

CRANFIELD UNIVERSITY

BENJAMIN GRIFFIN

DEVELOPMENT OF A COOPERATIVE MULTISTATIC
RADAR NETWORK BASED ON THE AVEILLANT
HOLOGRAPHIC RADAR

SCHOOL OF CRANFIELD DEFENCE AND SECURITY
PhD In Defence and Security

PhD
Academic Year: 2018–2021

Supervisors: Dr A. Balleri and Prof C. Baker

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November 11, 2022

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Abstract

Staring radars use a transmitting static wide-beam antenna and a directive digital array to form multiple simultaneous beams on receive. Because beams are static, the radar can employ long integration times that facilitate the detection of slow low-RCS targets, such as drones, which present a challenge to traditional air surveillance radar. Typical low-altitude trajectories employed by drones often result in low-grazing angle multipath effects, which are difficult to mitigate with a monostatic radar alone. Using multiple spatially separated receivers cooperating with the staring transmitters in a multistatic network allows multi-perspective target acquisitions that can help mitigate interference from the environment and improve the detection and localisation of drones. The term drone refers to small unmanned aerial systems (UASs) with a radar cross-section (RCS) of approximately 0.01 m^2 , such as the DJI Phantom/Inspire series.

This thesis presents the development of a cooperative radar network. The radar network consists of a staring transmitter and multiple bistatic passive receivers. The advantage of a staring transmitter is that the network can use long integration times to detect slow and low RCS targets such as drones. The staring transmitter ensures a constant signal at the receiver nodes within the radar's coverage and, therefore, can be simplified as they do not need to employ pulse-chasing techniques. Target position and velocity estimation algorithms have been developed for the cooperative radar network. The theoretical lower bound on the error of estimation algorithms is

used to determine the estimation performance of the radar network and is further used to optimise the network's geometry by minimising estimation error.

As part of the thesis, a passive bistatic dual-channel receiver has been developed and employed in a bistatic configuration with a staring transmitter of opportunity to test the theoretical algorithms and measure a drone in flight. The receivers are designed along passive radar principles and consist of two receive channels: a surveillance channel aligned with the transmitter and a reference channel aligned with the target. The receiver employs horn antennas with a wide 90° degree beamwidth and has been used to perform laboratory measurements of a rotating fan, the micro-Doppler signature of a remote-control helicopter, a moving car and a drone in flight.

Keywords

Radar Networks, Multistatic, Estimation, Drones, Aveillant Holographic Radar, Cooperative Network, Optimisation

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List of Abbreviations

CDS	Cranfield Defence and Security
UAV	Unmanned Aerial Vehicle
FIM	Fisher Information Matrix
CRLB	Cramér-Rao Lower Bound
BFIM	Bayesian Fisher Information Matrix
BCRLB	Bayesian-Cramér Rao Lower Bound
MLE	Maximum Likelihood Estimator
MAP	Maximum a Posteriori Estimator
FFT	Fast Fourier Transform
STFT	Short Time Fourier Transform
RDM	Range Doppler Map
RVM	Range Velocity Map
SNR	Signal to Noise Ratio
AOA	Angle of Arrival
RAM	Radar Absorbent Material
RMSE	Root Mean Square Error
GPS	Global Positioning System
UTC	Coordinated Universal Time
PRF	Pulse Repetition Frequency
PRI	Pulse Repetition Interval
RCS	Radar Cross Section

WSS	Weak-Sense Stationary
LTI	Linear Time-Invariant
AF	Ambiguity Function
LFM	Linear Frequency Modulation

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Contributions

Conference Papers

1. Optimal Receiver Placement in Staring Cooperative Radar Networks for Detection of Drones [1]
2. Advanced Cognitive Networked Radar Surveillance [2]
3. Prototyping a Dual-Channel Receiver for use in a Staring Cooperative Radar Network for the Detection of Drones [3]
4. Development of a Passive Dual Channel Receiver at L-Band for the Detection of Drones [4].

Journal Papers

1. Target Parameter Estimation and Optimal Receiver Placement in Cooperative Multistatic Radar networks with Staring Radars , *In preparation for IEEE Transactions on Aerospace and Electronic Systems.*

Chapter 1

Introduction

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1.1 Motivations

The robust detection of drones is an active area of research in the radar domain. Due to their small size, low flight profile and slow speed, drones present a unique challenge to traditional air surveillance radars. Studies into the use of radar networks are receiving increased interest as they have the potential to improve the detection of drones and drone-like targets [5]. A proposed list of capabilities for a radar system to be able to detect and track drones has been presented by Poitevin et al [6].

- Sensitivity for very small detectable signature
- Fast refresh rate
- Height measurement capability
- Large volumetric coverage requirements
- Nuisance elimination
- System cost

Staring array radar have been built that can detect small drones out to a range of 5 km and satisfy the sensitivity, fast refresh rate and height measurement capability [7]. The Aveillant Gamekeeper radar, shown in Figure 1.1, is a staring array radar operating in the L-Band designed to detect drones [8].



Figure 1.1: Aveillant Gamekeeper Radar

A cooperative radar network consists of a central staring radar and multiple passive bistatic receivers. The passive receivers use the transmissions of the central radar as a signal of opportunity.

The staring nature transmitted signal means that the passive receivers can record the reference signal without the need for pulse-chasing, greatly simplifying the receiver design and cost. The use of multiple spatially separated receivers cooperating with the staring transmitters in a multistatic network allows multi-perspective target acquisitions that can help to mitigate interference from the environment and improve the detection and localisation of drones. The spatially distributed nature of the networks aids the mitigation of signal multipath. As the receivers are spatially separated if the target's signal is fading in one receiver it is unlikely to be fading in the other receivers making the network more robust.

1.2 Objectives

The objective of this research is to investigate the performance of a cooperative radar network based on the Aveillant Gamekeeper radar. The proposed network consisting of a central staring array radar (Aveillant Gamekeeper) and multiple bistatic passive receivers.

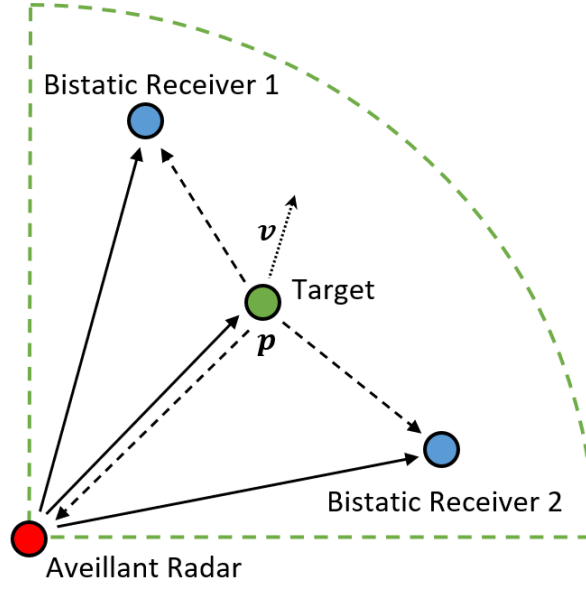


Figure 1.2: Cooperative Radar Network

The cooperative radar network is shown in Figure 1.2, the figure shows the central Aveillant radar, two bistatic passive receivers and a single target. The Aveillant radars field of regard is denoted by the dashed green line the transmitted signal is illustrated by the solid black arrows and the reflections from the target are illustrated by dashed black arrows.

There are three main parts to the research, theoretical, simulations and experimental measurements. The objective of the theoretical work is to develop a mathematical model of the radar network and drone. The theoretical work will focus on the detection and estimation performance of the network. Optimisation algorithms based on the model will be developed to determine the optimal configuration of the network and maximise its performance. The purpose of the simulations is verify the performance of the found optimal radar networks. This will be achieved by running Monte-Carlo simulations. Finally, experimental measurements will be used to further validate the results of the theoretical work and simulations. This will involve the construction of two low-cost passive receivers. The receivers will be designed to operate with the Aveillant Gamekeeper radar.

1.3 Achievements

Algorithms *Deterministic Target State*

Determined the CRLB for estimating a targets state using the range and Doppler measurements from a multistatic radar network. As the found CRLB is dependent on the network geometry, the lower bound can be used as a performance metric to optimise the network. Monte-Carlo simulations of the maximum likelihood estimator validate that the optimal network geometry attains the theoretical bound.

Algorithms *Random Target State*

The model was extended to consider a target modelled by an unknown random vector, generalising the previous work. The optimal geometry can then be defined against a probability distribution rather than a single state. The model was also extended to consider the effects of random amplitude fluctuations due to signal multipath effects. Monte-Carlo simulations of the MAP estimator were used to verify the performance of the optimal geometry.

Hardware *Dual Channel Passive Receiver*

Two dual-channel passive receivers have been designed and built, see Fig.1.3. These two receivers are designed with two channels a reference channel and a surveillance channel. The reference channel is used to record the signals from the Aveillant Gamekeeper radar, and the surveillance channel observes the target. Laboratory measurements have been taken of a rotating fan and a remote control helicopter's rotors. Larger scale measurements have been taken of a moving car and a drone in flight. These measurements employed the use of GPS to generate the target truth files necessary for the analysis of the receivers' performance.

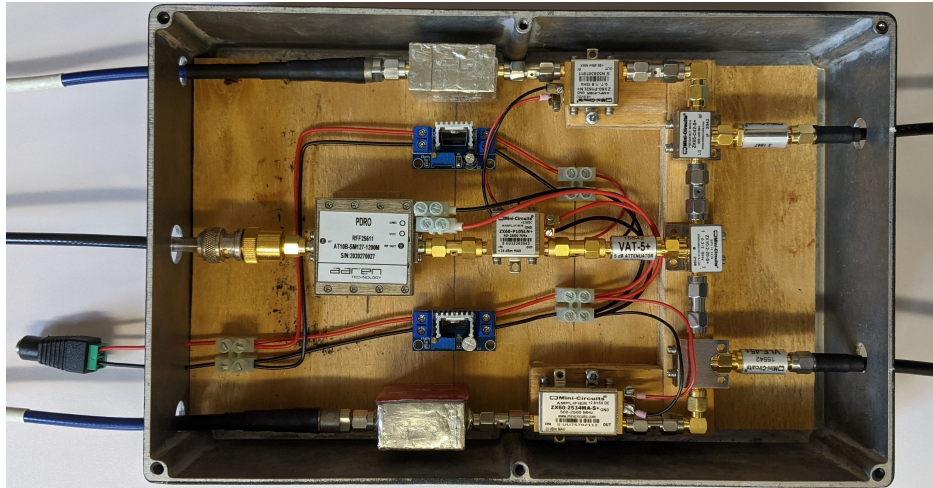


Figure 1.3: Dual Channel Passive Receiver Hardware

1.4 Thesis Structure

Chapter 1 - Introduction: This is a general introduction to the aims of the thesis and provides the initial motivation behind the project. A summary of the projects achievements is also presented.

Chapter 2 - Literature Review: The literature review explores the current state of the art of the literature focusing on Drone detecting radar, Radar networks, Network optimisation and the effects of signal multipath.

Chapter 3 - Background Theory: The background theory necessary for the development of the main results of the thesis is presented. The basics of Pulse-Doppler signal processing, random process and estimators are covered. Particular attention is given to the problem of estimating the time delay and Doppler shift from time continuous signal embedded in thermal noise.

Chapter 4 - Aveillant Holographic Radar: This chapter covers the design of the Aveillant Holographic radar and introduces the Gamekeeper radar, a staring array radar built by Aveillant for the detection of drones. Chapter 4 concludes with the fitting of a Gaussian pulse to a Square pulse used by the Aveillant radar. This is necessary due to the undesirable shape of the ambiguity function of the square pulse.

Chapter 5 - Network Algorithms: This chapter contains the main contributions, two network optimisation algorithms are presented. These algorithms minimise the joint estimation error of the target's position and velocity. The first algorithm is a distributed scheme in which the target's position and velocity are modelled as an unknown but constant parameter. Under the distributed scheme, the targets range and Doppler is estimated at each receiver. The range and Doppler measurements are centrally gathered, and the target's position and velocity are estimated from the

gathered measurements. The theoretical minimum covariance of this estimator is calculated and used to define a cost function describing the estimation performance of the network. The radar network geometry is optimised by minimising the cost function with respect to the positions of receivers and transmitters in the network. The performance of the optimal geometries is verified using Monte-Carlo simulations of the Maximum Likelihood Estimator.

The second algorithm extends the first by modelling the target's position and velocity as random vectors. The model is extended to consider fading across the network due to signal multipath. The extension to random vectors necessitates a centralised fusion process in which the raw data from all receivers in the network is centrally gathered and the target's position and velocity directly estimated. In a similar manner to the first estimator, the theoretical minimum covariance of this estimator is calculated and used to define a cost function describing the estimation performance of the network. The radar network geometry is optimised by minimising the cost function with respect to the positions of receivers and transmitters in the network. The performance of the found optimal geometries is verified using Monte-Carlo simulations of the Maximum a Posteriori estimator.

Chapter 6 - Dual Channel Passive Receiver: Chapter 6 covers the development of the dual channel passive receiver detailing the design principle and physical hardware. Measurements have been taken in the lab of a Remote Control Helicopter and a Rotating Fan. More complex measurements of a moving car and a drone in flight have also been carried out. These trials used GPS to correlate the targets position with the recorded data.

Chapter 7 - Hybrid Simulations: Chapter 7 combines data recorded by the Aveillant Gamekeeper radar with simulated data in a hybrid simulation. The aim

of this chapter is to combine the theoretical and experimental measurements to realistically model a cooperative radar network.

Chapter 8 - Conclusion: Chapter 8 draws an overall conclusion of the thesis and details the main contributions. The successes and limitations of the Dual-Channel receiver are reiterated, and a development path is suggested. The chapter concludes with a short summary of the future work.

Chapter 2

Literature Review

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Introduction

Small drones, such as the popular DJI series, have become widely used and easily obtained in recent years. Many of these are used for recreation and enjoyment. Others are used commercially, mainly for aerial photography and surveying. In the UK, one of the most widely covered incidents involving the misuse of drones occurred over Gatwick Airport in December 2018, when drones flew over and around the airport and saw flights cancelled for over 24 hours, causing widespread disruption and a significant financial impact.

Anyone wishing to operate a drone between 250 g and 20 kg in the UK has to register with the Civil Aviation Authority (CAA). Drones that use GPS also now have pre-programmed software, which prevents the drone from exceeding altitude limits and flying into sensitive areas such as airports, power plants or military bases. These rules only apply to drones sold in the UK, and while these rules may prevent accidental misuse, they do not prevent deliberate intrusions from more sophisticated actors.

Radars are all-weather sensors that are widely employed as surveillance systems in both commercial and military environments. Detecting drones is a significant challenge for traditional air surveillance radars; as such, the latest modern technology is needed to robustly detect and mitigate against the potentially dangerous misuse of drones. Radar networks can combine multi-perspective target acquisitions that can help mitigate interference from the environment and improve the detection and localisation of drones.

This review summarises the current research on detecting and estimating the properties of drones using radar and radar networks. The review indicates where the cooperative radar network concept can improve both detection and estimation performance.

The review starts with a look at state of the art for drone detecting radar and the characteristics suggested in the literature that a radar system should possess in order to detect drones successfully. Secondly, the review looks at developments in the field of radar networks, both active and passive networks. Thirdly the review looks at radar network optimisation techniques. Finally, the review looks at the effect of signal multipath on the received signals and how a network can mitigate these effects.

2.1 Drone Detecting Radar

The physical characteristics and flight profiles of drones make them a challenge to monitor in all domains, not only radar. Popular consumer multirotor drones such as the DJI series occupy a volume of 0.5 m^3 and are typically modelled as point targets with a RCS of 0.01 m^2 [9] [10]. Drones are similar to birds in both size and flight characteristics; drones are also capable of precise flight close to the ground. These similarities mean that drones and birds have a similar RCS and are likely to be misidentified as birds. Drone flight regulations in the UK limit the operation of drones to an altitude of 120 m. As drones can fly precisely low to the ground, the reflected radar signal will suffer interference from a strong ground reflection, as well as low angle grazing multipath. Multirotor drones are capable of slowly and hovering very precisely. The returned signal from a slow-moving target has a small Doppler shift; therefore, the return signal from the drone is likely to be masked by static and slow-moving ground clutter; all of the above combine to produce a challenging radar target.

A comprehensive list of required capabilities for a radar system to be able to detect and track drones is presented by Poitevin et al. [6].

- Sensitivity for very small detectable signature
- Fast refresh rate
- Height measurement capability
- Large volumetric coverage requirements
- Nuisance elimination
- System cost

The small size of drones necessitates a radar with high sensitivity. Drones, in particular multirotor drones, can be launched and operated from almost everywhere and therefore can appear at short distance leaving little time to react, a radar system operating at a high refresh rate can reduce the overall response time.

Nuisance elimination is defined as a radar's ability to remove unwanted reflections from the environment. The problem of clutter and thermal noise is a universal problem in radar and has been thoroughly investigated in the literature. For the drone problem, many elements contribute to clutter. Precipitation, terrain, urban landscape, sea surface, ground moving targets and birds all constitute clutter that will mask the target's signal and/or create false alarms. The similarities between modern drones and birds mean that discriminating between the two is not an easy task. In general, the small signatures and high manoeuvrability of modern drones are real challenges when designing a radar system. Poitevin et al suggests that a counter drone system requires a combination of sensors to be effective [6].

Continuous Wave Radar

Continuous Wave (CW) radar and Frequency Modulated Continuous Wave (FMCW) radars are able to put more energy onto a target than an equivalent pulsed radar,

thus improving the received target SNR and radar performance [11] [12]. By itself a CW radar unable to measure a targets range only the Doppler shift. FMCW radar applies additional periodic modulation to the carrier frequency of the transmitted signal. Measurements of a drone using an FMCW ubiquitous radar at X-band has shown detection ranges of 2 km and furthermore estimated the statistical properties of a DJI Phantom drone in flight, validating the drone as a Swerling 1 target with an average RCS of 0.01 m^2 [13].

Dogru et al. present an interesting approach in which target drones are detected and pursued by another drone [14]. This pursuer drone actively tracks a target drone using radar measurements. The airborne approach allows tracking a drone operating in non-line-of-sight and at longer distances than the detection range of a static system. Furthermore, the pursuer drone can follow the drone to determine its landing point, potentially helping to identify the operator. The sensor used is a commercial off the shelf (COTS), Frequency Modulated Continuous Wave (FMCW) radar employing a linear receiving array, the radar measures the targets range, Doppler-shift and azimuth. Results show that the pursuer drone successfully detect and start to follow the target drone at short ranges of 25 m. The authors developed the system for a single pursuer drone, this work can be potentially scaled to consider multiple pursuer that are able to share data and operate as moveable nodes in a network, improving coverage.

Staring Radar

Staring array radars also known as ubiquitous radars have been built that can detect drones with an RCS of 0.01 m^2 out to a range of 5 km [7]. The staring array radar has satisfies the sensitivity, fast refresh rate, and height measurement capabilities [7].

Staring array radar systems are characterised by a fixed transmitter that continu-

ally covers the radars field of regard. Compared with a rotating scanning radar, the staring radar has an effective 100% dwell time, putting more energy onto the target, thus improving SNR and performance. However, the fixed nature of the design is limited as multiple antenna faces are required to cover 360° around the radar; as such, currently deployed systems are limited in azimuth [7].

As the transmitter is fixed, scanning the environment is done by forming beams using a receiver array. With these fixed arrangements of beams, the sensor can continuously stare in all directions simultaneously, thus enabling very fine Doppler resolution via long and controllable dwell times. This fine Doppler resolution allows the detection of slow-moving targets against a background of high levels of stationary clutter encountered by all "near-to-ground" surveillance radars. The high-resolution sensing also exposes the micro-Doppler, which aids classification [15].

As the transmitter in a staring radar is required to cover the radars field continuously, the transmitter antenna gain will be considerably less than a scanning equivalent that can focus its beam on the currently selected point of interest. In theory, the longer integrations times will make up for the relative loss in transmitter gain. A subsequent consequence of longer coherent integration times is the substantial volume of data required to be processed; note that longer integration times requires longer Fourier transforms, necessitating more powerful hardware. The problem of processing such large data volumes in real-time is straightforward using current GPU processor technology [7]. Field programmable gate arrays (FPGA) can allow real-time radar signal processing such as digital beamforming and FFT [16].

Multiple experimental measurements of drones have been carried out using the Aveilant radar, the range of trials and scenarios are diverse and include drone flights over the ocean (sea clutter) and field trials conducted in a rural environment over flat

terrain with clusters of trees [15] [17].

2.2 Radar Networks

This section of the review looks specifically at the developments of radar networks, to detect drones. Broadly speaking radar networks can be split into two groups, active and passive. The defining characteristic between passive and active systems is the nature of the transmitted signal used. In passive radar the system doesn't have control over the nature of the transmitted signal, the term signal of opportunity highlights this. Conversely, an active system has control over the transmitter and transmitted signal, note that such systems can use multiple transmitters.

Formally the IEEE Standard for Radar Definitions defines bistatic radar as "A radar using antennas for transmission and reception at sufficiently different locations that the angles or ranges to the target are significantly different" and multistatic radar as "A radar system having two or more transmitting or receiving antennas with all antennas separated by large distances when compared to the antenna sizes" [18].

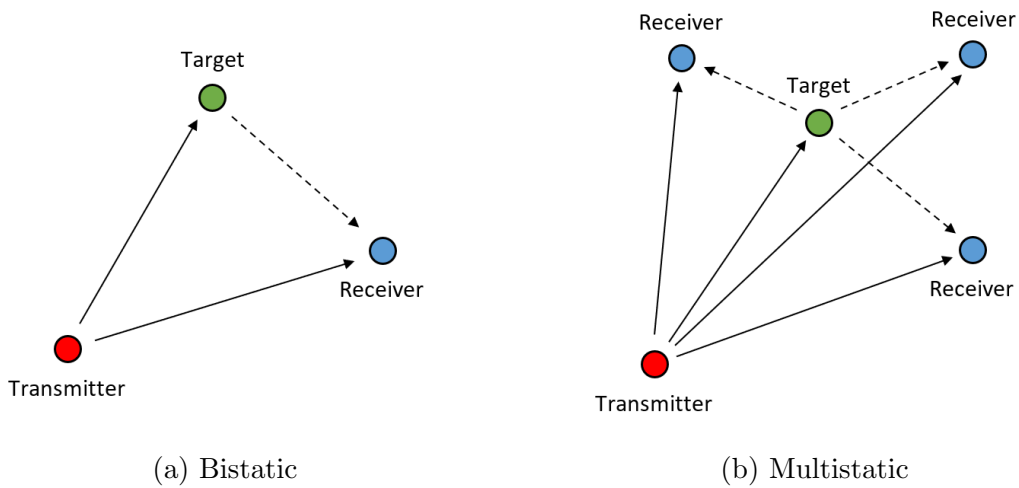


Figure 2.1: Illustrations of Bistatic and Multistatic Geometries

Fig. 2.1 illustrates both bistatic and multistatic geometries. The direct signal path

and reflected target signal paths are shown in both diagrams. Fig. 2.1a shows a bistatic radar consisting of a single transmitter, receiver and target. Fig. 2.1b shows a multistatic radar network consisting of a single transmitter, multiple receivers and a single target. Multistatic networks can also consist of more than one transmitter. Multistatic radar networks consisting of multiple transmitters and receivers have some clear benefits; firstly, a network will provide additional information to detect and track targets. Whilst a single geometry may suffer from obscuration or fading, it is unlikely to be the case with multiple, different transmitter-target-receiver propagation paths.

In all types of radar networks, the multiple signals gathered by the receiving nodes must be combined to generate a final result, a process known as data fusion. The types of fusion used depends on the design of the network. Central fusion requires all the data received by the network to be sent to a central processor, where all the radar signal processing is performed. Conversely, a distributed network performs data processing locally at each receiver node, and only a subset of the received data is sent to a central processor.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Location	Fixed	Fixed	Fixed	Fixed	Fixed and moving platforms	Nodes on moving platforms
Data level	Tracks	Tracks	Detections	Detections	Raw	Raw
Coherency	Incoherent	Incoherent	Incoherent	Coherent	Coherent	Coherent
Operation mode	N Tx, N Rx monostatic	1 Tx, N Rx multi-static	1 Tx, N Rx multi-static	M Tx, 1 Rx multi-static	M Tx, 1 Rx multi-static	M Tx, N Rx multi-static
Distribution	De-centralised	De-centralised	Semi De-centralised	Centralised	Centralised	Centralised
Assessment	Straight-forward	Multiple bistatics	Challenging	Complex	Very complex	Extremely complex

Figure 2.2: Development of Radar Networks (after Baker)

Baker and Griffith's detail how the development of a multistatic radar network could be achieved [5]. In Case 1, the tracks from several conventional, fixed monostatic radars are fused, which is straightforward. Case 2 is similar but with a single transmitter and several multistatic receivers. In Case 3, the data fusion is performed at the detection level rather than tracks. The level of complexity (and feasibility with present capabilities) increases at each stage. Case 6 has multiple transmitter nodes and multiple receiver nodes on moving platforms, processing the echoes coherently and delivering a high-resolution system as shown in Fig. 2.2.

Griffith's presents an itemised list of the advantages of a employing radar network [5, 19]. The relevant points are repeated here,

- Bistatic and multistatic radar have potential advantages in the detection of targets which are shaped to scatter energy in directions away from the monostatic.
- Many of the synchronisation and geolocation problems that were previously very difficult are now readily solvable using GPS.
- The extra degrees of freedom may make it easier to extract information from bistatic clutter for remote sensing applications.

Spatial diversity is the ability of a network to observe the target from multiple different aspect angles, increasing the amount of information the network receives on a target. In a particular example, the authors demonstrate that a multistatic system consisting of two receivers and two transmitters achieves the equivalent performance as four monostatic radars [20]. Performance, in this case, is the mean square error of the target location and velocity measurements. Ivashko et al. shows that given the same number of transmitting and receiving channels, a multistatic radar system possesses better measurement accuracy than a monostatic one regardless of the geometry of system nodes [20]. The minimum variance is computed for location and

velocity estimation using time of arrival and Doppler measurements for a monostatic and multistatic radar. The authors note that the multistatic system variance for target location and velocity estimation is smaller than the monostatic system, given the same number of channels. These results demonstrate one of the main advantages of a multistatic system, spatial diversity.

A target moving in a circular trajectory around a monostatic radar will appear in the zero Doppler region as the target's range is constant, and the returned signals will be masked by stationary and slow-moving clutter. A network with multiple spatially separated receivers can obtain multiple target acquisitions. For a moving target, at least one of the receivers in the network will detect the target with a non-zero Doppler shift. The main body returns from a hovering multi-rotor drone will be masked by stationary clutter. Advances in the detection and characterisation of the micro-Doppler signature means that drones can potentially still be tracked through the zero-Doppler clutter by associating the returns from the rotors with the returns from the main body. The small RCS from the rotor blades requires a high signal to noise ratio to detect them. The NetRad radar has detected and characterised the micro-Doppler signature of several quadrotor drones at ranges of up to 150 m [21].

A radar network by nature exhibits graceful degradation. If a single receiver in the network fails, the network coverage is reduced but not eliminated. Such a requirement is essential for critical systems which are required to function 100% of the time. Such requirements are common in the defence domain. Counter drone detection at airports is another example [22]. In the case of the sensor network, not only may the loss of one node of the network be tolerated, but also the network may be reconfigured accordingly in response.

A major challenge in bistatic radar is the direct signal interference, reflected signals

from the target can be buried under the sidelobes of the direct signal. A simple solution is to place the surveillance so that it is physically shielded from the direct path signal, using topography, buildings. If physical shielding is not possible radiation-absorbent material (RAM) can be used to shroud the surveillance channel.

A popular technique is to site directly between the transmitter and the surveillance region, the transmitter is then “looking over the receiver’s shoulder” [23]. If the surveillance antenna uses an array antenna then null-steering can be used to suppress the direct signal. These techniques combined with shielding from a large building directly behind the receiver was used in Howland’s FM radio-based bistatic radar NATO trials [24]

2.2.1 Passive Radar Networks

Passive radar sometimes referred to in the literature as passive bistatic radar (PBR), has become very popular in recent years, with a significant proportion of drone detecting radar research being in the passive radar network field. The use of passive radar to detect drones is an area of research that has seen much focus. This can be attributed to the availability of global positioning systems, namely, GPS, GLONASS, and Galileo, which can be used to solve synchronisation and location issues [19]. Since passive radar does not require a transmitter, these sensors offer several exciting advantages: low cost, reduced energy consumption, and covert operation. Passive radar has the advantage of not requiring space on the electromagnetic spectrogram; such systems are often referred to as green systems, ideal for urban environments [25].

An important benefits of passive radar is the significant number of signals available to passive receivers. These signals are known as signals of opportunity. Several

examples of commonly used signals are:

- Radio
- Television broadcast
- Mobile basestations
- WiFi
- Satellite communications
- Radio navigation

These signals can be characterised by their power density at the target, their coverage (spatial and temporal), and their waveform properties (range and Doppler resolution, range and Doppler ambiguities and sidelobe levels), via the ambiguity function. In general, such waveforms are not ideal for radar purposes, which given their design purpose is not radar should be expected. In particular, analogue broadcast modulation formats depend on the type content being broadcast, speech or music, and maybe strongly time-varying. Alternatively, digital modulation formats, which are currently being introduced in many countries as replacements for analogue radio and television broadcasting, are much more suitable since their signals are more constant with time and therefore have more suitable ambiguity functions [5]

Passive radar systems using WiFi signals in both 2D and 3D are studied by Milani et al. [26] [25]. The authors note that a passive radar is a green system ideal for urban area monitoring where the abundance of signals facilitates greater coverage. The wide bandwidth of the WiFi signals provides a good range resolution. A limitation of WiFi signals is low transmission power and subsequent short detection ranges. The setup employs three antennas with vertical and horizontal displacement employing interferometry. The system uses two horizontally displaced antennas to determine the target's position in the 2D plane and a third vertically offset antenna

to determine the elevation.

Detection of larger targets at greater ranges is possible using WiFi signals [25]. Using a similar set-up to the above, an ultralight aircraft is detected continuously along its trajectory up to a distance of about 100 m. Coverage and detection performance may be improved by using more than one WiFi base station.

Digital Terrestrial Multimedia Broadcast signals offer better coverage and are less likely to suffer from interference than WiFi. A passive bistatic radar system using digital television signals operating at 738 MHz as the transmitted signal is studied in [27]. This radar consists of a linear surveillance array and a separate reference antenna, and a DJI Phantom 4 employed as a target. The baseline distance between the transmitting tower and the receivers is 7.5 km, and the drone is successfully detected when flown at ranges of 50 m to 100 m from the surveillance receiver.

2.2.2 Active Radar Networks

Active radar networks consist of controllable transmitters and receivers. An active radar network has a number of advantages over a passive system. These primarily relate to the advantages of having control over the properties of the transmitted signal.

The ambiguity function of the complex envelope of the transmitted signal determines the range and Doppler resolution. With control of the waveform design, the signal ambiguity function can be shaped to the desired specifications, unlike in a passive system where by definition, the ambiguity function of the signal of opportunity is optimised for a different task.

Furthermore, the antennas used to broadcast the signals used by the passive receiver are often designed to focus the energy down towards ground based receiving anten-

nas. In surveillance radar, the desire is usually to focus the transmitted signal up at targets and avoid pointing the antennas at the ground to avoid interference from ground clutter. As the transmitted signal is known, the receivers can potentially operate without a surveillance channel requiring precise knowledge of the transmission times and synchronised clocks across the network. Future developments in quantum oscillators and atomic clocks may enable this form of operation.

The NetRAD radar is small scale active radar that has been used to look at drones, due to the high gain antennas used in the radar the network has a limit surveillance area. In the case of active networks, they are primarily smaller e.g UCL NetRad system which is only a bistatic radar system, [28] mentions a limited surveillance region. There is limited research into active radar networks in academic literature, the majority of drone detecting radar networks are passive.

NeXtRAD is the successor to the NetRAD system. While NetRAD uses cables to link the receivers to the central transmitter, NeXtRAD uses quartz-based GPS synchronised oscillators, allowing increased separation between nodes. The decoupling between the transmitter and receiver nodes introduces complexity to the design, furthermore the level of coherence attainable between the nodes will be reduced.

In general there is less focus on active radar networks in the literature, likely due to the added complexity and in particular increased costs over a passive system. Also, unlike passive radar systems an active system requires bandwidth in the already very congested electromagnetic spectrum. However, there are significant advantages, primarily in control over the waveform used as well as the coverage and power transmitted. In particular active networks have been built with the sensitivity to detect and characterise the micro-Doppler signature of drones and birds [28].

2.2.3 Cooperative Network

Define a cooperative radar network as a hybrid combination of active and passive networks in which multiple bistatic passive receivers operate alongside a central staring monostatic radar. The use of a staring transmitter ensures a constant signal which means that the passive receivers can always continually record the reference signal regardless of their position within the radars field of view, without any form of pulse-chasing, simplifying the receiver design. This also the receivers to use long integration times, improving the Doppler resolution which in turn aids the detection of slow moving targets against clutter.

Using a central monostatic radar as the signal of opportunity for the passive receivers means that the properties of the transmitted waveform are much better suited to detect targets, unlike other signals of opportunity, which by nature are designed for other purposes. Furthermore, the passive receivers will benefit from the coverage of the monostatic radar.

Passive receivers that use a dual-channel design with a reference channel and a surveillance channel can operate coherently without the requirement of a shared synchronisation signal between the network nodes, which is one of the limitations of traditional multistatic radar network solutions. The receiver nodes can self-synchronise using the transmitted signal to enable coherent processing of the received signal from the target, reducing the requirement for receivers to maintain synchronisation with the transmitter, which simplifies the system design. This also means there is no need for high precision clocks or separate timing references at the receivers.

Initial measurements of drones using passive receivers and the transmissions from a staring radar have been reported [29]. In these measurements, transmissions from an Aveillant Gamekeeper radar were used as signals of opportunity by a passive bistatic

receiver, the baseline distance between the Gamekeeper radar and the passive receiver was approximately 635 m. The passive receiver used here employs a dual channel design with a reference channel oriented towards the Gamekeeper radar and a surveillance channel oriented towards the target drone. The drone was detected at close range, within 100 m, and returns correlated with the drones on-board GPS log. Here the GPS log was used to obtain Cartesian coordinates for the targets as a function of time and the together, with the known coordinates of the transmitter and passive receiver, the Doppler-shift and radial velocity of the targets is estimated.

2.3 Network Optimisation

This section of the review looks at how multistatic radar networks can be optimised to improve detection and estimation performance. This is a very broad problem as many different parameters can be tuned, for example, the geometry of the network, the properties of the transmitted waveform, and how data is fused across the network. An optimisation problem is always defined with respect to a cost function, in the following section consider how data fusion, detection theory and estimation theory can be used to define the optimal radar network.

Data Fusion

A key choice in the design of a network is the manor in which signals from the receivers are combined across the network. Broadly these can be split into two categories, centralised or distributed scheme. Under a centralised fusion scheme all raw sensor data is sent to the central processor, alternatively under a distributed scheme data is processed at each receiver and a subset of the received data is sent to the central processor, significantly reducing the volume of data that the central processor needs to process. In practise the volume of data may be to great to be processed or even to be transferred across the network at high speeds, therefore

distributed processing is more widely used. A radar network operating under a centralised scheme might send the raw unprocessed complex time samples from the receivers to the central processor. A radar network operating under a distributed scheme might only send tracks to the central processor.

With the increase in the computational power of modern integrated circuits, data bandwidth and processing power are becoming less of a limiting factor in radar systems. In particular, graphics processing units, GPUs, have brought increased computational capabilities at a lower cost and scale. Bernaschi et al. developed a multi-channel, mobile dual-band passive radar system using GPUs to enable real-time processing in a compact and energy-efficient system [30]. Field-Programmable Gate Arrays, FPGAs, are highly configurable and capable of high data throughput, making complex radar systems achievable at a low cost allowing for their adoption by academia [31]. These advances in hardware can allow for deep levels of data fusion between network nodes and as noted by Baker and Griffith's can improve performance [5].

In practise the two most common ways of combining radar data are detection-to-track and track-to-track:

1. Detection to track fusion associates each detection to a central track, calculated using all detections from all the receivers. Therefore the entire stream of detections is potentially available to calculate the track state used to associate the next detections.
2. Track to track fusion, associates each detection to a single radar track state calculated using only detections from each receiver in the network. The single radar track states are then grouped to produce a fused track.

An example where detection-to-track fusion is preferred is when the receivers in

the network have a reduced probability of detection resulting potential gaps in the detection history or periods of no detections. In these cases, a more accurate track state can be calculated by using multiple data streams than one because multiple data streams will act to fill in the gaps in the detection and restore a constant data rate [32].

Track-to-track fusion is suitable in situations where the communication bandwidth between the sensors and the fusion centre is limited, transmitting a track list is often more efficient than transmitting a set of detections. This is particularly important for cases in which the set of tracks is sent at a reduced rate relative to the scan rate. When the number of sensors and detections is large, the computation complexity for a detection-to-track can be high, especially for detection assignment. The track-to-track fusion architecture can distribute the data assignment and track update/prediction workload to the sensor therefore reduces the computation complexity of the final fusion centre.

The relative sensor performance also affects the overall uncertainty in the fusion process. In general, sensor fusion produces the best results when the sensors have identical or similar characteristics. Conversely, if a network consists of a high-quality sensor and several low-quality sensors, the performance of the fused result may not be much greater than the performance of the single high-quality sensor [33].

Network Detection Optimisation

Target detection performance increases with the amount of available data. Detection performance depends on how detections from each receiving node are fused across the network [34]. Results from classical detection theory demonstrate that an increasing the number of receiving nodes and increasing the energy in the transmitted signal can improve the detection performance of the network [34]. The amount of

data available to the central processor and its relation to how detection performance changes with data rate are studied in [35]. The authors demonstrate that the detection performance increases monotonically with the data bit rate.

The problem of optimising the receiver positions based on the network coverage is studied by Jayaraman et al [36]. Here network coverage is defined as the region in which the probability of detection $P_d = 0.95$ is achieved for a false alarm rate $P_{fa} = 10^{-4}$. The authors explore how the coverage area can be increased whilst maintaining lower limits on the network SNR, probability of detection and probability of false alarm by changing the number of positions and number of nodes in the network. The model is based on the propagation of the signals through the environment via the radar range equation, the target RCS used in the analysis is $\sigma = 10\text{m}^2$, significantly larger than the measured RCS of a drone.

Yang et al. study the optimal receiver positions in a multistatic network, proposing a coverage ratio and an average geometric dilution of precision (GDOP) of the surveillance region as objective functions to take into account both the detection and localisation performance [37]. The authors note that initially increasing the number of network nodes available to the optimisation algorithm, the performance of the optimal geometry improves; however the improvement suffers from diminishing returns suggesting that there is scope for improvement of the algorithm by including the number of nodes as an additional optimisation parameter.

Tracker aware detection optimisation uses knowledge from the tracker to dynamically adjust spatial detection thresholds in the predicted region of the target to improve the probability of detection. For example knowledge of a targets position can be taken into account in distributed radar network with overlapping field's of view. Using knowledge of the target's position relative to one receiver the detection

thresholds corresponding to the same spatial location at the other receivers can be optimised to improve the probability of detection. Extremely low detection thresholds may cause many false alarms which may interfere with real target. Consider an extreme situation, if the target under tracking disappears suddenly, radar target tracker with the above threshold adjustment methods may still output tracks for a long time, in the form of false tracks. So when adjusting detection thresholds of signal fusion detection, it is necessary to guarantee a low false track level.

An adaptive detection threshold algorithm is presented in which the detection thresholds in the predicted region of the target are adjusted using information from the tracker. Results show both an improved detection performance and successful control of false track level in monostatic radar system [38] [39]. Similar results can be found for multistatic networks in which the detection threshold across all radar systems in the network is dynamical adjusted using the predicted target position fed to each node by a central fusion centre [40]. The algorithm presented by Yan et al. uses the predicted distribution of the target location, and then the location dependent detection threshold is decreased near where a target is expected to be and increased where it is unexpected.

Network Estimation Optimisation

Estimation theory aims to estimate the values of properties of a target from measurements that have a random component. There are many properties of a target a radar can estimate. Commonly these are a target position and velocity. A radar array can also estimate a signal's angle of arrival and determine the target's position through a combination of the angles of arrival and time delay of the signal. Other commonly estimated properties are the target's radar cross-section and the channel fading. This section will primarily focus on geometry optimisation of the radar network.

The theoretical minimum variance of an unbiased estimator is determined by the Cramér-Rao Lower Bound (CRLB). The lower bound states that an estimator's variance is bounded below by the inverse of the Fisher Information Matrix (FIM). This relationship is intuitive as the greater the FIM, the lower the theoretical bound. The CRLB is a powerful tool and is often used to solve optimisation problems.

There are several properties of the FIM that are used for optimisation. The most commonly used ones are

- A-Optimality: Optimise the trace of the inverse of the FIM
- D-Optimality: Optimise the determinant of the FIM
- E-Optimality: Optimise the eigenvalues of the FIM.

The determinant of the FIM is a measure of the estimation confidence region [41]. Often the literature will look at the determinant of the FIM, as it is not affected by scale changes in the parameters and linear transformations of the output [41]. The trace of the inverse of the FIM is the sum of the diagonal elements; the diagonal elements are the variances of the estimation parameters, the off diagonal elements are the covariances. Therefore minimising the trace minimises the sum of the variances.

There are many possible parameters that are estimated and can be studied and optimised using the CRLB. Analysis the Range and Doppler shift of a target are thoroughly covered in the existing literature [34] [42]. Typically, research focuses on estimating parameters such as a target's position and velocity using the range and Doppler estimates. If the receiver employs a receiving array then the the direction of arrival of the signal can also be estimated.

A geometry optimisation algorithm is presented by Nguyen et al in which a Doppler shift localisation scheme is used to estimate the position of the target in the 2D plane [41]. The network sensors are placed on moving platforms. The authors determine the CRLB of targets position using the initial position of the sensors. The determinant of the CRLB is used as a cost function, the trajectory of the sensors is determined by calculating the gradient of the cost function with respect to receivers position and steering in the direction of steepest descent. Over time the sensors settle into an optimal configuration.

Yi et al. present a combined geometry optimisation algorithm using three cost functions, involving detection performance, localisation accuracy, and cost efficiency using Pareto optimisation [43]. Detection performance is based on the probability of detection; localisation accuracy is represented by the determinant of the Fisher information matrix, and cost efficiency is defined based on the coverage area and cost attached to the deployment of each receiver. The use of Pareto optimisation ensures the optimal geometry does not bias a particular cost function. The proposed algorithm is unique in optimising the geometry with multiple aspects of a radar network.

As a target moves through a radar network, the network geometry with the best estimation performance will change. A multistatic radar network can be considered as consisting of multiple bistatic transmitter-receiver pairs. Rather than consider the optimal network geometry, the optimal transmitter-receiver pair can be found by considering the transmitter-receiver pair that minimises the trace of the CRLB of the targets position [44].

2.4 Signal Multi-Path

The IEEE defines multipath as the propagation of a wave from one point to another by more than one path. When multipath occurs in radar, it consists of a direct path and one or more indirect paths either by reflection from the surface of the earth or sea or from large man-made structures or atmospheric refraction. [18].

Multipath reflections can be separated into two types of reflections, specular and diffuse, and are generally a combination of both types. Specular multipath is a coherent reflection, which means that the phase of the reflected path is relatively constant in relation to the phase of the direct path signal. This type of reflection usually causes the most significant fading of the direct path signal. Diffuse multipath reflections are non-coherent with respect to the direct path signal. The diffuse multipath causes less distortion than the specular type of multipath because they reflects less energy toward the receiver and usually has a noise-like response due to the random dispersion of the reflection [45].

There are several commonly used methods to mitigating the effects of multipath [45]:

- Movement changes the specular multipath to provide an average error. Multipath is position sensitive, so movement will change the multipath nulls.
- Spread spectrum. Nulls caused by multipath are frequency dependent, so with a spread spectrum system spread across a wide band of frequencies, the multipath only affects a portion of the spectrum.
- Antenna diversity. Since multipath is dependent on position, antennas can be located at different positions so that as one antenna is in a multipath null, the other antenna is not.

In the radar, multipath usually consists of a direct path and one or more indirect paths. When the signals arrive at the receiver, the difference in phase due to the different path lengths results in interference. The interference can either add constructively or destructively to the direct signal. As drones operate at much lower altitudes than typical targets, and as a result, the returned signal is much more susceptible to interference from multipath signals.

Each receiver will measure a target moving through a radar network at different ranges; the signal at each receiver will fluctuate due to signal multipath. Each receiver measures the target at a different bistatic range. The received signal experiences different levels of signal multipath for different bistatic ranges. Centrally combining the measurements from the network allows more consistent detection, estimation and tracking of the target over a monostatic radar [46].

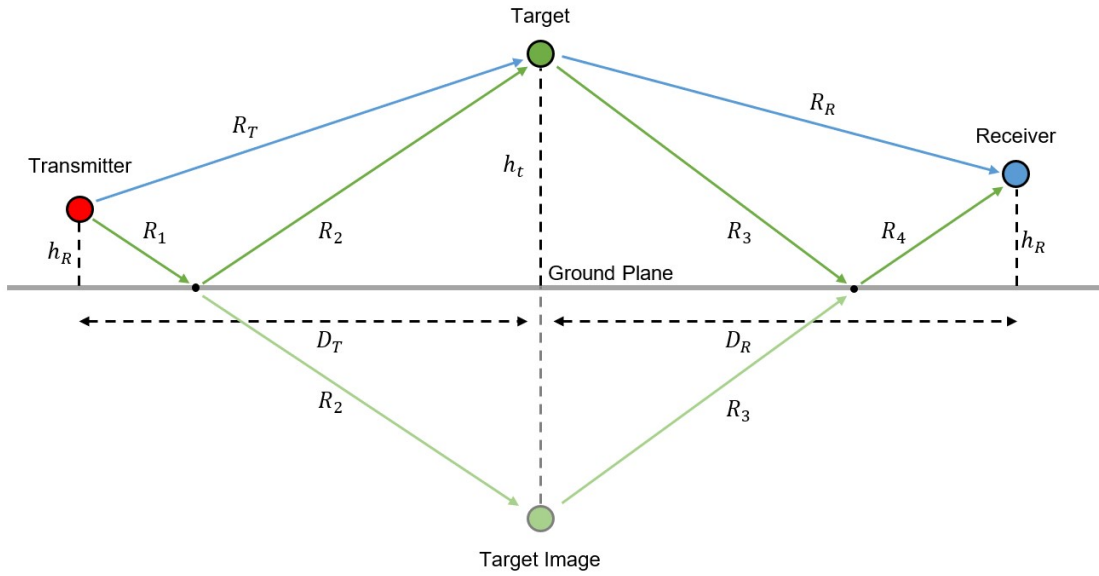


Figure 2.3: Signal multipath geometry

Figure 2.3 illustrates the direct and multipath signal for a specular reflection. For a flat surface, the interference due to the signal multipath can be significant. A more realistic rougher surface with multiple scattering points lessens the interference (diffuse multipath).

