIS	Solar System Object Image Search
CADC	Canadian Astronomy Data Centre
ESO	European Southern Observatory
IP	Impact Probability

Chapter 1

Introduction

Since the pre-historic era, humanity has been captivated by the intricate motion of celestial bodies. This fascination has spurred generations of physicists and mathematicians to derive planetary equations of motion, enabling astronomers to predict, track, and observe planets and Near-Earth Objects (NEOs) with remarkable precision.

The dramatic 1994 collision of Comet Shoemaker-Levy 9 with Jupiter, observed by astronomers worldwide, offered a stark, visceral reminder of the immense forces at play within our solar system. The sheer magnitude of that event forced pivotal questions into the public consciousness: What if such a collision were to occur on Earth? When will it occur? How would we defend ourselves? Today, these questions underpin the crucial field of Planetary Defence. (Strategy & Group, 2023, p. 1).

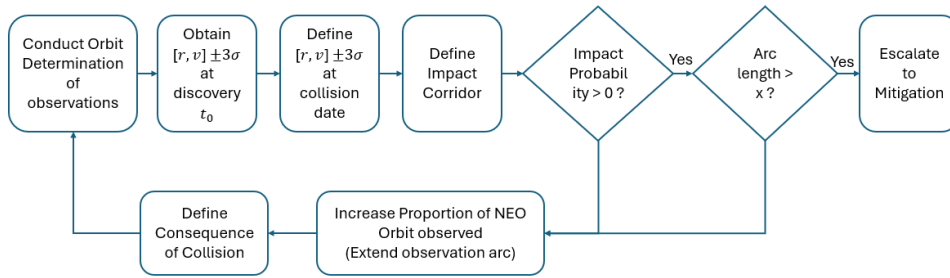
Critical to Planetary Defence is the execution of Mitigation Strategies. The success of the mitigation strategies may define the level of the destruction of an imminent impactor poses. Time is the most prominent variable influencing the success of a mitigation strategy. Time is needed to re-observe a possible impactor, determine its impact corridor and design an ad hoc mitigation strategy and implement it. For example, the Double Asteroid Redirection Test (DART) mission took 5 years to progress from conceptual design to an impact with its target Didymos (eoPortal, 2023, p. 1).

Mitigation development time is fixed. However, time required to accurately identify and determine a Near Earth Object's (NEO) trajectory is variable. It is dependant on how fast an observer can obtain independent observations. This Thesis aims to evaluate the impact of Differential Algebra, an alternative to traditional orbit determination techniques, on the time for re-observing a NEO.

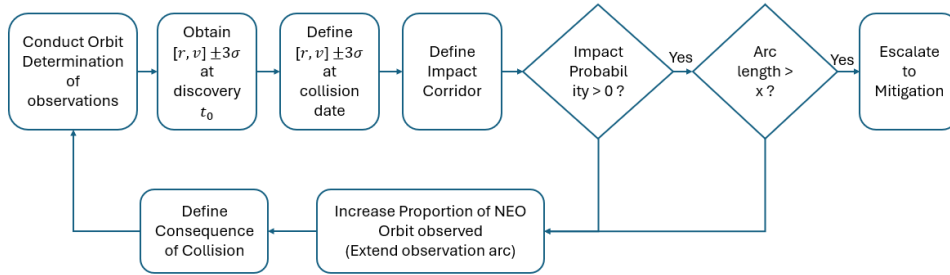
To fully understand the gap leading to this thesis, a long, albeit necessary, premise shall be explained. This will frame the gap within the wider context of Planetary Defence. Chiefly, the thesis will introduce key jargon, present the consequence of time within re-observation, and explain all factors that affect said time.

1.1 Planetary Defence

The field of Planetary Defence constituents three overarching activities: discovery, assessment, and if an impactor is identified, mitigation. Discovery refers to the observation, orbit determination, and cataloguing of NEOs. Assessment refers to the analysis of an impact corridor: the region defined by the overlap between the NEO's trajectory uncertainty and the Earth's orbital trajectory (Rumpf et al., 2019, p. 1). Mitigation refers to the strategies required to deflect an imminent impactor trajectory away from Earth. Figures 1.1 illustrates these processes are sequential; precise orbit determination enable representative collision assessment which enable mitigation strategy development.



(a) Workflow of NEO Discovery



(b) Workflow of NEO Assessment

Figure 1.1: NEO Discovery and Risk Assessment pipeline.

1.1.1 Observations, Discovery and Uncertainty

A NEO Discovery is the systematic process of observing NEOs, identifying NEO orbits, and cataloguing said orbits. Figure 1.2 shows an observation consists of a pair of right ascension and declination coordinates, (α, δ) , at an observation time, t . The precision

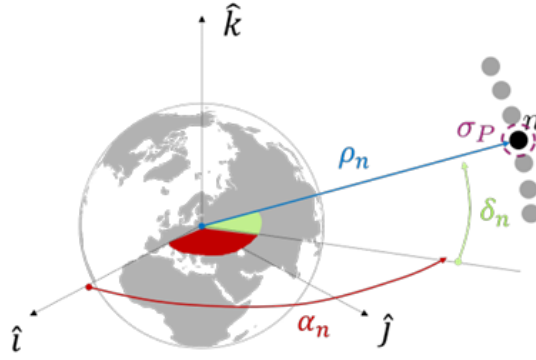


Figure 1.2: Observation coordinates: Right Ascension, α , Declination, δ of observation N at time t . Measurement Precision, σ , defines initial uncertainty. For Optical systems, range, ρ , is defined through Initial Orbit Determination Algorithms. Source: (Pirovano, 2020, p. 1)

of an observation, σ_p , is defined by the observatory telescope and observer capacity. σ_p is a measure of spatial uncertainty within the observation. A series of N observation, separated by time Δt , is an observation arc. The arc length refers to the degree of time separation between observations. The uncertainty of the observation arc is a function of i) the time separation of the observations and ii) the precision of the observation (Pirovano, 2020, p. 1).

A Discovery observation (arc) refers to the initial observations of an previously unidentified NEO. Initial observation arcs are characterised by short arcs, a set of $3 \leq N \leq 5$ observations along a $2h$ time span (Milani et al., 2004, p. 1). Short arcs dominate discovery due to orbital and operational constraints.

1. Operational constraints: dedicated NEO discovery telescopes prioritise quantity of sky scanned. Therefore, each observation arc has a limited survey time.
2. Orbital constraints: sufficient brightness to observe NEO requires near close approach observation. The high apparent angular velocity between objects result in NEO quickly moving outside the telescope Field of View.

From an observational arc, one conducts Initial Orbit Determination (IOD) to identify an orbit (Vallado, 2013, p. 1). The uncertainty within the observation (α, δ) result in uncertainty within the orbit (Vallado, 2013, p. 1). Shown in Figure 1.3, the orbit uncertainty results in a continuum of possible orbits rather than a singular orbit. The continuum of orbits is known as the Orbital Set (Pirovano, 2020, p. 1). For short arcs, the orbit uncertainty is significant due to the short time separation between measurements (Pirovano, 2020, p. 1). This means the orbit dynamics are less constrained and the number of possible

solutions, or orbit set, increases. Figure 1.3 shows the relationship between arc-length and orbit uncertainty for a constant measurement precision. The right-most figure illustrates that tightly spaced observations can lead to non-physical, hyperbolic, orbits for Asteroids. The extent of the orbital set is shown in light green. Henceforth, observation precision and arc length are known as observation-based uncertainty.

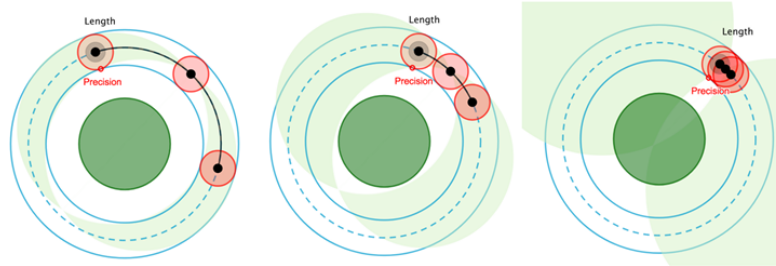


Figure 1.3: A planar view of the effect of observation arc length/ measurement precision on orbit uncertainty. The black dots are the observations. The red circles represent the measurement uncertainty. The light green region show the orbit uncertainty as a function of arc length. Source: (Pirovano, 2020, p. 1)

The large orbit uncertainties associated with the discovery short arc are insufficient for reliable NEO assessment (Pirovano, 2020, p. 1). To reduce the orbit uncertainty, a larger proportion of the NEO orbit must be observed. This constrains the region of possible solutions. This is known as arc extension. To achieve this observers traditionally wait 24 hours till the NEO has travelled further along its orbit, predict where the NEO will be in 24 hours, then re-observe it. This process is called follow up, while the observations obtained from follow up are called follow up observations. Through follow up, a relationship between time and orbit uncertainty is evident. The longer an observer waits, the further a NEO has travelled along the orbit, therefore a greater proportion of the NEO's orbit has been observed, the continuum of possible orbital solution are constrained and the orbit uncertainty decreases.

Alongside observation-based orbit uncertainty there is propagation uncertainty. Propagation uncertainty refers to the growth of initial uncertainties as the NEO is projected away from the time of observation. It arises due to unmodelled orbit perturbations as a result of few NEO observations (Vallado, 2013, p. 1). Observation-based and propagation uncertainties bound the so-called 'error ellipsoid' which represents the 6 dimensional uncertainty of an orbit. Due to propagation uncertainty the ellipsoid increases through time which is why follow up observations occur frequently.

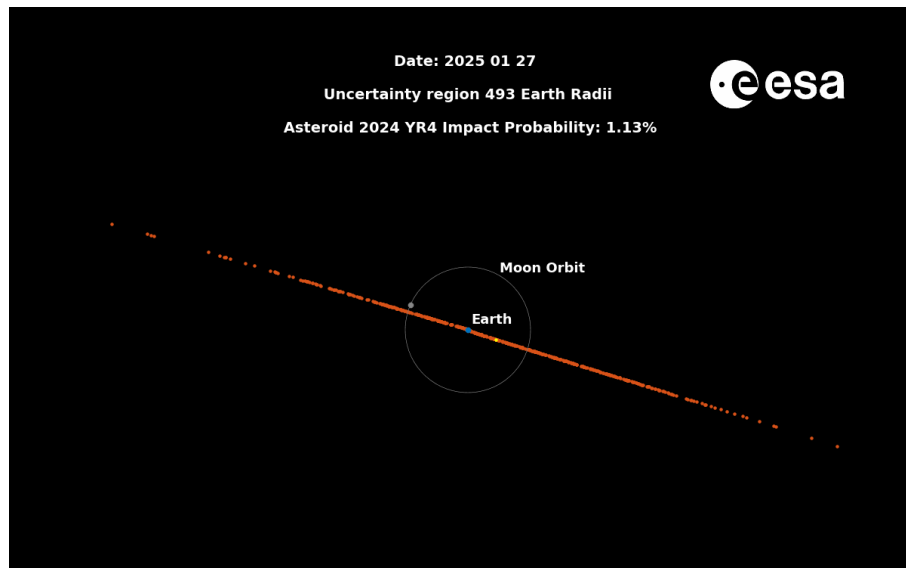


Figure 1.4: Evolution of Asteroid 2024 YR4 impact corridor. The red dots are the individual Virtual Asteroids. (Gif only works in pdf viewer) Source: (Agency, 2025a, p. 1)

1.1.2 The Relationship between Discovery, Assessment and Mitigation

Figure 1.1 outlines the relationship between NEO discovery, assessment and mitigation. The figure illustrates that orbit determination is at the root of the Planetary Defence pipeline. A fast orbit determination process enables sooner impact assessment. Discovery NEO observations are typically short arc observations (Gronchi, n.d., p. 1). This leads to an initial orbital set similar to the central or right-side diagram of 1.3. The large orbital set results in a wide distribution of possible orbits that may impact Earth. This results in a wide impact corridor at NEO discovery (see figure 1.4).

One cannot act on a wide impact corridor. The level of uncertainty can span 459 Earth radii. It is unlikely the NEO will impact Earth. To improve confidence in the likelihood of collision, one needs to reduce the uncertainty of collision. To achieve this, one needs to reduce the impact corridor. To achieve that, one needs to reduce the size of the orbital set. This can only reduce if the observation arc is extended.

Therefore the relationship between NEO observation and assessment is iterative. The discovery observations create a wide impact corridor, which becomes smaller over time as observations are associated to the NEO and the observation arc is extended (see figure 1.4). Once the impact corridor is deemed acceptable a mitigation strategy is acted upon.

However Mitigation strategies have long development times. For example, the only developed mitigation strategy, DART, had a development time of 5 years (eoPortal, 2023, p. 1). Mitigation strategies also required to be acted upon as soon as possible. A kinetic impact earlier within an imminent impactor ensures larger deflection than one close to earth (M. et al., 2020, p. 1). Therefore, a high confidence within impact corridor is required as soon as possible. Ultimately, this means the observation arc needs to be extended as soon as possible, the larger the observation arc the narrower the impact corridor.

As soon as possible requirement for mitigation, directly opposes the traditional follow-up approach observatories use for observation arc extension. Traditionally, a NEO is discovered with a short arc, the orbital set derived from this arc is propagated 24 hours in the future where the observer identifies the NEO. The approach, termed follow-up, means the arc is refined every 24 hours. The consequence of this approach is evident on Figure 1.4: The Impact corridor reduced everyday over the course of a month as new observations extended the YR4 2024 observation arc. This is a month which could've been used in subsequent planetary defence operations if the worst case was identified.

Therefore, a fast, ideally immediate, method of arc extension is required to ensure planetary defence operations can be actioned as soon as possible. This would prevent any possible casualties if a NEO presented itself as a imminent impactor.

1.2 The Importance of Pre-Discovery Observations

The fastest method to extend an observation arc extension is association of past pre-discovery observations. The initial arc is propagated backwards through time where archived images are associated to their past state. This is defined as 'Precovery' (Gwyn et al., 2016, p. 1).

To effectively minimize orbital uncertainty and, consequently ensure swift mitigation, the use of Precovery observations proves advantageous over relying solely on follow up observations. This is a result of:

1. **Time independence:** Precovery enables the association of an object's historical observations regardless of their original acquisition date. By definition, Precovery utilises already existent observations within the astronomical catalogue. This decouples arc extension from the real-time constraints of acquiring follow up observations.

2. **Rapid Uncertainty Reduction:** While traditional follow-up observations incrementally extend the observational arc through acquisition of future observations. Precovery immediately associates historical observations. These observations can predate the discovery arc by days, or weeks. The increased time separation between historical and discovery observations result in significant extension of the observation arc, which rapidly constrains the orbital dynamics. As previously discussed, this leads to a substantial reduction in orbital uncertainty far more swiftly than is possible through the sequential, real-time accumulation of new follow-up observations.

Thus, the use of Precovery enables the quick reduction of the orbit uncertainty and impact corridor. This accelerated certainty is critical. It provides the necessary lead time to confirm a potential NEO collision, and implement an effective mitigation strategy ensuring the safety of Earth if the worst-case scenario were to occur.

1.2.1 Gap within Precovery from Initial Arc Extension

While precovery offers significant theoretical advantages for rapidly reducing orbit uncertainty, its practical implementation for initial arc extension faces operation and computational challenges. Operational challenges are defined through the large orbit uncertainties at discovery, while computational challenges are defined through the chosen propagation methods.

Defined in Section 1.1.1, discovery short arcs are characterised by large initial orbit uncertainties. The short arc nature of discovery permit numerous orbital solutions compliant with the initial observations. Consequently, when this highly uncertain initial orbit set is propagated in search of Precovery observations the predicted position of the object on the celestial sphere covers a wide 'sky patch'. This defines an exceptionally large search space. Astronomers must meticulously examine all catalogued images within the sky patch.

This challenge is significant for Precovery due to the relatively long backward propagation times required to associate historical data to the discovery arc. Defined in Section 1.1.1, through propagation the orbit uncertainties grow. This exacerbates the search problem, initial high uncertainty at discovery expand further. This makes data association for long propagation times inherently difficult. Confidently linking a historical observation to the newly discovered object becomes akin to finding a needle in a haystack within this

ever-expanding uncertainty.

Essential to identification of potential Preccovery candidates is the propagation of the orbit uncertainty (Wittig et al., 2015, p. 1). Current methods employ Monte Carlo Simulations (MCS), which sample numerous initial positions and velocities within the $\pm 3\sigma$ orbital uncertainty distribution. Each sample, or 'Virtual Asteroid', is numerically propagated backward to the date of preccovery observation (Valvano et al., 2024, p. 1). For propagated uncertainty to reflect the true uncertainty, MCS require the thousands to millions of samples, ideally $N \rightarrow \infty$, to be propagated (Pirovano, 2020, p. 1). Therefore, the long numerical integration routine for a vast number of samples make the uncertainty propagation process computationally intensive. Computationally improved alternatives to MCS do exist, such as the Line of Variations (LOV), however still require the propagation of numerous samples (Milani et al., 2005, p. 1).

The combination of the operational and computational challenges results in prolonged computation times to complete Preccovery association. Occasionally, computational time may exceed that to obtain follow-up observations (Vavilov & Hestroffer, 2024, p. 1). There is a critical operational dilemma within the current Preccovery pipeline: the theoretical benefit of immediate arc extension is often offset by operational and computational challenges. As a result, decision and implementation on mitigation strategy may be delayed placing the Earth in increased danger.

1.3 Proposed Solution to Preccovery: Differential Algebra

This thesis aims to evaluate Differential Algebra (DA) as a possible framework to address the computational and operation challenges inherent to preccovery. DA provides a computationally efficient approach for initial orbit determination (IOD) (Pirovano, 2017, p. 1), precise uncertainty propagation (Wittig et al., 2015, p. 1), and robust data association (Pirovano, Santeramo et al., 2020, p. 1). Its application is suited to overcome the limitations of Preccovery through:

1. **Reduction of Initial Orbit Uncertainties:** DA-based IOD has demonstrated the capability to significantly reduce the initial orbital uncertainties derived from short arcs (Armellin et al., 2012, p. 1), directly narrowing the large search space for Preccovery associations.
2. **Ensuring Time-Efficient and Accurate Propagation:** DA methods facilitate accurate long-term orbit propagation compared to MCS, while achieving propagation

times orders of magnitude faster than MCS (Wittig et al., 2015, p. 1). Furthermore, its inherent ability to track and manage the growth of uncertainty through an Automatic Domain Splitter (ADS), ensures the accuracy of DA propagation even for the long durations critical to Precoverly.

3. **Facilitating Robust Data Association:** By precisely propagating both the nominal orbit and its associated uncertainty, DA enhances the reliability and efficiency of associating newly discovered objects with historical observations, even when dealing with initially high orbital uncertainty (Pirovano, Santeramo et al., 2020, p. 1).

Therefore, DA mathematical framework is a powerful candidate for overcoming the practical implement of the Precoverly pipeline. Ultimately, DA's fast propagation, and data association, enables fast mitigation action if an imminent impactor were to collide with Earth.

1.4 Case Study: Application of Precoverly for Apophis

Figure 1.5 illustrates the evolution of the Probability of Impact on Asteroid Apophis, the first NEO with a significant Impact Probability of 2.7% (Valvano et al., 2024, p. 1).

Discovered on June 19, 2004, Apophis is $\tilde{340}$ m across and was, soon after discovery, briefly given a 2.7% chance of impacting Earth in 2029. Precoverly occurred on the 28th of December where earlier March observations were associated to Apophis. The association of the March observations enabled the arc to extend from an 192 span to 274 days. As a result, the number of possible orbit solutions that fit the observations reduced, and the orbit uncertainty reduced. Propagated to the proposed 2029 collision date, the reduced $\pm 3\sigma$ orbit uncertainty led to a narrow Impact corridor and the previously thought imminent impactor was discarded as a hazard overnight.

1.5 Aims and Objectives

The strengths of Differential Algebra have been illustrated in orbit determination, orbit propagation, and data association. Differential Algebra may unlock the full theoretical benefits of Precoverly via its highly efficient computational structure. This would ensure as soon as possible mitigation via fast observation arc extension. The aim of this thesis is therefore to investigate the performance of Differential Algebra within the NEO Precov-

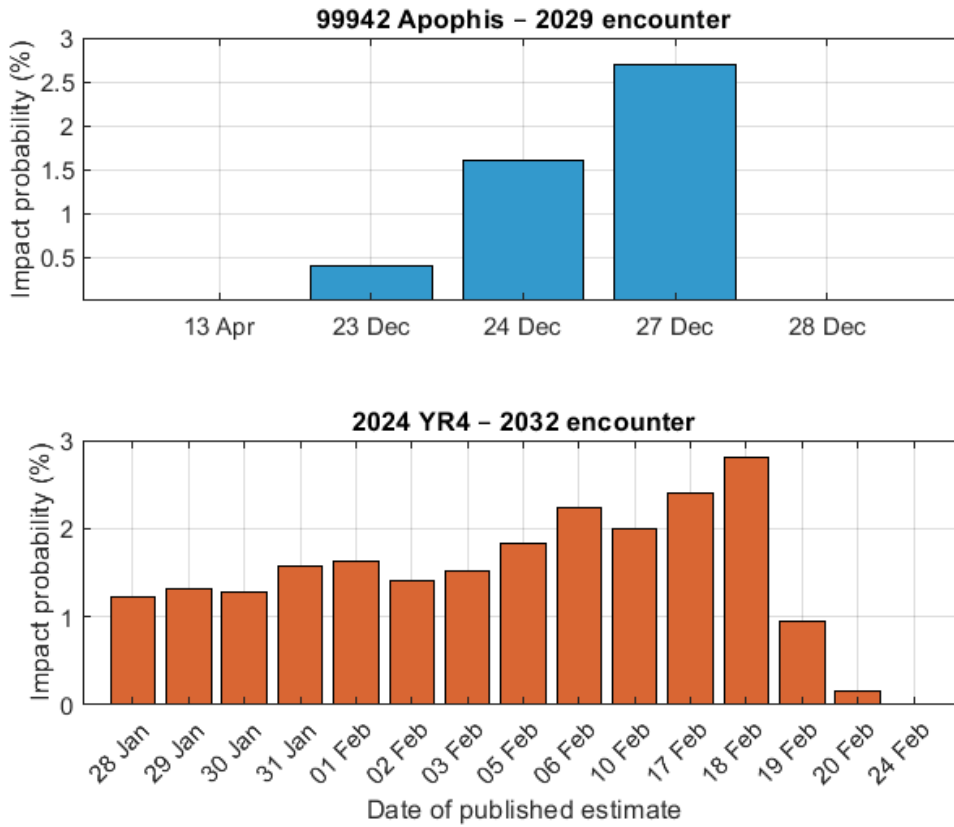


Figure 1.5: Evolution of Probability of Impact for Apophis (above) and YR4 2024 (Below). On the 27th December 65 observations over 192 days led to 2.7% Impact Probability (Valvano et al., 2024, p. 1). On 28th December Probability of Impact eliminated via Preccovery.

ery pipeline.

To achieve this aim, the Thesis will achieve the following objectives:

1. Define the relative computational cost of DA and Monte Carlo Orbit Determination techniques.
2. Assess the possibility of Apophis image recovery through existing DA-based Initial Orbit determination (DA-IOD) algorithms from initial observation arcs which span from 1 hour to 1 day.
3. Define the limit of the propagation horizon, via DA methods, and how it differs with classical Monte Carlo methods.
4. Assess the relationship between computational cost, arc length, and orbit uncertainty to evaluate the impact of DA and Automatic Domain Splitting (ADS) Algorithms to the Preccovery pipeline.

Through completion of the aim and objectives, this Thesis will conclude on the feasibility of the DA framework to the Precovery pipeline. The conclusions will inform the first step toward a possible DA integration within the Precovery pipeline.

Chapter 2

Literature Review

NEO Precovery consists of three process: Orbit Determination, Orbit Propagation and Orbit data association. To understand the potential of DA to improve Precovery Operations, one must assess the wider literature and evaluate the benefits and limitations of current methods.

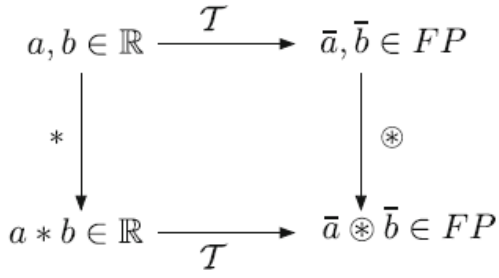
2.1 Differential Algebra Fundamentals

Differential Algebra (DA) is a computing technique which utilises Truncated Polynomials Series in place of floating point numbers. The technique was development by Martin Berz after the observation "it is possible to extract more information about a function than its mere value" (Martin, 1999, p. 1). For example, functions define relationships between input and output variables. The following section provides a mathematical overview, defined by Berz, and DA's application to Orbit Determination.

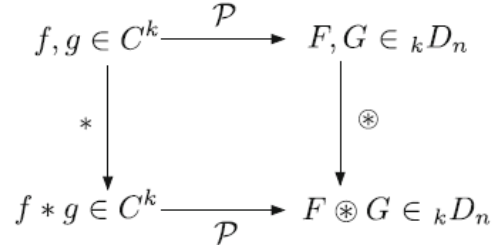
2.1.1 Principles of DA

Scientists and Engineers look for closed-form solutions to the equations of motion (EoM) (Pirovano, 2020, p. 1). Closed form solutions provide several advantages: i) it is efficient; One can obtain an specific output via simple function evaluation ii) They provide analytical insight into how states are sensitive to parameters and initial conditions.

However within complex dynamics closed-form solutions are rare. Instead EoM are solved through numerical means, where a specific set of initial conditions are integrated to define a specific output. If one were to change these initial conditions, one would have to restart the process. As a result, if a Monte Carlo Simulation is conducted the variation in initial conditions result in 1000s of numerical integration instances.



(a) Floating Point (FP) representation of real numbers within a computer environment



(b) Taylor Polynomial representation of functions within the computer environment

Figure 2.1: Analogy between floating point representation of real numbers in the computer environment, and the Taylor series representation of functions in a computer environment. Source: (Armellin et al., 2012, p. 1)

To tackle this problem, Berz devised Differential Algebra (DA). Differential Algebra extracts information on a function through approximating it as a higher order Taylor polynomials. Like a closed-form solution, this analytical equations describes the variation of input to an output through a simple function evaluation. Furthermore, the Taylor coefficients define the sensitivity of the solution with non-linear terms around the point of expansion. This is useful for sensitivity analysis. In the computer environment, the Taylor Polynomial Series Algebra (TPSA) represents functions in a similar way floating points arithmetic (FPs) represent real numbers (Martin, 1999, p. 1).

2.1.2 Truncated Polynomial Series Algebra

Figure 2.1a illustrates FP representation of real numbers. Consider two real numbers a, b , to operate on them within finite space they must be convert to a FP, a number with fixed number of decimal places $\bar{a}, \bar{b}$. This transformation is indicated by $\mathcal{T}$. An operation in real space, $*$, must have an equivalent in floating-point space, $\circledast$, such that performing the operation before or after the transformation yields the same floating-point result. This ensures that arithmetic on floating-point numbers remains a consistent approximation of real arithmetic within the limits of finite precision.

Figure 2.1b illustrates that TPSA are to functions, what floating points arithmetic is to real numbers. Suppose functions f, g , they are k -times differentiable across n variables. F and G represent their Taylor Series counterparts while $\mathcal{P}$ represent the transformation. TPSA defines the operator $\circledast$ in polynomial space so that the diagram in Figure 2.1b commutes. In other words, applying operations directly to the Taylor expansions F and G

yields the same result as first operating on the underlying functions f and g in real space and then Taylor expanding the outcome. Thus, $F \circledast G$ is the Taylor expansion of $f * g$

To guarantee that Taylor Polynomial arithmetic remains well defined, DA introduces nilpotent infinitesimal linear variation δx (Martin, 1999, p. 1). These are 'variables of variation' and describe an infinitesimal variation from a central number. These behave like algebraic variables under addition and multiplication, except that for a Taylor expansion order m , every infinitesimal δx satisfies the rule $\delta x^{m+1} = 0$ (Martin, 1999, p. 1). This nil potency rule enforces the truncation order of the Taylor series, and ensures that operations within TPSA never generate terms higher than the specified order m . This ensures that the community of TPSA evident in figure 2.1. Therefore, The nilpotent property of these infinitesimal parts provide the algebraic foundation for TPSA. It provides finite, consistent, representation of a functions behaviour analogous to how FP translates real arithmetic.

The relationship between the nilpotent infinitesimal variable and Taylor Expansion is explicit through the following example:

Suppose order $m = 1$ for function $f(x) = x^2$ where x is defined by a real part r and an infinitesimal part δx . Through expansion

$$f(x) = f(r + \delta x) = (r + \delta x)^2 = r^2 + 2r\delta x + \delta x^2 \quad (2.1)$$

However an order of $m = 1$ is imposed therefore $\delta x^2 = 0$

$$f(r + \delta x) = (r + \delta x)^2 = r^2 + 2r\delta x \quad (2.2)$$

Compare with the Taylor expansion of $f(x)$ around r :

$$f(x) = f(r) + f'(r)(x - r) + \frac{1}{2}f''(r)(x - r)^2 + \dots \quad (2.3)$$

Substitute $x = r + \delta x$, where δx is the linear variation term:

$$f(r + \delta x) = f(r) + f'(r)\delta x + \frac{1}{2}f''(r)\delta x^2 + \dots \quad (2.4)$$

But since $\delta x^2 = 0$, the expansion truncates:

$$f(r + d) = f(r) + f'(r)\delta x \quad (2.5)$$

Equating equation 2.1.2 and equation 2.1.2:

$$f(r) + f'(r)\delta x = r^2 + 2r\delta x \quad (2.6)$$

The nilpotent rule guarantees consistency: one can either (i) take the Taylor expansion first and then apply TPSA operations $\circledast$, as in equation 2.1.2, or (ii) evaluate the function first and then expand it. In both cases the result is identical, so the operations commute. A further implication is that evaluating a function in Taylor polynomial space automatically encodes its derivatives in the coefficients of the resulting series. For example, for $f(x) = x^2$, the derivative $f'(x) = 2x$ appears directly as the coefficient $2r$ in equation 2.1.2. This property becomes especially powerful when dealing with more complex composite functions.

Suppose a $f(x, y) = \sin(x^2 + y)$, let x_0, y_0 be the expansion points and $\delta x, \delta y$ the infinitesimal variations to an order $m = 1$. The Taylor expansion of the components of $f(x, y)$ written in DA terms are:

$$x^2 + y = (x_0 + \delta x)^2 + (y_0 + \delta y) = x_0^2 + y_0 + 2x_0\delta x + \delta y \quad (2.7)$$

Substitution of equation 2.1.2 into $\sin(x^2 + y)$ yields

$$\sin(x^2 + y) = \sin(x_0^2 + y_0) + \cos(x_0^2 + y_0)(2x_0\delta x + \delta y) \quad (2.8)$$

Re-arranging to make δx and δy to separate the $\delta x, \delta y$ terms:

$$\sin(x^2 + y) = \sin(x_0^2 + y_0) + 2x_0 \cos(x_0^2 + y_0)\delta x + \cos(x_0^2 + y_0)\delta y \quad (2.9)$$

$$f_x(x_0, y_0) = 2x_0 \cos(x_0^2 + y_0) \quad (2.10)$$

$$f_y(x_0, y_0) = \cos(x_0^2 + y_0) \quad (2.11)$$

Comparing equations 2.1.2 and 2.1.2 shows that the partial derivatives are recovered directly from the coefficients of the Taylor expansion. This is precisely the chain rule of the original function. In other words, evaluating a composite function on DA variables

automatically returns both the function value and its higher-order derivatives. To ensure the coefficients of the Taylor polynomial remain valid, the variation terms δx must be normalized to lie within the domain $[-1, 1]$

In the DA-based astrodynamics literature, variables are typically represented as a constant part x_0 (the floating point value) plus a linear variation δx . The square bracket notation $[x]$ denotes DA initialisation of the x variable (Pirovano, 2020, p. 1).

$$[x] = x_0 + \delta x \quad (2.12)$$

Once an initial variable is passed through a function f , the result is a Taylor expansion of arbitrary order k :

$$[x_f] = \mathcal{T}_{x_f}^k(x_0) \quad (2.13)$$

where x_f is Taylor polynomial, x_0 is the constant part of the input, and $\mathcal{T}_{x_f}^k$ represents x_f is the solution to the Taylor polynomial. For clarity, this thesis will follow this notation and not original convention of Berz.

2.1.3 Expansion of the solution of Parametric Implicit Equations

When looking to solve a for an implicit equation of type

$$f(x) = 0 \quad (2.14)$$

techniques such as Newton's method to identify the value of x which satisfies equation 2.1.3 within a desired tolerance. If x has an explicit dependance on p , the equation can be expressed in the form of a parametric implicit equation (Armellin et al., 2012, p. 1).

$$f(x, p) = 0 \quad (2.15)$$

To solve equation 2.1.3 one needs to find the value x , classical methods require p to be a fixed value. However, the relationship between the x and p within equation 2.1.3 may be of interest. If a relationship between x and p can be established, one can assess the relationship between p and solution of equation 2.1.3.

