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Optimal inspection policy for three-state systems monitored by control charts

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Abstract

This paper presents the formulations of the expected long-run cost per time unit for a system monitored by a static control chart and by an adaptive control chart respectively. The static chart has a fixed sampling interval and a fixed sample size. The adaptive chart has a fixed sample size but variable sampling intervals. The system is supposed to have three states, normal working state, failure delay time state, and failed state. Two levels of repair are used to maintain the system. A minor repair is used to restore the system if a detectable defect is confirmed by an inspection. A major repair will be performed if the system fails. The expected cost per time unit for maintaining such a system is obtained. The objective of such analysis is to find an optimal sampling policy for the inspection process. An artificially generated data example and a real data example are used to compare the expected cost per time unit for both the static and adaptive control charts.

Keywords: Maintenance policy; control chart; condition-based maintenance; multi-state system; inspection policy.

1 Introduction

The delay time concept introduced by Christer and Waller [1] regards the failure process as a two-stage process with the first stage (or called *normal working state*) being from new to the point of a detectable defect arising and the second stage (or called *failure delay time state*) from this point onward to the failure if no other maintenance interventions were taken. The concept of failure delay time can be used in planning condition-based maintenance policy (see [2], for example): if one can successfully detect the changing point, at which the process shifts from the first stage to the second stage, then we can conduct maintenance during the second stage (before the system fails).

Control charts, a tool from statistical process control (SPC), can be used to detect the changing point. If we use SPC terminologies, the normal working state is said an *in-control* state and the failure delay time state is said an *out-of-control* state. Control charts are commonly employed for the purpose of signalling the occurrence of assignable causes every time when the process parameter has shifted. As control charts may produce false signals that incorrectly indicate the state of the system, optimally designing the parameters of control charts to minimizing the cost incurred by the false signals is an important research topic. To this end, various scenarios have been considered by researchers. Tagaras and Lee [3] and Tagaras [4] separated the $\bar{X}$-chart into several zones and optimized the chart for monitoring a process whose deterioration can be classified into two states in which one state requires minor repair and the other requires major repair, and considered

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economic design of control charts. Cheng *et al* [5] used the p-chart to derive thresholds for aviation inspection. Cassady *et al* [6] considered economic design of control charts for optimization of preventive maintenance policies for systems. Chan and Wu [7] apply the concept of cumulative count of conforming chart (CCC chart) in inspection and maintenance planning for systems where minor inspection, major inspection, minor maintenance and major maintenance are available. Control charts have also been applied to condition-based maintenance for monitoring two-stage processes that was mentioned in the preceding paragraph (see [8] for example). Other examples of research in this area can also been seen in [9,10]. Typical application areas of control charts can be found in continuous manufacturing processes such as canning lines, high precision component machining centres, and electronic item assembly lines.

Commonly, the decision variables in designing control charts include the sampling interval between consecutive sampling points, sample sizes or control limits. The parameters in control charts can be fixed or variable, then we have two different control charts: *static* and *adaptive*. A static control chart has fixed parameters, including sample size n, sampling interval h, lower control limit (LCL) and upper control limit (UCL), whereas an adaptive control chart has at least one of its parameters (n, h, LCL and UCL) allowed to be changed based on the information about the state of the process. Compared with the static control charts, the adaptive control chart can utilise the inspection capacity more effectively for a better process control ([11,12,13,14]). Recent research on adaptive control charts can be found in [15,16,17,18].

However, in existing literature, there has been little work in investigating the potential of the adaptive control charts to monitor processes where a failure delay time exists. This paper presents the formulations of the expected long-run cost per time unit for such a process monitored by adaptive control charts with variable sampling intervals.

A system with a failure delay time is essentially a three-state system that is a typical case of a multistate system. Research in multistate systems is another interesting topic in reliability theory and engineering, and the reader is referred to [19,20] for more information in this area.

The novelty of this paper lies in the formulations and analysis of the expected cost per time unit for the scenarios where adaptive control charts are applied to monitor three-state systems.

This paper is structured as follows. Section 2 presents assumptions and notation that are used in the paper. In order to compare the efficiency of static and adaptive control charts, Section 3 formulates the expected long-run cost per unit for a system monitored by static and adaptive charts respectively. Section 4 discusses the relationship between the formulas of the expected long-run cost per unit of the static and adaptive charts. Section 5 gives numerical examples to study the sensitive analysis for various parameter settings. Section 6 concludes the findings of this paper.

2 General assumptions and notation

This section first presents the usage of the static and adaptive control charts, then makes assumptions for these two charts.

Consider a system with three states: in-control, out-of-control and failed. It produces outputs that can be monitored by a control chart: either a static or an adaptive chart.

