

Closed-form dynamic stiffness formulation for modal and dynamic response analysis of pile group foundations

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Abstract

A closed-form dynamic stiffness (DS) formulation is proposed for pile group foundations. This model consists of a rigid cap and an arbitrary number of flexible piles. The rigid cap, connecting to all piles, is modelled as a rigid body resting on an elastic foundation. For each pile, a longitudinal soil-pile model and a preloaded Timoshenko beam on viscoelastic Pasternak foundation are adopted to describe the longitudinal and transverse vibration, respectively. The piles can be either friction piles or end-bearing piles, and pile sections surrounded by different soil layers can be easily modelled by the assembly of pile elements with different parameters. The proposed DS formulations essentially use very few degrees of freedom to, nevertheless, describe the broadband dynamic behaviors of complex pile group foundations in a highly accurate manner. The DS formulation is then used for modal and dynamic response analysis. For modal analysis, the Wittrick-Williams algorithm is applied, with the key issue J_0 count problems resolved. The proposed method is verified against some existing results. This research provides an exact and highly efficient modal and response analysis tool for pile group foundations, which can be used for the dynamic analysis of pile foundations subjected to earthquake or traffic excitations.

Keywords: Pile group foundations; Viscoelastic Pasternak foundation; Dynamic stiffness method; Wittrick-Williams algorithm; Modal analysis; Frequency response analysis;

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1. Introduction

The widely used pile group foundation is composed of a cap connected to the tops of a group of piles. It has the characteristics of high bearing capacity, excellent stability, small and uniform settlement, strong earthquake resistance, and strong adaptability. The pile group foundations play an important role in many fields, e.g., offshore wind turbines (Bhattacharya and Adhikari, 2011), bridges (Xie and Sun, 2021), highway and railway structures (Dong et al., 2018), amongst many others. It is obvious that the dynamic modelling, modal and dynamic response analysis of the pile group foundations are of great significance to the engineering structure design. In terms of theoretical modelling for pile foundations considering pile-soil interaction, numerous studies have been conducted. For a single pile, different models have been proposed, such as the Winkler model (Winkler, 1867), the Pasternak model (Pasternak, 1954), the plane strain model (also known as the Novak model) (Novak, 1974; Novak et al., 1978), the Tajimi-type model (Tajimi, 1969, 1976), etc. Amongst these, the Winkler model is one of the most widely used models and have been improved by many researchers, e.g., (Nogami and Konagai, 1986; Novak et al., 1978; Novak and Beredugo, 1972). In the Winkler model, the pile is considered as a beam embedded in soil medium of foundation, and the interaction between the soil and the piles is modelled by using unconnected springs. Novak (1977) then considered the damping effect of the soil by assuming that the spring and damper were connected in parallel. Later, Nogami and Konagai (1986, 1987, 1988) proposed several improved models based on this, where both the damping coefficient and spring stiffness coefficient are frequency independent. To model the soil-pile kinematic interaction, the Winkler model was improved by assuming the spring and damper are frequency-dependent (Anoy-

52 atis et al., 2013a; Anoyatis and Mylonakis, 2012; Mylonakis and Gazetas, 2002; Nikolaou
53 et al., 2001; Novak and Aboul-Ella, 1978), such as the rod/beam on dynamic Winkler
54 foundation (R/BDWF) model. The frequency-dependent parameters can be determined
55 by other theoretical models (Gazetas and Dobry, 1984; Novak et al., 1978), by using
56 calibrations based on numerical simulations (Kavvads and Gazetas, 1993; Roesset, 1980)
57 or by experimental measurements. Different from the Winkler model, Pasternak (1954)
58 developed a two-parameter model by including the effect of cohesion and continuity of the
59 soil with a shear layer, which have been further developed and used in many researches
60 later (Filipich and Rosales, 2002; Zhang et al., 2018a,b). Different from the above equiva-
61 lent parameter models, the plane strain model (Novak, 1974; Novak et al., 1978) was first
62 proposed by Novak in 1974 by considering the soil around the pile as a series of infinitely
63 extended homogeneous and isotropic viscoelastic thin layers. Following Novak's work,
64 quite a few researchers have investigated the soil-pile interaction problems based on the
65 plane strain model (Gan et al., 2020; Militano and Rajapakse, 1999; Zheng et al., 2017).
66 However, the plane strain model does not consider the formation of standing waves in the
67 soil around the pile, resulting in its inability to involve high frequency loads (Zheng et al.,
68 2022a). The Tajimi-type model (Tajimi, 1969, 1976) is one of the particularly popular
69 continuum medium models, and it considers the soil as a linear visco-elastic layer with
70 hysteretic type damping, which has attracted much attention (Anoyatis et al., 2013b,
71 2017; Anoyatis and Lemnitzer, 2017; Nogami and Novák, 1976; Zheng et al., 2022a,b).
72 For example, Anoyatis et al. (2017) evaluated the behavior of a laterally loaded pile with
73 fixed-tip using improved Tajimi-type model, which were compared with the Winkler model
74 results. Zheng et al. (2022a) analysis the vertical dynamic response of single floating piles

in homogeneous viscoelastic soil based on Tajimi-type model. Meanwhile, a number of re-
searches have been conducted for the pile groups considering the soil–structure kinematic
interaction (Dai et al., 2017; Dezi et al., 2009; Di et al., 2017; Dobry and Gazetas, 1988;
Luan et al., 2020, 2019; Nozoe, 1999; Wen et al., 2015). For example, Dezi et al. (2009)
developed a finite element model for the soil–structure kinematic interaction of pile groups
in layered soils using Winkler-type medium. Di et al. (2017) investigated the influence
of the rotational component of the input motion caused by the kinematics interaction
of pile groups and the surrounding soil on the seismic behavior of the structure. Wen
et al. (2015) proposed a simplified method by replacing neighboring piles with a single
equivalent pile, and analyzed the impedance and foundation input motion of foundations
supported by pile groups. Luan et al. (2020) developed a coupled pile-to-pile interaction
model for the dynamic analysis of pile groups subjected to vertical loads, by considering
the secondary waves generated by the vibration of the receiver pile.

Apart from pile-foundation models, there are mainly two types of analysis methods for
the dynamic characteristics of single pile and pile group foundations, namely, simulation
analysis and experimental analysis. With reference to simulation analysis, many methods
have been applied, including the finite element method (FEM) (Avramidis and Morfidis,
2006; Iovino et al., 2019, 2022; Vassiliou et al., 2017; Wu and Finn, 1997a,b; Yokoyama,
1996), boundary element method (BEM) (Fattah et al., 2012; Mindlin, 1936; Nogami
and Novak, 1976), substructure method (Gazetas, 1984; Mylonakis et al., 2015; Wang
et al., 2021), differential transform method (DTM) (Attarnejad et al., 2010; Balkaya
et al., 2009; Vega-Posada et al., 2020), transfer matrix method (TMM) (Nogami and
Paulson, 1985; Zheng et al., 2021), etc. For example, Wu and Finn (1997a,b) proposed

an approximate three-dimensional finite element (FE) analysis method based on simple three-dimensional wave equation, which is used to calculate feasible dynamic response analysis of pile foundation in practical applications. Yokoyama (1996) applied the FEM to the vibration analysis of a uniform Timoshenko beam-column on a two-parameter elastic foundation. Vassiliou et al. (2017) presented a FE model for the seismic response analysis of deformable rock bodies and frames. Iovino et al. (2019, 2022) investigated the filtering effect in seismic response analysis and foundation design by means of a FE model. On the other hand, Mindlin (1936) firstly applied the BEM to solve the static problems of elastic half-space pile foundations. Then, Nogami and Novak (1976) used the BEM to obtain approximate solutions for the dynamic response of pile foundations under harmonic loading. Fattah et al. (2012) applied the BEM to analyze the pile-soil interaction of a group pile foundation consisting of nine piles, considering the effects of non-linear pile-soil interaction on the single pile force. Wang et al. (2021) developed a substructure method for simulating the pile-soil dynamic interaction during horizontal vibration in the time domain. Vega-Posada et al. (2020) proposed a simplified analytical method based on the DTM for analyzing soil-structure interactions for homogeneous or inhomogeneous Pasternak foundations. Nogami and Paulson (1985) analysed the nonlinear pile group response by using the TMM. Zheng et al. (2021) applied the TMM to the buckling analysis of end-bearing piles partially or fully embedded in a layered elastic medium. Chen et al. (2004) analysed the vibration of beams resting on a Pasternak elastic foundation by combining the state space method and the differential quadrature method. Tehrani et al. (2016) developed a semi-analytical method for the response of pile groups subjected to lateral loading in multi-layered elastic strata. Luan et al. (2022) presented an analytical

solution of the dynamic response of pile groups in layered soil under dynamic lateral loads. With reference to experimental aspects, the experiments are usually carried out in the field (Blaney et al., 1987; El-Marsafawi et al., 1992) or by using the shaking table or centrifuge experiments in the laboratory (Cao et al., 2022; Garala et al., 2020; Khari et al., 2015; Mizuno et al., 1984; Sun and Xie, 2019; Zhang et al., 2022). For instance, Sun and Xie (2019) investigated the effects of soil–structure interaction on a super long-span cable-stayed bridge model with pile group foundations by shaking table experiment. Zhang et al. (2022) conducted a centrifuge shaking table experiment to study the dynamic response of vertical and batter pile groups in saturated sand. Khari et al. (2015) carried out a series of experimental investigations on single and group pile subjected to lateral loadings. Garala et al. (2020) conducted a series of dynamic centrifuge experiments to investigate the kinematic effects on pile foundations. Cao et al. (2022) compared the dynamic stiffness of floating single pile and pile groups in multi-layered soil by a series of experimental results and analytical analysis results. Although these methods are all powerful approaches for the dynamic characteristics of single pile and pile group foundations, the experimental work requires specific experimental equipment and settings, and most of the simulations by numerical methods require a large amount of degrees of freedom, resulting in high computational costs and low efficiency, and simulations by analytical methods are only applicable to simple pile foundations.