DA techniques can solve the previous problem by identifying $x(p)$ as a high order Taylor expansion around a reference solution $x_0, x = x_0 + \mathcal{T}_x(p)$. The algorithm introduced to achieve this is detailed below (Armellin et al., 2012, p. 1):

1. The initial solution to equation 2.1.3, x_0 , is computed via Newton Iteration with a reference p_0 . It is conducted entirely in FP space.
2. x and parameter p are initialised as DA variables:

$$[x] = x_0 + \delta x \quad (2.16)$$

$$[p] = p_0 + \delta p \quad (2.17)$$

3. Substitution and evaluation of equations 2 into equation 2.1.3 results in a k^{th} order Taylor expansion of f in terms to $[x]$ and $[p]$.

$$f = \mathcal{T}_f(x, p) \quad (2.18)$$

4. $\mathcal{T}_f$ denotes the Taylor polynomial for f in terms of $[x]$, $[p]$. x_0 and p_0 are the solution to equation f , were the first terms to be initialised. Therefore the constant terms in $\mathcal{T}_f$, $f(x_0, p)$ are equal to zero. Thus, equation 3 represents the deviation of f from the nominal solution, that is it represents δf .
5. Equation 3 is augmented by an identity function to have as many functions as variables. This obtains a square system of equations.

$$\begin{bmatrix} f \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{T}_f \\ \mathcal{I} \\ \sqrt{} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \quad (2.19)$$

6. The k^{th} order map of equation 5 inverted to obtain

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{T}_f \\ \mathcal{I} \\ \sqrt{} \end{bmatrix}^{-1} \begin{bmatrix} f \\ p \end{bmatrix} \quad (2.20)$$

7. Remembering the constant term of f is 0

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{T}_f \\ \mathcal{I} \\ \sqrt{} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ p \end{bmatrix} \quad (2.21)$$

8. Through Expansion of the first row

$$\delta x = \mathcal{T}_x(p) \quad (2.22)$$

9. Substitution of equation 8 into equation 2 yields

$$[x] = x + \mathcal{T}_x(p) \quad (2.23)$$

Equation 9 is the k^{th} order Taylor expansion of the solution of the implicit equation, equation 2.1.3. For every value of p the approximate solution of $f(x, p) = 0$ can be computed by evaluating the Taylor expansion with a displacement δp from the reference p_0 . The accuracy of the approximation depends on i) the order of the Taylor expansion ii) the magnitude of displacement δp from p_0 (Pirovano, 2020, p. 1). Therefore, tuning of expansion order for ensure the δp range is accuracy is mandatory.

2.1.4 Differential Algebra effect on Precovery Operations

The key features of DA which are exploited are:

1. **Expansion of parametric implicit equations:** DA can represent the non-linear relationship between measurement uncertainty (e.g., 3σ in right ascension and declination) and orbital parameters via Taylor polynomials. Higher-order coefficients of the Taylor expansion capture these non-linear effects, providing a compact description of the uncertainty domain in $(p, \dot{p})$.
2. **Automatic differentiation of composite functions:** Because DA operations commute, both algebraic and differential equations can be evaluated entirely in Taylor space. The coefficients of the resulting polynomial directly encode higher-order derivatives, so Jacobians of composite functions are obtained automatically and at no additional computational cost. In orbit determination, this property allows the Jacobian needed for Newton iteration and differential correction to be extracted directly from the DA map.
3. **Model the Flow Expansion of Ordinary Differential Equations:** An ODE numerical integrator is a function just like any other. Therefore if numerical integration is carried out in TPSA the numerical scheme is embedded within the Taylor coefficients. As a result, the output is not a single trajectory, but Taylor polynomial

which maps the trajectory of the local neighbourhood of initial conditions. For any variation from the reference solution, the relationship, defined by the Taylor Polynomial, can be evaluated. The accuracy of this relationship depends on the order of the polynomial expansion.

2.2 Automatic Domain Splitting

The accuracy of DA degrades when the domain of interest becomes highly stretched in one direction (Pirovano, 2020, p. 1). This limitation is intrinsic to Taylor polynomials: the farther a point lies from the expansion reference, the larger the truncation error becomes. In orbit determination and propagation, such stretching arises from three factors: (i) strong non-linearities in the dynamics, (ii) large initial uncertainty sets (Pirovano, 2020, p. 1), and (iii) long-term propagation horizons (Wittig et al., 2015, p. 1). All three conditions occur in Preccovery problems: short observational arcs introduce non-linear effects, initial uncertainties can reach several arcseconds, and propagation over full NEO orbits is required. Without ADS, maintaining accuracy under these conditions would require very high-order DA expansions, which significantly increase computational cost (Wittig et al., 2015, p. 1) and slow down subsequent operations.

Automatic Domain Splitting (ADS) is a tool which enables an initial domain to be split into sub-domains. Sub-domains represent smaller partitions of the larger initial domain, so that each Taylor expansion remains within its region of validity. This allows lower-order polynomials to define the local solution space under better convergence properties (Wittig et al., 2015, p. 1). In effect, ADS is a "divide-and-conquer" approach which defines new reference points for smaller Taylor polynomials to be expanded around. By adaptively refining the domain, ADS mitigates the loss of accuracy that occurs when DA is applied over highly stretched regions, enabling accurate orbit determination and propagation for NEO Preccovery without resorting to prohibitively high expansion orders. ADS achieves this through an automatic procedure (Pirovano, 2020, p. 1).

The ADS procedure is fully automatic and follows these steps (Wittig et al., 2015, p. 1):

1. Estimate the truncation error of the current polynomial.
2. Identify the dimension along which the function exhibits the largest stretching.
3. Split the domain along that dimension, creating two new sub-domains with updated reference points and associated Taylor polynomials.

4. Repeat the process until the truncation error in every sub-domain falls below the prescribed tolerance.

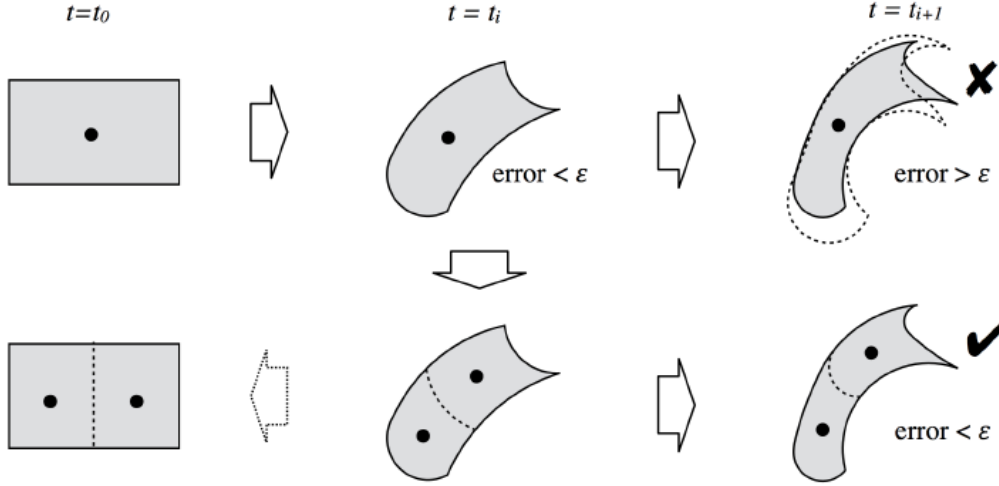


Figure 2.2: The mechanics of ADS in propagation (Wittig et al., 2015, p. 1)

Figure 2.2 is an example of ADS. Propagation starts on the top left. An initial step $t_i - t_0$ is performed, the truncation error is below the tolerance (top middles). Thus a subsequent step is performed such that $t = t_{i+1}$ (top right). At $t = t_{i+1}$ the truncation error above the threshold. Propagation is taken one step back (top middles), and the domain is halved (bottom middle). Propagation of this domain ensures accuracy at $t = t_{i+1}$. The split is stored on the initial domain.

Mathematically, let $\mathcal{T}_\oplus$ be the polynomial representation of the unique solution $\Phi_t^*(x)$ at time t^* within the domain $x \in [-1, 1]$. The split $\mathcal{T}_\oplus$ into $\mathcal{T}_{\oplus,\infty}$ and $\mathcal{T}_{\oplus,\epsilon}$ is

$$\mathcal{T}_{\oplus,\infty}(x) = \mathcal{T}_\oplus\left(\frac{1}{2}x - \frac{1}{2}\right) \quad (2.24)$$

$$\mathcal{T}_{\oplus,\epsilon}(x) = \mathcal{T}_\oplus\left(\frac{1}{2}x + \frac{1}{2}\right) \quad (2.25)$$

A halved domain reduces the truncation error of $\mathcal{T}_\oplus$ by 2^{2k+1} (Wittig et al., 2015, p. 1). This relationship is a result of Taylor's Theorem; Let f be a $k+1$ differentiable function $f \in C^{k+1}$ and let $\mathcal{T}_f$ be its k^{th} order Taylor expansion with domain radius $r > 0$. The following is defined

$$|f(\delta x) - \mathcal{T}_f| \leq C\delta x^{k+1} \leq Cr^{k+1} = e_r \quad (2.26)$$

If the radius of the domain r is halved

$$|f(\delta x) - \mathcal{T}_f| \leq C\delta x^{k+1} \leq C\left(\frac{r}{2}\right)^{k+1} = \frac{e_r}{2^{k+1}} \quad (2.27)$$

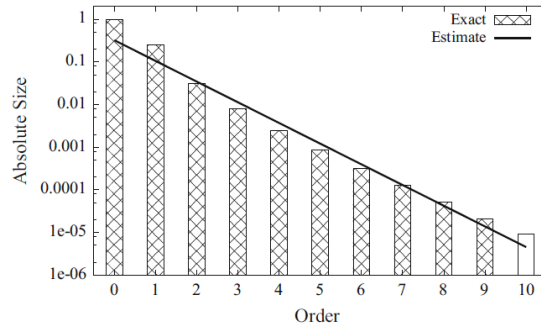


Figure 2.3: Estimation of truncation error of the 9th order Taylor expansion of $\sqrt{1 + \frac{x}{2}}$. S_i of each order is shown by the hatched bars and the fitted function is shown as a line. The true 10th order S_i is shown by the white bar. The difference between the true and predicted S_{10} is 2×10^{-6} (Wittig et al., 2015, p. 1)

Halving the domain reduces the truncation error of the Taylor polynomial. However, equation 2.2 requires the desired sub-domain size to be specified before the function is evaluated. In some cases, a single domain may already be sufficient. If the sub-domain size is fixed in advance, unnecessary splitting can occur, leading to wasted computational effort

Wittig devised a method to account for all the information available in the polynomial to approximate the size of the $k + 1$ order solution. This is used to estimate the truncation error. Given the coefficient of the same order i , the sum of the absolute values S_i is

$$S_k = \sum_{|\alpha|=k} |\alpha| \quad (2.28)$$

A least squares fit of the exponential function

$$f(i) = A \exp(Bk) \quad (2.29)$$

is used to determine coefficient A and B such that $f(k) = S_k$. $f(k + 1)$ is used to estimate the size of S_{k+1} of truncation order $k + 1$ of k^{th} Taylor polynomial $\mathcal{T}$. If S_{k+1} is too large with respect to the error threshold, the domain is split and evaluated within the new domain. The new polynomial coefficients will be smaller by a factor of 2^{k+1} and the truncation error will be reduced.

In the multi-variable case, this process is still conducted. Prior to the domain splitting the dimension with the largest truncation error is split first. The ADS logic is shown in

figure 2.4.

Implementation-wise, the ADS routine is defined by the truncation error and the maximum number of splits across the domain. The former, sets the threshold of the ADS routine, while the latter, places a upper threshold to number of splits to limit computational cost. Alongside all DA functions, the ADS routine within the DACEyPy repository were used (Purpura, 2024, p. 1).

2.2.1 Single-variable examples

This example is taken from Pirovano to aid understanding (Pirovano, 2020, p. 1).

Figure 2.5 shows the single polynomial expansion (left) and the ADS routine applied to the sine function (right). It is a 5^{th} order TPS with a truncation error threshold of 10^{-4} . ADS found 8 sub-domains were sufficient to meet the 10^{-4} accuracy requirement.

Figure 2.6 illustrates how ADS behaves with highly non-linear functions that approach singularities. The function is approximated by a 5^{th} order polynomial with a maximum truncation error of 10^{-4} and maximum number of splits is 7. In the right-hand figure, the function is approximated by 34 sets of polynomials. The frequency of the sub-domains (back dots) increase over significantly non-linear regions and at the singularity (Pirovano, 2020, p. 1). Near the singularity the ADS approaches the maximum number of splits. This shows the ADS tool cannot be used in functions which contain a discontinuity. For this reason, ADS is unsuitable for functions with true discontinuities, but it is highly effective for orbit determination and propagation problems, where the dynamics are continuous yet strongly non-linear.

2.2.2 ODE propagation

The following is an example of the ADS routine applied to orbit propagation. The routine starts at the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$. One time-step is integrated using DA based integrators offered by DaceyPy. At every propagation step the truncation error is checked. This results in one of two conditions:

1. Truncation error is below threshold tolerance.
2. Truncation error is larger than error tolerance.

The first condition results in no domain splits and the integrator continues as normal. The second condition forces the integrator to stop at timestep t_i . The domain correspond-

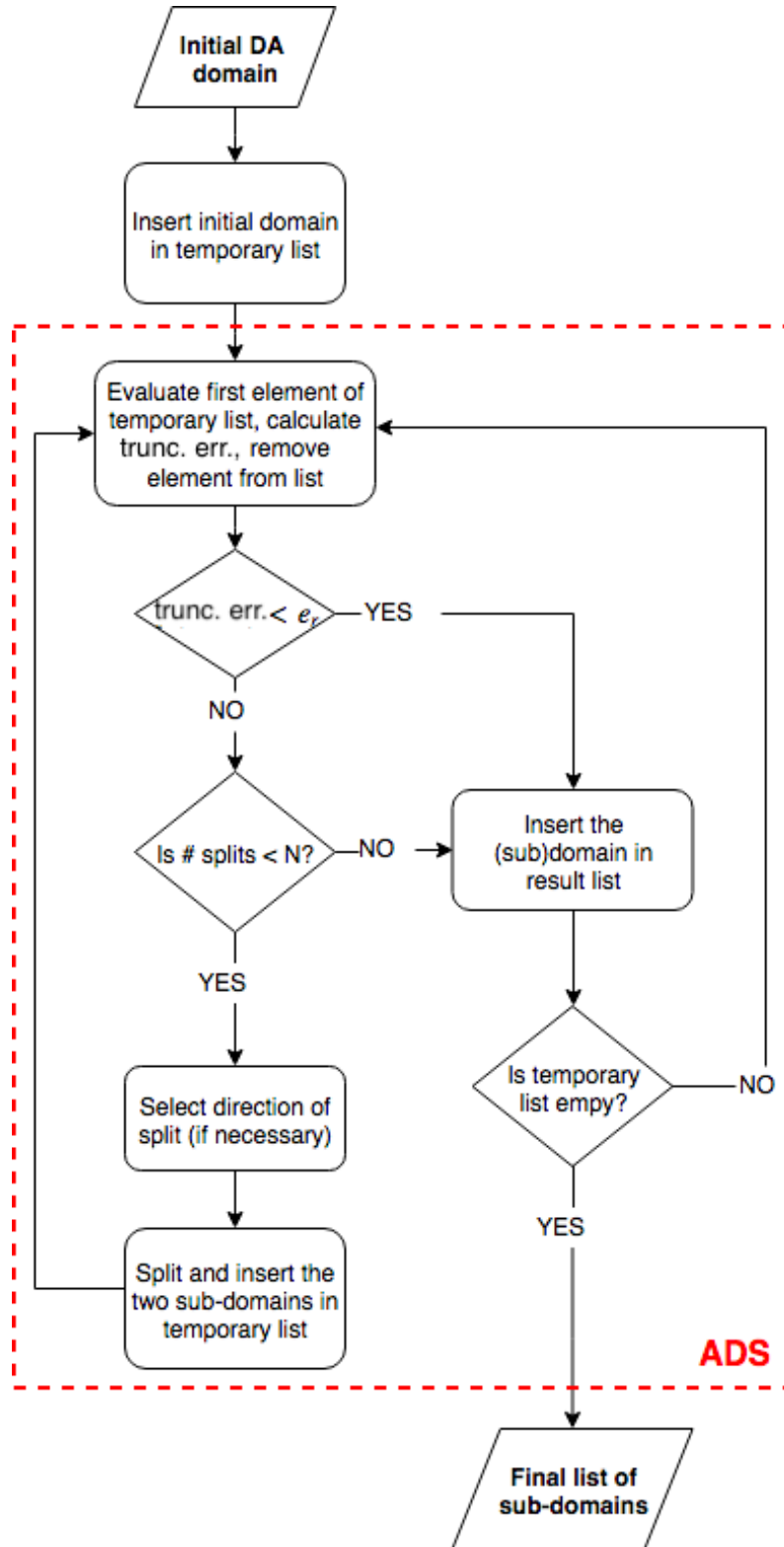
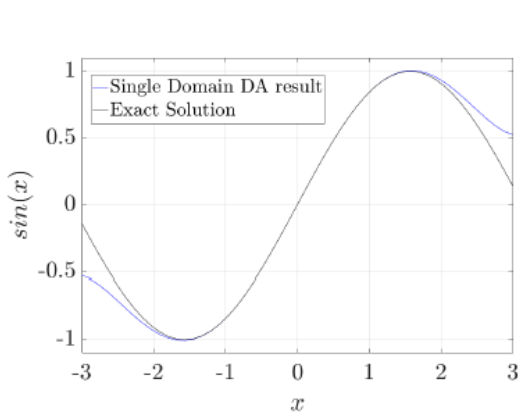
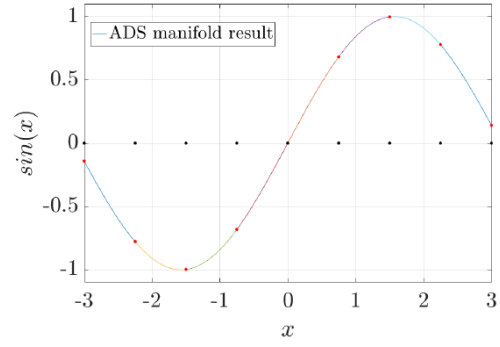


Figure 2.4: ADS logic: The initial domain is copied in a list, for each element of the list checks are performed to detect whether the truncation error is above the threshold and/or the maximum number of splits have been reached. If necessary the domain considered is split and inserted in the list. This list is checked again for further possible splits. Once the entire list is checked, the routine is ended (Pirovano, 2020, p. 1).



(a) Single Polynomial expansion vs exact solution



(b) Set of polynomials vs exact solution. The black dots on the x-axis define the centre of the sub-domains. The red dots are the centre of their Taylor polynomial expansion

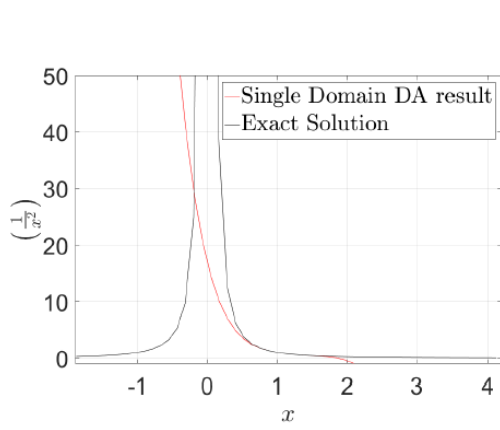
Figure 2.5: Polynomial approximate of the sine function (Pirovano, 2020, p. 1).

ing to $\mathbf{x}_i$ is split, and two new state vectors are generated at t_i . This continues till $t = t_{final}$. Figure 2.7 demonstrates the dependance between propagation time and number of split domains within ADS.

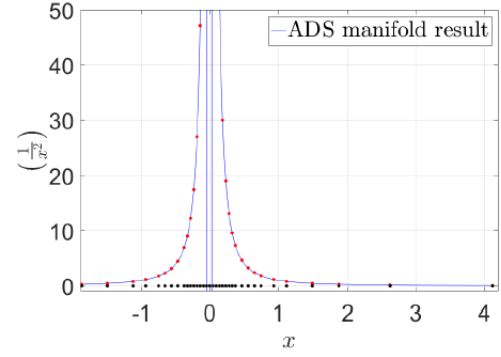
2.2.3 ADS Application on Precovery

Wittig (Wittig et al., 2015, p. 1) demonstrated that DA with Automatic Domain Splitting (ADS) can successfully propagate the uncertainty of Apophis across the highly non-linear dynamics of the 2029 close approach. Using a ninth-order expansion, the ADS routine maintained accuracy of 0.5×10^{-11} AU for nearly 1000 days after the encounter. Although the computation time was higher than point-wise propagation, Wittig showed that beyond a few thousand Monte Carlo samples, ADS becomes more efficient while still covering the full uncertainty domain. This established DA+ADS as a viable tool for long-horizon NEO propagation.

By contrast, Pirovano (Pirovano, 2017, p. 1) applied ADS to the short arc initial orbit determination (IOD) problem, but only in the context of Earth-resident space objects. Figure 2.8 illustrates splitting angular domain (α, δ) improved the 3σ boundary accuracy of Taylor polynomial within the $(\rho, \dot{\rho})$. The ρ maximum absolute error in the ADS figure is 50x smaller than the standard DA version. Similar to Wittig, the penalty of ADS is increased computational cost. Between the conclusions drawn by Pirovano and Wittig it was stated that the increase in computational cost was required for DA based bounding boxes to include ADS to handle non-linearities effectively.

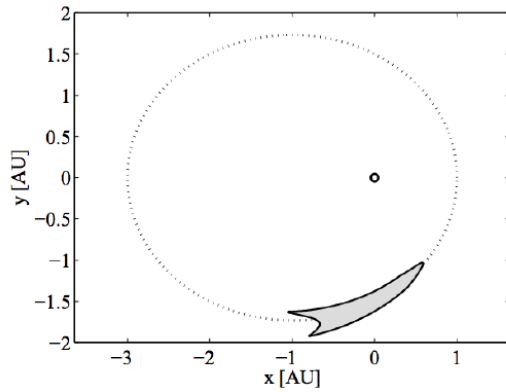


(a) Single Polynomial expansion vs exact solution

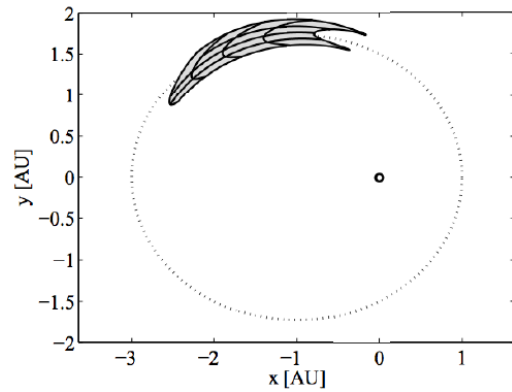


(b) Set of polynomials vs exact solution. The black dots on the x-axis define the centre of the sub-domains. The red dots are the centre of their Taylor polynomial expansion

Figure 2.6: Polynomial approximation of the inverse-square function (Pirovano, 2020, p. 1).

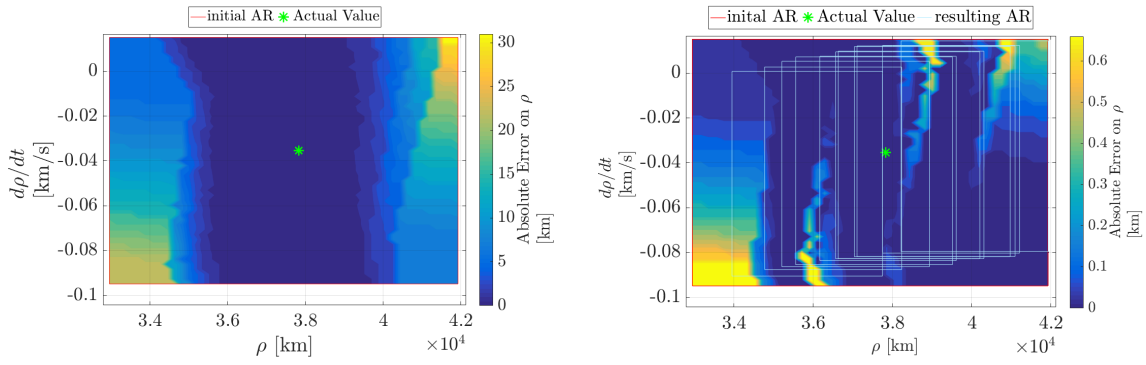


(a) 0.90 Nominal revolutions



(b) 2.25 nominal revolutions

Figure 2.7: Example of the evolution of ADS splits under two-body dynamics (Wittig et al., 2015, p. 1)



(a) Error within the Range Taylor Polynomial before ADS

(b) Error within the Range Taylor Polynomial ADS

Figure 2.8: The impact of ADS on Initial Orbit Determination accuracy (Pirovano, 2017, p. 1)

Therefore, DA with Automatic Domain Splitting (ADS) has been used to propagate NEO uncertainties (e.g., Apophis), demonstrating long-horizon accuracy under strong non-linearities (Wittig et al., 2015, p. 1). However, those works begin from a known orbit; they do not address initial orbit determination from short arcs. This contrasts with Precovery methods which require IOD then propagation. While DA/ADS IOD algorithms have been applied for Earth based satellites, no such algorithm has been applied for NEO Precovery. This motivates this study to integrate DA/ADS IOD and ADS propagation.

2.3 Orbit Determination

Orbit determination (OD) estimates and refines a space object's state ($\mathbf{r}, \mathbf{v}$) from observations (Vallado, 2013, p. 1). It comprises initial orbit determination (IOD)—obtaining a first state directly from sparse measurements—and refinement, typically via least squares. Because measurements contain instrument and atmospheric errors (Gronchi, n.d., p. 1), IOD from short arcs often admits multiple compatible orbits which force the need for orbit refinement. This chapter focuses on IOD methods relevant to NEO Precovery; as such IOD methods for short arc orbit determination are examined.

NEO observations are typically characterised by short arcs, or track-lets, obtained within a single telescope pointing. A track-let consists of $3 \leq N \leq 5$ detections spanning a timescale of order ~ 2 h (Gronchi, n.d.; Milani et al., 2004, p. 1). Over such short intervals, the apparent curvature of the orbit cannot be observed, and the line-of-sight vectors to the object are nearly collinear. As a result, the range and range-rate remain weakly constrained, and the initial orbital solution is highly uncertain (Pirovano, 2020, p. 1). These

characteristics make short-arc conditions the fundamental challenge in NEO initial orbit determination and Precovery operations.

NEO Precovery places unique demands on orbit determination. A suitable IOD technique must (i) define the initial uncertainty region with sufficient precision to enable reliable archival searches, (ii) their must be uncertainty representation; the initial solution space should be rigorously bounded (e.g., by measurement error) to balance true positives with false negatives, and (iii) do so with high computational efficiency, searches often involve year-long propagation over vast survey datasets (Valvano et al., 2024, p. 1) where time can be limited if the NEO is a possible impactor. In practice, this means balancing rigorous probabilistic modelling of the solution space with algorithms that can deliver results quickly enough to support timely risk assessment. Differential Algebra-based IOD methods are natural candidates, as they promise both compact analytic representations of uncertainty and reduced computational cost compared to large-scale sampling.

2.3.1 Classical Initial Orbit Determination

Classical orbit determination—as in Gauss’s solution for Ceres—was devised for a very different geometric regime (Gauss, 1809, p. 1). Nightly observations over several days provided an arc corresponding to 5% of the orbit. This arc holds observable curvature, allowing the assumption of a conic section and the use of geometric relations (coplanarity, Lagrange coefficients) to recover the state reliably (Vallado, 2013, p. 1). It assumes two-body dynamics where the Sun is central attractor.

Assume three observations $\alpha_i, \delta_i, i = 1, 2, 3$ of a celestial body at times t_i where $t_1 < t_2 < t_3$. Let $\mathbf{r}_i, \boldsymbol{\rho}_i$ denote the heliocentric and geocentric position of the body respectively, and let $\mathbf{q}_i$ be the heliocentric position of the observer.

$$\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{q}_i, \quad i = 1, 2, 3 \quad (2.30)$$

The heliocentric position of the observer, $\mathbf{q}_i$, is decomposed into the heliocentric position of Earth $\mathbf{R}_\oplus$ and the observer location on the Earths surface $\mathbf{R}_{obs}$.

$$\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{R}_\oplus + \mathbf{R}_{obs} \quad (2.31)$$

$\boldsymbol{\rho}_i$ is decomposed into the slant range, ρ_i , and the line-of-sight direction $\hat{\rho}_i$.

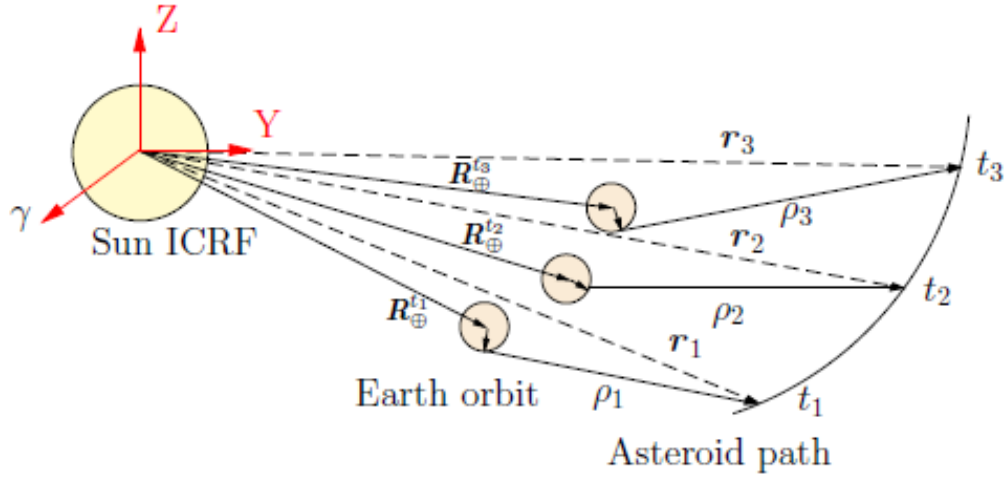


Figure 2.9: Vectors that define Gauss's Problem (Albini, 2010, p. 1)

$$\mathbf{r}_i = \rho_i \hat{\boldsymbol{\rho}}_i + \mathbf{R}_{\oplus i} + \mathbf{R}_{obs} \quad (2.32)$$

Where $\hat{\boldsymbol{\rho}}_i$ is the cosine direction vectors from the observation site to the Asteroid.

$$\hat{\boldsymbol{\rho}}_i = \cos(\delta_i) \cos(\alpha_i) \hat{i} + \cos(\delta_i) \sin(\alpha_i) \hat{j} + \sin(\delta_i) \hat{k} \quad (2.33)$$

The conversion of orbit angular momentum requires $\mathbf{r}_i$ to be coplanar.

$$\mathbf{r}_2 = c_1 \mathbf{r}_1 + c_3 \mathbf{r}_3 \quad (2.34)$$

Finally, the consequence of the two-body equations of motion is that the state vectors $(\mathbf{r}, \mathbf{v})$ are expressed in terms of the state vectors of any given time through Lagrange Coefficients f and g (Curtus, 2005, p. 1). Therefore $\mathbf{r}_1$ and $\mathbf{r}_3$ can be expressed in terms of $(\mathbf{r}_2, \mathbf{v}_2)$ when $|t_i - t_{i-1}|$ is sufficiently small (Curtus, 2005, p. 1). This is the reason for 5° separation requirement (Vallado, 2013, p. 1).

$$\mathbf{r}_1 = f_1 \mathbf{r}_2 + g_1 \mathbf{v}_2 \quad (2.35)$$

$$\mathbf{r}_3 = f_3 \mathbf{r}_2 + g_3 \mathbf{v}_2 \quad (2.36)$$

Through equations 2.3.1, 2.3.1 and 2.3.1 there are 18 equations in 18 unknowns.

Curtis illustrates the derivation the coefficients A, B, E to form the eighth order polynomial.

$$r_2^8 - (A^2 + 2AE + R_2^2)r_2^6 + (-2\mu B(A + E)r_2^3 - \mu^2 B^2) = 0 \quad (2.37)$$

The eighth order polynomial yields eight possible solutions. However, physically possible solutions are vetted under the following conditions: i) the root is a real solution ii) $r_2 > 0$

Once r_2 is known, the slant ranges are computed per Curtis' derivation, $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are computed through equation 2.3.1. Substitution of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and re-arrangement of equation 2.3.1 yields $\mathbf{v}_2$

$$\mathbf{v}_2 = \frac{1}{f_1 g_1 - f_3 g_1} (-f_3 \mathbf{r}_1 + f_1 \mathbf{r}_3) \quad (2.38)$$

Gauss's method produces an estimate of the asteroid's state vector $[\mathbf{r}_2, \mathbf{v}_2]$ at the central observation epoch t_2 . When the observation arc Δt is sufficiently long, the resulting position estimates can achieve accuracies better than 10^{-4} AU (Albini, 2010, p. 1). A key advantage is computational efficiency: because the method relies on closed-form analytical relations rather than iterative numerical schemes, the orbit can be recovered very quickly. In a planetary defence context, this speed is attractive, as it enables rapid impact assessment of a newly discovered object's trajectory.