Static chart The sampling interval h is fixed. n samples are taken at every h time units. Figure 1 shows a typical example of a control chart with two zones: Z_{f0} is a central zone, Z_{f1} is an action zone, and they are defined as: Z_{f0}=$\{0,$ UCL$\}$, Z_{f1} =$\{$UCL,$\infty\}$,

for example. The following policy is adopted.

- If the characteristics of the n samples falls in Z_{f0}, no further action will be taken and the next sampling interval will be h.
- If the characteristics of the n samples falls in Z_{f1}, then an inspection will be carried out to check the occurrence of assignable causes. If the occurrence is confirmed by the inspection, then a minor repair is performed. Otherwise no further action will be taken and the next sampling interval will be h.

Adaptive chart The first sampling interval is h_0 time units and the samples are taken immediately after the start of the system. After that, n samples are taken at every h_0 time units. Figure 2 shows a typical example of a control chart with three zones: Z_{a0} is a central zone, Z_{a1} is a warning zone, Z_{a2} is an action zone, and they are defined as: $Z_{a0} = \{0, w\}$, $Z_{a1} = \{w,\text{UCL}\}$, and $Z_{a2}=\{\text{UCL},\infty\}$, for example. The following policy is adopted.

- If the characteristics of the n samples falls in Z_{a0}, then the next sampling interval will be h_0 time units;
- If the characteristics of the n samples falls in Z_{a1}, then the next sampling interval will be h_1 time units, and an inspection will be carried out to check whether the system is in control or not. If the system is confirmed to be in the out-of-control state, then a minor repair is performed to restore the system, otherwise no further action will be taken and the next sampling interval will be h_0.
- If the characteristics of the n samples falls in Z_{a2}, then an inspection will be carried out to check the occurrence of the assignable causes. If occurrence is confirmed, then a minor repair is performed. Otherwise no further action will be taken and the next sampling interval will be h_0, which is the same as normal.

The following assumptions are also held for both the static and the adaptive control charts.

(1) Suppose that the system can shift either from an in-control state to an out-of-control state and then to a failure state; or directly from an in-control state to an failure state without going through an out-of-control state. The system itself in a failure state or an out-of-control state cannot restore back to the in-control state without any repair.

(2) When the system is in an in-control state or an out-of-control state, the defective probability of the product is p_0 or p_1, respectively, where $p_1 > p_0$.

(3) The inspection is assumed to be perfect in that it can reveal whether the system is in control or not. During the inspection, the system carries on running. As long as the system has been confirmed to be in an out-of-control state, repairmen will carry out a minor repair which can bring the system back to an in-control state. The system is as good as new immediately after the minor repair has been completed, and it is still operating while it is under a minor repair. Once the system fails, repairmen will conduct a major repair that can bring it back to the as-good-as-new state.

(4) For simplicity, times spent on an inspection, minor or major repair are small compared with the sampling interval, and therefore are negligible, but their costs are considered.

Th following notation is also used.

X_1: random time from the beginning of an in-control state to the occurrence of an assignable cause;

$f_1(x_1)$: pdf. of X_1, and $F_1(x_1) = \Pr(X_1 < x_1)$ is the cdf. of X_1;

X_2: random time from the beginning of the out-control state to failure;

$f_2(x_2)$: pdf. of X_2, and $F_2(x_2) = \Pr(X_2 < x_2)$, cdf. of X_2;

n: sample size;

h: sampling interval when a static control chart is used;

h_0: a longer sampling interval when an adaptive control chart is used;

h_1: a shorter sampling interval when an adaptive control chart is used;

α_i: the probability that the characteristics of the n samples falls in Z_{f0} when the system is in the in-control state for $i = 0$, or in the out-of-control state for $i = 1$. It is for a static control chart, where $\alpha_0 + \alpha_1 = 1$.

β_{ij}: the probability that the characteristics of the n samples falls in Z_{aj} $(j = 0, 1, 2)$ when the system is in the in-control state for $i = 0$, or in the out-of-control state for $i = 1$. It is for the adaptive control chart, where $\beta_{00} + \beta_{01} + \beta_{02} = 1$ and $\beta_{10} + \beta_{11} + \beta_{12} = 1$.

T_n: time to the first minor repair with an assignable cause when no control chart is used;

T_{f1}: the random time from the beginning of the in-control state to the occurrence of an assignable cause when a static control chart is used;

T_{f2}: time between the occurrence of an assignable cause and failure when a static control chart is used;

T_{a1}: expected time between the start of the system and a minor repair triggered by an inspection due to a warning signal in zone Z_{a1};

T_{a2}: expected time between the start of the system and a minor repair triggered by an inspection due to a warning signal in zone Z_{a2};

T_{a3}: expected time between the start of the system and a major repair;

c_s: sampling cost per item;

c_i: inspection cost for a possible assignable cause;

c_{r1}: cost for a minor repair;

c_{r2}: cost for a major repair;

C_n: time to failure when no control chart is used;

C_{f1}, C_{f2}: costs incurred within times T_{f1} and T_{f1}, respectively;

C_{a1}, C_{a2}, C_{a3}: costs incurred within times T_{a1}, T_{a2} and T_{a3}, respectively. They are for an adaptive chart.