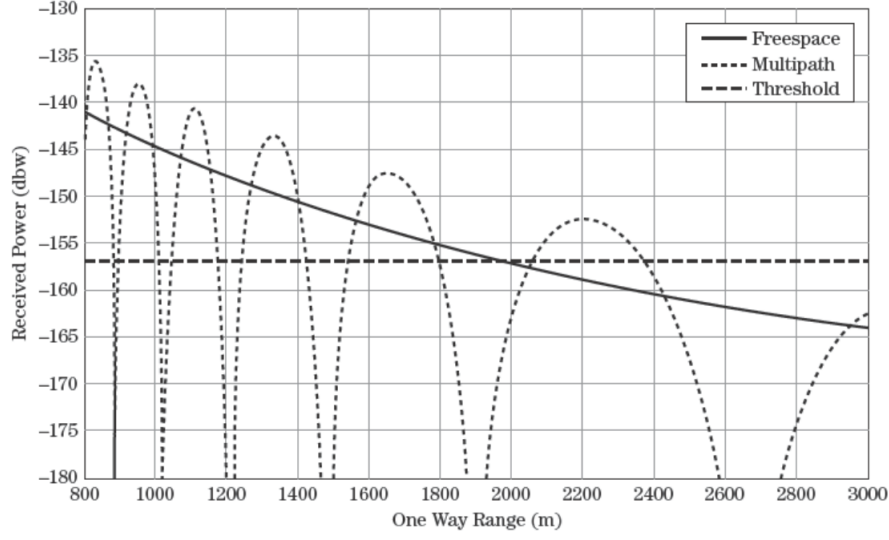


Figure 2.4: Signal multipath received power loss

Fig. 2.4 shows the signal loss due to the type specular multipath reflection illustrated in Fig. 2.3. Signal multipath can be modelled by extending the two signal approach to the multiple signals. The fading produced by signal multipath can be modelled as a random process [34]. The multipath can also be modelled using the method of the impulse response used to study linear systems; this technique is also used to simulate a noisy channel and is commonly used in communication systems.

Cancellation algorithms for signal multipath are explored in [47], which presents an adaptive multipath cancellation algorithm for passive radar. The problem of tracking a target in a multipath environment has been studied in [48, 49]. Okello looks at track registration for a bistatic radar with additional scattering from a known wall [48]. The author takes a deterministic approach to modelling the combined effect of the received signals. The tracking is then done using a modified, extended Kalman filter. The deterministic approach is limited as it requires exact knowledge of the wall/environment.

In passive radar, both the direct signal and multipath degrade the target signal

[49]. An effective cancellation filter for the passive radar can be obtained using the least square approach, to adaptively filter the received signals, cancelling multipath interference [50]. In an urban environment, the probability of detection of targets when the airborne radar is used has been shown to increase significantly by exploiting multipath [51].

2.5 Conclusion

Drone detecting radar is a highly active area of research; the majority of the systems studied in the literature are CW and FMCW radars. This is due to their low cost and simple design, which makes these types of radars more accessible to research labs. A CW radar can also put more energy on the target than an equivalent pulsed radar, improving detection performance, particularly for small drone-like targets. Staring radar that employs a digital array has been built as a counter-drone system. These systems can utilise long dwell times, which enable the detection of low RCS targets as well as fine Doppler resolution.

Broadly, most research into radar networks tends to be passive; there are only a hand full of active radar networks (e:g NetRAD). These networks are relatively small, only consisting of a couple of nodes, typically using only one transmitter and two to three receivers. None of the networks utilises a staring transmitter and multiple passive receivers in a cooperative network.

Of the many ways in which the properties of a network can be optimised, the thesis will focus on optimising the geometry of the network with respect to the positions of the passive receivers in the cooperative network. The optimisation will be based on an analysis of the CRLB of the joint position and velocity estimator. The fading effects of specular signal multipath on the received target's signal can be mitigated across the network by exploiting antenna diversity and combining the multiple target measurements from across the network.

Chapter 3

Background Theory

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This chapter introduces the necessary theory to develop the results presented in the latter part of the thesis. The chapter begins with a comment on the notation and conventions. The radar range equation for a bistatic radar system is introduced, and the fundamentals of pulse-Doppler signal processing are presented.

Next, estimators of constant and random parameters are developed and their corresponding lower bounds, the Cramér Rao Lower Bound (CRLB) and the Bayesian Lower Bound (BLB). Random processes and the Kosambi–Karhunen–Loève theorem are presented before moving on to the main result, which is the derivation of the lower bound of estimating the parameters of a signal contained in noise.

The Ambiguity Function is introduced, and its connection to the theoretical Cramér Rao Lower Bound of a time-delay and Doppler shift estimator from a received narrowband signal is presented.

The chapter concludes with a presentation of the CRLB for estimating the time delay and Doppler shift from a train of Square pulses and Gaussian Pulses, which is verified using Monte-Carlo simulations of the Maximum Likelihood Estimator.

3.1 Radar Range Equation

Consider a bistatic radar network with a single transmitter, receiver and a target, as shown in Fig. 3.1. The signal model and radar range equation are derived

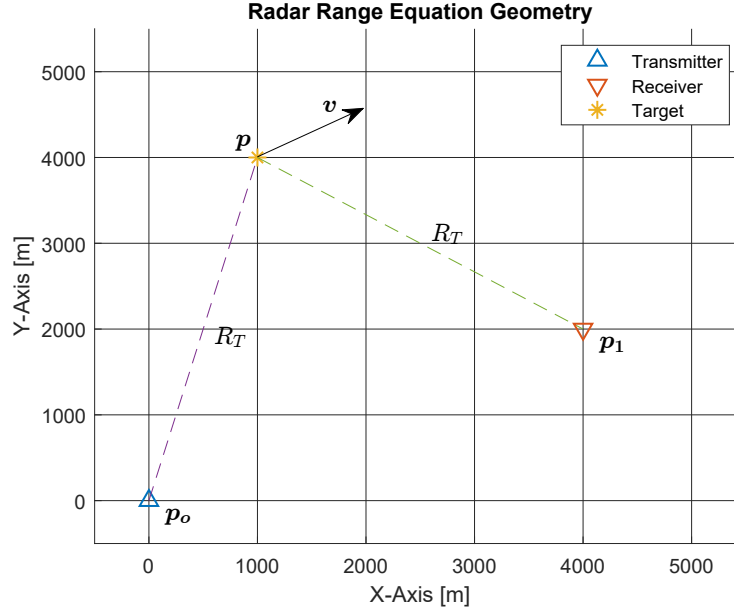


Figure 3.1: Radar Range Equation Geometry

Define the transmitter to target range as R_T and the receiver to target range as R_R , which, when written using the previously introduced notation, is $R_T = |\mathbf{p}_o - \mathbf{p}|$ and $R_R = |\mathbf{p}_1 - \mathbf{p}|$.

3.1.1 Received Signal Power

The radar range equation relates the received signal power to the transmitted power as well as properties of the radar (transmitter and receiver), target and the environment. Consider an isotropic antenna transmitting a waveform of power P_T W. As the transmission is isotropic and assuming a lossless environment, the power density at a range of R_T from the transmitter is the total transmitted power P_T divided by the surface area of a sphere of radius R_T

$$\text{Isotropic power density} = \frac{P_T}{4\pi R_T^2} \quad (3.1)$$

with units $\frac{\text{W}}{\text{m}^2}$. All radar systems use an antenna that has a directional beam pattern rather than an isotropic beam pattern. This means that the transmitted power is concentrated in a specific direction. Define the gain of the antenna as G_T , then the increased power density due to the use of a directional antenna is

$$\text{Peak transmitted power density} = \frac{P_T G_T}{4\pi R_T^2} \quad (3.2)$$

with units $\frac{\text{W}}{\text{m}^2}$. Consider a target at a range of R_T from the transmitter with a constant radar cross-section σ with units m^2 . The reflected power density at a range of R_R from the target is

$$\text{Reflected power density} = \frac{G_T P_T \sigma}{4\pi R_T^2} \frac{1}{4\pi R_R^2} \quad (3.3)$$

If the effective aperture of the radar antenna is A_e with units m^2 , then the total power received by the antenna is

$$P_R = \frac{G_T P_T \sigma}{4\pi^2 R_T^2 R_R^2} A_e \quad (3.4)$$

From antenna theory the effective aperture of an antenna is related to its gain and the wavelength of the signal, $A_e = \frac{G_R \lambda^2}{4\pi}$, therefore

$$P_R = \frac{G_T G_R \lambda^2 L \sigma}{(4\pi)^3 R_T^2 R_R^2} P_T. \quad (3.5)$$

including an atmospheric attenuation factor L , defined for $L \leq 1$. Equation (3.5) is known as the bistatic radar range equation, it relates the received signal power to the transmitted signal power when the transmitter and receiver are separate [52]. Table 3.1 lists the parameters and units of the bistatic radar range equation.

Equation parameters	Symbol	Unit
Received signal power	P_R	W
Transmitted signal power	P_T	W
Transmitter antenna gain	G_T	-
Receiver antenna gain	G_R	-
Atmospheric attenuation factor	L	-
Signal wavelength	λ	m
Radar cross section	σ	m ²
Transmitter to target range	R_T	m
Receiver to target range	R_R	m

Table 3.1: Bistatic Radar Range Equation Parameters

Furthermore, when the transmitter and receiver antenna are collocated, the transmitter to target range and the receiver to target range are equal, then (3.5) reduces to the monostatic radar range equation.

3.1.2 Noise Power

Thermal white noise power is uniformly distributed over all frequencies. It has a constant power spectral density and is modelled using a complex white Gaussian random process. Only noise signals within the range of frequencies capable of being detected by the radar's receiver will effect the performance of the radar. The range of frequencies for which the radar is susceptible to noise signals is determined by the receiver bandwidth, B . The mean power, P_N , of the thermal noise in the radar receiver is defined as

$$P_N = k_b T_0 F B \quad (3.6)$$

where

Equation parameters	Symbol	Unit
Receiver noise mean power	P_N	W
Standard temperature	T_0	K
Noise figure	F	-
Boltzmann Constant	k_b	J/K
Receiver bandwidth	B	Hz

Table 3.2: Receiver Thermal Noise Equation Parameters

The ratio of the received signal power to the receivers thermal noise mean power is known as the signal to noise ratio (SNR)

$$\text{SNR} = \frac{P_R}{P_N} = \frac{P_T G_T G_R \lambda^2 L \sigma}{(4\pi)^3 k_b T_0 F B R_T^2 R_R^2} \quad (3.7)$$

3.2 Pulse Doppler Signal Processing

The complex envelope of a signal $u(t)$ can be expressed as a complex unitary function $\int_{-\infty}^{\infty} |u(t)|^2 dt = 1$. The complex envelope of the signal is up-converted and amplified to the carrier frequency f_c to obtain the radio-frequency (RF) signal.

$$s_T(t) = \Re [A_T u(t) e^{2\pi i f_c t}] \quad (3.8)$$

Assume that a target is moving at a constant velocity $\mathbf{v}$, the received signal from target is a lower-amplitude and time-delayed version of the transmitted signal. Define $\tau(t)$ as the time delay between the signal being transmitted and received at time t . The received signal can be written as the transmitted signal at a time $t - \tau(t)$.

$$s_R(t) = \Re [A_R u(t - \tau(t)) e^{2\pi i f_c (t - \tau(t))}] \quad (3.9)$$

where A_R accounts for the reduced amplitude of the received signal which is described by the radar range equation. The interaction of the signal with the target also induces a phase shift which is absorbed into the A_R . As the target is moving

the targets range and therefore the time delay of the reflected signal vary with time.

$$\tau(t) = \frac{R_T(t) + R_R(t)}{c} \quad (3.10)$$

where $R_T(t)$ is the transmitter to target range and $R_R(t)$ is the receiver-target range.

$$R_T(t) = |\mathbf{p}_o - \mathbf{p}| - v_T t \quad (3.11a)$$

$$R_R(t) = |\mathbf{p}_1 - \mathbf{p}| - v_R t \quad (3.11b)$$

where $\mathbf{p}_o$ is the position of the transmitter, $\mathbf{p}_1$ is the position of the receiver, v_T is the target's velocity in the direction of the transmitter and v_R is the target's velocity in direction of receiver.

$$v_T = \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} \quad (3.12a)$$

$$v_R = \mathbf{v} \cdot \frac{(\mathbf{p}_1 - \mathbf{p})}{|\mathbf{p}_1 - \mathbf{p}|} \quad (3.12b)$$

therefore

$$\tau(t) = \tau_0 - \frac{v_T + v_R}{c} t \quad (3.13)$$

where $\tau_0 = \frac{|\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_1 - \mathbf{p}|}{c}$. The received signal is

$$s_R(t) = \Re \left[A_R u \left(t - \tau_0 + \frac{v_T + v_R}{c} t \right) e^{2\pi i f_c \left(t - \tau_0 + \frac{v_T + v_R}{c} t \right)} \right] \quad (3.14)$$

A moving target not only implies a shift in the carrier frequency known as Doppler shift but also a compression of the time scale of the complex envelope. For a pulse width of T the term $\frac{v_T + v_R}{c} t$ ranges between $[0, \frac{v_T + v_R}{c} T]$. This effect compression of the complex envelope can be neglected if $\frac{v_T + v_R}{c} T \ll \frac{1}{B}$ where B is the signal

bandwidth. In the radar domain this is general satisfied [34].

$$s_R(t) = \Re [A_R u(t - \tau_0) e^{2\pi i(f_c + f_D)t} e^{-2\pi i f_c \tau_0}] \quad (3.15)$$

where $f_D = \frac{1}{\lambda} (v_T + v_R)$ is the Doppler-shift. The velocity v_R can be written as $\frac{dR_R(t)}{dt}$ and similarly v_T can be written as $\frac{dR_T(t)}{dt}$ then $f_D = \frac{1}{\lambda} \left(\frac{dR_R}{dt} + \frac{dR_T}{dt} \right)$ which is the standard form of the Doppler shift for a bistatic geometry [52]. The term $e^{-2\pi i f_c \tau_0}$ is a constant complex number that can also be absorbed into the A_R . After down conversion the complex envelope is

$$s_R(t) = A_R u(t - \tau_0) e^{2\pi i f_D t} \quad (3.16)$$

3.3 Estimation Theory

This section introduces the foundations of estimation theory necessary to develop the results presented in the following chapters, Starting with some of the properties of estimators. Estimators of deterministic unknown parameters and unknown random variables are introduced. The Cramér Rao Lower Bound (CRLB) and Fisher Information Matrix (FIM) for both types of estimators are presented.

Consider two different types of estimation, deterministic estimation and Bayesian estimation. Deterministic or classical estimation is used when the estimation parameter is an unknown constant. Bayesian estimation is used when the estimation parameter is an unknown random variable.

3.3.1 Deterministic Estimation

An estimator is a rule for determining a parameter from a given set of observables. Define $\hat{\theta}$ as an estimator of the unknown deterministic parameter θ . The estimator $\hat{\theta}$ is a random variable and may assume more than one value. Ideally, there are some characteristics that $\hat{\theta}$ should possess; namely, it should be unbiased, consistent and efficient, with the minimum possible variance.

Unbiased Estimator

An unbiased estimator on average yields the true value of the unknown parameter. Since a parameter value may in general be anywhere in the interval $a \leq \theta \leq b$ unbiasedness asserts that the estimator will on average yield the true value regardless of its value. Formally, an estimator is unbiased if

$$\mathbb{E}[\hat{\theta}] = \theta, \quad \theta \in (a, b) \quad (3.17)$$

where (a, b) denotes the range of possible values of θ , [53]. There are two types of biased estimator

$$\mathbb{E}[\hat{\theta}] = \theta + b(\theta) \quad (3.18)$$

If $b(\theta)$ does not depend on θ , then the estimator has a known bias and $\hat{\theta} - b$ is an unbiased estimator. If the bias is dependent on θ then the estimator has unknown bias.

Consistent Estimator

An estimator $\hat{\theta}$ is a consistent estimator of θ , based on N observed samples, if

$$\lim_{N \rightarrow \infty} Pr(|\hat{\theta} - \theta| > \varepsilon) = 0, \quad \text{for all } \varepsilon > 0 \quad (3.19)$$

where $Pr(\cdot)$ denotes probability

Minimum Variance Unbiased Estimator

Within the class of unbiased estimators, the one with the minimum variance is sought. Such an estimator is known as the Minimum Variance Unbiased estimator or MVU estimator. However, in general, they do not exist. In order to search for an optimal estimator, an optimality criterion is needed. The natural choice is the mean square error or MSE

$$\text{mse}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2] \quad (3.20)$$

The MSE measures the average mean squared error of the estimator from the actual value. Unfortunately, the MSE leads to unrealisable estimators that cannot be

written solely as a function of the data.

$$\text{mse}(\hat{\theta}) = \mathbb{E} \left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}] \right)^2 \right] + \left(\mathbb{E}[\hat{\theta}] - \theta \right)^2 \quad (3.21)$$

Recognise the first term as the variance of the estimator and the second term as the square of the estimator's bias, which depends on the true value of the parameter. If the estimator $\hat{\theta}$ is unbiased then $\mathbb{E}[\hat{\theta}] = \theta$ and

$$\text{mse}(\hat{\theta}) = \mathbb{E} \left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}] \right)^2 \right] \quad (3.22)$$

Proceed by finding the Cramér-Rao Lower Bound (CRLB) and checking if some estimator satisfies it. The CRLB places a lower bound on the variance of an unbiased estimator. If an estimator exists whose variance equals the CRLB for each value of θ then it must be the MVU.

Consider a vector parameter $\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \theta_N \end{pmatrix}^\top$. Define the covariance matrix of $\hat{\boldsymbol{\theta}}$ as

$$\text{cov}(\hat{\boldsymbol{\theta}}) = \mathbb{E} \left[[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}][\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top \right] \quad (3.23)$$

where $\mathbf{z}$ is the data.

Theorem 3.1 (Cramér-Rao Lower Bound - Vector Parameter). *Assuming that the probability density function $p(\mathbf{z}; \boldsymbol{\theta})$ satisfies the regularity conditions*

$$\frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} \quad \text{and} \quad \frac{\partial^2 \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \quad (3.24)$$

exist and are absolutely integrable with respect to $\mathbf{z}$. Then the covariance matrix of any unbiased estimator $\hat{\boldsymbol{\theta}}$ is bound below by the inverse of the Fisher Information

Matrix (FIM)

$$\text{cov}(\hat{\boldsymbol{\theta}}) - \mathbf{J}_F^{-1}(\boldsymbol{\theta}) \geq \mathbf{0} \quad (3.25)$$

where $\geq \mathbf{0}$ means that the matrix is positive semi-definite. The FIM is defined as

$$\mathbf{J}_F = \mathbb{E} \left[[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})] [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] \quad (3.26a)$$

$$= -\mathbb{E} \left[\nabla_{\boldsymbol{\theta}} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \right] \quad (3.26b)$$

or elements wise

$$[\mathbf{J}_F]_{ij} = \mathbb{E} \left[\frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_j} \right] \quad (3.27a)$$

$$= -\mathbb{E} \left[\frac{\partial^2 \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (3.27b)$$

where the derivative is evaluated at the true value of θ and the expectation is taken with respect to $p(\mathbf{z}; \boldsymbol{\theta})$. The matrix inequality implies that the variance of the estimators elements are bounded below by the corresponding diagonal elements of the inverse of the Fisher information matrix [54].

$$\text{var}(\hat{\theta}_i) = \text{cov}(\hat{\boldsymbol{\theta}})_{ii} \geq [\mathbf{J}_F^{-1}(\boldsymbol{\theta})]_{ii} \quad (3.28)$$

The following calculations are a combination of the works of Kay, Van-Trees and Barkat [54], [53] and [55]. Beginning with the assumption that the estimator $\hat{\boldsymbol{\theta}}(\mathbf{z})$ is unbiased $\mathbb{E} [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] = \mathbf{0}$ using the definition of the expectation

$$\int_{-\infty}^{\infty} [\hat{\theta}_i(\mathbf{z}) - \theta_i] p(\mathbf{z}; \theta) d\mathbf{z} = 0 \quad (3.29)$$

for all $i \in [1, k]$. Differentiating both sides with respect to θ_j gives

$$-\delta_{ij} \int_{-\infty}^{\infty} p(\mathbf{z}; \theta) d\mathbf{z} + \int_{-\infty}^{\infty} [\hat{\theta}_i(\mathbf{z}) - \theta_i] \frac{\partial p(\mathbf{z}; \theta)}{\partial \theta_j} d\mathbf{z} = 0 \quad (3.30)$$

The first integral equals unity and that $\frac{\partial \ln p(\mathbf{z}; \theta)}{\partial \theta_j} = \frac{1}{p(\mathbf{z}; \theta)} \frac{\partial p(\mathbf{z}; \theta)}{\partial \theta_j}$ then

$$\int_{-\infty}^{\infty} [\hat{\theta}_i(\mathbf{z}) - \theta_i] p(\mathbf{z}; \theta) \frac{\partial \ln p(\mathbf{z}; \theta)}{\partial \theta_j} d\mathbf{z} = \delta_{ij} \quad (3.31)$$

which can be rewritten as

$$\int_{-\infty}^{\infty} \left[[\hat{\theta}_i(\mathbf{z}) - \theta_i] \sqrt{p(\mathbf{z}; \theta)} \right] \left[\frac{\partial \ln p(\mathbf{z}; \theta)}{\partial \theta_j} \sqrt{p(\mathbf{z}; \theta)} \right] d\mathbf{z} = \delta_{ij} \quad (3.32)$$

in matrix notation

$$\int_{-\infty}^{\infty} \left[[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \sqrt{p(\mathbf{z}; \theta)} \right] \left[\sqrt{p(\mathbf{z}; \theta)} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] d\mathbf{z} = \mathbf{I} \quad (3.33)$$

where $\mathbf{I}$ is the identity matrix. In order to simplify the application of the Cauchy-Schwarz inequality define two arbitrary vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ that do not depend on $\mathbf{z}$. Pre-multiply by $\mathbf{a}_1^\top$ and post-multiply by $\mathbf{a}_2$

$$\int_{-\infty}^{\infty} \left[\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \sqrt{p(\mathbf{z}; \theta)} \right] \left[\sqrt{p(\mathbf{z}; \theta)} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{a}_2 \right] d\mathbf{z} = \mathbf{a}_1^\top \mathbf{a}_2 \quad (3.34)$$

The terms in the brackets are now scalars, applying the Cauchy-Schwarz inequality to find

$$\begin{aligned} (\mathbf{a}_1^\top \mathbf{a}_2)^2 &\leq \int_{-\infty}^{\infty} \mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top \mathbf{a}_2 p(\mathbf{z}; \theta) d\mathbf{z} \\ &\quad \cdot \int_{-\infty}^{\infty} \mathbf{a}_1^\top [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})] [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \mathbf{a}_2 p(\mathbf{z}; \theta) d\mathbf{z} \end{aligned} \quad (3.35)$$

Recognising that the integrals are expectations of $\text{cov}(\hat{\boldsymbol{\theta}})$ and $\mathbf{J}_B$ also noting that both $\mathbf{a}_1$ and $\mathbf{a}_2$ are not dependent on $\mathbf{z}$ then

$$(\mathbf{a}_1^\top \mathbf{a}_2)^2 \leq [\mathbf{a}_1^\top \text{cov}(\hat{\boldsymbol{\theta}}) \mathbf{a}_1][\mathbf{a}_2^\top \mathbf{J}_F(\boldsymbol{\theta}) \mathbf{a}_2] \quad (3.36)$$

The choice $\mathbf{a}_2$ is arbitrary therefore let $\mathbf{a}_2 = \mathbf{J}_F^{-1} \mathbf{a}_1$

$$[\mathbf{a}_1^\top \text{cov}(\hat{\boldsymbol{\theta}}) \mathbf{a}_1][\mathbf{a}_1^\top \mathbf{J}_F^{-1} \mathbf{a}_1] \geq (\mathbf{a}_1^\top \mathbf{J}_F^{-1} \mathbf{a}_1)^2 \quad (3.37)$$

then

$$\mathbf{a}_1^\top [\text{cov}(\hat{\boldsymbol{\theta}}) - \mathbf{J}_F^{-1}] \mathbf{a}_1 \geq 0 \quad (3.38)$$

the arbitrary choice of the vector $\mathbf{a}_1$ implies

$$\text{cov}(\hat{\boldsymbol{\theta}}) - \mathbf{J}_F^{-1} \geq 0 \quad (3.39)$$

where the inequality means that the matrix is positive semi-definite, completing the proof.

Further Results - $\mathbf{J}_F$ Alternative Form

By definition of the probability density function $\int_{-\infty}^{\infty} p(\mathbf{z}; \boldsymbol{\theta}) d\mathbf{z} = 1$. Taking the derivative of the probability density function with respect to θ_i yields

$$\int_{-\infty}^{\infty} \frac{\partial p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} d\mathbf{z} = 0 \quad (3.40)$$

Noting that $\frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} = \frac{1}{p(\mathbf{z}; \boldsymbol{\theta})} \frac{\partial p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i}$ then

$$\int_{-\infty}^{\infty} \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} p(\mathbf{z}; \boldsymbol{\theta}) d\mathbf{z} = 0 \quad (3.41)$$

Differentiating with respect to θ_j

$$\int_{-\infty}^{\infty} \frac{\partial^2 \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} p(\mathbf{z}; \boldsymbol{\theta}) d\mathbf{z} + \int_{-\infty}^{\infty} \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_j} p(\mathbf{z}; \boldsymbol{\theta}) d\mathbf{z} = 0 \quad (3.42)$$

therefore

$$\mathbb{E} \left[\frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_j} \right] = -\mathbb{E} \left[\frac{\partial^2 \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (3.43)$$

for all possible combinations of $i, j \in [1, N]$ which in matrix notation

$$\mathbb{E} \left[[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})][\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \right] = -\mathbb{E} \left[\nabla_{\boldsymbol{\theta}} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \right] \quad (3.44)$$

Further Results - Equality

Furthermore equality in the Cauchy-Schwarz is achieved if and only if

$$\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \sqrt{p(\mathbf{z}; \boldsymbol{\theta})} = C(\boldsymbol{\theta}) \sqrt{p(\mathbf{z}; \boldsymbol{\theta})} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{a}_2 \quad (3.45)$$

where $C(\boldsymbol{\theta})$ is a scalar. The above may be simplified by dropping $\sqrt{p(\mathbf{z}; \boldsymbol{\theta})}$

$$\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] = C(\boldsymbol{\theta}) [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{a}_2 \quad (3.46)$$

If either $\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]$ or $[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{a}_2$ are zero then the Cauchy-Schwarz. If $C(\boldsymbol{\theta}) = 0$ then $\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] = 0$ and we also have equality.

Now assume that $\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}], [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{a}_2, C(\boldsymbol{\theta}) \neq 0$ and let $\mathbf{a}_2 = \mathbf{J}_F^{-1} \mathbf{a}_1$

$$\mathbf{a}_1^\top [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] = C(\boldsymbol{\theta}) [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{J}_F^{-1} \mathbf{a}_1 \quad (3.47)$$

Note that $\mathbf{a}_1$ and $\mathbf{a}_2$ are vectors that do not depend on $\boldsymbol{\theta}$ chosen arbitrarily in

order to simplify the application of the Cauchy Schwartz inequality, then the above simplifies to

$$[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] = C(\boldsymbol{\theta})[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top \mathbf{J}_F^{-1} \quad (3.48)$$

Rearranging

$$[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top = [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top \frac{\mathbf{J}_F}{C(\boldsymbol{\theta})} \quad (3.49)$$

Taking the gradient with respect to $\boldsymbol{\theta}$ yields

$$\nabla_{\boldsymbol{\theta}} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta})]^\top = -\mathbf{I} \frac{\mathbf{J}_F}{C(\boldsymbol{\theta})} + [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top \nabla_{\boldsymbol{\theta}} \frac{\mathbf{J}_F}{C(\boldsymbol{\theta})} \quad (3.50)$$

Multiplying by -1 and taking the expectation on both sides, the second term is equal to zero because the estimator is unbiased, then $\mathbf{J}_F(\boldsymbol{\theta}) = \frac{\mathbf{J}_F(\boldsymbol{\theta})}{C(\boldsymbol{\theta})}$ therefore $C(\boldsymbol{\theta}) = 1$ and

$$\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}; \boldsymbol{\theta}) = \mathbf{J}_F [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \quad (3.51)$$

The estimator will be the MVU estimator. An estimator which is unbiased and attains the CRLB is said to be efficient. Note that an MVU estimator may not be efficient as it may not necessarily attain the CRLB. The FIM has the useful property that it is additive for independent measurements. If the data $\mathbf{z}$ takes the form of a vector of Gaussian measurements $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta}))$ where both the mean and covariance matrix depend on $\boldsymbol{\theta}$, then the FIM has a standard form

$$\begin{aligned} [\mathbf{J}_F(\boldsymbol{\theta})]_{ij} &= \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i} \right]^\top \mathbf{C}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j} \right] \\ &+ \frac{1}{2} \text{tr} \left[\mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i} \mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_j} \right] \end{aligned} \quad (3.52a)$$

$$\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i} = \left[\frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_1}{\partial \theta_i} \quad \frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_2}{\partial \theta_i} \quad \dots \quad \frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_N}{\partial \theta_i} \right]^\top \quad (3.52b)$$

Maximum Likelihood Estimator

The probability density function $p(\mathbf{z}; \boldsymbol{\theta})$ when viewed as a function of $\boldsymbol{\theta}$ is known as the likelihood function. The maximum likelihood estimator is the value of $\boldsymbol{\theta}$ that maximises the likelihood function for the observed data $\mathbf{z}$. As the logarithm is a monotonically increasing function the log-likelihood function $\ln p(\mathbf{z}; \boldsymbol{\theta})$ may be used.

$$\hat{\boldsymbol{\theta}}_{\text{ml}} = \arg \max_{\boldsymbol{\theta}} \{\ln p(\mathbf{z}; \boldsymbol{\theta})\} \quad (3.53)$$

If the maximum is interior to the range of $\boldsymbol{\theta}$, and the log-likelihood function $\ln p(\mathbf{z}; \boldsymbol{\theta})$ has a continuous first derivative, then a necessary condition is obtained by differentiating $\ln p(\mathbf{z}; \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ and setting the result equal to zero

$$\left. \frac{\partial \ln p(\mathbf{z}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{\text{ml}}} = \mathbf{0} \quad (3.54)$$

If the derivatives of the log-likelihood function with respect to $\boldsymbol{\theta}$ exists and the fisher information matrix is non zero, then the MLE of the unknown parameter $\boldsymbol{\theta}$ is asymptotically distributed for large data records according to

$$\hat{\boldsymbol{\theta}} \sim \mathcal{N}(\boldsymbol{\theta}, \mathbf{J}^{-1}(\boldsymbol{\theta})) \quad (3.55)$$

where $\mathbf{J}(\boldsymbol{\theta})$ is the fisher information evaluated at the true value of the unknown parameter. The regularity conditions require the existence of the derivatives of the log-likelihood function as well as the fisher information matrix being non-zero. From the asymptotic distribution, the MLE is seen to be asymptotically unbiased and therefore asymptotically efficient.

3.3.2 Bayesian Estimation

Consider that the unknown parameter θ is a random variable. The estimation of random parameters is known as Bayes' estimation.

In Bayes estimation, a cost is assigned to all pairs of $(\hat{\theta}(\mathbf{z}), \theta)$. The cost is a non-negative real value function of the two random variables $C(\hat{\theta}(\mathbf{z}), \theta)$. The Bayes risk function is the average of the cost function, denoted by $\mathcal{R} = \mathbb{E}[C(\hat{\theta}(\mathbf{z}), \theta)]$. In many cases the cost function is assumed to only depend on the error of the estimate, defined as

$$\hat{\theta}_\epsilon(\mathbf{z}) = \hat{\theta}(\mathbf{z}) - \theta \quad (3.56)$$

The commonly used cost functions are the squared error C_s , absolute value of error C_{abs} , and the uniform or 1-0 cost function C_u

$$C_s(\hat{\theta}_\epsilon) = (\hat{\theta}_\epsilon)^2, \quad C_{abs}(\hat{\theta}_\epsilon) = |\hat{\theta}_\epsilon|, \quad C_u(\hat{\theta}_\epsilon) = \begin{cases} 1, & |\hat{\theta}_\epsilon| \geq \frac{\delta}{2} \\ 0, & |\hat{\theta}_\epsilon| < \frac{\delta}{2} \end{cases} \quad (3.57)$$

where $\delta > 0$. If δ is small, the uniform cost function assigns the same cost for any error and no penalty for no error. The Bayes estimate is defined as the estimate that minimises the Bayes' risk and is dependent on the type of cost function used.

If the squared error cost function is used, then the minimum mean square estimate (MMSE) is the Bayes' estimate. The MMSE is defined as the conditional mean of θ given $\mathbf{z}$, also known as the mean of the posteriori density. If the absolute value of the error is used as the cost function, then the Bayes estimate is the median of the posteriori density. Finally, if the uniform cost function is used, the Bayes estimator is the mode or the maximum of the posteriori density. This estimator is more commonly known as the maximum a posteriori estimate (MAP). Note that if

the posteriori is a Gaussian, the mean, mode, and median are equal.

Now consider a vector parameter $\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \theta_N \end{pmatrix}^\top$. Define the mean square error matrix of the vector estimator $\hat{\boldsymbol{\theta}}(\mathbf{z})$ as

$$\Upsilon(\hat{\boldsymbol{\theta}}(\mathbf{z})) = \mathbb{E} \left[[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top \right] \quad (3.58)$$

Bayesian Lower Bound - Vector Parameter

Assuming that the probability density function $p(\mathbf{z}, \boldsymbol{\theta})$ satisfies the regularity conditions

$$\frac{\partial p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} \quad \text{and} \quad \frac{\partial p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \quad (3.59)$$

are absolutely integrable with respect $\mathbf{z}$ and $\boldsymbol{\theta}$ for all possible values and combinations of $i, j \in [1, N]$. The conditional expectation of the error, given $\boldsymbol{\theta}$ is

$$B(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] p(\mathbf{z}|\boldsymbol{\theta}) d\mathbf{z} \quad (3.60)$$

with the assumption that

$$\lim_{\theta_i \rightarrow \pm\infty} B(\boldsymbol{\theta}) p(\boldsymbol{\theta}) = \mathbf{0} \quad (3.61)$$

for all values of $i \in [1, N]$. Then the mean square error matrix of any estimator $\hat{\boldsymbol{\theta}}(\mathbf{z})$ is bounded below by the inverse of the Bayesian Information Matrix (BIM)

$$\Upsilon(\hat{\boldsymbol{\theta}}) - \mathbf{J}_B^{-1} \geq \mathbf{0} \quad (3.62)$$

where $\geq \mathbf{0}$ means that the matrix is positive semi-definite. The Bayesian information matrix is defined as

$$\mathbf{J}_B = \mathbb{E} \left[[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})] [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] \quad (3.63a)$$

$$= -\mathbb{E} \left[\nabla_{\boldsymbol{\theta}} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] \quad (3.63b)$$

or element wise

$$[\mathbf{J}_B]_{ij} = \mathbb{E} \left[\frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_j} \right] \quad (3.64a)$$

$$= -\mathbb{E} \left[\frac{\partial^2 \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (3.64b)$$

The probability density is a joint density and the expectation is over both $\mathbf{z}$ and $\boldsymbol{\theta}$. The matrix inequality implies that the mean square error of the estimator's elements is bounded below by the corresponding diagonal elements of the inverse of the Bayesian information matrix.

$$\mathbb{E} \left[(\hat{\theta}_i(\mathbf{z}) - \theta_i)^2 \right] \geq [\mathbf{J}_B^{-1}]_{ii} \quad (3.65)$$

The Bayesian information matrix can be split into two parts

$$\mathbf{J}_B = \mathbf{J}_D + \mathbf{J}_P \quad (3.66)$$

where

$$\mathbf{J}_D = \mathbb{E}_{\boldsymbol{\theta}} [\mathbf{J}_F(\boldsymbol{\theta})] \quad (3.67a)$$

$$\mathbf{J}_{P_{ij}} = -\mathbb{E} \left[\frac{\partial^2 \ln p(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (3.67b)$$

$\mathbf{J}_D$ represents the contribution of the data and the $\mathbf{J}_P$ the contribution from the prior information.

The proof is similar to the deterministic case. The following calculations are a combination of the work of Van-Trees and Barkat [54] and [55]. Begin by multiplying the conditional expectation of the error by $p(\boldsymbol{\theta})$

$$p(\boldsymbol{\theta})\mathbf{B}^\top(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} p(\mathbf{z}|\boldsymbol{\theta})p(\boldsymbol{\theta})[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top d\mathbf{z} = \int_{-\infty}^{\infty} p(\mathbf{z}, \boldsymbol{\theta})[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top d\mathbf{z} \quad (3.68)$$

Differentiating with respect to $\boldsymbol{\theta}$

$$\nabla_{\boldsymbol{\theta}}[p(\boldsymbol{\theta})\mathbf{B}^\top(\boldsymbol{\theta})] = -\mathbf{I} \int_{-\infty}^{\infty} p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} + \int_{-\infty}^{\infty} \nabla_{\boldsymbol{\theta}}[p(\mathbf{z}, \boldsymbol{\theta})][\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top d\mathbf{z} \quad (3.69)$$

Now integrate with respect to $\boldsymbol{\theta}$. The ij -th element of the left-hand side is

$$\left[\int_{-\infty}^{\infty} \nabla_{\boldsymbol{\theta}}[p(\boldsymbol{\theta})\mathbf{B}^\top(\boldsymbol{\theta})] d\boldsymbol{\theta} \right]_{ij} = \int_{-\infty}^{\infty} \frac{\partial p(\boldsymbol{\theta})B_j(\boldsymbol{\theta})}{\partial \theta_i} d\boldsymbol{\theta} \quad (3.70)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(p(\boldsymbol{\theta})B_j(\boldsymbol{\theta}) \Big|_{\theta_i=-\infty}^{\theta_i=+\infty} \right) d\theta_1 \cdots d\theta_{i-1} d\theta_{i+1} \cdots d\theta_k \quad (3.71)$$

The assumption on the conditional expectation of the error the above zero for all i and j setting the right-hand side equal to $\mathbf{0}$, the zero matrix.

$$\mathbf{0} = -\mathbf{I} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla_{\boldsymbol{\theta}}[p(\mathbf{z}, \boldsymbol{\theta})][\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top d\mathbf{z} d\boldsymbol{\theta} \quad (3.72)$$

Noting that $\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta}) = \frac{1}{p(\mathbf{z}, \boldsymbol{\theta})} \nabla_{\boldsymbol{\theta}} p(\mathbf{z}, \boldsymbol{\theta})$ then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta}) p(\mathbf{z}, \boldsymbol{\theta}) [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^\top d\mathbf{z} d\boldsymbol{\theta} = \mathbf{I} \quad (3.73)$$

which can be written as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \sqrt{p(\mathbf{z}, \boldsymbol{\theta})} \right] \left[\sqrt{p(\mathbf{z}, \boldsymbol{\theta})} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^{\top} \right] d\mathbf{z} d\boldsymbol{\theta} = \mathbf{I} \quad (3.74)$$

In order to simplify the application of the Cauchy-Schwarz inequality define two arbitrary vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ that do not depend on $\mathbf{z}$. Pre-multiply by $\mathbf{a}_1^{\top}$ and post-multiply by $\mathbf{a}_2$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\mathbf{a}_1^{\top} [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \sqrt{p(\mathbf{z}, \boldsymbol{\theta})} \right] \left[\sqrt{p(\mathbf{z}, \boldsymbol{\theta})} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^{\top} \mathbf{a}_2 \right] d\mathbf{z} d\boldsymbol{\theta} = \mathbf{a}_1^{\top} \mathbf{a}_2 \quad (3.75)$$

The terms in the brackets are now scalars, applying the Cauchy-Schwarz inequality to find

$$\begin{aligned} (\mathbf{a}_1^{\top} \mathbf{a}_2)^2 &\leq \int_{-\infty}^{\infty} \mathbf{a}_1^{\top} [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] [\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}]^{\top} \mathbf{a}_1 p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} \\ &\quad \cdot \int_{-\infty}^{\infty} \mathbf{a}_2^{\top} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})] [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^{\top} \mathbf{a}_2 p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} \end{aligned} \quad (3.76)$$

Recognising that the integrals as $\boldsymbol{\Upsilon}(\boldsymbol{\theta})$ and $\mathbf{J}_B$. Also noting that both $\mathbf{a}_1$ and $\mathbf{a}_2$ are not dependent on $\mathbf{z}$ then

$$(\mathbf{a}_1^{\top} \mathbf{a}_2)^2 \leq [\mathbf{a}_1^{\top} \boldsymbol{\Upsilon}(\boldsymbol{\theta}) \mathbf{a}_1] [\mathbf{a}_2^{\top} \mathbf{J}_B(\boldsymbol{\theta}) \mathbf{a}_2] \quad (3.77)$$

The choice $\mathbf{a}_2$ is arbitrary therefore let $\mathbf{a}_2 = \mathbf{J}_B^{-1} \mathbf{a}_1$

$$[\mathbf{a}_1^{\top} \boldsymbol{\Upsilon}(\boldsymbol{\theta}) \mathbf{a}_1] [\mathbf{a}_1^{\top} \mathbf{J}_F^{-1} \mathbf{a}_1] \geq (\mathbf{a}_1^{\top} \mathbf{J}_F^{-1} \mathbf{a}_1)^2 \quad (3.78)$$

then

$$\mathbf{a}_1^{\top} [\boldsymbol{\Upsilon}(\boldsymbol{\theta}) - \mathbf{J}_B^{-1}] \mathbf{a}_1 \geq 0 \quad (3.79)$$

the arbitrary choice of the vector $\mathbf{a}_1$ implies

$$\Upsilon(\boldsymbol{\theta}) - \mathbf{J}_B^{-1} \geq 0 \quad (3.80)$$

where the inequality means that the matrix is positive semi-definite, completing the proof.

Further Results - $\mathbf{J}_B$ Alternate Form

From the definition of the probability density function $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} d\boldsymbol{\theta} = 1$, taking the derivative with respect to θ_i then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} d\mathbf{z} d\boldsymbol{\theta} = 0 \quad (3.81)$$

Noting that $\frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} = \frac{1}{p(\mathbf{z}; \boldsymbol{\theta})} \frac{\partial p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i}$ then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} = 0 \quad (3.82)$$

Differentiating with respect to θ_j

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} d\boldsymbol{\theta} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_j} p(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} d\boldsymbol{\theta} = 0 \quad (3.83)$$

and therefore

$$\mathbb{E} \left[\frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_j} \right] = -\mathbb{E} \left[\frac{\partial^2 \ln p(\mathbf{z}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (3.84)$$

for all possible combinations of $i, j \in [1, k]$, which can be expressed as

$$\mathbb{E} \left[[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})][\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] = -\mathbb{E} \left[\nabla_{\boldsymbol{\theta}} [\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta})]^\top \right] \quad (3.85)$$

Further Results - Equality

Furthermore, equality in the Cauchy-Schwarz is achieved if and only if

$$\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}, \boldsymbol{\theta}) = \mathbf{J}_B[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \quad (3.86)$$

Furthermore the equality statement can be written in terms of the posteriori density.

From Bayes theorem $\ln p(\mathbf{z}, \boldsymbol{\theta}) = \ln p(\boldsymbol{\theta}|\mathbf{z}) + \ln p(\mathbf{z})$ also $\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}) = 0$ then

$$\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}|\boldsymbol{\theta}) = \mathbf{J}_B[\hat{\boldsymbol{\theta}}(\mathbf{z}) - \boldsymbol{\theta}] \quad (3.87)$$

This is also a statement that the posteriori probability density of $\boldsymbol{\theta}$ given $\mathbf{z}$ must be Gaussian in order for an efficient estimate to exist.

Maximum A Posteriori Estimator

The MAP estimate is the value of $\boldsymbol{\theta}$ that maximises the a posteriori density, denoted by $\hat{\mathbf{a}}_{\text{map}}(\mathbf{z})$. Bayes theorem states that $p(\boldsymbol{\theta}|\mathbf{z}) = \frac{p(\mathbf{z}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{z})}$, then the MAP estimate can be written as

$$\hat{\boldsymbol{\theta}}_{\text{map}}(\mathbf{z}) = \arg \max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\mathbf{z})\} \quad (3.88a)$$

$$= \arg \max_{\boldsymbol{\theta}} \{\ln p(\mathbf{z}|\boldsymbol{\theta}) + \ln p(\boldsymbol{\theta})\} \quad (3.88b)$$

The term $\ln p(\mathbf{z})$ does not depend on $\boldsymbol{\theta}$ therefore it may be dropped without affecting the estimate. The term $\ln p(\boldsymbol{\theta})$ accounts for the distribution of $\boldsymbol{\theta}$.

MAP Equation

Because the logarithm is a monotone function, it is often much more convenient to find the maximum of $\ln p(\boldsymbol{\theta}|\mathbf{z})$. If the maximum is interior to the allowable range of $\boldsymbol{\theta}$ and the derivative of $\ln p(\boldsymbol{\theta}|\mathbf{z})$ with respect to $\boldsymbol{\theta}$ is continuous then a necessary

but not sufficient condition for a maximum is

$$\left. \frac{\partial p(\boldsymbol{\theta}|\mathbf{z})}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}(\mathbf{z})} = 0 \quad (3.89)$$

3.3.3 Random Processes

A random processes may be viewed as a collection of random variables, with time t as a parameter running through all real numbers. Formally, a random process $X(t)$ is a mapping of elements of the sample space, S , into functions of time. Each element of the sample space is associated with a time function as shown in Fig. 3.2. Associating a function of time to each element of the sample space results in a family of functions known as the ensemble.