However, Gauss's method for NEO Precovery is limited for Precovery through two key reasons:

- **Gauss takes a deterministic approach to a probabilistic problem:** Each observed position $[\alpha_i, \delta_i]$ is subject to astronomical uncertainty σ . Thus, the true position $[\alpha_{i,true}, \delta_{i,true}]$ satisfy:

$$\alpha_{i,true} \in [\alpha_i - 3\sigma, \alpha_i + 3\sigma] \quad (2.39)$$

$$\delta_{i,true} \in [\delta_i - 3\sigma, \delta_i + 3\sigma] \quad (2.40)$$

However, Gauss assumes:

$$[\alpha_{i,true}, \delta_{i,true}] = [\alpha_i, \delta_i] \quad (2.41)$$

Therefore Gauss Method assumes $(\mathbf{r}_2, \mathbf{v}_2)$ is the true solution. In reality it is one out of multiple possible solution bounded by the measurement noise. As a result, Gauss's method defines a singular orbit a region possible orbit solutions. The omission of the solution space reduces the possibility of correctly identifying true-positive images and, consequently, significantly diminishes the likelihood of successful Precovery. Monte Carlo simulation Gauss IOD offer a solution to modelling the measurement uncertainty, however at great computational performance (section 2.4.1).

- $[\mathbf{r}_2, \mathbf{v}_2]$ deviates from true solution if $\Delta t \rightarrow 0$, in other words the observation is a short arc: If Δt becomes very small the line of sight vectors $\hat{\rho}_i$ are nearly co-linear. The elements within the denominator matrix D_{ij} are the cross-product of the line of sight vectors, $\hat{\rho}_i$. If $\hat{\rho}_i$ are co-linear $D_{ij} \rightarrow 0$ and denominator matrix becomes near-singular. The geometric assumptions break, and Gauss' method produces erroneous results for short arc observations. Short arc observations are apparent in NEO observations, therefore Gauss's method cannot complete NEO IOD reliably.
- **Deterministic treatment of a probabilistic problem:** Each observed position $[\alpha_i, \delta_i]$ is affected by measurement uncertainty σ , so the true values lie within an interval:

$$\alpha_{i,true} \in [\alpha_i - 3\sigma, \alpha_i + 3\sigma], \quad (2.42)$$

$$\delta_{i,true} \in [\delta_i - 3\sigma, \delta_i + 3\sigma]. \quad (2.43)$$

Gauss's method assumes instead that

$$[\alpha_{i,true}, \delta_{i,true}] = [\alpha_i, \delta_i], \quad (2.44)$$

producing a single orbit solution $[\mathbf{r}_2, \mathbf{v}_2]$. In reality, many possible solutions are consistent with the noisy observations. By ignoring this solution space, Gauss reduces the chance of correctly identifying true precovery images. Monte Carlo sampling of Gauss IOD can model this uncertainty, but only at significant computational cost.

- **Breakdown under short arcs:** When the observation interval $\Delta t \rightarrow 0$, the line-of-sight vectors $\hat{\rho}_i$ become nearly collinear. Since the denominator matrix D_{ij} is formed from cross-products of these vectors, $D_{ij} \rightarrow 0$ and the system becomes ill-conditioned. The geometric assumptions underlying Gauss's method fail, yielding unreliable results for short arcs. Because NEO observations are typically short-arc tracklets, this limitation directly undermines its use for NEO IOD.

For NEO Precovery, the combination of a deterministic state solution and the high level of error inherent to short-arc observations results in a severe degradation of Precovery reliability. The assumption of a unique orbit ignores the admissible region of possible orbits created by the measurement uncertainty. While the geometric approach results in poor short arc performance. Together, these limitations restrict Gauss' method from providing robust initial orbit determinations for NEO Precovery.

Berz and Healy concluded that measurement uncertainties within Gauss's method can only be quantified by Monte Carlo techniques (Martin, 1999, p. 1). The principle of a Monte Carlo (MC) simulation is to repeatedly perturb the observational inputs within their statistical error distributions, typically assuming Gaussian noise at the level of the astrometric uncertainty σ . Each perturbed set of observations is then passed through Gauss's IOD algorithm, and produces an distribution of orbit solutions. This ensemble approximates the probability distribution of the true orbit, with its spread reflecting the effect of measurement noise on the orbital elements.

While conceptually simple, this approach is computationally expensive. For reliable statistics, thousands to tens of thousands of samples may be required, each demanding a full Gauss IOD solution followed by orbital propagation. In the context of Precovery, where candidate images from years of survey data must be searched and compared against predicted positions, this becomes prohibitive: every perturbed orbit must be propagated across the entire archival span. Consequently, while Monte Carlo-augmented Gauss IOD can in principle account for observational uncertainty, its high computational cost makes it impractical for timely NEO Precovery operations.

In summary, Gauss's method is efficient and accurate when applied to long arcs such as Ceres, but its assumptions break down for modern NEO discovery. Short-arc track-lets lack observable curvature, making the line-of-sight vectors nearly collinear and the solution unstable. In addition, Gauss treats noisy data deterministically, yielding only a single orbit rather than a distribution. A probabilistic extension via Monte Carlo sampling can in principle recover the solution space, but only at the cost of propagating thousands of candidate orbits, which is prohibitive for Precovery operations that require rapid searches over years of archival data. For these reasons, classical geometric methods like Gauss have been left in the NEO context by approaches that explicitly model the region of possible solutions, such as the Admissible Region (AR) and Differential Algebra-based IOD techniques.

2.3.1.1 Lambert's Problem

Lambert's theorem states that the time of flight, Δt , to travel from $\mathbf{r}_1$ to $\mathbf{r}_2$ along a keplerian orbit is a function of: the orbit's Semi Major Axis, the sum of the two position $\mathbf{r}_1 + \mathbf{r}_2$, and the chord of the ellipse that join those two points (Izzo, 2014, p. 1). The theorem is used to define the velocities at the points $\mathbf{r}_1$ and $\mathbf{r}_2$

Lambert's theorem defines many algorithms which differ in mathematical implementation (for example Battin's algorithm and universal variables formulation (Curtus, 2005, p. 1)). In this thesis Izzo's algorithm is used (Izzo, 2014, p. 1). This is because existing literature on DA based Initial Orbit Determination choose this form. Henceforth it will be referred to as Lambert's Algorithm. For more information see Curtis (Curtus, 2005, p. 1)

Shown in section 3.3, this utilises Thesis Lamberts Algorithm is used to refine the central observation velocity. This is achieved through enforcement of condition

$$\mathbf{v}_2^- - \mathbf{v}_2^+ = 0 \quad (2.45)$$

Where $\mathbf{v}_2^-$ is the velocity at position $\mathbf{r}_2$ calculated via Lambert's algorithm between $\mathbf{r}_1$ and $\mathbf{r}_2$, and $\mathbf{v}_2^+$ is the velocity of position $\mathbf{r}_2$ calculated via Lambert's algorithm between $\mathbf{r}_2$ and $\mathbf{r}_3$.

2.3.2 Admissible Region

The Admissible Region (AR) approach begins with a measured attributable,

$$\mathcal{A} = [\alpha, \delta, \dot{\alpha}, \dot{\delta}] \in [-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^2 \quad (2.46)$$

where α, δ are the right ascension and declination, and $\dot{\alpha}, \dot{\delta}$ are their angular rates, typically estimated from at least three observations (Milani et al., 2004, p. 1).

Since the attributable does not determine the observer range ρ or range rate $\dot{\rho}$, the AR method constrains these unknowns using the physical requirements of a valid orbit (e.g., bound to the Solar System, not a satellite of Earth) (Milani et al., 2004, p. 1). Each admissible pair $[\rho, \dot{\rho}]$ corresponds to a distinct candidate orbit, and the AR defines the set of all possible orbital states consistent with the initial measurements. It assumes $\mathcal{A}$ is known from a series of 3 or more individual observations.

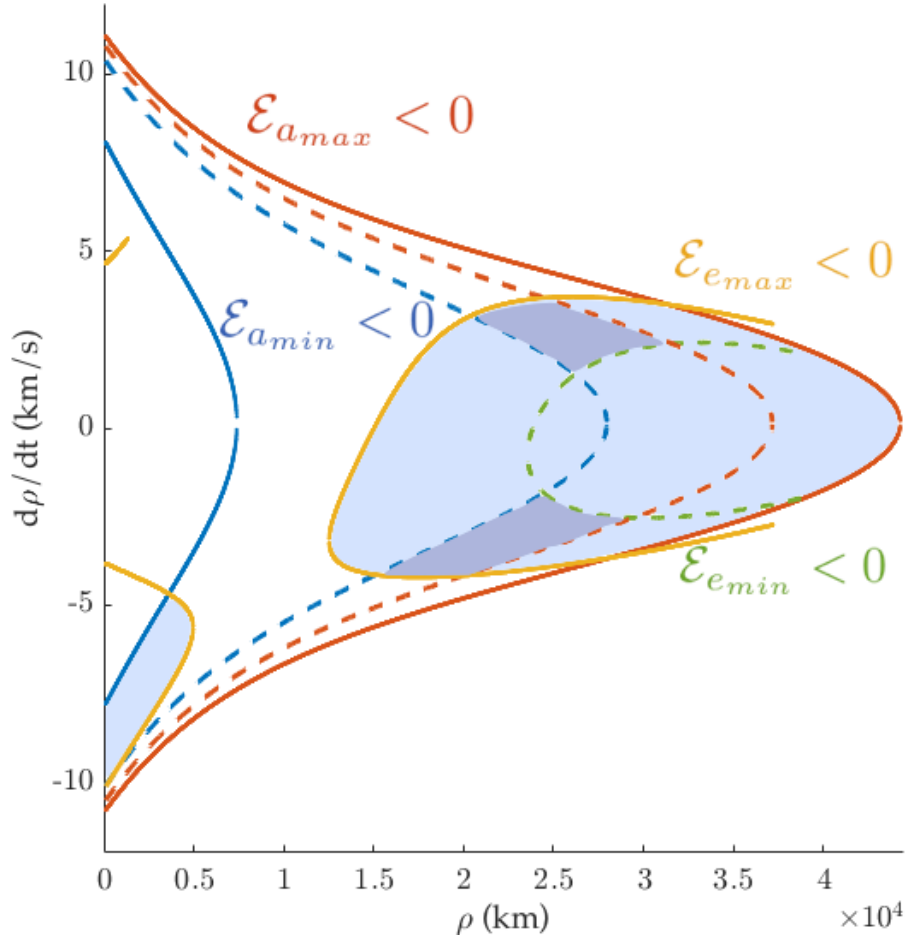


Figure 2.10: Admissible region of a satellite within a Geosynchronous Transfer Orbit

NEO $(\rho, \dot{\rho})$ constraints are defined through the conditions based on physical laws of a plausible orbit (Milani et al., 2004, p. 1):

1. A is not a satellite of the Earth $\epsilon_{\oplus} \geq 0$
2. The orbit of A is not controlled by the Earth $\rho \geq R_{SI}$
3. A belongs to the Solar System $\epsilon_{\odot} \leq 0$
4. A is outside of Earth $\rho \geq R_{\oplus}$

The application of these constraints avoid the use of Gauss's algorithm to define a NEO's state. The performance of AR for short arcs is much greater (Milani et al., 2004, p. 1). The AR covers a set of $(\rho, \dot{\rho})$ solutions, these contains are constrained by the laws above (Milani et al., 2004, p. 1).

The AR is sampled repeated. Dense sample grids that best represent the region of possible solutions are computationally expensive. Delaunay triangulation is used to provide

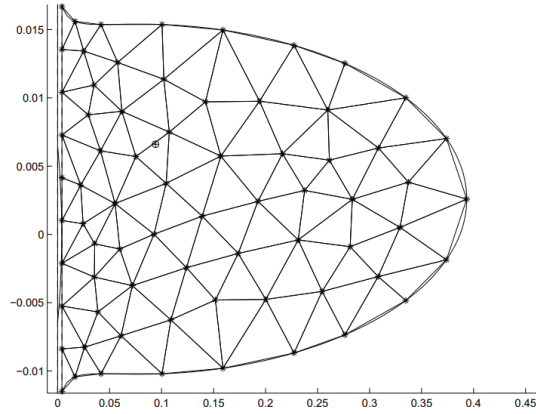


Figure 2.11: Admissible Region: Delaney Triangulation

stable, efficient and flexible mesh for the irregular 2D region of feasible solutions (Gronchi, n.d., p. 1). Each grid point defines a specific $(\rho, \dot{\rho})$ and when combined with the attributable corresponds to a virtual asteroid (VA), a full six-dimensional state vector at the observation epoch (Gronchi, n.d., p. 1). VAs are propagated forward to time of the subsequent attributable, where their predicted

$$(\alpha, \delta) \tag{2.47}$$

are compared to the second set of observations. If the residual are within tolerance the candidate virtual asteroid is retained. This is recursive process is repeated over a sequence of attributables, progressive refining the local mesh after each successful association (Gronchi, n.d., p. 1).

For NEO Precovery, the AR method has the advantage of guaranteeing that all physically feasible solutions are included, ensuring that no valid orbit possibility is missed and thus avoiding false negatives. This includes if the observations are short arcs. This is because only physical constraints are considered, not the geometry of the observed arc. However, this comes at the cost of a very large solution space: a vast number of $(\rho, \dot{\rho})$ combinations are typically admissible (see Figure 2.17). Each combination corresponds to a candidate Virtual Asteroid (VA), which must be propagated and tested against archival data.

This creates the so-called identification problem: associating track-lets across multiple nights requires searching over a large $(\rho, \dot{\rho})$ domains (Gronchi, n.d., p. 1). In practice, this means propagating thousands of VAs forward to compare their predicted (α, δ) with subsequent observations. While effective for handling short-arc data, this process

is computationally intensive and produces many false positives, especially when archival searches span years of survey data (Valvano et al., 2024, p. 1).

This means orbit association and impact risk assessment can be significantly delayed. Moreover, NEO data association is conducted manually, therefore an pure AR approach would force a significant number of images to be pruned manually. This would delay subsequent even longer. Finally, the AR method constructs a grid deterministically. This means any 1 out of the 1000s of VAs cannot be prioritised based on likelihood statistics such as the Mahalanobis miss distance (Harris et al., 2024, p. 1). Therefore, between the large number of false positives, the deterministic VA generation, and the thousands of VAs generated the AR method is too cumbersome to ensure fast impact risk assessment due to slow IOD.

Within the categorisation of identification problems (Gronchi, n.d., p. 1), Precovery corresponds most closely to an attribution case: a newly determined short-arc attributable must be linked to candidate detections in archival images that predate discovery. This is challenging because the admissible region produces thousands of Virtual Asteroids, each of which must be propagated and compared to archival data. The result is a large number of false positives and heavy computational cost, both of which slow down timely impact risk assessment.

In summary, the Admissible Region framework provides a rigorous way to capture all feasible orbital solutions from short-arc data, ensuring that no valid orbit is overlooked. However, its deterministic sampling of the $(\rho, \dot{\rho})$ plane leads to thousands of Virtual Asteroids that must each be propagated and compared against archival images. This exhaustive approach, while effective, is computationally intensive and produces many false positives, limiting its practicality for Precovery operations where rapid turnaround is essential. To address these challenges, more efficient methods are needed that can represent the measurement uncertainty domain while avoiding brute-force sampling. Differential Algebra (DA)–based Initial Orbit Determination offers such a pathway: by encoding the uncertainty directly into high-order Taylor expansions, DA captures the non-linear mapping from observations to orbital states with far fewer samples, making it a natural candidate for Precovery applications.

2.3.3 Differential Algebra based Initial Orbit Determination

Gauss’s method offers either a unique solution that disregards the solution space of possible orbits, or a computationally intensive Monte Carlo. Both fail under short arcs. AR

defines a broad solution space which holds high computational cost and time to conduct Orbit determination. Differential Algebra based Initial Orbit Determination (DAIOD) offers a compromise between high computational cost and uncertainty quantification. It achieves this through the construction of a smaller AR, where the boundaries are defined by the 3σ measurement uncertainty (Pirovano, 2020, p. 1). The DAIOD algorithm takes as input the observation of an object, (α_i, δ_i) at t_i for $i = 1, 2, 3$. It outputs Taylor polynomial series of the object state at the central time of the observation, t_2 (Pirovano, 2017, p. 1). The benefits of DAIOD stem from the fundamental properties of Differential Algebra explain in section 2.1:

1. **A reduction within in the search space:** DAIOD models the effect of measurement uncertainty on the orbit solution through Taylor polynomial expansions (Pirovano, 2020, p. 1). The variational terms, δx , encode the 3σ measurement uncertainty directly. For NEO Preccovery, this reduces the initial search space and, consequently, the number of false positives that must be evaluated. The result is faster data association and lower computational effort.
2. **Analytical Description of Uncertainty:** The Taylor polynomial representation defines an analytical equation relating measurement uncertainty and orbit state. Definition of the uncertainty regions is created through a simple evaluation of the final Taylor polynomial (TPS) 3σ . Monte Carlo Gauss or AR handle uncertainty through numerical techniques; Orbit determination algorithms are continuously repeated till for N samples (Pirovano, 2020, p. 1). The DA approach defines an analytical equation which can be propagated in one-shot and evaluated post propagation. The smaller number of integration cycles mean DA is more efficient than numerical integration techniques.
3. **Improved Short Arc performance over Gauss Algorithm:** To construct the Taylor expansion, DAIOD begins with Gauss's IOD solution and then refines the topocentric range ρ by solving Lambert's problem until the difference in velocity at the centre point is zero (Pirovano, 2020, p. 1). This ensures that the nominal orbit satisfies physical Keplerian constraints rather than relying solely on geometric approximations. As a result, DAIOD has demonstrated reliable orbit determination for tracklets as short as 2 hours (Armellini et al., 2012, p. 1). This falls under Milani's short arc definition for NEOs (Milani et al., 2004, p. 1).

Armellini et al, defined the first DA-based Initial Orbit Determination algorithm for NEOs. He defined the Gauss-Lambert solver methodology through the DA framework and illustrated DAIOD application to short arc orbit determination (Armellini et al., 2012,

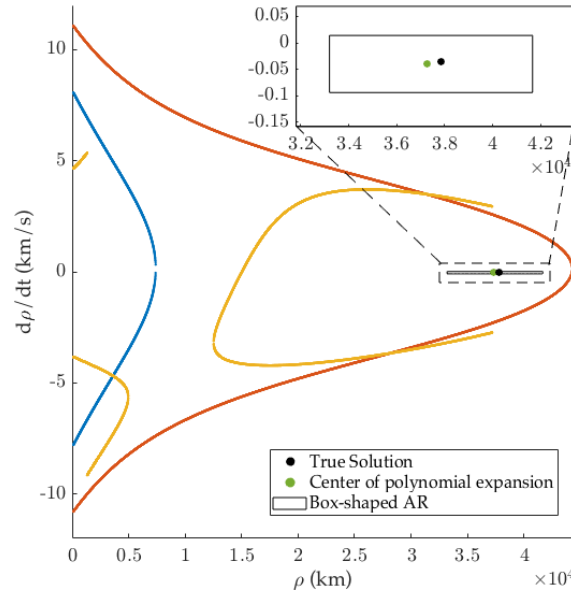


Figure 2.12: Comparison between AR solution space and the DA-IOD solution space. Note 1) the containment of True Solution within the DA-IOD black coloured box 2) The compared to standard AR (Pirovano, 2020, p. 1)

p. 1). The method accounted for the effect of 3σ measurement uncertainty on an orbits state. The framework took a Monte Carlo approach, thus suffered from the same computational burden as a standard Gauss Monte Carlo simulations. He illustrated the logarithmic relationship between the magnitude error of the DA IOD solution and the order of TPS expansion. He validated the DAIOD algorithm within a Monte Carlo Gauss Control simulation (Armellin et al., 2012, p. 1).

Pirovano advanced the DA-IOD algorithm by introducing a bounding box approach. The approach assumed that observations noise followed a gaussian distribution (Pirovano, 2020, p. 1). The box is defined by the 3σ measurement uncertainties, ensuring that 99.7% of the possible observations are contained within its limits. As it defined a 3σ bounding box around 99.7% of the possible orbits. This eliminates the need for thousands of repeated DA evaluations, as required in Armillen's Monte Carlo scheme. Instead, a single run of the DAIOD algorithm produces an analytical polynomial representation that fully characterises the solution space. As a result, the bounding box method provides a dramatic reduction in computational cost while still capturing the complete set of admissible orbits.

The DAIOD Method is included as it will be used later within the Methodology Chapter, it can be decomposed into three parts:

The first part of DAIOD aims to obtain the nominal solution for Taylor Polynomial

Series:

An initial position vector $\mathbf{r}_i$ and slant range ρ_i are defined via Gauss's Algorithm as per section 2.3.1.

$$(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \rho_1, \rho_2, \rho_3) \quad (2.48)$$

The Velocities vectors are computed via two runs of Izzo's Algorithm to Lambert's problem. Run A which is between, $[\mathbf{r}_1, \mathbf{r}_2]$ where $\Delta t = t_2 - t_1$, and run B which is between $[\mathbf{r}_2, \mathbf{r}_3]$ where $\Delta t = t_3 - t_2$. The result is complete state vector at t_i .

$$\mathbf{X}_i = (\mathbf{r}_i, \mathbf{v}_i) \quad (2.49)$$

However, Gauss's Algorithm does not ensure that the three position vectors define a unique Keplerian orbit (Pirovano, 2017, p. 1). As seen in figure 2.13a, it does not ensure that the two $\mathbf{v}_2$ vectors derived from Lambert's algorithm, $\mathbf{v}_2^-$ and $\mathbf{v}_2^+$ coincide (Pirovano, 2020, p. 1).

$$\mathbf{v}_2^- = f(\mathbf{r}_1, \mathbf{r}_2) \quad (2.50)$$

$$\mathbf{v}_2^+ = f(\mathbf{r}_2, \mathbf{r}_3) \quad (2.51)$$

To conduct Lambert differential correction the slant range magnitude, ρ_1, ρ_2, ρ_3 , is initialised as independent DA variables. $\delta\rho_i$ represents the variational term in slant range.

$$[\rho_1] = \rho_{1,Gauss} + \delta\rho_1 \quad (2.52)$$

$$[\rho_2] = \rho_{2,Gauss} + \delta\rho_2 \quad (2.53)$$

$$[\rho_3] = \rho_{3,Gauss} + \delta\rho_3 \quad (2.54)$$

Substitution of $[\rho_1]$ into equation 2.3.1 result in

$$\mathbf{r}_i = [\rho_1]\hat{\rho} + \mathbf{q} \quad (2.55)$$

Therefore, $\mathbf{r}_i = f([\rho_i])$. Substitution of $\mathbf{r}_i = f([\rho_i])$ into equation 2.3.3 results in

$$[\mathbf{v}_2^-] = f(\mathbf{r}_1([\rho_1]), \mathbf{r}_2([\rho_2])) \quad (2.56)$$

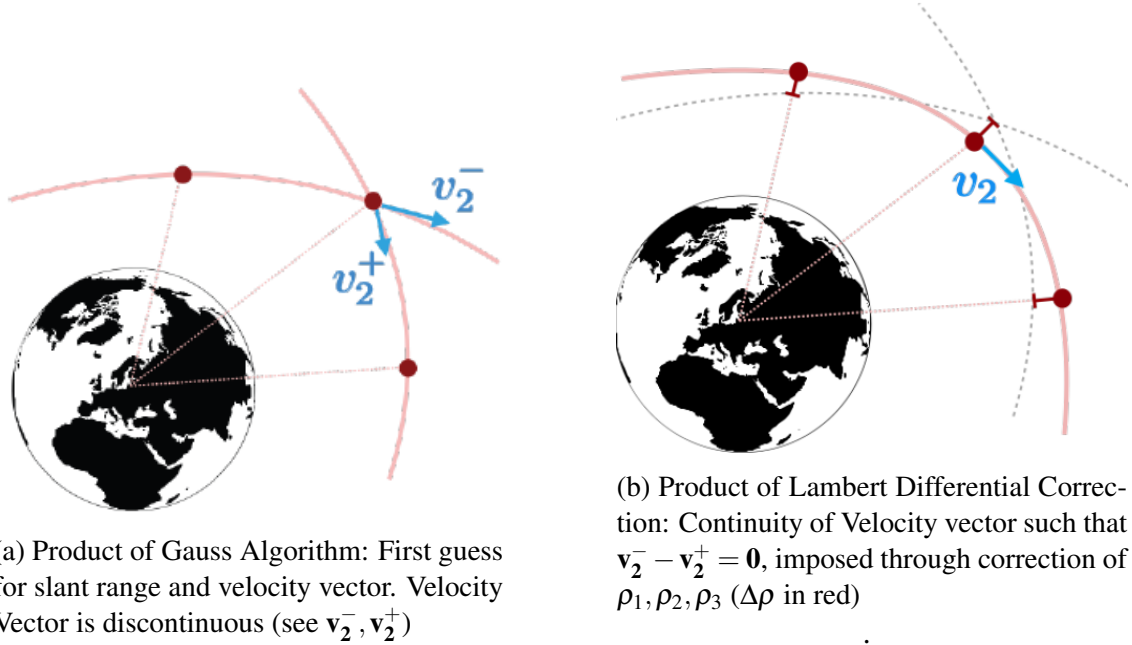


Figure 2.13: First step of the DAIOD Algorithm: initial range guess found via Gauss's algorithm, refine solution by imposing $\mathbf{v}_2^- - \mathbf{v}_2^+ = \mathbf{0}$ within Lambert's Algorithm (Pirovano, 2020, p. 1).

$$[\mathbf{v}_2^+] = f(\mathbf{r}_2([\rho_2]), \mathbf{r}_3([\rho_3])) \quad (2.57)$$

where f is lamberts algorithm. With the goal of solving the discontinuity in t_2 , the $\Delta\mathbf{v}_2 = 0$ can be expressed as

$$[\Delta\mathbf{v}_2] = [\mathbf{v}_2^-] - [\mathbf{v}_2^+] = \mathbf{0} \quad (2.58)$$

Through equation 2.3.3 and 2.3.3 it is evident

$$\Delta\mathbf{v}_2([\rho_1], [\rho_2], [\rho_3]) = \mathbf{0} \quad (2.59)$$

Through Partial Inversion, one can find the solution to the implicit parameters $\delta\rho_1, \delta\rho_2, \delta\rho_3$ by imposing $\Delta\mathbf{v}_2 = 0$. Therefore $\Delta\mathbf{v}_2$ can be defined by equation 2.3.3.

$$\Delta\mathbf{v}_2(\rho_1, \rho_2, \rho_3; \delta\rho_1, \delta\rho_2, \delta\rho_3) = \mathbf{0} \quad (2.60)$$

The constant value, post partial inversion, range magnitudes are defined as $(\rho_{1,L1}, \rho_{2,L1}, \rho_{3,L1})$.

Through a substitution of $\rho_{L1,1}, \rho_{L1,2}, \rho_{L1,3}$ into equation 2.3.1, the outcome of the

lambert DA iterative improvement is state vector in the form of a Taylor Polynomial Series t_2 .

$$\mathbf{r}_2 = \mathbf{r}_2(\rho_{2,L1}) + \mathcal{T}(\delta\rho_1, \delta\rho_2, \delta\rho_3) \quad (2.61)$$

$$\mathbf{v}_2 = \mathbf{v}_2(\rho_{2,L1}) + \mathcal{T}(\delta\rho_1, \delta\rho_2, \delta\rho_3) \quad (2.62)$$

Knowledge of an objects state with respect to the slant ranges is not useful for optical orbit determination. For an optical system measurement uncertainties are in the right ascension and declination components (Vallado, 2013, p. 1). Therefore to assess the variation of the NEO's state in terms measurement uncertainty the variational components must be in Right Ascension and Declination (Pirovano, 2020, p. 1).

The second stage of the DAIOD algorithm defines the right ascension α and declination δ in terms of a observed angle, $(\alpha_{obs}, \delta_{obs})$ and the angle variational $(\delta\alpha_i, \delta\delta_i)$ component. Equation 2.3.3

$$[\alpha_i] = \alpha_{i,obs} \pm 3\sigma \delta\alpha_i \quad (2.63)$$

$$[\delta_i] = \delta_{i,obs} \pm 3\sigma \delta\delta_i \quad (2.64)$$

The variational parts are scaled by the measurement uncertainty, 3σ , so that by evaluating the final solution within the required $[-1, 1]$ interval, one can find the 3σ deviation from $[\alpha_{i,obs}, \delta_{i,obs}]$. Evaluation of a DA polynomial series must be in be within the range $(-1, 1)$ (Pirovano, 2020, p. 1).

A first order DA Newton Scheme is required to compute the following:

$$\delta\rho = \mathcal{T}_\rho(\alpha, \delta) \quad (2.65)$$

The initial guess for ρ within the newton scheme must be an output of ρ such that $\Delta\mathbf{v}_2(\rho) = 0$. This is the nominal solution to the lambert differential correction conduct in the first part, ρ_{L1} .

This is fed into equation 2.3.3, to compute the $\delta\rho$ in terms of $\delta\alpha, \delta\delta$.

$$\rho_{i+1} = \rho_i - J_{\Delta\mathbf{v}(\rho_0)}^{-1} \Delta\mathbf{v}(\rho_i) \quad (2.66)$$

The Jacobian part, $J_{\Delta\mathbf{v}(\rho_0)}$, is defined such that:

$$J_{\Delta\mathbf{v}(\rho_0)} = J_{\Delta\mathbf{v}(\rho_{L1})}^{-1} \times J_{\Delta\mathbf{v}(\rho_{L1} + \mathcal{T}([\alpha], [\delta]))} \quad (2.67)$$

$J_{\Delta\mathbf{v}(\rho_{L1})}$ is obtained from the linear part of the $[\rho_i]$ such that $\Delta\mathbf{v}([\rho_i]) = 0$ (Equation 2.3.3). $J_{\Delta\mathbf{v}(\rho_{L1})} + \mathcal{T}([\alpha], [\delta])$ is the linear part of the second a lambert iteration where $\rho = \rho_{L1}$ are initialised as DA variables accounting to 2.3.3.

The Newton iteration of equation 2.3.3 is carried out till $i = \text{MaxOrder}$, this ensures the highest order of DA variable is achieved (Martin, 1999, p. 1). The result is the state of an object, at a central time, as a map on a domain defined by three Right Ascension and Declination angles (Pirovano, 2020, p. 1):

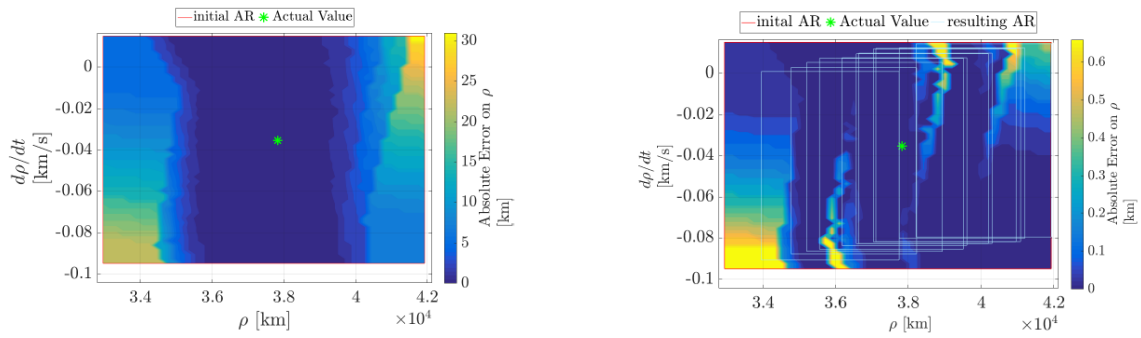
$$f_{IOD} \approx \mathcal{I}_{IOD} : (\alpha, \delta) \quad (2.68)$$

An important outcome of this method is that one not only obtains the point solution of the three observations, but can calculate the variation of solution within the uncertainty region through the evaluation of the TPS. This ensures the continuum of possible orbits can be assessed directly.

Pirovano et al, identified DAIOD's poor performance at the 3σ boundary of uncertainty region. When the value evaluated is outside the convergence radius of the Taylor polynomials, error between the DAIOD solution and the true solution is large. Pirovano proposed a combined DAIOD+ADS algorithm to offset this. As explained in Section 2.2, ADS enables a 'divide and conquer' strategy where the (α, δ) domain is automatically split into subdomains that house separate nominal $([\alpha], [\delta])$ values. This results in multiple Taylor Polynomial Series that describe the smaller, more accurate, local region. They are defined such their union covers the original $\pm 3\sigma$ domain (Pirovano, 2020, p. 1). This is particular useful for highly non-linear orbit dynamics, such as short-arc observations, or NEO observations near Perihelion or Aphelion. These conditions have smaller convergence radii, thus ADS is essential to conduct accurate DA based initial orbit determination.

$$f_{IOD+ADS} = \bigcup_j \mathcal{T}_j : (\alpha, \delta) \quad (2.69)$$

Shown in Figure 2.14, the implementation of the ADS into the $(\rho, \dot{\rho})$ uncertainty region reduced the DAIOD's bounding box absolute error by approximately 50x. This resulted in a absolute error of 1×10^{-6} , similar to Armellin's Monte Carlo based DAIOD algorithm (Armellin et al., 2012, p. 1). Figure 2.14 also demonstrates the DAIOD+ADS algorithm can reduce the $(\rho, \dot{\rho})$ admissible region further. For Preccovery operations, this means less false negatives need to be processed.