Throughout the modelling development, we use the renewal reward theorem, which simply

states that the expected long-run cost per time unit is the ratio between the expected renewal cycle cost and expected renewal cycle length [21].

3 Expected long-run cost per time unit

This section derives the expected long-run cost per time unit for both the static control chart scenario and the adaptive control chart scenario.

3.1 No control chart scenario

In the case there is no control chart used to monitor the system, the expected long-run time is given by

$$E(T_n) = \int_0^\infty \int_0^x x f_1(x_1) f_2(x - x_1) dx_1 dx, \tag{1}$$

and the expected cost is

$$E(C_n) = c_{r2} \int_0^\infty \int_0^x f_1(x_1) f_2(x - x_1) dx_1 dx. \tag{2}$$

The expected long-run cost per time unit is given by

$$E_n(T, C) = \frac{E(C_n)}{E(T_n)} \tag{3}$$

3.2 Static control chart scenario

In the case when a static control chart is employed to monitor the system, the expected time length and expected cost are derived as follows.

3.2.1 Expected time length

There can be two renewals: a renewal after a minor repair is conducted, and a renewal after a major repair is completed. The expected renewal cycle length is $E(T_f) = E(T_{f1}) + E(T_{f2})$.

Then we have

$$E(T_{f1}) = \sum_{i=1}^{\infty} ih \sum_{j=1}^{i} \alpha_1^{i-j}(1-\alpha_1) \int_{(j-1)h}^{jh} f_1(x_1)[1 - F_2(ih - x_1)]dx_1. \tag{4}$$

Proof. In this case, a renewal occurs after a minor repair is conducted at the ih time point. There are $i - j$ false signals followed by one true signal. The false signals wrongly indicate that the system is in an in-control state whereas it is actually in an out-of-control state. The time length is ih that includes i sampling intervals in both the in-control state and the out-of-control state. □

The expected value of T_{f2} is given by

$$E(T_{f2}) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_1^{i-j} \int_{(i-1)h}^{ih} \int_{(j-1)h}^{\tau_{ij}} x f_1(x_1) f_2(x - x_1) dx_1 dx, \tag{5}$$

where $\tau_{ij} = \begin{cases} jh \text{ if } j < i; \\ x \quad \text{if } j = i. \end{cases}$

Proof. In this scenario, There are $i - j$ false signals followed by a failure. □

The expected renewal cycle length $E(T_f)$ is the summation of $E(T_{f1})$ and $E(T_{f2})$, and it is given by

$$E(T_f) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} ih\alpha_1^{i-j}(1-\alpha_1)\int_{(j-1)h}^{jh} f_1(x_1)[1-F_2(ih-x_1)]dx_1 + \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_1^{i-j}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{\tau_{ij}} x f_1(x_1) f_2(x-x_1)dx_1 dx. \quad (6)$$

3.2.2 Expected renewal cycle cost

Denote $c_{ij} = (i-1)nc_s + (1-\alpha_0)(j-1)c_i$ which is the expected cost of samplings and false alarm inspections before a renewal at ih or x, $x \in ((i-1)h, ih]$, and the assignable cause occurs at x_1, $x_1 \in ((j-1)h, jh]$, then summing over all possible sampling intervals and events, we have

$$\begin{aligned} & E(C_f) \\ =& \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_1^{i-j}(1-\alpha_1)\int_{(j-1)h}^{jh}(c_{ij}+nc_s+c_i+c_{r1}) f_1(x_1)[1-F_2(ih-x_1)]dx_1 \\ +& \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_1^{i-j}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{\tau_{ij}}(c_{ij}+c_{r2}) f_1(x_1) f_2(x-x_1)dx_1 dx. \end{aligned} \quad (7)$$

Hence, the expected long-run cost per time unit is given by

$$E_f(T,C) = \frac{E(C_f)}{E(T_f)}. \quad (8)$$

3.3 Adaptive control chart scenario

There are two possible kinds of transition of the system: a transition from the in-control state to the out-of-control state detected by signals either in the warning zone or in the action zone, and a transition from the out-of-control state to failure. We assume that any level of maintenance can renew the system.

3.3.1 Expected renewal cycle length

The expected renewal cycle length is $E(T_a) = E(T_{a1}) + E(T_{a2}) + E(T_{a3})$, which is explained as follows.