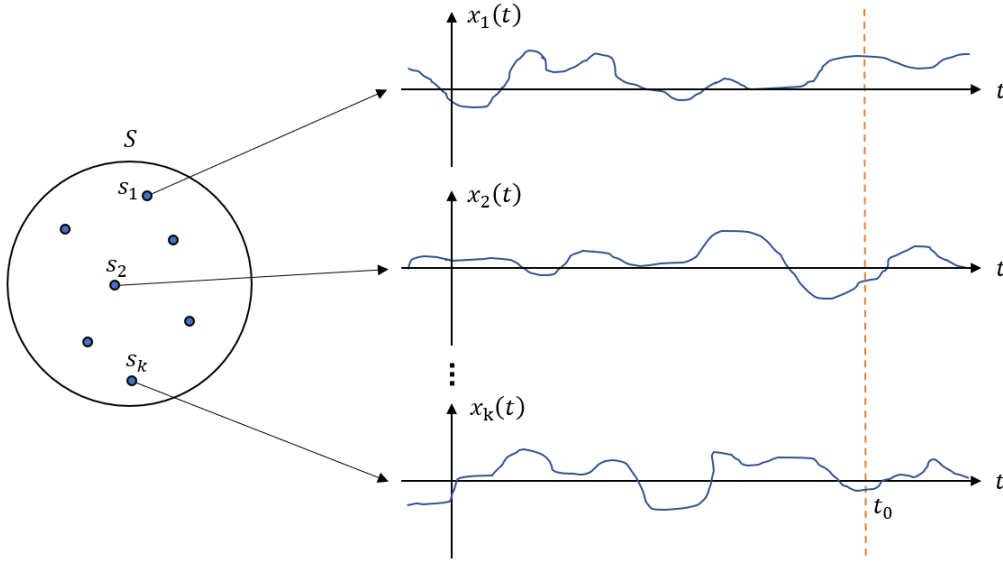


Figure 3.2: Ensemble of Sample Functions

Note that by convention the dependence of the random process on s is omitted. A sample function is denoted by $x(t)$. When the parameter $t = t_0$ is fixed $X(t_0)$ is a random variable and $x(t_0)$ is a sample value of the random variable. A continuous random process is a random process that is continuous in both time and in the sample space.

Expectations

Given a random process $X(t)$ its mean value function is

$$m_x(t) = \mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f_X(x, t) dx \quad (3.90)$$

The autocorrelation function is defined as

$$R_{xx}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)^*] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2^* f_{X_1 X_2}(x_1, x_2; t_1, t_2) dx_1 dx_2 \quad (3.91)$$

A random process $X(t)$ is wide-sense stationary (WSS) if its mean value function and autocorrelation is time invariant. This means that the mean value function is constant $m_x(t) = m_x$ and the autocorrelation $R_{xx}(t_1, t_2)$ is only a function of the time difference $t_2 - t_1$. A random process is strictly stationary if its statistics are unchanged by a time shift. A strictly stationary process is also wide-sense stationary, but not the opposite.

Power Spectral Density

If $X(t)$ is wide sense stationary then the power spectral density $S_{xx}(f)$ and the autocorrelation function $R_{xx}(\tau)$ are Fourier pairs.

$$S_{xx}(f) = \mathcal{F}[R_{xx}] \quad (3.92a)$$

$$R_{xx}(\tau) = \mathcal{F}^{-1}[S_{xx}] \quad (3.92b)$$

The relation between the autocorrelation function and the power spectral density is known as the Wiener–Khinchine theorem.

Gaussian Random Process

A random process $X(t)$ is a Gaussian random process if, for all $t_1, t_2, \dots, t_n$, the random variables $X(t_1), X(t_2), \dots, X(t_n)$ are jointly Gaussian. Further more if a Gaussian process is wide-sense stationary, it is also strict-sense stationary. If the input $X(t)$ of a linear system is a Gaussian process, the output will also be a Gaussian process.

White Gaussian Random Process

A random process $X(t)$ is a white Gaussian random process if $X(t)$ is a stationary Gaussian random process with zero mean, $m_x = 0$, and flat power spectral density

$$S_{xx}(f) = \sigma_x \quad (3.93)$$

Since the power spectral density of a white Gaussian random process is constant, its autocorrelation function $R_{xx}(\tau) = \sigma_x \delta(\tau)$ where δ is the Dirac delta function. This implies that the process has infinite power and as $R_{xx}(\tau) = 0$ for any $\tau \neq 0$ then $X(t_1)$ and $X(t_2)$ are independent Gaussian random variables for any $t_1 \neq t_2$. The white Gaussian random process is used to model the effects of thermal noise in electronic systems.

Complex White Gaussian Random Process

A random process $X(t)$ is a complex white Gaussian random process if

$$X(t) = X_r(t) + iX_i(t) \quad (3.94)$$

where $X_r(t)$ and $X_i(t)$ are independent jointly Gaussian, each with a flat power spectral density $\frac{N_0}{2}$. The power spectral density of $X(t)$ is therefore N_0 . The samples

$X(t_n)$ are complex Gaussian random variables with probability density

$$p(X(t_n)) = \frac{1}{\pi N_0} \exp\left(-\frac{|X(t_n)|^2}{N_0}\right) \quad (3.95)$$

3.3.4 Series Representation of Random Processes

Kosambi–Karhunen–Loève theorem allows the decomposition of a random process into a basis in a similar manner to the expansion of the deterministic energy functions.

$$x(t) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^N x_i \phi_i(t), \quad 0 \leq t \leq T \quad (3.96)$$

where $x_i = \int_0^T x(t) \phi_i(t)^* dt$. The notation l.i.m. denotes a limit in the mean

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left(x(t) - \sum_{i=1}^N x_i \phi_i(t) \right)^2 \right] = 0, \quad 0 \leq t \leq T. \quad (3.97)$$

3.4 Time Delay and Doppler Shift Estimation

We now wish to estimate the time delay and Doppler shift of a received signal in noise. The first step is to find the likelihood function of the time-delay and Doppler shift with respect to the received signal.

The complex envelope of a signal received by the radar may be written using a narrowband approximation as

$$z(t) = \beta\sqrt{E}u(t - \tau)e^{-2\pi ift} + w(t) \quad (3.98)$$

$w(t)$ is an independent zero-mean white complex Gaussian random process

$$\mathbb{E}[w(t)w^*(u)] = N_0\delta(t - u) \quad (3.99)$$

and $u(t)$ is the complex envelope of the received signal. The complex envelope is normalised $\int_{-\infty}^{\infty} |u(t)|^2 dt = 1$ and has energy E . Also, β is a zero-mean complex Gaussian random variable $\mathbb{E}[|\beta|^2] = 2\sigma_b^2$. The FIM depends on the likelihood function, to find the likelihood function note the equivalence between the likelihood function and the likelihood ratio test, the received waveform under each hypothesis is

$$z(t) = \begin{cases} \beta\sqrt{E}u(t - \tau)e^{-2\pi ift} + w(t) : \mathcal{H}_1 \\ w(t) : \mathcal{H}_0 \end{cases} \quad (3.100)$$

The first step is to find a sufficient statistic. Since the noise is white, we can expand using any complete orthonormal set of functions and obtain statistically independent coefficients. Choose the desired signal as the first orthonormal function and the resulting coefficient will be a sufficient statistic. In the complex case the resulting

coefficient is

$$z_1 = \int_{-\infty}^{\infty} z(t)u^*(t - \tau)e^{2\pi ift} dt \quad (3.101)$$

Substituting 3.98 into the above and simplifying

$$z_1 = \begin{cases} \beta\sqrt{E} + w_1 : \mathcal{H}_1 \\ w_1 : \mathcal{H}_0 \end{cases} \quad (3.102)$$

where $w_1 = \int_{-\infty}^{\infty} w(t)u^*(t - \tau)e^{2\pi ift} dt$. As both β and w_1 are zero mean random variables then $\mathbb{E}[z_1] = 0$ under both hypothesis. The variance under each hypothesis is

$$\text{var}[z_1] = \begin{cases} 2E\sigma_\beta^2 + N_0 : \mathcal{H}_1 \\ N_0 : \mathcal{H}_0 \end{cases} \quad (3.103)$$

The likelihood ratio is $\Lambda(z_1) = \frac{p(z|\mathcal{H}_1)}{p(z|\mathcal{H}_0)}$

$$\frac{p(z|\mathcal{H}_1)}{p(z|\mathcal{H}_0)} = \frac{\frac{1}{\pi(2E\sigma_\beta^2 + N_0)} e^{-\frac{1}{2E\sigma_\beta^2 + N_0}|z_1|^2}}{\frac{1}{\pi N_0} e^{-\frac{1}{N_0}|z|^2}} \quad (3.104)$$

which simplifies to

$$\frac{p(z|\mathcal{H}_1)}{p(z|\mathcal{H}_0)} = \frac{N_0}{2E\sigma_\beta^2 + N_0} \exp\left(\frac{2E\sigma_\beta^2}{N_0(2E\sigma_\beta^2 + N_0)}|z_1|^2\right) \quad (3.105)$$

Taking the logarithm

$$\ln\left(\frac{p(z|\mathcal{H}_1)}{p(z|\mathcal{H}_0)}\right) = \ln\left(\frac{N_0}{2E\sigma_\beta^2 + N_0}\right) + \frac{2E\sigma_\beta^2}{N_0(2E\sigma_\beta^2 + N_0)}|z_1|^2 \quad (3.106)$$

Note that $z_1 = z_1(\tau, f)$ then desired log-likelihood function is

$$\Lambda(\tau, f) = \frac{2E\sigma_\beta^2}{N_0(2E\sigma_\beta^2 + N_0)} \left| \int_{-\infty}^{\infty} z(t)u^*(t - \tau)e^{2\pi ift} dt \right|^2 \quad (3.107)$$

note that as $\ln\left(\frac{N_0}{2E\sigma_\beta^2 + N_0}\right)$ is a constant, it can be dropped. The FIM of the signals time-delay and Doppler shift is

$$\mathbf{J}_F(\tau, f) = -\mathbb{E} \left[\begin{pmatrix} \frac{\partial^2 \Lambda(\tau, f)}{\partial \tau \partial \tau} & \frac{\partial^2 \Lambda(\tau, f)}{\partial \tau \partial f} \\ \frac{\partial^2 \Lambda(\tau, f)}{\partial f \partial \tau} & \frac{\partial^2 \Lambda(\tau, f)}{\partial f \partial f} \end{pmatrix} \right] \quad (3.108)$$

Now look at the structure of the log-likelihood function, expanding the definition of $\Lambda(\tau, f)$ substituting the definition of the received signal (3.98) into (3.107) neglecting the constant terms. Define τ' and f' as the actual time-delay and Doppler shift of the received signal $z(t)$ then τ and f are the parameters of the log-likelihood function

$$\begin{aligned} \Lambda(\tau, f) = & E|\beta|^2 \left| \int_{-\infty}^{\infty} u(t - \tau')u^*(t - \tau)e^{2\pi i(f - f')t} dt \right|^2 \\ & + 2\Re \left[\int_{-\infty}^{\infty} \beta\sqrt{E}u(t - \tau')u^*(t - \tau)e^{2\pi i(f - f')t} dt \int_{-\infty}^{\infty} w(t)u^*(t - \tau)e^{2\pi ift} dt \right] \\ & + \left| \int_{-\infty}^{\infty} w(t)u^*(t - \tau)e^{2\pi ift} dt \right|^2 \end{aligned} \quad (3.109)$$

The first term in (3.109) is due only to the signal with time delay τ' and Doppler shift f' and is the only term that would be present in the absence of noise, this term is known as the signals Ambiguity Function.

3.4.1 Ambiguity Function

Define the Ambiguity Function (AF) of the complex envelope of a narrowband signal as

$$\chi(\tau, f) = \int_{-\infty}^{\infty} u(t)u^*(t - \tau)e^{2\pi ift} dt \quad (3.110)$$

The AF is a measure of the degree of similarity between a complex envelope and a replica of it that is shifted in time and frequency. Note that this definition of the AF is equivalent to the first term in (3.109) merely shifted such that the peak occurs at τ' and f' . An important property of the AF is that for a unitary signal $|\chi(0, 0)|^2 = 1$ and from the Cauchy–Schwarz inequality $|\chi(\tau, f)|^2 \leq |\chi(0, 0)|^2$.

Define $\mathbf{J}_{AF}$ as the matrix of the second-order derivatives of the squared amplitude of the AF.

$$\mathbf{J}_{AF} = \left(\begin{array}{cc} \frac{\partial^2 |\chi(\tau, f)|^2}{\partial \tau^2} & \frac{\partial^2 |\chi(\tau, f)|^2}{\partial \tau \partial f} \\ \frac{\partial^2 |\chi(\tau, f)|^2}{\partial f \partial \tau} & \frac{\partial^2 |\chi(\tau, f)|^2}{\partial f^2} \end{array} \right) \bigg|_{\tau, f=0} \quad (3.111)$$

Then the time-delay and Doppler shift FIM, $\mathbf{J}_F(\tau, f)$, can be expressed in terms of the SNR and the AF of the signal.

$$\mathbf{J}_F(\tau, f) = -\frac{\text{SNR}}{2} \mathbf{J}_{AF} \quad (3.112)$$

Where $\text{SNR} = \frac{4\sigma_\beta^2 E}{N_0} \left(\frac{2\sigma_\beta^2 E}{2\sigma_\beta^2 E + N_0} \right)$, if β is constant, $\text{SNR} = \frac{2E|b|^2}{N_0}$ [54]. Examining the structure of the time-delay and Doppler shift CRLB (3.112) and taking the logarithm.

$$\log(\text{CRLB}(\tau)) = -\log(\text{SNR}) + \log \left(\frac{-2\mathbf{J}_{AF}(2, 2)}{\det(\mathbf{J}_{AF})} \right) \quad (3.113a)$$

$$\log(\text{CRLB}(f)) = -\log(\text{SNR}) + \log\left(\frac{-2\mathbf{J}_{AF}(1,1)}{\det(\mathbf{J}_{AF})}\right) \quad (3.113b)$$

From the above equations the graph of the theoretical CRLB against SNR when plotted on a log-log axis is a straight line with a negative gradient. Furthermore, the properties of the ambiguity function of the signal determines only the y-intercept of the CRLB.

3.5 Ambiguity Functions of the Square and Gaussian Pulses

The main waveforms of interest in this research is the Square pulse and the Gaussian pulse. The square pulse is base waveform used by Aveillant, however the ambiguity function of the square pulse is not smooth at the origin which means that the joint range and Doppler CRLB doesn't exist. In order to proceed with the analytical calculations a fitted Gaussian pulse is used to approximate the Aveillant signal.

3.5.1 Square Pulse

Define a square pulse with pulse width ε centred around the origin and a train of κ pulses as

$$u_s(t) = \frac{1}{\sqrt{\varepsilon}} \text{Rect}\left(\frac{t}{\varepsilon}\right), \quad \text{Rect}\left(\frac{t}{\varepsilon}\right) = \begin{cases} 1 & \text{if } -\frac{\varepsilon}{2} \geq t \geq \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases} \quad (3.114a)$$

$$u_{\kappa,s} = \frac{1}{\sqrt{\kappa}} \sum_{m=1}^{\kappa} u_g(t - (m-1)\text{pri}) \quad (3.114b)$$

Note that centring the square pulse at the origin allows for a simplification of the analysis and the following results are valid for any constant time offset.

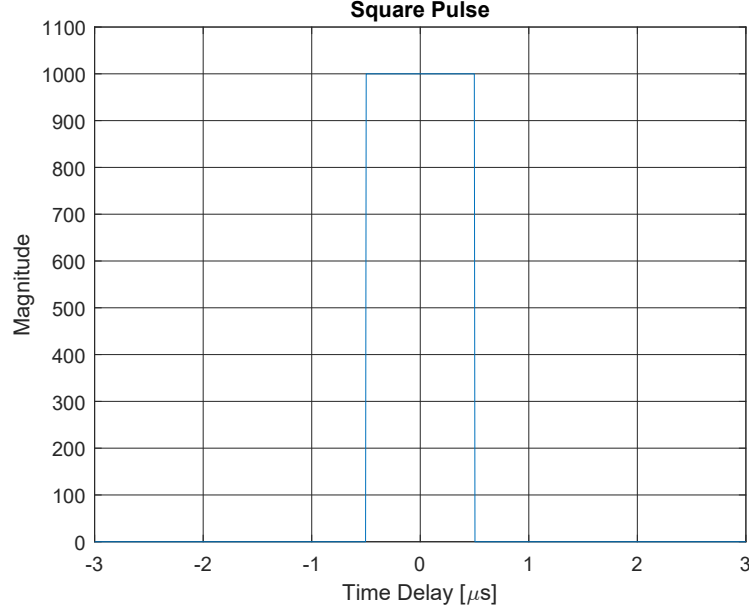


Figure 3.3: Square Pulse

Fig. 3.3 shows the Square pulse as defined by (3.114a) with pulse width $\varepsilon = 1\mu s$.

The Ambiguity function of a single square pulse is

$$\chi_s(\tau, f) = \int_{-\infty}^{\infty} u_s(t) u_s^*(t - \tau) e^{2\pi i f t} dt \quad (3.115a)$$

$$= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \text{Rect}\left(\frac{t}{\varepsilon}\right) \text{Rect}\left(\frac{t - \tau}{\varepsilon}\right) e^{2\pi i f t} dt \quad (3.115b)$$

Given the piecewise nature of the square pulse, split the integral into 4 cases. Firstly note that for $\tau < -\varepsilon$ and $\tau > \varepsilon$ there is zero overlap between the two pulses therefore the integrand and ambiguity function is zero. For $-\varepsilon \leq \tau \leq 0$, the overlap between the two rect's is bounded by $-\frac{\varepsilon}{2}$ and $\tau + \frac{\varepsilon}{2}$ therefore

$$\chi_s(\tau, f) = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\tau + \frac{\varepsilon}{2}} e^{2\pi i f t} dt, \quad \text{for } -\varepsilon \leq \tau \leq 0 \quad (3.116a)$$

$$= \frac{1}{2\pi i f \varepsilon} (e^{2\pi i f (\tau + \frac{\varepsilon}{2})} - e^{-2\pi i f \frac{\varepsilon}{2}}) \quad (3.116b)$$

$$= \frac{1}{\pi f \varepsilon} e^{\pi i f \tau} \sin(\pi f (\tau + \varepsilon)) \quad (3.116c)$$

$$= \left(1 + \frac{\tau}{\varepsilon}\right) e^{\pi i f \tau} \frac{\sin(\pi f(\tau + \varepsilon))}{\pi f(\tau + \varepsilon)} \quad (3.116d)$$

$$= \left(1 + \frac{\tau}{\varepsilon}\right) e^{\pi i f \tau} \text{sinc}(\pi f(\tau + \varepsilon)) \quad (3.116e)$$

For $0 \leq \tau \leq \varepsilon$, the overlap between the two rect's is bounded by $\tau - \frac{\varepsilon}{2}$ and $\frac{\varepsilon}{2}$ therefore

$$\chi_s(\tau, f) = \frac{1}{\varepsilon} \int_{\tau - \frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} e^{2\pi i f t} dt, \quad \text{for } 0 \leq \tau \leq \varepsilon \quad (3.117a)$$

$$= \frac{1}{2\pi i f \varepsilon} (e^{2\pi i f \frac{\varepsilon}{2}} - e^{2\pi i f(\tau - \frac{\varepsilon}{2})}) \quad (3.117b)$$

$$= \frac{1}{\pi f \varepsilon} e^{\pi i f \tau} \sin(\pi f(\varepsilon - \tau)) \quad (3.117c)$$

$$= \left(1 - \frac{\tau}{\varepsilon}\right) e^{\pi i f \tau} \frac{\sin(\pi f(\varepsilon - \tau))}{\pi f(\varepsilon - \tau)} \quad (3.117d)$$

$$= \left(1 - \frac{\tau}{\varepsilon}\right) e^{\pi i f \tau} \text{sinc}(\pi f(\varepsilon - \tau)) \quad (3.117e)$$

Note that it is possible to combine the above using the absolute value function

$$\chi_s(\tau, f) = \begin{cases} \left(1 - \frac{|\tau|}{\varepsilon}\right) e^{\pi i f \tau} \text{sinc}(\pi f(\varepsilon - |\tau|)) & \text{for } -\varepsilon \leq \tau \leq \varepsilon \\ 0 & \text{otherwise} \end{cases} \quad (3.118)$$

and

$$|\chi_s(\tau, f)|^2 = \begin{cases} \left(1 - \frac{|\tau|}{\varepsilon}\right)^2 \text{sinc}(\pi f(\varepsilon - |\tau|))^2 & \text{for } -\varepsilon \leq \tau \leq \varepsilon \\ 0 & \text{otherwise} \end{cases} \quad (3.119)$$

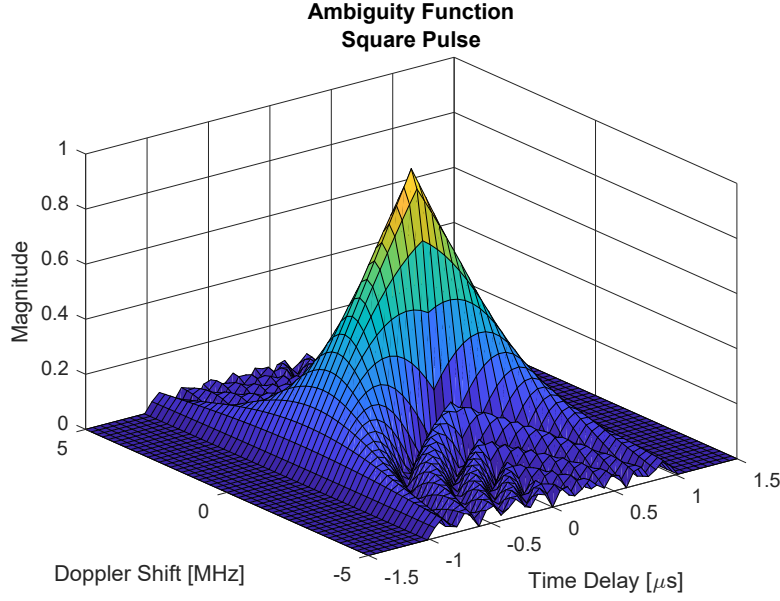


Figure 3.4: Magnitude of the Square Pulse Ambiguity Function

Fig. 3.4 shows a surface plot of the Ambiguity Function of a single Square Pulse with scaling parameter $\varepsilon = 1 \mu\text{s}$.

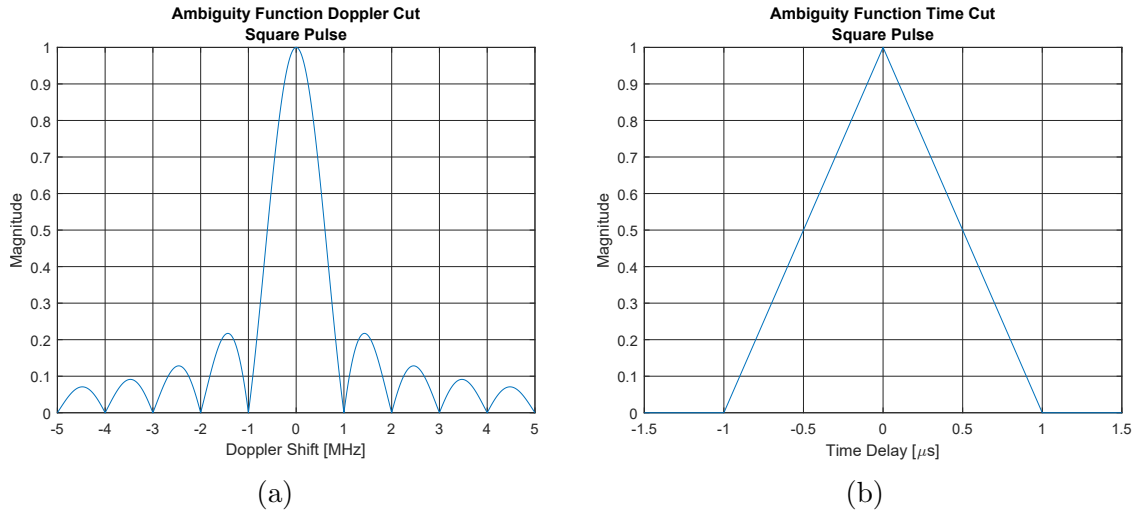


Figure 3.5: Zero Time and Doppler slices of the Square Pulse Ambiguity Function

Fig. 3.5 shows both the Doppler cut $|\chi_s(0, f)|$ and the time cut $|\chi_s(\tau, 0)|$ of the magnitude of the square pulse AF. The Doppler cut Fig. 3.5a is a sinc function with nulls at integer multiples of $\frac{1}{\varepsilon}$. The time cut Fig. 3.5b is a triangle function with base 2ε . The Square Pulse AF is not smooth at the origin, therefore the CRLB of an unbiased joint estimator of time-delay and Doppler-shift estimator does not

exist, thus the need for the Gaussian pulse.

3.5.2 Gaussian Pulse

Deriving the estimation performance of the Gaussian pulse, for use in the remaining sections. The Gaussian pulse is defined by equation (3.120a)

$$u_g(t) = \left(\frac{1}{\pi\nu^2} \right)^{\frac{1}{4}} \exp \left(-\frac{t^2}{2\nu^2} \right) \quad (3.120a)$$

$$u_{\kappa,g}(t) = \frac{1}{\sqrt{\kappa}} \sum_{m=1}^{\kappa} u_g(t - (m-1)\text{pri}) \quad (3.120b)$$

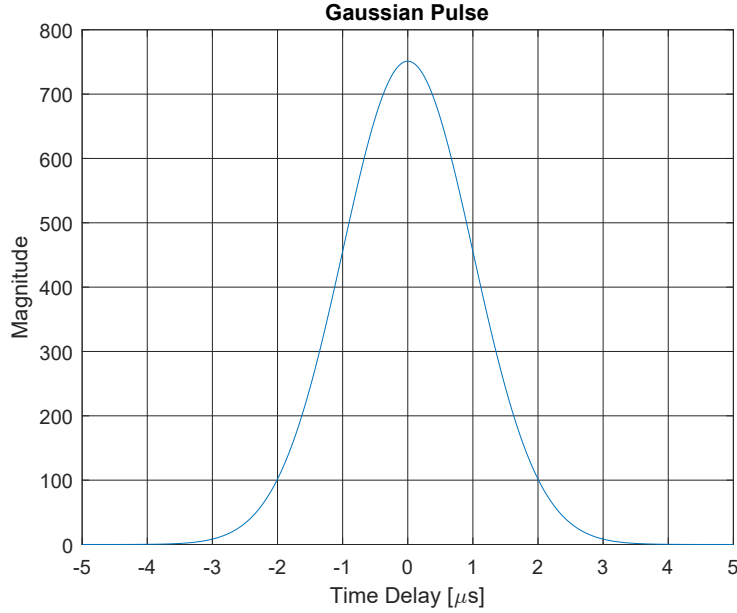


Figure 3.6: Gaussian Pulse

Fig. 3.6 shows the Gaussian pulse defined by (3.120a) with parameter $\nu = 1\mu$ s.

The Ambiguity function of a single Gaussian pulse is

$$\chi_g(\tau, f) = \int_{-\infty}^{\infty} u_g(t) u_g^*(t - \tau) e^{2\pi i f t} dt \quad (3.121a)$$

$$= \frac{1}{\nu\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{\nu^2}t^2 + \frac{(\tau+2\pi i\nu^2 f)}{\nu^2}t - \frac{\tau^2}{2\nu^2}} dt \quad (3.121b)$$

Which can be generalised as

$$\chi_g(\tau, f) = \frac{1}{\nu\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-at^2 - bt - c} dt \quad (3.122)$$

Where $a = \frac{1}{\nu^2}$, $b = -\frac{(\tau+2\pi i\nu^2 f)}{\nu^2}$ and $c = \frac{\tau^2}{2\nu^2}$ if $a \geq 0$ then the integral has the following solution.

$$\int_{-\infty}^{\infty} e^{-at^2 - bt - c} dt = \frac{\sqrt{\pi}}{\sqrt{a}} e^{-\frac{4ac-b^2}{4a}}, \quad a > 0 \quad (3.123)$$

The scaling parameter $\nu > 0$ therefore $\frac{1}{\nu^2} > 0$ therefore the standard solution can be used. Substituting in the definitions of a, b and c into equation (3.123) and simplifying

$$4ac - b^2 = 4 \left(\frac{1}{\nu^2} \right) \left(\frac{\tau^2}{2\nu^2} \right) - \left(-\frac{(\tau + 2\pi i\nu^2 f)}{\nu^2} \right)^2 \quad (3.124a)$$

$$= \frac{\tau^2 - 4\tau\pi i\nu^2 f + 4\pi^2\nu^4 f^2}{\nu^4} \quad (3.124b)$$

$$\frac{4ac - b^2}{4a} = \frac{\tau^2 - 4\tau\pi i\nu^2 f + 4\pi^2\nu^4 f^2}{4\nu^2} \quad (3.124c)$$

Combining the above, the ambiguity function of a single Gaussian pulse and its squared amplitude the complex ambiguity function is

$$\chi_g(\tau, f) = e^{\pi i \tau f - \pi^2 \nu^2 f^2 - \frac{\tau^2}{4\nu^2}} \quad (3.125a)$$

$$|\chi_g(\tau, f)|^2 = e^{-2\pi^2 \nu^2 f^2 - \frac{\tau^2}{2\nu^2}}. \quad (3.125b)$$

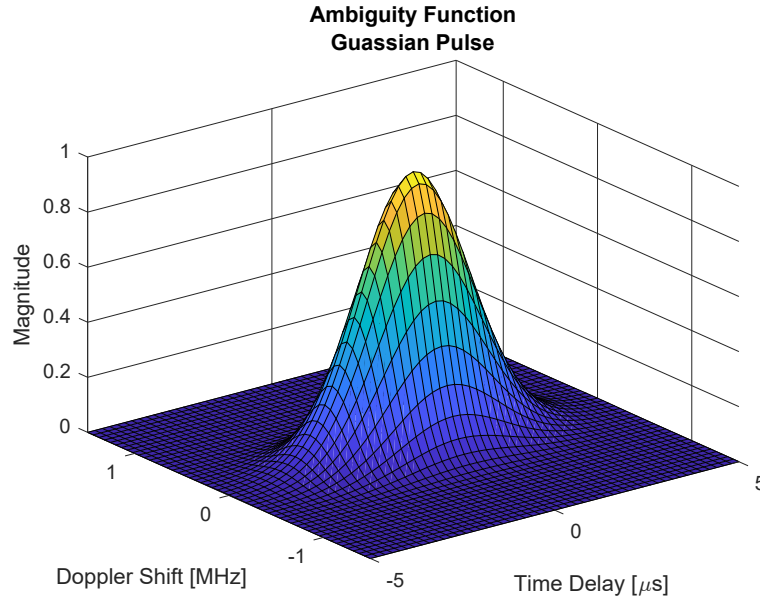


Figure 3.7: Magnitude of the Gaussian Pulse Ambiguity Function

Fig. 3.7 shows a surface plot of the magnitude of the Gaussian Pulse AF (3.125a) with parameter $\nu = 1 \mu$ s.

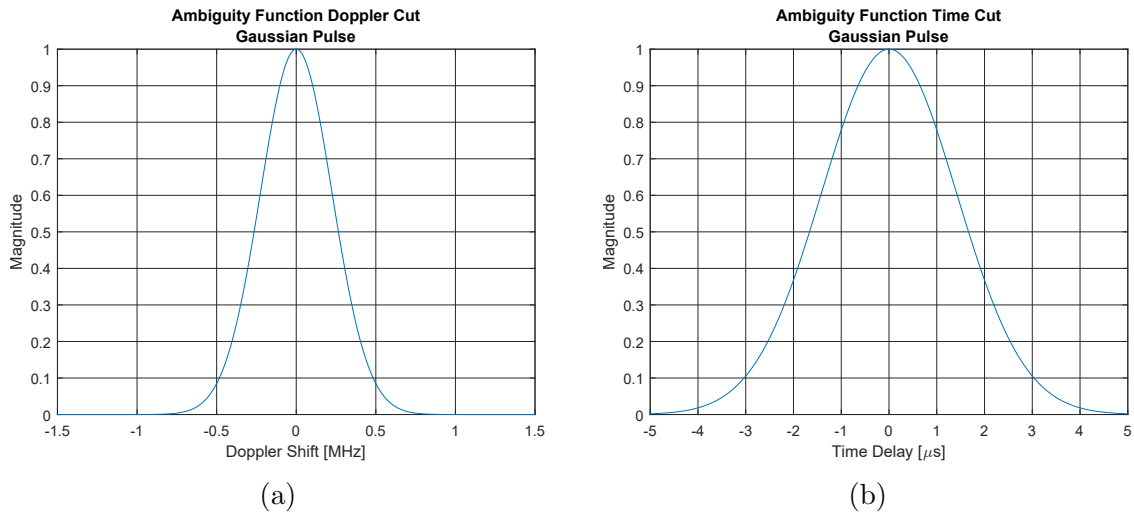


Figure 3.8: Zero Time and Doppler cuts of the Gaussian Pulse Ambiguity Function

Fig. 3.8 shows the Doppler cut $|\chi_g(0, f)|$ and the time cut $|\chi_g(\tau, 0)|$, in both cases the cuts are Gaussian curves.

3.6 Estimating Time-Delay and Doppler from the Gaussian Pulse

Proceeding with calculating the elements of $\mathbf{J}_{AF}$ for the Gaussian Pulse directly from the Ambiguity Function (3.125b) and (3.125a). Starting with the derivatives with respect to τ

$$\frac{\partial |\chi_g(\tau, f)|^2}{\partial \tau} = -\frac{\tau}{\nu^2} e^{-2\pi^2 \nu^2 f^2 - \frac{\tau^2}{2\nu^2}} \quad (3.126a)$$

$$= -\frac{\tau}{\nu^2} |\chi_g(\tau, f)|^2 \quad (3.126b)$$

$$\frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial \tau^2} = \frac{1}{\nu^2} \left(\frac{\tau^2}{\nu^2} - 1 \right) |\chi_g(\tau, f)|^2. \quad (3.126c)$$

Differentiating with respect to f

$$\frac{\partial |\chi_g(\tau, f)|^2}{\partial f} = -4\pi^2 \nu^2 f e^{-2\pi^2 \nu^2 f^2 - \frac{\tau^2}{2\nu^2}} \quad (3.127a)$$

$$= -4\pi^2 \nu^2 f |\chi_g(\tau, f)|^2 \quad (3.127b)$$

$$\frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial f^2} = 4\pi^2 \nu^2 (4\pi^2 \nu^2 f^2 - 1) |\chi_g(\tau, f)|^2. \quad (3.127c)$$

With cross derivatives

$$\frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial \tau \partial f} = -4\pi^2 \nu^2 f \frac{\partial |\chi_g(\tau, f)|^2}{\partial \tau} \quad (3.128a)$$

$$= 4\pi^2 \tau f |\chi_g(\tau, f)|^2 \quad (3.128b)$$

$$\frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial f \partial \tau} = -\frac{\tau}{\nu^2} \frac{\partial |\chi_g(\tau, f)|^2}{\partial f} \quad (3.128c)$$

$$= 4\pi^2 \tau f |\chi_g(\tau, f)|^2. \quad (3.128d)$$

Evaluating the above expressions at the origin ($\tau, f = 0$) and noting that $u(t)$ is unitary implying that $|\chi_g(0, 0)|^2 = 1$

$$\left. \frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial \tau^2} \right|_{\tau, f=0} = -\frac{1}{\nu^2} \quad (3.129a)$$

$$\left. \frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial f^2} \right|_{\tau, f=0} = -4\pi^2 \nu^2 \quad (3.129b)$$

$$\left. \frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial f \partial \tau} \right|_{\tau, f=0} = 0 \quad (3.129c)$$

$$\left. \frac{\partial^2 |\chi_g(\tau, f)|^2}{\partial \tau \partial f} \right|_{\tau, f=0} = 0. \quad (3.129d)$$

Combining the above with equation (3.112) the time-delay and Doppler shift FIM for the Gaussian pulse is given by

$$\mathbf{J}(\tau, f) = \frac{\text{SNR}}{2} \begin{pmatrix} \frac{1}{\nu^2} & 0 \\ 0 & 4\pi^2 \nu^2 \end{pmatrix} \quad (3.130)$$

The FIM is evaluated for only a single pulse, the FIM is extended to a train of κ pulses [56].

$$\mathbf{J}(\tau, f) = \frac{\text{SNR}}{2} \begin{pmatrix} \frac{1}{\nu^2} & 0 \\ 0 & 4\pi^2 \nu^2 + \frac{2}{3}\pi^2 \text{pri}^2(\kappa^2 - 1) \end{pmatrix} \quad (3.131)$$

The bistatic range of the target is $r = \tau c$ then $\text{var}[r] = c^2 \text{var}[\tau]$

$$\mathbf{J}(r, f) = \frac{\text{SNR}}{2} \begin{pmatrix} \frac{1}{c^2 \nu^2} & 0 \\ 0 & 4\pi^2 \nu^2 + \frac{2}{3}\pi^2 \text{pri}^2(\kappa^2 - 1) \end{pmatrix} \quad (3.132)$$

3.7 Monte-Carlo Simulations of the Gaussian Pulse

Having developed both the MLE and CRLB for estimating the time-delay and Doppler shift of a Gaussian pulse train, now demonstrate how the MLE tends to the CRLB for high SNR by running Monte-Carlo simulations.

$$z(t) = \sqrt{E}u_{64,g}(t)(t - \tau)e^{-2\pi ift} + w(t) \quad (3.133)$$

The noise term $w(t)$ is modelled as complex white Gaussian random process with a power spectral density N_0 . The signal to noise ratio is defined as $\text{SNR} = \frac{2E}{N_0}$. The simulation ran 5000 Monte-Carlo trials for each of the 16 different SNR values. The simulation SNR ranged from 0 dB to 30 dB. Further simulation parameters are tabulated in Table. 3.3.

Simulation Parameter	Value	Unit
Pulse repetition frequency	7200	Hz
Carrier frequency	L-Band	GHz
Number of pulses	64	-
Gaussian pulse scaling parameter (ν)	1	μs
Receiver bandwidth	1	MHz
Number of Monte-Carlo trials	5000	-
SNR range	0 – 30	dB
Number of SNR sample points	16	-

Table 3.3: Monte-Carlo Simulation Parameters

The targets bistatic range is defined as $r = c\tau$ where τ is the signal time-delay and c is the speed of light in a vacuum.

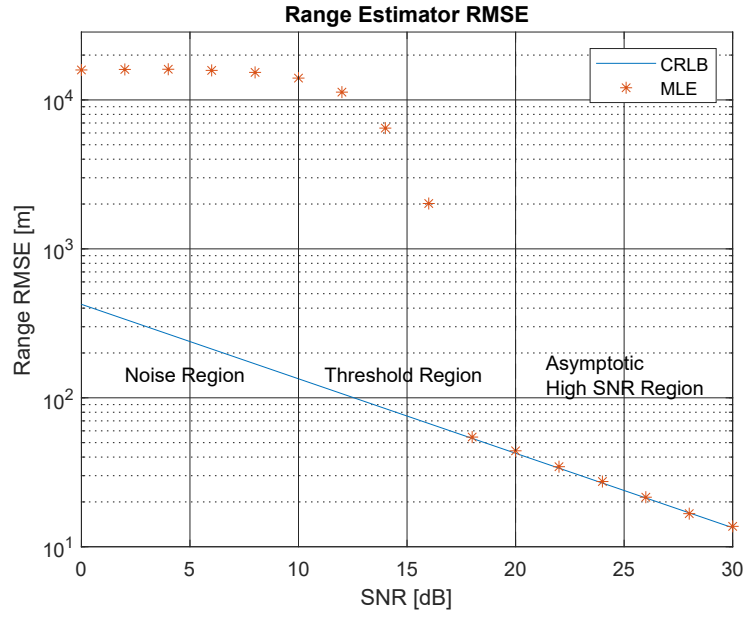


Figure 3.9: Monte-Carlo Simulations of Range Estimator

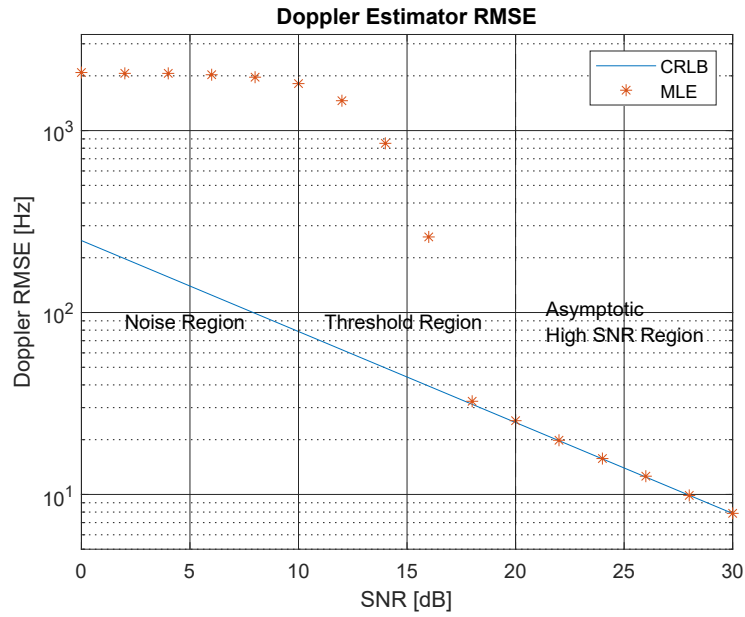


Figure 3.10: Monte-Carlo Simulations of Doppler Estimator

In both Fig. 3.9 and Fig. 3.10 the performance of the joint range and Doppler MLE is shown against SNR. The plots are split into three SNR regions, noise, threshold and asymptotic/ high SNR region.

In the noise region, the signal power is less than the noise floor, therefore in each

Monte-Carlo trial the MLE is sampling the highest point on the noise floor which changes location every iteration. The estimator error in this case is limited by the dynamic range of the simulation.

In the threshold region the estimator is starting to sample the peak, but there is still considerable interference from the noise. As the SNR increases from 10 dB to 20 dB the estimator is sampling the peak of the AF correctly more often and the estimator error decreases.

In the asymptotic/ high SNR region, the estimator is highly likely to be correctly sampling the peak of the AF and the error attains the CRLB.

When running these Monte-Carlo simulations care must be taken when deciding on the dynamic range of the simulations. In both Fig. 3.9 and Fig. 3.10 in the noise region the RMSE of the estimator is limited by the maximum extent of the range Doppler map. As the SNR increases and the estimator is in the asymptotic region the minimum attainable RMSE is limited by the cell size of the range Doppler map. The cell size can then be set by computing the relevant lower bound for a given SNR and setting the cell size to be a tenth of found bound.

3.8 Conclusion

In this section the background theory necessary for the development of the theoretical results presented in the remaining chapters. The key points of this chapter are the presentation of both the Cramér Rao Lower Bound (CRLB) and the Bayesian Lower Bound (BLB) as well as the Maximum Likelihood Estimator (MLE) and the Maximum a Posteriori (MAP) estimator. The Ambiguity Function is also introduced, and its connection to the theoretical CRLB of a time-delay and Doppler shift

estimator from a received narrowband signal is presented. The ambiguity function and theoretical lower bounds are necessary for the development of the theoretical algorithms presented in chapter 5 Network Algorithms.

Chapter 4

Aveillant Holographic Radar

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Staring or ubiquitous radar is described by Skolnik as a radar utilising a low gain or quasi-omnidirectional transmitting antenna and a fixed receiver array, capable of performing real-time digital beamforming [57]. Such techniques require significant computational resources that have previously not been available. Commercially available high-performance GPUs and FPGAs can now perform these tasks. Staring array radars employ long coherent integration times, enabling fine Doppler resolution and increased detection of low RCS targets.

The Aveillant Holographic radar is a staring radar, sometimes referred to in the literature as ubiquitous, utilising a static transmitter and receiving array to perform digital beamforming. The receiver array is fixed and constructed using a modular arrangement of tiles, each containing four antenna elements in a grid pattern. The Aveillant Holographic radar currently exists in two forms, the Theia radar, designed as an infill radar for wind farm mitigation and the Gamekeeper radar, designed as a counter-drone radar. The 2D array provides vertical separation of beams improving the resolution of low flying targets. The long integration time and resulting fine Doppler resolution improves the detection of slow-moving objects close to the ground and non-stationary clutter. The staring principle results in an effective 100% dwell time on the target, increasing the energy on the target, allowing the detection of low RCS targets at greater ranges.

4.1 Aveillant Gamekeeper Radar

The Gamekeeper radar is designed to detect and track drones, targets with a nominal RCS of 0.01m^2 out to a range of 5 km. The receiving array contains 64 elements arranged in a 4×16 vertical configuration.



Figure 4.1: Aveillant Gamekeeper Radar

Fig. 4.1 shows the Gamekeeper radome containing the transmit and receiver antennas and the service container housing the signal processing and data logging equipment. The Gamekeeper radar transmits a pulsed waveform in the L-band (1-2 GHz) with a pulse repetition interval of approximately 7.5 kHz. As previously noted, the radar does not rotate; therefore, coverage is limited to a 90° sector in azimuth and 30° in elevation. Table 4.1 contains a summary of the general properties of the Gamekeeper radar [8].

Gamekeeper Parameter	Value	Unit
Frequency	L-Band	Hz
Azimuth coverage	90°	-
Elevation coverage	30°	-
Receiver array	4×16	-
Pulse repetition frequency	~ 7.5	kHz
Update rate	4	Hz
Polarisation	Vertical	-
Instrumented range	7.5	km

Table 4.1: Gamekeeper Radar Parameters

The radar is capable of detecting of drones, targets with an RCS of 0.01 m^2 out to a range of 5 km. The radar nominally coherently integrates 2048 pulse at a pulse repetition rate of 7500 Hz resulting in a coherent processing interval of approximately 0.27 s. After each coherent processing interval the radar generates new detections

which are applied to the tracker and therefore updates at a rate of approximately 4 Hz. The performance of the Gamekeeper radar has been well reported in the literature, notably the involvement of the radar in the EU-SESAR projects aimed allowing the safe and controlled operation of drones within an already congested airspace. The radar aims to detect non-cooperative drones that would otherwise not normally be detected by existing primary surveillance radar [58]

One of the disadvantages of the increased detection sensitivity leads to a classification challenge as birds and drones have similar RCS and occupy the same volume of airspace. Robust classification and suppression of bird tracks is crucial to ensuring that the radar is able to provide reliable reporting of drone tracks [7] [59] [60].

The fine Doppler resolution of the radar improves the velocity resolution and acceleration characterisation of the target, which aids discrimination between drones and birds due to the differences resulting from their flight mechanics [15]. The fine Doppler resolution also enables extraction of a drone's micro-Doppler signature produced by the rotation of the drones rotor blades [15]. It is possible to associate the micro-Doppler signature with the main body return of the drone aiding classification and potentially tracking the target through the zero-Doppler clutter.

4.2 Gamekeeper Power Budget

Component	Property	Symbol	Value	Unit
Transmitter	Transmit power	P_T	2	kW
	Transmitter antenna gain	G_T	12.5	dBi
	Carrier frequency	f_c	L-Band	Hz
Environment	Atmospheric attenuation	L	0	dB
Target	Radar cross-section	σ	0.01	m ²
	Nominal range	R	0.3 – 5	km
Receiver	Receiver antenna gain	G_R	5.5	dBi
	Receiver noise power	P_N	-	W
Signal processing	Matched filter SNR gain	G_{MF}	10	dB
	Integration SNR gain	G_I	31.7	dB
	Beamforming SNR gain	G_{BF}	18	dB

Table 4.2: Gamekeeper Power Budget

The Gamekeeper radar integrates 2048 pulses using a Hamming window for side-lobe suppression. The final SNR before the detector is

$$\text{SNR}_D = \frac{P_T G_T G_R \lambda^2 L \sigma}{(4\pi)^3 P_N R^4} G_{MF} G_I G_{BF} \quad (4.1a)$$

$$= \frac{P_T G_T G_R \lambda^2 L \sigma}{(4\pi)^3 P_N R_T^2 R_R^2} G_{MF} G_I G_{BF} \quad (4.1b)$$

where R_T and R_R represent the target-transmitter range and target-receiver range. Equation (4.1a) is the monostatic version of the radar range equation and equation (4.1b) is the bistatic version. Note that in both cases the targets RCS is assumed to be the same.