(a) Plot of the DAIOD defined AR (red box), the absolute error within the AR

(b) Plot of the DAIOD+ADS AR (white boxes) and absolute error in ρ .

Figure 2.14: Plots of the error distribution for the single Taylor polynomial expansion and Taylor Polynomial expansion which has undergone ADS (Pirovano, 2017, p. 1).

Wittig et al defined the computational cost of an DA+ADS algorithm lay in the number of domain splits required to meet the truncation tolerance (Wittig et al., 2015, p. 1). Explain in section 2.2, this is because a domain split requires another iteration of a DA algorithm, with a new (α, δ) domain, to define a new TPS so that equation 2.3.3 holds.

Figure 2.15 illustrates the inverse logarithmic relationship between the number of splits and the arc length of the observation interval. This illustrates that the why DAIOD+ADS holds higher computational cost than the DAIOD algorithm.

The core of the DAIOD+ADS algorithm assumes the solution produced by Gauss algorithm is within the neighbourhood of the true solution, and provides an adequate initial guess for lambert's algorithm. The Lambert Algorithm also assumes all points within the uncertainty region are elliptical (Pirovano, 2020, p. 1). For the first case, section 2.3.1 established that Gauss estimates may be erroneous due to poor arc curvature. This would impede the lambert algorithm to reach convergence of the point ρ solution. For the latter case, suppose the DAIOD algorithm reached a point solution, large IOD uncertainties

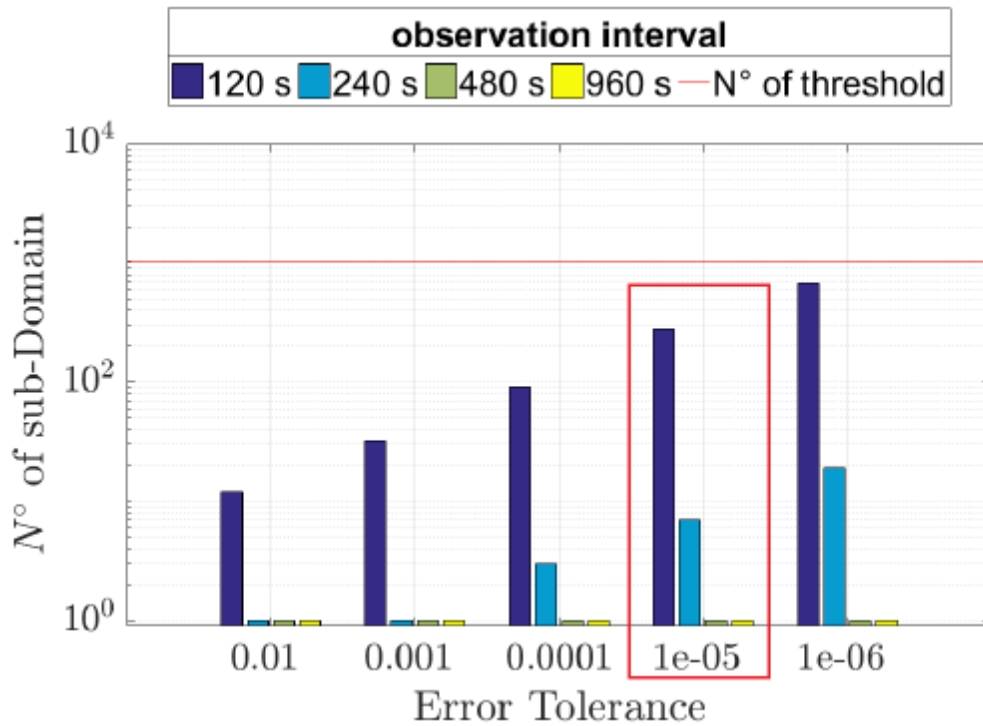


Figure 2.15: Number of sub-domains with 6-th order expansion. The red line indicates the split threshold (Pirovano, 2017, p. 1).

of NEOs would cross into hyperbolic solution (Pirovano, 2020, p. 1). This would cause the DA-based newton iteration to fail and result in failure to achieve the higher order terms of the Taylor polynomial series. Figure 2.16 illustrates the likelihood of success for DAIOD+ADS for satellite observations.

2.3.4 DAIOD for Preccovery

Thus far, DAIOD and DAIOD+ADS have been developed and validated primarily in the context of satellite short-arc association (Pirovano, 2020, p. 1). Their efficiency relative to Monte Carlo sampling, the rigorous definition of bounded uncertainty regions, and the ability to operate under strong non-linearities have made them powerful tools for Resident Space Object cataloguing (Principe, 2020, p. 1). Unlike Gauss IOD, which yields only a point estimate with no associated uncertainty region (Curtus, 2005, p. 1), DAIOD produces both the nominal solution and its variation across the observational uncertainty domain σ . Compared to the grid-search AR method DAIOD+ADS generates compact probabilistic sets, splitting only when necessary. This may enable efficient propagation and, ultimately, efficient data association for Preccovery as the challenge largely lies in constraining the large VA clouds for archival search.

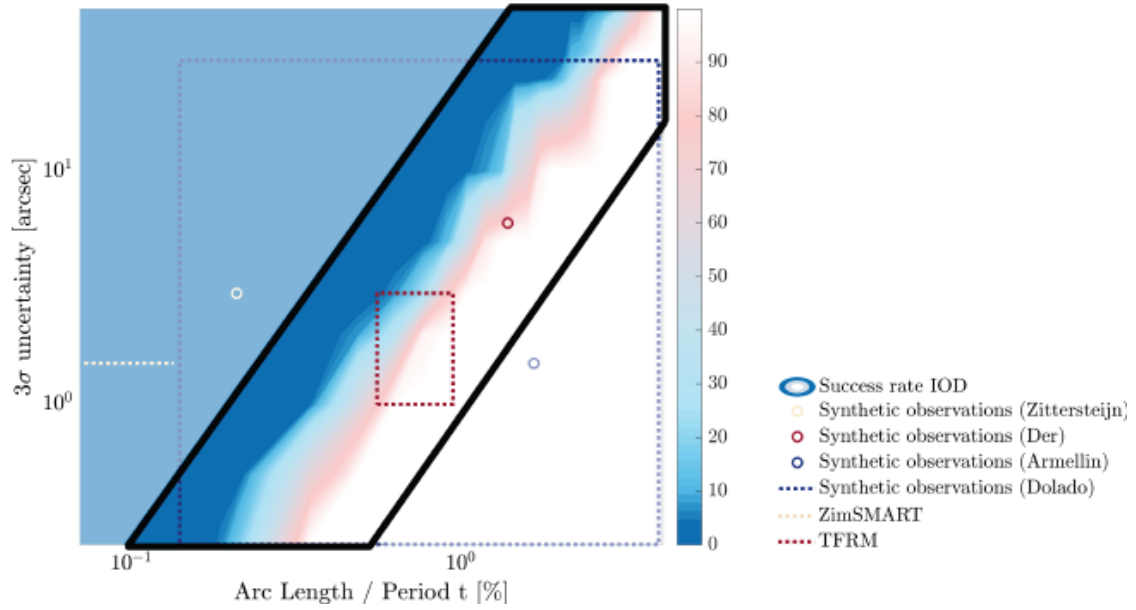


Figure 2.16: Success rate (contour) of the DA-IOD algorithm (black-box) depending on Arc length and Uncertainty (Pirovano, 2020, p. 1).

It is important to note, however, that DA-based IOD was originally conceived for asteroid problems. Armellin et al. (Armellin et al., 2012, p. 1) first applied a DA Gauss Lambert solver to the Apophis 2029 encounter, which demonstrated the feasibility of DA techniques for NEOs. Since then, development has shifted toward satellite and RSO applications, where key improvements such as Automatic Domain Splitting (ADS) and bounding-box formulations (Pirovano, 2020; Principe, 2020, p. 1). These refinements, however, have not yet been systematically extended back to NEO discovery or Precovery, where the challenges of short arcs and large uncertainty clouds remain critical. The limits of the new DAIOD + ADS algorithm is not known.

This gap motivates the present study. Specifically, with the aim to re-integrate the ADS-enhanced DAIOD algorithm into the NEO orbit determination pipeline and to assess its feasibility for Precovery. Successful Precovery is driven by fast computational time, propagation accuracy and IOD accuracy. This ensures maximum possible arc extension which will not delay possible mitigation strategies.

2.4 Orbit Propagation

Within orbit propagation there exists a fundamental trade-off between computational cost and state uncertainty growth. Computational cost refers to the resources required to complete each propagation step, while state uncertainty growth is expressed through the co-

variance matrix of the propagated state (Vallado, 2013, p. 1). Growth of uncertainty is driven by unmodelled orbit dynamics (e.g., neglected perturbations) and errors within the measurement model (Luo & Yang, 2017, p. 1). Different approaches to uncertainty propagation exist: Monte Carlo, linearised, and fully nonlinear methods (Vallado, 2013, p. 1). Each method varies in cost and accuracy.

For Near-Earth Object (NEO) Precovery, both efficiency and accuracy are critical. Propagation must extend over timescales of months to years, as demonstrated in the case of Apophis (Valvano et al., 2024, p. 1). Sparse discovery arcs make long propagation necessary to connect present-day detections with archival observations, maximising arc length and reducing orbit uncertainty. At the same time, accurate propagation is required to model the non-Gaussian "banana-shaped" uncertainty regions that arise due non-linear orbital dynamics. A poorly modelled ellipsoid, over long propagation timescales, can lead to propagated uncertainty regions that span the entire celestial sphere (Vavilov & Hestroffer, 2024, p. 1). This produces impractically large number of candidates during archive queries and makes Precovery impractical. By contrast, accurate and computationally efficient propagation produces compact, realistic uncertainty regions that focus the Precovery search, reduce false candidates, and accelerate orbit refinement through significant arc extension. This, in turn, enables rapid impact risk assessment, which is the ultimate goal of planetary defence operations.

To evaluate this trade-off, one must assess Monte Carlo, linearised and non-linear propagation methods through the perspective of computational cost, ability to capture non-linear dynamics and overall suitability for NEO Precovery.

2.4.1 Monte Carlo Propagation

Monte Carlo Propagation sample an initial, typically gaussian, probability density function $p(\mathbf{x}_0, t_0)$ to define a mean, $\mathbf{m}$ and Covariance Matrix, $\mathbf{P}$ (Luo & Yang, 2017, p. 1). Monte Carlo propagation is IOD agnostic. It is compatible with Gauss IOD, Admissible Region grid sample strategies (Milani et al., 2004, p. 1) are Differential Algebra based methods (Armellin et al., 2012, p. 1). For example, it captures Gauss IOD measurement uncertainty through perturbing α, δ by $(\Delta\alpha, \Delta\delta)$ according a probabilistic distribution.

$$\mathbf{m}^i(t) = \frac{1}{N} \sum_{k=1}^N \Phi^i(t; \mathbf{x}_0^k, t_o) \quad (2.70)$$

$$\mathbf{P}^{ij}(t) = \frac{1}{N-1} \sum_{k=1}^N [\Phi^i(t; \mathbf{x}_0^k, t_0) - \mathbf{m}^i(t)][\Phi^j(t; \mathbf{x}_0^k, t_0) - \mathbf{m}^j(t)] \quad (2.71)$$

where N represent the number of random samples, $\mathbf{x}_0^k$ represents the k^{th} random sample from the PDF (Luo & Yang, 2017, p. 1). MC simulation approach the true probabilistic distribution where $N \rightarrow \infty$. Therefore, to accurately model the 99.7% uncertainty region, $\pm 3\sigma$, with high confidence, 1000 - 10,000s of samples are required (Luo & Yang, 2017, p. 1). For deterministic IOD, such as Gauss's Algorithm, MC requires 1000 - 10000 angle perturbation and subsequent execution of the IOD algorithm. To estimate the NEOs future state alongside the uncertainty distribution 1000 - 10,000 samples require propagation. Therefore, MC propagation techniques are computationally intensive. Each sample is termed a 'Virtual Asteroid' (VA) within the NEO literature (Milani et al., 2004, p. 1). All samples are VAs no matter if its AR based grid searching, or MC sampling.

MC propagation is highly accurate for modelling propagation uncertainty under non-linear dynamics (Luo & Yang, 2017, p. 1). MC propagation only assume a PDF at the time of initial orbit association. As the optical instruments PDF is known at the time of observation, this is known. Evident in equation 2.4.1, the evolution of the PDF is a direct function of the non-linear orbital dynamics. The lack of assumption required for the MC propagation is why it is highly accurate. Typically MC are used to assess the validity of linear or non-linear propagation techniques (Luo & Yang, 2017, p. 1).

Accurate non-linear modelling of the uncertainty region lead MC techniques to be used in NEO Precovery and follow-up operations. For example, NASA's Sentry Earth Impact Monitoring (SEIM) system utilises the VA approach (Aeronautic & Administration, 2025, p. 1). This is because the minimal assumptions of the evolution of the PDF minimises the uncertainty region to search for the NEO over longer propagation times. Therefore, this is favoured for data association over long propagation timescales.

Sampling VAs is integral to MC propagation, modern NEO system improve computational efficiency through minimising the number of samples through new sample techniques. The line-of-variations (LOV) is a sampling technique which samples along a one dimensional curve instead of the full 6 dimensional space (Milani et al., 2005, p. 1). The dimension with the maximum uncertainty dimension, the range, is sampled. If 100 samples in each dimension were required to accurately estimate the NEOs uncertainty classical MC sampling would require 100^6 samples, LOV only requires 100 (Vigna, 2020, p. 1). For highly non-linear observations, such as short arc observations, the assumption

of an initial Gaussian PDF is no longer adequate. The LOV expands to the Manifold of Variations (MOV) where the range and range-rate distribution is modelled. This is used by NASA JPL's SCOUT (Park, 2025, p. 1). If 100 samples were required to model each dimensions distribution, 100^2 samples would be required instead of 100^6 .

However, 100^2 VAs still require 100^2 separate orbit determination and propagation cycles to model the orbit uncertainty over time. Therefore, no matter the sampling technique used, MC propagation requires vast computational resources which require long periods to complete propagation. Therefore, MC-based Precovery is characterised by long propagation times and long wall clock times. Therefore significant arc extension can be achieved, however the costs is long wall clock time. This delays subsequent planetary defence operations such as impact risk assessments and mitigation strategies. However, the accuracy of MC techniques establish MC propagation as the ground truth to assess other propagation techniques (Luo & Yang, 2017, p. 1).

2.4.2 Linearised Propagation

Linear uncertainty propagators propagate the initial covariance matrix analytically through a linearisation to create a State Transition Matrix (STM) (Luo & Yang, 2017, p. 1). Two assumptions are required for this i) a linearised model sufficiently approximates the dynamics of the neighbouring solutions ii) the uncertainty, and its evolution, can be completely characterised by a Gaussian probability distribution (Luo & Yang, 2017, p. 1). As a result, only the nominal trajectory and a single covariance matrix is propagated.

Linear propagators were originally developed for on-board navigation of spacecraft (Luo & Yang, 2017, p. 1). For this use case the gaussian probability distribution assumption is acceptable, spacecraft observations are frequent. However, linear propagators hold low accuracy for highly non-linear systems where long-duration uncertainty propagation can break the gaussian assumption (Luo & Yang, 2017, p. 1). Typically, observations of NEOs are highly non-linear and long propagation periods, characteristic of Precovery operations, introduce non-gaussian probability distributions. Therefore, the use of linear propagators would result in impractically large (α, δ) search regions which would delay Precovery operations. As a result, it is not considered in this thesis.

2.4.3 Non-Linear Propagation

Non-linear uncertainty propagators aim to preserve the non-gaussian nature of the evolution of the Probability Density function (PDF) (Luo & Yang, 2017, p. 1). Accurate characterisation of the PDF is critical to understand the propagated uncertainty boundary where one can complete NEO Preccovery.

Non-linear propagators are characterised by analytical and semi-analytical functions which result in improved computational efficiency over MC propagation. Non-linear propagators fall into one of three categories: i) Sample specific points to be propagated then reconstruct a PDF ii) Constrain dynamics to deliver semi-analytical solutions which do not require sampling iii) Approximate the PDF.

There are two sample based method; unscented transformation (UT) and polynomial chaos expansions (Luo & Yang, 2017, p. 1). The unscented transformation relies on selected points, which are chosen and weighted such that the initial mean and covariance matrix are exactly captured. Instead of random samples, UT method requires no more than $2N+1$ samples (Luo & Yang, 2017, p. 1) to reconstruct the final mean and covariance. The UT cannot define the higher moments of the propagated probability distribution, such as skewness. Therefore it is inadequate for the long timescales which define NEO Preccovery propagation. Polynomial chaos expansion provide higher moments of an arbitrary order, thus full determination of the PDF and uncertainty region is obtained. The main drawback is that the number of terms varies exponentially with the number of dimensions within the uncertainty input (Luo & Yang, 2017, p. 1).

State Transition Tensors (SST) and Differential Algebra (DA) are the major semi-analytical methods. Their use of Higher order Taylor expansions enable accurate description of non-linearities within the dynamics. SST require higher order Taylor series terms to describe localised non-linear motion and analytically map the initial uncertainties. This method has been shown to work with an arbitrary degree of acting forces (Luo & Yang, 2017, p. 1). Over long-term propagation, it is required to derive higher orders of the SSTs accurately represent the governing dynamics (Pirovano, 2020, p. 1). Numerical differentiation is computationally costly.

Similarly, DA utilises the Taylor polynomial Series (TPS), section 2.1, however function derivatives are automatically computed within the TPS (Pirovano, 2020, p. 1). Moreover, DA can automatically split the domain into sub-domains, via ADS see section 2.2. This ensure propagation accuracy throughout highly non-linear regimes. Most importantly,

DA propagation only needs to be completed once. As the Taylor polynomials employed describe the neighbourhood of initial conditions, a single propagation of the TPS and then evaluation post-propagation leads to significant reduction in computational cost compared to MC methods. Luo and Zhen suggested that the smooth function requirement of Taylor polynomials means DA-based propagation can only model a restricted set of dynamics (Luo & Yang, 2017, p. 1). Pirovano stated this is easily overcome through a comparison between the DA based propagation and Monte Carlo based propagation (Pirovano, 2020, p. 1).

Finally, Gaussian Mixture Models (GMM) and the solution to the Plank-Fokker equation (PFE) are the methods to estimate a simplified PDF post propagation. They're conceptually appealing but impractical. GMM attempts to reconstruct a non-gaussian PDF as the sum of several smaller gaussian PDFs. If a PDF is strongly non-gaussian, then more gaussian PDFs are used (Luo & Yang, 2017, p. 1). Long term propagation results in highly non-gaussian PDFs, therefore for Precovey this would greatly increase computational burden. The PFE governs the evolution of the PDF (Luo & Yang, 2017, p. 1). The solution of the PFE has not been adequately solved in six dimensional space (Pirovano, 2020, p. 1) so it not applicable for this study.

In summary, the three classes of uncertainty propagation techniques each present distinct advantages and limitations for NEO Precovey. Monte Carlo methods provide the most accurate representation of uncertainty growth under non-linear dynamics, but their prohibitive computational cost restricts their use to benchmarking other approaches. Linearised propagators offer fast computation, but their reliance on Gaussian assumptions and local linearity makes them unsuitable for the long, non-linear propagation intervals characteristic of Precovey operations. Non-linear methods extend capability beyond linearisation, yet sample-based approaches (e.g., Unscented Transformation, Polynomial Chaos) and density approximations (e.g., Gaussian Mixtures, Fokker-Planck formulations) remain either dimensionally limited or computationally demanding. Among non-linear techniques, Differential Algebra (DA), particularly when combined with Automatic Domain Splitting (ADS), stands out as the most promising. DA automatically generates high-order Taylor polynomial maps with rigorous error control, while ADS mitigates truncation error in highly non-linear regimes. This combination delivers accurate, compact uncertainty regions at a fraction of the computational cost of Monte Carlo, making DA-based propagation the most suitable framework for efficient and reliable NEO Precovey. The following study therefore assesses DA with ADS as a core propagation method, and benchmarks its performance against Monte Carlo to establish reliability for NEO Precov-

ery applications.

2.5 Precovery Data Association

Orbit Determination identifies the Asteroid, orbit propagation enables scientists to predict its position, however data association is what is used to extend an observational arc. Data association fits an observation instance, or series of observations, to a known Asteroid. Multiple Observation associations constrain the orbit further till a single unique orbit is identified (Pirovano, Santeramo et al., 2020, p. 1). However, short arcs are common for NEOs observations. Section 2.3.1 established that short arc observations lead to larger IOD uncertainty, while section 2.4.1 explained why this increases propagation uncertainty. As a result, NEO data association is conducted between two sets of highly uncertain states. Thus, it is difficult to identify whether multiple independent sets of observations belong to the same NEO. This is known as observation-to-track association (Pirovano, Principe & Armellin, 2020, p. 1) or Attribution (Gronchi, n.d., p. 1). Precovery conducts initial orbit determination on an arc, propagates the object backwards in time, and highlight possible observations that comply with the initial orbit. In other words, it is an observation-to-track association.

Precovery Data Association is the product of two steps:

1. **Initial Orbit Determination:** A set of observations defines an set of orbits at t_1 and t_2
2. **Orbit Propagation:** The set of orbits at t_1 is propagated to t_2 and level of correlation is observed within a fixed region of the α, δ plane.

Deterministic and Probabilistic approaches to data association are available. Figure 2.17 illustrates the deterministic approach. It is the discretisation of the AR through a uniform or delaunay triangulated mesh (Gronchi, n.d., p. 1). The semi-major axis and eccentricity physical constraints are substituted for orbit energy and angular momentum (DeMars & Jah, 2013, p. 1). The deterministic approach prevents the inclusion of measurement uncertainty, a probabilistic function, into the AR. It also prevents a classification of the most likely candidate observations to be associated to an Asteroid. The best method to define an NEOs state is through the nominal state and its PDF. Therefore, there has been a recent push for a probabilistic approach.

This first took the form of a uniform PDF across the AR (J.L. & M.J, 2015, p. 1), where the extend of the AR was dictated by the uncertainty within measurements and

time uncertainty. However, the assumed PDF was gaussian. This assumption does not hold for short arcs (Milani et al., 2004, p. 1), and is not evident in the deterministic approach which relies on physical laws to constrain the AR. Two years later the probabilistic AR approach was iterated upon by inclusion of a Gaussian Mixture Model to define a generic PDF as the sum of gaussian PDFs (DeMars & Jah, 2013, p. 1).

To aid Precovery association the definition of the NEOs PDF is desirable. It enables the ranking of the most likely observations whether it is σ , 2σ , or 3σ from an nominal state. This provides a figure of merit where different candidate observations can be prioritised based on their degree of likelihood. This is encapsulated in the Mahalanobis distance metric (Harris et al., 2024, p. 1) which describes the goodness of fit, but requires the uncertainties to be well-characterised (Pirovano et al., 2021, p. 1). This is difficult under short-arc observations, thus it is difficult for NEO Precovery.

DA enables the measurement uncertainty to directly link to the NEOs state, association is completed by the degree of intersection between the track-let and observation uncertainty (Pirovano et al., 2021, p. 1). DA methods have been defined for too short arcs, Admissible states region, and short arc methods, DAIOD+ADS. As Precovery concerns the association of short arc, this method is recommended.

This thesis does not complete data association of candidate arcs, nor does it rank Precovery candidates through the degree of uncertainty intersection. This is because precovery data association is a lengthy process where observers are required to manually inspect candidate images. It is not feasible to complete this within the timeline of the study. Instead, this thesis evaluates the feasibility of DA based techniques through the ratio between of the true position of Apophis (via SSOIS) and the total number of images (via CADC and ESO database). The total number of images acts as a proxy to traditional data association techniques, a high image count corresponds to highly uncertainty association while a low image count corresponds to low uncertainty associations.

2.5.1 CADC, ESO and SSOIS

Candidate images for Precovery are archived within large observatory databases. The most comprehensive collections of Apophis are hosted by the Canadian Astronomy Data Centre (CADC) and the European Southern Observatory (ESO) (Agency, n.d.; of Canada, 2025, p. 1). These databases provide fixed sky-plane pointing (α, δ) from which candidate images can be queried and associated

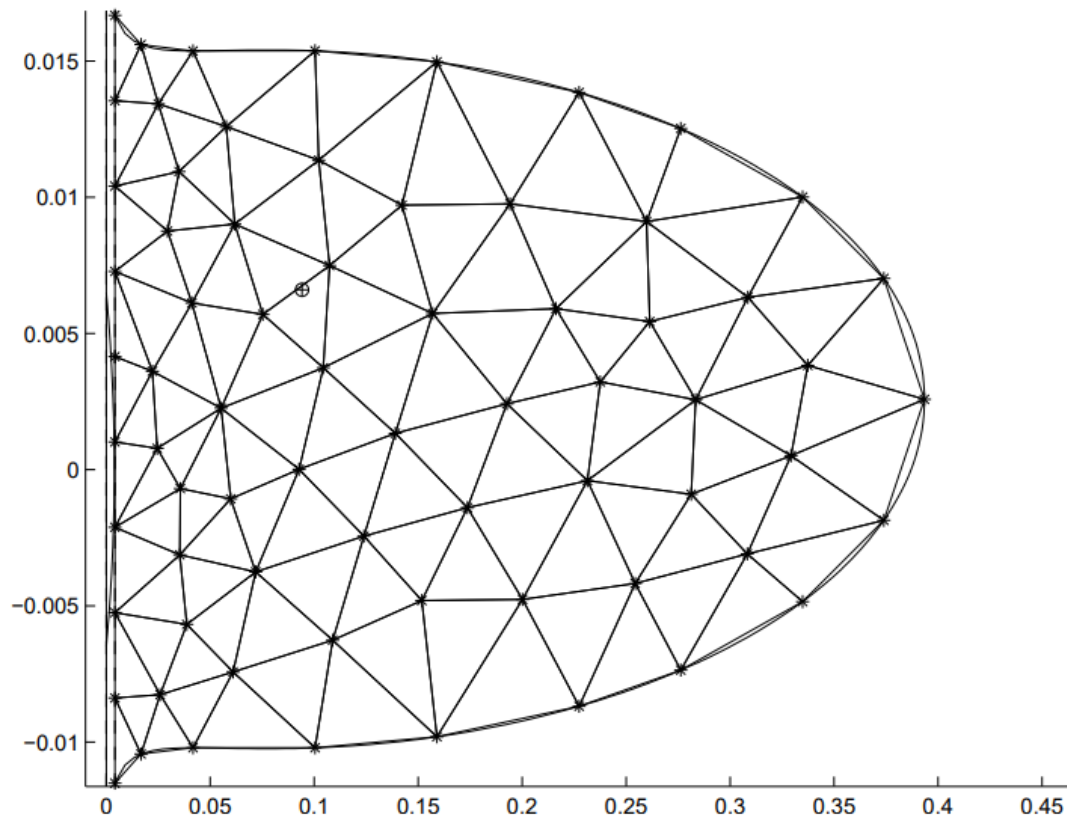


Figure 2.17: Triangulated admissible region in the plane $(\rho, \dot{\rho})$ for the asteroid 2003 BH84. The units are AU, AU/yr and the coordinates of the asteroid are marked with a circled cross (Gronchi, n.d., p. 1)

Another widely used tool is the Solar System Object Image Search (SSOIS), which allows users to retrieve candidate images associated with a specific NEO, an observation arc, or a set of orbital elements. SSOIS aggregates data from CADC, ESO, the ESA NEO Coordination Centre, Pan-STARRS, and many other survey archives (Gwyn et al., 2016, p. 1). Candidate Precovery images are identified by comparing a NEO ephemeris with image footprints: if the predicted position falls within an image boundary, the image is returned with the ephemeris location within it (Gwyn et al., 2016, p. 1).

However, SSOIS models positional uncertainty only in a simplified way. Users may specify a fixed error margin, but this is represented as a cuboid buffer around the nominal ephemeris and does not evolve dynamically with propagation time or method (Gwyn et al., 2016, p. 1). As a result, measurement uncertainty is included, but growth of uncertainty under orbital dynamics is neglected. Using SSOIS directly would therefore bypass the core aim of this study: to evaluate the effect of DA-based propagation on Precovery uncertainty. For this reason, SSOIS is employed here only to identify the “true” images containing Apophis, using a zero-uncertainty query. Candidate searches, by contrast, are performed through raw CADC and ESO queries, ensuring that uncertainty growth from propagation methods is explicitly represented in the evaluation.

To assess the impact of DA-based propagators and IOD on Precovery, candidate images are identified manually from raw CADC and ESO queries. These databases provide only the telescope pointing direction (α, δ) , without the full image footprint $(\alpha_{\text{point}} \pm \Delta\alpha, \delta_{\text{point}} \pm \Delta\delta)$. Consequently, some images may be missed. However, this limitation applies equally to all techniques evaluated in this study, ensuring that relative performance comparisons remain valid.

2.6 Apophis

Should I place the first half section of this section in the Introduction? Apophis was discovered in 19 June 2004 (Valvano et al., 2024, p. 1). In December 2004 it had a 1.7 day arc and held an impact probability (IP) of increased from 2×10^{-4} , this increased to 2% after the 1.7 day arc was extended into a 6-month arc via the association of the June observations. However, the June observations had significant error due to clock inaccuracies (Valvano et al., 2024, p. 1). After corrections, a 187 day arc with 55 follow-up observations were collected. The majority were taken in December. The extension of the observation arc to reduce the probability of impact to 0.4%. The next day 10 new

observations increased the IP to 1.6%, over the next four days it peaked at 2.7% on 27 December. Finally, association of Precovery observation (March 2004) eliminated any possibility of Impact in 2029 (Valvano et al., 2024, p. 1).

The case of Apophis illustrates the limitation of follow-up observations and the impact of Precovery association on IP. The former is evident through the volatility of the IP metric; follow-up observations only extend the arc by a day, many orbits fit these observations therefore the uncertainty of an orbit is poorly constrained. The latter is evident as once the March 2004 observations were associated the possibility of impact was eliminated. Apophis is used for this study i) to observe if the DA based algorithm can complete Precovery from December 2004 to March 2004 ii) to investigate how DA based algorithms perform compared to Monte Carlo techniques. It also enables future work to build on the DAIOD - Monte Carlo method through an assessment of Impact Probability.

Apophis has also been extensively studied under the DA/ADS framework (Wittig et al., 2015, p. 1) and (Armellin et al., 2012, p. 1). These studies have define important DA parameters such as expansion orders settings and ADS error thresholds for propagation. This enables the author of this thesis to complete the study within the set time frame.

2.7 Research Gap and Motivation

DAIOD combined with ADS has been demonstrated as an efficient, accurate tool for short-arc orbit determination of Earth satellites. However, to date there has been no equivalent application to the Precovery of Near-Earth Objects. Precovery observations are defined by a short arcs, like Earth Satellites, however longer propagation times are required to extent the arc over a significant proportion of the NEOs orbit. This gap motivates the present study: to quantify the performance limits of DAIOD+ADS for NEOs, compared against Monte Carlo Gauss baselines, in terms of arc length, propagation horizon, and operational utility for Precovery.

Chapter 3

Methodology

This study analyses the feasibility of DA based methods for NEO Precovery from short arcs. Figure 3.1 provides a overview. First, a virtual observational arc is generated, then Initial Orbit determination defines a (α, δ) uncertainty region. It propagates the uncertainty region backwards by 6 months. At a 1 day time-step, the uncertainty region is queried for archival images within the $(\alpha \pm 3\sigma_\alpha, \delta \pm 3\sigma_\delta)$. Precovery precision is then estimated by comparing images returned by the query with the catalogue of known Apophis detections. The end to end Orbit determination propagation query pipeline is timed to characterise the computational load. See figure 3.1 and 3.2 for a visual high level overview.

To isolate the benefits of DA over pointwise methods, a pointwise variant of the DAIOD algorithm is used to separate the DA techniques from the lambert refinement. Furthermore, a Monte Carlo Gauss-IOD experiment across multiple arc lengths was used to establish a baseline Precovery performance. DA-based techniques are not compared with the admissible region (AR) approach; this is a limitation to be addressed in future work (see Section 7).

The implementation employs the DaceyPy library (Purpura, 2024, p. 1), a Python wrapper of the C++ DACE library (Wittig et al., 2016, p. 1). DACE has been applied to orbit determination by Pirovano, Armellin, and Di Lizia (Pirovano, Santeramo et al., 2020, p. 1), and its effectiveness for orbit propagation was demonstrated by Wittig (Wittig et al., 2015, p. 1). Python is adopted for its usability and versatility. Because C++ is substantially faster than Python, computational timings reported in C++ based literature are not directly comparable. As a result, this study times the initial orbit determination (IOD) and propagation stages to compare the relative performance of the examined schemes.