The expected time between the start and a minor repair triggered by an inspection due to a signal in zone Z_{a1} is given by

$$E(T_{a1}) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \left\{ \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \beta_{11} (1-\beta_{10}) H_0 \int_{H_1-h_0-h_1}^{H_1-h_1} f_1(x_1)(1-F_2(H_0-x_1))dx_1 \right.$$
$$\left. + \beta_{01} \beta_{10}^{k_1-k_2+1} \beta_{11}^{k_3+1} (1-\beta_{10})(H_0+h_0+h_1) \int_{H_1-h_1}^{H_1} f_1(x_1)(1-F_2(H_0+h_0+h_1-x_1))dx_1 \right\}$$
$$+ \sum_{k_2=0}^{\infty} \left\{ \beta_{01}(1-\beta_{10}) H_2 \int_{H_2-h_1}^{H_2} f_1(x_1)(1-F_2(H_2-x_1))dx_1 \right\}, \quad (9)$$

where $H_0 = k_1 h_0 + (\beta_{01}(k_2-1) + k_3 + 1)h_1$, $H_1 = k_2 h_0 + (\beta_{01}(k_2-1)+1)h_1$, and $H_2 = (k_2+1)h_0 + (\beta_{01}k_2 + 1)h_1$.

Proof. The description of three units in Eq (9) is given below. There are three scenarios for the system transiting from an in-control state to an out-of-control state. These three states correspond to the following the three units in Eq (9).

Denote k_1 as the total number of longer sampling intervals (ie. h_0) in both the in-control and out-of-control states, k_2 as the total number of longer sampling intervals in an in-control state, and k_3 as the number of false signals followed by true ones in the out-of-control state.

Unit 1 The system transits from the in-control state to the out-of-control state in a longer sampling interval h_0. Apparently, the effectiveness of the control chart is associated with the length of an out-of-control state, but independent of the length of the in-control state.

When the system is in the out-of-control state, there might be false signals (with

a probability of $\beta_{10}^{k_1-k_2-k_3}$) in h_0, or true signals in h_0 but followed by false signals in h_1 (with a probability of $(\beta_{11}\beta_{10})^{k_3}$). These two scenarios make up an event with a probability of $(\beta_{10})^{k_1-k_2-k_3}(\beta_{11}\beta_{10})^{k_3}$(or $\beta_{10}^{k_1-k_2}\beta_{11}^{k_3}$), and take time $(k_1-k_2)h_0+k_3h_1$. Eventually, a correct signal in h_0 is followed by another correct signal in h_1, which has a probability of $\beta_{11}(1-\beta_{10})$ and a time length of h_0+h_1.

Before the system has transited from the in-control state to the out-of-control state, the time length is $(k_2-1)h_0+\beta_{01}(k_2-1)h_1$. Hence, the total length is $k_1h_0+(\beta_{01}(k_2-1)+k_3+1)h_1=H_0$. The transition occurs in the time interval $((k_2-1)h_0+\beta_{01}(k_2-1)h_1, k_2h_0+\beta_{01}(k_2-1)h_1)$, or $(H_1-h_0-h_1, H_1-h_1)$.

Unit 2 The system might also transit from the in-control state to the out-of-control state within a short sampling interval h_1 after a false signal appears in the in-control state, but a correct signal follows. This event has a probability of $\beta_{01}\beta_{10}$. A correct signal can appear in the sample of this short sampling interval. This event has a probability of $\beta_{01}(1-\beta_{10})$ and a time length of h_0+h_1. The time length of the system in the in-control state is $(k_2-1)h_0+\beta_{01}(k_2-1)h_1$, then the transition from the in-control state to the out-of-control state occurs in (H_1-h_1, H_1) (where $H_1=k_2h_0+\beta_{01}(k_2-1)h_1+h_1$). After the system has transited to the out-of-control state, the probability of the appearance of a correct signal is given by $\beta_{10}^{k_1-k_2-k_3}(\beta_{11}\beta_{10})^{k_3}\beta_{11}(1-\beta_{10})$ (or $\beta_{10}^{k_1-k_2}\beta_{11}^{k_3+1}(1-\beta_{10})$) and has a time length of $H_0+h_0+h_1$.

Unit 3 When the system is in the in-control state, a false signal appears in a longer sampling interval h_0. Then a short sampling interval is followed and the system transits to the out-of-control state in this short sampling interval, and then a true signal appears. This event has a probability of $\beta_{01}(1-\beta_{10})$.□

The expected time between the start and a minor repair triggered by an inspection due

to a signal in zone Z_{a2} is given by

$$E(T_{a2}) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \left\{ \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \beta_{12} (H_0 - h_1) \int_{H_1-h_0-h_1}^{H_1-h_1} f_1(x_1)(1 - F_2(H_0 - h_1 - x_1)) dx_1 \right\}. \tag{10}$$

Proof. The proof is similar to that of $E(T_{a1})$, apart from the appearance of the out-of-control signals in a longer interval h_0 in this case. □