Another powerful analysis tool that has shown potential in the dynamic analysis of structures is the dynamic stiffness method (DSM). The method is an analytical and exact method since its shape functions are based on the exact general solution of the frequency-dependent governing differential equations. It can also reduce the number of degrees of

freedom for the dynamic characteristics of single pile and pile group foundations, and can be solved by an efficient and reliable eigenvalue algorithm. The DSM has been widely applied to the free vibration and forced vibration of rod (Liu et al., 2022), beam (Banerjee et al., 2021; Sun et al., 2022), plate (Boscolo and Banerjee, 2011; Chauhan et al., 2022; Li et al., 2018), multi-layered structures (Ba et al., 2022), amongst others. For example, Sun et al. (2022) developed the DS matrix for free vibration analysis of partial-interaction composite beams. Chauhan et al. (2022) applied the DSM to the free vibration analysis of sigmoid functionally graded plates embedded on Winkler-Pasternak foundation. Ba et al. (2022) developed the seismograms synthesis for layered transversely isotropic half-space based on the DSM. Moreover, the DSM can also be extended to the modal and response analysis of pile foundations. For example, Ruge and Birk (2007) derived the translational and rotational dynamic stiffnesses of Timoshenko and Euler-Bernoulli beams on Winkler foundations and performed a dynamic analysis of the infinitely long beam model. Bozyigit et al. (2020) and Bozyigit (2021) conducted the free vibration analysis of soil-structures coupling system and analysed the seismic response of pile supported frames using the DSM. They also demonstrated the accuracy and high efficiency of the DSM. Cairo and Conte (2006) studied the nonlinear dynamic response of a group pile under vertical loads by using the DSM. However, few researches have combined the longitudinal and transverse vibration together to investigate the dynamic behaviours of pile group foundations and more importantly, analytical model of pile group foundations with a rigid cap has not been reported. Therefore, the main purpose of this paper is to establish a dynamic stiffness model that combines a longitudinal soil-pile model and a preloaded Timoshenko beam on viscoelastic Pasternak foundation, and to extend the DSM to the modal and response

analysis of pile group foundations with a rigid cap by integrating the exact analytical dynamic stiffness formulations and the Wittrick-Williams algorithm.

The paper is organized as follows. Following this Introduction, Section 2 develops the DS matrix for a general pile group foundation. The longitudinal and transverse vibrations of a pile are demonstrated by a vertical soil-pile model combined with a Timoshenko beam on elastic foundation, respectively. The matrices are then assembled with the rigid cap to form the DS matrix for a pile group foundation. Then, Section 3 describes the solution technique for both modal and response analysis. Section 4 first validates the proposed method, then conducted the modal and response analysis for a pile group foundation. Finally, Section 5 concludes the paper.

2. Dynamic stiffness formulation for a pile group foundation

This section gives the detailed dynamic stiffness formulation for a pile group foundation. Fig.1 shows a pile group foundation with a rigid cap. The cap is considered as a rigid body with lumped mass m and moment of inertia I_α . The piles are modelled as beams with longitudinal and transverse vibrations, i.e., each node of the element has three degrees of freedom. Appropriate soil-pile models are considered for both longitudinal and transverse vibrations of the pile, as illustrated in Fig.1(c). Meanwhile, the bottom ends of the piles can be either fixed (Fig.1(d)) or constrained by springs (Fig.1(e)).

Section 2.1 gives the explicit derivation of the dynamic stiffness (DS) formulation for a single pile-foundation element. Section 2.2 describes the DS formulation for a pile group foundation and the application of the boundary conditions.

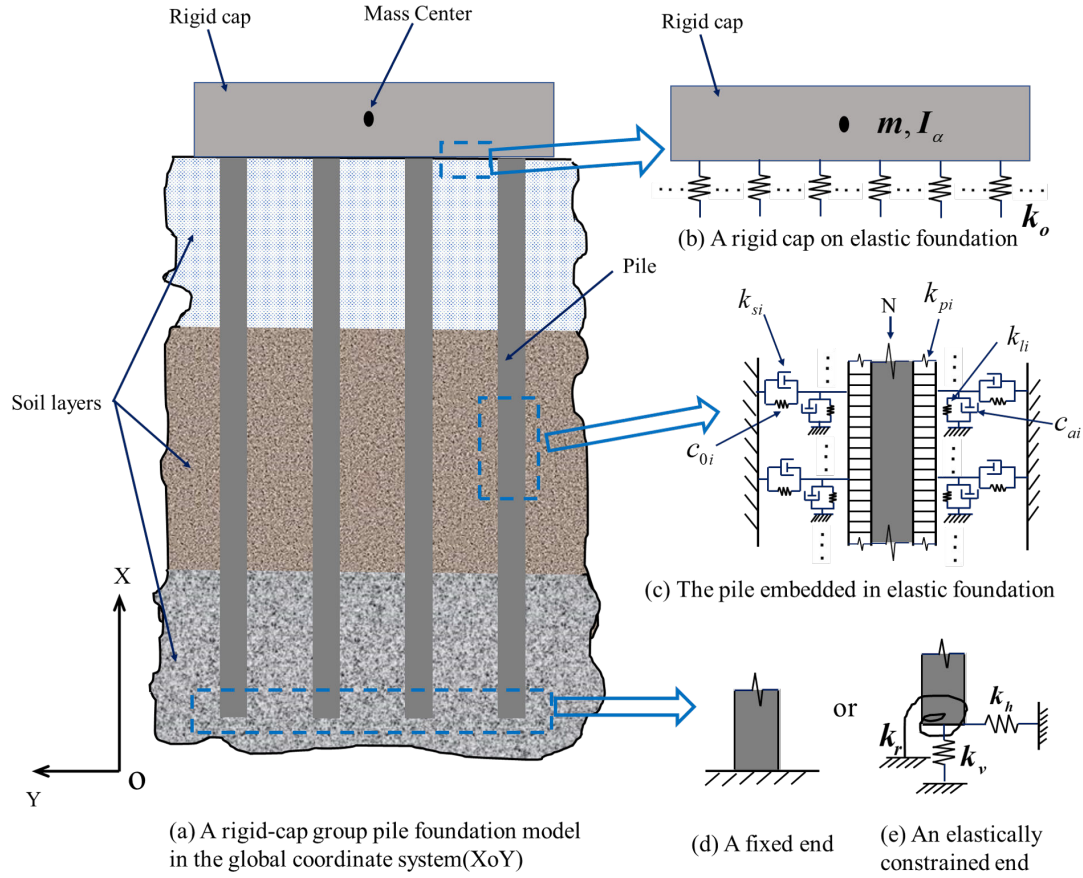


Fig. 1. A pile group foundation model and the boundary conditions.

2.1. DS formulation for a single pile element

This section develops the DS formulations for the longitudinal and transverse vibration of a pile element, shown as Fig.1(c). To consider the pile-soil interaction, classical rod theory with spring and viscoelastic damping is adopted for longitudinal vibration Fig.2(a) and a preloaded Timoshenko beam on three-parameter viscoelastic Pasternak foundations is used for transverse vibration Fig.2(b).

2.1.1. DS formulation for longitudinal vibration of a single pile element

A longitudinal soil-pile model used to describe the longitudinal vibration of a pile is shown in Fig.2(a). In this model, rods are assumed to be constrained by distributed spring (with stiffness k_l) and viscoelastic damping (c_a) in the longitudinal direction. The

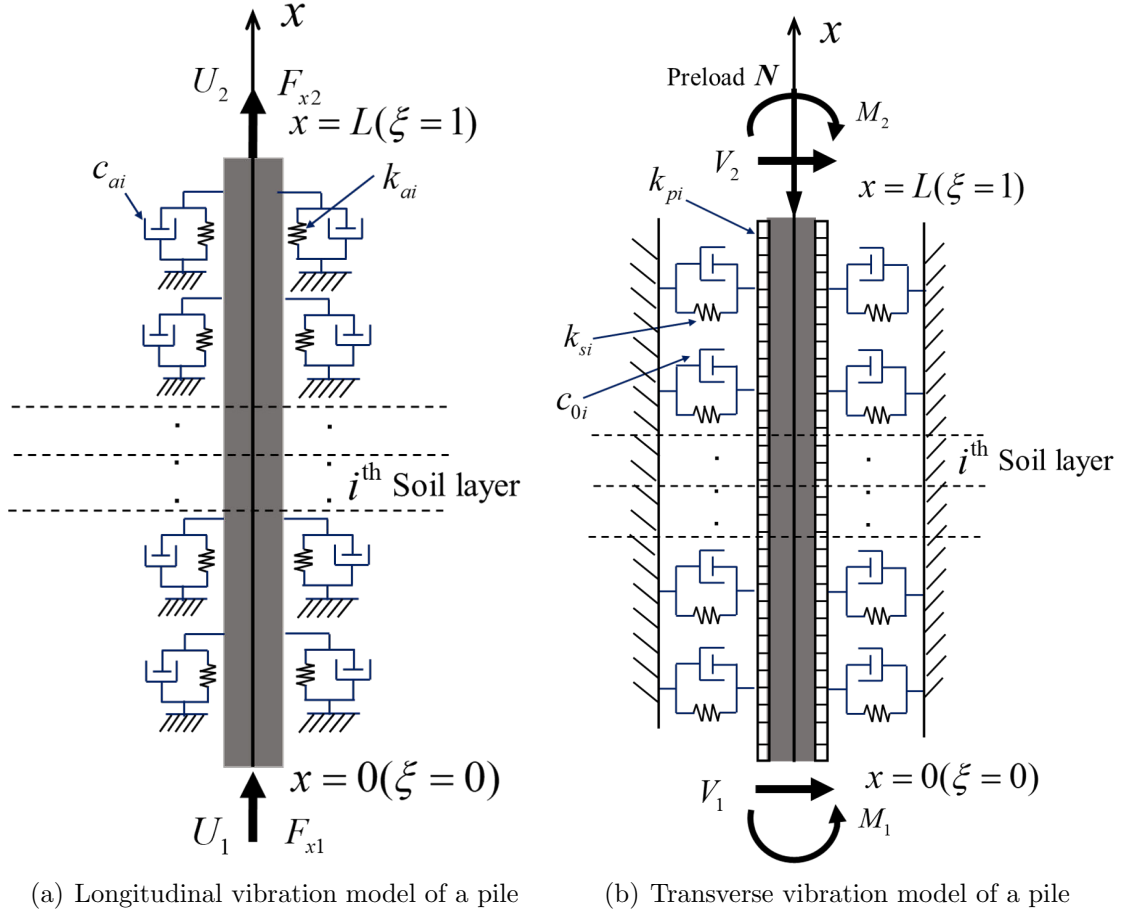


Fig. 2. A pile foundation model undergoing both longitudinal and transverse vibrations.