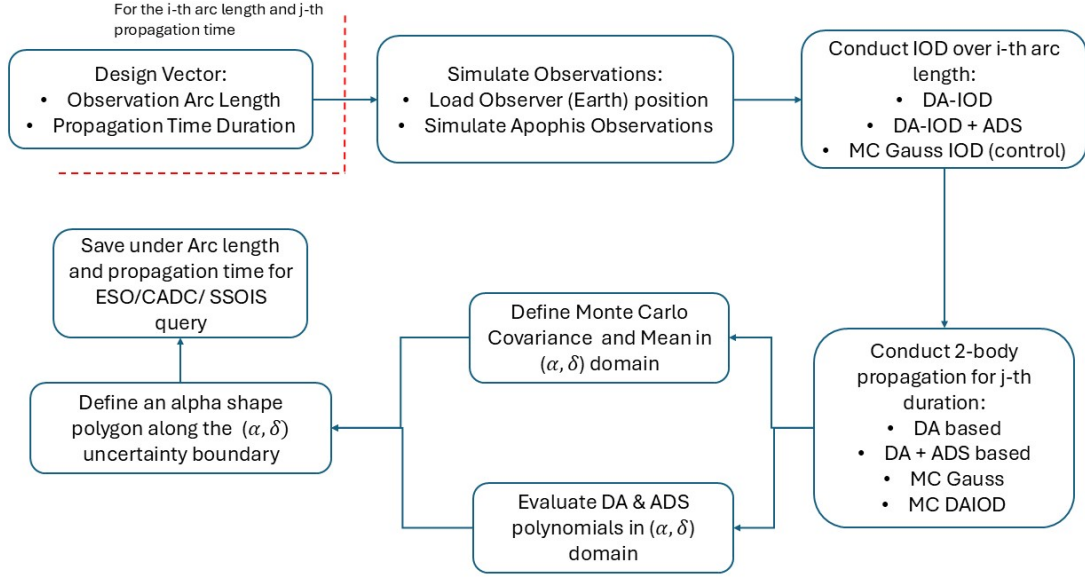


Figure 3.1: High level logic of $(\alpha \pm 3\sigma, \delta \pm 3\sigma)$ region generation

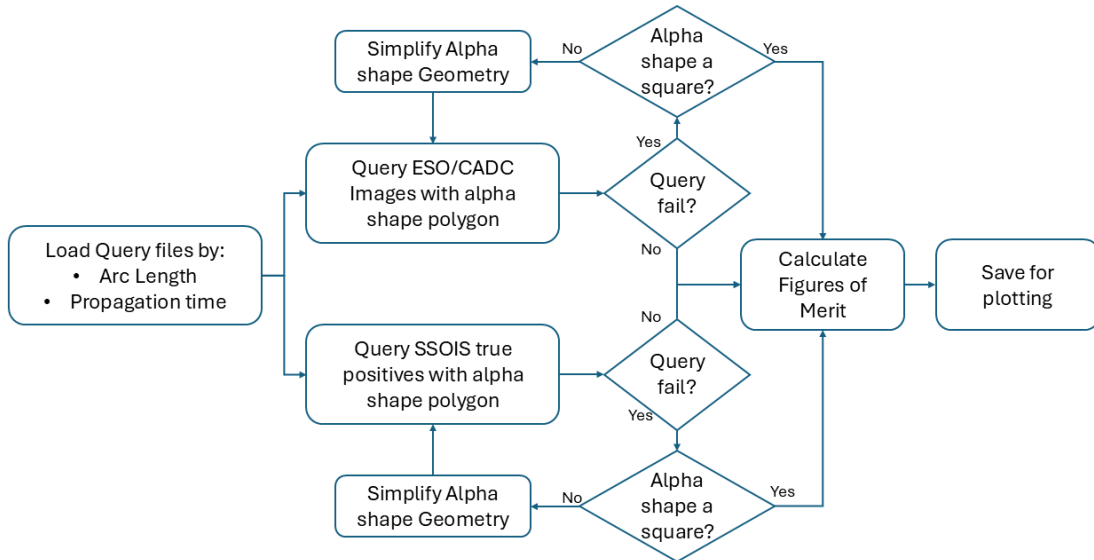


Figure 3.2: High level logic of observatory database query and Figure of Merit generation

Parameters	Values	units
Arc Length Vector	[0.1, 0.15, 0.18, 0.25, 0.3, 0.5, 1]	days
Central Observation time	2005-06-17 23:41:07	UTC
Maximum propagation time	183	days
Propagation time-step	0.5	days
Monte Carlo Samples	10000	(-)
Taylor Expansion order	6	(-)

Table 3.1: Model Inputs

3.1 Figures of Merit

Precovery performance is defined by the following figures of merit:

- **Precision:** The definition of Precovery precision is in equation 3.1. It is a proxy for the magnitude of the (α, δ) uncertainty region. If the uncertainty region span a significant portion of the Celestial Sphere the total number of archival images that lie within that region is high. Therefore, the IOD and propagation scheme has a low precision. A low precision requires a high number of candidate images to be vetted for data association, therefore this metric should be maximised.

$$Precision = \frac{True\ Positives}{Total\ Number\ of\ Images} \quad (3.1)$$

The Precision cannot be modelled within the study due computational constraints, as a result the (α, δ) area seen on the celestial sphere will be used as an analogue to the precision figure of merit. A large area results in numerous images. Therefore it should be minimised to minimise the number of false data associations required.

- **Wall clock time:** This refers to the computational time required complete the orbit determination and propagation pipeline. This metric should be minimised; a low wall clock time results in sooner archive image retrieval, this means potentially means sooner data association. This would enable accurate impact risk quantification sooner.

3.2 Input vector

Identification of the arc length limit is a key research aim therefore the arc length is varied from 2 h to 24 h . A short observation arc is defined by either $3 \leq N \leq 5$ observations over the course of 2 h (Milani et al., 2004, p. 1), or it is 0.2% of a bodies orbit (Principe, 2020, p. 1). Apophis has an orbital period of 324 days, 0.2% of this orbit is 15.5 h . An arc

ranging from 1 hour to 24 hours covers both definitions.

The propagation time is varied from 0.5 days after the epoch observation to 6 months with a time step of 0.5 days. Successful Precovery of Apophis was achieved one month (Valvano et al., 2024, p. 1) after discovery, while other NEOs have associated two months old pre-discovery observations (Agency, 2025b, p. 1). The propagation interval chosen ensures this can be investigated. The maximum propagation time was chosen so that one can identify the propagation time limit (Wittig et al., 2015, p. 1).

For every arc length, the Initial Orbit Determination and propagation routine was timed. This defined the wall clock time.

3.3 Orbit Determination

The DA and ADS based orbit determination followed Pirovano's DAIOD+ADS method (Pirovano, 2020, p. 1). A Monte Carlo, pointwise, version of the DAIOD method was used to isolate the effects of DA on computational performance. A Gauss-based Monte Carlo Simulation was used as a baseline, see Section 3.7.

3.3.1 Assumptions

All Initial Orbit Determination schemes assume the following about the orbits dynamics:

- No knowledge of the orbit is known prior to observation; therefore IOD must be conducted with a series of three points. This is drawn from the fundamental aim of the study, to assess success of Precovery based on discovery arcs.
- Sun is only acting gravitational force (two-body dynamics); the Earth-based observer has not gravitational effect on the Asteroid. This is a common assumption for discovery arcs, observers do not the degree of perturbations acting on the orbits as the orbit is not defined.
- Observer is placed at the Earth's geocentre in an Earth Centred Inertial Frame of Reference.

It assumes the following about the sensor characteristics are:

- Angle-only IOD; NEO discovery is conducted through optical systems (Aeronautic & Administration, 2025, p. 1). As a result an $(\rho, \dot{\rho})$ region must be defined through (α, δ) .

- Measurement Noise is White

$$\alpha_i \sim \mathcal{N}(\alpha, \sigma^2) \quad (3.2)$$

$$\delta_i \sim \mathcal{N}(\delta, \sigma^2) \quad (3.3)$$

- The measurement probability is a Gaussian distribution (Pirovano, Santeramo et al., 2020, p. 1), where the observed (α, δ) is the mean.
- Right Ascension and Declination measurement uncertainties are independent
- No astronomical corrections have been applied.
- The measurement uncertainty $3\sigma_\alpha, 3\sigma_\delta$ are assumed to be

$$3\sigma_\alpha = 0.3'' \quad 3\sigma_\delta = 0.3'' \quad (3.4)$$

This is taken from the discovery observation uncertainty of Apophis according to ESO-NTT.

- All uncertainties, besides measurement uncertainty, are discarded.

The Observer is assumed to be at the origin of a, Equatorial, Earth Centred Inertial frame of reference, at an epoch of J2000. Therefore, the direction of the line of sight vector is with respect to the centre of the Earth and not a specific observatory. Equation 2.3.1 demonstrates the omission of the topocentric observatory position will reduce the accuracy of the solution. However, this applied all IOD algorithms (i.e. the inaccuracy is applied across the DA, DA+ADS and Monte Carlo pointwise schemes), therefore the error is insignificant for comparative analysis.

To obtain the Earth position at the time of observation, Astropy's 'builtin' solar system ephemeris function. This function uses polynomial estimates for the Earth Ephemeris ('Astropy', 2025, p. 1). Use of this function departs from the traditional literature. For example Pirovano uses JPL Spice kernels (Pirovano, 2020, p. 1), This is far more accurate ('Astropy', 2025, p. 1). However, as this error is applied for DA-based, DA+ADS-based and pointwise algorithms. Therefore any the error will not obstruct accurate comparative analysis.

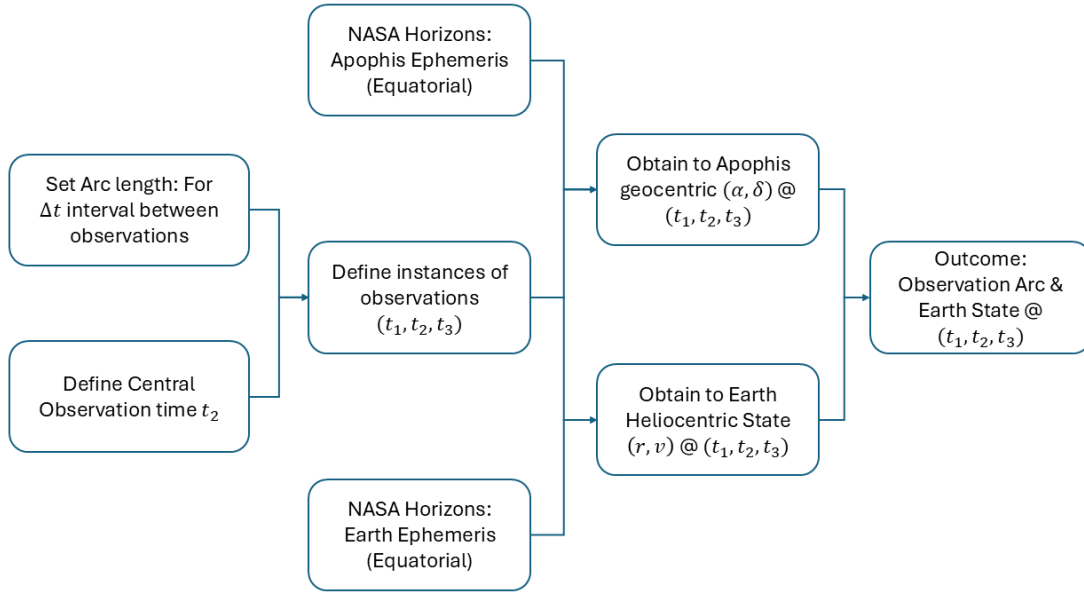


Figure 3.3: Pre-conditioning for Initial Orbit Determination

3.3.2 Virtual Observations, Observer Position and Orbit dynamics

Figure 3.3 illustrates that virtual observations are a result of three components: The arc length, the central observation time and the NASA Horizons Ephemeris generator. The specified arc length and central observation time define the 'observation' instances (t_1, t_2, t_3) .

$$t_1 = t_2 - \Delta t \quad t_3 = t_2 + \Delta t \quad (3.5)$$

(t_1, t_2, t_3) are used to define the Heliocentric state of Earth and Apophis. Both states are defined relative to the Equatorial plane. One can define the position of Apophis relative to Earth through the vector sum. Apophis' geocentric Cartesian position is converted to spherical coordinates Right Ascension, Declination and Range (α, δ, ρ) . The calculated Right Ascension and Declination are used to simulate the observation arc.

The arc length and central observation is user defined. The values used are in table 3.1. A central observation time of 2005-06-17 23:41:07 UTC was chosen because between 2004-2006 there was an observation campaign (Search, 2025, p. 1).

3.3.3 DAIOD

The DA-based IOD algorithm is composed of three stages, it is identical to Pirovano's DAIOD algorithm (Pirovano, 2017, p. 1):

1. **Gauss angle-only IOD** determined state as a product of (α, δ) . its serves as the

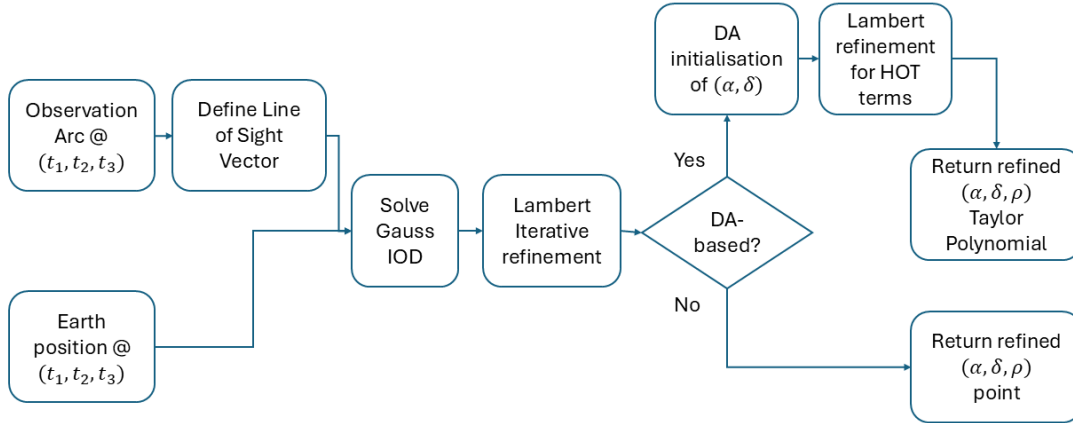


Figure 3.4: Differential Algebra Initial Orbit Determination Logic Diagram

initial guess

2. **Floating point refinement** Differential correction of the Gauss estimated range estimate. Izzo's solution to Lambert's problem is used to solve for (ρ_1, ρ_2, ρ_3) when $\Delta \mathbf{v}_2(\rho_1, \rho_2, \rho_3) = 0$. Under this condition (ρ_1, ρ_2, ρ_3) represent a Keplerian orbit (Pirovano, 2017, p. 1).
3. **Differential Algebra expansion** Initialise (α, δ) as DA variables. Let the higher order variation terms in the Taylor polynomial be expressed in terms of $(3\sigma_\alpha, 3\sigma_\delta)$. Define (ρ_1, ρ_2, ρ_3) in terms of α, δ Taylor polynomials. Solve for (ρ_1, ρ_2, ρ_3) such that all terms in the Taylor polynomial $\Delta \mathbf{v}_2(\rho_1, \rho_2, \rho_3)$

As a result the geocentric range is expressed as a Taylor polynomial in the form:

$$\boldsymbol{\rho} = \mathcal{T}_{\Delta \mathbf{v}_2(\boldsymbol{\rho})}(\delta \alpha, \delta \delta) \quad (3.6)$$

The geocentric range is used to define Apophis' Heliocentric State for propagation.

Gauss's angles-only IOD algorithm is defined in section 2.3.3. It assumes:

- observations lie on an orbit plane
- the target orbit is well approximated by a conic section
- the observational arc can be well approximated by a first order Lagrange series (Curtus, 2005, p. 1).

Gauss's 8th degree polynomial can return more than 1 real root (Curtus, 2005, p. 1). Therefore additional constraints are imposed to ensure the returned root is physically feasible:

1. Geocentric slant range must be positive, $\rho > 0$.
2. The Heliocentric position of Apophis must not exceed the its apoapsis, $r_{Gauss} < r_{apoapsis}$.
3. Apophis' orbit is be elliptic, therefore the orbital energy < 0 (Vallado, 2013, p. 1).

The Gauss first estimate is refined via partial inversion. The constraint $\Delta \mathbf{v}_2(\rho_1, \rho_2, \rho_3) = 0$ is imposed and (ρ_1, ρ_2, ρ_3) is solved. The values (ρ_1, ρ_2, ρ_3) are initialised as DA variables.

$$[\rho_1] = \rho_{1,guass} + \delta \rho_1 \quad (3.7)$$

$$[\rho_2] = \rho_{2,guass} + \delta \rho_2 \quad (3.8)$$

$$[\rho_3] = \rho_{3,guass} + \delta \rho_3 \quad (3.9)$$

The output of Lambert's algorithm are the velocities which depend on the variations of the slant ranges:

$$[\mathbf{v}_2^-] = \mathcal{T}_{\mathbf{v}_2^-}(\rho_1, \rho_2) \quad (3.10)$$

$$[\mathbf{v}_2^+] = \mathcal{T}_{\mathbf{v}_2^+}(\rho_2, \rho_3) \quad (3.11)$$

$$(3.12)$$

The goal is to solve the discontinuity in $\mathbf{v}_2$, therefore $\Delta \mathbf{v}_2(\rho_1, \rho_2, \rho_3) = 0$. Written in DA notation

$$[\mathbf{v}_2] = \mathbf{v}_2^+ - \mathbf{v}_2^- = \mathcal{T}_{\Delta \mathbf{v}_2}(\delta \rho_1, \delta \rho_2, \delta \rho_3) = 0 \quad (3.13)$$

Enforcement of $\Delta \mathbf{v}_2 = 0$ will yield the ρ_i and $\delta \rho_i$ necessary to remain in a Keplerian orbit. Partial Inversion and Newtons method for DA, explain in section 2.3.3 is used. The function to iterate over is $\Delta \mathbf{v}_2$, the parameter $\mathbf{x}$ is ρ_i and the variation δp are $\delta \rho_i$. As a result, one obtains the Taylor polynomial:

$$\mathcal{T}_{\Delta \mathbf{v}_2}(\rho_1, \rho_2, \rho_3; \delta \rho_1, \delta \rho_2, \delta \rho_3) = 0 \quad (3.14)$$

This implementation assumes:

1. Utilises Izzo's solution as Lambert's algorithm

2. The orbit of interest is prograde; Apophis' orbit is inclined by 3.3° (Aeronautical & Laboratory, 2022, p. 1).
3. No orbit revolution completed within the observed arc.

Therefore, (ρ_1, ρ_2, ρ_3) has been solved so that the states of the object define a single unique orbit. However, one wants to understand the relationship between the object state and observation (α, δ) variations. The independent DA variables are re-initialised so that $3\sigma_\alpha, 3\sigma_\delta$ represent the maximum variation within the Taylor polynomial

$$\alpha_{1,DA} = \alpha_1 + 3\sigma_\alpha \delta \alpha_1 \quad (3.15)$$

$$\alpha_{2,DA} = \alpha_2 + 3\sigma_\alpha \delta \alpha_2 \quad (3.16)$$

$$\alpha_{3,DA} = \alpha_3 + 3\sigma_\alpha \delta \alpha_3 \quad (3.17)$$

$$\delta_{1,DA} = \delta_1 + 3\sigma_\delta \delta \delta_1 \quad (3.18)$$

$$\delta_{2,DA} = \delta_2 + 3\sigma_\delta \delta \delta_2 \quad (3.19)$$

$$\delta_{3,DA} = \delta_3 + 3\sigma_\delta \delta \delta_3 \quad (3.20)$$

$$(3.21)$$

where α_i, δ_i are the observed right ascension and declination and $3\sigma_\alpha, 3\sigma_\delta$ are measurement uncertainty. $3\sigma_\alpha, 3\sigma_\delta$ are DA coefficients so when $\delta\alpha, \delta\delta$ are evaluated between $[1, -1]$, the Taylor polynomial returns $\alpha \pm 3\sigma_\alpha$ and $\delta \pm 3\sigma_\delta$.

$\alpha_{i,DA}, \delta_{i,DA}$ are used to express $\boldsymbol{\rho}$ in Taylor polynomial terms

$$\boldsymbol{\rho} = \mathcal{T}_{\boldsymbol{\rho}}(\alpha_{i,DA}, \delta_{i,DA}) \quad (3.22)$$

The final Taylor polynomial, such that $\Delta \mathbf{v}_2 = 0$ across all higher order terms, is computed via first order DA Newton's iteration scheme:

$$\boldsymbol{\rho}_{j+1} = \boldsymbol{\rho}_j - J_{\Delta \mathbf{v}_2(\boldsymbol{\rho}_0)}^{-1} \Delta \mathbf{v}_2(\boldsymbol{\rho}_j) \quad (3.23)$$

Where the first guess is defined by output of the first lambert based differential correction. The Jacobian $J_{\Delta \mathbf{v}_2(\boldsymbol{\rho}_0)}$ is extracted from linear term of Taylor polynomial equation 3.3.3. $\Delta \mathbf{v}_2(\boldsymbol{\rho}_j)$ is the output of the Izzo lambert solver. $\boldsymbol{\rho}_j$ is written in terms of equation 3.3.3. The iteration is carried out till $j = \text{MaxOrder}$. This ensures all orders of the Taylor polynomial fit the $\Delta \mathbf{v}_2 = 0$ condition (Martin, 1999, p. 1).

The result is full state as a function of right ascension and declination angles via a Taylor polynomial. The Taylor polynomial contains:

1. The nominal state; through the Gauss-Lambert differential correction
2. A non-linear description of how the state is affected by $3\sigma_\alpha, 3\sigma_\delta$; through equation 3.3.3

$$f_{IOD} \approx \mathcal{T}_{IOD} : (\alpha_i, \delta_i) \quad (3.24)$$

This yields both the nominal trajectory and the continuum of admissible solutions as a function of measurement uncertainty. Similar DA-based IOD methods have been explored by Armellin et al. (Ramírez-Montoya et al., 2025, p. 1), Pirovano et al. (Pirovano, 2017, p. 1), and Principe (Principe, 2020, p. 1). However the one used in this Thesis is Pirovano's as it focuses on short arc IOD and has ADS compatibility.

Pirovano found that after a Taylor polynomial order of 6 there were diminishing returns between absolute error and Taylor polynomial order. Therefore, the DA variables were initialised to the 6th order.

3.3.4 DAIOD+ADS

The DA+ADS IOD algorithm utilises the baseline DAIOD algorithm to generate the Taylor polynomials. Indeed, the ADS only splits to domain such that the union of the ADS domain is equal to initial, pre-split, domain (Pirovano, 2020, p. 1). As a result, the DAIOD+ADS routine follows the three core steps of the DAIOD algorithm to define the orbital set as Taylor polynomials in terms of measurement uncertainty.

$$f_{IOD+ADS} = \cup_j \mathcal{T}_j : (\alpha, \delta)_j \rightarrow (\mathbf{r}_2, \mathbf{v}_2)_j \quad (3.25)$$

The ADS routine, however, requires:

1. A split tolerance; the difference between the maximum k^{th} order term and the approximated $k + 1$ order term. This can be thought of as the convergence radius, the higher the tolerance is set, the higher the maximum degree of error till a split occurs.
2. A maximum number of splits per dimension.

Wider satellite based DAIOD+ADS literature has enforced a position and velocity split tolerance of 1 km and $1 \times 10^{-3} \text{ km/s}$ respectively (Pirovano, 2020, p. 1). A split number of 5 is also drawn from the literature (Pirovano, 2020, p. 1), however is limited by the computational resources available.

The ADS Algorithm extension to DAIOD algorithm is as follows:

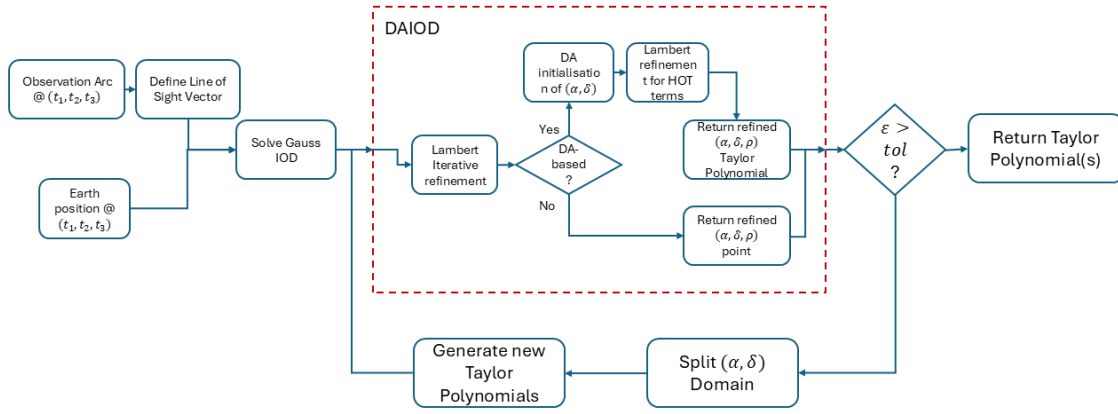


Figure 3.5: DAIOD+ADS Orbit Determination Logic Diagram

Parameter	Value	Unit
Split tolerance (position)	10	km
Split tolerance (velocity)	0.1	km/s
Number of Splits	5	(-)

Table 3.2: ADS Parameters

1. Complete DAIOD Algorithm to generate:

$$\boldsymbol{\rho}_i = \mathcal{T}(\alpha_i, \delta_i) \quad (3.26)$$

2. Evaluate the $\boldsymbol{\rho}_i$ truncation error of the Taylor polynomial
3. If truncation error is larger than the tolerance, split the (α_i, δ_i) domain
4. Pass the new split (α_i, δ_i) domain through the DAIOD algorithm.
5. Repeat till maximum number of splits achieved or all $\boldsymbol{\rho}_i$ solutions within the truncation error

The DaceyPy has in-built tools so that one can execute on the ADS splitting logic with a high level understanding (Purpura, 2024, p. 1). This is done through the ADS.eval() function, and has been used extensively throughout the Thesis. If one were interested in the implementation of the ADS algorithm for this study see the GitHub repository (Dawson, 2025, p. 1).

3.4 Orbit Propagation

Initial Orbit Determination established 4 solutions for any given arc length based on the computation method:

1. DA based IOD
2. DA + ADS based IOD
3. Pointwise Monte Carlo Simulation to Gauss's algorithm
4. Pointwise Monte Carlo Simulation of the Gauss-Lambert refinement algorithm

All possible combinations of IOD and propagation scheme were considered (Table 3.4).

IOD techniques	Propagation technique
DA based	DA based
DA + ADS based	DA based
Pointwise Gauss	Pointwise
Pointwise DAIOD	Pointwise

Table 3.3: IOD and Propagation combination that were tested

As with Monte Carlo IOD, Pointwise propagation method is described within validation subsection, subsection 3.7.

All propagation schemes were defined by the following steps:

1. Initialise the propagation wall clock time outside of the loop.
2. Define the IOD solution as the epoch state, $t_0 = t_i$
3. Pass the epoch state through a DA/ADS/Pointwise variant of the Runge Kutta 4 numerical integrator. Advance with a time-step of 1 day, $t_{i+1} = t_i + \Delta t$.
4. Save previous state into an array
5. Equate the new state to the previous state
6. Repeat steps 2-5
7. Save the wall clock time once the loop has finished

The Runge Kutta 4 numerical integrator was used for its computational efficiency. This is useful as the computational resources of the project. However, previous long-term DA and ADS propagations have used a variable step Runge Kutta 78 numerical integration scheme to ensure their study remained accurate over decades (Pirovano, 2020; Principe, 2020; Wittig et al., 2015, p. 1).

3.4.1 Assumptions

The following assumptions were used in all propagation schemes:

- Two body dynamics; There is no information of a newly discovered NEO's orbit. Therefore, perturbations cannot be modelled within the numerical integration steps.
- Heliocentric Equatorial Reference Frame; Propagation stayed within Heliocentric equatorial reference frame to ensure resultant (α, δ) uncertainty region was compatible with the ICRS reference frame of the observation databases.
- Sun is the only gravitational body acting on Apophis.
- Propagation in Cartesian coordinate system sufficient.

3.4.2 DA-based

The DA-based propagator propagates the Taylor polynomial itself, not points which have been evaluated within the IOD defined (α, δ) domain. As discussed in section 2.1, the propagator achieves this through propagating the reference state and the higher order Taylor coefficients. This way, the numerical integration scheme is embedded within the Taylor polynomial and a simple evaluation of 3σ , on the propagated polynomial, would yield a propagated state deviated from the reference orbit by 3σ .

if

$$\mathbf{X}_i = \mathcal{T}_{\mathbf{X}_i}^k(\delta\alpha, \delta\delta) \quad (3.27)$$

where k is the order of the polynomial, $\mathbf{X}_i$ is Apophis' state at t_i , and $\mathcal{T}_{\mathbf{X}_i}$ represents the Taylor coefficients of the state at t_i . Then the propagated state, $\mathbf{X}_{i+1}$, is

$$\mathbf{X}_{i+1} = \mathcal{T}_{\mathbf{X}_{i+1}}^k(\delta\alpha, \delta\delta) \quad (3.28)$$

where $\mathcal{T}_{\mathbf{X}_{i+1}}$ is a function of Taylor coefficients $\mathcal{T}_{\mathbf{X}_i}$ and the numerical integration scheme of Runge-Kutta 4. As a result, a function evaluation of $\mathcal{T}_{\mathbf{X}_{i+1}}$ defines any future state as a function of the measurement uncertainty 3σ .

A pseudo-code of the DA propagation scheme is shown in Algorithm 3.4.2:

Its important to note this function is called once. The Taylor polynomial only needs to be called once for the coefficients to encode the numerical integration scheme.

In line with the DA-based IOD literature, the DA order for propagation is a 6-th order Taylor polynomial (Pirovano, 2020, p. 1).

Algorithm 1 DA Propagation under Two-Body Dynamics

-
- 1: Define number of time-steps T_s
 - 2: Initial Δt , the time-step
 - 3: Initialize $X_{propagated}$ as an empty 2D matrix of shape $(6 \times T_s)$
 - 4: Store $\mathbf{X}_0$
 - 5: **for** $i = 1$ to $T_s - 1$ **do**
 - 6: $t_{i+1} = t_i + \Delta t$
 - 7: Propagate dynamics with two-body acceleration Φ :

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$$

- 8: $\mathbf{X}_{i+1} \leftarrow RK4(\mathbf{X}_i, t_i, t_{i+1}, \Phi)$
 - 9: $\mathbf{X}_{propagated, i} \leftarrow \mathbf{X}_{i+1}$
 - 10: **end for**
 - 11: **return** $\mathbf{X}_{propagated}$
-

3.4.3 ADS-based

Automatic domain splitting identifies whether the original (α, δ) domain should be split if the truncation error exceeds a user defined tolerance. The ADS-based propagator compares the two values at every time step. If the truncation error does exceed the user defined tolerance the previous time-step domain is split. This is shown in figure 2.5.

The pseudo-code for the ADS propagator is shown in Algorithm 3.4.3

Therefore the original Taylor polynomial is split into multiple Taylor polynomial sequentially. The $tol < \varepsilon$ boolean operation is conducted at every time-step of propagation, therefore splits occur sequentially.

Literature of ADS propagation of Apophis found that a split tolerance of $1 \times 10^{-9} AU$ and $1 \times 10^{-9} AU/day$ is required for accurate long-term propagation (Wittig et al., 2015, p. 1). As this study, and general Precovery, require long propagation timescales. The same split tolerance is used. Moreover, a 6-th order Taylor polynomial was found to be the best comprise between split accuracy and computation time (Wittig et al., 2015, p. 1).

However, the limited computational resources prevented ADS propagation. A split polynomial requires all polynomials to be propagated. Under the available computational resources this resulted in 1 observation requiring $\tilde{33}$ hours to propagate 6 months. This was unfeasible given the time constraints.

Algorithm 2 ADS Propagation within DAIOD

```

Initialize timer  $t_{start}$ 
2: Define Maximum Split Number  $N$ 
   Define split tolerance
4: Define time grid  $tgrid = \{t_0, t_1, \dots, t_i\}$ 
   Create initial ADS domain:  $init\_domain \leftarrow X_0$ 
6: Set  $init\_list \leftarrow [init\_domain]$  to store split Taylor Polynomials in
   Initialize  $final\_lists \leftarrow [init\_list]$  as the final, split, Taylor polynomial
8: for  $i = 0$  to length of  $tgrid$  do
   Propagate each ADS domain from  $t_i$  to  $t_{i+1}$  via algorithm 3.4.2.
10:  while  $tol > \epsilon$  do
   Split Taylor polynomial at  $t_i$ 
12:   Save Split domain
   Propagate each ADS domain from  $t_i$  to  $t_{i+1}$  via algorithm 3.4.2
14:  if  $tol < \epsilon$  then
   break
16:  end if
  end while
18:  Append all split  $t_i$  polynomials into  $[final\_lists]$ 
   set  $t_i \leftarrow t_{i+1}$ 
20: end for
return  $final\_lists$ 

```

3.4.4 ADS Propagation of multiple domains

Propagation of the multiple initial domains are required for the DA+ADS based Initial Orbit determination algorithm. This propagation Algorithm is nearly identical to the ADS propagation algorithm. The only difference is that the *initial_domain* variable holds multiple domains defined by the DA+ADS algorithm.

3.5 Differential Algebra Image Search Preparation

The key strength of Differential Algebra is that a single numerical integration routine can define the relationship between a set of initial conditions to a set of final conditions. However, this is still in polynomial form. In order to search archival Images the propagated polynomial needs to be evaluated such that a (α, δ) bounding box can be defined. Moreover, the propagation was conducted within the Heliocentric Equatorial frame whereas the image catalogues of ESO and CADC are in the geocentric equatorial frame (Gwyn et al., 2016, p. 1). Finally, propagation occurred within Cartesian coordinates however images are defined by their Right Ascension and Declination on the Celestial Sphere. Therefore three process need to occur:

1. Reference Frame change: Translate Heliocentric state to Geocentric state.
2. Coordinate Transform: Convert from Cartesian vectors to Spherical.
3. Resultant Taylor Polynomial Evaluated.