The expected time between the start and a major repair is given by

$$\begin{aligned} E(T_{a3}) = & \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \left\{ \int_{H_0-h_0-h_1}^{H_0-h_1} \int_{H_1-h_0-h_1}^{\tau_{k_1k_2k_3}} x f_1(x_1) f_2(x - x_1) dx_1 dx \right. \\ & + \beta_{11} \int_{H_0-h_1}^{H_0} \int_{H_1-h_0-h_1}^{\tau_{k_1k_2k_3}} x f_1(x_1) f_2(x - x_1) dx_1 dx \\ & + \int_{H_0-h_0-h_1}^{H_0-h_1} \int_{H_1-h_1}^{\tau'_{k_1k_2k_3}} x f_1(x_1) f_2(x - x_1) dx_1 dx \\ & \left. + \beta_{11} \int_{H_0-h_1}^{H_0} \int_{H_1-h_1}^{\tau'_{k_1k_2k_3}} x f_1(x_1) f_2(x - x_1) dx_1 dx \right\}. \end{aligned} \tag{11}$$

where $\tau_{k_1k_2k_3} = \begin{cases} H_1 - h_1 \text{ if } k_1 - k_2 \neq 0 \\ x \qquad \text{if } k_1 - k_2 = 0, \end{cases}$ and $\tau'_{k_1k_2k_3} = \begin{cases} H_1 \text{ if } k_1 - k_2 \neq 0 \\ x \quad \text{if } k_1 - k_2 = 0. \end{cases}$

Proof. The system might transit from the in-control state to the out-of-control state either in a longer sampling interval h_0 or in a shorter sampling interval h_1, and the system can then fail in either h_0 or h_1 as well, which creates four scenarios. The first two components in Eq (11) correspond to the scenarios when the transition from the in-control state to the out-of-control state occurs in a h_0, and the last two components in Eq (11) correspond to the scenarios when the transition from the in-control state to the out-of-control state occurs in a h_1.

The first component in Eq (11) is the scenario when the two transitions (i.e., from the in-control state to the out-of-control state and then fail) occur in longer sampling intervals.

The second component means that a correct signal appears in a longer sampling interval h_0 (with a probability β_{11}) followed by a shorter sampling interval h_1 for confirmation, but the system fails within this h_1. The third component means that the transition from the in-control state to the out-of-control state occurs (with a probability β_{01} followed by a shorter sampling interval). The last component means that the two scenarios occur in short sampling intervals. □

3.3.2 Expected renewal cycle cost

The costs incurred during periods $E(T_{f1})$, $E(T_{f2})$, and $E(T_{f3})$ are derived in the following.

$$\begin{aligned} E(C_{a1}) = & \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \left\{ \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \beta_{11}(1-\beta_{10}) \int_{H_1-h_0-h_1}^{H_1-h_1} C_0 f_1(x_1)(1-F_2(H_0-x_1))dx_1 \right. \\ & \left. + \beta_{01}\beta_{10}\beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \beta_{11}(1-\beta_{10}) \int_{H_1-h_1}^{H_1} C_1 f_1(x_1)(1-F_2(H_0+h_0+h_1-x_1))dx_1 \right\} \\ & + \sum_{k_2=0}^{\infty} \left\{ \beta_{01}(1-\beta_{10}) \int_{H_2-h_1}^{H_2} C_2 f_1(x_1)(1-F_2(H_2-x_1))dx_1 \right\}, \end{aligned} \tag{12}$$

where $C_0 = (k_1 + \beta_{01}(k_2-1) + k_3 + 1)nc_s + ((\beta_{02} + \beta_{01}(1-\beta_{00}))(k_2-1)+1)c_i + c_{r1}$, $C_1 = C_0 + 2nc_s$, and $C_2 = k_2 nc_s + \beta_{01}k_2 nc_s + 2nc_s + (\beta_{02} + \beta_{01}(1-\beta_{00}))k_2 c_i + c_i + c_{r1}$.

Proof. After the system transited from an in-control state to an out-of-control state within a longer interval h_0, there will be two possible scenarios before two warning signals appear consecutively in a longer interval and shorter interval, respectively. The first scenario is incorrect signals (with a probability of β_{10}) appears, the second scenario is a correct signal followed by an incorrect signal (with a probability of $\beta_{11}\beta_{10}$). These two scenarios makes up an event with a probability of $(\beta_{10})^{k_1-k_2-k_3}(\beta_{11}\beta_{10})^{k_3}\beta_{11}(1-\beta_{10})$, and the event incurs sampling cost $(k_1-k_2+1)nc_s + (k_3+1)nc_s$. Before the system has transited from the in-control state to the out-of-control state, the sampling cost is $k_2 nc_s + \beta_{01}k_2 nc_s$, inspection cost $(\beta_{02} + \beta_{01}(1-\beta_{00})k_2 + 1)c_i$ and level 1 repair cost c_{r1}. Hence, the sub-total cost is C_0.