governing differential equation (GDE) in the time domain is

$$EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} - k_a u - c_a \frac{\partial u}{\partial t} = 0 \quad (1)$$

where EA is the longitudinal stiffness of the rod, ρA is the mass per unit length of the rod, $u(x, t)$ is the longitudinal displacement of the rod. k_l and c_a are the stiffness coefficient and damping coefficient describing the longitudinal friction of the soil layer, which can be assumed to be a constant or frequency dependent. The expression for internal longitudinal force $f(x, t)$ is

$$f(x, t) = EA \frac{\partial u}{\partial x} \quad (2)$$

Introducing harmonic oscillation assumption $u(x, t) = U(x)e^{i\omega t}$ to Eq.(1) gives the governing differential equation (GDE) in the frequency domain.

$$EA \frac{d^2 U}{dx^2} + (\rho A \omega^2 - k_a - i\omega c_a) U = 0 \quad (3)$$

Also, the longitudinal force amplitude of Eq.(2) becomes

$$F(x) = EA \frac{dU}{dx} \quad (4)$$

Then the exact shape function can be obtained

$$U(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) \quad (5)$$

where $\alpha = \sqrt{\frac{\rho A \omega^2 - k_l - i\omega c_a}{EA}}$, and C_1 and C_2 are unknown constants. Applying the boundary conditions for displacements and forces at both ends yields

$$U(0) = U_1, F(0) = -F_1, (x = 0) \quad (6a)$$

$$U(L) = U_2, F(L) = F_2, (x = L) \quad (6b)$$

Substituting Eq.(6) into Eqs.(4) and (5) and eliminating the unknown coefficients C_1 and C_2 , we can obtain the relationship between the force and displacement amplitudes of longitudinal vibration at both ends of the pile element, i.e.

$$\mathbf{f}_l = \mathbf{K}_l \mathbf{d}_l \quad (7)$$

where $\mathbf{f}_l = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^T$ and $\mathbf{d}_l = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$ represent respectively the column vectors of the force and displacement amplitudes at both ends of the rod element. $\mathbf{K}_a$ is the dynamic stiffness matrix for the longitudinal soil-pile model in the local coordinate system, namely

$$\mathbf{K}_l = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix} \quad (8)$$

where $e_1 = EA\alpha \cot(\alpha L)$, $e_2 = -EA\alpha \csc(\alpha L)$.

2.1.2. DS formulation for transverse vibration of a single pile element

In this section, a preloaded Timoshenko beam on three-parameter viscoelastic Pasternak foundations is used to describe the transverse vibration of a pile element, as shown in Fig.2(b) and the DS formulation is developed. The governing differential equation (GDE) is expressed in the following form (Chen, 1993)

$$-\rho A \frac{\partial^2 w}{\partial t^2} + \kappa AG \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} - \theta \right) - N \frac{\partial^2 w}{\partial x^2} - c_0 \frac{\partial w}{\partial t} - k_s w + k_p \frac{\partial^2 w}{\partial x^2} = 0 \quad (9)$$

$$-\rho I \frac{\partial^2 \theta}{\partial t^2} + EI \frac{\partial^2 \theta}{\partial x^2} + \kappa AG \left(\frac{\partial w}{\partial x} - \theta \right) = 0 \quad (10)$$

where I is the moment of inertia of cross section, G is the shear modulus, κ is the shear coefficient or shape factor, N is the axial preload (being positive when compressive), $w(x, t)$ and $\theta(x, t)$ are the transverse deflection and bending slope, respectively. The transverse foundation stiffness k_s , foundation shear stiffness k_p and damping parameter c_0 can be assumed to be a constant or frequency dependent. The associated natural boundary conditions are

$$M = -EI \frac{\partial \theta}{\partial x} \quad (11a)$$

$$V = -\kappa AG \left(\frac{\partial w}{\partial x} - \theta \right) + N \frac{\partial w}{\partial x} - k_p \frac{\partial w}{\partial x} \quad (11b)$$

By introducing harmonic oscillation assumption,

$$w(x, t) = W(x) e^{i\omega t} \quad (12a)$$

$$\theta(x, t) = \Theta(x) e^{i\omega t} \quad (12b)$$

where $W(x)$ and $\Theta(x)$ are the amplitudes of the bending displacement and rotation, respectively. Introducing $\xi = x/L \in [0, 1]$ and substituting Eq.(12) into Eqs.(9) and (10)

232 give

$$\begin{bmatrix} \frac{1}{L^2}(\kappa AG + k_p - N)\lambda^2 + \rho A\omega^2 - k_s - i\omega c_0 & -\frac{1}{L}\kappa AG\lambda \\ \frac{1}{L}\kappa AG\lambda & \frac{1}{L^2}EI\lambda^2 - \kappa AG + \rho I\omega^2 \end{bmatrix} \begin{bmatrix} W \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13)$$

233 where $\lambda = \frac{d}{d\xi} = L \frac{d}{dx}$. Setting the determinant of the coefficient matrix of Eq.(13) equal

234 to zero gives the characteristic equation

$$\lambda^4 + a_1\lambda^2 + a_2 = 0 \quad (14)$$

235 where

$$\begin{aligned} a_1 &= \frac{b^2(r^2 + s^2) - s^2K_s^2 + (K_p^2 - p^2)(b^2r^2s^2 - 1)}{1 + s^2(K_p^2 - p^2)}, \\ a_2 &= \frac{(b^2 - K_s^2 - i\omega C_0^2)(b^2r^2s^2 - 1)}{1 + s^2(K_p^2 - p^2)} \\ 236 \quad b^2 &= \frac{\rho A\omega^2 L^4}{EI}, r^2 = \frac{I}{AL^2}, s^2 = \frac{EI}{\kappa AGL^2} \\ p^2 &= \frac{NL^2}{EI}, K_s^2 = \frac{k_s L^4}{EI}, K_p^2 = \frac{k_p L^2}{EI}, C_0^2 = \frac{c_0 L^4}{EI} \end{aligned}$$

237 The roots of Eq.(14) is given by

$$\lambda^2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \quad (15)$$

238 Therefore the general solutions of Eq.(13) can be obtained, but subjected to the following

239 four different cases.

240 Case 1: $a_1^2 - 4a_2 > 0$ and $a_2 < 0$

$$W(\xi) = A_1 \cos \Phi \xi + A_2 \sin \Phi \xi + A_3 \cosh \Lambda \xi + A_4 \sinh \Lambda \xi \quad (16a)$$

$$\Theta(\xi) = B_1 \cos \Phi \xi + B_2 \sin \Phi \xi + B_3 \cosh \Lambda \xi + B_4 \sinh \Lambda \xi \quad (16b)$$

241 Case 2: $a_1^2 - 4a_2 \geq 0$ and $a_2 > 0$ and $a_1 > 0$

$$W(\xi) = A_1 \cos \Phi \xi + A_2 \sin \Phi \xi + A_3 \cos \Lambda \xi + A_4 \sin \Lambda \xi \quad (17a)$$

$$\Theta(\xi) = B_1 \cos \Phi \xi + B_2 \sin \Phi \xi + B_3 \cos \Lambda \xi + B_4 \sin \Lambda \xi \quad (17b)$$

242 Case 3 $a_1^2 - 4a_2 \geq 0$ and $a_2 > 0$ and $a_1 < 0$

$$W(\xi) = A_1 \cosh \Phi \xi + A_2 \sinh \Phi \xi + A_3 \cosh \Lambda \xi + A_4 \sinh \Lambda \xi \quad (18a)$$

$$\Theta(\xi) = B_1 \cosh \Phi \xi + B_2 \sinh \Phi \xi + B_3 \cosh \Lambda \xi + B_4 \sinh \Lambda \xi \quad (18b)$$

243 where $i\Phi^2 = \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2}$ and $j\Lambda^2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}$, with $i = 1, j = 1$ for Case 1; $i = 1, j =$
 244 -1 for Case 2; and $i = -1, j = 1$ for Case 3. and $A_1 - A_4$ and $B_1 - B_4$ are unknown
 245 constants, and the relationships between them obtained based on Eq.(13) are as follows.

$$B_1 = \frac{k_\Phi}{L} A_2; B_2 = -i \frac{k_\Phi}{L} A_1; B_3 = \frac{k_\Lambda}{L} A_4; B_4 = j \frac{k_\Lambda}{L} A_3 \quad (19)$$

246 where

$$k_\Phi = \frac{(\kappa AG + k_p - N)\Phi^2 - i(\rho A \omega^2 - k_s - i\omega c_0)L^2}{\kappa AG \Phi} \quad (20a)$$

$$k_\Lambda = \frac{(\kappa AG + k_p - N)\Lambda^2 + j(\rho A \omega^2 - k_s - i\omega c_0)L^2}{\kappa AG \Lambda} \quad (20b)$$

247 Case 4 $a_1^2 - 4a_2 < 0$

$$W(\xi) = D_1 \cosh \alpha \xi \cos \beta \xi + D_2 \cosh \alpha \xi \sin \beta \xi + D_3 \sinh \alpha \xi \cos \beta \xi + D_4 \sinh \alpha \xi \sin \beta \xi \quad (21a)$$

$$\Theta(\xi) = R_1 \cosh \alpha \xi \cos \beta \xi + R_2 \cosh \alpha \xi \sin \beta \xi + R_3 \sinh \alpha \xi \cos \beta \xi + R_4 \sinh \alpha \xi \sin \beta \xi \quad (21b)$$