3.5.1 Heliocentric to Geocentric conversion

To convert from a Heliocentric to Geocentric frame one must complete the origin translation

$$\mathbf{X}_{geocentric} = \mathbf{X}_{heliocentric} - \mathbf{X}_{\oplus} \quad (3.29)$$

where $\mathbf{X}_{heliocentric}$ is the 6-dimensional state vector of Apophis post propagation and $\mathbf{X}_{\oplus}$ is the 6-dimensional heliocentric state of the earth. As both the propagated and image search reference frames are in the Equatorial plane no plane change is required.

The state of Earth changes throughout propagation. Therefore, one must calculate the heliocentric state of the Earth after each propagation time step. This is done through the AstroPy 'solar system ephemeris' function with the 'builtin' setting ('Astropy', 2025, p. 1). The time of propagation is used to call the Earth's position within its Ephemeris.

3.5.2 Coordinate Transform

Once the origin translation is complete a coordinate transform between Cartesian and Spherical is conducted. First the position components where ρ is the geocentric range, α is the Right Ascension, δ is the Declination (Curtus, 2005, p. 1)

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \alpha = \arctan 2(x, y) \quad \delta = \arcsin\left(\frac{z}{\sqrt{\rho}}\right) \quad (3.30)$$

Then the range-rate, right ascension rate and declination rate are:

$$\dot{\rho} = \frac{xv_x + yv_y + zv_z}{\rho} \quad \dot{\alpha} = \frac{xv_y - yv_x}{x^2 + y^2} \quad \dot{\delta} = \frac{\rho v_z - z\dot{\rho}}{\rho^2 \cos \delta}. \quad (3.31)$$

This coordinate conversion and reference frame transform is required for each DA/ADS Taylor polynomial and every pointwise Monte Carlo sample.

3.5.3 DA/ADS Evaluation

Post Propagation, the DA and ADS solutions are Taylor Polynomials.

$$\mathbf{X} = \mathcal{T}_{\mathbf{X}}^k(\delta\alpha, \delta\delta) \quad (3.32)$$

To define the propagated (α, δ) boundary, from which images can be searched in, one must evaluate $\delta\alpha, \delta\delta$ as combinations of $[-1, 1]$ (Pirovano, 2020, p. 1). Combinations of $[-1, 1]$ are required as DA variables were initialised with 3σ measurement uncertainty as a DA coefficient, see equation 2. This means the 3σ uncertainty is encoded into every coefficient of the higher order Taylor polynomial. Therefore, evaluating the $\delta\alpha, \delta\delta$ as combinations of $[-1, 1]$ ensure that the propagated α, δ boundary are defined as $(\alpha \pm 3\sigma_\alpha, \delta \pm 3\sigma_\delta)$. In other words, if the Taylor polynomial was evaluated with $\delta\alpha, \delta\delta = 2$ then $(\alpha \pm 6\sigma_\alpha, \delta \pm 6\sigma_\delta)$.

The propagated (α, δ) can appear non-linear (Wittig et al., 2015, p. 1). Therefore the number of samples taken along the α, δ boundary define the precision of the bounding box to search Images within. Long-term propagation forces non-linear terms to dominate (Wittig et al., 2015, p. 1), therefore corner samples $[-1, 1]$ across all six dimension. See this projects GitHub repository for the evaluation algorithm for DA and ADS variants.

The result of the evaluation is a (α, δ) bounding box at every propagation time step. The boundaries are defined by a $\pm(3\sigma_\alpha, 3\sigma_\delta)$, numerical integration scheme, coordinate conversions, orbital dynamics model and Initial Orbit determination algorithm. This is because these operations were applied to the Taylor polynomial coefficients before the evaluation step.

3.6 Fixed Region Query

A $(\alpha \pm 3\sigma_\alpha, \delta \pm 3\sigma_\delta)$ boundary for all time steps have been constructed. There are $36, 6^2$, samples along all six dimensions. One must query the ESO and CADC databases so that any archival image within this boundary is returned. The query strategy is identical across DA, ADS and pointwise initial orbit determination and propagation schemes.

The ESO and CADC databases utilise the Astronomical Data Query Language (ADQL) to search the Image Archive (Agency, n.d., p. 1). This is accessed through its Python API available within the Astropy module ('Astropy', 2025, p. 1). For a successful ADQL query the following conditions must be met:

1. Coordinates must be ordered anti-clockwise
2. There must be no duplicate points. Duplicate points mean duplicate vertices within the alpha shape. Duplicate vertices are not redundant and will cause the ADQL query to fail (Dowler, 2018, p. 1).

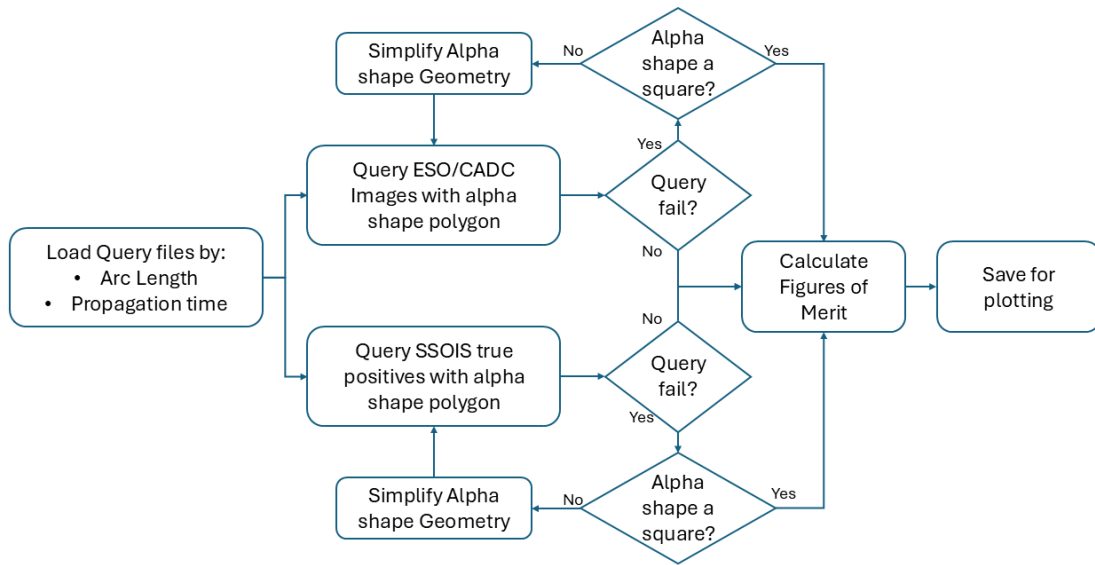


Figure 3.6: Logic Flow chart of the Query

3. Must be a shapely polygon type. Within an archival search it is required a Polygon, defined by a series of points, to be used. This polygon must be recognised by Python as a shapely polygon type, it cannot be a series of (α, δ) points.

In section 3.6.1 this methodology is shown further.

A shapely polygon type can be constructed by two means:

1. Take the Convex Hull of a series of points.
2. Define the Alpha shape over a series of points.

The Convex Hull of a series of points is the smallest convex shape that completely encloses all the points within the 2D space (Grima & Márquez, 2001, p. 1). For simple geometries with no curves, such as a rectangle, it is an accurate approximation. However for irregular and curved geometries it is a poor approximation (Grima & Márquez, 2001, p. 1), it often overestimates the size of the shape (see Figure 3.7). Therefore, an alpha shape algorithm is used.

The Alpha Shape is the generalisation of the Convex Hull concept (Fischer, 2011, p. 1). Indeed, an alpha shape of 0 alpha coefficient is a Convex Hull of a set of points. As a result, an alpha shape construction may be more accurate at defining the propagated α, δ boundary from a set of points. The alpha coefficient, however, must be found prior to the alpha shape construction.

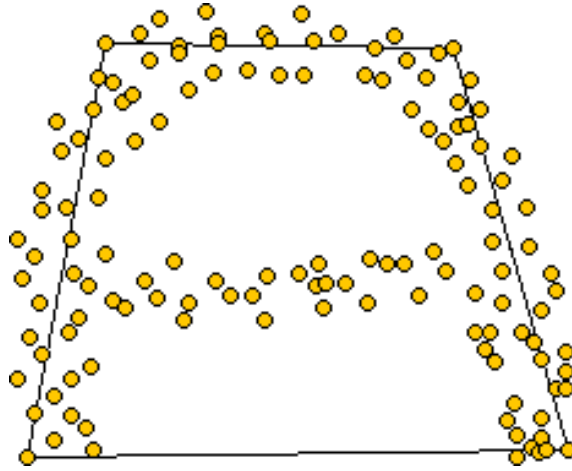


Figure 3.7: The Convex Hull of a set of points

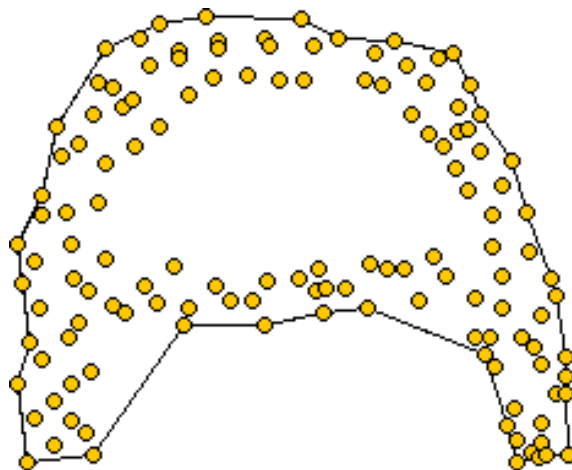


Figure 3.8: The alpha shape of a set of points

3.6.1 Database Query Strategy

First, the (α, δ) boundary coordinates need to be sorted anti-clockwise. To do this the centroid of the (α, δ) region is calculated

$$\alpha_{centroid} = \frac{\sum_1^N \alpha_N}{N} \quad \delta_{centroid} = \frac{\sum_1^N \delta_N}{N} \quad (3.33)$$

The the angle from an (α_i, δ_i) coordinate and the centroid coordinate is calculated

$$\theta = \arctan 2(\alpha_i - \alpha_{centroid}, \delta_i - \delta_{centroid}) \quad (3.34)$$

Sorts the (α, δ) coordinate by smallest to largest θ .

Once the list of coordinates are sorted anti-clockwise, any duplicate coordinates are removed through element-wise comparison of the array of coordinates.

A Python alpha shape Python library is used to generate the shapely polygon (Bellock et al., n.d., p. 1). First the 'optimizealpha' function is used to define an alpha coefficient. This finds an capable alpha coefficient which are within a set of predetermined bound. The maximum alpha coefficient was chosen to 10, while the minimum was chosen to be 0.1. The 'optimizealpha' function selects an alpha coefficient that defines a single enclosed polygon (Bellock et al., n.d., p. 1), typically at the maximum coefficient limit. The maximum coefficient of 10 was enforced through trial and error. See the projects GitHub repository for the python implementation (Dawson, 2025, p. 1).

Once an alpha shape approximation of the (α, δ) uncertainty region was defined, the propagated fixed regions were queried in ESO and CADC databases via the python API. This was chosen as raw ESO image archives identify an 'image' (α, δ) as point defined by the telescope pointing direction. All images returned were stored for later analysis. For the same propagation time step, the true positive image was obtained through querying the SSOIS database for Apophis.

The ADQL querying fails if the polygon vertex count is too high (Dowler, 2018, p. 1), or if the vertices define a highly irregular shape. If either of these conditions are in the ADQL query it would return nil images, even if the SSOIS returned a true image. To overcome this, a Convex Hull fallback was added. If the vertex count of the original polygon is above 20, the Convex Hull of the set of points overwrites the original alpha shape then the uncertainty region is queried with the Convex Hull polygon. The Convex Hull fall back is key limitation within the methodology. This is discussed in the results section.

Once the fixed (α, δ) region has been queried, the number of images within the uncertainty region, and the true positive SSOIS image have been returned. The figure of merits defined in section 3.1 are calculated.

An important distinction was found between the Monte Carlo and the DAIOD alpha shapes. The ADQL failed for all propagated Alpha shapes defined through the DAIOD algorithm. For this reason the Fixed Image Query was omitted from the final methodology and the alpha shape area was assessed as an analogue.

3.7 Validation

The DA/ADS Taylor Polynomials were validated through a pointwise Monte Carlo Simulation. A Monte Carlo simulation represent the entire PDF and is the current approach of incorporating sensor uncertainty into Initial Orbit Determination (Aeronautic & Administration, 2025, p. 1). The Monte Carlo method mirrors the DA/ADS approach, each step is independently validated:

1. Orbit Determination; the evaluated Taylor polynomial within the $(\rho, \dot{\rho})$ plane is compared to the pointwise Monte Carlo $(\rho, \dot{\rho})$ distribution.
2. Orbit Propagation; over each time step, the Taylor polynomial is evaluated and compared to the pointwise Virtual Asteroid propagation.
3. Alpha Shape Construction; Obtain alpha shape plots at random time-step, compare the two plots.
4. CADC/ESO/SSOIS Query; observe the number of images returned between the Monte Carlo and DA/ADS Taylor polynomials are consistent.

All inputs such as the time of observation, the arc length and the propagation time step are consistent with the DA/ADS algorithms. Furthermore, the pointwise algorithm is used to assess the Computational time and Precovery performance relative to the DA/ADS approach. This allows for comparative analysis between the traditional methods and a DA/ADS approach.

3.7.1 Orbit determination

The Monte Carlo simulation tests 10,000 Virtual Asteroids. NASA states that 10,000 is sufficient for a coarse estimation of the size of the orbital set (Aeronautic & Administration, 2025, p. 1). After the Earth's position has been defined at the time of observation,

see section 3.3, the Monte Carlo simulation begins. Gauss's and a pointwise variant of the DAIOD algorithm is executed. The $3\sigma_\alpha, 3\sigma_\delta$ distributions are modelled as random perturbations in α, δ . The distribution is assumed to Gaussian where the observed (α, δ) are the mean values.

First, a covariance matrix is defined:

$$cov = \begin{bmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_\delta^2 \end{bmatrix} \quad (3.35)$$

Equation 3.35 shows the measurement uncertainties are assumed to be independent. The covariance and observed (α, δ) form a multi-variable gaussian distribution. A sample from this distribution defines a perturbed (α, δ) of a Virtual Asteroid. The line-of-sight vector, $\hat{\rho}$, is calculated with the Virtual Asteroid's (α, δ) . This is used to define a position and velocity via Gauss's algorithm. This process is repeated 10,000 times as 10,000 samples are drawn from the distribution. This defines the control Monte Carlo distribution.

A pointwise algorithm which follows the Gauss - Lambert iterative refinement is used to i) validate the DA operations have been implemented successfully ii) Assess the impact of DA methods on computational performance.

Like the DAIOD algorithm, the pointwise Gauss - Lambert refinement algorithm passes the Gauss determined slant range into a Lambert iterative solver. The iterative process solves (ρ_1, ρ_2, ρ_3) such that $\Delta \mathbf{v}_2(\rho_1, \rho_2, \rho_3) = 0$. Unlike the DAIOD algorithm, there is no higher order Taylor expansion, the Gauss-lambert solver is repeated across 10,000 Virtual Asteroids. Each asteroid has a perturbed (α, δ) , therefore a perturbed $\hat{\rho}$, to capture the measurement uncertainty. As a result, the states across 10,000 Virtual Asteroids are defined and corrected to ensure they're Keplerian. This is the same method used in the first part of the DAIOD algorithm, see section 3.3. A computational timer starts once the $(\sigma_\alpha, \sigma_\delta)$ covariance matrix has been formed. It marks once the 10,000 Virtual Asteroids have completed the Gauss algorithm, then ends once the 10,000 Virtual Asteroid complete Gauss - Lambert refinement.

The state velocity components are used to define the $\dot{\rho}$ distribution and the $(\rho, \dot{\rho})$ plane is plotted. The DA-IOD plots and the pointwise Monte Carlo plots are compared. For the DA-IOD algorithm to be valid, the 3σ bounding box must bound 99.7% of the Virtual Asteroids generated by the Monte Carlo Gauss-Lambert algorithm. An identical method is used to validate the DAIOD+ADS algorithm.

3.7.2 Orbit Propagation

The propagation timescale and number of time-steps are identical to the DA/ADS approach. Propagation is repeated 10,000 times in order to propagate the 10,000 Virtual Asteroids. The computational time required to complete pointwise propagation is measured from the start of the first Virtual Asteroid, to the end of the 10,000th Virtual Asteroid.

The pointwise propagation is identical to the DA/ADS propagation: a Runge Kutta 4 numerical integration scheme with two-body dynamics, where the central gravitational force is the sun. It is propagated in the Heliocentric equatorial plane of reference till the final time-step

The validation strategy followed the same rationale as the initial orbit determination, instead of the $(\rho, \dot{\rho})$ plane the (α, δ) plane was assessed. The DAIOD (α, δ) must contain 99.7% of the Virtual Asteroids from the Gauss-Lambert refinement IOD algorithm. All time-steps were assessed. The same approach was taken for the ADS variant

The propagated position of Apophis relative to the HORIZONS generated Apophis ephemeris was also compared:

1. The euclidean norm between centre of the Taylor expansion and the Horizons Apophis ephemeris
2. The euclidean norm between mean state of the VAs and Horizons Apophis ephemeris
3. The euclidean norm between the centre of the Taylor expansion and the mean state of the VAs will be calculated

The euclidean norm symbolised the straight line displacement between two points. It is used to quantify propagation error, where the ground truth is Horizon generated ephemeris.

These strategies ensure: i) The 3σ distribution in DA/ADS polynomials are preserved in ODE propagation, ii) The propagation is similar to an independent source.

3.7.3 Fixed Region Query

If the initial orbit determination and propagation algorithms are correct, then the alpha shape construction is assumed correct. This can be assumed as the alpha shape is con-

structured through an external python module which has been independently verified.

Section 3.6 states the vertex polygon limit for an ADQL query. If a DAIOD or DAIOD+ADS alpha shape is over the 20 vertex count the Convex Hull of the set of points is taken. If this happens, the Convex Hull of its pointwise equivalent is also taken. This ensure the search region is identical, and the comparison between the number of Images identified is a fair verification method. Similar to the DA based algorithms, the query is timed.

Chapter 4

Validation Results

4.1 Orbit Determination

Figure 4.2 demonstrates the DA based algorithm has successfully constructed the State Taylor Polynomial of Apophis in terms of Right Ascension and Declination. Both algorithms contain the true position of Apophis within their respective uncertainty distributions. The DA based algorithm contains 100% of the Monte Carlo VAs within its distribution. These characteristics ensure the DA based algorithm is implemented correctly.

The top right of the figure illustrates that the DA-based algorithm overestimates the maximum $(\rho, \dot{\rho})$ relative to the Monte Carlo simulation. This due to the differing sample strategies rather than a flaw within the DA implementation. As figure 4.1 illustrates, the DA method samples a bounding box within the (α, δ) plane with side lengths $(\pm 3\sigma_\alpha, \pm 3\sigma_\delta)$. The corners of the 2D plane are $\pm 4.23\sigma$ from the mean. Assuming all (α, δ) measurements are independent (Pirovano, 2020, p. 1), each corner of the 6 dimensional hypercube is $\pm 7.35\sigma$ from the mean (Harris et al., 2024, p. 1). This is determined through calculation of the Mahalanobis distance of independent variables.

By contrast, The Monte Carlo method generates 10,000 VAs from a Gaussian distribution. The probability of a 7.35σ VA is

$$P(|X| > 7.35\sigma) \approx 2 \times 10^{-13} \quad (4.1)$$

where X is the state and σ is the measurement uncertainty. This would require on the order of 5×10^{12} samples to expect a single occurrence.

$$N \approx \frac{1}{2 \times 10^{-13}} = 5 \times 10^{12} \quad (4.2)$$

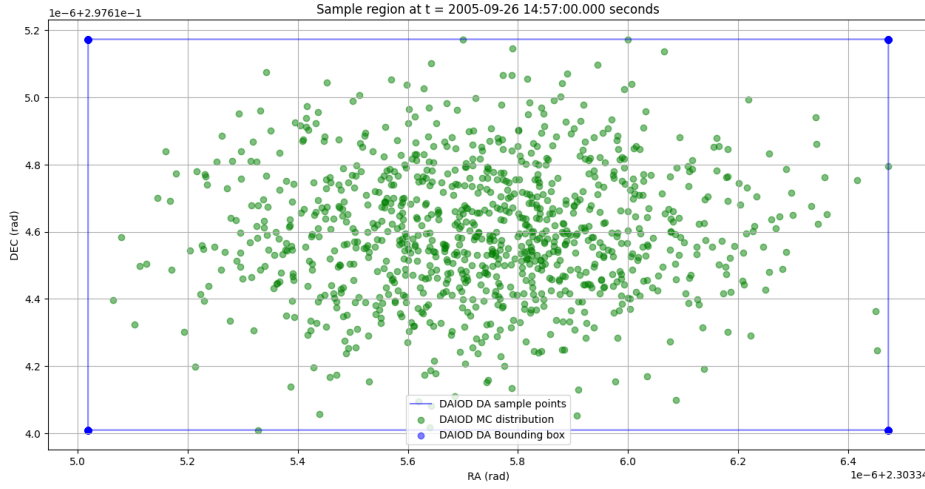


Figure 4.1: 2D bounding box representation of the (α, δ) at epoch

With 10,000 VAs were generated, the 7.35σ VA case is not applicable, whereas the DA method always generates them by the 6 Dimensional bounding box construction (Pirovano, 2020, p. 1).

In addition, the DA formulation inherently treats the Right Ascension and Declination measurements at each observation epoch as independent variables. The DA construction assumes there are three independent $(\delta\alpha, \delta\delta)$ pairs. While this guarantees enclosure of the entire uncertainty ellipsoid, it neglects that true observational uncertainties are correlated within Right Ascension and Declination (same optical system). This forces the Mahalanobis distance, the measure of statistical spread, to increase from the 2 dimensional domain to 6 a dimensional domain. This increases the corner σ sample from 4.24σ to 7.35σ which forces a overwhelming conservative admissible region.

As a result, the DA approach overestimates the admissible $(\rho, \dot{\rho})$ domain compared to the Monte Carlo approach. However, this is solely through the sampling method rather than the operations of the Differential Algebra. In the context of Preccovery, the bounding box sampling strategy may increase the number of candidate images flagged for potential detection, potentially lengthening the data-association process and ultimately Preccovery itself. A quantitative assessment of this trade-off is left for future work.

The constituent functions of the Monte Carlo and DA based algorithms (Gauss's Algorithm and Izzo's algorithm to Lamberts problem) are validated through textbook examples within Orbital Mechanics for Engineering Students (Curtus, 2005, p. 1).

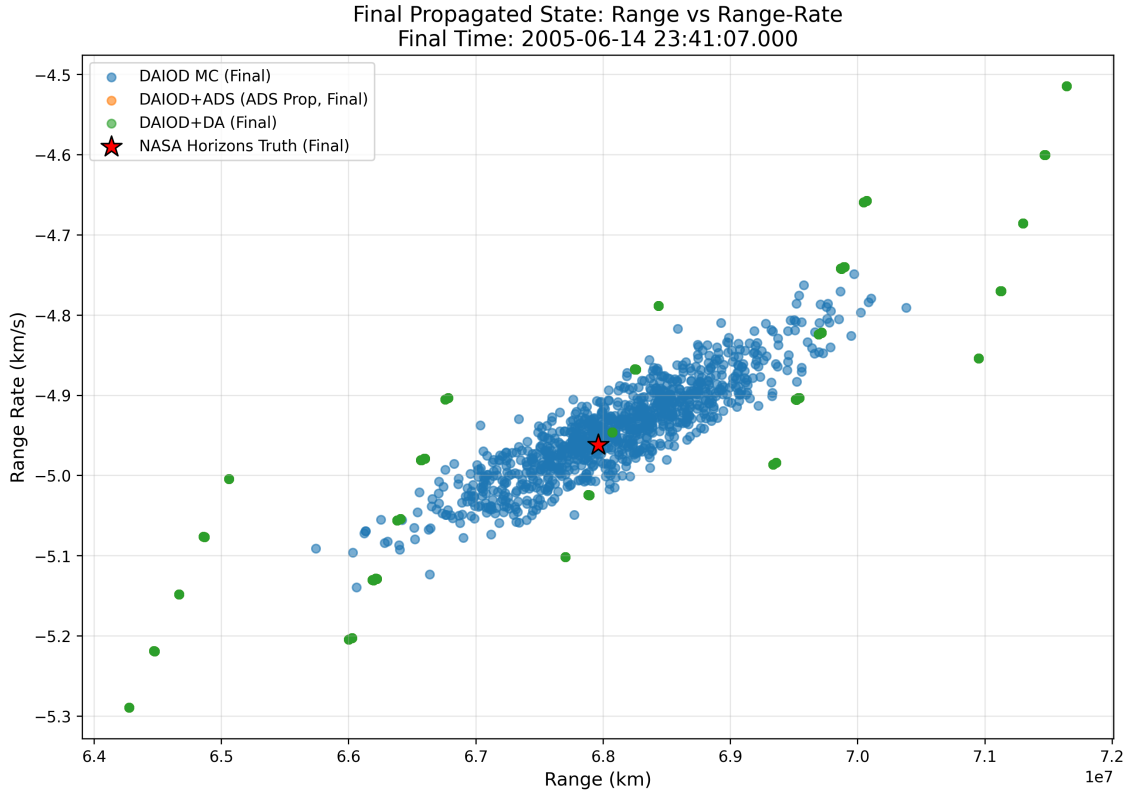


Figure 4.2: Propagated state of an 2005 06 17 23:41:07 epoch state

4.2 Orbit Propagation

Figure 4.2 shows the $(\rho, \dot{\rho})$ phase space of Apophis after three days of propagation with two-body dynamics. Consistent with figure 4.2, the Monte Carlo samples and the NASA Horizons truth state are fully contained within the propagated DA bounding box. This confirms that the DA representation preserves the admissible region through time.

The residual error between the mean Monte Carlo sample and the truth state is 139029.489 km in range and 0.018277 km/s in range-rate. These offsets are expected, the simplified two-body model neglects perturbations included in the Horizons ephemerides. In particular, non-conservative effects such as the Yarkovsky acceleration (Aeronautical & Laboratory, 2022, p. 1). This is not a limitation of the methodology: discovery arcs can only assume gravitational forces act on the NEO. Therefore Preccovery via NEO discovery arcs can only assume gravitational forces.

Figures 4.3 illustrate the $(\rho, \dot{\rho})$ phase space of Apophis at the initial observation epoch and after three days of propagation. At the IOD epoch, the DA based bounding box is broader than the VA cloud, due to the sampling method. Both the distribution enclose

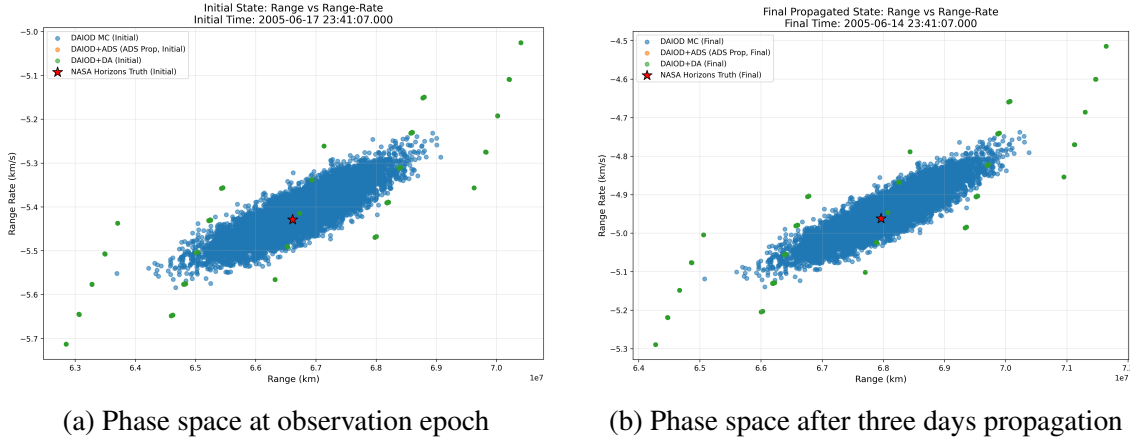


Figure 4.3: Comparison of DA and MC methods at initial Orbit determination and through propagation

the NASA Horizons true state. After propagation, this relationship is preserved: the DA bounding box continues to overestimate the admissible region relative to the cloud of VAs. The consistent geometry between the two methods before and after propagation demonstrates that the DA structure is preserved under two-body dynamics.

4.3 Image Query

Initial Orbit determination of a $(\alpha \pm 3\sigma_\alpha, \delta \pm 3\sigma_\delta)$ region, similar to figure 4.1 was queried for images. The DA based algorithm had a cuboid bounding box while the MC based algorithm followed VA's from a Gaussian distribution. At the epoch of the initial orbit determination (2005-06-17 23:41:07 UTC) the image query script returned a unique Image: EMMI.2005-06-17T23:40:07.359. This is the same image which is returned by the SSOIS at time 2005-06-17 23:41:07. No additional Images were identified within the small search space. This confirms the query returned a unique and valid detection. As a result, the image query script are correctly linked to the orbit determination outputs and consistent with independant archival image systems.

In summary, the orbit determination, propagation, and image query stages have each been validated against independent truth sources (NASA Horizons ephemerides and SSOIS archival queries). This demonstrates that the implemented methodology is both accurate and representative of real-world Precovery scenarios for short-arc NEO discoveries.

Chapter 5

Results and Discussion

5.1 Differential Algebra Performance

The study tested the feasibility of precovery of short arcs; arcs which are shorter than two hours (Milani et al., 2004, p. 1). For a observation arc of less than 2 hours both the DA and the Monte Carlo simulation failed. This was because both techniques rely on Gauss Angles-only IOD as the first estimate. Within Gauss's Algorithm, short arcs cause the line of sight (LOS) vectors to approach co-linearity, this results the cross product of the LOS to approach zero, thus Gauss fails.

Between the region of 2-4 hours the Lambert solver failed. This was because the lambert solver failed to converge: the gauss initial solution, the seed for the root finding algorithm, was near the hyperbolic regime. This forced the lambert solver to switch between hyperbolic and elliptical states within the differential correction process and a solution never converged. The logic behind these findings is consistent to why DA-based IOD fails for spacecraft operations (Pirovano, 2020, p. 1).

It was found the minimum arc length which could complete orbit determination was a 0.18 days observation or approximately 4 hours. The following discussion refers to the 4 hour observation as the 'short arc' while a 'long arc' refers to the maximum observation period of 1 day.

5.1.1 Assessment of the Taylor Series Construction

Describe in section 2.3.3, DA based IOD algorithms construct a $(\rho, \dot{\rho})$ Taylor polynomial as a product of the Right Ascension and Declination. As a result, all variation with the six components of the state are defined via the angle observation uncertainty. The Right

Ascension, Declination and their rates are directly observable, while the accuracy of $\rho, \dot{\rho}$ are defined via the algorithm. Through analysis of $\rho, \dot{\rho}$ phase space one can determine the accuracy of the orbit state as a result of the Differential Algebra Algorithms. The following section is an assessment of the initial $(\rho, \dot{\rho})$ phase space. Specifically via the location of the true solution and the distribution of the DA defined bounding box.

The $\rho, \dot{\rho}$ uncertainties are integrated over time. The $\rho, \dot{\rho}$ error compound, and affect the validity of the (α, δ) domain over several propagation steps. To understand DA's propagation error, one must assess the accuracy of the Taylor Polynomial. The Taylor Polynomial defines the $\rho, \dot{\rho}$, therefore an assessment of the propagated $\rho, \dot{\rho}$ is vital to understand the source of error in propagated (α, δ) .

This is important for precovery, α, δ are searched within image archives so one can make data association. The effect of an incorrect α, δ , or a highly uncertain α, δ , on precovery is discussed further in Section 5.1.2.