The system might also transit from the in-control state to the out-of-control state within a short interval h_1 after a false signal appear in the in-control state. This event incurs cost $nc_s + c_i + c_{r1}$. The cost incurred before the transition is $k_2 nc_s + \beta_{01} k_2 nc_s + \beta_{01}\beta_{01} k_2 c_i$. The sub-total cost is C_1.

A similar explanation to the third component in Eq (12) can be given. □

Similarly, we have

$$E(C_{a2}) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \left\{ \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \beta_{12} \int_{H_1-h_0-h_1}^{H_1-h_1} C_3 f_1(x_1)(1 - F_2(H_0 - h_0 - x_1))dx_1 \right\} \tag{13}$$

where $C_3 = C_0 - nc_s$. Finally, we have

$$\begin{aligned} E(C_{a3}) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} \sum_{k_3=0}^{k_1-k_2} \beta_{10}^{k_1-k_2} \beta_{11}^{k_3} \Bigg\{ & \int_{H_0-h_0-h_1}^{H_0-h_1} \int_{H_1-h_0-h_1}^{\tau_{k_1k_2k_3}} C_4 f_1(x_1) f_2(x - x_1) dx_1 dx \\ & + \beta_{11} \int_{H_0-h_1}^{H_0} \int_{H_1-h_0-h_1}^{\tau_{k_1k_2k_3}} C_5 f_1(x_1) f_2(x - x_1) dx_1 dx \\ & + \int_{H_0-h_0-h_1}^{H_0-h_1} \int_{H_1-h_1}^{\tau'_{k_1k_2k_3}} C_6 f_1(x_1) f_2(x - x_1) dx_1 dx \\ & + \beta_{11} \int_{H_0-h_1}^{H_0} \int_{H_1-h_1}^{\tau'_{k_1k_2k_3}} C_7 f_1(x_1) f_2(x - x_1) dx_1 dx \Bigg\}. \end{aligned} \tag{14}$$

where $C_4 = (k_1 + \beta_{01}(k_2 - 1) + k_3 - 1)nc_s + ((\beta_{02} + \beta_{01}(1 - \beta_{00}))(k_2 - 1))c_i + c_{r2}$, $C_5 = C_4 + nc_s + c_i$, $C_6 = C_4$, and $C_7 = C_5$.

Hence, the expected cost per time unit is given by

$$E_a(T, C) = \frac{E(C_{a1}) + E(C_{a2}) + E(C_{a3})}{E(T_{a1}) + E(T_{a2}) + E(T_{a3})}. \tag{15}$$

4 Discussion

4.1 Comparison between the two types of control charts

For the static control chart, if we set UCL=∞ and $n = 0$, then Eq. (6) becomes

$$\begin{aligned} E(T_f) &= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \int_{(i-1)h}^{ih} \int_{(j-1)h}^{\tau_{ij}} x f_1(x_1) f_2(x - x_1) dx_1 dx \\ &= \int_0^{\infty} x \int_0^{x} f_1(x_1) f_2(x - x_1) dx_1 dx \end{aligned} \tag{16}$$

Eq. (7) becomes

$$\begin{aligned} E(C_f) &= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \int_{(i-1)h}^{ih} \int_{(j-1)h}^{\tau_{ij}} c_{r2} f_1(x_1) f_2(x - x_1) dx_1 dx, \\ &= c_{r2} \int_0^{\infty} \int_0^{x} f_1(x_1) f_2(x - x_1) dx_1 dx. \end{aligned} \tag{17}$$

Then the expected cost per time unit, $\frac{E(C_f)}{E(T_f)}$ based on Eq (17) and Eq (16), is the same as Eq (3) where no control chart is employed.

Similarly, if we set $Z_{a1} = \phi$, or $\beta_{01} = \beta_{11} = 0$, then the adaptive np chart becomes the static np control chart. Then we have $E(T_{a1}) = E(C_{a1}) = 0$, $E(T_{a2}) = E(T_{f1})$, $E(T_{a3}) = E(T_{f2})$, and $E(C_{a2}) + E(C_{a3}) = E(C_f)$. The expected costs per time unit from the adaptive and static np control charts are the same.

By minimising Eq. (8) and Eq. (15), we can obtain the optimal h when using static control charts and h_0 and h_1 when using adaptive control charts, respectively. For given cdf's ($F_1(x_1)$ and $F_2(x_2)$) and costs, h, h_0 and h_1 can be obtained by using heuristic search algorithms such as genetic algorithms and simulated annealing [19].

4.2 Design procedure

When designing an optimal inspection policy, one can adopt the following steps.