248 where $\alpha = \sqrt{\frac{1}{2} \left(-\frac{a_1}{2} + \sqrt{a_2} \right)}$ and $\beta = \sqrt{\frac{1}{2} \left(\frac{a_1}{2} + \sqrt{a_2} \right)}$, $D_1 - D_4$ and $R_1 - R_4$ are unknown

249 constants with the following relationships.

$$R_1 = (k_\alpha D_3 + k_\beta D_2)/L, R_2 = (k_\alpha D_4 - k_\beta D_1)/L \quad (22)$$

$$R_3 = (k_\alpha D_1 + k_\beta D_4)/L, R_4 = (k_\alpha D_2 - k_\beta D_3)/L$$

250 with

$$k_\alpha = \frac{(\kappa AG + k_p - N)(\alpha^2 + \beta^2) + (\rho A \omega^2 - k_s - i\omega c_0)L^2}{\kappa AG(\alpha^2 + \beta^2)} \alpha \quad (23a)$$

$$k_\beta = \frac{(\kappa AG + k_p - N)(\alpha^2 - \beta^2) - (\rho A \omega^2 - k_s - i\omega c_0)L^2}{\kappa AG(\alpha^2 + \beta^2)} \beta \quad (23b)$$

251 The natural boundary conditions of the shear force (V) and bending moment (M) are

252 as follows

$$V = -\frac{\kappa AG + k_p - N}{L} \frac{\partial W}{\partial \xi} + \kappa AG \Theta \quad (24a)$$

$$M = -\frac{EI}{L} \frac{d\Theta}{d\xi} \quad (24b)$$

253 Now referring to Fig.2(b), the boundary conditions for displacement and force amplitudes

254 of transverse vibration at both ends of the pile can be applied as follows

$$W(0) = W_1, \Theta(0) = \Theta_1, V(0) = V_1, M(0) = M_1 \quad (25a)$$

$$W(1) = W_2, \Theta(1) = \Theta_2, V(1) = -V_2, M(1) = -M_2 \quad (25b)$$

255 In the following, only the derivations for Cases 1-3 are provided for conciseness since

256 that for Case 4 can be developed in a similar manner. For Cases 1-3, by substituting

257 Eq.(25) into Eqs.(16),(17) and (18), the displacement BCs can be written in the form

$$\begin{bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & k_\Phi/L & 0 & k_\Lambda/L \\ C_1 & S_1 & C_2 & S_2 \\ -ik_\Phi S_1/L & k_\Phi C_1/L & jk_\Lambda S_2/L & k_\Lambda C_2/L \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (26)$$

258 i.e.

$$\mathbf{d}_t = \mathbf{D}_t \mathbf{c} \quad (27)$$

259 Similarly, the force BCs can be written as

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} (\kappa A G k_\Phi - e \Phi) & 0 & \frac{1}{L} (\kappa A G k_\Lambda - e \Lambda) \\ i \frac{EI}{L^2} k_\Phi \Phi & 0 & -j \frac{EI}{L^2} k_\Lambda \Lambda & 0 \\ \frac{i}{L} (\kappa A G k_\Phi - e \Phi) S_1 & \frac{1}{L} (-\kappa A G k_\Phi + e \Phi) C_1 & -\frac{j}{L} (\kappa A G k_\Lambda - e \Lambda) S_2 & \frac{1}{L} (-\kappa A G k_\Lambda + e \Lambda) C_2 \\ -i \frac{EI}{L^2} k_\Phi \Phi C_1 & -i \frac{EI}{L^2} k_\Phi \Phi S_1 & j \frac{EI}{L^2} k_\Lambda \Lambda C_2 & j \frac{EI}{L^2} k_\Lambda \Lambda S_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (28)$$

260 i.e.

$$\mathbf{f}_t = \mathbf{F}_t \mathbf{c} \quad (29)$$

261 where $e = \kappa A G + k_p - N$

262 $S_1 = \sin \Phi; C_1 = \cos \Phi$ for $i = 1$;

263 $S_1 = \sinh \Phi; C_1 = \cosh \Phi$ for $i = -1$;

264 $S_2 = \sinh \Lambda; C_2 = \cosh \Lambda$ for $j = 1$;

265 $S_2 = \sin \Lambda; C_2 = \cos \Lambda$ for $j = -1$.

266 By eliminating the constant vector $\mathbf{c}$ from Eqs.(27) and (29), the DS formulation for

267 a preloaded Timoshenko beam on Pasternak elastic foundations can be represented as

$$\mathbf{f}_t = \mathbf{K}_t \mathbf{d}_t \quad (30)$$

268 where $\mathbf{f}_t = \begin{bmatrix} V_1 & M_1 & V_2 & M_2 \end{bmatrix}^T$, $\mathbf{d}_t = \begin{bmatrix} W_1 & \Theta_1 & W_2 & \Theta_2 \end{bmatrix}^T$ and $\mathbf{K}_t = \mathbf{F}_t \mathbf{D}_t^{-1}$ repre-
269 sents the DS matrix, and

$$\mathbf{K}_t = \begin{bmatrix} d_1 & d_2 & d_4 & d_5 \\ d_2 & d_3 & -d_5 & d_6 \\ d_4 & -d_5 & d_1 & -d_2 \\ d_5 & d_6 & -d_2 & d_3 \end{bmatrix} \quad (31)$$

270 The closed-form expressions of $d_1 \sim d_6$ are given in the Appendix A for four different cases.

271 2.1.3. DS matrix for a single pile embedded in different soil layers

272 After developing the explicit DS formulation, we can couple Eqs.(8) and (31) together
 273 to obtain the DS matrix for a pile section embedded in soil foundation in the following
 274 form

$$\mathbf{K}_{pile} = \begin{bmatrix} e_1 & 0 & 0 & e_2 & 0 & 0 \\ 0 & d_1 & d_2 & 0 & d_4 & d_5 \\ 0 & d_2 & d_3 & 0 & -d_5 & d_6 \\ e_2 & 0 & 0 & e_1 & 0 & 0 \\ 0 & d_4 & -d_5 & 0 & d_1 & -d_2 \\ 0 & d_5 & d_6 & 0 & -d_2 & d_3 \end{bmatrix} \quad (32)$$

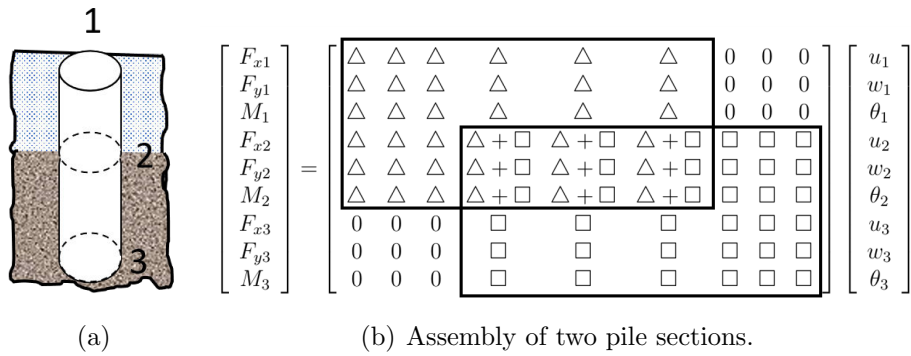


Fig. 3. Assembly procedure for a single pile embedded in two soil layers.

For a single pile surrounded by different soil layers with different properties, the DS matrix can be obtained by the assembly of pile sections. Take Fig.3(a) as an example, where a single pile is embedded in two soil layers, the assembly procedure of these two pile sections is straightforward as shown in Fig.3(b).

2.2. DS formulation for a pile group foundation

In general, the pile cap is connected to the building above and its stiffness is much larger than that of the pile, which can be idealised as a rigid body. Thus, the rigid-cap group piles structure can be regarded as a multibody system consisting of a rigid-body (⑤) and flexible beams (①-④) as shown in Fig.4. The dynamic stiffness formulation procedure of multibody systems consisting of beams and rigid-bodies is similar, which is briefly introduced here with that in (Liu et al., 2021).

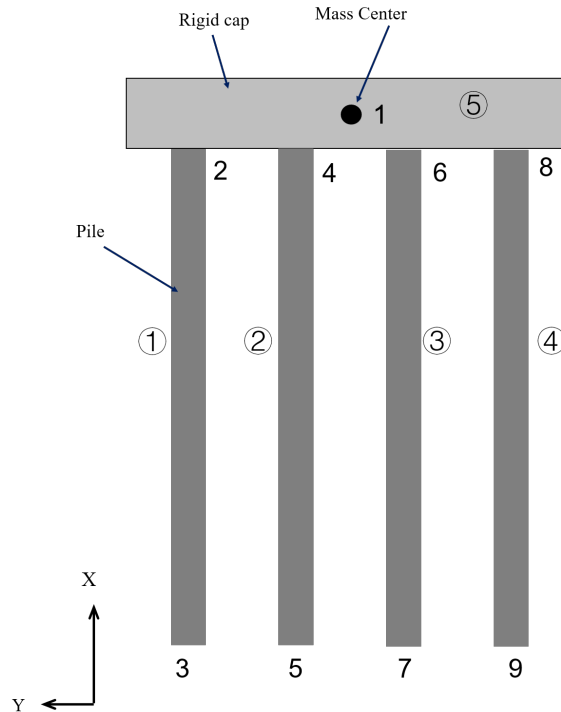


Fig. 4. A pile group foundation model.

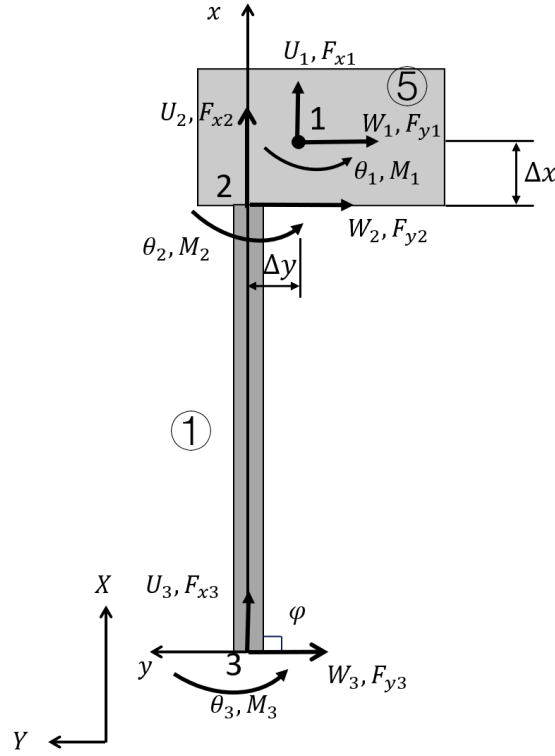


Fig. 5. A rigid cap (with mass center at node 1) connected to a flexible pile at node 2.