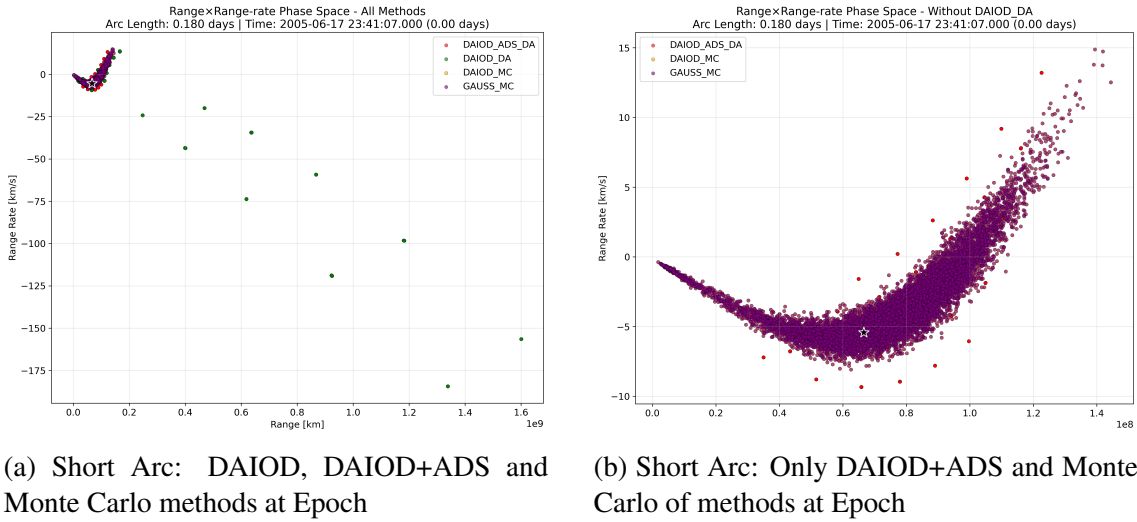


Figure 5.1: Range - Range rate Phase space's at observation Epoch

Figure 5.1a demonstrates the poor performance of DA for short arcs. The solely DA based admissible region spans multiple Astronomical units (1×10^9 km). At best this is only possible if Apophis is directly opposite the Earth, on the other side of the Sun. This is Impossible to observe. At worst, the range of 1×10^9 km places Apophis far out of its orbital plane, as the apoapsis is 1.14×10^8 km. Both of these scenarios are impossible. In contrast, there is a the strong correlation between the DAIOD+ADS and Monte Carlo Simulation, see figure 5.1b. This suggests the pure DA solution does not contain a sufficiently high Taylor polynomial order to describe the non-linearities of the arc at the edge of the domain. The DAIOD+ADS is accurate as splits within α, δ domain result in a

smaller radius of convergence. Therefore multiple lower order Taylor polynomials accurately describe the edge of the domain.

This is reinforced by Figure 5.2a; a observational arc of 0.5 days result in the arc to be sufficiently linear so that the edge of the domain is described by a six order Taylor Polynomial expansion.

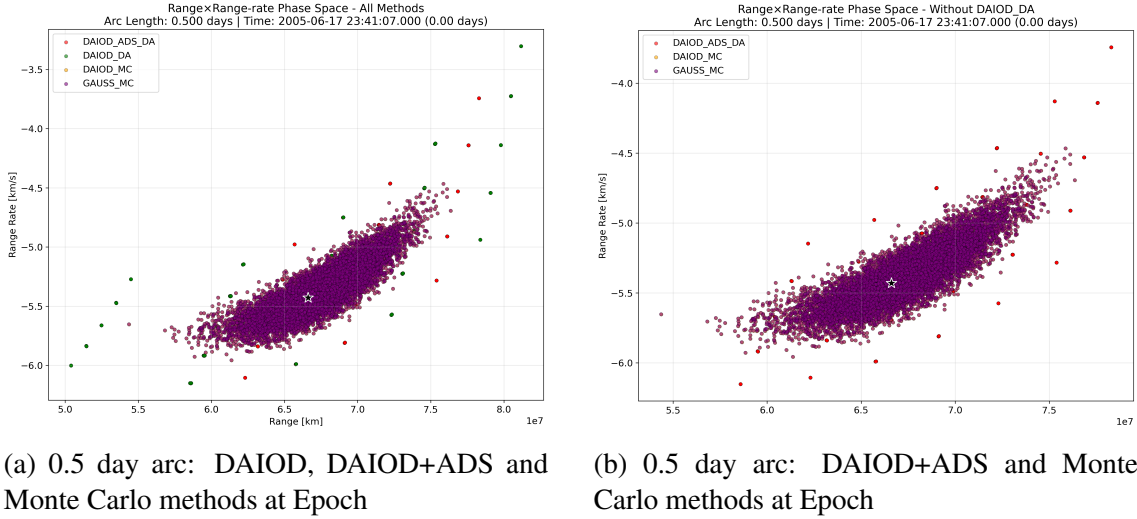


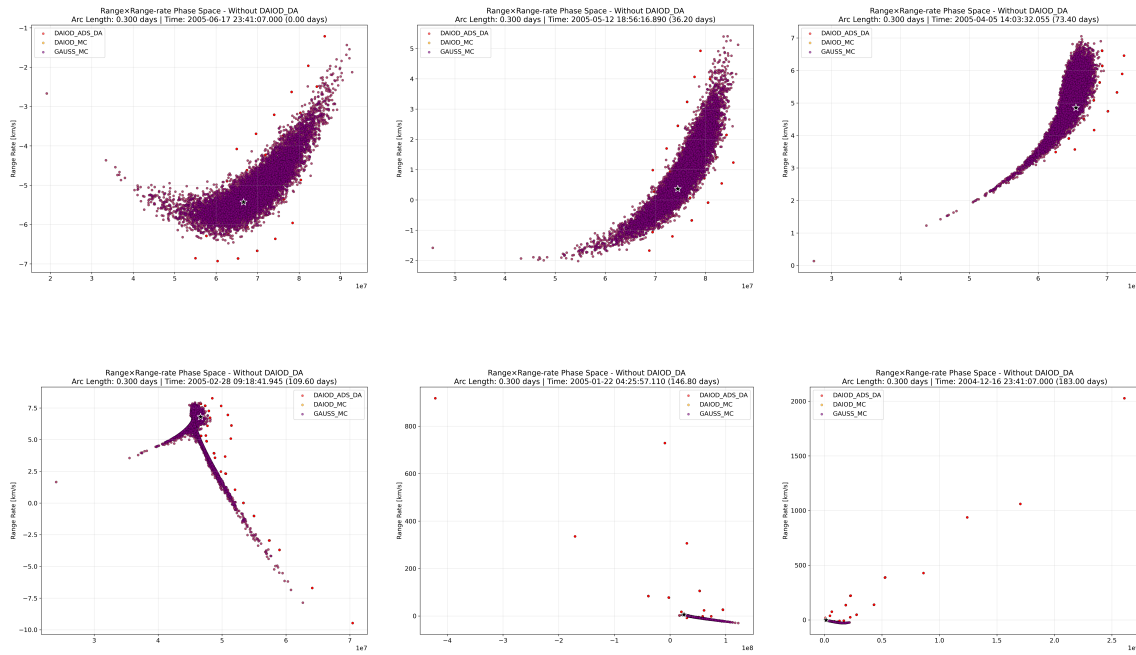
Figure 5.2: Long Arc Range and Range rate phase space

Observe Figure 5.1.1 and 5.1.1, across all arc lengths the initial DA Taylor polynomial defines a wider $(\rho, \dot{\rho})$ region. As a result, the pure DA Taylor polynomial presents a widely distributed initial state uncertainty. Through propagation this results in a wider α, δ search region as the large $\dot{\rho}$ bounds are integrated forward. This is evident in Section 5.1.2, where the propagated pure DA solutions cover half the celestial sphere over the duration of a month.

In contrast, the epoch $(\rho, \dot{\rho})$ defined by the DAIOD+ADS algorithm is tightly constrained to the Monte Carlo simulation across all arc lengths, see Figure 5.1.1 and 5.1.1. Like the Monte Carlo, the DAIOD+ADS is centred on the true solution of Apophis. For shorter arcs, 0.18 - 0.3 days, the epoch DAIOD+ADS $\rho, \dot{\rho}$ covers a smaller region of uncertainty compared to the Monte Carlo simulation. Discussed in Section 5.1.2, the reduced $(\rho, \dot{\rho})$ translate to reduced $(\alpha, \delta)_{t_0+1}$, therefore search area. However the absence of the full (α, δ) uncertainty region result in different propagated α, δ alpha shapes centred on the true solution. The disfigured DAIOD+ADS search region may hold implications of the number of false positive detections, but this left for future work.

Over propagation this trend remains: the propagated ADS state continues to contain

the true Apophis state for a narrower range and range rate distribution till the 109.60 th day of propagation. For the the 0.18 day case, the 109.60 th day of propagation defines when DAIOD+ADS is less accurate than the Monte Carlo while the 0.3 day case demonstrates this condition is met on the 141 st day of propagation. This relationship with arc length suggests that this is a result of the higher order non-linearities which are not modelled for a 6th order Taylor polynomial that spans the split domain. For precovery, this means one would have to fine-tune the Taylor polynomial order and domain split size to the propagation duration and expected arc length. The effect of DAIOD+ADS increased accuracy for certain time steps is discussed further in section 5.1.2.



(a) Caption 6

Figure 5.3: The Evolution of $\rho, \dot{\rho}$ across 6 months of propagation

Figure 5.3 shows the monthly evolution of the uncertainty in $(\rho, \dot{\rho})$ for the Monte Carlo and DAIOD+ADS solutions. At early epochs (e.g., Figures 5.17b and ??) the DAIOD+ADS cloud is notably tighter than the MC distribution, indicating that ADS has removed strongly non-linear “tails” that lie outside the local validity of the split Taylor map. Because DA propagation at t_{k+1} is the composition of the DA map built at t_k ($T_{k+1} \approx \Phi_{\Delta t} \circ T_k$), any early under-representation of the state support carries forward: regions that were truncated or insufficiently split at t_k cannot re-enter the support at t_{k+1} . Consequently, the DAIOD+ADS distribution can remain overly compact for several steps, and when the dynamics later become more non-linear the map inflates primarily along a specific direction that does not fully match the MC distribution $(\rho, \dot{\rho})$, producing the wide,

uncertain phase space seen at later epochs. This behaviour is consistent with ADS missing parts of the high-curvature manifold when the split threshold or maximum split count is too small for the local degree of non-linearity.

This illustrates that the future work requires an investigation of i) the optimal ADS split tolerance / split count for IOD and ii) a ADS propagator which can refine the Taylor polynomial map as the NEO is propagated forward.

The characteristics of the phase space define the key α, δ trends in the next section. Propagated $\rho, \dot{\rho}$ uncertainties result in regions of α, δ uncertainty, which ultimately effect the success of precovery.

5.1.2 Coverage of the Celestial Sphere

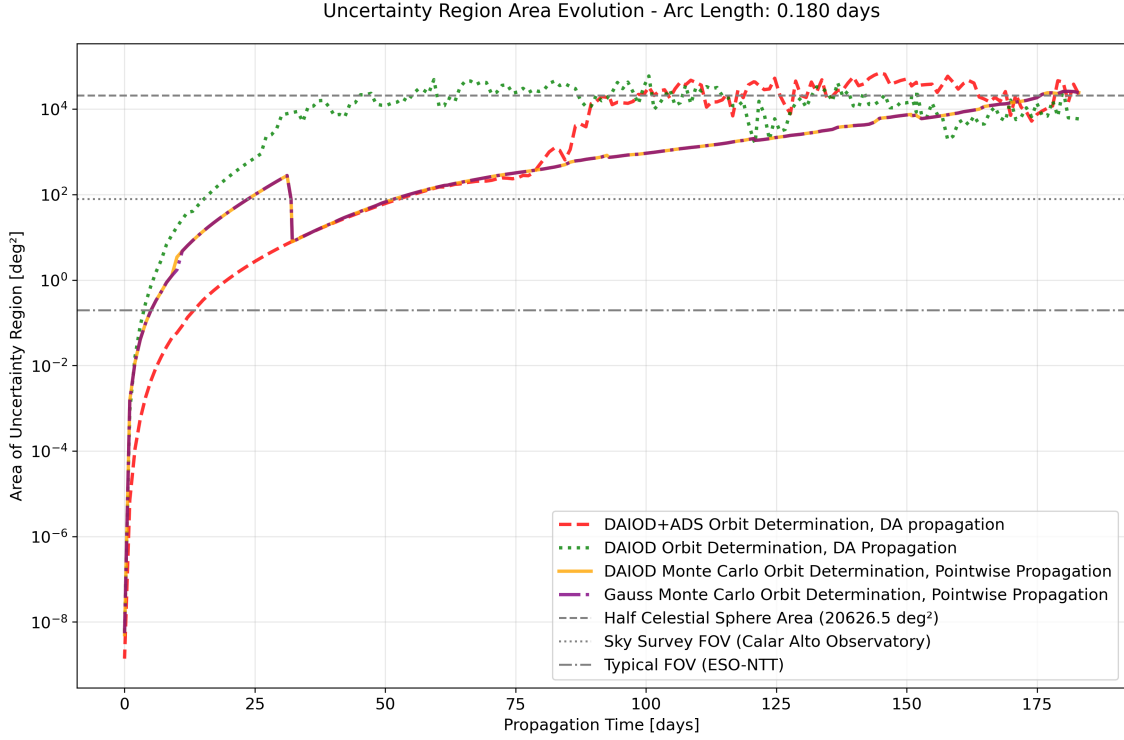
During orbit propagation both the observed state and its associated uncertainty must be propagated to the next observation epoch (Vallado, 2013, p. 1). This ensures association between the new observation and the original NEO. Because orbital mechanics are inherently non-linear (Vallado, 2013, p. 1), the continuum of possible orbital solutions growth over time. This represented by the uncertainty region. This expansion results in a continuous increase in the predicted α, δ search area on the Celestial Sphere.

Telescopes have a maximum field of view (FOV), so the growth of the uncertainty region limits the ability to recover the NEO. As the entire arc of the NEO is typically required to be contained within the FOV, the growth of the uncertainty region places inherent propagation time limits before a NEO is unrecoverable.

For precovery, the evolution of the uncertainty region has two major effects:

1. A large propagated uncertainty region results in a large number of archival images that must be retrieved, inspected, and associated with the NEO. This significantly increases the computational cost and slows down the arc-extension process. This increases the risk that a newly discovered NEO is lost before the orbit can be further constrained. Therefore, the size of the uncertainty region defines the feasibility of precovery.
2. The inflation of the uncertainty region limits the useful propagation time: to ensure reliable data association (Gronchi, n.d., p. 1), the entire uncertainty region should remain within the telescope FoV. Once it exceeds the FoV, association is no longer guaranteed. Hence, the uncertainty region defines a maximum propagation time.

Hence the temporal evolution of the uncertainty region governs both the efficiency and feasibility of NEO precovery. As a result, it defines whether NEO precovery can rapidly reduce the NEOs hazard level.



Figures 5.4 and 5.1.1 illustrate the evolution of the 3σ (α, δ) uncertainty growth for the short arc and long arc cases. The y axis defines the area of uncertainty in degrees squared, while the x axis is the propagation time in days. The grey horizontal lines show the coverage region of a wide angle telescope (Calar Alto Observatory), and the telescope that has observed Apophis (ESO-NTT) (Gwyn et al., 2016, p. 1).

The figures compare the the effect of DA-based IOD algorithms (DAIOD), DA and ADS based IOD algorithm (DAIOD+ADS) to Monte Carlo algorithms. Across all arcs the initial uncertainty at epoch is non-zero, this is the measurement uncertainty of the observation. Explained in Section 4, DA and ADS based (α, δ) begin with a larger initial area at epoch. This reflects the bounding box approach of the DA-based methods compared to the Monte Carlo methods.

Within all arc lengths, the DAIOD algorithm initially matches the MC growth curve for the first 2-3 days. The accurate performance is a result of the α, δ samples instead of the Taylor polynomial construction. This is evident as after 3-15 days, the specific

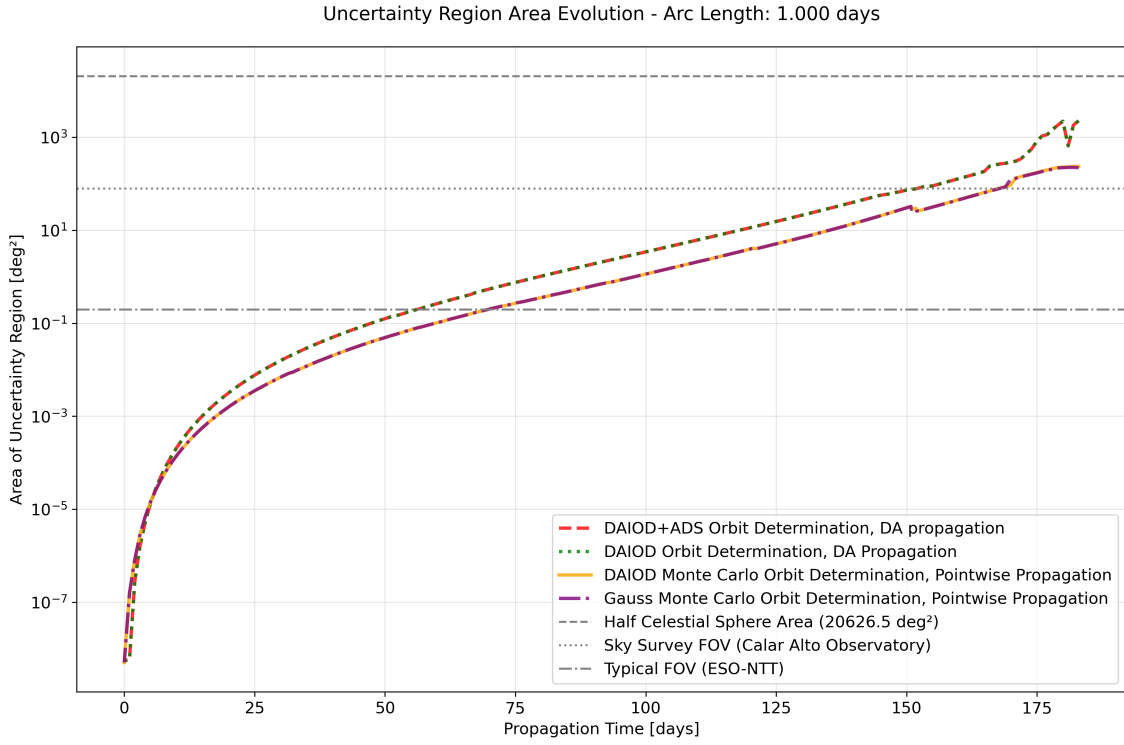
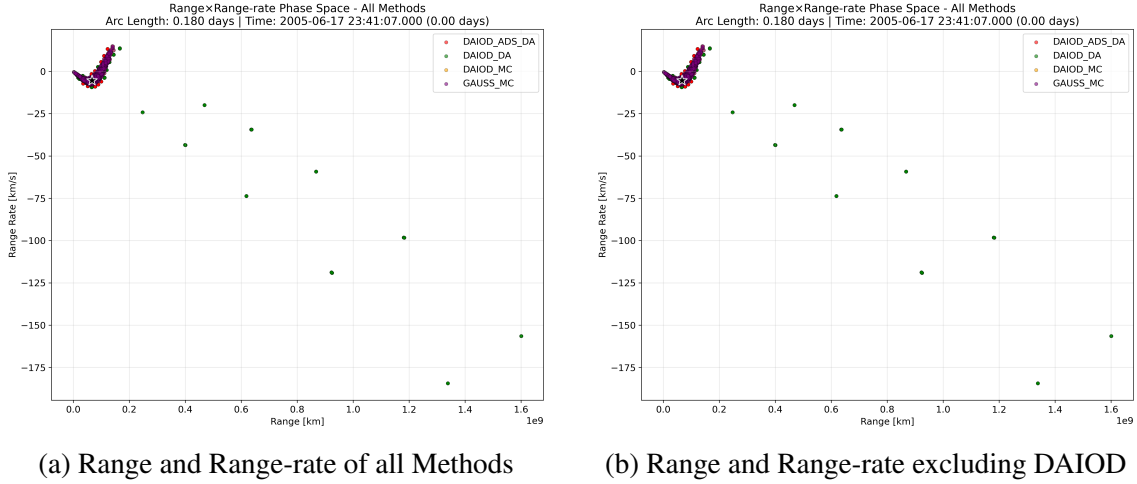


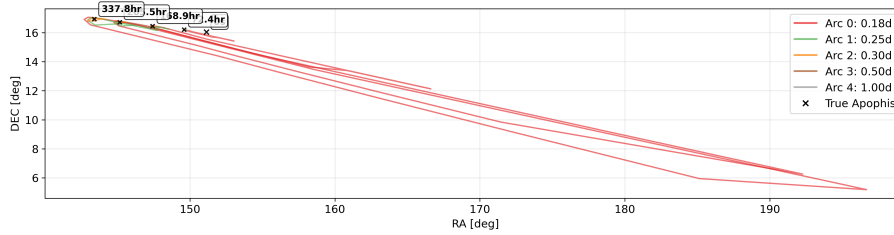
Figure 5.5: Long Observational Arc (24 day)

time depends on arc length, the DA deviate significantly from the MC baseline and the DAIOD+ADS algorithm. The most significant deviation occur at the 0.18 arc length where the DA α, δ region of possible solutions is 1 order of magnitude larger by the 14 th day, see Figure 5.4.

This is because of the poorly defined initial $\rho, \dot{\rho}$ state, or in other words the definition of the Taylor polynomial. This is supported by Figure 5.1a where the pure DA solution spans range magnitude of 10 AU and range-rates up to -175 km/s. In contrast, the DAIOD+ADS and Monte Carlo algorithms span 1 AU in range variation and 10 km/s in range rate. As discussed in Section 5.1.1, this is because the short arc is highly non-linear and the 6th order Taylor expansion omits capture these effects. Figure 5.7 demonstrates how the pure DA 14th day propagated $\alpha\delta$ search regions increase from a 0.3 arcseconds to 80 degrees uncertainty within Right Ascension. It is not feasible for Apophis to become fully contained within an archival Image. Indeed a wide telescope, such as ones used at the Calar Alto Observatory, span a Right Ascension of 3.65 degrees. As a result, the maximum propagation time for DA is 7.0375 days before the search area is larger than Calar Alto Observatory Telescope FOV. This value is only applicable to wide survey telescopes, such as Calar Alto. This reduces the probability of identifying a true positive image, thus DAIOD is unsuitable for precovery.

Figure 5.6: The epoch $(\rho, \dot{\rho})$ for an 4 hour observation arc

DAIOD Orbit Determination, DA Propagation - RA×DEC Alpha Shapes
Figure 1/11 | Time range: 2005-06-17 23:41:07 to 2005-06-03 21:50:20

Figure 5.7: The evolution of the (α, δ) projected on the Celestial Sphere. The 0.18 day Arc results in large uncertainty regions within 3 days

However, the long observation arcs, for example 1 day, show that DA has similar uncertainty growth to the Monte Carlo baseline. Moreover, Figure 5.7 illustrates the α, δ search area is well constrained in propagation for arcs greater than 4 hours. As discussed in Section 5.1.1, it is suspected this is due to more linear relationship exhibited in long arcs. Discussed in section 5.2, DAIOD is 5 times faster than DAIOD+ADS. This makes a 6th order DA defined Taylor polynomial attractive for a long observation arc use-case, where there are reduced non-linearities.

The DAIOD+ADS consistently outperforms the DA algorithm across all arc lengths. The single Taylor map is subdivided into smaller domains, and the split sixth order polynomial better represents the local dynamics. As a result the long term divergence point

from the Monte Carlo simulation occurred on the 75 th day for the short arc case. The relationship between arc lengths is identical to the DAIOD and the Monte Carlo simulation.

Over arc lengths of 0.18-0.3 days, the DAIOD+ADS has a slower rate of (α, δ) growth than the Monte Carlo simulation. As a result, a DAIOD+ADS based precovery pipeline can propagate the NEO to 13.5 days to ensure Apophis observation arc is fully contained in a typical ESO-NTT Image, and 52.08 days to ensure Apophis is fully enclosed in a Sky Survey observatory. This is improved performance over the Monte Carlo Simulation where propagation is constrained to 5.1 days and 23.4 days respectively.

Figures 5.8 and 5.9 illustrates that the DAIOD+ADS 0.180 day arc still contains the true solution at the 52.08 days and 13.5 days. The DAIOD+ADS solutions result in uniform search regions which decompose into teardrop shapes on propagation time. The level of decomposition is dependant on the arc length, this relationship is expected. Explained in Section 5.1.1, DAIOD+ADS is not fully representative of the $\rho, \dot{\rho}$ plane. As such, the extreme values (such as observations at 7σ) define a large propagated search region.

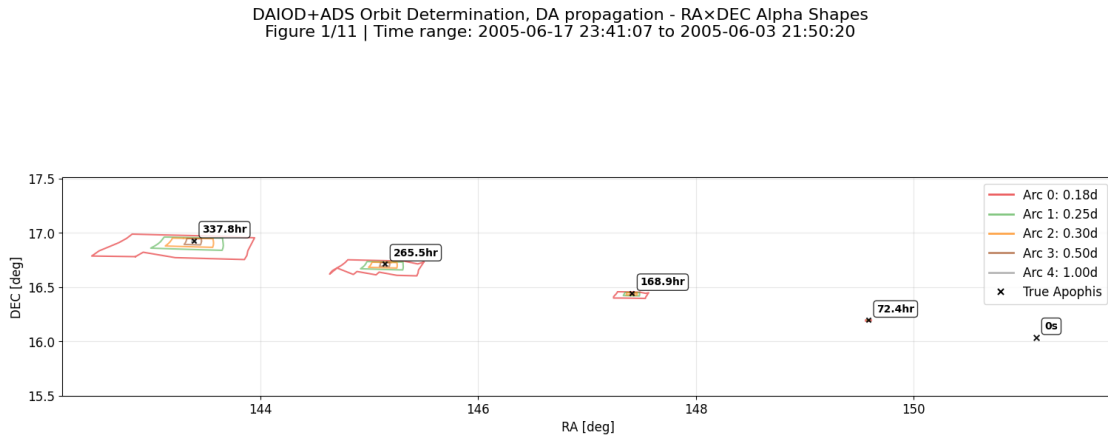


Figure 5.8: Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 13 days (312 hours)

Figures 5.10 and 5.11 illustrate the evolution of the Gauss defined α, δ search region. In contrast to the DAIOD+ADS algorithm, the MCS's 13 day α, δ search region initially covers a wide teardrop shape which shrinks into a jagged teardrop at approximately 50 days. The wide area is because the MCS defines the full 3σ distribution of the $\rho, \dot{\rho}$ plane,

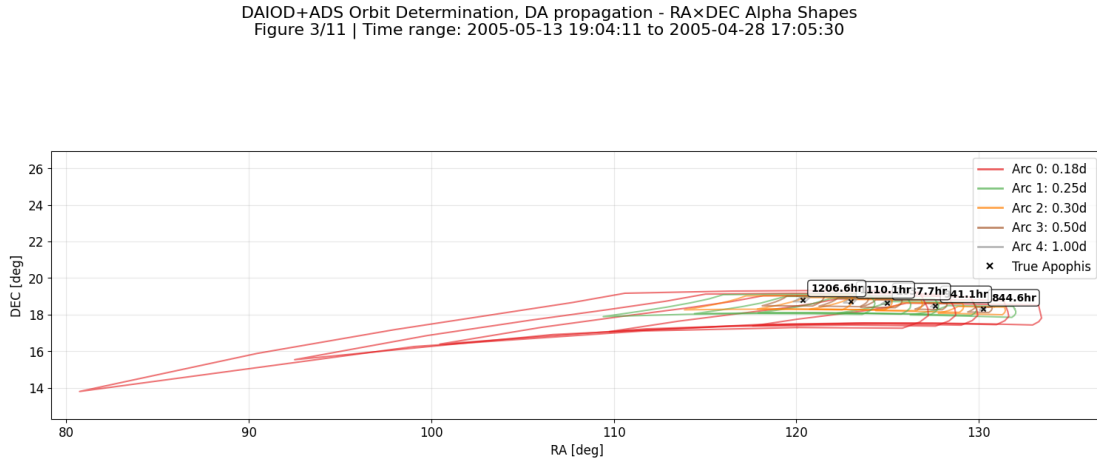


Figure 5.9: Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 52.08 days (1249.92 hours)

including highly non-linear outliers (see figure 5.3), whereas the DAIOD+ADS algorithm eliminates these outliers under the 6th order polynomial. The smaller search region results in DAIOD+ADS short term propagation to be preferred over MCS for precovery association.

Comparing Figures 5.11 and 5.9, it is evident that DAIOD+ADS results in a smaller dimensions for the search space than the Gauss MCS solution. This, combined with the smaller search area, means DAIOD+ADS is preferred for precovery accuracy till a 52.08 day propagation duration (for a 4 hour arc).

This relationship is consistent till the 0.5 day arc, where the IOD solution is becomes linear, see figure 5.1.1. As discussed in Section 5.1.1 the slower area growth is a result of the smaller $(\rho, \dot{\rho})$ uncertainty distribution. Figure 5.9 demonstrates that despite the smaller uncertainty, the DAIOD+ADS algorithm consistently contains the true Apophis state within the (α, δ) plane till specific propagation time. That propagation time depends on the arc length; the shorter the arc length, the sooner this characteristic of the DAIOD+ADS algorithm fails. The consequence of this under-representation of the non-linear region is shown in the later propagation steps, where the uncertainty region area is much larger than the Monte Carlo area.

For precovery, the DAIOD+ADS algorithm reduced search region for short arcs is advantageous. It enables longer maximum propagation times till the NEO is no longer in the

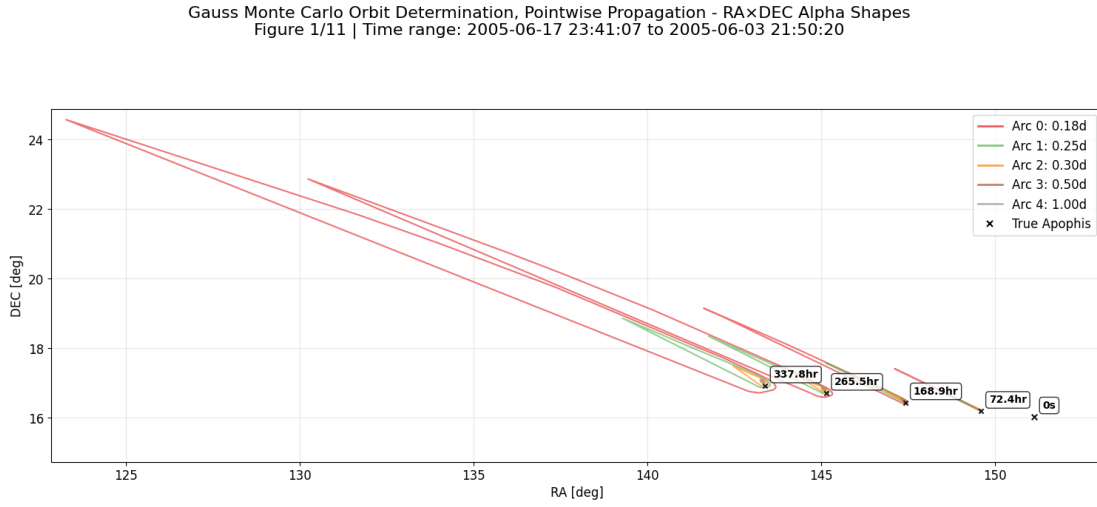


Figure 5.10: Evolution of (α, δ) Search Region: Gauss Monte Carlo 13 day (312 hour)

FOV of a telescope, therefore the archival image. It also ensures less computational effort for a given propagation time. The uncertainty area is less, therefore the number of less false candidates are identified that require trial association. Moreover the DAIOD+ADS algorithm identifies Apophis α, δ at intermediate propagation times, where the Monte Carlo algorithms do not, see Figure. This solidifies the DAIOD+ADS as the most accurate for the short arc short duration use case.

The inconsistency between the DAIOD+ADS and the Monte Carlo scheme in propagation suggests an ADS propagator, and a study on the split tolerance, should be conducted within the NEO precovery context. This would enable the DAIOD+ADS algorithm to accurately represent the initial uncertainty and therefore the evolution of the uncertainty. This should be investigated for future work.

The search space defined by the DAIOD+ADS algorithm fully departs from the MCS defined search space past the 83rd day of propagation time, see Figures 5.12 and 5.13. After this time, failure to capture the full extent of the $(\rho, \dot{\rho})$ plane at IOD results in a disfigured search space not representative of the Monte Carlo solution. It is suspected this is because the DAIOD+ADS fails to capture the complete $\rho, \dot{\rho}$ phase space with the DA defined Taylor polynomial at IOD. This results in the absence of orbital states within IOD, and therefore at the propagated time step. At the 83rd day propagated time step, the non-linear orbital dynamics take hold and the propagated final solution exaggerate the errors of the DAIOD+ADS IOD $\rho, \dot{\rho}$. As a result, the Monte Carlo solution is preferred for the short arc, long term propagation case. However, for precovery this is impractical

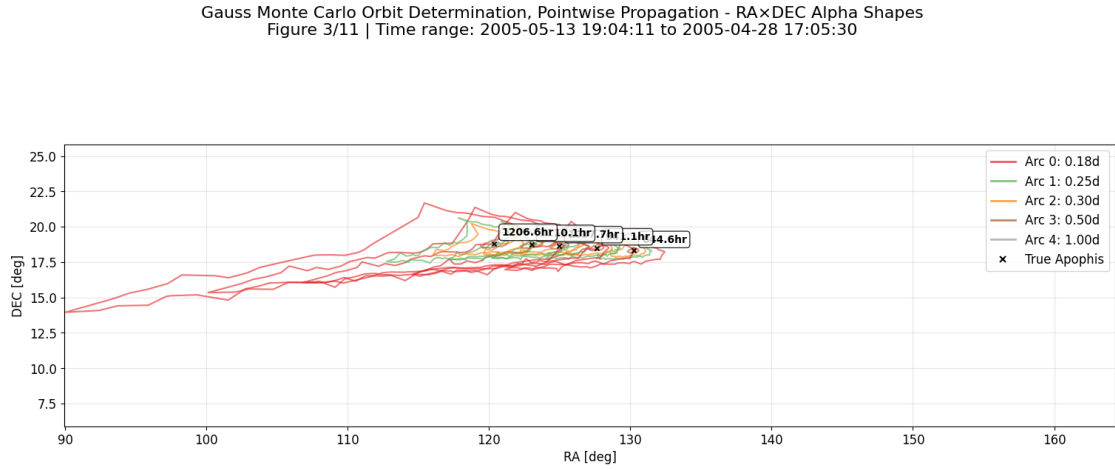


Figure 5.11: Evolution of (α, δ) Search Region: Gauss Monte Carlo 52.08 day (1249.92 hours)

as the MCS defined α, δ search area is greater than any observatory with archival Images.