4.3 Fitting a Gaussian Pulse

As previously noted, the ambiguity function of a train of square pulses is not smooth at the origin, which means that the joint time delay and Doppler shift derivatives of

the ambiguity function do not exist. Therefore the theoretical CRLB also does not exist. In order to progress with the theoretical analysis an alternative signal, the Gaussian pulse, must be used. The parameters of the Gaussian pulse are selected to best approximate the properties of the Aveillant signal.

Now proceed by fitting the Gaussian pulse based on the energy contained within the square pulse. A pulse width of $2 \mu\text{s}$ is used to model the Aveillant signal. The energy of a Gaussian pulse contained in the interval $t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is

$$E_{u_g} = \left(\frac{1}{\pi\nu^2} \right)^{\frac{1}{2}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} e^{-\frac{t^2}{\nu^2}} dt \quad (4.2a)$$

$$= \text{erf} \left(\frac{\varepsilon}{2\nu} \right) \quad (4.2b)$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the Gaussian error function. For a given pulse width ε select the value of ν that ensures that the desired amount of energy is contained on $t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$.

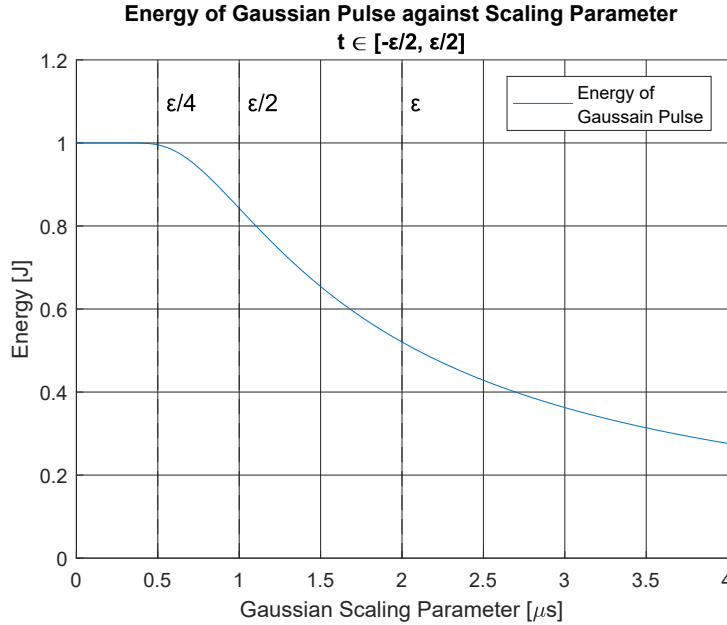


Figure 4.2: Energy of Gaussian Pulse against Scaling Parameter

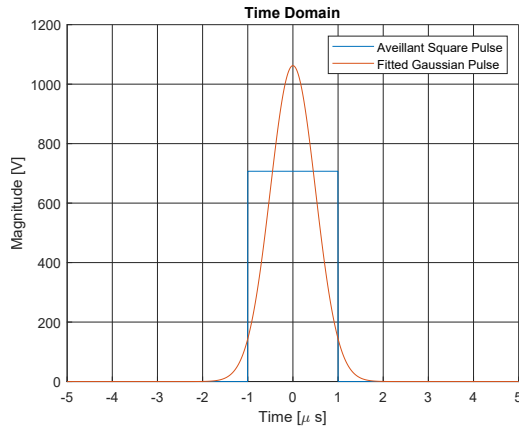
Fig. 4.2 shows how the energy of the Gaussian pulse contained within the square

pulse width changes as ν varies. As ν decreases the width of the Gaussian pulse decreases and more energy is contained within the square pulse. Specific values of ν are marked and tabulated in Table 4.3

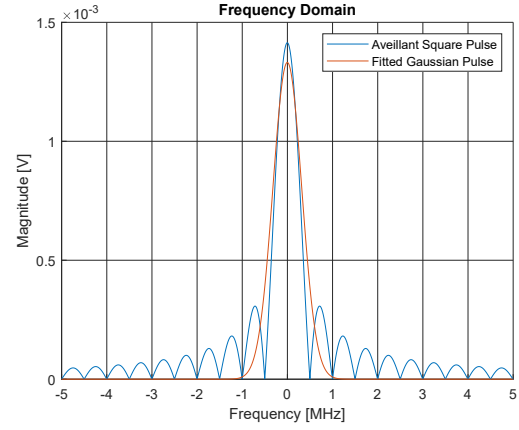
Value of ν	Ratio ε/ν	Percentage of signal energy
$2 \mu s$	1	50%
$1 \mu s$	0.5	84%
$0.5 \mu s$	0.25	99.5%

Table 4.3: Energy of Gaussian Pulse against Scaling Parameter

Selecting $\nu = 0.5 \mu s$ ensures that 99.5% of the energy is contained within the pulse width of the Aveillant square pulse.



(a) Time Domain Comparison



(b) Frequency Domain Comparison

Figure 4.3: Aveillant Square Pulse and Fitted Gaussian Pulse Comparison

Fig. 4.3a and Fig. 4.3b shows the time domain and Doppler domain comparison of the Aveillant square pulse and the fitted Gaussian pulse respectively. Both pulses are normalised to have unit energy.

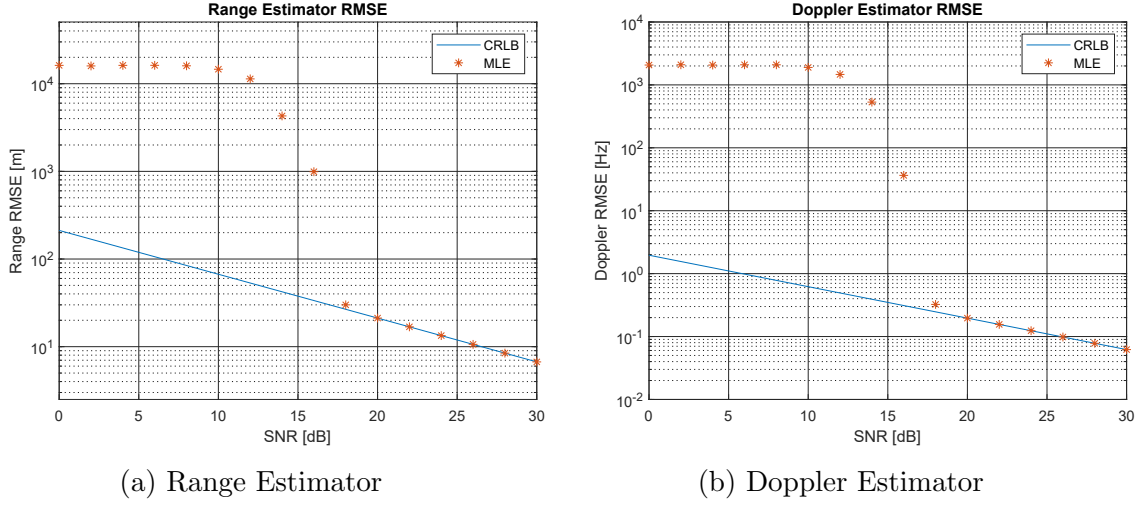


Figure 4.4: Range and Doppler Estimation Performance of the Fitted Gaussian Pulse

Fig. 4.4a and Fig. 4.4b show Monte-Carlo simulations of the range and Doppler estimation performance of 2048 fitted Gaussian pulses. The simulation ran 5000 samples for each of the 16 SNR points. The estimator attains the CRLB at an SNR of 20 dB in both cases.

4.4 Conclusion

This chapter introduces the Aveillant Gamekeeper radar and presents the power budget. The key section of this chapter is the fitting of the Gaussian pulse. This is necessary because the ambiguity function square pulse, used by Aveillant, has a sharp peak at zero, this means that the derivative of the ambiguity function in the range dimension does not exist, and therefore, the joint theoretical lower bound for estimating the range and Doppler-shift of a square pulse also does not exist. The ambiguity function of the Gaussian pulse does have continuous derivatives at zero, and therefore the theoretical lower bound does exist. The Gaussian pulse is fitted to ensure that the subsequent theoretical matches the Aveillant radar as much as possible.

Chapter 5

Network Algorithms

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5.1 Data Fusion

Data fusion is the process of combining the measurements recorded across the radar network. This chapter develops two algorithms, the first uses a distributed data fusion scheme, and the second uses a centralised data fusion scheme. Under each scheme, the theoretical bound on the network's estimation performance relative to the target's position and velocity is calculated and used to determine the optimal geometry with respect to the estimation errors. The performance of the optimal geometry is verified using Monte-Carlo simulations of the MLE and MAP estimators.

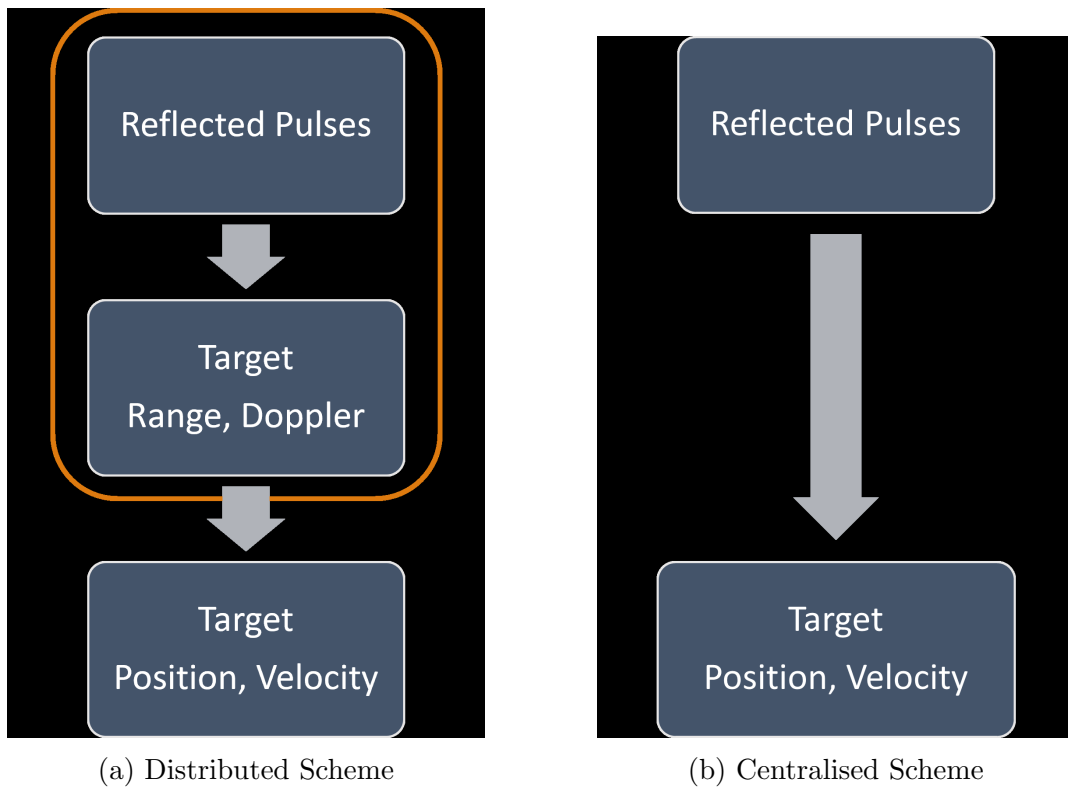


Figure 5.1: Data Fusion Schemes

Fig. 5.1 illustrates two data fusion schemes, on the left a distributed scheme and on the right a centralised scheme. A distributed network estimates the targets range and Doppler-shift locally at each receiver, indicated by the orange perimeter. The targets range and Doppler-shift estimates are gathered centrally where the targets position and velocity are estimated. Crucially, only a subset of the data is sent to

the central processor. A network employing a centralised scheme gathers all the available data from each receiver before jointly estimating the targets position and velocity.

5.2 Distributed Data Fusion

Under the distributed data fusion scheme, the targets range and Doppler is estimated locally at each receiver. The range and Doppler estimates are centrally gathered, and the targets state is estimated from the gathered measurements. The target state vector $\boldsymbol{\theta}$ contains the targets position $\mathbf{p}$ and velocity $\mathbf{v}$ in two dimensions, with out loss of generality these vectors are

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \quad (5.1)$$

Note the target's state and receiver position's are defined in 2D, in order to reduce the dimension of the target's state and simplify the implementation of the MLE. Consider a multistatic radar network containing a single static staring transmitter and N multistatic receivers. Denote the position of the transmitter as $\mathbf{p}_o$ and the position of the n -th receiver as $\mathbf{p}_n$ for $n \in [1, N]$.

$$\mathbf{p}_o = \begin{pmatrix} p_{xo} \\ p_{yo} \end{pmatrix}, \quad \mathbf{p}_n = \begin{pmatrix} p_{xn} \\ p_{yn} \end{pmatrix} \quad (5.2)$$

The transmitter $\mathbf{p}_o$ and first receiver $\mathbf{p}_1$ are co-located and model the central staring monostatic radar. The remaining receivers $\mathbf{p}_n$ for $n \in [2, N]$ model the bistatic receivers that make up the remainder of the network.

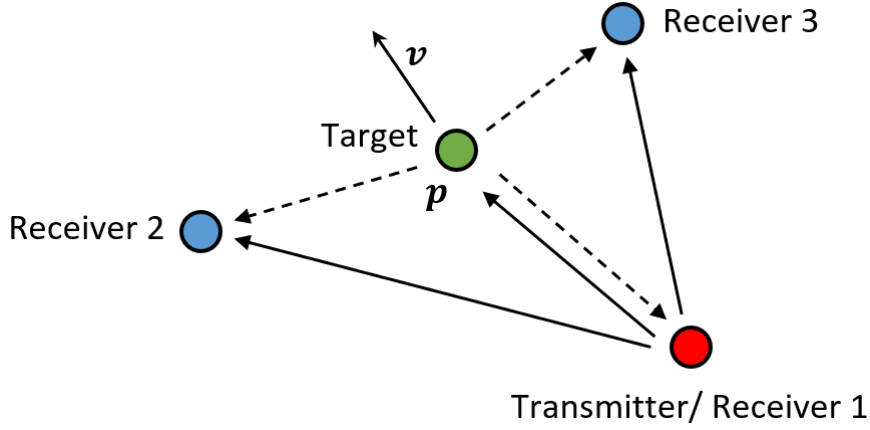


Figure 5.2: Illustrative Network Geometry

Fig. 5.2 illustrates a multistatic radar network consisting of a single central staring radar and two bistatic receivers. A key concept of the cooperative radar network are the passive bistatic receivers which complement the performance of the central radar.

Fig. 5.2 shows how the bistatic receivers measures both the transmitted signal and reflections from the target. In practice, this is accomplished using two separate channels, a reference channel, which records the transmitted signal, and a surveillance channel, which records the reflections from the target.

5.2.1 Signal Model

Each receiver in the network receives a noisy, time-delayed and Doppler-shifted version of the transmitted signal with complex envelope $u(t)$ which, after down-conversion, may be expressed as

$$z_n(t, \boldsymbol{\theta}) = \sqrt{E_n} u(t - \tau_n(\boldsymbol{\theta})) e^{2\pi i f_n(\boldsymbol{\theta})t} + w_n(t) \quad (5.3)$$

where $\tau_n(\boldsymbol{\theta})$ and $f_n(\boldsymbol{\theta})$ represent the range and the Doppler shift of the target's signal received at the n th receiver respectively. The complex envelope, of the signal, $u(t)$ is normalised and E_n is the received signals energy. The signal noise is a zero-mean

complex symmetric white Gaussian random process with power spectral density N_0 . Note that as in the previous example the effects of clutter are not considered in order to simplify the model, as such it should be noted that in the presence of clutter the found optimal geometry and estimated performance presented in this chapter may not be applicable. The relative bistatic range of the target, defined as $r_n(\boldsymbol{\theta}) = c\tau_n(\boldsymbol{\theta})$ where c is the speed of light, is used instead of τ_n .

$$r_n(\boldsymbol{\theta}) = |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \quad (5.4a)$$

$$f_n(\boldsymbol{\theta}) = \frac{v}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.4b)$$

Equations (5.4a) and (5.4b) explicitly state how the measured range and Doppler shift depend on the target's state and the geometry of the network. Estimating the target's range and Doppler shift from the received signal (5.3) is dependent on the properties of the ambiguity function of the complex envelope and the signal to noise ratio at the receiver [34].

The distribution of the estimates of the maximum likelihood estimator asymptotically approaches a Gaussian distribution with covariance equal to the inverse of the Fisher information matrix. Furthermore the estimates of the MLE approaches this bound when the signal to noise ratio is high [34]. In practise, this is achieved for signal to noise ratios of +20 dB, see section 3.6 through to 3.7 for further details. The measurement model at the n -th receiver is

$$\mathbf{z}_n = \boldsymbol{\mu}_n(\boldsymbol{\theta}) + \mathbf{e} = \begin{pmatrix} r_n(\boldsymbol{\theta}) \\ f_n(\boldsymbol{\theta}) \end{pmatrix} + \begin{pmatrix} e_{r_n} \\ e_{f_n} \end{pmatrix} \quad (5.5)$$

The distribution of the measurement vector $\mathbf{z}_n$ is asymptotically a multivariate

Gaussian, $\mathbf{z}_n \sim \mathcal{N}(\boldsymbol{\mu}_n(\boldsymbol{\theta}), \Sigma_n)$ where the covariance is the inverse of the range and Doppler Fisher information matrix (3.112).

$$\Sigma_n = -\frac{2}{\text{SNR}_n} \mathbf{J}_{AF}^{-1} \quad (5.6)$$

where $\text{SNR}_n = \frac{2E_n}{N_0}$. These measurements are dependent on the target parameters $\boldsymbol{\theta}$ and the geometry of the radar network.

5.2.2 Fusion of the Range and Doppler measurements

The measurements from each receiver z_n in the radar network are gathered and combined into a single vector $\mathbf{z}$.

$$\mathbf{z} = \boldsymbol{\mu}(\boldsymbol{\theta}) + \mathbf{e} = \begin{pmatrix} r_1(\boldsymbol{\theta}) \\ f_1(\boldsymbol{\theta}) \\ \vdots \\ r_N(\boldsymbol{\theta}) \\ f_N(\boldsymbol{\theta}) \end{pmatrix} + \begin{pmatrix} e_{r_1} \\ e_{f_1} \\ \vdots \\ e_{r_N} \\ e_{f_N} \end{pmatrix} \quad (5.7)$$

The combined data $\mathbf{z}$ asymptotically approaches a Gaussian random vector with mean $\boldsymbol{\mu}(\boldsymbol{\theta})$ and covariance matrix Σ . The mean value $\boldsymbol{\mu}(\boldsymbol{\theta})$ contains the targets range and Doppler shift. The vector $\mathbf{e}$ contains the estimation error.

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Sigma_N \end{pmatrix} \quad (5.8)$$

The covariance matrix Σ is a block diagonal matrix as the measurements from each

receiver in the network are statistically independent as the noise at each receiver are statistically independent. Each diagonal block contains the range and Doppler error covariance matrix from each individual receiver in the network. The general expression of the FIM for a vector of Gaussian measurements of the type of (5.7) is given by [53]

$$[\mathbf{J}_F(\boldsymbol{\theta})]_{ij} = \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i} \right]^\top \boldsymbol{\Sigma}^{-1} \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j} \right] + \frac{1}{2} \text{tr} \left[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{\Sigma}(\boldsymbol{\theta})}{\partial \theta_i} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{\Sigma}(\boldsymbol{\theta})}{\partial \theta_j} \right] \quad (5.9)$$

Equation (5.9) is simplified by assuming that the covariance matrix $\boldsymbol{\Sigma}$ is independent of the targets state, this implies that SNR_n is constant at each receiver. Given the block diagonal structure of $\boldsymbol{\Sigma}$ equation (5.9) can be written as a summation where each term in the sum depends on the measurements from a single receiver.

$$[\mathbf{J}(\boldsymbol{\theta})]_{ij} = \sum_{n=1}^N \left[\frac{\partial \boldsymbol{\mu}_n(\boldsymbol{\theta})}{\partial \theta_i} \right]^\top \boldsymbol{\Sigma}_n^{-1} \left[\frac{\partial \boldsymbol{\mu}_n(\boldsymbol{\theta})}{\partial \theta_j} \right] \quad (5.10a)$$

$$\left[\frac{\partial \boldsymbol{\mu}_n(\boldsymbol{\theta})}{\partial \theta_i} \right] = \begin{bmatrix} \frac{r_n(\boldsymbol{\theta})}{\partial \theta_i} \\ \frac{f_n(\boldsymbol{\theta})}{\partial \theta_i} \end{bmatrix} \quad (5.10b)$$

Equation (5.10) depends on the derivatives of the measurements (target's relative range and Doppler shift) with respect to the target state (position and velocity). The derivatives are presented below. See Appendix A.1 for further details.

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_y} = \frac{p_y - p_{y0}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (5.11a)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_x} = \frac{p_x - p_{x0}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (5.11b)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_x} = 0 \quad (5.11c)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_y} = 0 \quad (5.11d)$$

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_y} = \frac{1}{\lambda} \left(\frac{p_{yo} - p_y}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{yn} - p_y}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.11e)$$

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_x} = \frac{1}{\lambda} \left(\frac{p_{xo} - p_x}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{xn} - p_x}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.11f)$$

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} = \frac{1}{\lambda} & \left(\frac{v_x(p_{xo} - p_x)(p_{yo} - p_y) - v_y(p_{xo} - p_x)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \right. \\ & \left. + \frac{v_x(p_{xn} - p_x)(p_{yn} - p_y) - v_y(p_{xn} - p_x)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (5.11g)$$

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = \frac{1}{\lambda} & \left(\frac{v_y(p_{yo} - p_y)(p_{xo} - p_x) - v_x(p_{yo} - p_y)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \right. \\ & \left. + \frac{v_y(p_{yn} - p_y)(p_{xn} - p_x) - v_x(p_{yn} - p_y)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (5.11h)$$

The resulting FIM is a 4D matrix, the inverse of which is by definition the CRLB for estimating $\boldsymbol{\theta}$, using the range and Doppler measurements from the network. For radar networks with more than two receivers, numerical methods are required to determine the position and velocity CRLB. The CRLB depends on the geometry of the network; the network can be optimised by minimising the CRLB with respect to the positions of the receivers for a given target state.

5.2.3 Maximum Likelihood Estimator

The MLE is an estimator with performance that asymptotically attains the theoretical CRLB at high SNR. Each receiver estimates the targets range and Doppler shifts are estimated by coherently integrating the received κ pulses at each receiver and sampling the peak of the range-Doppler map. Define $\ln \Lambda(\boldsymbol{\theta}; \mathbf{z})$ as the log-likelihood of the parameters $\boldsymbol{\theta}$ taking the values given by the estimated target range and Doppler shift from the network receivers $\mathbf{z}$.

$$\ln \Lambda(\boldsymbol{\theta}; \mathbf{z}) = -2N \ln(2\pi) - \ln |\boldsymbol{\Sigma}| - \frac{1}{2} \sum_{n=1}^N (\mathbf{z}_n - \boldsymbol{\mu}_n(\boldsymbol{\theta}))^\top \boldsymbol{\Sigma}_n^{-1} (\mathbf{z}_n - \boldsymbol{\mu}_n(\boldsymbol{\theta})) \quad (5.12)$$

The MLE estimate $\hat{\boldsymbol{\theta}}$ is the value of $\boldsymbol{\theta}$ which maximises the likelihood function $\ln \Lambda(\boldsymbol{\theta}; \mathbf{z})$.

$$\hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta}} \{\ln \Lambda(\boldsymbol{\theta}; \mathbf{z})\} \quad (5.13a)$$

$$= \arg \max_{\boldsymbol{\theta}} \left\{ \sum_{n=1}^N (\mathbf{z}_n - \boldsymbol{\mu}_n(\boldsymbol{\theta}))^\top \boldsymbol{\Sigma}_n^{-1} (\mathbf{z}_n - \boldsymbol{\mu}_n(\boldsymbol{\theta})) \right\} \quad (5.13b)$$

Equation (5.13b) has been simplified by removing the constant terms as they do not effect the location of the estimate $\hat{\boldsymbol{\theta}}$.

The CD algorithm initialises by randomly selecting a value $\boldsymbol{\theta}^0$ within the search space $D \subset \mathbb{R}^d$ where $d = \dim(\boldsymbol{\theta})$. The algorithm then selects a single variable and minimises along the direction of the chosen variable, keeping all other variables constant. The algorithm terminates when the sequence $(\boldsymbol{\theta}^k)$ converges.

Algorithm 1 Coordinate Descent

- 1: Set $k = 0$ and initialise $\boldsymbol{\theta}^0 \in D$
 - 2: **repeat**
 - 3: Choose index $i_k \in \{1, 2, \dots, d\}$
 - 4: Update $\theta_{i_k}^k = \arg \max_{\theta_{i_k}} \{\ln \Lambda(\theta_1^{k-1}, \dots, \theta_{i_k}, \dots, \theta_d^{k-1}; \mathbf{z})\}$
 - 5: Set $\theta_j^k = \theta_j^{k-1}$ for $j \neq i_k$
 - 6: Set $k = k + 1$
 - 7: **until** $(\boldsymbol{\theta}^k)$ converges or maximum iterations reached
 - 8: Set $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^k$
-

In order to ensure that the coordinate descent algorithm correctly converges to the peak of the MLE the algorithm performs multiple runs, each time with a different randomly selected initial value for $\boldsymbol{\theta}^0 \in D$. The estimate and value that the log-likelihood function attains at the estimate from each run is collected. The estimated from the run that attains the maximum log-likelihood out of the all the runs is selected as the final estimate.

5.2.4 Example Geometry

The network has to estimate four target parameters, and as each receiver in the network generates two measurements, at minimum of two receivers are required. Therefore consider a radar network consisting of a single transmitter and four identical bistatic receivers.

Label	Symbol	x [m]	y [m]
Tx	$\mathbf{p}_o$	0	0
Rx 1	$\mathbf{p}_1$	0	0
Rx 2	$\mathbf{p}_2$	-500	-2500
Rx 3	$\mathbf{p}_3$	-2000	0

Table 5.1: Static Network Geometry

The network is arranged around a centrally located monostatic radar, two static receivers, and one receiver tethered to the monostatic radar. The positions of the static components are given in Table 5.1. The tethered receiver can be placed on a circle of radius 1 km around the monostatic radar (labelled by Rx 4). The geometry of the network is varied by changing the angle of the tethered receiver. The target's position and velocity is.

$$\mathbf{p} = \begin{pmatrix} 2000 \text{ m} \\ 500 \text{ m} \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 10 \text{ ms}^{-1} \\ 18 \text{ ms}^{-1} \end{pmatrix} \quad (5.14)$$

The targets velocity has a magnitude of approximately 20 ms^{-1} which is typical of modern multi-rotor drone. Fig. 5.3 shows the initial geometry and possible positions for the 4th receiver.

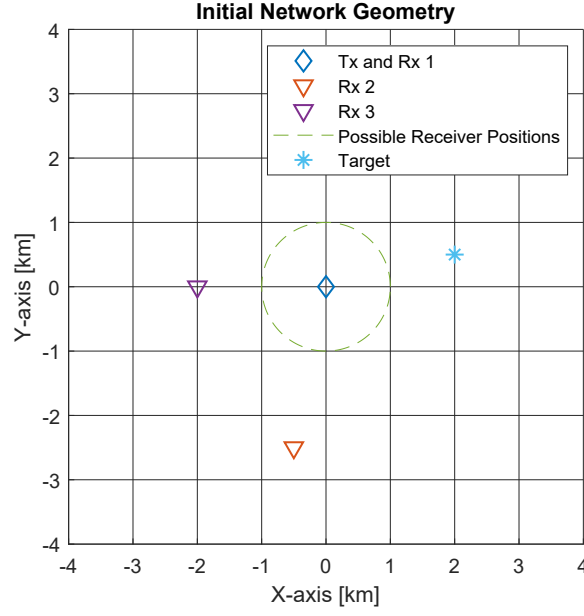


Figure 5.3: Initial Network Geometry

Signal Properties

Table 5.2 lists the parameters of a Gaussian pulse.

Signal parameter	Value	Unit
Pulse repetition frequency (pri)	7200	Hz
Carrier frequency	L-Band	GHz
Number of pulses (κ)	64	-
Gaussian pulse scaling parameter (ν)	0.5	μ s
Receiver bandwidth	1	MHz
Sampling frequency	20	MHz

Table 5.2: Signal Parameters

Using the above notation the components of the CRLB for estimating the range and Doppler shift from (5.3) for a complex envelope consisting of κ Gaussian pulses is

$$\mathbf{J}_F(r, f) = \frac{\text{SNR}_n}{2} \begin{pmatrix} \frac{1}{c^2 \nu^2} & 0 \\ 0 & 4\pi^2 \nu^2 + \frac{2}{3}\pi^2 \text{pri}^2 (\kappa^2 - 1) \end{pmatrix} \quad (5.15)$$

The energy of the signal received at each receiver is assumed to be independent of

the targets state, $\boldsymbol{\theta}$, therefore the signal to noise ratio at each receiver, SNR_n can be freely chosen. In order for the lower bound to apply to the estimator the SNR_n must be sufficiently high, see section 3.7, the range and Doppler bound is achieved when the SNR_n exceeds 20 dB, therefore choose the SNR at the monostatic radar, receiver 1, to be $\text{SNR}_1 = 30$ dB. It is also possible to intuitively chose the SNR of the static receivers using the radar range equation, assuming that the receivers are identical then

$$\text{SNR}_n = \text{SNR}_1 \left(\frac{|\mathbf{p}_o - \mathbf{p}|}{|\mathbf{p}_n - \mathbf{p}|} \right)^2 \quad (5.16)$$

where $|\mathbf{p}_o - \mathbf{p}|$ is the target-transmitter range and $|\mathbf{p}_n - \mathbf{p}|$ is the target-receiver range for $n = 2, 3$. Applying (5.16) to the static receivers, the SNR across the network is

$$\text{SNR}_1 = 30 \text{ dB} \quad (5.17)$$

$$\text{SNR}_2 = 24.45 \text{ dB} \quad (5.18)$$

$$\text{SNR}_3 = 24.18 \text{ dB} \quad (5.19)$$

The SNR at the tethered receiver (receiver 4) varies as the position of the network geometry is optimised.

5.2.5 Geometry Optimisation

The network model is constrained to one parameter, the angle of the tethered receiver, therefore the optimal geometry is defined as the angle of the tethered receiver that minimises the trace of the position and velocity CRLB. Fig. 5.4 shows the trace of the position and velocity CRLB against the angle of the tethered receiver.

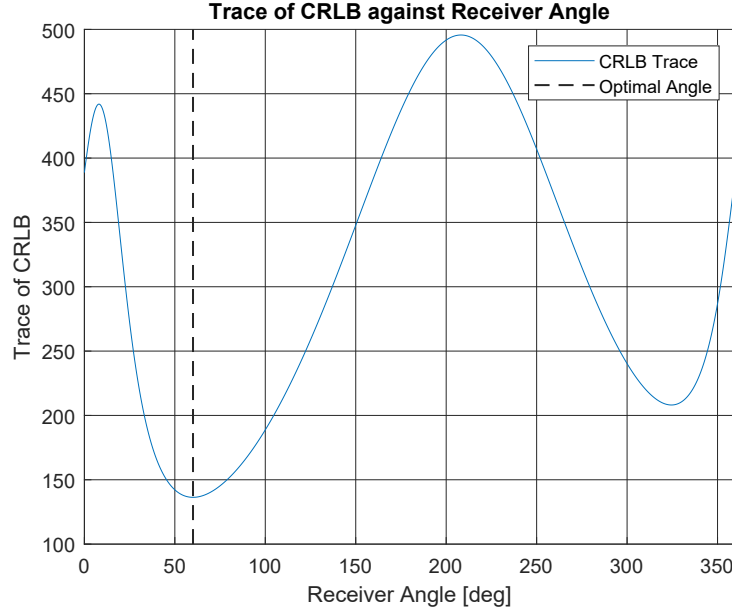


Figure 5.4: Trace of CRLB against Receiver Angle Variable SNR

Fig. 5.4 shows the trace of the position and velocity CRLB against the angle of the tethered receiver whilst allowing the SNR to vary with the receivers position. Fig. 5.4 shows two local minima at 60° and 324° . This places the tethered receiver at $\mathbf{p}_4 = \begin{pmatrix} 499 & 866 \end{pmatrix}^\top$.

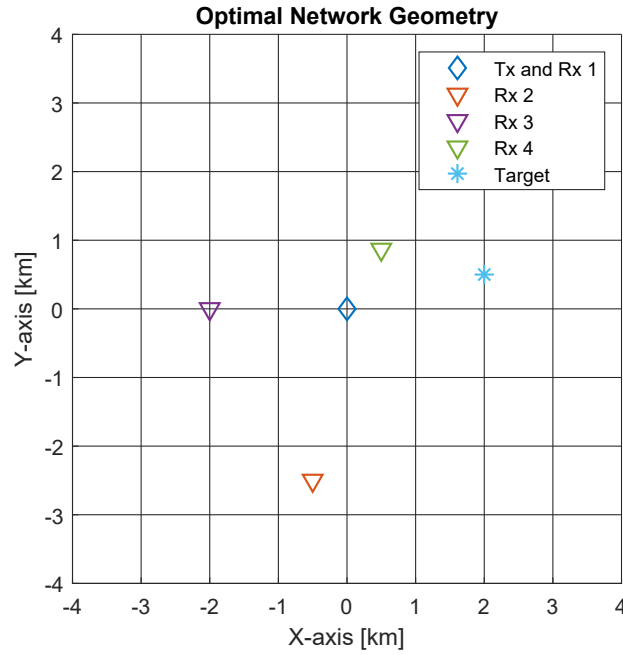
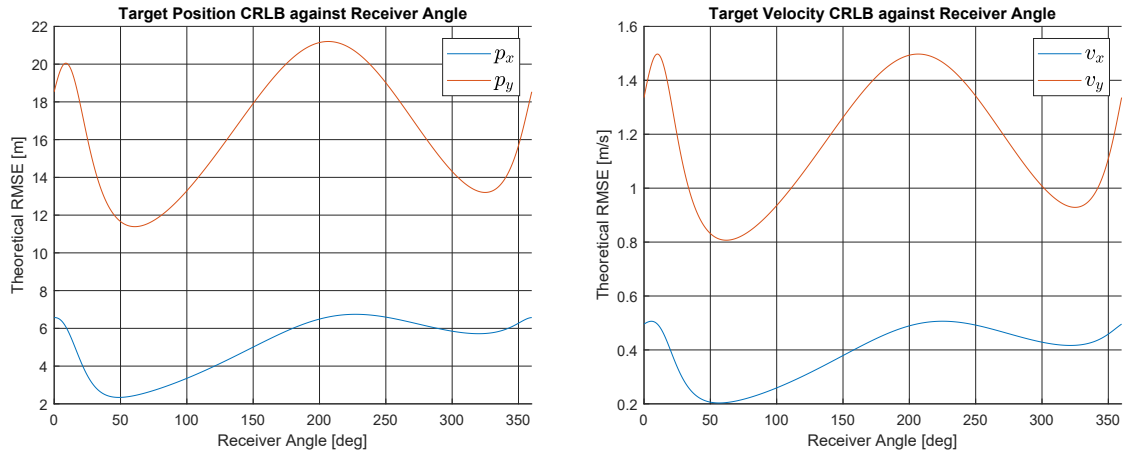


Figure 5.5: Optimal Network Geometry

Fig. 5.5 shows the optimal radar geometry relative to the position of the target and the static receivers.



(a) Target Position CRLB against Receiver Angle

(b) Target Velocity CRLB against Receiver Angle

Figure 5.6: Breakdown of Target State CRLB against Receiver Angle

From Fig. 5.6a and Fig. 5.6b the network is able to estimate the targets x position (p_x) and the targets velocity in the x direction (v_x) with less error than the targets y position (p_y) and the targets velocity in the y direction v_y .

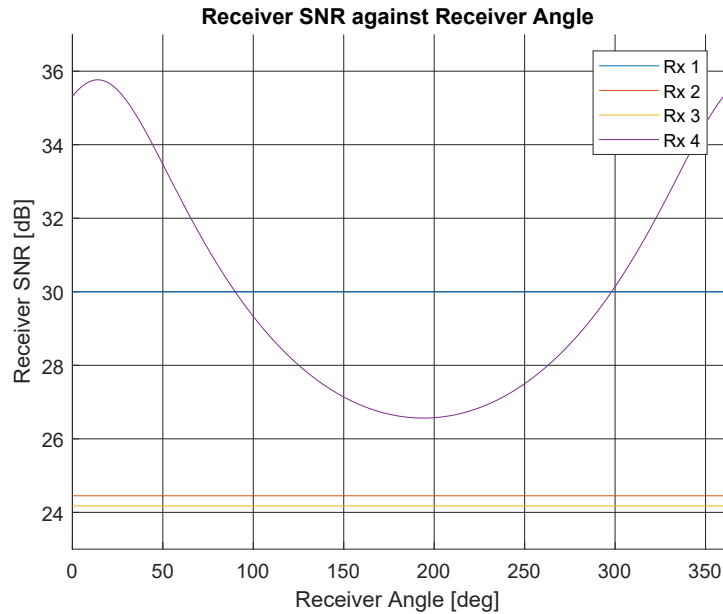


Figure 5.7: Receiver SNR against Receiver Angle

Fig. 5.7 shows how the received SNR varies across the network as the angle of the

receiver changes.

5.2.6 Monte-Carlo Simulations of the Maximum Likelihood Estimator

Monte Carlo simulations of the MLE are used to verify that the performance of the found optimal network geometry attains the theoretical lower bound. The estimator consists of two parts. Firstly, the targets range and Doppler shift at each receiver are estimated. These estimates are centrally gathered. Then, the target's position and velocity are estimated using the gathered range and Doppler estimates and knowledge of the network's geometry.

Simulation Parameter	Value	Unit
SNR dynamic range	0 – 30	dB
Number of SNR sample points	30	-
Number of Monte-Carlo trials per SNR sample point	5000	-

Table 5.3: Monte-Carlo Simulation Parameters

With simulation geometry

Label	Symbol	x [m]	y [m]
Tx	$\mathbf{p}_o$	0	0
Rx 1	$\mathbf{p}_1$	0	0
Rx 2	$\mathbf{p}_2$	-500	-2500
Rx 3	$\mathbf{p}_3$	-2000	0
Rx 4	$\mathbf{p}_4$	499	866

Table 5.4: Simulation Geometry

The targets range and Doppler shifts are estimated by coherently integrating the received pulses at each receivers and sampling the peak of the range-Doppler map. Fig. 5.8 - 5.11 shows the RMSE of the targets range and Doppler shift MLE estimator and CRLB at each receiver.

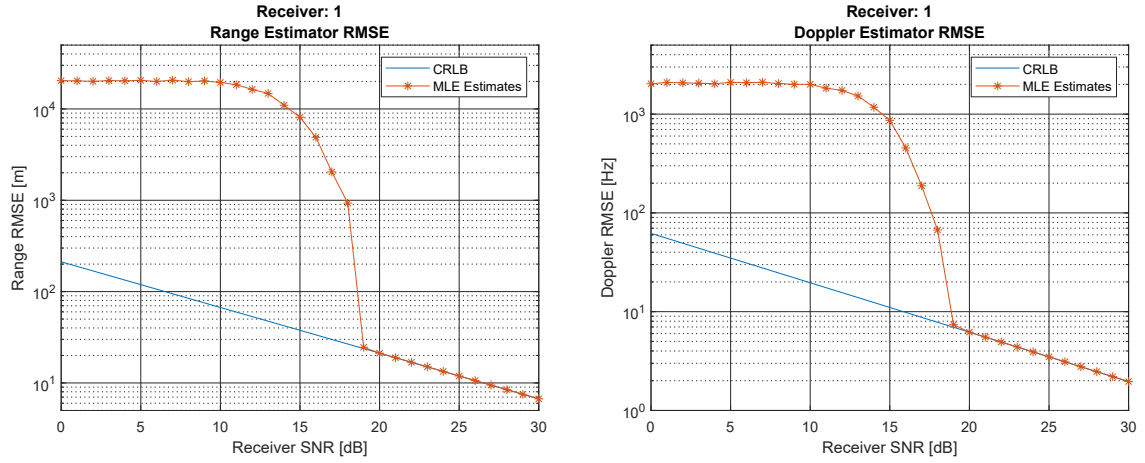


Figure 5.8: Receiver 1 Range and Doppler RMSE

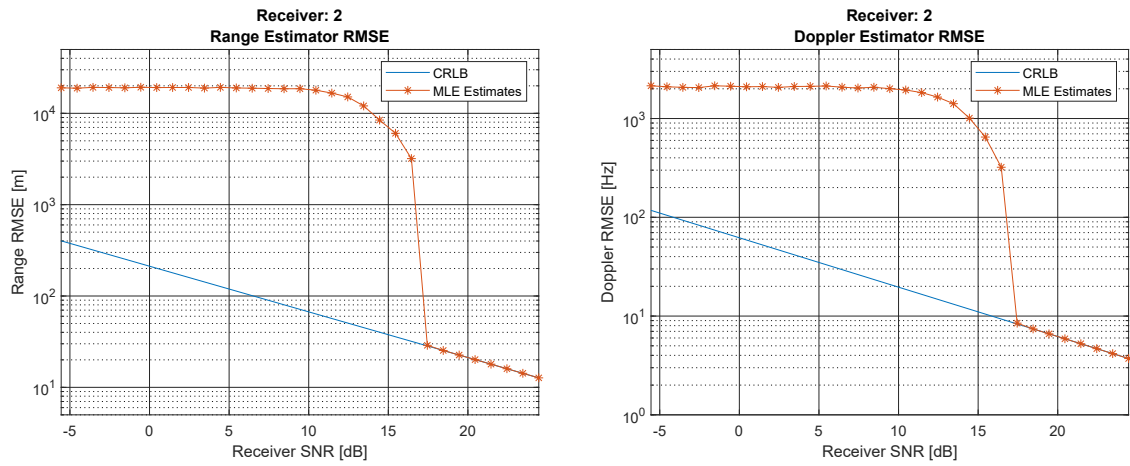


Figure 5.9: Receiver 2 Range and Doppler RMSE

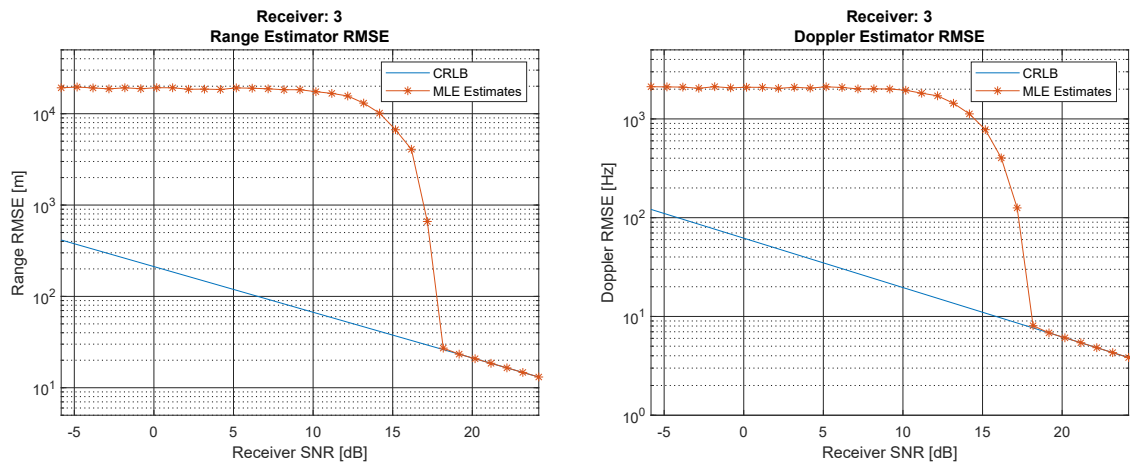


Figure 5.10: Receiver 3 Range and Doppler RMSE

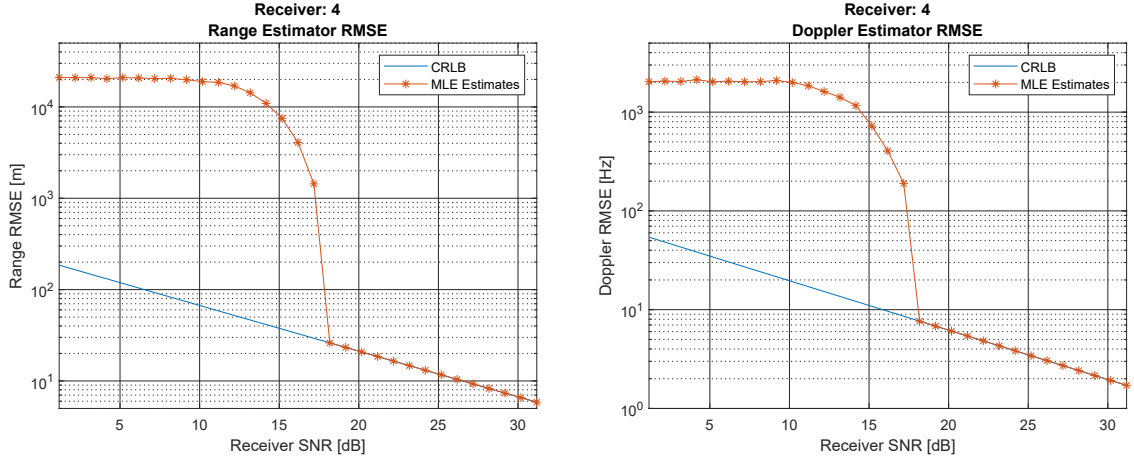


Figure 5.11: Receiver 4 Range and Doppler RMSE

These results are similar to the results presented in section 3.7 with variation in the exact lower bound due to differences in received SNR across the network. Notice that at low SNR the estimator is randomly sampling the noise of the range-Doppler map, the constant error is a result finite dynamic range in the simulations. The dynamic range of the simulations were set to ensure that the MLE was not constrained. Note that in all of the above figures the estimator attains its theoretical bound.