It is reasonable to assume that in practice we can obtain cost parameters c_s, c_i, c_{r1}, c_{r2}, cumulative distribution functions $F_1(x_1)$ and $F_2(x_2)$.

If the values of α_0, α_0 (where $\alpha_0 + \alpha_1 = 1$), β_{00}, β_{01}, and β_{02} (where $\beta_{00} + \beta_{01} + \beta_{02} = 1$), β_{10}, β_{11}, and β_{12} (where $\beta_{10} + \beta_{11} + \beta_{12} = 1$) are given, we can minimise the expected costs in Eq. (8) and Eq. (15), respectively, to obtain the optimal values h_0, h_1, and n.

5 Numerical experiments

In this section, we use both artificial generated data and real data for validating the approaches proposed in this paper.

5.1 Artificial generated data

Assume that $F_1(x_1) = 1 - \exp(-(\frac{x_1}{300})^{2.5})$, and $F_2(x_2) = 1 - \exp(-(\frac{x_2}{200})^4)$. We also assume the parameter values in Table 1 in this example.

With the parameter values listed in Table 1, we can compare the values of the expected cost per time unit derived by Eq (8) and Eq (15). Table 2 shows the expected cost per time unit against sampling intervals h for the system monitored with the static control chart. Table 3 shows the expected cost per time unit against longer sampling intervals h_0 and shorter sampling intervals h_1. For example, when h_0 is 80, then $h_1 = 48$ makes the expected value per time unit minimised (the minimised value is 3.510). This experiment offers the

following optimal solutions for the static and adaptive control charts, respectively.

- When h is 88, the expected cost per time unit achieves the minimal value 3.531847 for the static control chart.
- When $h_0 = 104$, and $h_1 = 9$, the expected long-run cost per time unit is 3.306189 for the adaptive control chart.

It can be seen from the above solutions that the adaptive control chart is superior over the static control chart. However, for given sampling intervals, for example, $h_0 = h_1 = h = 40$, then the optimal expected cost per time unit, $E_a(T, C)$, incurred in the static control chart case can be smaller than in the adaptive control chart case. This is because there is cost added by an additional sampling interval h_1 immediately followed by sampling interval with another length h_0 in the adaptive control chart case.

Table 3 also indicates that for minimising the expected long-run cost per time unit a larger value h_0 should be followed by a smaller value h_1, whereas a smaller value h_0 should be followed by a larger value h_1, although in both cases $h_0 > h_1$. For example, when h_0 is smaller than 60, every long sampling interval h_0 should be followed by the same length of *short* sampling interval h_1: the values of the two columns h_1 and h_0 are the same. When h_0 is larger than 60, every long sampling interval h_0 should be followed by the smaller length of *short* sampling interval h_1.

5.2 Vibration data of bearings

Figure 3 shows the data of the overall vibration level in root mean square (RMS) value of six bearings. The data were collected from an accelerated laboratory experiment. It has been generally accepted in the case of vibration monitoring that a consistent increase in the overall vibration level is a good indicator of the presence of a hidden defect which

indicates clearly a two-stage process [2,8].

If we use RMS=4.15 as a threshold: the RMS larger than 4.15 implies that the process is in the failure delay time and the RMS less than 4.15 suggests that the process is in the normal working stage. We can obtain the exact time before each bearing passes over the threshold. By treating these data as time-to-failure data and using maximum likelihood estimation, we can obtain that $F_1(x_1) = 1 - \exp(-(\frac{x_1}{161.58})^{0.72})$. Similarly, we can obtain $F_2(x_2) = 1 - \exp(-(\frac{x_2}{78.62})^{1.80})$. Assume c_s=0.08, c_i=80, c_{r1}=1500, c_{r2}=5000, and $n = 5$, and the rest parameter values are the same as those in Table 1. Then we can obtain the optimal solution $E_a(T, C) = 7.75$ when an adaptive control chart with $h_0 = 13$ and $h_1 = 1$ is used, whereas the optimal value $E_f(T, C) = 7.95$ when a static control with a sample interval $h = 9$ is used.

6 Conclusions

In this paper, we formulated the expected long-run cost per time unit for a three-state system monitored by an adaptive control chart with variable parameters and a control chart with static parameters. The adaptive control chart has variable sampling intervals: if the control chart signals that the system is near out-of-control at a time point, the subsequent sampling will be conducted in a shorter time period.

The numerical example shows that applying an adaptive control chart to monitor a three-state system is more cost effective than applying a static one. It also shows that for economically maintaining a three state system, an adaptive control chart should be designed to be that the length of the longer sampling interval should be inverse proportional to the length of the shorter sampling interval.