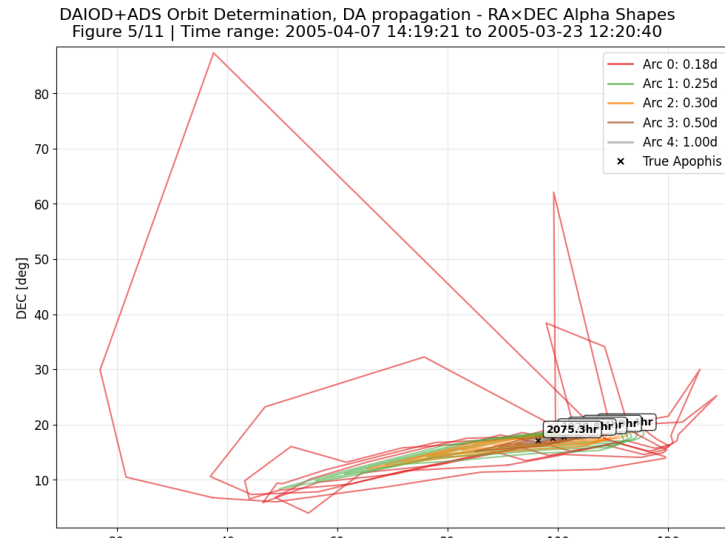


Figure 5.12: Evolution of (α, δ) Search Region: DAIOD+ADS 0.18 day observation arc approximately 83 day (1992 hours)

5.2 Computational Performance

The computational efficiency is vital to precovery, it influences whether an observer should conduct precovery or a follow-up observation. To enhance precovery, the al-

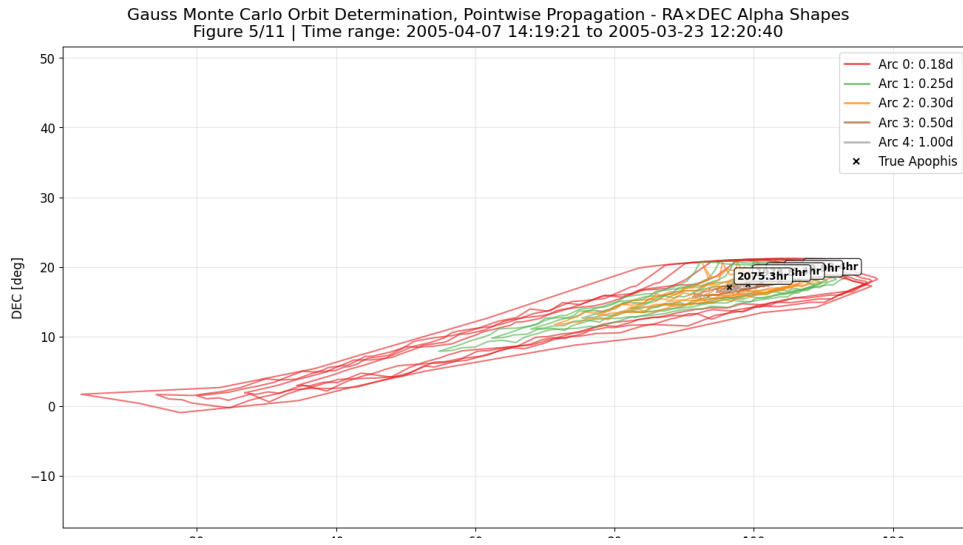


Figure 5.13: Evolution of (α, δ) Search Region: Gauss Monte Carlo 83 day (1992 hours)

gorithm must minimise the computational time required to propagate a NEO to a specific time step (Section 3).

5.2.1 Differential Algebra comparison

There are five stages to the DA based algorithms: Initial Orbit determination, propagation, reference frame handling, Evaluation of the propagated Taylor polynomial and Alpha shape creation. Alpha shape creation references to the α, δ search space constructed from the set of points defined by the alpha shape algorithms.

Figure 5.14 shows that the pointwise algorithms 102.333 times slower than the DAIOD and 21.9 times slower than DAIOD+ADS algorithm. Figure 5.14 demonstrates that the alpha shape creation contributes the largest to the computational time for the pointwise algorithms, where the DA evaluation contributes the most to computational time from the Taylor polynomial methods.

Shown in Figure 5.1a, Monte Carlo IOD is characterised by many individual points that represent the probability distribution. While the the DA methods are defined by a bounding box. The complex Monte Carlo geometry results in the long execution of the alpha-shape algorithms. The is because numerous loops are tested to find the optimal alpha coefficient within the 'optimizealpha' Python function (Bellock et al., n.d., p. 1). The DA methods are characterised by a bounding box, less DA evaluation points result in more regular geometry, thus, a faster alpha-shape construction time.

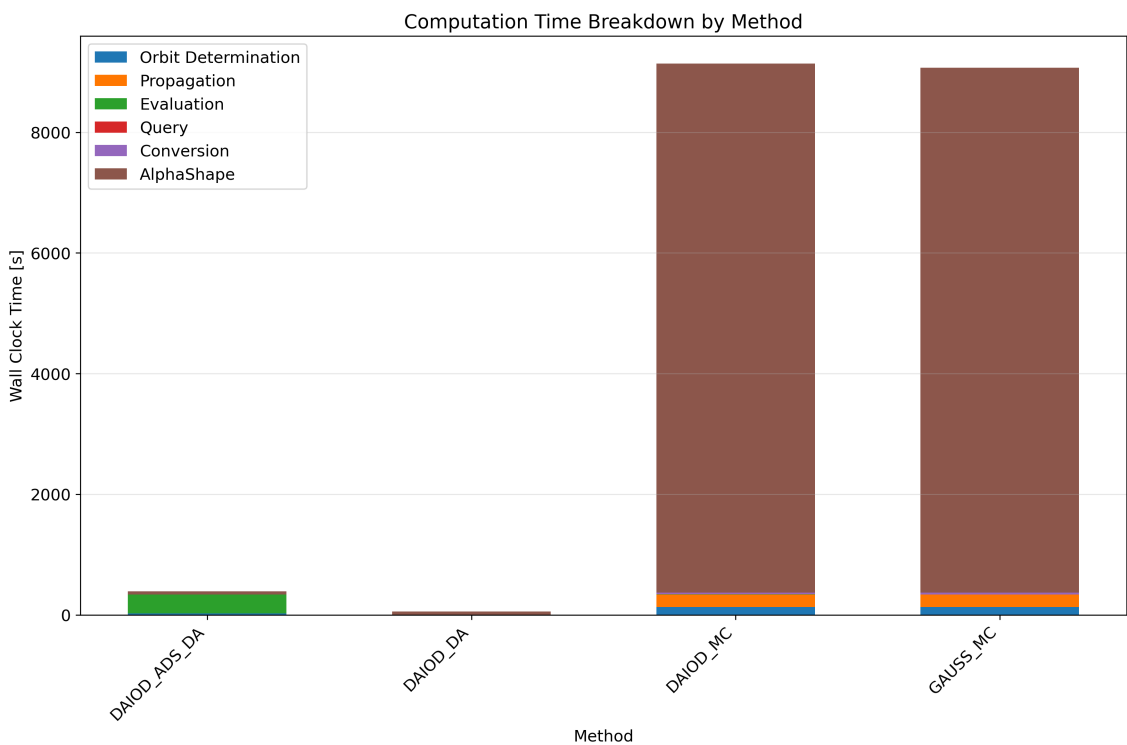


Figure 5.14: Breakdown of the Mean total computational time across all observation arcs.

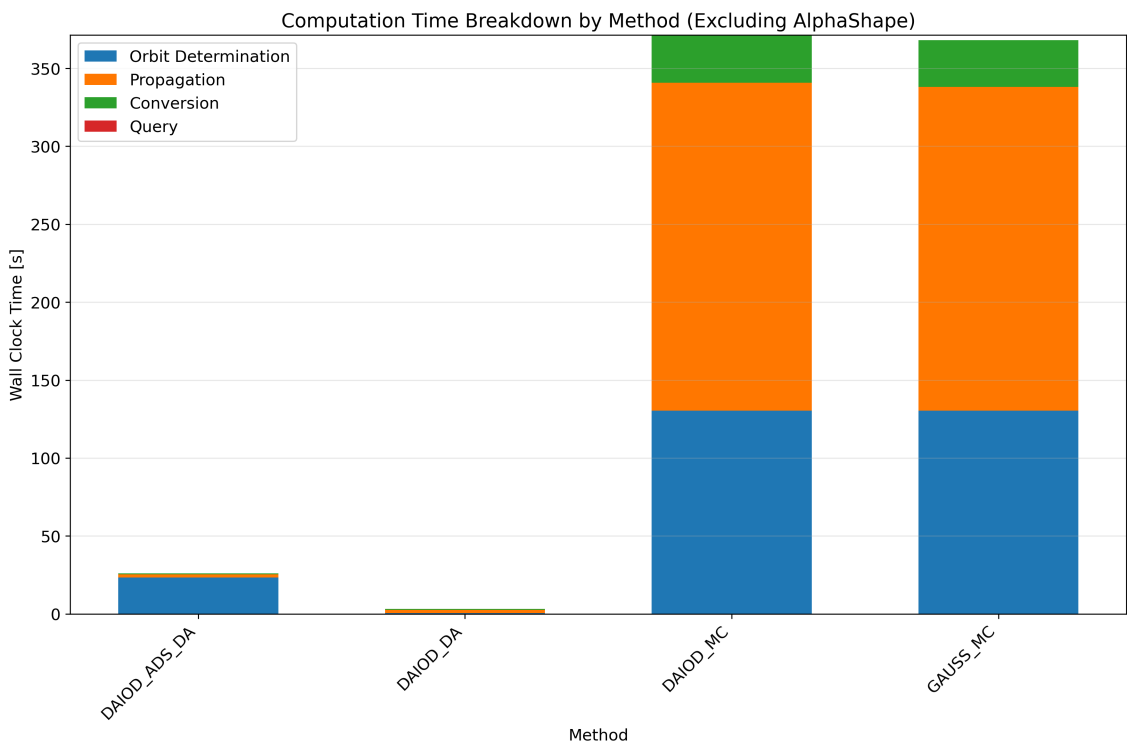


Figure 5.15: Total mean computational time, alpha shape generation removed.

Furthermore, propagation process are longer for the Monte Carlo algorithm. The time taken to complete propagation is 207.6 seconds, while the DAIOD+ADS algorithm is 2.7 seconds. The DAIOD algorithm is 2.01 seconds. This is expected; the 10,000 individual points need to be propagated where only a single, or a few in the case of DAIOD+ADS, Taylor polynomials need to be propagated.

Excluding alpha shape generation, the DAIOD+ADS has a computational time of 26 seconds, while the DAIOD algorithm required 3.31 seconds. This is a 7.86 times increase. This is because of the multiple splits required in the IOD section from the ADS algorithm. The splits result in 7.86 times more Taylor polynomial evaluations that must occur, however the propagation time only differs by 0.69 seconds. Through the differing computational times under different algorithms, there is a relationship between computational time and α, δ uncertainty region. More splits result in an accurate initial orbit determination but requires more evaluation steps to evaluate the multiple Taylor polynomials and long IOD time.

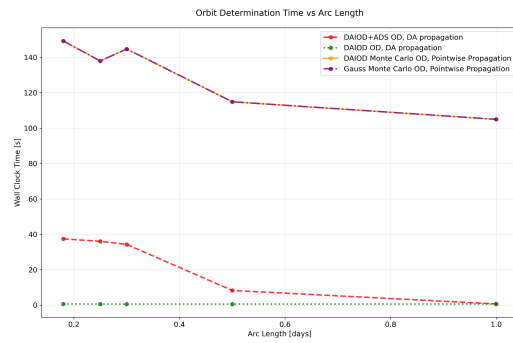
It is clear that DA based algorithms are faster than Monte Carlo ones. This primarily due to alpha shape creation processes. However, figure 5.15 demonstrates that if the alpha shape algorithm was dismissed the DAIOD+ADS algorithm is 17.16 times faster than the Monte Carlo methods, while the DAIOD is 110 times faster.

This improved computational performance of DA based algorithms result in faster IOD and propagation, hence faster precovery. This enables quick data association with candidate images, and ultimately fast precovery operations. This confirms that DAIOD+ADS algorithms are optimal for precovery operations, both due to the relative α, δ accuracy and the rapid computational time. Ultimately, this means use of DAIOD+ADS will enable fast risk assessment of newly discovered NEOs.

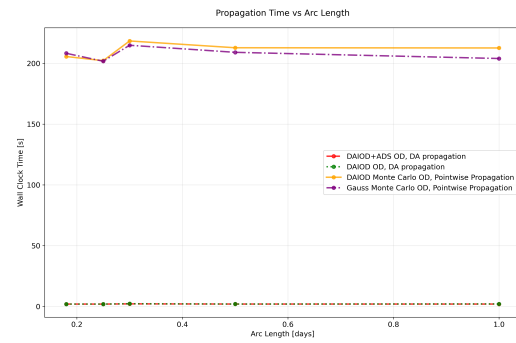
5.2.2 Computational time scaling

Figures 5.16a - 5.16d demonstrate the affect of arc length on the computation time. Observing the Figures, it is evident that the duration of pure DA methods are independent of arc length. This is result of the Taylor polynomial description of the NEO state, a fixed number of Taylor polynomials describe the state therefore the required iterations to conduct orbit determination and propagate is fixed. This is often referred to as 'fixed effort' within the literature.

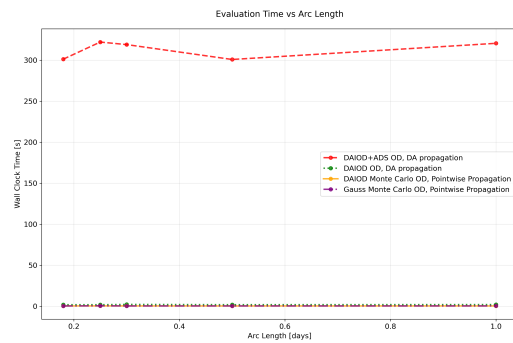
The computational time of the DAIOD+ADS algorithm follow dependance on Arc



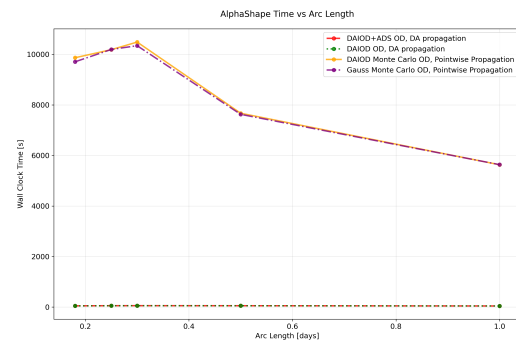
(a) The effect of Arc length on Orbit Determination time



(b) The effect of Arc length on Orbit Propagation time



(c) The effect of Arc length on Taylor Polynomial Evaluation time



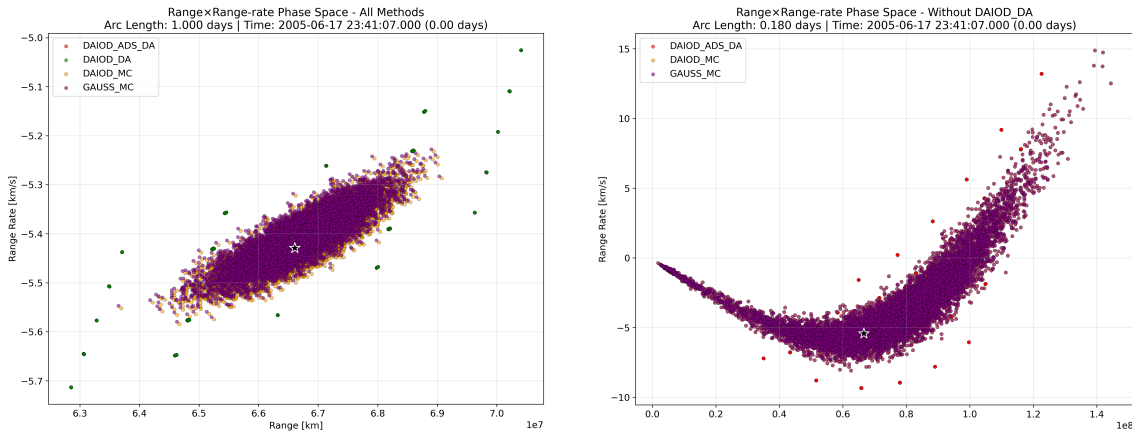
(d) The effect of Arc length on Alpha shape creation

Figure 5.16: The effect of Arc length on Computational time

length within the Orbit Determination. This is the result of the high non-linearities within shorter arcs. As a result, the ADS algorithm is required to split the large initial domain into smaller, accurate, domains.

For the Monte Carlo simulation, the time required to complete orbit determination and determine the alpha shape decreases with arc length. This confirms the hypothesis that Alpha shape construction is a result of the complex geometry defined by the non-linear distribution of the VAs. Figure 5.17 demonstrates the complexity of the 0.18 day arc compared to the 1 day arc.

For precovery, this means the duration of precovery pipeline is predictable and independent of the NEO arc length. This can streamline operations and promote faster precovery. For example, a fixed computation time means an operator can better predict whether to conduct precovery or do a follow up observation. The fixed time enables an informed choice, which allows for operators to make informed decision to ensure a fast, accurate NEO hazard assessment can be guaranteed.



(a) The Range and Range-rate phase space at Epoch for a long arc

(b) The Range and Range-rate phase space at Epoch for a short arc

Figure 5.17: Notice the distribution of the Monte Carlo points. The short arc forms a complex shape due to the non-linearities while the longer arc is symmetrical so a less complex alpha shape is required.

5.3 Precover Performance

Overall, the DAIOD+ADS method is shown to be the best algorithm for precovery.

- It effectively controls the α, δ search region for all arc lengths. This minimises the number of candidate images that need to be associated, quicken the precovery process.
- It ensures the maximum propagation time till the α, δ uncertainty region is outside a non sky survey telescopes FOV. This enables longer propagation times, thus greater arc extension if the archival image is associated.
- It is computationally efficient. The DAIOD+ADS algorithm is approximately 10 times faster than standard Monte Carlo techniques and provides a fixed runtimes. This enables more precovery searches for multiple NEOs, independent of the input data. This enables informed decisions that can ensures effective use of precovery operations.

Where the DAIOD+ADS does fail is the long term propagation of the NEO however, by this time, the α, δ search region across all techniques so large that no observatory could contain the observation arc within a single image. Therefore, no possible archival Images that could be used to complete precovery. As a result, DAIOD+ADS is feasible, and in may be preferred, over classical MCS for precovery.

The conclusions made on the DAIOD+ADS method is limitation in propagation performance. As a result, the future work should include exploration of an ADS propagation scheme, an investigation of the required ADS propagation tolerances, and the consequence ADS propagation has on the precovery pipeline.

Chapter 6

Limitations and Future Work

This study demonstrates the potential of Differential Algebra (DA) and Automatic Domain Splitting (ADS) for Near-Earth-Object (NEO) precovery; however, several factors limit the generality of the findings. These limitations arise from computational constraints, methodological assumptions, and the short project time frame. This forces this study to remain a feasibility study and the raw findings shouldn't be used to construct a precovery operations without further investigation.

6.1 Computational Limitations

The implementation relied on the Python wrapper for the C++ DACE library, DACEyPy. DACEyPy supports DA arithmetic and ADS logic but Python prioritises user simplicity over computational speed. For example, C++ is approximately 18 times faster than Python (ÜNSAL et al., 2022, p. 1). Although this does not affect the conclusions drawn between the four algorithms, it does mean that the runtime metrics cannot be extrapolated to coding platforms used within the literature (e.g. C++).

ADS-based propagation was included in the original project scope; however, its computational cost was too great. During testing, the number of sub-domains increased from 64 at the initial epoch (t_0) to approximately 1,500 at the final epoch (t_f) for a 4-hour IOD arc. Each sub-domain required an independent propagation step at every time-step, causing total runtimes to approach a week on the study's computational resources. Due to these time constraints, ADS was applied only during the Initial Orbit Determination stage. Previous work by Wittig has shown that ADS propagation can substantially improve accuracy under strongly non-linear dynamics (e.g. short discovery arcs) over long-term propagation horizons; the omission of it therefore likely reduced accuracy within the uncertainty evolution, this is particularly relevant for precovery scenarios where propagation

horizons are long.

Longer propagation horizons were initially planned as part of the study. However, due to the short project timescale and the limitations of the available hardware, the maximum propagation period had to be reduced from two years to six months. This reinforces the scope of results as a feasibility assessment as operational precovery scenarios benefit from multi-year propagation horizons. The further one propagates, the more likely one would find a True Positive Candidate. As a result, results must be extrapolated cautiously if propagation horizons greater than 6 months is considered.

Finally, the limited computational resources limited the investigation to six arc lengths. More arc lengths would enable identification of a precise limit for the DA-based IOD.

In summary, the reliance on Python and the restricted computational resources shaped both the scope and depth of this study. The slower Python implementation constrained propagation horizons, limited the number of arc lengths tested, and eliminated the inclusion of ADS in the propagation phase. While these factors do not compromise the comparative findings among the four algorithms, they mean that the reported runtimes and scalability cannot be directly generalised to high-performance C++ implementations or to full operational precovery scenarios.

Future work should conduct a similar study in C++, through DACE, will enable fast computational times. This would enable the inclusion of ADS propagation, as well as longer propagation horizons to be investigated. Results from a DACE simulation would enable explicit comparison of results from outside this thesis. Computational improvements would enable the assessment of the ADS propagator.

6.2 Methodology

Several methodological choices constrain the direct application of the data to an operational precovery setting.

The archive image search was omitted due to failure to implement the SSOIS precovery tool for DA defined α, δ search region. This forces the author to conduct assessments based on alpha shape area and conclude a large search area results in higher false positive archival images. Moreover, failure to query the relevant image data bases means the query stage of the precovery pipeline cannot be assessed over the four algorithms.

The study relied on analogue models to assess the feasibility of DA to NEO precov-ery. While the work successfully implemented Initial Orbit Determination and uncertainty propagation, it did not include association of propagated states with archive images. As a result, the study only represents a part of the larger operational precovery pipeline. Future work would assess metrics such as the association of the orbital set (Pirovano, Santeramo et al., 2020, p. 1) or the Mahalanobis distance (Harris et al., 2024, p. 1). This would enable an end to end assessment of the precovery operational pipeline.

In summary, these methodological constraints mean that the present work cannot be applied directly to an operational precovery scenario. The (α, δ) search area assessment results in the exclusion of the Image query phase of the precovery pipeline, while an image-association stage is required to define an end to end pipeline performance. As a result, the findings presented here should be viewed as a feasibility demonstration of DA-based IOD and propagation rather than a ready-to-deploy precovery solution.

Future work should focus on the development of a DA compatible data association algorithm. This would provide an end to end simulation, and therefore results which are directly applicable to precovery operations.

6.3 Time Constraint

Only four months were allocated to complete this study, this meant simplifying assumptions were made that limit scope.

Taylor expansion order and ADS split tolerance were based on DA literature for a spacecraft use case. As such, the ADS split tolerance was on the order of meters and the maximum number of splits threshold was high to ensure meter level precision. This thesis adapted the ADS tolerance for a NEO use-case by a flat 100 times multiplication factor and halved the maximum number of splits to represent acceptable NEO position tolerances. Section ?? shows this was a valid assumption. However, a further study should be conducted to define an optimal expansion order and ADS tolerance for NEO geometry. This would optimise result precision and computational time (Wittig et al., 2015, p. 1).

Runge–Kutta 4 (RK4) integration was enforced as the baseline ODE solver without comparison to higher-order or variable time-step propagators. Use of a higher order ODE would ensure the Differential Algebra were the sole factor in the accuracy of the propa-

ated state. However, investigation of the correct ODE solver is not feasible within the given time frame.

Finally, the Admissible States Region was not included within the study. The results of the study show that the DA-IOD algorithm is not successful for observation arcs shorter than 4 hours. Therefore, it does not meet criteria 2 hour criteria for a short arc (Milani et al., 2004, p. 1). To further investigate DA's feasibility for short arcs, the admissible states region should be investigated (Pirovano, 2020, p. 1). The admissible states region applies DA to the admissible region method described in Chapter 2.3.3, and improves the computational performance over traditional methods (Pirovano, 2020, p. 1).

Ultimately the short project duration required assumptions. These assumptions were equally implemented across all algorithms. As a result, all precision penalties were equally applied across all algorithms. This ensures the conclusions drawn from comparative analysis are valid, however the data should therefore be applied cautiously outside the conditions tested.

Future work should build on the results. Strict deadlines prevented further investigation of results. For example, the DA-based IOD algorithms failed to conduct orbit determination of observation arcs less than 4 hours. This does not meet the 2 hour criteria for a short arc (Milani et al., 2004, p. 1). Future work should focus on investigation of the admissible states region Initial Orbit Determination which expands on the AR theory presented in Section 2.3.3 through DA (Pirovano, 2020, p. 1). The admissible states region is suggested as it avoids conventional IOD algorithms while being computationally efficient (Pirovano, 2020, p. 1).

Despite these limitations, the comparative results presented in this work remain valid within the defined assumptions. By addressing the integration of ADS into propagation, the optimisation of DA parameters for NEO geometries, and the development of a DA-compatible precovery association module, the approach demonstrated here can progress beyond a feasibility study toward an operationally deployable pipeline for NEO precovery.

Chapter 7

Conclusion

This thesis investigated the potential of Differential Algebra and Automatic Domain Splitting to improve the precovery of Near Earth Objects (NEOs). This study was driven by the need to have earlier, more confident, impact assessments. As a result, the feasibility of DA and ADS were assessed in terms of their effect on NEO uncertainty, computational cost and observation arc length required relative to traditional Monte Carlo methods.

Although, the DA based Initial Orbit Determination did not identify orbits from shorter arcs than the Monte Carlo simulation, the DA formulation defined a compact Taylor polynomial which could complete the initial stages of the precovery pipeline faster than the Monte Carlo method. When coupled with an ADS, DA was comparable to Monte Carlo propagation in terms of α, δ . In fact, the DAIOD+ADS reduced the rate of α, δ uncertainty growth with respect to propagation time. As a result, DAIOD+ADS solutions can be propagated further before the (α, δ) uncertainty region is larger than the coverage of an archival image. It was found this phenomena was the consequence of poor initial order determination of short arcs within the DAIOD+ADS algorithm, and long term propagation resulted DAIOD+ADS failure.

A key outcome was that DA methods do not enable initial orbit determination of arcs less than two hours (short arcs). In fact, Monte Carlo techniques and DA techniques failed for similar arc lengths. This means the DA methods do not improve precovery via observing shorter arcs, rather DA offers reduced computational time by roughly an order of magnitude compared to pointwise algorithms for equivalent propagation horizons. Although the required of α, δ splits increase DAIOD+ADS computational time relative to baseline DAIOD the additional cost was offset by a vastly improved accuracy. This enabled propagation lengths of 52.08 days DAIOD+ADS algorithm, where the DAIOD saw 3-5 days dependant on arc length. Furthermore, the computational performance of

DAIOD/DAIOD+ADS methods scales independently from arc length. This is advantageous for precovery operations where foresight of computational time can streamline operations.

Overall, this thesis concludes that DA-based ADS algorithms are a promising alternative to traditional Monte Carlo methods for precovery. They offer an improved trade-off between computational speed and uncertainty control, making them particularly attractive to the Planetary Defence scenarios where both variables are required. The findings suggest that future precovery pipelines could integrate DA methods to reduce arc-extension delays after discovery, thus enhancing the responsiveness of subsequent hazard assessments.

The future work should aim to further understand the impact of Automatic Domain splitting methods in the precovery pipeline. Specifically, a propagation ADS may enable the DAIOD+ADS algorithm to have the long term performance of the Monte Carlo algorithm. A study which assessed the impact of ADS parameters, such as truncation tolerance, and the effect on precovery should be refined. Optimisation of these parameters may see DAIOD+ADS have improved performance. Finally, the precovery pipeline should be completed through the investigation of data association algorithms. In this way, the end to end precovery pipeline can be assessed and the effect of DA and ADS can be studied holistically.

By demonstrating the practical feasibility and limitations of the DA and ADS in the precovery context, this research provides a foundation of future improvements in rapid NEO response. Ultimately, it contributes to more responsive and effective planetary defence infrastructure.

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Chapter 8

Appendix

8.1 Appendix A: CURES



15 May 2025

Dear Mr Dawson ,

Reference: CURES/25157/2025

Project ID: 28648

Title: Short Arc Orbit Determination

We are pleased to inform you that you have successfully declared that your research project is a **Literature Review – based solely on openly available literature which is in the public domain, and you are undertaking desk-based research not involving any other form of data or information.**

You have also confirmed that your project **does not meet** any of the literature review specified exceptions, listed both within the relevant section of the CURES form and below:

1) Your supervisor has requested that you apply for approval through CURES because:

- The journals or data are in a sensitive area (please discuss this with your supervisor)
- The project will be embargoed **i.e. will not be publicly available via the Cranfield library immediately or longer term**

2) Approval is/will be specifically required by another external body e.g. journal publishers

Therefore, **you do not require ethical approval** and your CURES application will be automatically closed.

Please keep a copy of this letter safe, if this exception is in relation to your thesis project, you will need to include a copy with your final thesis submission.

If you have any queries, please contact CURES Support.

We wish you every success with your project.

Regards,

CURES Team

8.2 Appendix B: Example of DA as Taylor polynomials

Suppose the function $f(x_0)$ is approximated as a Taylor polynomial $F(x_0)$ of the 3rd order. The Taylor Series dictates:

$$F(x_0) = f(x_0) + f'(x_0)(x - x_0) - \frac{f''(x_0)}{2!}(x - x_0)^2 - \frac{f'''(x_0)}{3!}(x - x_0)^3$$

If $f(x_0) = \sin(x_0)$ the Taylor Polynomial $F(x_0)$ is equivalent to:

$$F(x_0) = \sin(x_0) + \cos(x_0)(x - x_0) - \frac{\sin(x_0)}{2!}(x - x_0)^2 - \frac{\cos(x_0)}{3!}(x - x_0)^3$$

Suppose x_0 is initiated as a DA variable of order 3. Where x_0 is the nominal value, and δx_0 is the variation from the nominal solution, $\delta x_0 = (x - x_0)$. By definition (Martin, 1999, p. 1):

$$[x_0] = x_0 + \delta x_0 + 0\delta x_0^2 + 0\delta x_0^3 = x_0 + \delta x_0$$

Substitution of $[x_0]$ into $F([x_0])$ yields $F([x_0]) = F(x_0 + \delta x_0)$. Equation 8.2 becomes

$$F(x_0 + \delta x_0) = \sin(x_0) + \cos(x_0)\delta x_0 - \frac{\sin(x_0)}{2!}\delta x_0^2 - \frac{\cos(x_0)}{3!}\delta x_0^3$$

Where $\sin(x_0)$ represents the function around nominal point, while $\cos(x_0)\delta x_0 - \frac{\sin(x_0)}{2!}\delta x_0^2 - \frac{\cos(x_0)}{3!}\delta x_0^3$ represents the variational parts. Equation 8.2 is commonly written in equation 8.2. Where $\tau_f^k(x_0)$ represents the Taylor polynomial of the function f with respect to x_0 at expansion order k . In this example $f = \sin(x_0)$ and $\tau_f^k(x_0)$ is defined by equation 8.2, where expansion order $k = 3$.

$$F(x_0 + \delta x_0) = \tau_f^3(x_0)$$

The initial first order DA variable, $[x_0] = x_0 + \delta x_0$, is promoted to the 3rd order through the Taylor polynomial function $F([x_0])$.

If $x_1 = x_0 + 0.2$ by inspection $\delta x_0 = 0.2$. Substitution of $\delta x_0 = 0.2$ into equation 8.2 yields

$$F(x_0 + 0.2) = \sin(x_0) + \cos(x_0)0.2 - \frac{\sin(x_0)}{2!}0.2^2 - \frac{\cos(x_0)}{3!}0.2^3$$

Therefore $F(x_f)$ is:

$$F(x_1) = \sin(x_0) + \cos(x_0)0.2 - \frac{\sin(x_0)}{2!}0.2^2 - \frac{\cos(x_0)}{3!}0.2^3$$

Comparing equations 8.2 and 8.2, a variation of an output with respect to x_0 is defined through a simple evaluation of $F([x_0])$ where $\delta x_0 = 0.2$.

More generally and written in DA notation:

$$[x_f] = \tau_{x_f}^k(x_0)$$

Where τ is the Taylor Polynomial expansion of function x_f expanded around x_0 with order k . Besides the initial conditions, $\delta p = 0$, the solution $[x_f]$ can be expanded with respect to any parameter δp that appears in the dynamical model. Thus, one can assess the variation of δp on the solution through a single evaluation step.