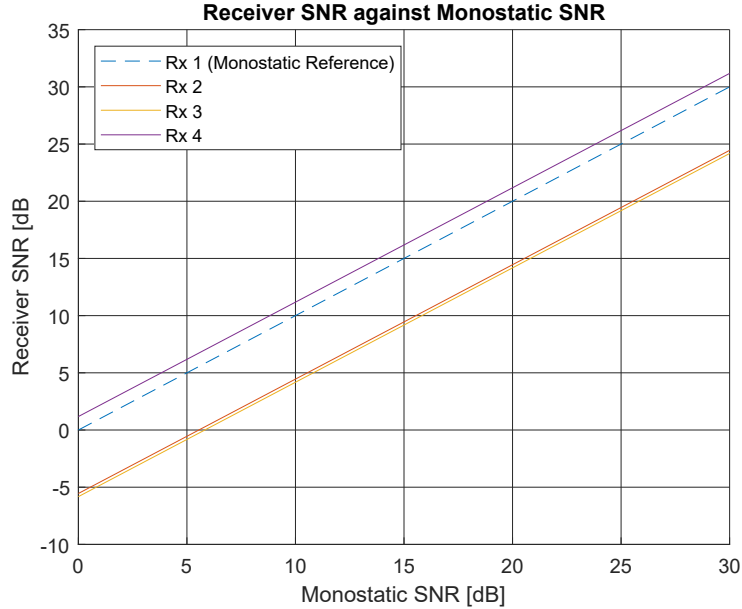


Figure 5.12: Receiver SNR against Monostatic SNR

Fig. 5.12 shows the relative differences in SNR across the network. Both receivers 3

and 4 –5 dB less than the monostatic reference due to the increased distance from the target.

Fig. 5.13-5.16 shows the root mean square error (RMSE) of the MLE estimates for each of the four target state parameters as the SNR varies from 0 dB to 30 dB. The RMSE for each data point was calculated using 5000 samples.

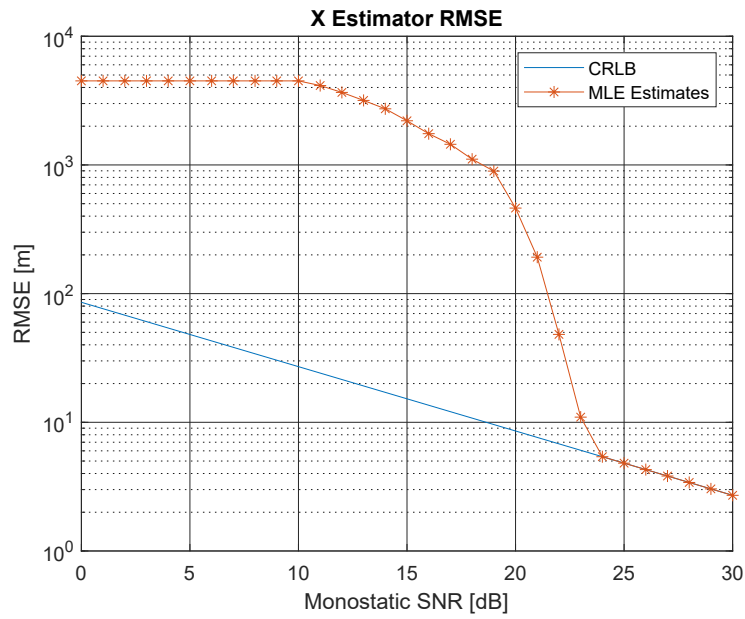


Figure 5.13: Target X Position (p_x) RMSE Vs SNR

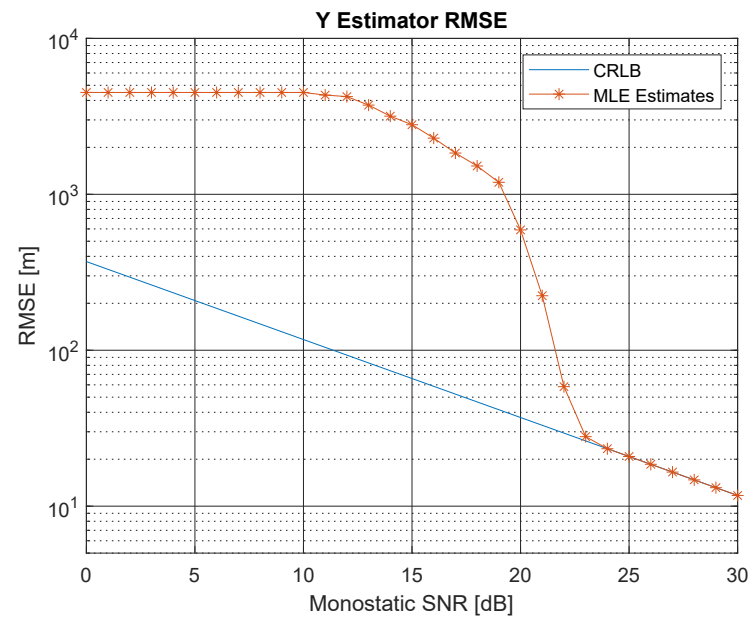


Figure 5.14: Target Y Position (p_y) RMSE Vs SNR

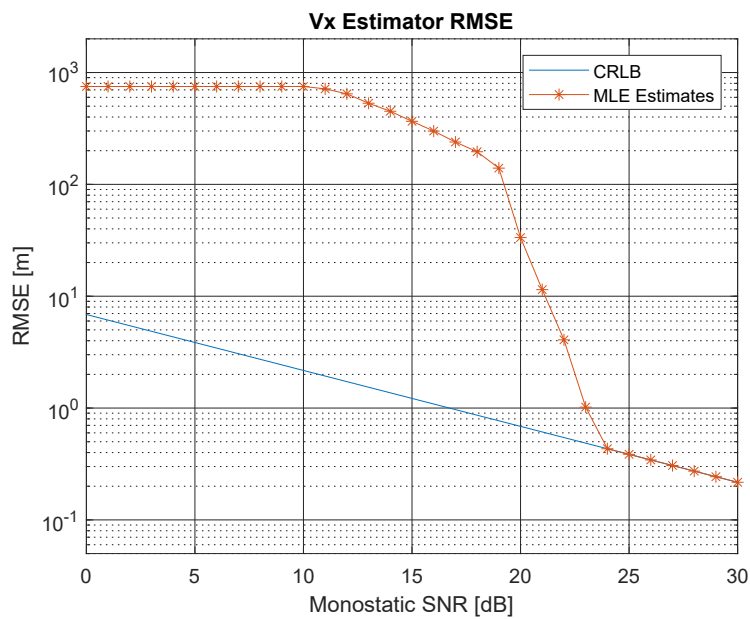
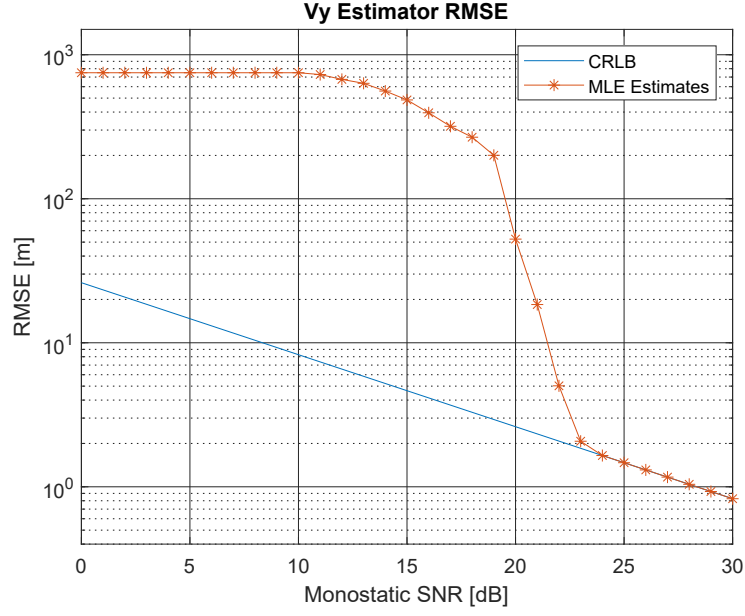


Figure 5.15: Target X Velocity (v_x) RMSE Vs SNR

Figure 5.16: Target Y Velocity (v_y) RMSE Vs SNR

The results of the simulations show that the RMSE of the MLE attains the theoretical CRLB for each of the four target parameters, at high SNR and verifies the accuracy of optimal network geometry. Fig. 5.13-5.16 shows the root mean square error (RMSE) of the MLE estimates for each of the four target state parameters as the SNR varies from 0 dB to 30 dB. The RMSE for each data point was calculated using 5000 samples.

5.2.7 Optimal Receiver Selection

Consider a multistatic radar network consisting of a single transmitter and multiple coherent receivers. This section looks at how the range and Doppler measurements generated by the network can be best combined to estimate a targets position and velocity with the smallest possible error.

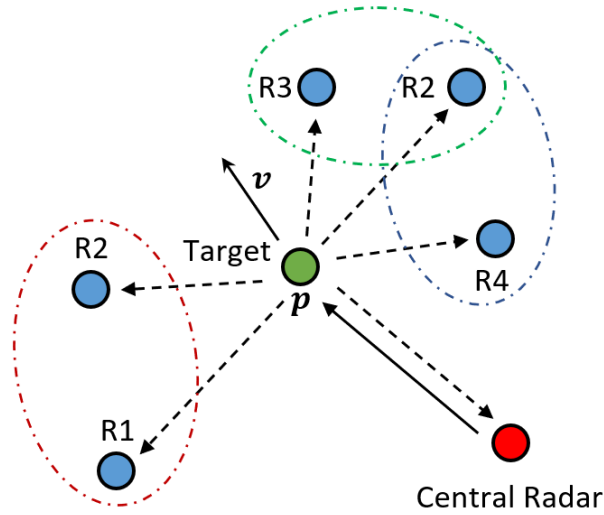


Figure 5.17: Optimal Receiver Selection - Illustrative Diagram

This section looks at an optimisation method for selecting which receivers measurements should be combined in the fusion process. Fig. 5.17 illustrates a radar network consisting of a central radar and four smaller bistatic receivers. The central radar is based on the performance of the Aveillant Gamekeeper radar. The bistatic receivers is based on the performance of a single Gamekeeper tile.

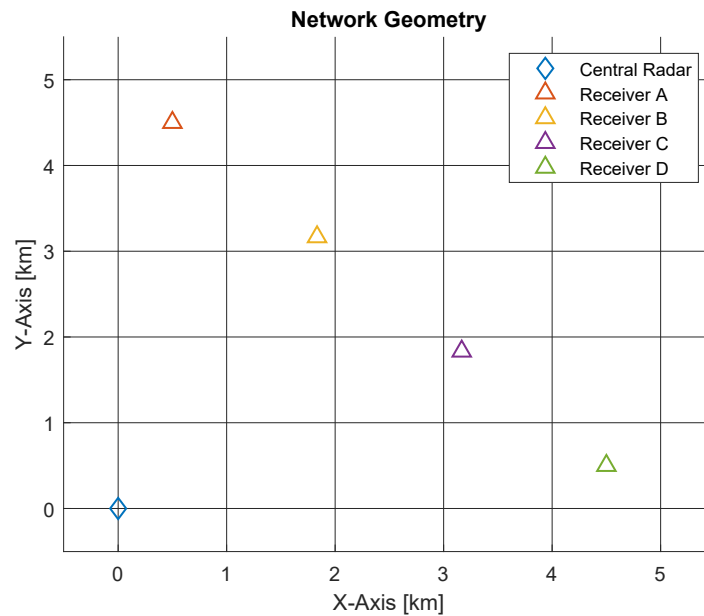


Figure 5.18: Network Geometry

In this geometry, as shown in 5.18, there are 6 possible receiver pairs to choose from,

these are

$$\{AB, AC, AD, BC, BD, CD\} \quad (5.20)$$

The network geometry is fixed so the CRLB depends on the position of the target and its velocity. For all possible target states (target position and velocity) calculate the trace of the position and velocity CRLB using the range and Doppler measurements from each pair of receivers in combination with the measurements from the central radar. Then compare the trace of the CRLB for each target state and select the receiver pair that minimises the trace of the CRLB. The output is a 4D map specifying the optimal receiver pair for all possible target states.

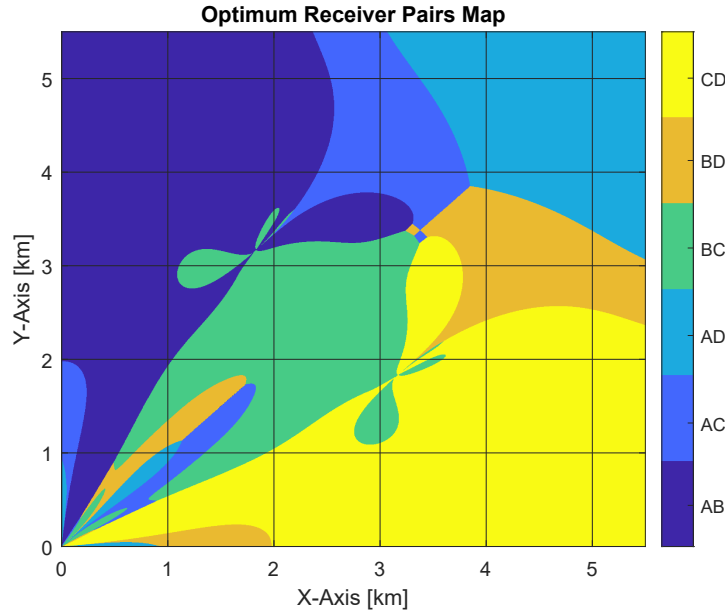


Figure 5.19: Optimum Pairs for Position Velocity Estimation

Fig. 5.19 shows a colour coded slice of the 4D map for a target that has a velocity of -5 ms^{-1} in the x direction and -5 ms^{-1} in the y direction. The colour map is split into 6 colours each colour corresponds to the pair of receivers. Consider a simulated trajectory, for each point in the targets trajectory the optimal receiver pairing is calculated based on the targets current position and velocity relative to

the network.

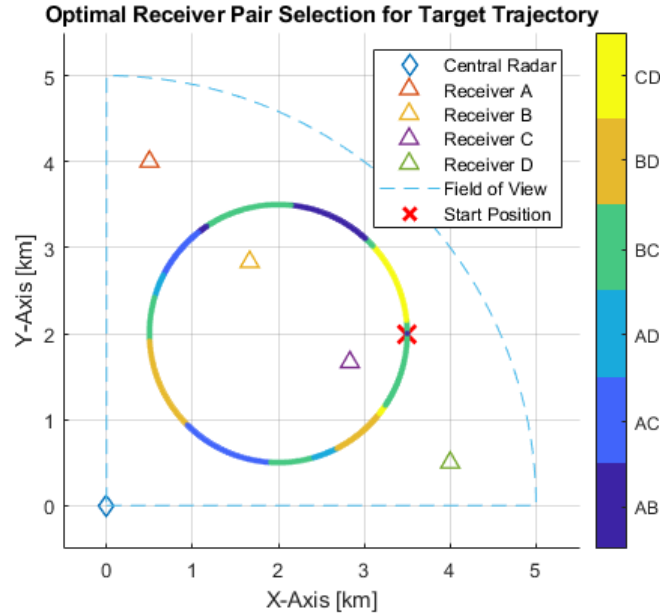


Figure 5.20: Optimal Receiver Pair Selection for Target Trajectory

Fig. 5.20 shows the target trajectory, color coded according to the optimal receiver pairings. The target flies a circular trajectory centred at the point (2, 2) km with a radius of 1 km at a speed of 20 ms^{-1} . It's clear that as the targets position and velocity change the optimal pairing of receivers also changes.

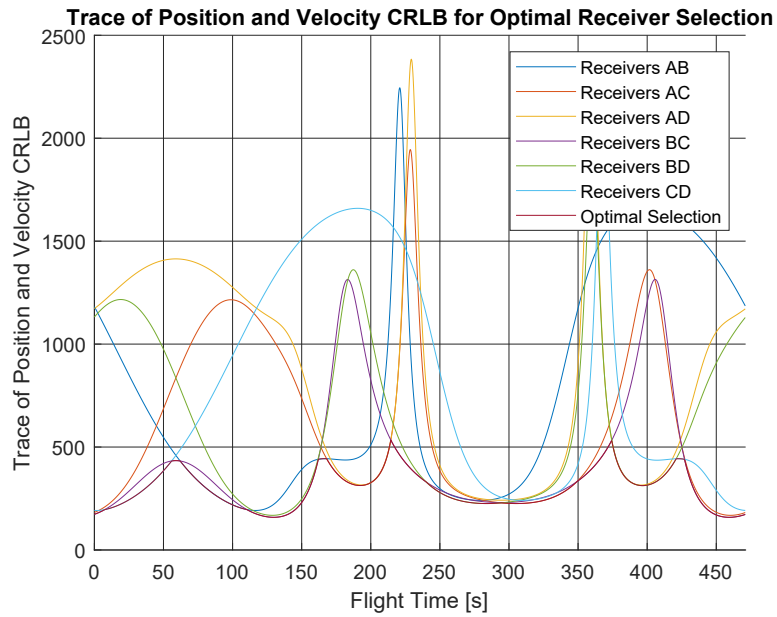


Figure 5.21: Trace of Position and Velocity CRLB for Optimal Receiver Selection

Fig. 5.21 shows how the trace of the position and velocity CRLB changes as the targets position and velocity changes and how the optimal receiver pairing is selected. The trace of the CRLB using the optimal receiver pairing reaches a maximum of 533 and a minimum value of 158.

5.3 Centralised Data Fusion

This section presents the second geometry optimisation algorithm. The previous approach optimised a radar network geometry for a given target's position and velocity. As the previous model treated the target's state as a deterministic unknown parameter, the resulting calculations and subsequent optimal geometry depend on the real parameter values, which in practice are not known. To overcome this limitation, the target's state is modelled as a multivariate random vector.

As the target's position and velocity are now modelled as multivariate random vectors, the Bayesian-Cramér Rao Lower Bound (BCRLB) is needed to describe the theoretical lower bound. The BCRLB depends on the distribution of the estimation variable. The previous approach used a two-step estimator in which the target's relative range and Doppler shift at each receiver are estimated. The target's position and velocity are then estimated from the centrally gathered range, and Doppler measurements, these range and Doppler measurements are random vectors with non-trivial distributions

In order to calculate the theoretical lower bound for estimating the target's position and velocity from the range and Doppler measurements, exact knowledge of the distribution of the range and Doppler measurements are required. The equations defining the relative range and Doppler shift of a target are complex and non-linear, and therefore, it is not possible to analytically calculate the distributions and hence not possible to use a distributed scheme. However, it is possible to estimate the target's state directly from the combined received signals, requiring a centralised data fusion scheme.

5.3.1 Signal Model

Consider a cooperative multistatic radar network consisting of a single transmitter and N passive receivers. The network geometry is identified by the set $\Gamma = \{\mathbf{p}_o, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\}$ where $\mathbf{p}_o$ is the transmitter position and $\mathbf{p}_n$ indicates the position of the n -th receiver, for $n = 1, 2, \dots, N$. Note that here the signal model is extended from the 2D case previously studied to 3D to provide a more complete analysis of the problem. Also, note that in order to determine the target's in 3D requires that at least the signals from 3 receivers are combined.

$$\mathbf{p}_o = \begin{pmatrix} p_{xo} \\ p_{yo} \\ p_{zo} \end{pmatrix} \quad \mathbf{p}_n = \begin{pmatrix} p_{xn} \\ p_{yn} \\ p_{zn} \end{pmatrix} \quad (5.21)$$

Assume a target is present in the scene with position $\mathbf{p}$ and velocity $\mathbf{v}$, both the target's position and velocity are modelled as multivariate Gaussian random vectors.

$$\mathbf{p} \sim \mathcal{N}(\boldsymbol{\mu}_p, \Sigma_p), \quad \mathbf{v} \sim \mathcal{N}(\boldsymbol{\mu}_v, \Sigma_v) \quad (5.22)$$

The target state is the joint vector of the target's position and velocity

$$\boldsymbol{\theta} = \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}, \quad \boldsymbol{\theta} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \quad (5.23)$$

with mean value and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_p \\ \boldsymbol{\mu}_v \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_p & 0 \\ 0 & \Sigma_v \end{pmatrix} \quad (5.24)$$

Where $\mathbf{p}$ and $\mathbf{v}$ are assumed to be independent. The transmitter $\mathbf{p}_o$ and first receiver $\mathbf{p}_1$ are co-located and model the central staring monostatic radar. The remaining

receivers $\mathbf{p}_n$ for $n \in [2, N]$ model the bistatic receivers that make up the remainder of the network, Fig. 5.22 shows an illustrative geometry, with the target's state covariance matrix confidence ellipsis shown in green.

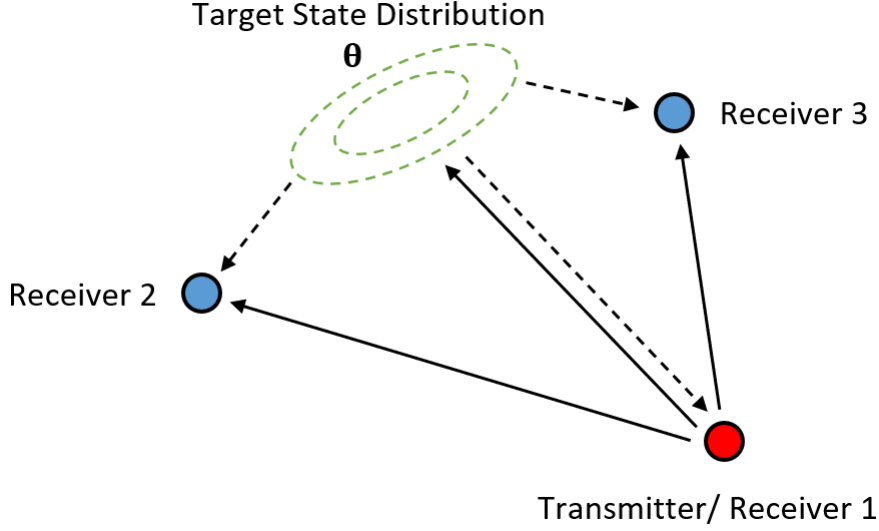


Figure 5.22: Illustrative Network Geometry

The n -th receiver in the network receives a noisy, time-delayed and Doppler-shifted version of the transmitted signal with complex envelope $u(t)$ which, after down-conversion, may be written as

$$z_n(t) = \beta_n \sqrt{E_n} u(t - \tau_n(\boldsymbol{\theta})) e^{2\pi i f_n(\boldsymbol{\theta})t} + w_n(t) \quad (5.25)$$

The energy of the complex envelope $u(t)$ is unity, and β_n is modelled as a zero mean complex Gaussian random variable with $\mathbb{E}[|\beta_n|^2] = 2\sigma_{\beta_n}^2$ that accounts for the target reflection properties and channel propagation fluctuations, such as fading multipath. The noise $w_n(t)$ is modelled as a complex zero-mean Gaussian white random process with power spectral density N_0 [W/Hz]. Note that as in the previous example the effects of clutter are not considered in order to simplify the model, as such it should be noted that in the presence of clutter the found optimal geometry and estimated performance presented in this chapter may not be applicable. At each receiver, the time delay and Doppler shift of the received signal depend on the network geometry

and the position and velocity of the target as

$$\tau_n(\boldsymbol{\theta}) = \frac{(|\mathbf{p} - \mathbf{p}_o| + |\mathbf{p} - \mathbf{p}_n|)}{c} \quad (5.26a)$$

$$f_n(\boldsymbol{\theta}) = \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p} - \mathbf{p}_o}{|\mathbf{p} - \mathbf{p}_o|} + \frac{\mathbf{p} - \mathbf{p}_n}{|\mathbf{p} - \mathbf{p}_n|} \right). \quad (5.26b)$$

where c denotes the speed of light in a vacuum.

5.3.2 Fusion of the Received Signals

Consider a centralised data fusion process in which the entire received signals are combined into a single vector $\mathbf{z}(t)$ which can be written as the sum of the two vectors containing the signal and the noise.

$$\mathbf{z}(t) = \begin{pmatrix} \beta_1 \sqrt{E_1} u(t - \tau_1(\boldsymbol{\theta})) e^{2\pi i f_1(\boldsymbol{\theta}) t} \\ \beta_2 \sqrt{E_2} u(t - \tau_2(\boldsymbol{\theta})) e^{2\pi i f_2(\boldsymbol{\theta}) t} \\ \vdots \\ \beta_N \sqrt{E_N} u(t - \tau_N(\boldsymbol{\theta})) e^{2\pi i f_N(\boldsymbol{\theta}) t} \end{pmatrix} + \begin{pmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{pmatrix} \quad (5.27)$$

Assume that the random variables $\beta_1, \dots, \beta_N$ are independent complex Gaussian random variables and the noise $w_n(t)$ is also independent the received signals are independent. The Bayesian lower bound states that the mean square error of the estimators elements are bounded below by the corresponding diagonal elements of the inverse of the Bayesian information matrix.

$$\mathbb{E} \left[(\hat{\theta}_i(\mathbf{z}) - \theta_i)^2 \right] \geq [\mathbf{J}_B^{-1}]_{ii} \quad (5.28)$$

The Bayesian information matrix is written as a sum of two parts $\mathbf{J}_B = \mathbf{J}_D + \mathbf{J}_P$ where

$$\mathbf{J}_D = \mathbb{E}_{\boldsymbol{\theta}} [\mathbf{J}_F(\boldsymbol{\theta})] \quad (5.29a)$$

$$\mathbf{J}_{P_{ij}} = -\mathbb{E} \left[\frac{\partial^2 \ln p(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (5.29b)$$

the first term $\mathbf{J}_D$ represents the contribution of the data and the $\mathbf{J}_P$ the contribution from the prior information. Starting with the matrix $\mathbf{J}_F(\boldsymbol{\theta})$

$$\mathbf{J}_{F_{ij}} = -\mathbb{E}_{\mathbf{z}|\boldsymbol{\theta}} \left[\frac{\partial^2 \ln p(\mathbf{z}|\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right] \quad (5.30)$$

the expectation is taken over $\mathbf{z}$ given $\boldsymbol{\theta}$. Recognise the term $\ln p(\mathbf{z}|\boldsymbol{\theta})$ as the log-likelihood function of $\boldsymbol{\theta}$. The received signal is a vector of independent time continuous signals, therefore

$$\ln p(\mathbf{z}|\boldsymbol{\theta}) = \sum_{n=1}^N \ln p(\mathbf{z}_n|\boldsymbol{\theta}) \quad (5.31)$$

Define $\Lambda_n(\boldsymbol{\theta}) = \ln p(\mathbf{z}_n|\boldsymbol{\theta})$. Proceeding as in section 3.4 by expanding the received signal using a complete orthonormal basis.

$$\Lambda_n(\boldsymbol{\theta}) = \frac{2E\sigma_\beta^2}{N_0(2E\sigma_\beta^2 + N_0)} \left| \int_{-\infty}^{\infty} z_n(t) u^*(t - \tau_n(\boldsymbol{\theta})) e^{-2\pi i f_n(\boldsymbol{\theta})t} dt \right|^2 \quad (5.32)$$

Writing $\Lambda_n(\boldsymbol{\theta}) = \Lambda_n(\tau_n(\boldsymbol{\theta}), f_n(\boldsymbol{\theta}))$ then applying the chain rule to differentiate the log-likelihood twice, first with respect to θ_i and then with respect to θ_j

$$\begin{aligned} \frac{\partial^2 \Lambda_n(\tau_n(\boldsymbol{\theta}), f_n(\boldsymbol{\theta}))}{\partial \theta_i \partial \theta_j} &= \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n^2} \frac{\partial \tau_n}{\partial \theta_i} \frac{\partial \tau_n}{\partial \theta_j} + \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n \partial f_n} \frac{\partial f_n}{\partial \theta_i} \frac{\partial \tau_n}{\partial \theta_j} \\ &+ \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial f_n^2} \frac{\partial f_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} + \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial f_n \partial \tau_n} \frac{\partial \tau_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} \\ &+ \frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \frac{\partial^2 \tau_n}{\partial \theta_i \partial \theta_j} + \frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \frac{\partial^2 f_n}{\partial \theta_i \partial \theta_j} \end{aligned} \quad (5.33)$$

Note, that the dependence of τ_n and f_n on $\boldsymbol{\theta}$ is omitted for notational clarity. Taking expectation of (5.33) with respect to the data z_n yields

$$\begin{aligned}
\mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \theta_i \partial \theta_j} \right] &= \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n^2} \right] \frac{\partial \tau_n}{\partial \theta_i} \frac{\partial \tau_n}{\partial \theta_j} \\
&+ \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n \partial f_n} \right] \frac{\partial f_n}{\partial \theta_i} \frac{\partial \tau_n}{\partial \theta_j} \\
&+ \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial f_n^2} \right] \frac{\partial f_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} \\
&+ \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial f_n \partial \tau_n} \right] \frac{\partial \tau_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} \\
&+ \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right] \frac{\partial^2 \tau_n}{\partial \theta_i \partial \theta_j} \\
&+ \mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right] \frac{\partial^2 f_n}{\partial \theta_i \partial \theta_j}
\end{aligned} \tag{5.34}$$

A demonstration follows to show that equation (5.34) can be simplified because $\mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right]$ and $\mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right]$ are zero. Firstly, it is observed that the autocorrelation of the received data is

$$\mathbb{E}_{z_n|\boldsymbol{\theta}} [z_n(t) z_n^*(\varepsilon)] = 2\sigma_n^2 E_n u(t - \tau_n) u^*(\varepsilon - \tau_n) e^{2\pi i f_n(t - \varepsilon)} + N_0 \delta(t - \varepsilon) \tag{5.35}$$

Which follows from noting that the expectation of the noise is zero and the autocorrelation $\mathbb{E} [w_n(t) w_n^*(\varepsilon)] = N_0 \delta(t - \varepsilon)$, where δ is the Dirac delta function and by definition $\mathbb{E} [|\beta_n|^2] = 2\sigma_{\beta_n}^2$. Beginning with $\mathbb{E}_{z_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right]$

$$\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} = 2\Re \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_n(t) z_n^*(\varepsilon) u^*(t - \tau_n) \frac{\partial u(\varepsilon - \tau_n)}{\partial \tau_n} e^{2\pi i f_n(\varepsilon - t)} dt d\varepsilon \right] \tag{5.36}$$

Taking the expectation with respect to the received data z_n and interchanging the order of integration

$$\mathbb{E}_{z_n|\theta} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right] = 2\Re \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}_{z_n|\theta} [z_n(t) z_n^*(\varepsilon)] u^*(t - \tau_n) \frac{\partial u(\varepsilon - \tau_n)}{\partial \tau_n} e^{2\pi i f_n(\varepsilon - t)} dt d\varepsilon \right] \quad (5.37)$$

Substituting equation (5.35) into the above and simplifying

$$\mathbb{E}_{z_n|\theta} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right] = 2(2\sigma_n^2 E_n + N_0) \Re \left[\int_{-\infty}^{\infty} u^*(\varepsilon - \tau_n) \frac{\partial u(\varepsilon - \tau_n)}{\partial \tau_n} d\varepsilon \right] \quad (5.38)$$

By definition the complex envelope of the signal is normalised therefore $\int_{-\infty}^{\infty} |u(t - \tau)|^2 dt = 1$ for all $\tau \in \mathbb{R}$, Differentiating with respect to τ yields

$$\Re \left[\int_{-\infty}^{\infty} u^*(\varepsilon - \tau) \frac{\partial u(\varepsilon - \tau)}{\partial \tau} d\varepsilon \right] = 0 \quad (5.39)$$

Combining the above, $\mathbb{E}_{z_n|\theta} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial \tau_n} \right] = 0$. Now to show that $\mathbb{E}_{z_n|\theta} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right] = 0$

$$\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} = 2\Re \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_n(t) z_n^*(\varepsilon) (2\pi i \varepsilon) u^*(t - \tau_n) u(\varepsilon - \tau_n) e^{2\pi i f_n(\varepsilon - t)} dt d\varepsilon \right] \quad (5.40)$$

Taking the expectation with respect to the received data z_n and interchanging the order of integration

$$\mathbb{E}_{z_n|\theta} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right] = 2\Re \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}_{z_n|\theta} [z_n(t) z_n^*(\varepsilon)] (2\pi i \varepsilon) u^*(t - \tau_n) u(\varepsilon - \tau_n) e^{2\pi i f_n(\varepsilon - t)} dt d\varepsilon \right] \quad (5.41)$$

Substituting (5.35) into the above and simplifying

$$\mathbb{E}_{\mathbf{z}_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right] = 2\Re \left[2\pi i (2\sigma_b^2 E_n + N_0) \int_{-\infty}^{\infty} \varepsilon |u(\varepsilon - \tau_n)|^2 d\varepsilon \right] \quad (5.42)$$

The expectation is on the real part of integral is real valued therefore the term in the curly brackets is always imaginary hence the expectation is zero, $\mathbb{E}_{\mathbf{z}_n|\boldsymbol{\theta}} \left[\frac{\partial \Lambda_n(\tau_n, f_n)}{\partial f_n} \right] = 0$. Combining both of the above results, equation (5.34) is simplified

$$\mathbb{E}_{\mathbf{z}_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \theta_i \partial \theta_j} \right] = \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_i} \\ \frac{\partial f_n}{\partial \theta_i} \end{pmatrix}^\top \mathbb{E}_{\mathbf{z}_n|\boldsymbol{\theta}} \left[\begin{pmatrix} \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n^2} & \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n \partial f_n} \\ \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \tau_n \partial f_n} & \frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial f_n^2} \end{pmatrix} \right] \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_j} \\ \frac{\partial f_n}{\partial \theta_j} \end{pmatrix} \quad (5.43)$$

Noting that the matrix is $\mathbf{J}_F(\tau, f)$ and recalling the relation between the ambiguity function and the range and Doppler information matrix, then

$$\mathbb{E}_{\mathbf{z}_n|\boldsymbol{\theta}} \left[\frac{\partial^2 \Lambda_n(\tau_n, f_n)}{\partial \theta_i \partial \theta_j} \right] = -\frac{\text{SNR}_n}{2} \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_i} \\ \frac{\partial f_n}{\partial \theta_i} \end{pmatrix}^\top \mathbf{J}_{AF} \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_j} \\ \frac{\partial f_n}{\partial \theta_j} \end{pmatrix} \quad (5.44)$$

The matrix is the FIM of time delay and Doppler from the received signals which depends on the ambiguity function and SNR of the received signal at the n th receiver, denote by $\mathbf{J}_{\tau f}$. Related to the ambiguity function.

$$\mathbf{J}_{F_{ij}} = -\frac{1}{2} \sum_{n=1}^N \text{SNR}_n \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_i} \\ \frac{\partial f_n}{\partial \theta_i} \end{pmatrix}^\top \mathbf{J}_{AF} \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_j} \\ \frac{\partial f_n}{\partial \theta_j} \end{pmatrix} \quad (5.45)$$

where $\text{SNR}_n = \frac{1}{N_0} \frac{2\sigma_n^2 E_n}{N_0 + 2\sigma_n^2 E_n}$. Note that $\mathbf{J}_{F_{ij}}$ is equal to the fisher information found in the previous section (5.10). The derivatives of the bistatic range and Doppler shift with respect to the targets position and velocity are

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_x} = \frac{p_x - p_{x0}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (5.46a)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_y} = \frac{p_y - p_{y0}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (5.46b)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_z} = \frac{p_z - p_{z0}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_z - p_{zn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (5.46c)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_x} = 0 \quad (5.46d)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_y} = 0 \quad (5.46e)$$

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_z} = 0 \quad (5.46f)$$

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = & -\frac{1}{\lambda} \left(\frac{v_x[(p_{y0} - p_y)^2 + (p_{z0} - p_z)^2]}{|\mathbf{p}_0 - \mathbf{p}|^3} \right. \\ & + \frac{v_x[(p_{yn} - p_y)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\ & + \frac{(p_x - p_{x0})[v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)]}{|\mathbf{p}_0 - \mathbf{p}|^3} \\ & \left. + \frac{(p_x - p_{xn})[v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)]}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (5.46g)$$

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} = & -\frac{1}{\lambda} \left(\frac{v_y[(p_{x0} - p_x)^2 + (p_{z0} - p_z)^2]}{|\mathbf{p}_0 - \mathbf{p}|^3} \right. \\ & + \frac{v_y[(p_{xn} - p_x)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\ & + \frac{(p_y - p_{y0})(v_x(p_{x0} - p_x) + v_z(p_{z0} - p_z))}{|\mathbf{p}_0 - \mathbf{p}|^3} \\ & \left. + \frac{(p_y - p_{yn})(v_x(p_{xn} - p_x) + v_z(p_{zn} - p_z))}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (5.46h)$$

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_z} = & -\frac{1}{\lambda} \left(\frac{v_z[(p_{x0} - p_x)^2 + (p_{y0} - p_y)^2]}{|\mathbf{p}_0 - \mathbf{p}|^3} \right. \\ & + \frac{v_z[(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\ & + \frac{(p_z - p_{z0})[v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y)]}{|\mathbf{p}_0 - \mathbf{p}|^3} \\ & \left. + \frac{(p_z - p_{zn})[v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)]}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (5.46i)$$

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_x} = \frac{1}{\lambda} \left(\frac{p_{x0} - p_x}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_{xn} - p_x}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.46j)$$

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_y} = \frac{1}{\lambda} \left(\frac{p_{y0} - p_y}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_{yn} - p_y}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.46k)$$

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_z} = \frac{1}{\lambda} \left(\frac{p_{z0} - p_z}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{p_{zn} - p_z}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (5.46l)$$

Combining the above and taking the expectation with respect to $\boldsymbol{\theta}$

$$\mathbf{J}_{D_{ij}} = -\frac{1}{2} \sum_{n=1}^N \mathbb{E}_{\boldsymbol{\theta}} \left[\text{SNR}_n \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_i} \\ \frac{\partial f_n}{\partial \theta_i} \end{pmatrix}^\top \mathbf{J}_{AF} \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_j} \\ \frac{\partial f_n}{\partial \theta_j} \end{pmatrix} \right] \quad (5.47)$$

The expectation over $\boldsymbol{\theta}$ is a 6 dimensional integral which requires numerical methods to evaluate. The second term of the Bayesian information matrix encapsulates the information gained from knowledge of the prior distribution of $\boldsymbol{\theta}$. The targets state is modelled as a multivariate Gaussian random vector, with mean value $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$

$$\ln(p(\boldsymbol{\theta})) = -\ln(2\pi\sqrt{|\boldsymbol{\Sigma}|}) - \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta} - \boldsymbol{\mu}) \quad (5.48)$$

therefore

$$\mathbb{E}_{\boldsymbol{\theta}} \left[\frac{\partial^2 \ln(p(\boldsymbol{\theta}))}{\partial \theta_i \partial \theta_j} \right] = \boldsymbol{\Sigma}_{ij}^{-1} \quad (5.49)$$

read as the ij element of the matrix inverse of $\boldsymbol{\Sigma}$. Combining the above the elements of the Bayesian fisher information matrix are

$$\mathbf{J}_{B_{ij}} = \boldsymbol{\Sigma}_{ij}^{-1} - \frac{1}{2} \sum_{n=1}^N \text{SNR}_n \mathbb{E}_{\boldsymbol{\theta}} \left[\begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_i} \\ \frac{\partial f_n}{\partial \theta_i} \end{pmatrix}^\top \mathbf{J}_{AF} \begin{pmatrix} \frac{\partial \tau_n}{\partial \theta_j} \\ \frac{\partial f_n}{\partial \theta_j} \end{pmatrix} \right] \quad (5.50)$$

The Bayesian lower bound of the targets state is the inverse of the Bayesian fisher information matrix $\text{BCRLB}(\boldsymbol{\theta}) = \mathbf{J}_B^{-1}$. Note that the inverse of $\mathbf{J}_D = \mathbb{E}_{\boldsymbol{\theta}} [\mathbf{J}_F^{-1}]$, is the expectation of the conditional Cramér-Rao Lower Bound, denoted by $\text{ECRLB}(\boldsymbol{\theta})$.

5.3.3 Maximum a Posteriori Estimator

The MAP estimator of the random vector $\boldsymbol{\theta}$ from the data $\mathbf{z}$ is $\hat{\boldsymbol{\theta}}_{map}(\mathbf{z})$ where

$$\hat{\boldsymbol{\theta}}_{map}(\mathbf{z}) = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \{ \ln p(\mathbf{z}|\boldsymbol{\theta}) + \ln p(\boldsymbol{\theta}) \} \quad (5.51)$$

The CD algorithm initialises by randomly selecting a value $\boldsymbol{\theta}^0$ within the search space $D \subset \mathbb{R}^d$ where $d = \dim(\boldsymbol{\theta})$. The algorithm then selects a single variable and minimises along the direction of the chosen variable, keeping all other variables constant. The algorithm terminates when the sequence $(\boldsymbol{\theta}^k)$ converges.

Algorithm 2 MAP Coordinate Descent

- 1: Set $k = 0$ and initialise $\boldsymbol{\theta}^0 \in D$
 - 2: **repeat**
 - 3: Choose index $i_k \in \{1, 2, \dots, d\}$
 - 4: Update $\theta_{i_k}^k = \underset{\theta_{i_k}}{\operatorname{argmax}} \{ \ln p(\mathbf{z}|\theta_1^{k-1}, \dots, \theta_{i_k}, \dots, \theta_d^{k-1}) + \ln p(\theta_1^{k-1}, \dots, \theta_{i_k}, \dots, \theta_d^{k-1}) \}$
 - 5: Set $\theta_j^k = \theta_j^{k-1}$ for $j \neq i_k$
 - 6: Set $k = k + 1$
 - 7: **until** $(\boldsymbol{\theta}^k)$ converges or maximum iterations reached
 - 8: Set $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^k$
-

In order to ensure that the coordinate descent algorithm correctly converges to the global peak of the MAP estimator, the algorithm performs multiple runs. Each run randomly selects a different initial starting value for $\boldsymbol{\theta}^0 \in D$ while keeping $\mathbf{z}$ constant. The estimate and value that the attains at the estimate from each run is collected.

5.3.4 Geometry

Consider a multistatic radar network containing a central monostatic radar and two bistatic receivers. This geometry is represented by a transmitter and three receivers, the transmitter and first receiver are co-located representing the central radar with the bistatic receivers represented by the remaining two receivers.

The targets position and velocity are modelled by multivariate Gaussian random vectors with mean and covariance

$$\boldsymbol{\mu}_p = \begin{pmatrix} 2000 \\ 3000 \\ 1000 \end{pmatrix} \quad \Sigma_p^{\frac{1}{2}} = \begin{pmatrix} 500 & -10 & 5 \\ 0 & 350 & 10 \\ 0 & 0 & 380 \end{pmatrix} \quad (5.52a)$$

$$\boldsymbol{\mu}_v = \begin{pmatrix} 10 \\ 5 \\ 5 \end{pmatrix} \quad \Sigma_v^{\frac{1}{2}} = \begin{pmatrix} 10 & 2 & 1 \\ 0 & 15 & 3 \\ 0 & 0 & 15 \end{pmatrix} \quad (5.52b)$$

The goal is to determine the positions of the two bistatic receivers that minimise the estimation errors of the target's state. The placement of each receiver is constrained, receiver 1 is constrained to a vertical line and receiver 2 to a horizontal line, as shown in Fig. 5.23. The search space is constrained to a 2D plane and can therefore the optimal geometry can be easily calculated using a direct search for the optimal point. These constraints were chosen to avoid optimal geometries in which the target is in the forward scattering region between the transmitter and receivers. As the forward scattering effect is not contained within the model this is a practical consideration.

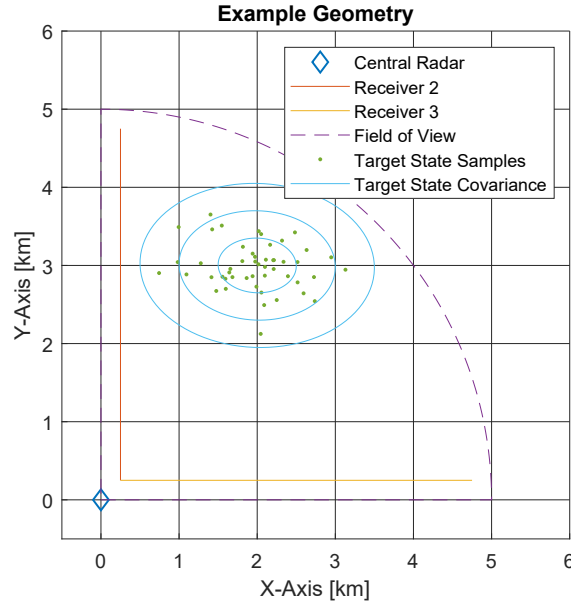


Figure 5.23: Example Initial Network Geometry

Fig. 5.23 illustrates the example geometry. The possible positions of the first bistatic receiver, receiver 1, is shown as a vertical line in the y-axis in red. Likewise, the possible positions of the second bistatic receiver, receiver 2, is shown as a horizontal line in the x-axis in orange. The covariance of the target's position is illustrated by the blue ellipses, each ellipse showing the first, second and third standard deviations of the covariance of the target's state. Samples of the distribution of the target are shown as green dots.

The signal used is the Gaussian pulse train containing 64 pulses with the following properties

Signal parameter	Value	Unit
Pulse repetition frequency (pri)	7200	Hz
Carrier frequency	L-Band	GHz
Number of pulses (κ)	64	-
Gaussian pulse scaling parameter (ν)	0.5	μs

Table 5.5: Signal Parameters

Using the above notation the components of the CRLB for estimating the range and Doppler shift from (5.3) for a complex envelope consisting of κ Gaussian pulses, is

$$\mathbf{J}_F(r, f) = \frac{\text{SNR}_n}{2} \begin{pmatrix} \frac{1}{c^2 \nu^2} & 0 \\ 0 & 4\pi^2 \nu^2 + \frac{2}{3}\pi^2 \text{pri}^2 (\kappa^2 - 1) \end{pmatrix} \quad (5.53)$$

The model is derived assuming that the signal energy E is constant and independent of the targets state, $\boldsymbol{\theta}$. Define the energy as $E = 1 \times 10^{-15}$ J and the noise power spectral density as $N_0 = 2 \times 10^{-18}$ W/Hz. The variance of β_n determines the amount, on average, of signal fading that occurs at the n-th receiver, for each of the 3 receiver in the radar network define the following variances.

$$\sigma_{\beta_1}^2 = 0.5, \quad \sigma_{\beta_2}^2 = 0.25, \quad \sigma_{\beta_3}^2 = 1.5 \quad (5.54)$$

The equation for the SNR at the n th receiver is

$$\text{SNR}_n = \frac{4\sigma_{\beta_n}^2 E}{N_0} \left(\frac{2\sigma_{\beta_n}^2 E}{2\sigma_{\beta_n}^2 E + N_0} \right) \quad (5.55)$$

Substituting the defined values for the signal fading into the SNR equation, noting that the noise power spectral density is assumed to be the same at all receivers. The SNR at each receivers is therefore

$$\text{SNR}_1 = 30 \text{ dB}, \quad \text{SNR}_2 = 26.9 \text{ dB}, \quad \text{SNR}_3 = 34.7 \text{ dB} \quad (5.56)$$

The Bayesian lower bound describes the theoretical estimation error of a given radar geometry Γ . Define the optimal radar network geometry as the geometry which minimises the trace of the Bayesian lower bound for a given target distribution. The geometry is optimised by first generating a number of samples of the target's state, in this example 1 million samples were generated. The targets state is distributed as a multivariate Gaussian with mean and covariance given by (5.52). Each of the two receivers are constrained to a line therefore the space of all possible geometries is 2D. For all possible configurations of the two receivers the trace of the Bayesian lower bound (5.50) is estimated.