2.2.1. DS formulation for the coupling of a flexible pile and a rigid cap

A single pile cap element is demonstrated in Fig.5. Node 1 is the mass centre of the rigid cap. Nodes 2 and 3 are the two ends of the pile element. First, we shift the force and displacement BCs of the pile at node 2 to the mass center of rigid cap at node 1 through a transform matrix $\mathbf{T}$, namely,

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\Delta y & \Delta x & 1 \end{bmatrix} \begin{bmatrix} F_{x2} \\ F_{y2} \\ M_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} F_{x2} \\ F_{y2} \\ M_2 \end{bmatrix} \quad (33)$$

$$\begin{bmatrix} U_2 \\ W_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta y \\ 0 & 1 & \Delta x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \\ \Theta_1 \end{bmatrix} = \mathbf{T}^T \begin{bmatrix} U_1 \\ W_1 \\ \Theta_1 \end{bmatrix} \quad (34)$$

where Δx and Δy are the coordinate difference between beam node and rigid body mass center in the local coordinate system. Then we can obtain the relationship between the forces and displacements at node 1 and node 3, in the form

$$\mathbf{f} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \mathbf{K}_{pile} \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}^T \mathbf{u} = \mathbf{K}_{cappile} \mathbf{u} \quad (35)$$

where $\mathbf{f} = \begin{bmatrix} F_{x1} & F_{y1} & M_1 & F_{x3} & F_{y3} & M_3 \end{bmatrix}^T$, $\mathbf{u} = \begin{bmatrix} U_1 & W_1 & \Theta_1 & U_3 & W_3 & \Theta_3 \end{bmatrix}^T$, $\mathbf{E}$ is a 3×3 identity matrix and $\mathbf{K}_{pile}$ is the DS matrix of the pile element ① with nodes 2-3. $\mathbf{K}_{cappile}$ is the local DS matrix for the pile ① connected to the rigid cap ⑤ (without considering cap mass and rotatory inertia, which are considered later) with two nodes 1-3.

2.2.2. Assembly procedure and the application of boundary conditions

Then the element DS matrices can be assembled to form the global DS matrix of the pile group foundation. For a structure shown in Fig.4, we can obtain the element dynamic stiffness matrices of segments 1-3, 1-5, 1-7, 1-9, represented by $\mathbf{k}^{①}$, $\mathbf{k}^{②}$, $\mathbf{k}^{③}$, $\mathbf{k}^{④}$, respectively.

$$\mathbf{k}^{①} = \begin{bmatrix} \mathbf{k}_{11}^{①} & \mathbf{k}_{13}^{①} \\ \mathbf{k}_{31}^{①} & \mathbf{k}_{33}^{①} \end{bmatrix}, \mathbf{k}^{②} = \begin{bmatrix} \mathbf{k}_{11}^{②} & \mathbf{k}_{15}^{②} \\ \mathbf{k}_{51}^{②} & \mathbf{k}_{55}^{②} \end{bmatrix}, \mathbf{k}^{③} = \begin{bmatrix} \mathbf{k}_{11}^{③} & \mathbf{k}_{17}^{③} \\ \mathbf{k}_{71}^{③} & \mathbf{k}_{77}^{③} \end{bmatrix}, \mathbf{k}^{④} = \begin{bmatrix} \mathbf{k}_{11}^{④} & \mathbf{k}_{19}^{④} \\ \mathbf{k}_{91}^{④} & \mathbf{k}_{99}^{④} \end{bmatrix} \quad (36)$$

Assembling $\mathbf{k}^{①}$ - $\mathbf{k}^{④}$ at the mass center of the rigid cap of node 1 gives

$$\mathbf{K}(\omega) = \begin{bmatrix} \mathbf{k}_{11}^g & \mathbf{k}_{13}^{①} & \mathbf{k}_{15}^{②} & \mathbf{k}_{17}^{③} & \mathbf{k}_{19}^{④} \\ \mathbf{k}_{31}^{①} & \mathbf{k}_{33}^{①} & 0 & 0 & 0 \\ \mathbf{k}_{51}^{②} & 0 & \mathbf{k}_{55}^{②} & 0 & 0 \\ \mathbf{k}_{71}^{③} & 0 & 0 & \mathbf{k}_{77}^{③} & 0 \\ \mathbf{k}_{91}^{④} & 0 & 0 & 0 & \mathbf{k}_{99}^{④} \end{bmatrix} \quad (37)$$

where $\mathbf{K}(\omega)$ is the global dynamic stiffness matrix for the pile group foundation. It is notable that $\mathbf{k}_{11}^g$ should consider the mass m and moment of inertia I_α of the rigid cap, which is as follows

$$\mathbf{k}_{11}^g = \mathbf{k}_{11}^{(1)} + \mathbf{k}_{11}^{(2)} + \mathbf{k}_{11}^{(3)} + \mathbf{k}_{11}^{(4)} + \begin{bmatrix} -m\omega^2 & 0 & 0 \\ 0 & k_o - m\omega^2 & 0 \\ 0 & 0 & h^3 k_o / 12 - I_\alpha \omega^2 \end{bmatrix} \quad (38)$$

where h is the width of the rigid cap. After formulating the global DS matrix of the structure, the corresponding boundary conditions can be applied.

If the ends of all piles in Fig.4 are fastened to the bedrock as shown in Fig.1(d), the boundary conditions can be applied by deleting the rows and columns corresponding to the relevant degrees of freedom, which results in only a 3×3 matrix $\mathbf{k}_{11}^g$ as given in Eq.(38). This is a very concise but exact formulation for the dynamic analysis of pile groups, i.e.,

$$\mathbf{K}_f(\omega) = [\mathbf{k}_{11}^g]_{3 \times 3} \quad (39)$$

For friction piles shown as Fig.1(e) where the piles ends are subjected to three springs, the stiffness of those end springs can be superposed to the global DS matrix of Eq.(37) leading to

$$\mathbf{K}_f(\omega) = \mathbf{K}(\omega) + \begin{bmatrix} \mathbf{0}_{3 \times 3} & & & \\ & \mathbf{k}_{b3} & & \\ & & \mathbf{k}_{b5} & \\ & & & \mathbf{k}_{b7} \\ & & & & \mathbf{k}_{b9} \end{bmatrix} \quad (40)$$

where

$$\mathbf{k}_{bi} = \begin{bmatrix} k_v & 0 & 0 \\ 0 & k_h & 0 \\ 0 & 0 & k_r \end{bmatrix} \quad (41)$$

where k_v , k_h , k_r are respectively the stiffnesses in the vertical, horizontal and rotational directions, as shown in Fig.1(e). Then the final DS matrix can be used for modal and response analysis.

3. Modal and dynamic response analysis

3.1. Wittrick-Williams algorithm for modal analysis

For modal analysis, the main object now is to extract the eigensolutions from the determinant of the DS matrix, which is a transcendental function of the frequency. The Wittrick-Williams algorithm (Williams and Wittrick, 1970; Wittrick and Williams, 1971) is an efficient and reliable technique to solve such a transcendental function. The Wittrick-Williams algorithm stems from the Rayleigh theorem and monitors the properties of the Sturm sequence, which guarantees that no natural frequency is missed. Also, the natural frequency can be pin-pointed by the bisection method to any desired accuracy. In the Wittrick-Williams algorithm, the mode count J when ω is lower than the trial frequency ω^* is given by

$$J = \sum_i J_{0i} + s(\mathbf{K}^\nabla(\omega^*)) \quad (42)$$

where $J_K = s(\mathbf{K}(\omega^*))$ is the number of negative diagonal elements of $\mathbf{K}^\nabla(\omega^*)$, and $\mathbf{K}^\nabla(\omega^*)$ is the upper triangular matrix by performing a Gaussian elimination process of $\mathbf{K}(\omega)$ evaluated at $\omega = \omega^*$. J_{0i} is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for an individual element with both ends fully clamped.

It is apparent that the last part of Eq.(42) is easy to be calculated using the global DS matrix of the pile group foundations when ω^* is given, while calculating J_{0i} is relatively difficult but is a crucial part in the Wittrick-Williams algorithm. In the following, the derivation of J_{0i} for both longitudinal (J_{0l}) and transverse vibration (J_{0t}) will be provided in detail, which finally gives the J_{0i} count for a pile element with both longitudinal and transverse vibrations as follows

$$J_{0i} = J_{0l} + J_{0t} \quad (43)$$

3.1.1. J_{0l} count for longitudinal vibration

For the longitudinal vibration of a pile, the fixed-fixed mode count J_{0i} can be directly obtained by setting the determinant of the square matrix of Eq.(8) to be zero, yielding the following equation

$$\sin(\alpha L) = \sin(n\pi) = 0 \quad (44)$$

Then J_{0l} can be easily obtained, given by Banerjee and Ananthapuvirajah (2019)

$$J_{0l}(\omega) = \text{floor}\left(\frac{\omega L}{\alpha \pi}\right) \quad (45)$$

where $\alpha = \sqrt{\frac{\rho A \omega^2 - k_a}{EA}}$, and $\text{floor}(\bullet)$ represents the the integer not larger than ' $\bullet$ '.