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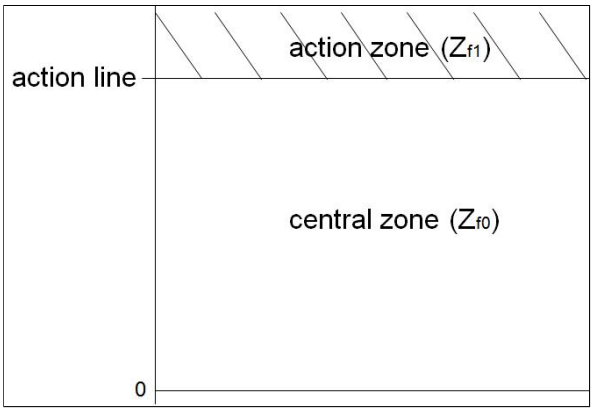


Fig. 1. A control chart with two zones

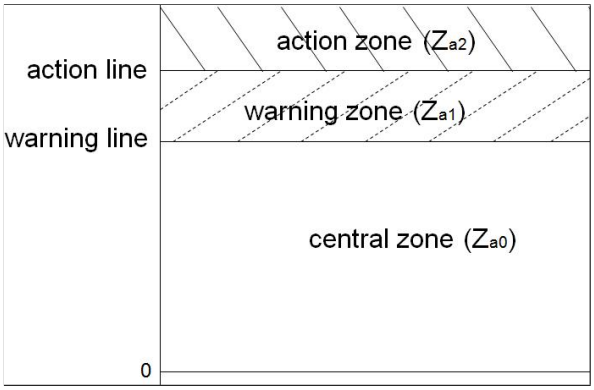


Fig. 2. A control chart with three zones

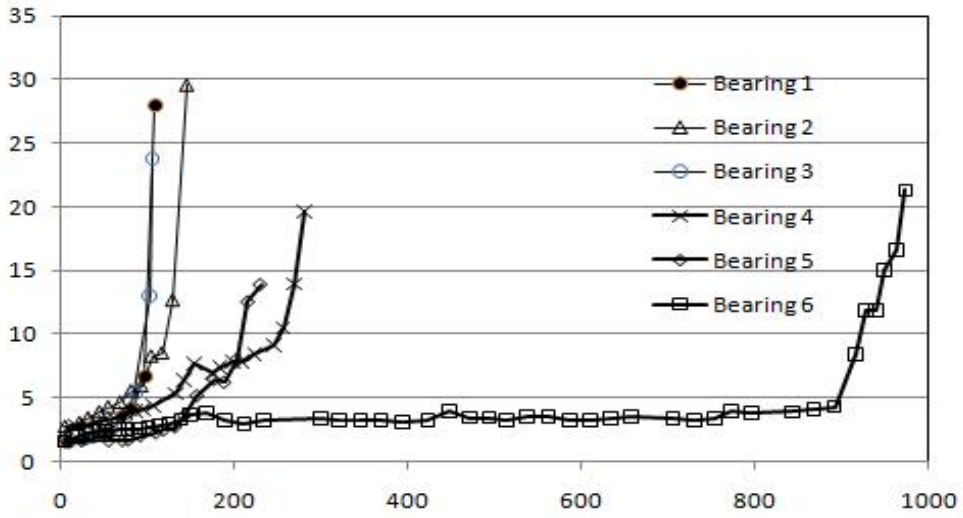


Fig. 3. Vibration data of six bearings. (X-axis: operating hours since new; Y-axis: vibration levels (rms))

Table 1

Parameters used in the numerical example.

α_0	α_1	β_{00}	β_{01}	β_{02}	β_{10}	β_{11}	β_{12}	c_s	c_i	c_{r1}	c_{r2}	n
0.95	0.05	0.833	0.147	0.02	0.02	0.08	0.9	1	100	500	5000	100

Table 2

The expected cost per time unit of different sampling intervals.

h	cost	h	cost	h	cost	h	cost	h	cost	h	cost
40	4.659	60	3.842	80	3.586	100	3.613	120	3.805	140	4.092
45	4.367	65	3.742	85	3.572	105	3.649	125	3.870	145	4.174
50	4.145	70	3.668	90	3.573	110	3.694	130	3.940	150	4.259
55	3.974	75	3.617	95	3.587	115	3.746	135	4.014		

Table 3

The expected cost per time unit with changing longer and shorter sampling intervals.

h_0	h_1	$E_a(T,C)$	h_0	h_1	$E_a(T,C)$	h_0	h_1	$E_a(T,C)$	h_0	h_1	$E_a(T,C)$
40	36	4.684	70	49	3.646	100	10	3.374	130	3	3.431
45	45	4.391	75	52	3.572	105	9	3.354	135	3	3.471
50	50	4.162	80	48	3.510	110	4	3.351	140	3	3.520
55	55	3.983	85	34	3.457	115	4	3.357	145	3	3.577
60	60	3.842	90	27	3.422	120	4	3.372	150	3	3.642
65	58	3.730	95	19	3.387	125	3	3.398			



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