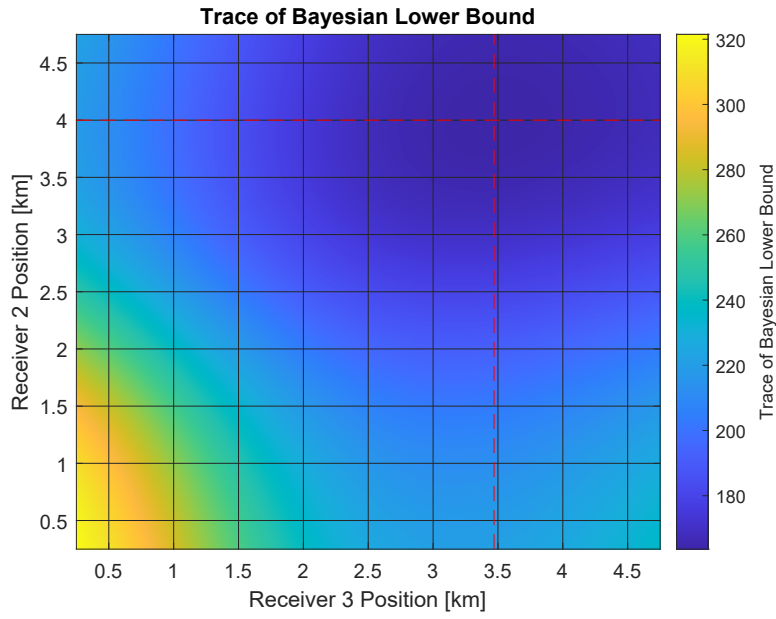


Figure 5.24: Trace of Bayesian Lower Bound

Fig. 5.24 shows the trace of the Bayesian lower bound as the geometry is varied. A step size 10 m was used. The minimum and therefore optimal geometry is indicated by the dashed red lines. Therefore the optimal positions of the two receiver are

$$\mathbf{p}_2 = \begin{pmatrix} 250 \\ 4000 \\ 0 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 3470 \\ 250 \\ 0 \end{pmatrix} \quad (5.57)$$

Fig. 5.25 illustrates the resulting optimal geometry with respect to the central radar and the distribution of the target.

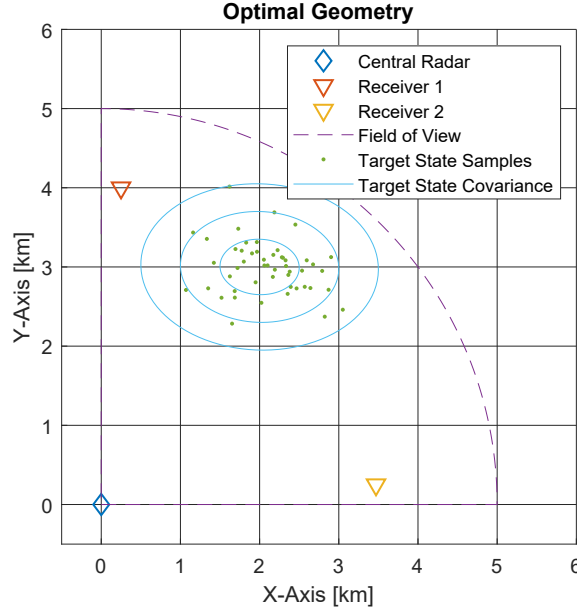


Figure 5.25: Network Geometry

Monte-Carlo Simulations

Monte-Carlo simulations of the MAP estimator are used to verify that the performance of the radar network attains the theoretical lower bound. The simulation geometry is previously found optimal geometry Fig. 5.25.

The simulations consists of 5000 Monte-Carlo simulations per SNR point. 16 SNR points were run, The value of E is controlled in order to vary the average SNR at the central radar -30 dB and $+30$ dB. The simulations initialised by generating 5000 samples of the targets state using the prior mean and prior covariance. For each sample the data vector $\mathbf{z}$ is generated. The data contains the received signals at each receiver that a target in the given state would create. The coordinate descent algorithm then finds the MAP estimator by iterating over the search space.

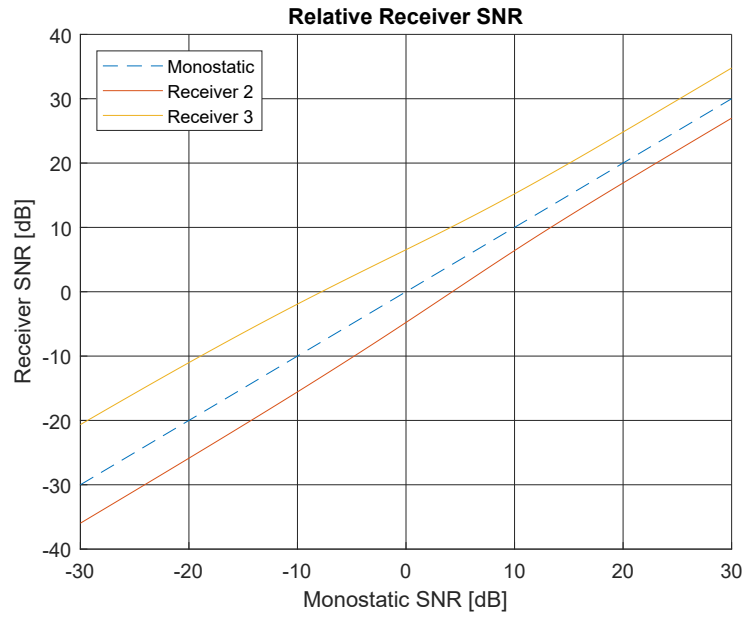


Figure 5.26: Receiver SNR against Central Radar SNR

Fig. 5.26 shows the SNR at each of the bistatic receivers relative to the SNR at the central monostatic radar. Note that the relative SNR curves are not linear, this is due to the second term in Eq. 5.55.

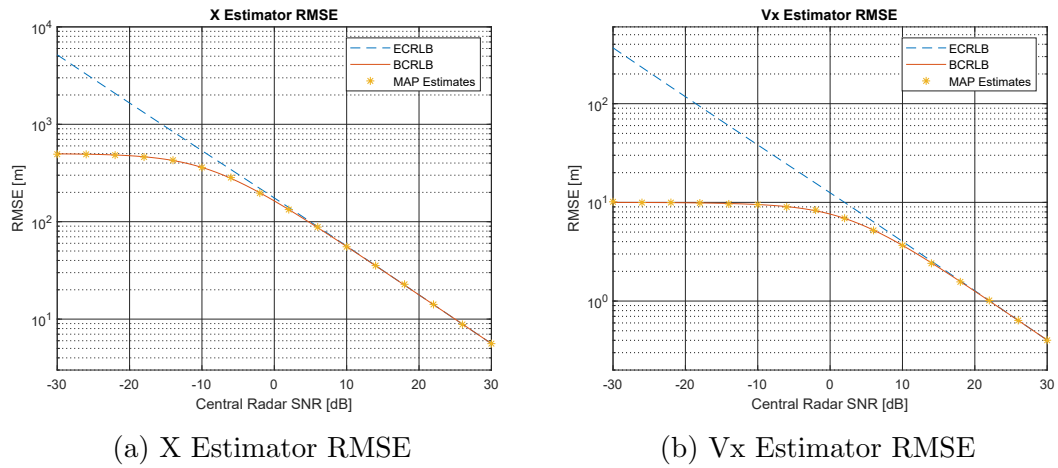


Figure 5.27: RMSE of the MAP Estimator against SNR (X Components)

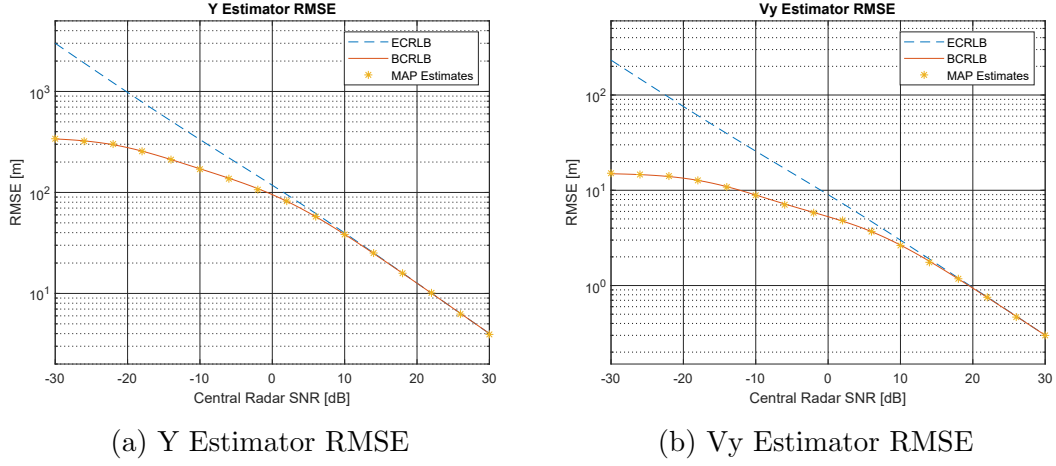


Figure 5.28: RMSE of the MAP Estimator against SNR (Y Components)

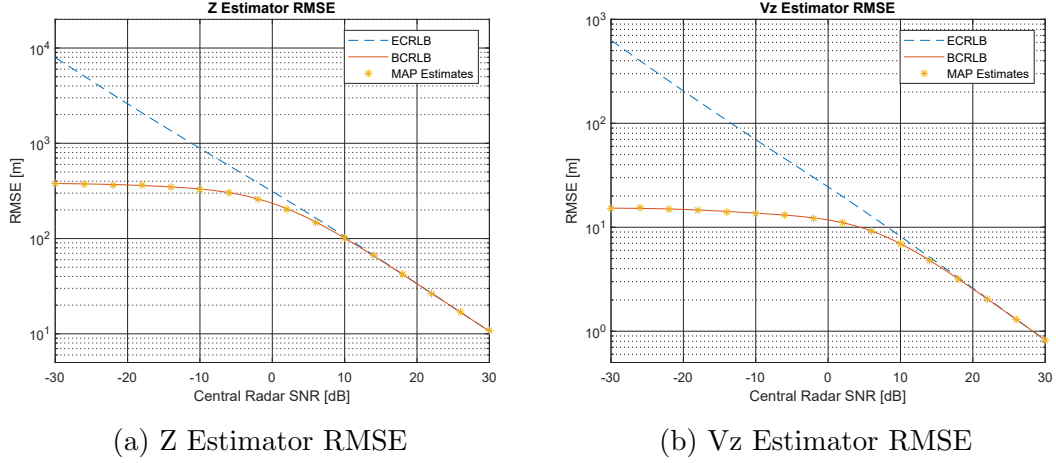


Figure 5.29: RMSE of the MAP Estimator against SNR (Z Components)

Fig. 5.27-5.29 shows the root mean square error of the MAP estimates for each of the six target state parameters as the SNR varies from -30 dB to 30 dB. The BCRLB depends on the prior information and the data from the network. When the SNR is low the BCRLB is equal to the square root of the variance of each component of the targets distribution. For example the variance of the targets v_x velocity was defined as 10^2 m² in Fig. 5.27 the BCRLB at SNR = -30 dB is 10m. When the SNR is sufficiently high, in this case 10 dB the BCRLB is bounded by the ECRLB which only depends on the data. In all cases the variance of the MAP estimator attains the BCRLB for all SNR. This is unlike the previous Monte-Carlo simulations of the

MLE which attains the theoretical bound only when the SNR is high, the difference is due to the prior information. These results validate the model and theoretical lower bound, and by extension the optimality of the found network geometry.

5.3.5 Variable SNR

The theoretical model developed in the previous section assumes that the signal energy E in (5.3) is constant, only time delay and Doppler shift of the signal depend on the target's state. In this section, the same model is used but the signal energy E is defined as a function of the target's state, via the received power and radar range equation. The purpose of this is to test the robustness of the model, as in practice the signal energy is dependent on the position of the target.

The results of the found geometry will be compared with the optimal geometry from the previous section and the found lower bound will be tested using Monte-Carlo simulations of the MAP estimator.

Due to the spatially distributed nature of the network, the received signals at each receiver have different multipath effect. The variance of β_n determines the amount, on average, of signal fading that occurs at the n -th receiver. or each receiver in the radar network define the following variances.

$$\sigma_{\beta_1}^2 = 0.5, \quad \sigma_{\beta_2}^2 = 0.1, \quad \sigma_{\beta_3}^2 = 1 \quad (5.58)$$

The SNR at the n th receiver is

$$\text{SNR}_n(\boldsymbol{\theta}) = \frac{4\sigma_{\beta_n}^2 E_n(\boldsymbol{\theta})}{N_0} \left(\frac{2\sigma_{\beta_n}^2 E_n(\boldsymbol{\theta})}{2\sigma_{\beta_n}^2 E_n(\boldsymbol{\theta}) + N_0} \right) \quad (5.59)$$

for $n \in [1, N]$. The energy of a Gaussian pulse train depends on the shape of the

Gaussian pulse, the number of pulses and the peak received power.

$$E_n(\boldsymbol{\theta}) = \sqrt{\pi\nu\kappa}P_{R_n}(\boldsymbol{\theta}) \quad (5.60)$$

The peak signal power is determined by the bistatic radar range equation

$$P_R(\boldsymbol{\theta}) = \frac{G_T G_R \lambda^2 L \sigma}{(4\pi)^3 |\mathbf{p}_o - \mathbf{p}|^2 |\mathbf{p}_n - \mathbf{p}|^2} P_T. \quad (5.61)$$

With the following values

Component	Property	Symbol	Value	Unit
Transmitter	Transmit power	P_T	2	kW
	Antenna gain	G_T	40	dBi
	Pulses	κ	64	-
	Frequency	f_c	L-Band	Hz
Environment	Atmospheric attenuation	L	1	-
Target	Radar cross-section	σ	0.01	m ²
Receiver	Antenna gain	G_R	45	dBi
	Bandwidth	B	1	MHz
	Noise PSD	N_0	-150	dB W/Hz
	Noise power	P_N	-90	dBW

These values were chosen to ensure that the SNR at the central radar is at least 15 dB across the radars field of view.

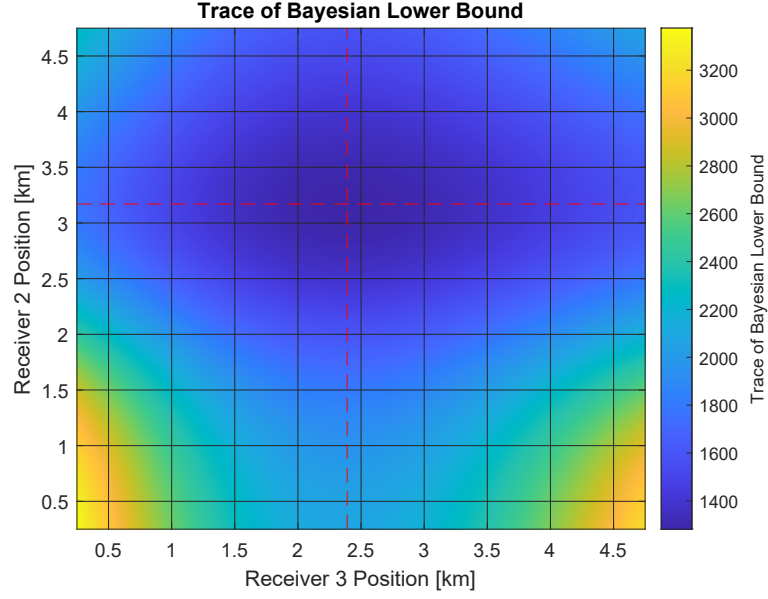


Figure 5.30: Trace of Bayesian Lower Bound

Fig. 5.30 shows the trace of the Bayesian lower bound against the position of the two receivers. From Fig. 5.30 the optimal positions of the two receivers are

$$\mathbf{p}_2 = \begin{pmatrix} 250 \\ 3170 \\ 0 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 2390 \\ 250 \\ 0 \end{pmatrix}. \quad (5.62)$$

Fig. 5.31 shows the expectation of the SNR against position for the receivers 2 and 3. The optimal positions of the receivers are indicated by the dashed vertical lines, note that they occur close to but not at the peak of the SNR curves. This suggests that the optimal geometry is not necessarily the one that maximises the receiver SNR.

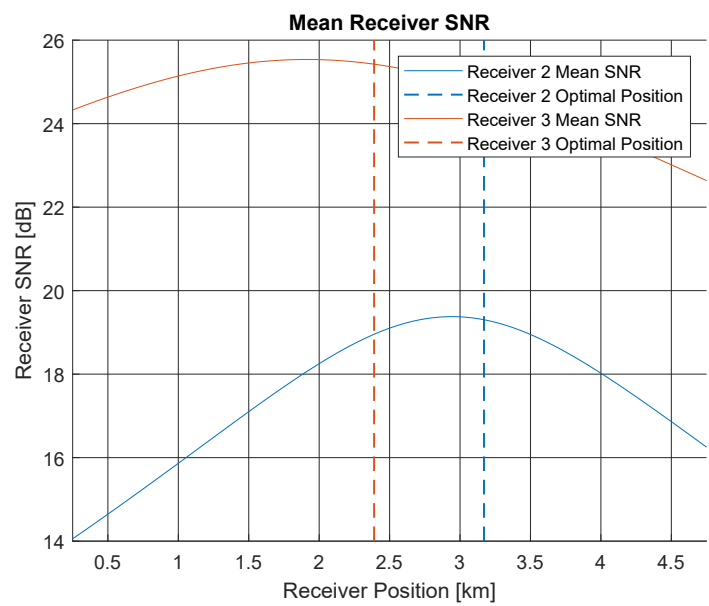


Figure 5.31: Expectation of Receiver SNR

Fig. 5.32 illustrates the found optimal network geometry with respect to the targets distribution

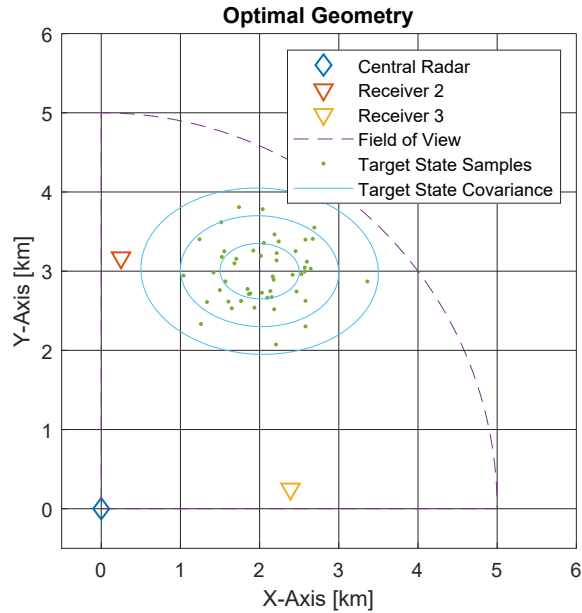


Figure 5.32: Network Geometry

Monte-Carlo Simulations

Monte-Carlo simulations of the MAP estimator are used to compare the performance of the radar network with the found lower bound. Note that as the model was developed for E constant, the performance of the MAP estimator is not expected to go to the found lower bound.

The simulation geometry is optimal geometry found above, as shown in Fig. 5.32. The value of P_T is controlled to ensure that the average SNR at the central radar varies between -30 dB and $+30$ dB.

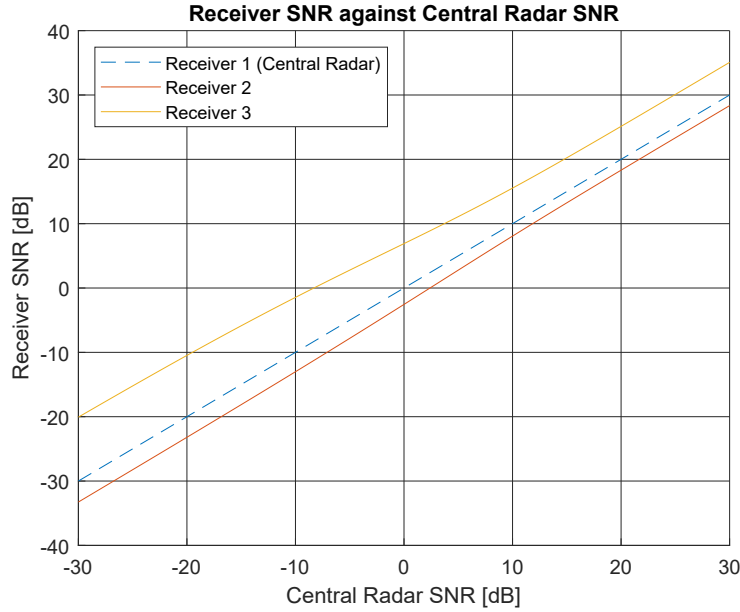
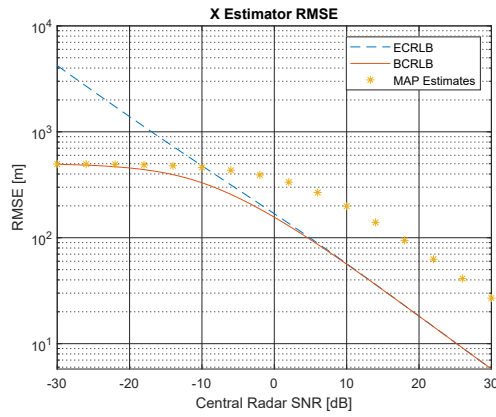
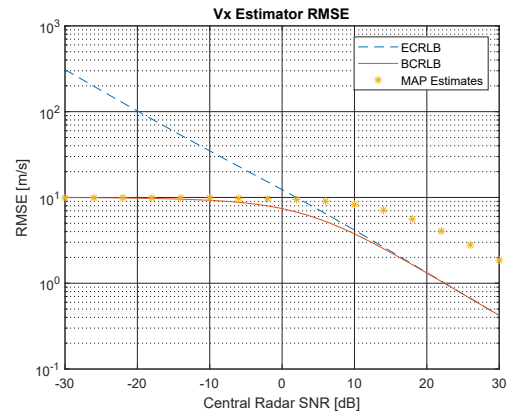


Figure 5.33: Receiver SNR against Central Radar SNR

Fig. 5.33 shows the relative SNR at each receiver, note that the relative SNR is no longer a line of constant slope on a log-log plot due to the second part of equation (5.55). The following figures show the root mean square estimation errors of the elements of the target estimation vector using 5000 Monte-Carlo trials for each of the 8 SNR sample point. In order to ensure that the CD algorithm correctly converged to the peak of the MAP the algorithm was ran multiple times for each trial, with a randomly selected initial starting point.

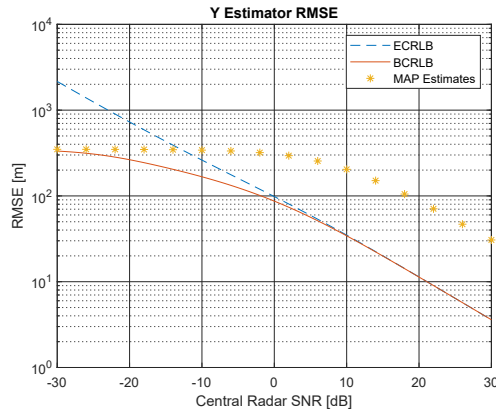


(a) X Estimator RMSE

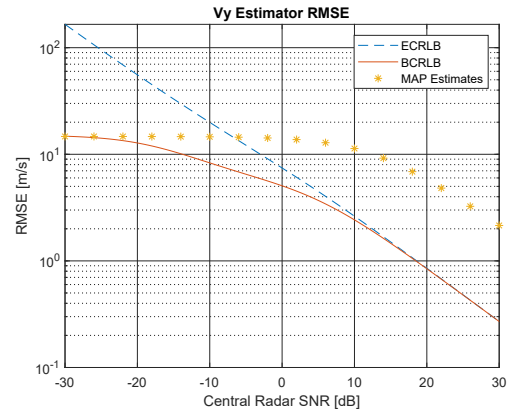


(b) Vx Estimator RMSE

Figure 5.34: RMSE of the MAP Estimator against SNR (X Components)

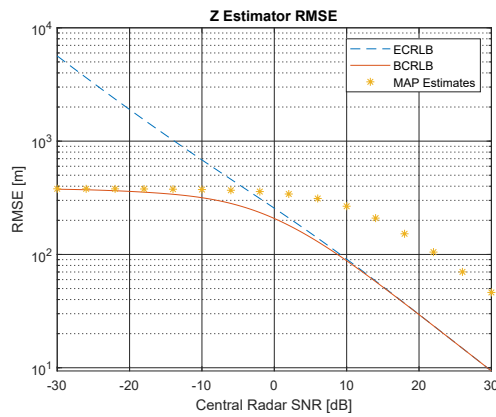


(a) Y Estimator RMSE

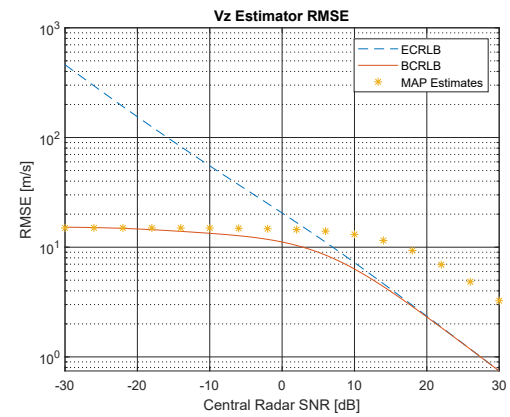


(b) Vy Estimator RMSE

Figure 5.35: RMSE of the MAP Estimator against SNR (Y Components)



(a) Z Estimator RMSE



(b) Vz Estimator RMSE

Figure 5.36: RMSE of the MAP Estimator against SNR (Z Components)

Fig. 5.34-5.36 shows the root mean square error of the MAP estimates for each of the six target state parameters as the SNR varies from -30 dB to 30 dB. At low SNR the variance of the MAP estimator attains the lower bound as in the previous example. Unlike the previous case the estimator does not attain the bound at high SNR. This can be explained by looking definition of the MAP estimator.

$$\hat{\boldsymbol{\theta}}_{map}(\mathbf{z}) = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \{ \ln p(\mathbf{z}|\boldsymbol{\theta}) + \ln p(\boldsymbol{\theta}) \} \quad (5.63)$$

The MAP estimator depends on two terms the first depends on the data, $\mathbf{z}$ and the second depends on the properties of the probability density function of $\boldsymbol{\theta}$.

When the SNR is low the returns from the target are below the noise floor therefore the first term contributes only noise. At this point the MAP estimator is behaving exactly as the previous example in which the signal energy was not dependent on the target's state. Therefore it is expected that the RMSE is the same as the previous model and attains the lower bound, as at low SNR the variable energy extension fits the model used to derive the lower bound. It is at high SNR that the variable energy extension diverges from the model and it is expected that the estimator does not attain the lower bound. The estimator does show convergence, suggesting that a lower bound does exist, this is possible future extension to the thesis.

5.4 Conclusion

In this chapter two geometry optimisation algorithms are developed, the first for a target with an unknown but constant position and velocity which uses a distributed data fusion scheme and the second for a target with an unknown random position and velocity which uses a centralised data fusion scheme.

The first algorithm models the target's position and velocity as an unknown de-

terministic parameter. An estimator of the target's state is developed under a distributed data fusion scheme, in which the relative range and Doppler shift of the target are estimated locally at each receiver. The estimated range and Doppler shift from each receiver are centrally gathered and the target's state is estimated. The theoretical Cramér-Rao Lower Bound for the position and velocity estimator is calculated. The calculated bound depends on the network geometry, properties of the transmitted signal and the relative position and velocity of the target. The geometry optimisation algorithm uses the trace of the lower bound as a cost function. The algorithm varies the positions of the receivers to find the geometry that minimises the lower bound and, therefore, the estimation errors. Monte-Carlo simulations of the Maximum Likelihood Estimator have been developed to verify that the optimal geometry attains the theoretical lower bound. The MLE requires a search over a 4D space, as it is not computationally feasible to search every point; the Coordinate Descent algorithm was used to iteratively search for the maximum likelihood estimate for each Monte-Carlo trial. The results show that the estimator attains the theoretical bound for high SNR (+20 dB).

The second algorithm extends the first by modelling the target's position and velocity as a multivariate random vector. The model also takes into account the effect of a fluctuating received power due to signal multipath effects. The shift to modelling the target's position and velocity as random vectors requires the use of a centralised data fusion scheme in which the target's position and velocity are directly estimated using all the received signals in one step. The theoretical Bayesian-CRLB for the position and velocity estimator is calculated. The geometry optimisation algorithm uses the trace of the lower bound as a cost function; the algorithm varies the positions of the receivers in the network to find the geometry that minimises the bound and, therefore, the estimation errors. Monte-Carlo simulations of the Maximum a Posteriori Estimator are used to verify that the performance of the found optimal

geometry attains the theoretical lower bound. The MAP estimate requires a search over 6D space, as it is not computationally feasible to search every point; the Coordinate Descent algorithm was used to iteratively search for the maximum a posteriori estimate for each Monte-Carlo trial. The results show that the estimator attains the theoretical bound.

Chapter 6

Dual Channel Passive Receiver

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The dual channel receiver is designed to be used as a multi-static receiver, designed to cooperate with the Aveillant radar as part of a multistatic radar network. With the ultimate goal of the experimentally validating the theoretical results developed in the previous chapters.

6.1 Design Principle

The networks receiver nodes need to be capable of operating in a multistatic configuration with the central Aveillant radar; therefore the receivers are designed using passive radar principles with two receive channels, as illustrated in Fig. 6.1.

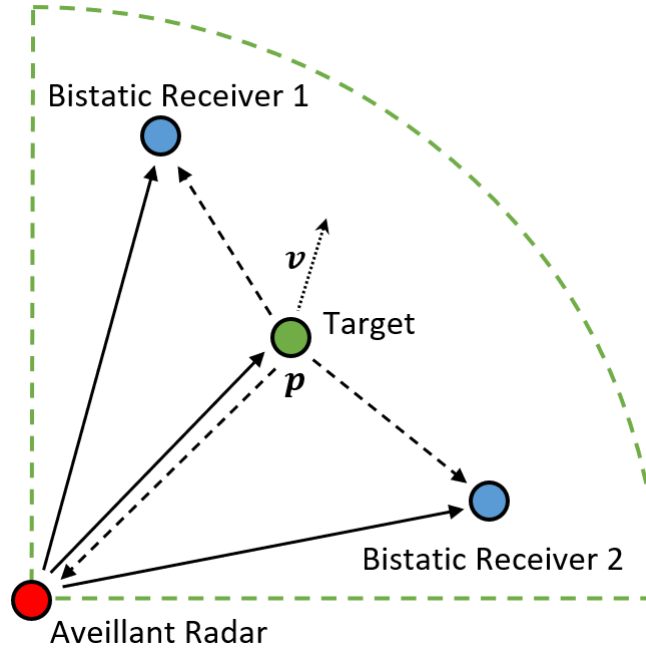


Figure 6.1: Cooperative Multistatic Network

The use of two channels allows the receiver to record the transmitted signal and reflections from the target simultaneously and separately. Ideally the reference channel should be highly directional and oriented towards the transmitter to ensure that only the transmitted signal is recorded. The surveillance should be as omni-directional as possible and be oriented such that the returns from the target are recorded but not the transmitted signal. Here, two horn antennas with a 3 dB beam-width of 90°

will be used for both channels. A future extension of the receiver should explore the use of a phased array antenna for the surveillance channel as it would permit steering in the direction of the target while maintaining a null in the direction of the transmitter. When carrying out measurements the positions of transmitter, target and receiver will be carefully considered to ensure that the reference channel only records the transmitted signal and that the surveillance channel only records the returns from the target.

The use of a staring transmitter (Aveillant radar) simplifies the design of the receivers in several ways, firstly it means that the receiver can continually record the reference signal regardless of it's position within the radar's field of view. This also means that the receiver does not need to perform any form of pulse chasing. To illustrate this consider that the transmitter is steered either mechanically or electronically across it's field of regard, in order to determine the range of the target, the receiver would need to know the steered direction of the transmitted signal at all times, requiring an additional communications link between the receivers and transmitter. The use of a reference channel allows the receivers to operate coherently locally but incoherently between nodes, as the receiver clocks aren't locked.

The theoretical derivations presented in the previous chapters assume that the reference signal is free from interference or noise, in reality this won't be case. The receiver's reference channel will still suffer from thermal noise, however we assume that the signal to noise ratio is sufficiently large that these effects are negligible. Note that while the received target signal propagates from the transmitter to the target to the receiver, the received reference signal propagates directly from the transmitter to the receiver and therefore the reference signal power decrease according to the inverse square law.

The Global Positioning System (GPS) is used to synchronise measurements across the network. Each of the networks receivers has an attached GPS antenna, which the computer uses to record the receivers position and the Coordinated Universal Time (UTC) of each measurement. This allows the recorded data from the receivers to be combined post measurement as knowledge of the receivers position is crucial otherwise returns from different spatial regions will be incorrectly combined. Most modern drones also contain a GPS unit used to record aspects of the drone's flight such as position, speed and heading. After a trial is complete, the drone's position can be more accurately correlated with the data recorded at each receiver. For targets not equipped with GPS receivers, light-weight GPS tags can also be used.

The dual-channel receiver prototype consists of two receiving channels, a voltage controlled oscillator (VCO) block, digitizer, GPS antenna and computer, as shown in Fig. 6.2. The VCO is an Aaren Technology Electronics RFF25611 which generates the local oscillator (LO) signal at L-band, necessary to match the carrier frequency used by the Aveillant radar. The LO signal is amplified by a low noise amplifier (LNA) (mini-circuit ZX60-P105LN+) and attenuated to 10 dBm. The amplification and attenuation of the LO signal is required to ensure that the power input to both mixers is 7 dBm, therefore the power of the LO signal going into the splitter must be twice this at 10 dBm. An RF splitter (mini-circuit ZX10-2-20-S+) splits the VCO signal into two signals at 7 dB, necessary to drive the mixers on the receiving channels.

Receiver Block	Component	Model ID	Quantity
VCO Block	Voltage Controlled Oscillator	ZX95-1300-S+	1
	Attenuator	VAT-5+	2
	RF Splitter	ZX10-2-20-S+	1
Receiving Block	Bandpass filter	-	2
	Low Noise Amplifier (Reference)	ZXP60-P162	1
	Low Noise Amplifier (Surveillance)	ZX60-2534MA-S+	1
	RF Mixer	ZX05-C42-S+	2
	Low Pass Filter	VLF-45+	2

Table 6.1: Dual Channel Receiver Components

Table 6.1 details the components of the receiver, the impedance is 50 Ω for all components.

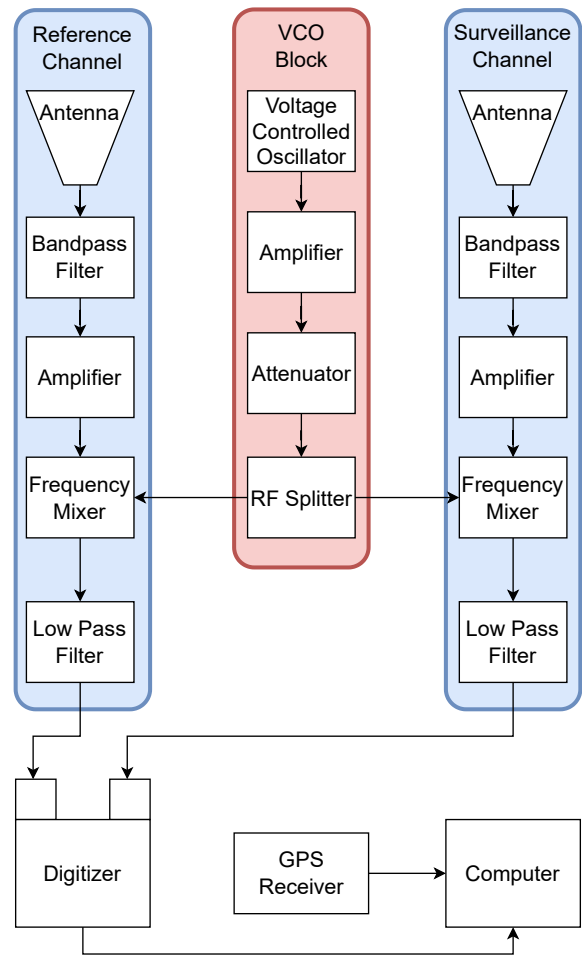


Figure 6.2: Dual Channel Receiver Block Diagram

The received signal from each antenna is filtered by a bandpass filter and then is

amplified by a low noise amplifier (LNA). The LNA used on the reference channel is a Mini-Circuit ZX60-P162LN+ with a gain of 22 dB. The LNA used on the surveillance channel is a Mini-Circuit ZX60-2534MA+ with a gain of 43.25 dB. The amplified signal is mixed down (mini-circuit ZX10-2-20-S+) using the LO signal from the VCO block. The received signals, now at baseband, is filtered by a 45 MHz low pass filter (mini-circuit VLF-45+) and digitized using a TiePie Handyscope digital oscilloscope. The data from the digitizer is recorded by the computer along with the current UTC time as well as the receiver's latitude and longitude. The timing and positioning data is obtained from the attached GPS, a GlobalSat BU-353-S4.

6.2 Signal Processing

The radar operates in a block data collection cycle. The computer reads the receivers current latitude, longitude and UTC time from from the attached GPS receiver and triggers the digitizer. The digitizer is preconfigured to record a continuous block of data 250–300ms from both channels when triggered. Once the digitizer has finished recording, the computer saves the data to disk along with the information from the GPS and the digitizer's settings. Once the data is saved, the next measurement is triggered.

Each block of data is loaded and processed independently. The data from both channels is loaded as two vectors, the mean value from each channel is subtracted, and the Hilbert transform is used to create the the IQ components of the signal. The peak of the reference channel in the frequency domain is selected and used to place a bandpass filter over the main lobe of the signal, filter width of 2 – 3 MHz. The filtered signals are transformed back to the time domain and each channel is divided into pulses to form two Fast-Time Slow-Time frames. The surveillance frame is matched filtered with the frame data from the reference channel. The short-time

Fourier transform and the Range-Doppler map can then be generated. Moving Target Indication (MTI) is performed on the Range-Doppler map to suppress the direct transmitter to receiver signal and stationary clutter.

Fig. 6.3 shows the components of the dual-channel receiver integrated into an RF shielded enclosure. The two blue cables on the left are the inputs from the antennas. Each of these inputs connects to a low noise amplifier connected to the mixers and low pass filters.

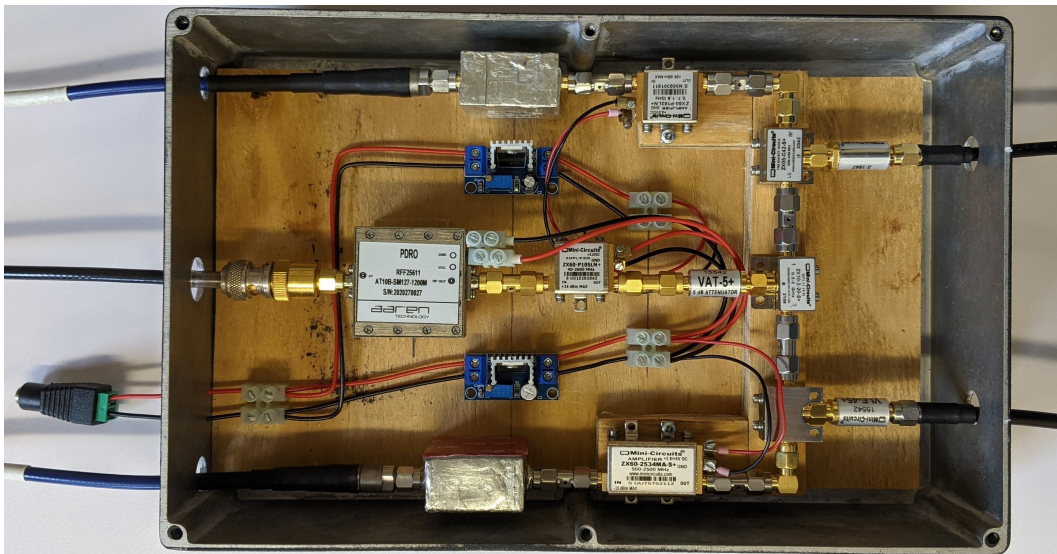
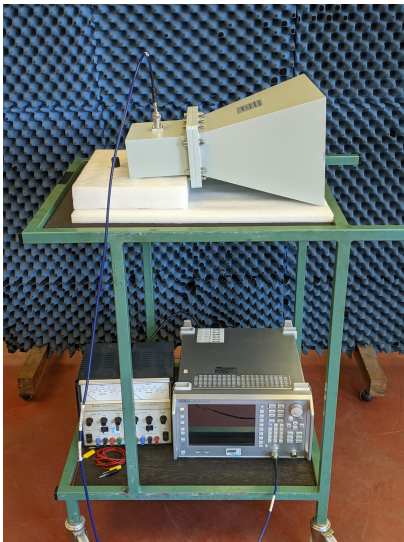


Figure 6.3: Dual Channel Receiver Hardware

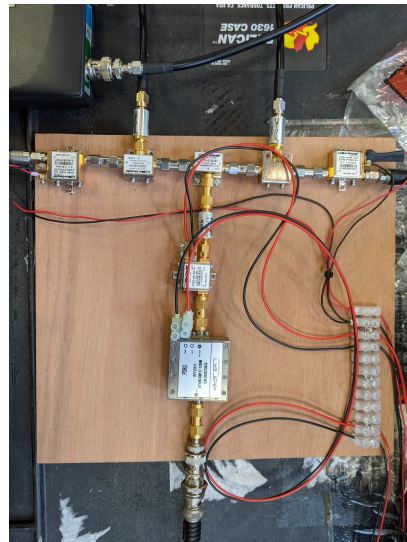
The two black cables on the right are the output cables to the digitiser. The central black cable on the left provides the tuning voltage for the VCO, the output from which is connected to an amplifier, attenuator and splitter. The output of the splitter connects to the mixers. The remaining components are the two voltage regulators which convert the input 12V DC to the 5V and 4V required by the individual components.

6.3 Measurements

Initial measurements of a rotating fan, remote control helicopter and a moving vehicle have been carried out to evaluate the performance of the dual-channel receiver prototype, in the radar laboratory. During the measurements, a rotating fan with metal reflectors and a Helicopter was placed in front of a panel of Radar Absorbent Material (RAM) as shown in Fig. 6.5. A separate transmitter, an Anritsu MS2691A Signal generator and antenna, was placed off to the side oriented towards the target Fig. 6.4a. The prototype receiver was placed such that the surveillance channel could observe the target without picking up the transmitter's direct signal; the reference channel was then aligned with the transmitter.



(a)



(b)

Figure 6.4: Transmitter (a) and Dual Channel Receiver Hardware (b)

Experimental results were obtained using an integration time of 10 ms sampled at 20 MHz (200 thousand total samples). The frequency of the local oscillator signal, used to convert the received signal from the carrier frequency to baseband, was tuned to 5 MHz below the frequency of the transmitted signal.

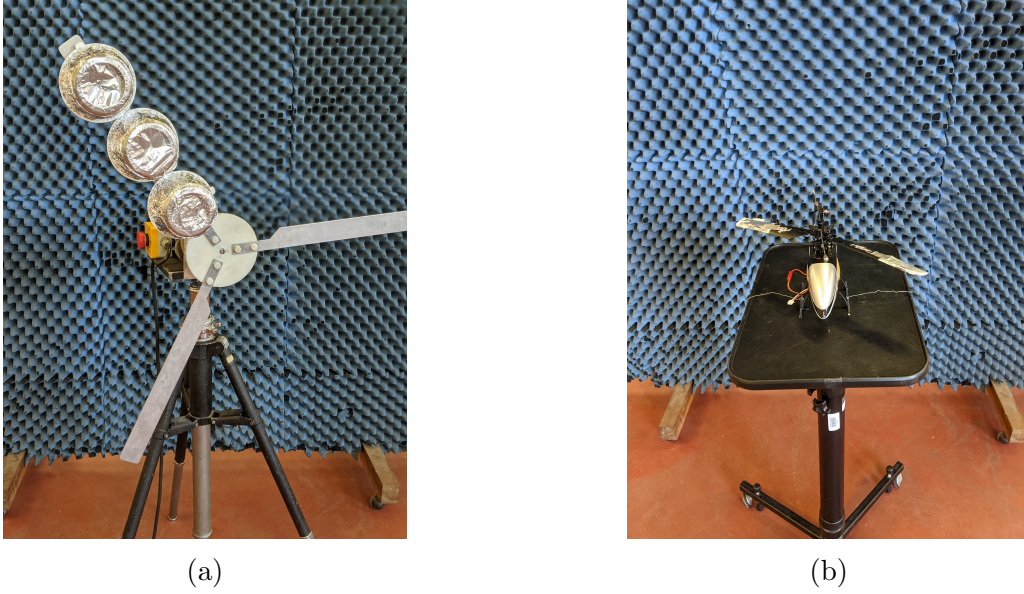


Figure 6.5: Rotating Fan (a) and Helicopter (b)

6.3.1 Measurements of a Rotating Fan

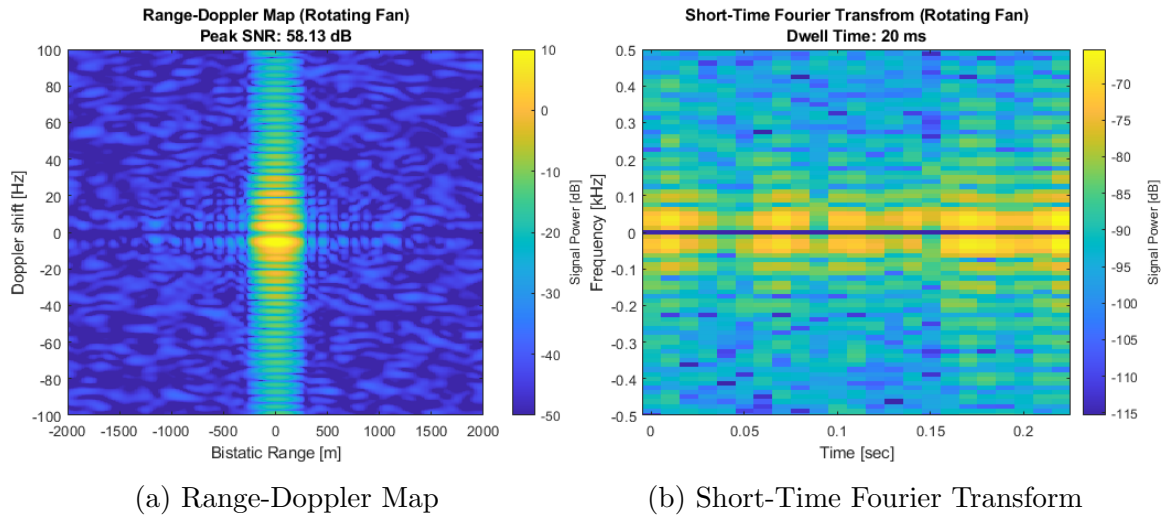


Figure 6.6: Measurements of a Rotating Fan

Fig. 6.6a and Fig. 6.6b shows the measurements of a rotating fan taken by the dual-channel receiver. The fan consists of three 46 cm long blades rotating at approximately 120 rotations per minute, as shown in Fig. 6.5b. Fig. 6.6a shows the range-Doppler response of the rotating fan. The fan is detected with a peak SNR of 54 dB at a range of 7.5m. Two peaks are visible at ± 5 Hz. Sidelobes are visible in

both the range and Doppler. Fig. 6.6b shows the STFT of the target using a dwell time of 20 ms, which is starting to show the periodic motion of the fan, however further work is required to improve this image.