3.1.2. J_{0t} count for a preloaded Timoshenko beam on elastic Winkler/Pasternak foundations

For a preloaded Timoshenko beam on elastic foundations, it is difficult to determine the frequency number J_{0t} of fixed supports at both ends directly and can not be found in the literature. Thus, we adopt an indirect method by using the mode count J_s of the element with both ends simply supported according to the Wittrick-Williams algorithm, and J_{0t} can be given by

$$J_{0t} = J_s - s(\mathbf{K}_s(\omega)) \quad (46)$$

in which $\mathbf{K}_s(\omega)$ is the DS matrix for an element with both ends simply supported, i.e.,
 $\mathbf{K}_s(\omega) = \begin{bmatrix} d_3 & d_6 \\ d_6 & d_3 \end{bmatrix}$, where d_3, d_6 have been given in Eq.(31). Next, we will derive the
 explicit expression for J_s . For a preloaded Timoshenko beam simply supported on both
 ends with length L , we have

$$W(0) = W(L) = M(0) = M(L) = 0 \quad (47)$$

We can assume

$$\begin{aligned} W(x) &= \sin\left(\frac{m\pi}{L}x\right) \\ \Theta(x) &= C \cos\left(\frac{m\pi}{L}x\right) \end{aligned} \quad (48)$$

where $m = 0, 1, 2, 3, \dots$. Substituting Eq.(48) into Eq.(13) leads to a fourth polynomial
 equation with respect to m , i.e.

$$\left(\frac{\pi}{L}m\right)^4 - a_1\left(\frac{\pi}{L}m\right)^2 + a_2 = 0 \quad (49)$$

where a_1 and a_2 have been given in Eq.(14). Four roots of Eq.(49) are

$$m_{1,2} = \pm \frac{L}{\pi} \sqrt{\frac{a_1 - \sqrt{a_1^2 - 4a_2}}{2}}, \quad m_{3,4} = \pm \frac{L}{\pi} \sqrt{\frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2}} \quad (50)$$

According to the physical sense, the wavenumber m should be a nonnegative integer value.
 Therefore, the two negative roots m_2 and m_4 should be omitted. However, there are still
 five situations for roots m_1 and m_3 .

(i) If $a_1^2 - 4a_2 < 0$, then both m_1 and m_3 are complex value, we have

$$J_s = 0 \quad (51)$$

(ii) If $a_1^2 - 4a_2 > 0$ and $a_1 - \sqrt{a_1^2 - 4a_2} < 0$ and $a_1 + \sqrt{a_1^2 - 4a_2} < 0$, then both m_1
 and m_3 are complex value, so

$$J_s = 0 \quad (52)$$

372 (iii) If $a_1^2 - 4a_2 > 0$ and $a_1 - \sqrt{a_1^2 - 4a_2} < 0$ and $a_1 + \sqrt{a_1^2 - 4a_2} > 0$, then only m_3 is
 373 positive real, so

$$J_s = \text{floor}(m_3) \quad (53)$$

374 (iv) If $a_1^2 - 4a_2 > 0$ and $a_1 - \sqrt{a_1^2 - 4a_2} > 0$, then both m_1 and m_3 are positive real.

375 Thus, by taking the above (i)-(iv) cases, we have

$$J_s = \text{floor}(m_1) + \text{floor}(m_3) \quad (54)$$

376 Finally, the J_{0t} of an axially loaded Timoshenko beam on elastic foundations can be
 377 determined by Eq.(46).

378 3.2. Method for dynamic response analysis

379 Following the procedure for DS formulation discussed above, the DS model for a
 380 structure can be derived in the form of

$$\mathbf{K}_f(\omega)\mathbf{d}_f = \mathbf{f}_f \quad (55)$$

381 where $\mathbf{K}_f(\omega)$ is the final DS matrix, $\mathbf{d}_f$ is the final nodal displacement vector, and $\mathbf{f}_f$ is
 382 the final force vector acting on the structure. The frequency-domain dynamic response
 383 $\mathbf{d}_f$ can be computed by

$$\mathbf{d}_f = \mathbf{K}_f^{-1}(\omega)\mathbf{f}_f \quad (56)$$

384 The dynamic response at any point of all of the piles and the rigid cap can then be
 385 recovered by using exact shape functions of Eqs.(5) and (16)-(21).

386 4. Results and discussion

387 The theoretical framework described above has been implemented into a Matlab pro-
 388 gram to analyse the free vibration and dynamic response of a pile group foundation. The

accuracy of the method is validated in Section 4.1 by published results and commercial software ANSYS. Section 4.2 conducts the free vibration analysis of a single pile-cap structure and analyses the change of the natural frequency of a end-bearing pile-cap considering the variation of foundation stiffness. In Section 4.3, we compute the natural frequencies of a pile group foundation, and present the dynamic response analysis of a pile group foundation in the frequency domain. Section 4.4 investigated the effect of different boundary conditions on the lower end of the piles. It is notable that the foundation parameters are assumed to be a constant rather than frequency dependent in the following analysis and that the damping is not taken into consideration in the modal analysis.

4.1. Validation and comparison

In order to verify the accuracy of the proposed method, this part compares the results obtained by the proposed method with some published results and commercial software ANSYS.

4.1.1. Validation for transverse vibration of a single pile element

First, we compare results computed by the proposed theory with published results for three types of beams, i.e., classical Timoshenko beam, Timoshenko beam on Winkler foundation and Timoshenko beam on Pasternak foundation without damping. All beams are simply supported on both ends. For comparison, we choose the same material and geometric parameters used in (Attarnejad et al., 2010; De Rosa and Lippiello, 2021; Soares et al., 2017), shown as follows.

For a single classical Timoshenko beam: Young's modulus $E = 2.1 \times 10^{11}$ Pa, Poisson's ratio $1/3$, mass density $\rho = 7850$ kg/m³, the length of the beam is 0.4m, the cross-sectional shape factor $\kappa = 2/3$ and its cross-section is rectangular with 0.08m height and 0.02m

width.

For a Timoshenko beam on Winkler foundation: Young's modulus of pile $E = 2.06 \times 10^{11}$ Pa, Poisson's ratio 0.3, mass density $\rho = 7850 \text{ kg/m}^3$, the length of pile is $L = 1 \text{ m}$, and the cross-sectional shape factor $\kappa = 13/15$ and square cross-section $b = 0.05 \text{ m}$, $h = 0.05 \text{ m}$, the Winkler foundation parameter k_s is $12 \times 10^6 \text{ Pa}$.

For a Timoshenko beam on Pasternak foundation: Young's modulus of pile $E = 2.1 \times 10^{11}$ Pa, Poisson's ratio 0.3, mass density $\rho = 7850 \text{ kg/m}^3$, the length of pile is $L = 0.5 \text{ m}$, and its cross-section is circle with $r = 0.04 \text{ m}$, and the cross-sectional shape factor $\kappa = 5/6$. The parameters for Pasternak foundation is $k_s = 4.223 \times 10^7 \text{ Pa}$, $k_p = 1.056 \times 10^7 \text{ N/m}$.

Table 1. The first four natural frequencies of three types of beams under simply-simply supported boundary conditions computed by DSM and solutions in (Attarnejad et al., 2010), (De Rosa and Lippiello, 2021) and (Soares et al., 2017)

Mode No.	Natural frequency (Hz)					
	Timoshenko without foundation		Timoshenko on Winkler foundation		Timoshenko on Pasternak foundation	
	Attarnejad et al. (2010)	DSM	De Rosa and Lippiello (2021)	DSM	Soares et al. (2017)	DSM
1	6838.8333	6838.8335	1066.6700	1066.9734	5209.2300	5209.2304
2	23190.8265	23190.8271	2976.2700	2976.1718	15965.2080	15965.2079
3	43443.4923	43443.4931	6387.5200	6387.5214	31051.6890	31051.6894
4	64939.1842	64939.1848	11017.2000	11017.2325	48571.4480	48571.4478

The first four natural frequencies of the three kinds of beams are compared with Attarnejad et al. (2010), De Rosa and Lippiello (2021) and Soares et al. (2017) respectively, as reported in Table 1. According to Table 1, the results of the DSM are in excellent agreement with the published results, which verifies the proposed DSM formulations for the Timoshenko beam on elastic foundations.

4.1.2. Validation for a rigid-cap pile group without foundation

In this subsection, we compare the natural frequencies of a rigid-cap pile group without foundation as shown in Fig.4 computed by DSM and commercial software ANSYS. The

distance between piles is 3m. The ends of all piles are fixed. The material and dimensions properties for the piles are: total length $L = 10\text{m}$, radius $r = 0.1\text{m}$, Young's modulus $E = 3.8 \times 10^{11}\text{Pa}$, mass density $\rho = 2350\text{kg/m}^3$, Poisson's ratio $\nu = 0.17$ and cross-sectional shape factor $\kappa = 0.857$. The mass and mass moment of inertia of the rigid cap are $m = 22.5\text{kg}$ and $I_\alpha = 1.875\text{kgm}^2$. The mass centre of the cap is (4.5m, 10.5m). In addition, the foundations are not considered in this example, thus $k_l = k_s = k_p = 0$.

Table 2 shows the first nine natural frequencies obtained from DSM and commercial software ANSYS. The FEM results are computed by using different numbers of elements. The computation of both DSM and FEM results was performed on the same PC equipped with a 2.40 GHz Intel core i7-5500U CPU processor and 8 GB of memory. It can be clearly observed from the results in Table 2 that the relative error between the results computed by DSM and FEM is within 0.1%, which validates the analytical DS model. What's more, the computational efficiency of DSM is much higher than FEM, since only one element for each member is used in the DS model, which means that the matrix size in the dynamic stiffness model is much smaller than that in the FE model.

4.2. Free vibration analysis of a single pile-cap structure

In this subsection, free vibration analysis is performed for a single pile-cap structure with the lower end clamped. Here the longitudinal spring parameter k_l is adopted to be $1 \times 10^5\text{N/m}$ and the effect of the Pasternak foundation stiffness is analysed. The specific parameters of the piles, cap and foundation are as follows: For the pile: Young's modulus $E = 3.8 \times 10^{11}\text{Pa}$, Poisson's ratio $\nu = 0.17$, mass density $\rho = 2350\text{kg/m}^3$, cross-sectional shape factor $\kappa = \pi^2/12$, length $L = 1\text{m}$, and radius $r = 0.1\text{m}$. For the cap: Mass $m = 22.5\text{kg}$, moment of inertia $I_\alpha = 1.875\text{kgm}^2$, the location of the mass centre of the

Table 2. The first nine natural frequencies in Hz and computational time(Comp. Time) of the rigid-cap pile group without foundation of Fig.4 using the finite element method(FEM) by using different numbers of elements(N) for each pile and by the dynamic stiffness method(DSM), % Error in FEM relative to DSM shown in the parenthesis

Mode No.	FEM					DSM
	N = 1	N = 5	N = 10	N = 50	N = 100	
1	5.995 (7.00%)	5.600 (0.05%)	5.599 (0.06%)	5.599 (0.07%)	5.599 (0.07%)	5.603
2	306.850 (1257.83%)	22.621 (0.10%)	22.586 (0.06%)	22.584 (0.06%)	22.584 (0.06%)	22.599
3	316.660 (1301.24%)	22.621 (0.10%)	22.586 (0.06%)	22.584 (0.06%)	22.584 (0.06%)	22.599
4	383.310 (1596.17%)	22.621 (0.10%)	22.586 (0.06%)	22.584 (0.06%)	22.584 (0.06%)	22.599
5	383.310 (1165.16%)	30.365 (0.22%)	30.283 (0.05%)	30.278 (0.06%)	30.278 (0.06%)	30.297
6	383.310 (516.83%)	62.849 (1.14%)	62.150 (0.01%)	62.103 (0.06%)	62.102 (0.06%)	62.142
7	383.310 (516.83%)	62.849 (1.14%)	62.150 (0.01%)	62.103 (0.06%)	62.102 (0.06%)	62.142
8	383.310 (516.83%)	62.849 (1.14%)	62.150 (0.01%)	62.103 (0.06%)	62.102 (0.06%)	62.142
9	383.310 (413.55%)	75.828 (1.59%)	74.674 (0.05%)	74.592 (0.06%)	74.592 (0.06%)	74.640
Comp. Time(s)	2.40	3.00	3.90	4.20	4.50	0.23

cap is (0, 1.05m), i.e., there is no eccentricity between the mass center of the rigid cap and the pile.

Fig.6 demonstrates the variation of the first natural frequencies of the pile-cap model with respect to both nondimensional Pasternak foundation stiffness. It can be seen from Fig.6 that the first natural frequency of the structure increases with the increasing of the nondimensional Pasternak foundation parameters, as expected.

4.3. Free vibration and dynamic response analysis of a pile group foundation

In this subsection, free vibration and dynamic response analysis are performed for a pile group foundation, as shown in Fig.4. All piles have the same parameters, and the Pasternak foundation model is chosen for analysis. The specific parameters of the piles,

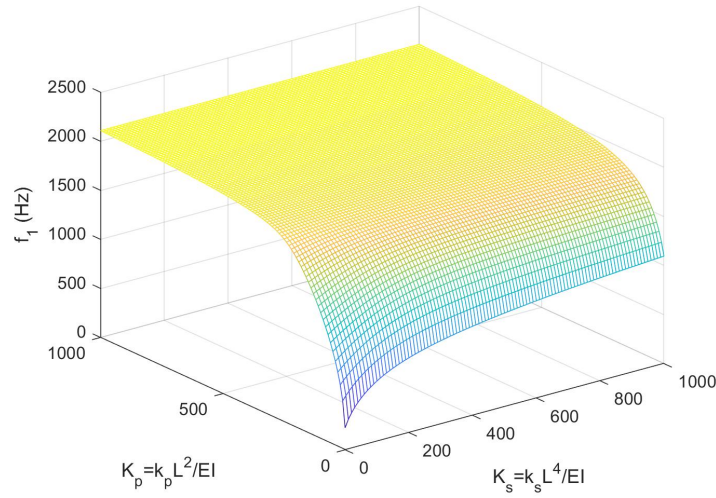


Fig. 6. Influence of nondimensional foundation parameters K_s and K_p on the first natural frequency of a single pile-cap element.

cap and foundation are as follows:

For the piles: Young's modulus $E = 3.8 \times 10^{11} \text{ Pa}$, Poisson's ratio 0.17, mass density $\rho = 2350 \text{ kg/m}^3$, cross-sectional shape factor $\kappa = 0.857$, length $L = 40 \text{ m}$, and radius $r = 0.5 \text{ m}$. The distance between piles is 3m. For the cap: Mass $m = 225 \text{ kg}$, moment of inertia $I_\alpha = 18.75 \text{ kgm}^2$, the location of mass centre is (4.5m, 40.5m). For the longitudinal soil stiffness: $k_l = 1 \times 10^6 \text{ N/m}$. For the Pasternak foundation: $k_s = 1 \times 10^6 \text{ Pa}$, $k_p = 1 \times 10^7 \text{ N/m}$.

Table 3. The first seven natural frequencies with the lower ends of the piles fixed and the elastic foundation for the transverse vibration is described by Winkler model ($k_s = 1 \times 10^6 \text{ Pa}$) and Pasternak model ($k_s = 1 \times 10^6 \text{ Pa}$, $k_p = 1 \times 10^7 \text{ N/m}$), respectively.

Mode No.	Natural frequency (Hz)	
	Winkler foundation	Pasternak foundation
1	4.09739	4.12940
2	7.96787	8.03364
3	7.96787	8.03364
4	7.96787	8.03364
5	10.1668	10.2703
6	19.7236	19.8230
7	19.7236	19.8230

Table 3 gives the first seven natural frequencies with the lower ends of the piles fixed and the transverse foundation is described by both Winkler and Pasternak models. It is obvious that compared with Winkler foundation, the Pasternak parameter k_p increases the natural frequencies. Fig.7 illustrates the 1st to 10th mode shapes of the rigid-cap group piles structure embedded in Pasternak foundation. The undeformed shapes and the deformed modal shapes of the group pile foundation are plotted as dashed and continuous lines, respectively. It is notable that the 2nd-4th mode shapes of the structure are given by Fig.7(b) since they are of coincident natural frequencies.

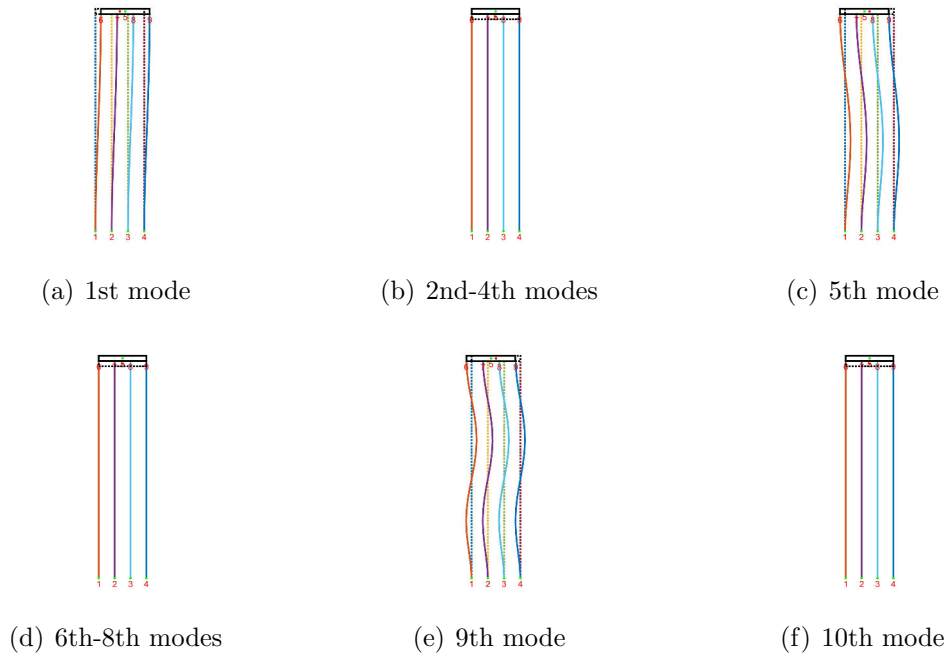


Fig. 7. First ten mode shapes of a rigid-cap group piles.

Dynamic response of group pile foundation can also be performed following the procedure described in Section 3.2. We take the group pile foundation shown in Fig.4 as an example, where the lower ends of the piles are fixed and a concentrated excitation of 1000N is applied at the mass center of the rigid cap. The amplitude of the frequency

response function (FRF) is shown in Fig.8, where four kinds of foundation are adopted. The response is computed up to 30Hz covering about the first seven natural modes of the structure. It can be seen from Fig.8 that considering the damping of the foundations reduces the amplitude of the response to a large extent. Fig.9 illustrates the amplitudes of dynamic responses at node 5 with different foundation damping values using the viscoelastic Pasternak foundation. It is apparent that with the increase of the damping, the amplitudes of dynamic responses decrease greatly.