6.3.2 Measurements of a Remote Control Helicopter's Rotors

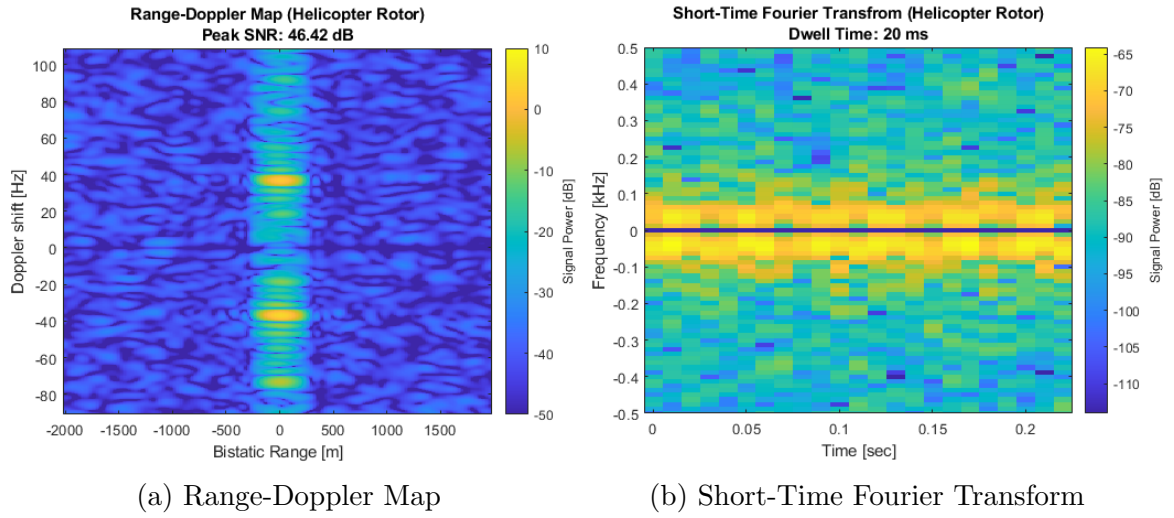


Figure 6.7: Measurements of the Remote Control Helicopters Rotors

Fig. 6.7 shows the measurements of a remote control helicopters rotors taken by the dual-channel receiver. The helicopter with 18.5 cm long rotors was placed on a stand with the rotors spinning in a stationary position Fig. 6.5. Results in Fig. 6.7a shows the range-Doppler response of the Helicopters rotors. Two peaks are visible with a Doppler shift of ± 40 Hz, the rotors are detected with a peak SNR of 46 dB at a range of 7.5 m. Fig. 6.7 (b) shows the STFT of the target using a dwell time of 20 ms, the periodic signature of the rotors is visible showing multiple rotor rotations, indicating a maximum Doppler shift of 625 Hz.

6.3.3 Measurements of a Car

Measurements of a moving car were taken using the dual channel receiver acting as a bistatic receiver to a separate transmitter. The car was driven towards and away from the receiver. Experimental results were obtained using blocks of 250 ms each sampled at 20 MHz (5M total samples). The frequency of the LO signal was tuned to 3 MHz below the frequency of the transmitted signal.

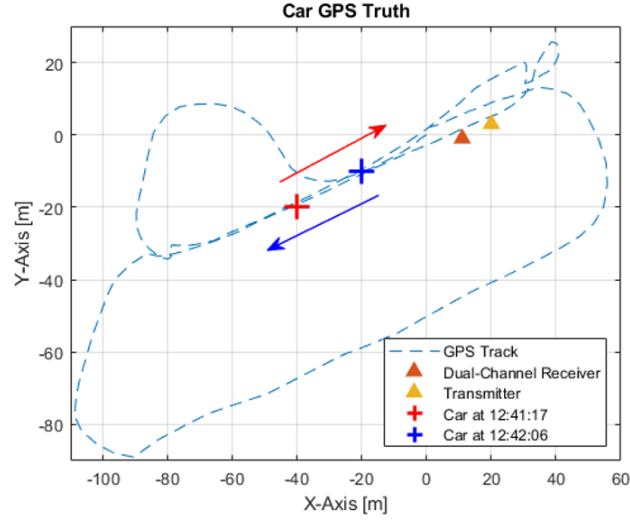


Figure 6.8: Car GPS Truth

Fig. 6.8 shows the GPS track of the car as well as the positions of the dual-channel receiver and transmitter. The receiver was placed directly between the target and the transmitter. The reference channel was aligned with the transmitter, while the surveillance channel was pointing directly away from the transmitter towards the target. The car was driven towards and away from the receiver at constant speeds. The two markers indicate the position of the car at 12:41:17 and 12:42:06, the range-velocity maps at these times are shown in Fig. 6.9a and Fig. 6.10a.

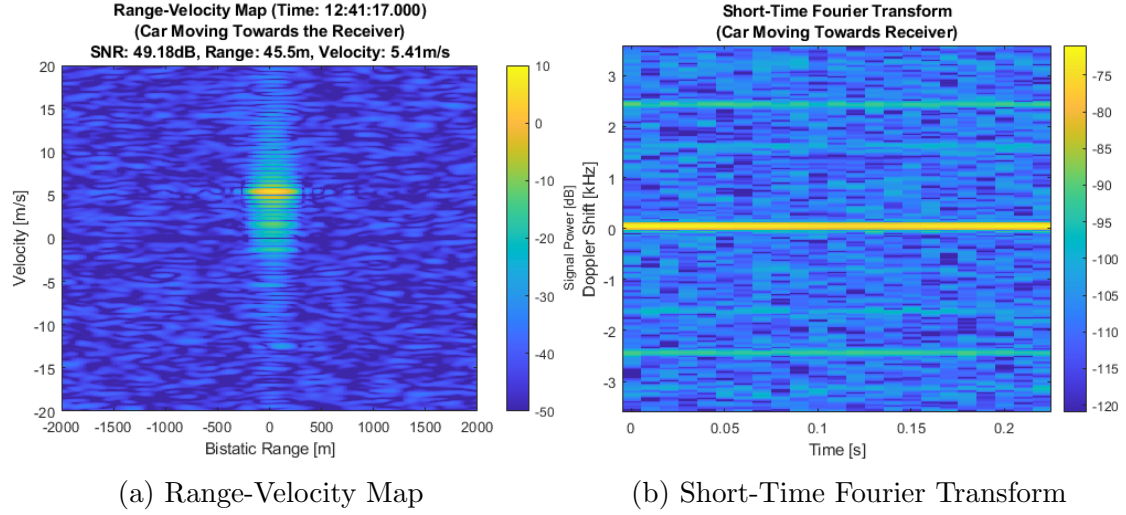


Figure 6.9: Range-Velocity Map and Short-Time Fourier Transform of the car towards the radar

Fig. 6.9a shows the Range-Velocity map and 6.9b Short-Time Fourier transform of the car moving towards the receiver. The target is detected with an SNR of 49 dB at a range of 46 m with a velocity of 5.4 m/s. The GPS recorded the car at a range of 50 m with a velocity of 6.6 m/s.

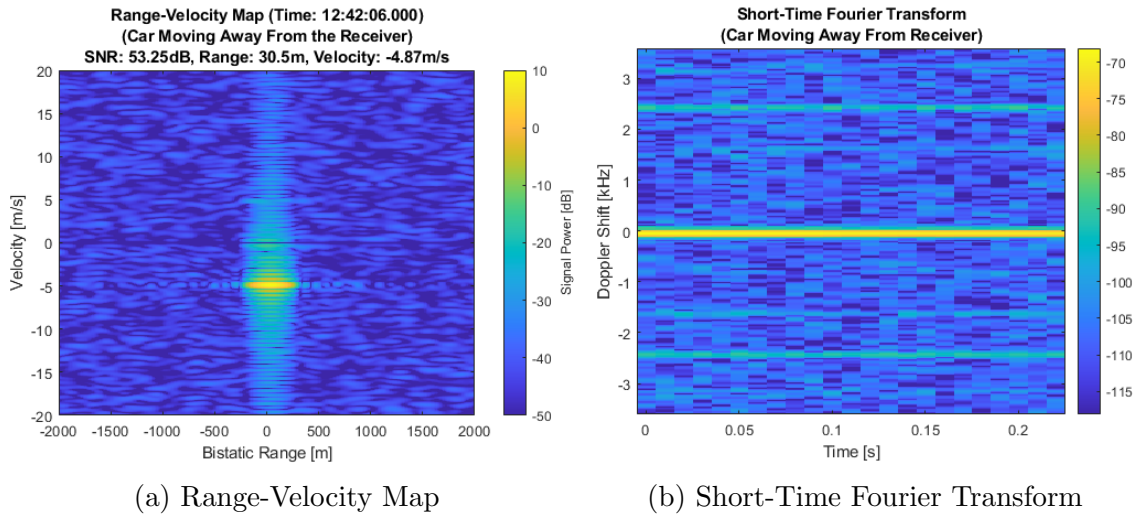


Figure 6.10: Range-Velocity Map and Short-Time Fourier Transform of the car moving away from radar

Fig. 6.10a shows the Range-Velocity map and 6.10b Short-Time Fourier transform of the car moving away from the receiver. The target is detected with an SNR of 52 dB at a range of 22.5 m with a velocity of -5 m/s. The GPS recorded the car

at range of 32 m with a velocity of -5.2 m/s. In both cases the noise floor of the range-Doppler map is estimated at -40 dB.

6.3.4 Measurements of a Drone

Measurements of a drone using the dual channel receiver have been recorded. The DJI Matrice 600 Pro is a hexacopter drone measuring $1.6\text{m} \times 1.5\text{m} \times 0.7\text{m}$, shown in Fig. 6.11. The drone weighs 10kg and is capable of reaching a top speed of 40mph.



Figure 6.11: DJI Matrice 600 Pro

The transmitter and a single dual channel receiver were set up in a bistatic configuration with a baseline of 20 m. As in the previous measurements the transmitter consisted of an Anritsu signal generator/ spectrum analyser connected to a horn antenna.

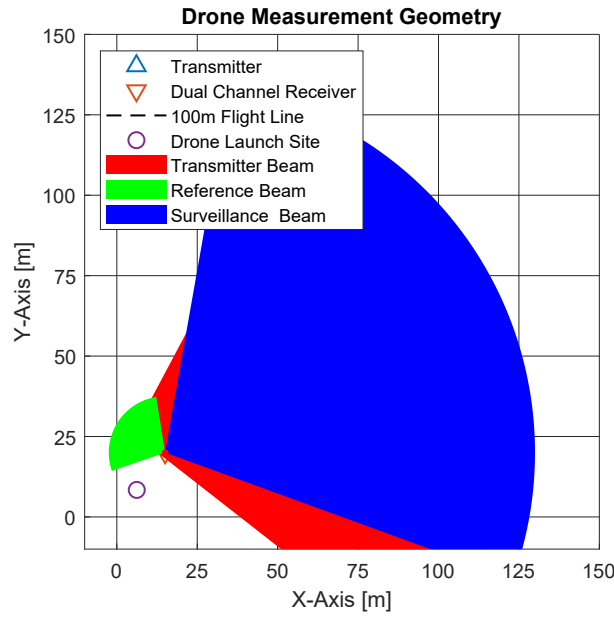


Figure 6.12: Drone Measurement Geometry

Fig. 6.12 illustrates the relative positions of the transmitter and dual-channel receiver. The approximate direction of the transmit and receive antenna beams are shown. The transmit beam, shown in red, was aligned on the target flight line and the dual-channel receiver. The reference beam, shown in green, was aligned with the transmitter. The surveillance beam was aligned with the target's flight line. The purple area shows the overlap of the transmit beam and the surveillance beam and indicates the region in which the target can be detected. Broadband horn antennas, A-Info LB-7180 and A-Info LB-10180, were used for both the transmitter and receiver, these antennas have a very broad beam-widths of 90° .



Figure 6.13: Photo of Dual Channel Receiver Setup

Fig. 6.13 is a photo of the dual channel receiver setup. The photo is taken from behind receiver. The line of cones marks out 10m increments along the flight line. The antenna on the left is the reference channel which is aligned with the transmitter. The antenna on the right is the surveillance channel which is aligned with the target flight line.

During the measurement, the drone took off and hovered in front of the receiver at a height of 5m, then maintaining an altitude of 5m, flew a distance of 100m away from the receiver, stopped and flew back towards the receiver. The drone flew two flights, the results of which are presented below.

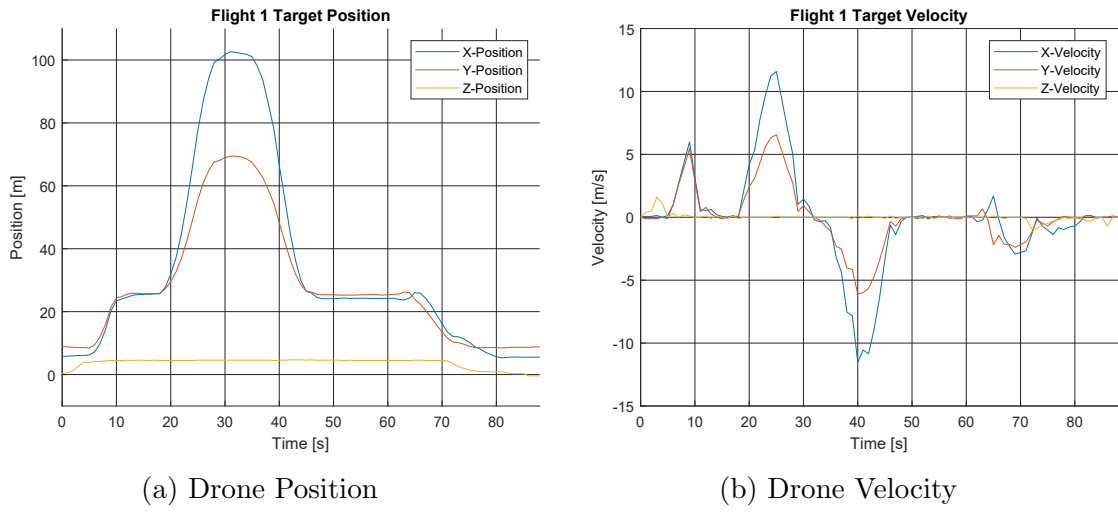
Flight 1

Figure 6.14: Target Position and Velocity Flight 1 (GPS Truth)

Fig. 6.14 shows the drones position and velocity as recorded by the onboard GPS. The flight took 34 s during which the drone reached a top speed of 12 m/s.

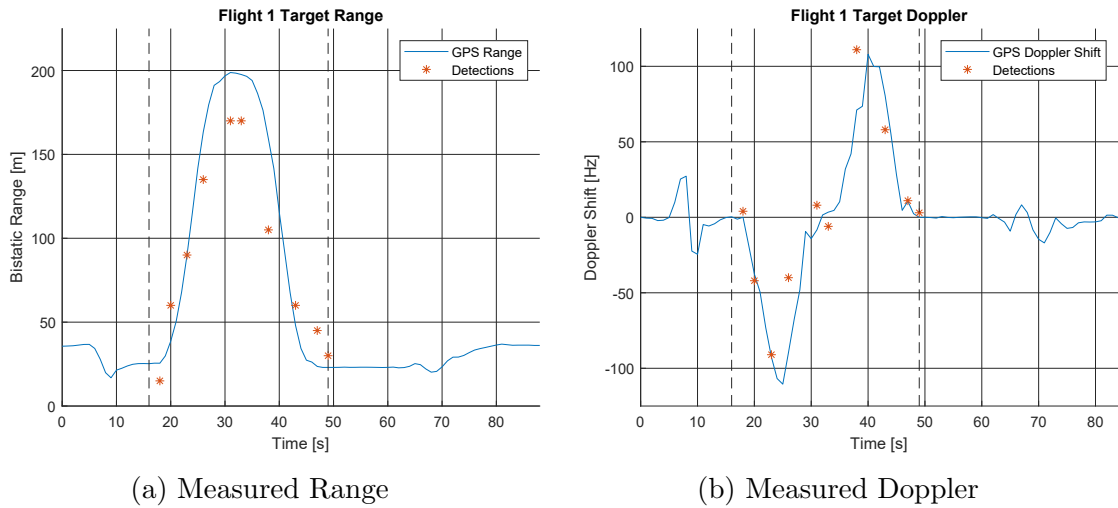


Figure 6.15: Flight 1 Receiver Detections against GPS Track

Fig. 6.15 shows the receiver detections from flight 1 against the targets range and Doppler, calculated from the GPS track. The on-board GPS records the drone latitude, longitude and altitude which is used to determine the drones position in a Cartesian reference system as a function of time. The positions of the transmitter

and receiver are also recorded using GPS and transformed to the same Cartesian reference system. Together, with the known coordinates of the transmitter and dual-channel receiver, the relative range and Doppler shift of the target can be accurately found using equations (5.4b) and (5.4a). A detection SNR threshold of 25 dB was used to avoid detecting the substantial interference around zero. The interference to noise ratio is approximately 22 dB. The range measurements have an average error of 9m with a root mean square error of 26m, while the Doppler measurement has an average error of 26Hz with a root mean square error of 22Hz.

Flight: 2

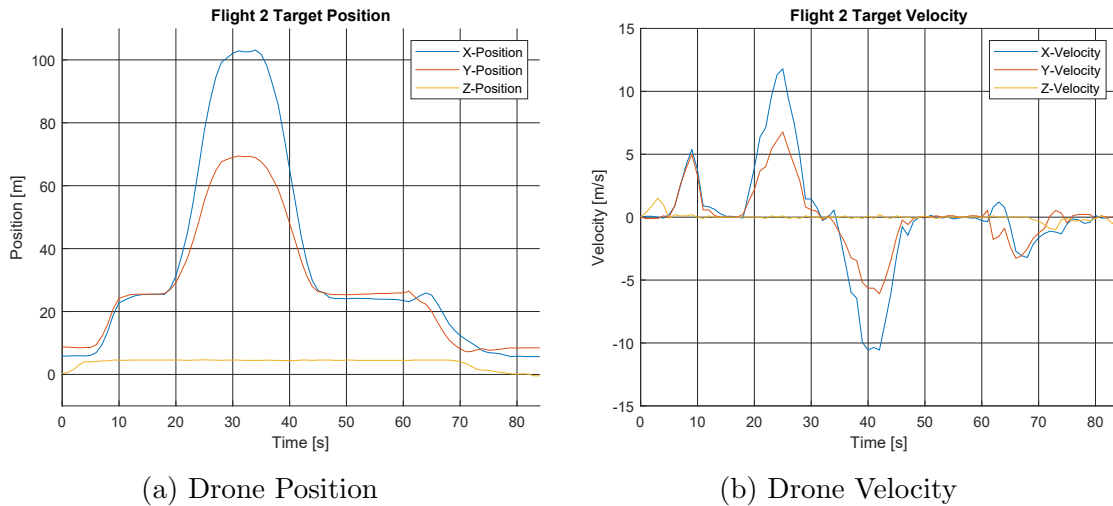


Figure 6.16: Target Position and Velocity Flight 2 (GPS Truth)

Fig. 6.14 shows the drones position and velocity as recorded by the onboard GPS. The flight took 33 s during which the drone reached a top speed of 13.5 m/s.

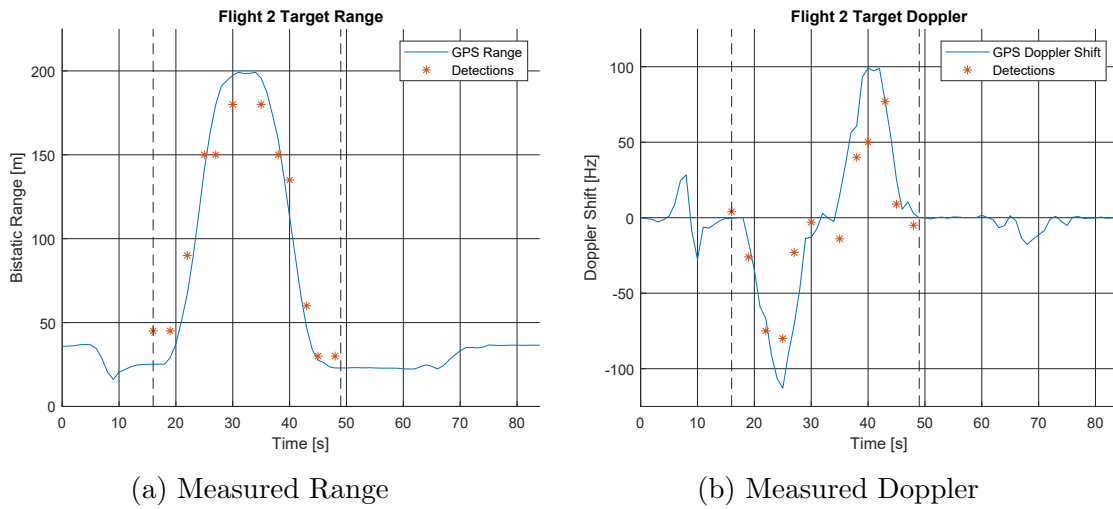


Figure 6.17: Flight 2 Receiver Detections against GPS Track

Fig. 6.17 shows the receiver detections from flight 2 against the targets range and Doppler, calculated from the GPS track. The range measurements have an estimated average error of 13 m with a root mean square error of 19 m, while the Doppler measurement has an average error of 7 Hz with a root mean square error of 26 Hz, when compared to the GPS trajectory. Over both flights the mean error in range was 11 m with a root mean square error of 23 m, the mean error in Doppler was 8 Hz with a root mean square error of 24 Hz.

6.4 Conclusion

The receiver has successfully detected a drone in flight over a range of 100 m. In both flights, the detections show a good correlation with the GPS truth; however, there are several improvements that need to be made before network measurements can be carried out. Currently, the receiver operates in a block data collection cycle, in which a continuous block of data is recorded and then sent to the computer. The digitiser then waits for the computer to save the data before the next block of data is recorded. It takes approximately 3 s for the receiver to record and save 300 ms of data; therefore, the receiver is only recording data 10% of the time. Reducing the

time taken to record blocks would lead to a significant improvement in the number of detections and build a more continuous track, improving the correlation with the GPS truth.

Currently, the signal processing does not take into account interference from the direct signal in the surveillance channel. Throughout the measurement campaign attempts have been made to mitigate this, mainly by placing a physical object, such a large panel of radar absorbent material (RAM), between the reference channel and the transmitter antenna. However, it was not possible to isolate the reference channel effectively during the measurements of drones. The strong interference that the receiver sees around the zero point is likely to be partly due to this interference.

There are also some general improvements that can be made to the receiver. The antennas used are broadband antenna which operate over large frequency range, only 6-7 dB of gain was possible. These antenna's were used for convenience and to reduce cost, any further measurements with the receiver should use antennas with higher gain at L-Band. The stability of the voltage controlled oscillator is always something that can be improved, for these receivers it would be possible to implement a phase locked loop which uses feedback to more precisely control the frequency of the local oscillator.

Chapter 7

Hybrid Simulations

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7.1 Introduction

In the previous chapters, algorithms for optimising the geometry of a cooperative network and a passive receiver for use in a cooperative network have been developed. This chapter aims to combine both the theoretical and some experimental measurements, by using measurements of a drone recorded by the Aveillant Gamekeeper radar in conjunction with simulated passive receivers to create a hybrid simulation of a cooperative radar network.

Unfortunately, attempts to record a drone in flight using the dual-channel receivers using the Aveillant Gamekeeper radar as a cooperative transmitter were unsuccessful. This was partially due to travel restrictions implement due to COVID-19 and the limited options for deploying the receivers at Cranfield Airport which hosts an Aveillant Gamekeeper radar. Cranfield Airport is active airport and therefore the locations for deploying the dual-channel receiver are limited and unsuitable.

In this chapter real data collected by the Aveillant radar is with simulated data to create a hybrid simulation of a cooperative radar network. The recordings of a drone are used to characterise the signal fading, which is used to refine the parameters of the signal model. The signal fading characterisation along with the GPS track the drone flew are used to set up a more realistic scenario using simulated passive receivers, to which the geometry optimisation algorithm is applied.

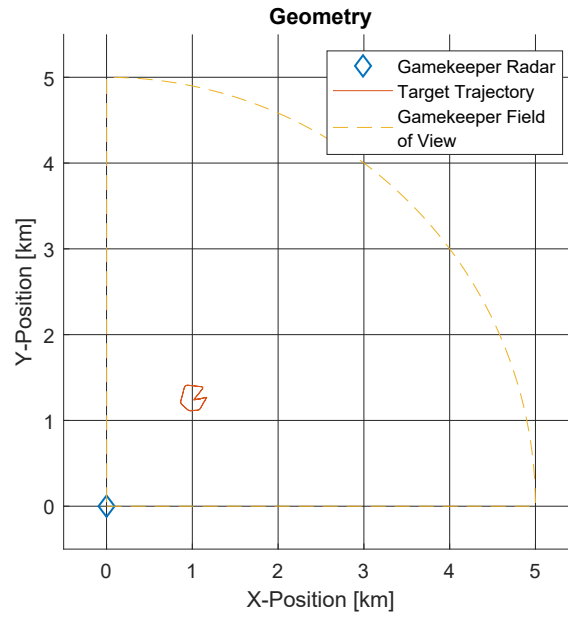


Figure 7.1: Geometry

Fig. 7.1 shows the GPS track of the drone, a DJI Inspire, flight within the field of view of the radar. The measurements were carried out at Cranfield Airport using an Aveillant Gamekeeper radar, these measurements were carried out as part of a wider trial involving the University of Birmingham in which drones and several species of birds were flown. The GPS track of the drone is converted to cartesian co-ordinates for use as a truth file. The target starts on the ground, climbs to an altitude of 100 m and flies a complete circuit, before pausing for 15s and flying a second circuit.

7.2 Target Trajectory

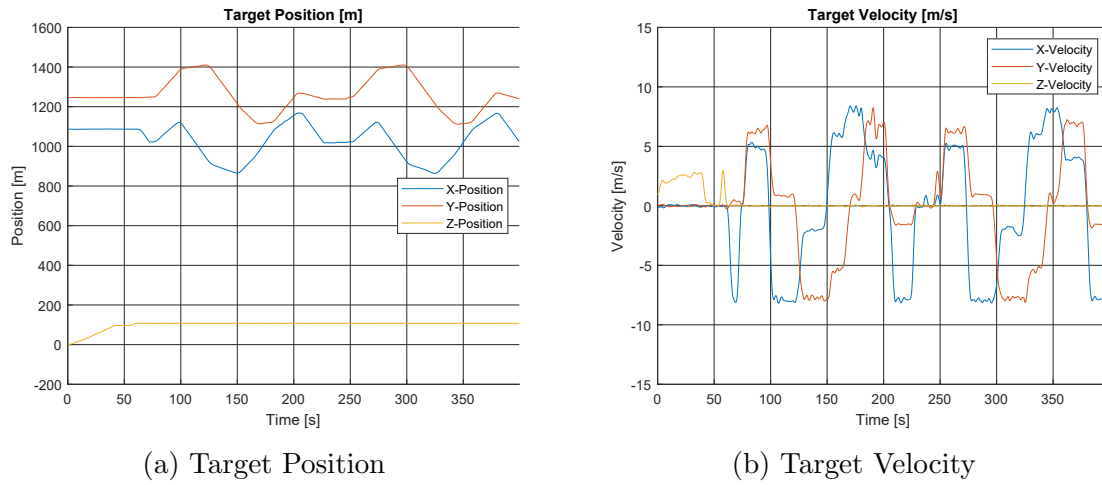


Figure 7.2: Target Position and Velocity

Fig. 7.2a and Fig. 7.2b break down the target's trajectory into its position and velocity. From Fig. 7.2a the initial climb to the flight altitude takes 60 s, the drone takes approximately 150 s to complete one circuit.

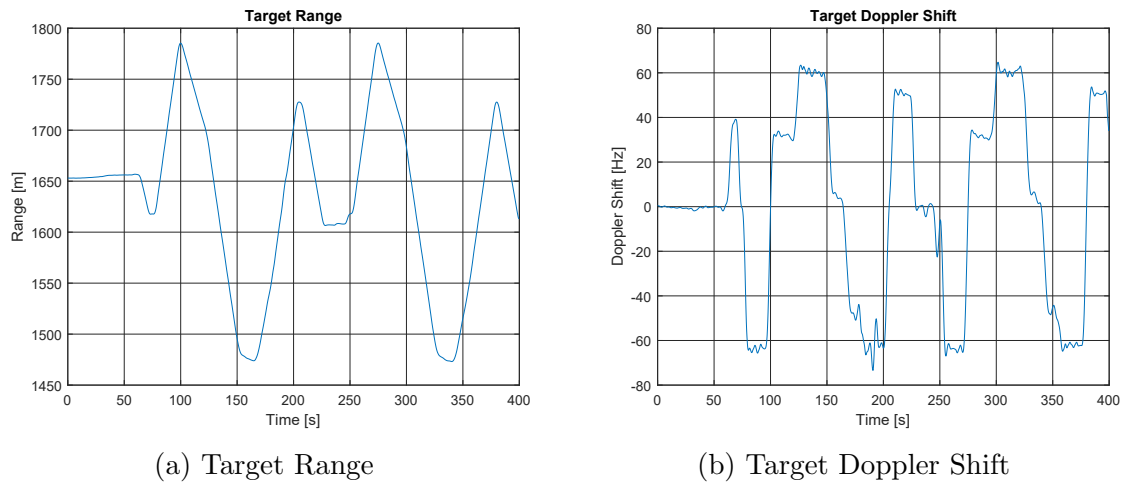


Figure 7.3: Target Position and Velocity

Fig. 7.3a and Fig. 7.3b shows the targets relative range and Doppler shift over the full flight, as seen by the Gamekeeper radar.

Assume that the targets position and velocity can be modelled by two independent

multivariate Gaussian random vectors

$$\mathbf{p} \sim \mathcal{N}(\boldsymbol{\mu}_p, \Sigma_p) \quad (7.1a)$$

$$\mathbf{v} \sim \mathcal{N}(\boldsymbol{\mu}_v, \Sigma_v) \quad (7.1b)$$

The targets state is therefore assumed to be distributed according to a multivariate Gaussian distribution $\boldsymbol{\theta} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_p \\ \boldsymbol{\mu}_v \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_p & 0 \\ 0 & \Sigma_v \end{pmatrix} \quad (7.2)$$

Where the vector $\boldsymbol{\mu}$ is a 6×1 column vector and Σ is a 6×6 block diagonal matrix. The distribution of the target is estimated from the GPS truth file as

$$\boldsymbol{\mu}_p = \begin{pmatrix} 1032 \\ 1260 \\ 100.6 \end{pmatrix} \quad \Sigma_p = \begin{pmatrix} 7229 & -84.69 & -390.3 \\ -84.69 & 6960 & 104.1 \\ -390.3 & 104.1 & 470 \end{pmatrix} \quad (7.3a)$$

$$\boldsymbol{\mu}_v = \begin{pmatrix} -0.1548 \\ -0.01413 \\ 0.2723 \end{pmatrix} \quad \Sigma_v = \begin{pmatrix} 26.52 & 6.134 & 0.0187 \\ 6.1340 & 19.61 & -0.0044 \\ 0.0187 & -0.0044 & 0.5559 \end{pmatrix} \quad (7.3b)$$

7.3 Geometry

The cooperative network consists of a central staring monostatic radar, colocated transmitter and receiver, and two passive bistatic receivers. Here, the central monostatic radar is the Aveillant Gamekeeper radar and the two bistatic receivers are modelled on a single Aveillant receiver tile (4 channels), as defined in chapter 4. Consider six possible receiver positions for the two bistatic receivers arranged equidistant around the centre point of the target's trajectory.

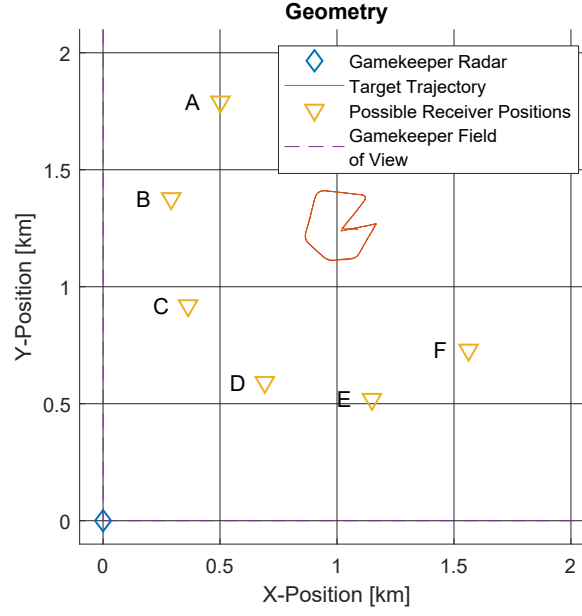


Figure 7.4: Possible Receiver Geometry

Fig. 7.4 shows the possible receiver positions with respect to the targets trajectory. The six possible positions are labelled A to F and are placed on a circle of radius 750 m centred on the targets trajectory. The receivers position are chosen to avoid the target moving into the forward scattering region. The forward scattering region is not part of the signal model but consideration is given to it here as it has a significant effect on radar systems in practise, because the returns from the target would be hidden in the stationary clutter. The transmitter $\mathbf{p}_0$ and first receiver $\mathbf{p}_1$ are co-located at the origin. The possible positions of the two bistatic receivers are given in Table. 7.1.

Label	Symbol	x [m]	y [m]	z [m]
A	$\mathbf{p}_2$	502	1790	0
B	$\mathbf{p}_3$	291	1377	0
C	$\mathbf{p}_4$	364	920	0
D	$\mathbf{p}_5$	692	592	0
E	$\mathbf{p}_6$	1149	519	0
F	$\mathbf{p}_7$	1562	730	0

Table 7.1: Possible Receiver Positions

7.4 Signal Model and Power Budget

The m -th receiver in the network receives the following signal

$$z_m(t) = \beta_m \sqrt{E_m} u(t - \tau_m(\boldsymbol{\theta})) e^{2\pi i f_m(\boldsymbol{\theta})t} + w_m(t) \quad (7.4)$$

The complex envelope $u(t)$ is normalised and has unit energy. The term β_m is a zero mean complex Gaussian random variable where $\mathbb{E}[|\beta_m|^2] = 2\sigma_{\beta_m}^2$ that accounts for the target reflection properties and channel propagation fluctuations, such as fading multipath. The average received signal energy is $\tilde{E}_m = 2\sigma_{\beta_m}^2 E_m$. The deterministic contribution to the received signal energy, E_m , depends on the shape of the Gaussian pulse, the number of pulses in the pulse train and the peak received power.

$$E_m = \sqrt{\pi} \nu \kappa P_{R_m} \quad (7.5)$$

Where ν is scaling factor that determines the shape of the Gaussian pulse, κ is the number of pulses in the pulse train and P_R is the received power. The received power P_{R_m} is derived from the radar range equation and the receiver's power budget.

The signal to noise ratio, against which the theoretical results are derived, at the m -th receiver is defined as

$$\text{SNR}_m = \frac{2\tilde{E}_m}{N_0} \left(\frac{\tilde{E}_m}{\tilde{E}_m + N_0} \right), \quad \tilde{E}_m = 2\sigma_{\beta_m}^2 E_m \quad (7.6)$$

In order to accurately simulate the received signals alongside the recorded data from the Gamekeeper radar the power budgets needs to be aligned to the real data. Proceed by using properties of the noise floor.

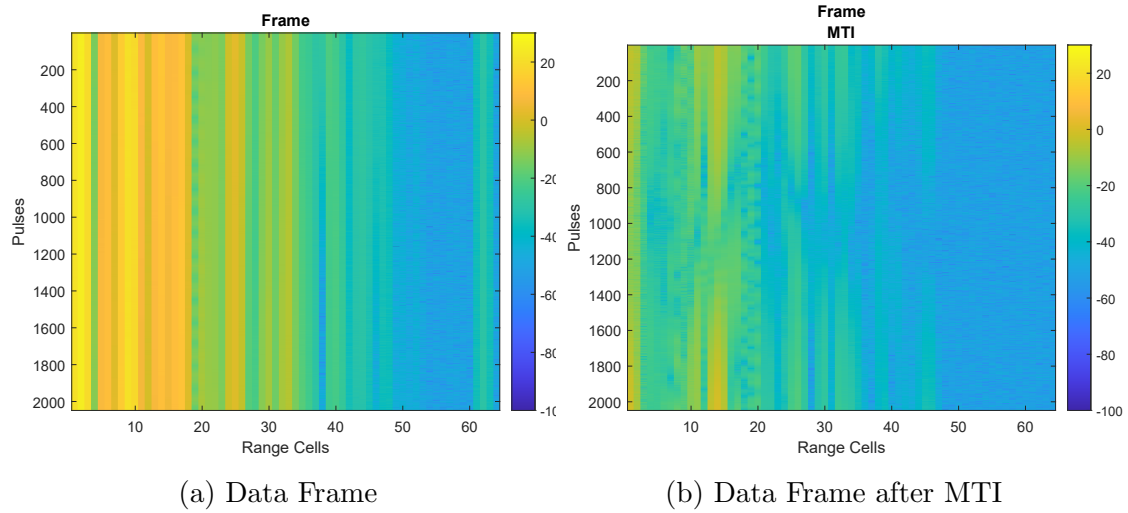


Figure 7.5: Frame data before and after MTI

Fig. 7.5a and Fig. 7.5b show the fast-time slow-time data, or "frame" for one coherent processing interval, before and after MTI. The MTI was performed by subtracting the mean value down each range cells. Fig. 7.5b appears to show the noise floor from range cell 50 to 64. The power of the noise floor is estimated between range cell 55 to range cell 64 across all 2048 pulses. The power of the noise is estimated at -49 dB. The characteristics of the noise floor can be checked by coherently integrating the frame.

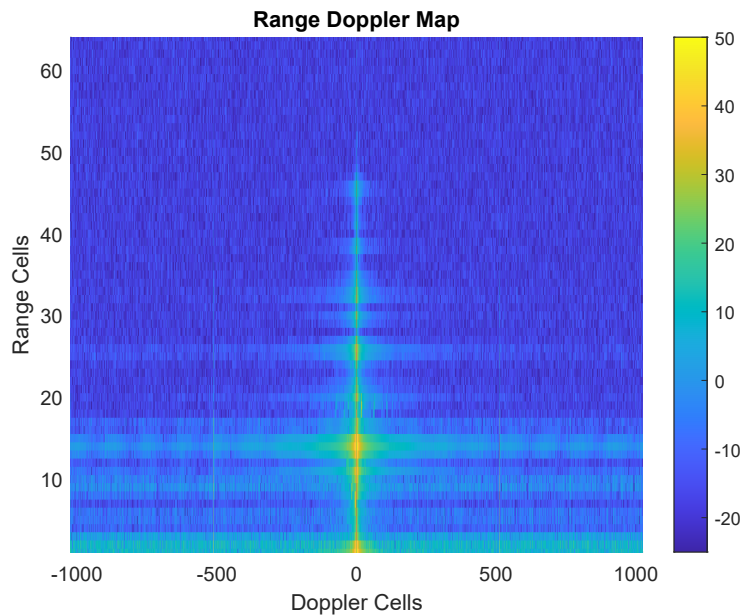


Figure 7.6: Range Doppler Map [dB]

Fig. 7.6 shows the Range-Doppler map produced by coherently integrating the frame data. The power of the noise floor is estimated between range cell 55 to range cell 64 across all 2048 Doppler cells. The power of the noise is estimated at -16 dB. The theoretical difference between the noise power before and after integrating 2048 pulses is 33 dB, the estimated difference is 33 dB. Therefore conclude that the behaviour of the noise floor in the real data is consistent with the theoretical behaviour of white noise, and therefore can be used to reliably align the power budget with the recorded data.

The frame data in Fig. 7.5a is after the signals have been through the matched filter and then beam-formed. The remaining step is the integration of the pulses. The signal given by (7.4) models the received signal after beamforming but before the signal has been filtered. The increase in noise power due to beamforming depends on the number of channels in the receiving array in this case 64, therefore the noise power increases by 64 or 18.0618 dB. Assuming that the impulse response of the matched filter is unitary then the noise power after the matched filter is equal to the noise power spectral density before the matched filter. Using these properties the power budget can be fitted to the recorded data.

Component	Property	Symbol	Value	Unit
Transmitter	Transmit power	P_T	2	kW
	Antenna gain	G_T	17.78	-
	Carrier frequency	f_c	L-Band	Hz
Environment	Atmospheric attenuation	L	1	-
Target	Radar cross-section	σ	0.01	m ²
	Range	R	1	km
Receiver	Antenna gain	G_R	3.55	-
	Noise PSD	N_0	-199.5	dB W/Hz
	Bandwidth	B	10	MHz
	Signal power	P_s	-134.4	dBW
	Noise power	P_n	-129.5	dBW
RF Amplifier	Gain	G_{amp}	60	dB
	Signal power	P_s	-74.4	dBW
	Noise power	P_n	-69.251	dBW
Analogue to Digital Converter	Gain	G_{adc}	72.1511	dB
	Signal power	P_s	-2.2489	dBW
	Noise power	P_n	2.6511	dBW
Beamforming	Elements	n	64	-
	Signal power	P_s	33.8747	dBW
	Noise power	P_n	20.7129	dBW
Matched Filter	Filter energy	-	1	J
	Signal power	P_s	-26.1253	dBW
	Noise power	P_n	-49.2871	dBW
Integration	Number of Pulses	κ	2048	-
	Signal power	P_s	40.1013	dBW
	Noise power	P_n	-16.1738	dBW
	SNR	-	56.2751	dB

Table 7.2: Fitted Power Budget

Table. 7.2 details the fitted power budget for a drone, target with an RCS of 0.01 m², at a range 1 km from the Gamekeeper radar. Note that the noise power before integration in the power budget is aligned with the noise power measured in the real data before integration. Also note that the final SNR reported in Table. 7.2 is the ratio of the signal power and noise power after integration, not the SNR defined by (7.6).

The energy E_m before the matched filter depends on the power and the scaling factor of the Gaussian pulse. The power before the matched filter is 33.8747 dB, the scaling factor of the Gaussian pulse was defined in chapter 4 as $\nu = 0.5 \times 10^{-6}$ then the energy at the Gamekeeper radar, $m = 1$, is

$$E_1 = \sqrt{\pi\nu\kappa}P_{R_1} \quad (7.7a)$$

$$= 4.4294 \text{ J} \quad (7.7b)$$

The noise power spectral density before the matched filter is -49.2871 dB. Just for the this short example assume that $\sigma_\beta^2 = 0.5$ then $\tilde{E}_1 = E_1$ then the SNR defined by (7.6) is

$$\text{SNR} = \frac{2E_1}{N_0} \left(\frac{E_1}{E_1 + N_0} \right) \quad (7.8a)$$

$$= 58.76 \text{ dB} \quad (7.8b)$$

7.5 Characterising Channel Fading

It is possible to characterise the level of fading in the measurements recorded by the Gamekeeper radar, by sampling the energy of the returns from the target at different ranges and comparing them with the fitted power budget. To ensure the characterisation is as clean as possible, the target must be clear of any clutter/interference, therefore only a section of the target's flight is used. Here a 16 s period containing 64 detections, starting after 78 s into the flight is used.

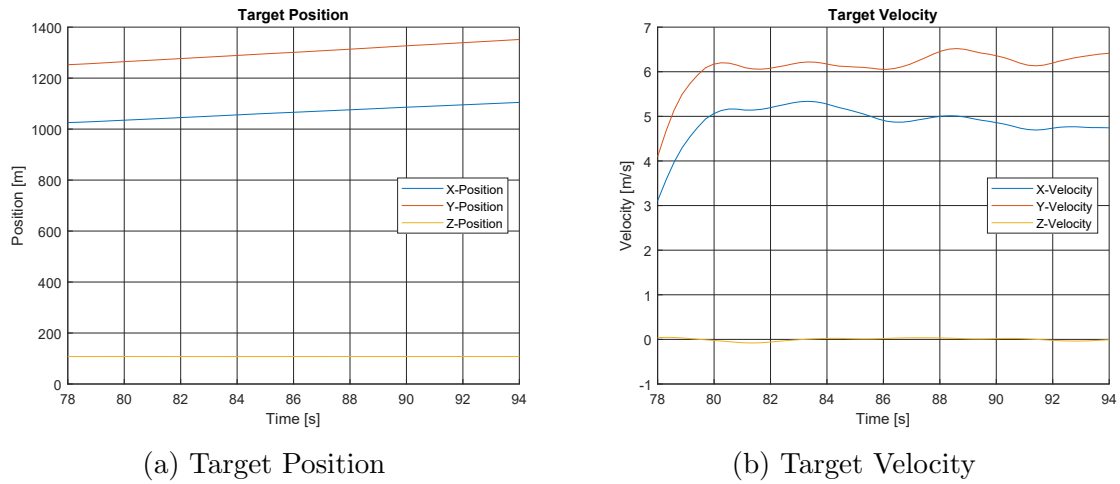


Figure 7.7: Target Position and Velocity

Fig. 7.7a and 7.7b shows the targets position and velocity during the 16 s sampling period. Fig. 7.7a shows the target moving away from the radar in both the x and y directions, while the targets z component (altitude) remains constant at 100 m. 7.7b shows the targets x and y component of velocity increasing initially but broadly remaining constant, while the targets z velocity component remains constant at 0 ms^{-1} .