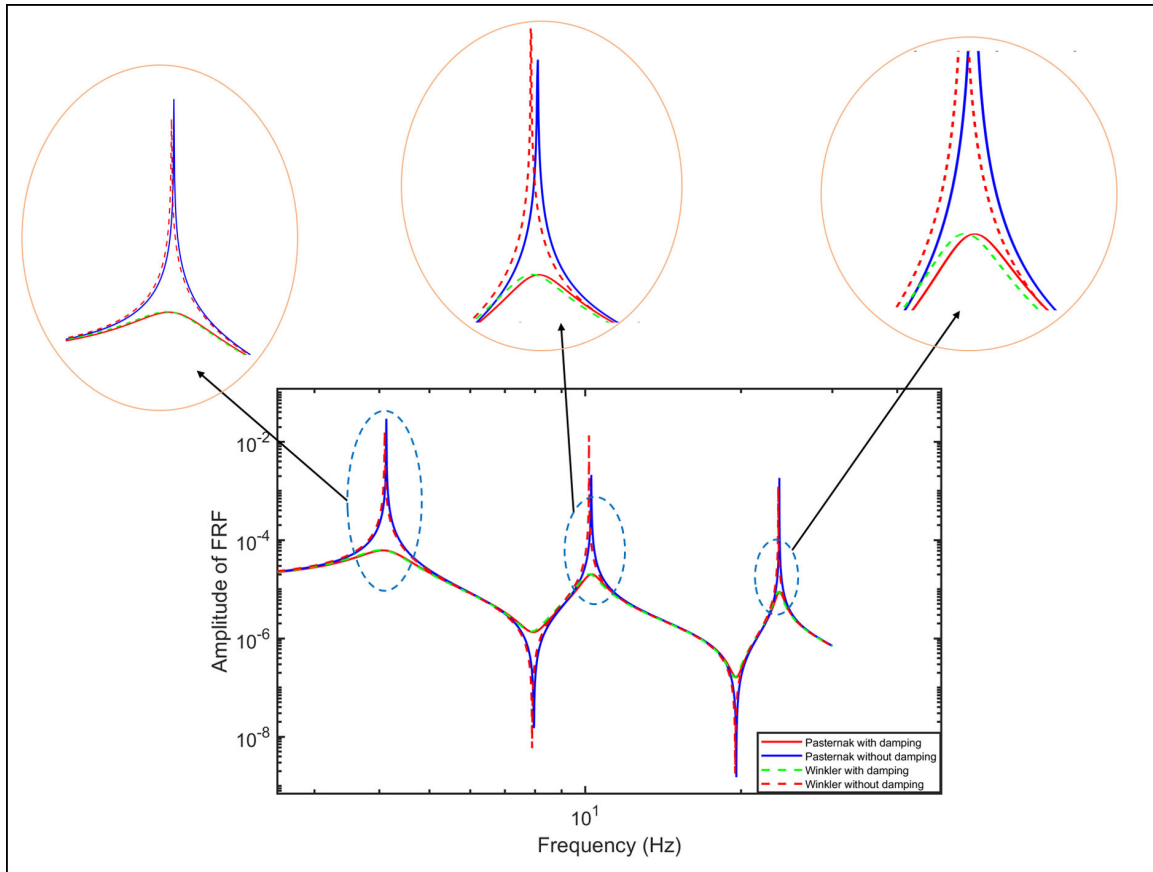


Fig. 8. The amplitudes of dynamic responses at node 5 under a concentrated harmonic excitation.

4.4. Effect of different boundary conditions on the lower end of the piles

The effect of different boundary conditions on the lower end of the piles are also investigated. The parameters for the rigid cap, piles and Pasternak foundation are the

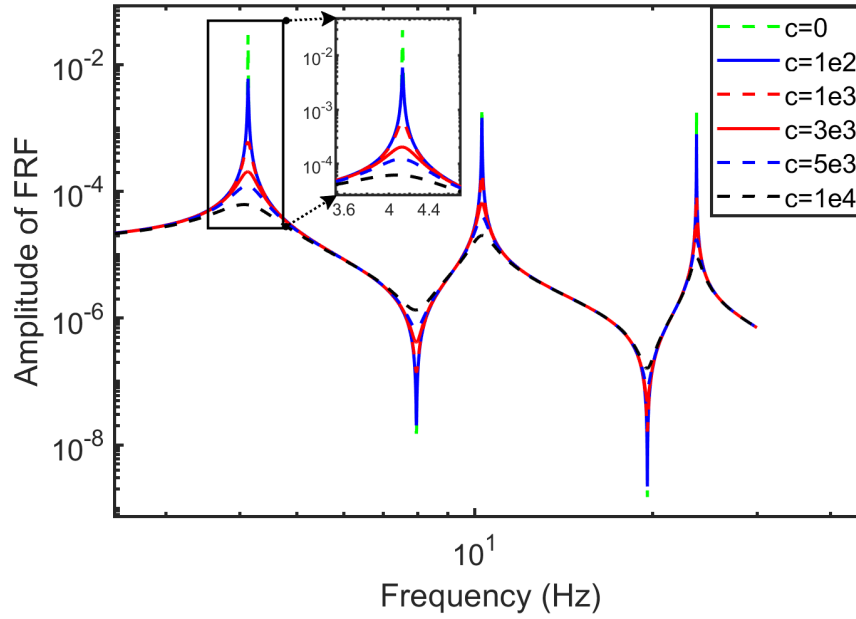


Fig. 9. The amplitudes of dynamic responses at node 5 with different foundation damping values. ($c_a = c_s = c$, and the unit of damping is $\text{N}/(\text{m}/\text{s})$)

same as those in Section 4.3. Table 4 demonstrated the first seven natural frequencies of the rigid-cap group pile foundation as shown in Fig. 4 with different boundary conditions, where the lower ends of the piles are chosen to be fixed or constrained by elastic springs in the horizontal, vertical and rotational directions with different stiffness. The results indicate that with the increase of the elastic spring stiffness of the pile end, the natural frequencies increase and approach those when pile lower ends fully clamped.

Table 4. The first seven natural frequencies of the rigid-cap group pile foundation as shown in Fig. 4 with piles ends constrained by different stiffnesses ($k_v = k_h = k_r = k$)

Mode No.	Natural Frequency (Hz)						
	fixed	k=1e3	k=1e5	k=1e6	k=1e7	k=1e8	k=+∞
1	4.1294	3.70318	3.70705	3.72261	3.73975	3.78392	4.1294
2	8.03364	3.70324	3.70782	3.74919	4.1401	6.05977	8.03364
3	8.03364	3.82529	3.83833	3.9677	4.8307	6.05977	8.03364
4	8.03364	3.91768	3.93445	4.07891	4.98734	6.05977	8.03364
5	10.2703	3.91768	3.93445	4.07891	4.98734	6.33284	10.2703
6	19.823	3.91768	3.93445	4.07891	4.98734	6.91699	19.823
7	19.823	6.50838	6.51598	6.58623	7.33856	10.369	19.823

5. Conclusion

In this paper, closed-form dynamic stiffness (DS) formulation for pile group foundation consisting of a rigid cap and arbitrary numbers of flexible piles has been proposed. The proposed model considers both the longitudinal vibration as well as the transverse vibration of the pile. Using the Wittrick-Williams algorithm with the key issue J_0 count resolved, the DS formulation has been verified against some existing exact results and modal and dynamic response analysis are conducted for a group piles-cap foundation. Some novel aspects of the proposed method are as follows:

- The analytical dynamic stiffness matrix of a pile group foundation with a rigid cap is developed for the first time incorporating the longitudinal and transverse vibration of a pile.
- The proposed DS formulations essentially use very few degrees of freedom to, nevertheless, describe the broadband dynamic behaviors of complex pile group foundations very accurately. In particular, a pile group foundation with arbitrary number of ending bearing piles connecting to a rigid cap can be modelled accurately by only a 3×3 DS matrix.
- The Wittrick-Williams algorithm with the key issue J_0 resolved is extended to the modal analysis of pile group foundations, which gives accurate solutions with high computational efficiency.

This research provides an exact and highly efficient tool for analysis of modal property and dynamic response of pile foundations, which can be further extended to the

dynamic analysis of pile group foundations subjected to traffic or seismic excitations, or
the dynamic instability analysis of pile-supported structures during soil liquefaction.

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Appendix A. Explicit expressions for the dynamic stiffness elements $d_1 \sim d_6$

For case 1-3, the explicit expressions for $d_1 \sim d_6$ are as follows.

$$\begin{aligned}
 d_1 &= e(k_\Lambda \Phi - k_\Phi \Lambda)(jk_\Lambda C_1 S_2 + ik_\Phi S_1 C_2)/(L\delta) \\
 d_2 &= -EI k_\Phi k_\Lambda (jk_\Lambda \Lambda C_1^2 + C_1 C_2 (ik_\Phi \Phi - jk_\Lambda \Lambda) - ik_\Phi \Phi C_2^2 + ij(k_\Lambda S_1 - k_\Phi S_2)(\Lambda S_1 - \Phi S_2))/(L^2 \delta) \\
 d_3 &= EI(k_\Lambda S_1 C_2 - k_\Phi C_1 S_2)(ik_\Phi \Phi + jk_\Lambda \Lambda)/(L\delta) \\
 d_4 &= e(k_\Phi \Lambda - k_\Lambda \Phi)(ik_\Phi S_1 + jk_\Lambda S_2)/(L\delta) \\
 d_5 &= e(k_\Phi \Lambda - k_\Lambda \Phi)(C_1 - C_2)/(L\delta) \\
 d_6 &= EI(k_\Phi S_2 - k_\Lambda S_1)(ik_\Phi \Phi + jk_\Lambda \Lambda)/(L\delta) \\
 \delta &= k_\Phi k_\Lambda C_1^2 - 2k_\Phi k_\Lambda C_1 C_2 + C_2^2 k_\Phi k_\Lambda + (k_\Lambda S_1 - k_\Phi S_2)(ik_\Phi S_1 + jk_\Lambda S_2)
 \end{aligned} \tag{A.1}$$

For case 4, the explicit expressions for $d_1 \sim d_6$ are as follows.

$$\begin{aligned}
 d_1 &= e(k_\beta \alpha - k_\alpha \beta)(k_\alpha \sin(2\beta) + k_\beta \sinh(2\alpha))/(L\delta) \\
 d_2 &= -EI(k_\alpha^2 + k_\beta^2)(k_\beta \beta - k_\alpha \alpha + k_\alpha \alpha \cos(2\beta) - k_\beta \beta \cosh(2\beta))/(L^2\delta) \\
 d_3 &= EI(k_\beta \alpha + k_\alpha \beta)(-k_\alpha \sin(2\beta) + k_\beta \sinh(2\alpha))/(L\delta) \\
 d_4 &= -2e(k_\beta \alpha - k_\alpha \beta)(k_\beta \cos \beta \sinh \alpha + k_\alpha \sin \beta \cosh \alpha)/(L\delta) \\
 d_5 &= 2e(k_\beta \alpha - k_\alpha \beta) \sinh \alpha \sin \beta/(L\delta) \\
 d_6 &= -2EI(k_\beta \alpha + k_\alpha \beta)(k_\beta \cos \beta \sinh \alpha - k_\alpha \sin \beta \cosh \alpha)/(L\delta) \\
 \delta &= -k_\alpha^2 - k_\beta^2 + k_\alpha^2 \cos(2\beta) + k_\beta^2 \cosh(2\alpha)
 \end{aligned} \tag{A.2}$$

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