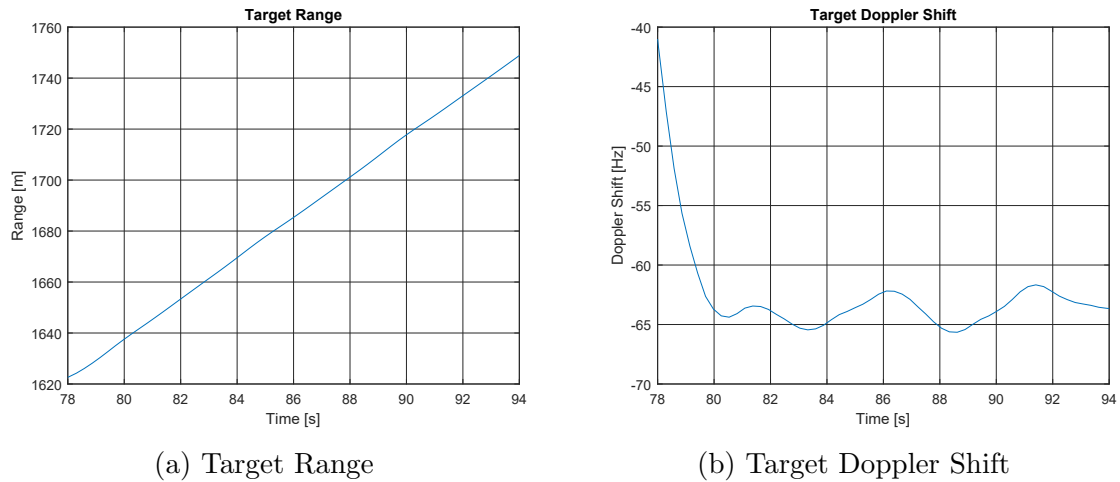


Figure 7.8: Target Position and Velocity

Fig. 7.8a and 7.8b show the target's range and Doppler shift during the 16 s sampling period. Fig. 7.7a shows the target's range increasing from 1622 m to 1748 m. Fig. 7.7b shows the target's Doppler shift, averaging -63 Hz .

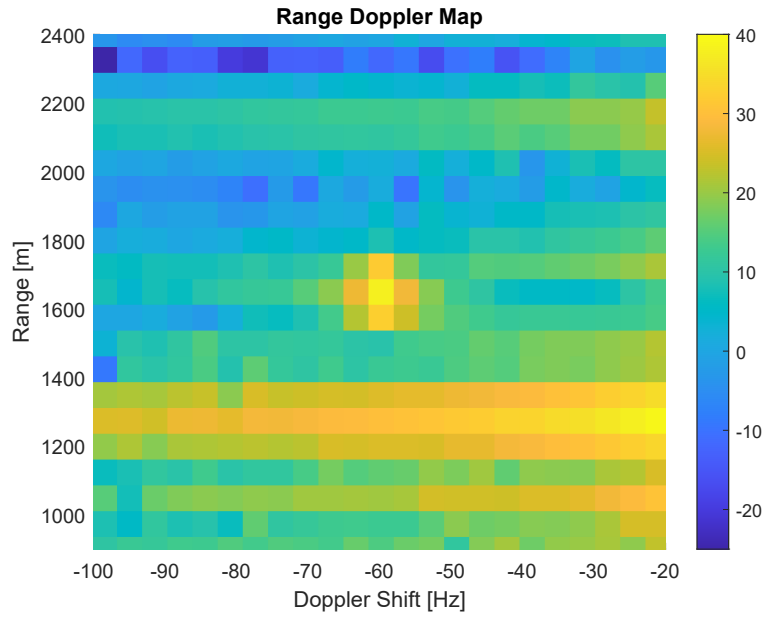


Figure 7.9: Range Doppler Map [dB]

Fig. 7.9 shows the target during the sampling period, due to the targets speed, the returns are broadly clear of interference. For each of the frames within the sampling period the peak power of the target is sampled and the received signal energy is calculated. The received signal energy is compared with the fitted power budget and the fading in the channel is estimated.

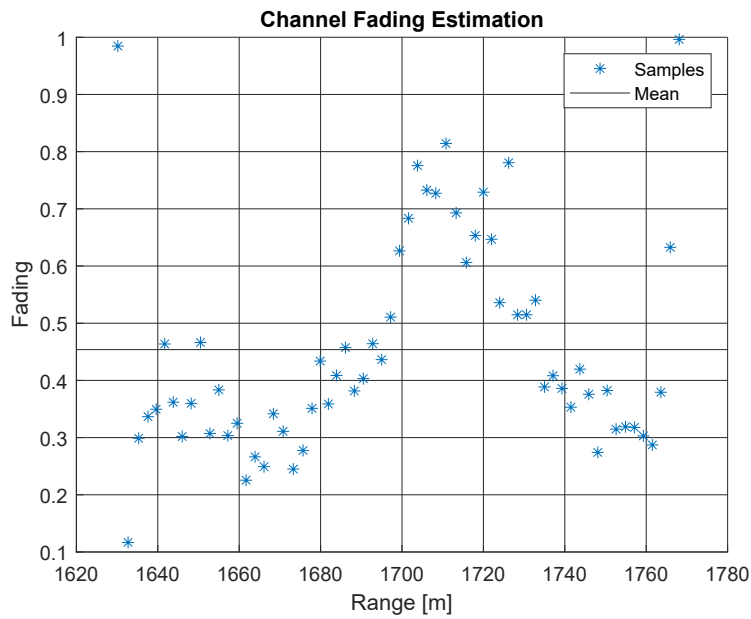


Figure 7.10: Channel Fading Estimation

Fig. 7.10 shows the estimated values of $\sigma_{\beta_1}^2$ for the received data. The samples were obtained by estimating $\tilde{E}_1$ from the target's peak power in the range-Doppler map. The deterministic part of the received energy E_1 was calculated using the target's range from the GPS truth file and the fitted power budget. The sample values of $\sigma_{\beta_1}^2$ are equal to the ratio $\frac{\tilde{E}}{2E}$, these values are the samples shown in Fig. 7.10. The mean value is estimated at $\sigma_b^2 = 0.4539$. It should be noted that some of the signal fading is likely due to a change in the aspect angle of the drone with respect to the radar which as the drone's RCS is not uniform will result in a varying amount of backscattered energy towards the radar. As a part of a future extension it would be possible to account for signal fading due changes in the drones aspect if the both the drone's RCS map, and the heading of the drone with respect to the radar was known.

7.6 Geometry Optimisation

Having characterised the fading at the Gamekeeper radar, proceed to set-up the remaining part of the network. The receiver positions shown in Fig. 7.4 are placed equidistantly on a circle of radius 750 m centred on μ_p . The values of $\sigma_{\beta_n}^2$ can be chosen for each of the possible receiver positions. Using the power budget and modelling the receivers using 4 channels the SNR at each receiver is calculated.

Label	Symbol	x [m]	y [m]	z [m]	$\sigma_{\beta_n}^2$	SNR [dB]
A	$\mathbf{p}_2$	502	1790	0	1.5	35.1344
B	$\mathbf{p}_3$	291	1377	0	0.5	30.3579
C	$\mathbf{p}_4$	364	920	0	0.05	20.2868
D	$\mathbf{p}_5$	692	592	0	0.025	17.1988
E	$\mathbf{p}_6$	1149	519	0	1	30.3579
F	$\mathbf{p}_7$	1562	730	0	2	36.3845

Table 7.3: Possible Receiver Positions, Multipath Fading and SNR

Using the values given in Table. 7.3, the fitted power budget and the expression for

the BCRLB detailed in Chapter 5, the optimal receiver positions can be determined. For each of the 16 possible pairs the trace of the BCRLB is calculated. The receiver pairing that minimises the trace is selected as the optimal.

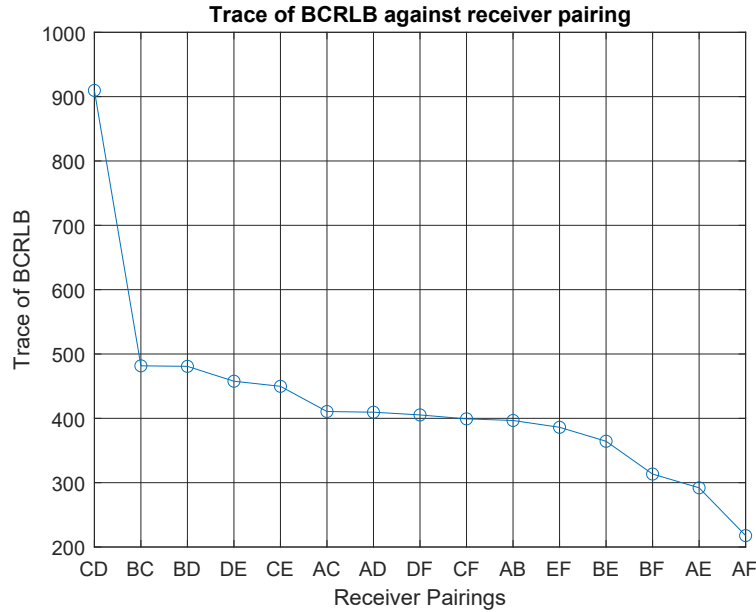


Figure 7.11: Receiver Pairing

Fig. 7.11 shows the trace of the BCRLB against each of the 15 possible receiver pairings. The receiver pairings are sorted based on the trace of the BCRLB. The worst pairing is receivers C and D, the optimal pairings are receivers A and F.

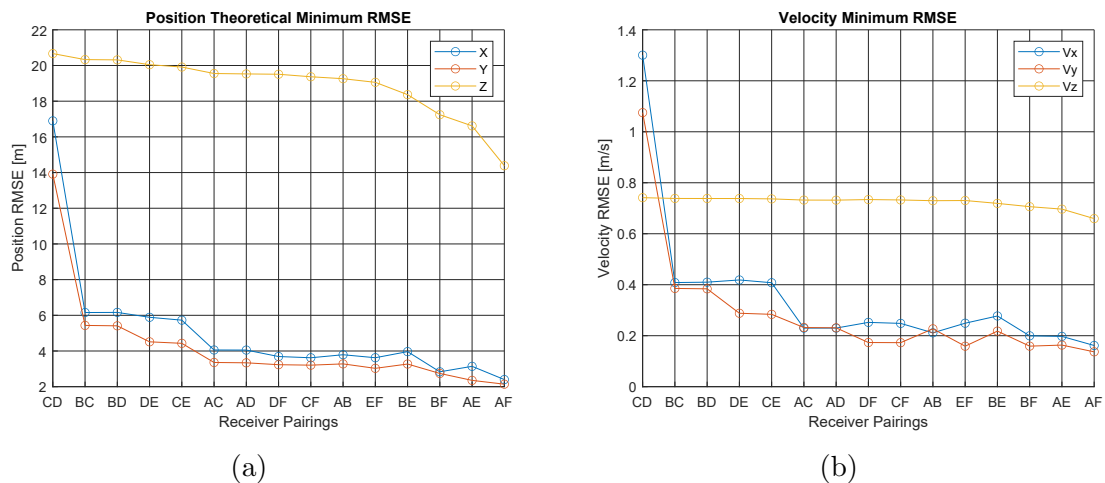


Figure 7.12: Position and Velocity Theoretical Minimum RMSE

Fig. 7.12a and Fig. 7.12b show the theoretical minimum root-mean-square-error of

the individual targets state with respect to the receiver pairings.

Observe from Fig. 7.4 the optimal receiver pairing A and F maximise the relative bistatic angle between the target's distribution and the transmitter (Gamekeeper radar), while the worst receiver pairing C and D are the two receivers that minimise the relative bistatic angle.

7.7 Conclusion

In this chapter, a step towards a more realistic network is taken. In the first part, the GPS data from a drone flight recorded by the Aveillant Gamekeeper is presented, from which the distribution of the target's position and velocity is estimated. Next, the network geometry is set up with six possible positions for the two passive receivers. The signal model used in Chapter 5 is restated, and the properties of the recorded data are characterised. These properties are used to align the theoretical power budget, used to determine energy and noise power spectral density required by the signal model, with the recorded data. The fading in the Gamekeeper channel is characterised using the aligned power budget.

The optimal receiver geometry is calculated using the fitted power budget and the expression for the BCRLB detailed in Chapter 5 and the characterisation of the fading at the Gamekeeper radar.

Further work comparing the estimation performance of the optimal geometry with the other possible geometries will be carried out. This will be achieved using Monte-Carlo simulations of the MAP estimator in a similar manner to the previous chapters.

Chapter 8

Summary

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8.1 Conclusion

The research presented in this thesis contains theoretical, simulation and experimental results of a cooperative multistatic radar network.

Chapter 5 of this thesis presents the optimisation algorithms developed to determine the optimal configuration of a cooperative radar network. The first algorithm models the target's position and velocity as an unknown deterministic parameter. An estimator of the target's state is developed under a distributed data fusion scheme, in which the relative range and Doppler shift of the target are estimated locally at each receiver. The estimated range and Doppler shift from each receiver are centrally gathered and the target's state is estimated. The theoretical Cramér-Rao Lower Bound of an unbiased estimator of the target's position and velocity is presented. The calculated bound depends on the network geometry, properties of the transmitted signal and the relative position and velocity of the target. The geometry optimisation algorithm uses the trace of the lower bound as a cost function. The algorithm varies the positions of the receivers to find the geometry that minimises the lower bound and, therefore, the estimation errors. Monte-Carlo simulations of the Maximum Likelihood Estimator have been developed to verify that the optimal geometry attains the theoretical lower bound. The MLE requires a search over a 4D space, as it is not computationally feasible to search every point; the Coordinate Descent algorithm was used to iteratively search for the maximum likelihood estimate for each Monte-Carlo trial. The results show that the estimator attains the theoretical bound for high SNR (+20 dB).

The second algorithm extends the first by modelling the target's position and velocity as a multivariate random vector. The model also takes into account the effect of a fluctuating received power due to signal multipath effects. The shift to modelling the target's position and velocity as random vectors requires the use of a centralised

data fusion scheme in which the target's position and velocity are directly estimated using all the received signals in one step. The theoretical Bayesian-CRLB for the position and velocity estimator is calculated. The geometry optimisation algorithm uses the trace of the lower bound as a cost function, the algorithm varies the positions of the receivers in the network to find the geometry that minimises the bound and, therefore, the estimation errors. Monte-Carlo simulations of the Maximum a Posteriori Estimator are used to verify that the performance of the found optimal geometry attains the theoretical lower bound. The MAP estimate requires a search over 6D space, as it is not computationally feasible to search every point, the Coordinate Descent algorithm was used to iteratively search for the maximum a posteriori estimate for each Monte-Carlo trial. The results show that the estimator attains the theoretical bound.

Chapter 6 details the developments of a dual-channel passive receiver. The receivers are designed to cooperate with a staring transmitter, in this case, the Aveillant Gamekeeper radar. The receivers use a dual-channel design with a separate reference and a surveillance channel which enables the receivers to operate coherently without the requirement of a shared synchronisation signal, which is a limitation of other traditional multistatic radar network solutions.

Measurements of a remote-control helicopter's rotor blades and a rotating fan have been taken in the laboratory, as well as more complex measurements involving the receiver operating in a bistatic geometry measuring a drone flight. The results show that the receiver is able to detect drones at short ranges, up to 100 m.

The current receiver design is not able to continually stream the recorded data from the digitiser to the computer and save at a rate of 20MHz, therefore the receiver records a continuous block data approximately 300ms and waits while the data is

transferred over USB to the host computer and saved. It takes approximately 3 s for the receiver to record and save 300 ms of data; therefore, the receiver is only recording data 10% of the time.

8.2 Summary of Contributions

This section details the main contributions presented in this thesis in a brief form.

The main contributions are:

Chapter 5 - Network Algorithms: Two methods of optimising the geometry of a multistatic radar network are presented. The first is for a target with a deterministic state and uses a distributed data fusion scheme in which the relative range and Doppler shift of the target are estimated locally at each receiver. The estimated range and Doppler shift from each receiver are centrally gathered, and the target's state is estimated. The theoretical CRLB for the estimator has been derived, the trace of which is used to define a cost function for the optimisation of the network's geometry. The second algorithm models the target state as a multivariate random vector and uses a centralised data fusion scheme in which the target's state is directly estimated using all the received signals in one step. The theoretical Bayesian-CRLB for the estimator has been derived, the trace of which is used to define a cost function for the optimisation of the network's geometry.

Chapter 6 - Dual Channel Passive Receiver Chapter 6 details the developments of a dual-channel passive receiver. The passive receiver was designed and built to operate using the transmissions of the Aveillant radar. Measurements of targets in the laboratory as well as more complex measurements involving the receiver operating in a bistatic geometry measuring a drone flight. The results show the receiver's capability to detect drones at short ranges, up to 100 m.

8.3 Future Work

8.3.1 Network Measurements

This thesis presents the development of a dual-channel passive receiver and initial measurements. Two of these receivers have been built with the purpose of deploying a small scale multistatic network. Future work should prioritise the deployment of these receivers in a cooperative radar network with the Aveillant Gamekeeper radar. Furthermore, by deploying the receivers in different geometries, the geometry optimisation algorithms can be fully tested.

8.3.2 Signal Model

The signal model currently only considers the effect of thermal noise on the received signal; while important, interference from clutter can often be much greater than the thermal noise in radar systems. Developing a more realistic clutter model would improve the relevance of the model. A crucial part of the Aveillant Gamekeeper radar is the digital receiving array. Currently, the model considers only one channel per receiver; adding beamforming would improve network performance as the azimuth and elevation of the target contain information about the target's position. The addition of beamforming to the model would allow for a more thorough analysis of the cooperative network. Additionally, the theoretical lower bounds are derived assuming that the signal used to perform matched filtering of the target's returned signal are free of noise and interference. In practise this is not the case, as the reference channel of the receiver will be degraded by thermal noise, and warrants further investigation.

The optimisation algorithms are used to determine the optimal network geometry, the model assumes that the transmitters and receivers are static so the algorithms are used only once to place the network, however the dual-channel receivers are

low-cost and a simple design potentially they can be deployed on moving platforms, such as other drones. The geometry optimisation algorithms can then be used to optimise the network in real-time by adjusting the trajectory of the moving platform.

The model assumes only one target is present this is a simplification as it is not necessary to consider the problem of data association. Realistically, multiple targets will be present and therefore determining which target to assign the received signals to is crucial and an future area of research.

8.3.3 Dual Channel Passive Receiver

Currently, the receiver operates in a block data collection cycle, in which a continuous block of data is recorded and then sent to the computer. It takes approximately 3 s for the receiver to record and save 300 ms of data; therefore, the receiver only records data 10% of the time. Reducing the time taken to record blocks would lead to a significant improvement in the number of detections and build a more continuous track, improving the correlation with the GPS truth. There are also some general improvements that can be made to the receiver. The antennas used are broadband antenna which operate over large frequency range, only 6-7 dB of gain was possible at L-Band. These antenna's were used for convenience and to reduce cost, any further measurements with the receiver should use antennas with higher gain at L-Band. The surveillance channel can be extended to an array of antennas and beamforming implemented.

The passive receiver assumes that the signal in the reference channel is of sufficient quality to allow complete extraction of the transmitted signal without any losses. In reality, the reference channel contains noise, and the transmitted signal will suffer from propagation through the environment. In general, it is assumed that for sufficiently high SNR, it will be possible to extract the transmitted signal, including

noise, and other effects in the model of the transmitted signal will allow further insight into what precisely defines a sufficiently high SNR.

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Appendix A

Range and Doppler Derivatives

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A.1 2D Range and Doppler Derivatives

The range and Doppler measurements from the radar are

$$r_n(\boldsymbol{\theta}) = |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \quad (\text{A.1a})$$

$$f_n(\boldsymbol{\theta}) = \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.1b})$$

The derivatives are the rates of change of the measurements with respect to the target state vector,

$$\boldsymbol{\theta} = \begin{pmatrix} p_x & p_y & v_x & v_y \end{pmatrix}^\top. \quad (\text{A.2})$$

The range and Doppler measurements from the radar can be rewritten as

$$r_n(\boldsymbol{\theta}) = |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \quad (\text{A.3a})$$

$$f_n(\boldsymbol{\theta}) = \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.3b})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to p_x

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_x} = \frac{\partial}{\partial p_x} \{ |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \} \quad (\text{A.4a})$$

$$\begin{aligned} &= \frac{\partial}{\partial p_x} \left\{ \sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2} \right. \\ &\quad \left. + \sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2} \right\} \quad (\text{A.4b}) \end{aligned}$$

$$= \frac{-2(p_{xo} - p_x)}{2\sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2}} + \frac{-2(p_{xn} - p_x)}{2\sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2}} \quad (\text{A.4c})$$

$$= \frac{p_x - p_{xo}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (\text{A.4d})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to p_y

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_y} = \frac{\partial}{\partial p_y} \{ |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \} \quad (\text{A.5a})$$

$$= \frac{\partial}{\partial p_y} \left\{ \sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2} + \sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2} \right\} \quad (\text{A.5b})$$

$$= \frac{-2(p_{yo} - p_y)}{2\sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2}} + \frac{-2(p_{yn} - p_y)}{2\sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2}} \quad (\text{A.5c})$$

$$= \frac{p_y - p_{yo}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (\text{A.5d})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to v_x

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_x} = 0 \quad (\text{A.6})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to v_y

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_y} = 0 \quad (\text{A.7})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to p_x

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = \frac{\partial}{\partial p_x} \left\{ \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \right\} \quad (\text{A.8a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_x} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.8b})$$

$$\begin{aligned}
&= \frac{1}{\lambda} \frac{\partial}{\partial p_x} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)}{|\mathbf{p}_o - \mathbf{p}|} \right. \\
&\quad \left. + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.8c})
\end{aligned}$$

The quotient differentiation formula is $\frac{\partial}{\partial p_x} \left\{ \frac{h(p_x)}{g(p_x)} \right\} = \frac{g(p_x) \frac{\partial h(p_x)}{\partial p_x} - h(p_x) \frac{\partial g(p_x)}{\partial p_x}}{g(p_x)^2}$, define $h(p_x) = v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)$ and $g(p_x) = |\mathbf{p}_o - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_x)}{\partial p_x} = -v_x$ and $\frac{\partial g(p_x)}{\partial p_x} = \frac{p_x - p_{xo}}{|\mathbf{p}_o - \mathbf{p}|}$.

$$= \frac{-v_x |\mathbf{p}_o - \mathbf{p}| - [v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)] \frac{p_x - p_{xo}}{|\mathbf{p}_o - \mathbf{p}|}}{|\mathbf{p}_o - \mathbf{p}|^2} \quad (\text{A.9a})$$

$$= \frac{-v_x |\mathbf{p}_o - \mathbf{p}|^2 - [v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)](p_x - p_{xo})}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.9b})$$

$$= \frac{v_y(p_{yo} - p_y)(p_{xo} - p_x) - v_x(p_{yo} - p_y)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.9c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_x} \left\{ \frac{h(p_x)}{g(p_x)} \right\} = \frac{g(p_x) \frac{\partial h(p_x)}{\partial p_x} - h(p_x) \frac{\partial g(p_x)}{\partial p_x}}{g(p_x)^2}$, define $h(p_x) = v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)$ and $g(p_x) = |\mathbf{p}_n - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_x)}{\partial p_x} = -v_x$ and $\frac{\partial g(p_x)}{\partial p_x} = \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|}$.

$$= \frac{-v_x |\mathbf{p}_n - \mathbf{p}| - [v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)] \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|}}{|\mathbf{p}_n - \mathbf{p}|^2} \quad (\text{A.10a})$$

$$= \frac{-v_x |\mathbf{p}_n - \mathbf{p}|^2 - [v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)](p_x - p_{xn})}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.10b})$$

$$= \frac{v_y(p_{yn} - p_y)(p_{xn} - p_x) - v_x(p_{yn} - p_y)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.10c})$$

Combining the above

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = \frac{1}{\lambda} \left(\frac{v_y(p_{yo} - p_y)(p_{xo} - p_x) - v_x(p_{yo} - p_y)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \right)$$

$$+ \frac{v_y(p_{yn} - p_y)(p_{xn} - p_x) - v_x(p_{yn} - p_y)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \Big) \quad (\text{A.11})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to p_y

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} = \frac{\partial}{\partial p_y} \left\{ \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \right\} \quad (\text{A.12a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_y} \left\{ \frac{\mathbf{v} \cdot (\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{v} \cdot (\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.12b})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_y} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)}{|\mathbf{p}_o - \mathbf{p}|} + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.12c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_y} \left\{ \frac{h(p_y)}{g(p_y)} \right\} = \frac{g(p_y) \frac{\partial h(p_y)}{\partial p_y} - h(p_y) \frac{\partial g(p_y)}{\partial p_y}}{g(p_y)^2}$, define $h(p_y) = v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)$ and $g(p_y) = |\mathbf{p}_o - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_y)}{\partial p_y} = -v_y$ and $\frac{\partial g(p_y)}{\partial p_y} = \frac{p_y - p_{yo}}{|\mathbf{p}_o - \mathbf{p}|}$.

$$= \frac{-v_y |\mathbf{p}_o - \mathbf{p}| - [v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)] \frac{p_y - p_{yo}}{|\mathbf{p}_o - \mathbf{p}|}}{|\mathbf{p}_o - \mathbf{p}|^2} \quad (\text{A.13a})$$

$$= \frac{-v_y |\mathbf{p}_o - \mathbf{p}|^2 - [v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)](p_y - p_{yo})}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.13b})$$

$$= \frac{v_x(p_{xo} - p_x)(p_{yo} - p_y) - v_y(p_{xo} - p_x)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.13c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_y} \left\{ \frac{h(p_y)}{g(p_y)} \right\} = \frac{g(p_y) \frac{\partial h(p_y)}{\partial p_y} - h(p_y) \frac{\partial g(p_y)}{\partial p_y}}{g(p_y)^2}$, define $h(p_y) = v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)$ and $g(p_y) = |\mathbf{p}_n - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_y)}{\partial p_y} = -v_y$ and $\frac{\partial g(p_y)}{\partial p_y} = \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|}$.

$$= \frac{-v_y |\mathbf{p}_n - \mathbf{p}| - [v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)] \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|}}{|\mathbf{p}_n - \mathbf{p}|^2} \quad (\text{A.14a})$$

$$= \frac{-v_y |\mathbf{p}_n - \mathbf{p}|^2 - [v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)](p_y - p_{yn})}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.14b})$$

$$= \frac{v_x(p_{xn} - p_x)(p_{yn} - p_y) - v_y(p_{xn} - p_x)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.14c})$$

Combining the above

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} &= \frac{1}{\lambda} \left(\frac{v_x(p_{xo} - p_x)(p_{yo} - p_y) - v_y(p_{xo} - p_x)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \right. \\ &\quad \left. + \frac{v_x(p_{xn} - p_x)(p_{yn} - p_y) - v_y(p_{xn} - p_x)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (\text{A.15})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to v_x

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_x} = \frac{1}{\lambda} \frac{\partial}{\partial v_x} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.16a})$$

$$\begin{aligned} &= \frac{1}{\lambda} \frac{\partial}{\partial v_x} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)}{|\mathbf{p}_o - \mathbf{p}|} \right. \\ &\quad \left. + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \end{aligned} \quad (\text{A.16b})$$

$$= \frac{1}{\lambda} \left(\frac{p_{xo} - p_x}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{xn} - p_x}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.16c})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to v_y

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_y} = \frac{1}{\lambda} \frac{\partial}{\partial v_y} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.17a})$$

$$\begin{aligned} &= \frac{1}{\lambda} \frac{\partial}{\partial v_y} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)}{|\mathbf{p}_o - \mathbf{p}|} \right. \\ &\quad \left. + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \end{aligned} \quad (\text{A.17b})$$

$$= \frac{1}{\lambda} \left(\frac{p_{y_o} - p_y}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{y_n} - p_y}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.17c})$$

A.2 3D Range and Doppler Derivatives

The measurements from the radar are

$$r_n(\boldsymbol{\theta}) = |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \quad (\text{A.18a})$$

$$f_n(\boldsymbol{\theta}) = \frac{v}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.18b})$$

The derivatives are the rates of change of the measurements with respect to the target state vector,

$$\boldsymbol{\theta} = \begin{pmatrix} p_x & p_y & p_z & v_x & v_y & v_z \end{pmatrix}^\top. \quad (\text{A.19})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to p_x

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_x} = \frac{\partial}{\partial p_x} \{ |\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}| \} \quad (\text{A.20a})$$

$$= \frac{\partial}{\partial p_x} \left\{ \sqrt{(p_{x_o} - p_x)^2 + (p_{y_o} - p_y)^2 + (p_{z_o} - p_z)^2} \right. \\ \left. + \sqrt{(p_{x_n} - p_x)^2 + (p_{y_n} - p_y)^2 + (p_{z_n} - p_z)^2} \right\} \quad (\text{A.20b})$$

$$= \frac{-2(p_{x_o} - p_x)}{2\sqrt{(p_{x_o} - p_x)^2 + (p_{y_o} - p_y)^2 + (p_{z_o} - p_z)^2}} \\ + \frac{-2(p_{x_n} - p_x)}{2\sqrt{(p_{x_n} - p_x)^2 + (p_{y_n} - p_y)^2 + (p_{z_n} - p_z)^2}} \quad (\text{A.20c})$$

$$= \frac{p_x - p_{xo}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (\text{A.20d})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to p_y

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_y} = \frac{\partial}{\partial p_y} \{|\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}|\} \quad (\text{A.21a})$$

$$= \frac{\partial}{\partial p_y} \left\{ \sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2 + (p_{zo} - p_z)^2} \right. \\ \left. + \sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2 + (p_{zn} - p_z)^2} \right\} \quad (\text{A.21b})$$

$$= \frac{-2(p_{yo} - p_y)}{2\sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2 + (p_{zo} - p_z)^2}} \\ + \frac{-2(p_{yn} - p_y)}{2\sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2 + (p_{zn} - p_z)^2}} \quad (\text{A.21c})$$

$$= \frac{p_y - p_{yo}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (\text{A.21d})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to p_z

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial p_z} = \frac{\partial}{\partial p_z} \{|\mathbf{p}_o - \mathbf{p}| + |\mathbf{p}_n - \mathbf{p}|\} \quad (\text{A.22a})$$

$$= \frac{\partial}{\partial p_z} \left\{ \sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2 + (p_{zo} - p_z)^2} \right. \\ \left. + \sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2 + (p_{zn} - p_z)^2} \right\} \quad (\text{A.22b})$$

$$= \frac{-2(p_{zo} - p_z)}{2\sqrt{(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2 + (p_{zo} - p_z)^2}} \\ + \frac{-2(p_{zn} - p_z)}{2\sqrt{(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2 + (p_{zn} - p_z)^2}} \quad (\text{A.22c})$$

$$= \frac{p_z - p_{zo}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_z - p_{zn}}{|\mathbf{p}_n - \mathbf{p}|} \quad (\text{A.22d})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to v_x

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_x} = 0 \quad (\text{A.23})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to v_y

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_y} = 0 \quad (\text{A.24})$$

Derivative of $r_n(\boldsymbol{\theta})$ with respect to v_z

$$\frac{\partial r_n(\boldsymbol{\theta})}{\partial v_z} = 0 \quad (\text{A.25})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to p_x

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = \frac{\partial}{\partial p_x} \left\{ \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \right\} \quad (\text{A.26a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_x} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.26b})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_x} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)}{|\mathbf{p}_o - \mathbf{p}|} + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.26c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_x} \left\{ \frac{h(p_x)}{g(p_x)} \right\} = \frac{g(p_x) \frac{\partial h(p_x)}{\partial p_x} - h(p_x) \frac{\partial g(p_x)}{\partial p_x}}{g(p_x)^2}$, define $h(p_x) = v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)$ and $g(p_x) = |\mathbf{p}_o - \mathbf{p}|$. Differentiating,

$$\frac{\partial h(p_x)}{\partial p_x} = -v_x \text{ and } \frac{\partial g(p_x)}{\partial p_x} = \frac{p_x - p_{x0}}{|\mathbf{p}_0 - \mathbf{p}|}.$$

$$= \frac{-v_x |\mathbf{p}_0 - \mathbf{p}| - (v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)) \frac{p_x - p_{x0}}{|\mathbf{p}_0 - \mathbf{p}|}}{|\mathbf{p}_0 - \mathbf{p}|^2} \quad (\text{A.27a})$$

$$= \frac{-v_x |\mathbf{p}_0 - \mathbf{p}|^2 - (v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z))(p_x - p_{x0})}{|\mathbf{p}_0 - \mathbf{p}|^3} \quad (\text{A.27b})$$

$$= \frac{-v_x(p_{x0} - p_x)^2 - v_x(p_{y0} - p_y)^2 - v_x(p_{z0} - p_z)^2}{|\mathbf{p}_0 - \mathbf{p}|^3} + \frac{v_x(p_{x0} - p_x)^2 + v_y(p_{y0} - p_y)(p_{x0} - p_x) + v_z(p_{z0} - p_z)(p_{x0} - p_x)}{|\mathbf{p}_0 - \mathbf{p}|^3} \quad (\text{A.27c})$$

$$= -\frac{v_x[(p_{y0} - p_y)^2 + (p_{z0} - p_z)^2]}{|\mathbf{p}_0 - \mathbf{p}|^3} - \frac{(p_x - p_{x0})[v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)]}{|\mathbf{p}_0 - \mathbf{p}|^3} \quad (\text{A.27d})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_z} \left\{ \frac{h(p_z)}{g(p_z)} \right\} = \frac{g(p_z) \frac{\partial h(p_z)}{\partial p_z} - h(p_z) \frac{\partial g(p_z)}{\partial p_z}}{g(p_z)^2}$, define $h(p_z) = v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)$ and $g(p_z) = |\mathbf{p}_n - \mathbf{p}|$. Differentiating,

$$\frac{\partial h(p_x)}{\partial p_x} = -v_x \text{ and } \frac{\partial g(p_x)}{\partial p_x} = \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|}.$$

$$= \frac{-v_x |\mathbf{p}_n - \mathbf{p}| - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)) \frac{p_x - p_{xn}}{|\mathbf{p}_n - \mathbf{p}|}}{|\mathbf{p}_n - \mathbf{p}|^2} \quad (\text{A.28a})$$

$$= \frac{-v_x |\mathbf{p}_n - \mathbf{p}|^2 - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z))(p_x - p_{xn})}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.28b})$$

$$= \frac{-v_x(p_{xn} - p_x)^2 - v_x(p_{yn} - p_y)^2 - v_x(p_{zn} - p_z)^2}{|\mathbf{p}_n - \mathbf{p}|^3} + \frac{v_x(p_{xn} - p_x)^2 + v_y(p_{yn} - p_y)(p_{xn} - p_x) + v_z(p_{zn} - p_z)(p_{xn} - p_x)}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.28c})$$

$$= -\frac{v_x[(p_{yn} - p_y)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} - \frac{(p_x - p_{xn})[v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)]}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.28d})$$

Combining the above

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_x} = & -\frac{1}{\lambda} \left(\frac{v_x[(p_{y0} - p_y)^2 + (p_{z0} - p_z)^2]}{|\mathbf{p}_0 - \mathbf{p}|^3} \right. \\ & + \frac{v_x[(p_{yn} - p_y)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\ & + \frac{(p_x - p_{x0})[v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)]}{|\mathbf{p}_0 - \mathbf{p}|^3} \\ & \left. + \frac{(p_x - p_{xn})[v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)]}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (\text{A.29a})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to p_y

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} = \frac{\partial}{\partial p_y} \left\{ \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_0 - \mathbf{p}}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \right\} \quad (\text{A.30a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_y} \left\{ \frac{\mathbf{v} \cdot (\mathbf{p}_0 - \mathbf{p})}{|\mathbf{p}_0 - \mathbf{p}|} + \frac{\mathbf{v} \cdot (\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.30b})$$

$$\begin{aligned} = & \frac{1}{\lambda} \frac{\partial}{\partial p_y} \left\{ \frac{v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)}{|\mathbf{p}_0 - \mathbf{p}|} \right\} \\ & + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \end{aligned} \quad (\text{A.30c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_y} \left\{ \frac{h(p_y)}{g(p_y)} \right\} = \frac{g(p_y) \frac{\partial h(p_y)}{\partial p_y} - h(p_y) \frac{\partial g(p_y)}{\partial p_y}}{g(p_y)^2}$, define

$h(p_y) = v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)$ and $g(p_y) = |\mathbf{p}_0 - \mathbf{p}|$. Differentiating,

$$\frac{\partial h(p_y)}{\partial p_z} = -v_y \quad \text{and} \quad \frac{\partial g(p_y)}{\partial p_z} = \frac{p_y - p_{y0}}{|\mathbf{p}_0 - \mathbf{p}|}.$$

$$= \frac{-v_y |\mathbf{p}_0 - \mathbf{p}| - (v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z)) \frac{p_y - p_{y0}}{|\mathbf{p}_0 - \mathbf{p}|}}{|\mathbf{p}_0 - \mathbf{p}|^2} \quad (\text{A.31a})$$

$$= \frac{-v_y |\mathbf{p}_0 - \mathbf{p}|^2 - (v_x(p_{x0} - p_x) + v_y(p_{y0} - p_y) + v_z(p_{z0} - p_z))(p_y - p_{y0})}{|\mathbf{p}_0 - \mathbf{p}|^3} \quad (\text{A.31b})$$

$$\begin{aligned} = & \frac{-v_y(p_{x0} - p_x)^2 - v_y(p_{y0} - p_y)^2 - v_y(p_{z0} - p_z)^2}{|\mathbf{p}_0 - \mathbf{p}|^3} \\ & + \frac{v_x(p_{x0} - p_x)(p_{y0} - p_y) + v_y(p_{y0} - p_y)^2 + v_z(p_{z0} - p_z)(p_{y0} - p_y)}{|\mathbf{p}_0 - \mathbf{p}|^3} \end{aligned} \quad (\text{A.31c})$$

$$\begin{aligned}
&= \frac{-v_y[(p_{xo} - p_x)^2 + (p_{zo} - p_z)^2]}{|\mathbf{p}_o - \mathbf{p}|^3} \\
&- \frac{(p_y - p_{yo})(v_x(p_{xo} - p_x) + v_z(p_{zo} - p_z))}{|\mathbf{p}_o - \mathbf{p}|^3}
\end{aligned} \tag{A.31d}$$

The quotient differentiation formula is $\frac{\partial}{\partial p_y} \left\{ \frac{h(p_y)}{g(p_y)} \right\} = \frac{g(p_y) \frac{\partial h(p_y)}{\partial p_y} - h(p_y) \frac{\partial g(p_y)}{\partial p_y}}{g(p_y)^2}$, define $h(p_y) = v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)$ and $g(p_y) = |\mathbf{p}_n - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_y)}{\partial p_z} = -v_y$ and $\frac{\partial g(p_z)}{\partial p_z} = \frac{p_y - p_{yn}}{|\mathbf{p}_o - \mathbf{p}|}$.

$$= \frac{-v_y |\mathbf{p}_n - \mathbf{p}| - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)) \frac{p_y - p_{yn}}{|\mathbf{p}_n - \mathbf{p}|}}{|\mathbf{p}_n - \mathbf{p}|^2} \tag{A.32a}$$

$$= \frac{-v_y |\mathbf{p}_n - \mathbf{p}|^2 - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z))(p_y - p_{yn})}{|\mathbf{p}_n - \mathbf{p}|^3} \tag{A.32b}$$

$$\begin{aligned}
&= \frac{-v_y(p_{xn} - p_x)^2 - v_y(p_{yn} - p_y)^2 - v_y(p_{zn} - p_z)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \\
&+ \frac{v_x(p_{xn} - p_x)(p_{yn} - p_y) + v_y(p_{yn} - p_y)^2 + v_z(p_{zn} - p_z)(p_{yn} - p_y)}{|\mathbf{p}_n - \mathbf{p}|^3}
\end{aligned} \tag{A.32c}$$

$$\begin{aligned}
&= \frac{-v_y[(p_{xn} - p_x)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\
&- \frac{(p_y - p_{yn})(v_x(p_{xn} - p_x) + v_z(p_{zn} - p_z))}{|\mathbf{p}_n - \mathbf{p}|^3}
\end{aligned} \tag{A.32d}$$

Combining the above

$$\begin{aligned}
\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_y} = & -\frac{1}{\lambda} \left(\frac{v_y[(p_{xo} - p_x)^2 + (p_{zo} - p_z)^2]}{|\mathbf{p}_o - \mathbf{p}|^3} \right. \\
& + \frac{v_y[(p_{xn} - p_x)^2 + (p_{zn} - p_z)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \\
& + \frac{(p_y - p_{yo})(v_x(p_{xo} - p_x) + v_z(p_{zo} - p_z))}{|\mathbf{p}_o - \mathbf{p}|^3} \\
& \left. + \frac{(p_y - p_{yn})(v_x(p_{xn} - p_x) + v_z(p_{zn} - p_z))}{|\mathbf{p}_n - \mathbf{p}|^3} \right)
\end{aligned} \tag{A.33a}$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to p_z

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial p_z} = \frac{\partial}{\partial p_z} \left\{ \frac{\mathbf{v}}{\lambda} \cdot \left(\frac{\mathbf{p}_o - \mathbf{p}}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{p}_n - \mathbf{p}}{|\mathbf{p}_n - \mathbf{p}|} \right) \right\} \quad (\text{A.34a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_z} \left\{ \frac{\mathbf{v} \cdot (\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \frac{\mathbf{v} \cdot (\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.34b})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial p_z} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)}{|\mathbf{p}_o - \mathbf{p}|} \right\} \\ + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.34c})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_z} \left\{ \frac{h(p_z)}{g(p_z)} \right\} = \frac{g(p_z) \frac{\partial h(p_z)}{\partial p_z} - h(p_z) \frac{\partial g(p_z)}{\partial p_z}}{g(p_z)^2}$, define $h(p_z) = v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)$ and $g(p_z) = |\mathbf{p}_o - \mathbf{p}|$. Differentiating, $\frac{\partial h(p_z)}{\partial p_z} = -v_z$ and $\frac{\partial g(p_z)}{\partial p_z} = \frac{p_z - p_{zo}}{|\mathbf{p}_o - \mathbf{p}|}$.

$$= \frac{-v_z |\mathbf{p}_o - \mathbf{p}| - (v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)) \frac{p_z - p_{zo}}{|\mathbf{p}_o - \mathbf{p}|}}{|\mathbf{p}_o - \mathbf{p}|^2} \quad (\text{A.35a})$$

$$= \frac{-v_z |\mathbf{p}_o - \mathbf{p}|^2 - (v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z))(p_z - p_{zo})}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.35b})$$

$$= \frac{-v_z(p_{xo} - p_x)^2 - v_z(p_{yo} - p_y)^2 - v_z(p_{zo} - p_z)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \\ + \frac{-v_x(p_{xo} - p_x)(p_z - p_{zo}) - v_y(p_{yo} - p_y)(p_z - p_{zo}) + v_z(p_{zo} - p_z)^2}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.35c})$$

$$= - \frac{v_z[(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2]}{|\mathbf{p}_o - \mathbf{p}|^3} \\ - \frac{(p_z - p_{zo})[v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)]}{|\mathbf{p}_o - \mathbf{p}|^3} \quad (\text{A.35d})$$

The quotient differentiation formula is $\frac{\partial}{\partial p_z} \left\{ \frac{h(p_z)}{g(p_z)} \right\} = \frac{g(p_z) \frac{\partial h(p_z)}{\partial p_z} - h(p_z) \frac{\partial g(p_z)}{\partial p_z}}{g(p_z)^2}$, define $h(p_z) = v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)$ and $g(p_z) = |\mathbf{p}_n - \mathbf{p}|$. Differentiating,

$$\frac{\partial h(p_z)}{\partial p_z} = -v_z \text{ and } \frac{\partial g(p_z)}{\partial p_z} = \frac{p_z - p_{zn}}{|\mathbf{p}_n - \mathbf{p}|}.$$

$$= \frac{-v_z |\mathbf{p}_n - \mathbf{p}| - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)) \frac{p_z - p_{zn}}{|\mathbf{p}_n - \mathbf{p}|}}{|\mathbf{p}_n - \mathbf{p}|^2} \quad (\text{A.36a})$$

$$= \frac{-v_z |\mathbf{p}_n - \mathbf{p}|^2 - (v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z))(p_z - p_{zn})}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.36b})$$

$$= \frac{-v_z(p_{xn} - p_x)^2 - v_z(p_{yn} - p_y)^2 - v_z(p_{zn} - p_z)^2}{|\mathbf{p}_n - \mathbf{p}|^3} + \frac{-v_x(p_{xn} - p_x)(p_z - p_{zn}) - v_y(p_{yn} - p_y)(p_z - p_{zn}) + v_z(p_{zn} - p_z)^2}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.36c})$$

$$= -\frac{v_z[(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} - \frac{(p_z - p_{zn})[v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)]}{|\mathbf{p}_n - \mathbf{p}|^3} \quad (\text{A.36d})$$

Combining the above

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\theta})}{\partial p_z} = & -\frac{1}{\lambda} \left(\frac{v_z[(p_{xo} - p_x)^2 + (p_{yo} - p_y)^2]}{|\mathbf{p}_o - \mathbf{p}|^3} + \frac{v_z[(p_{xn} - p_x)^2 + (p_{yn} - p_y)^2]}{|\mathbf{p}_n - \mathbf{p}|^3} \right. \\ & + \frac{(p_z - p_{zo})[v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y)]}{|\mathbf{p}_o - \mathbf{p}|^3} \\ & \left. + \frac{(p_z - p_{zn})[v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y)]}{|\mathbf{p}_n - \mathbf{p}|^3} \right) \end{aligned} \quad (\text{A.37})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to v_x

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_x} = \frac{1}{\lambda} \frac{\partial}{\partial v_x} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.38a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial v_x} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)}{|\mathbf{p}_o - \mathbf{p}|} + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.38b})$$

$$= \frac{1}{\lambda} \left(\frac{p_{xo} - p_x}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{xn} - p_x}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.38c})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to v_y

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_y} = \frac{1}{\lambda} \frac{\partial}{\partial v_y} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.39a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial v_y} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)}{|\mathbf{p}_o - \mathbf{p}|} + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.39b})$$

$$= \frac{1}{\lambda} \left(\frac{p_{yo} - p_y}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{yn} - p_y}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.39c})$$

Derivative of $f_n(\boldsymbol{\theta})$ with respect to v_z

$$\frac{\partial f_n(\boldsymbol{\theta})}{\partial v_z} = \frac{1}{\lambda} \frac{\partial}{\partial v_z} \left\{ \mathbf{v} \cdot \frac{(\mathbf{p}_o - \mathbf{p})}{|\mathbf{p}_o - \mathbf{p}|} + \mathbf{v} \cdot \frac{(\mathbf{p}_n - \mathbf{p})}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.40a})$$

$$= \frac{1}{\lambda} \frac{\partial}{\partial v_z} \left\{ \frac{v_x(p_{xo} - p_x) + v_y(p_{yo} - p_y) + v_z(p_{zo} - p_z)}{|\mathbf{p}_o - \mathbf{p}|} + \frac{v_x(p_{xn} - p_x) + v_y(p_{yn} - p_y) + v_z(p_{zn} - p_z)}{|\mathbf{p}_n - \mathbf{p}|} \right\} \quad (\text{A.40b})$$

$$= \frac{1}{\lambda} \left(\frac{p_{zo} - p_z}{|\mathbf{p}_o - \mathbf{p}|} + \frac{p_{zn} - p_z}{|\mathbf{p}_n - \mathbf{p}|} \right) \quad (\text{A.40c})$$