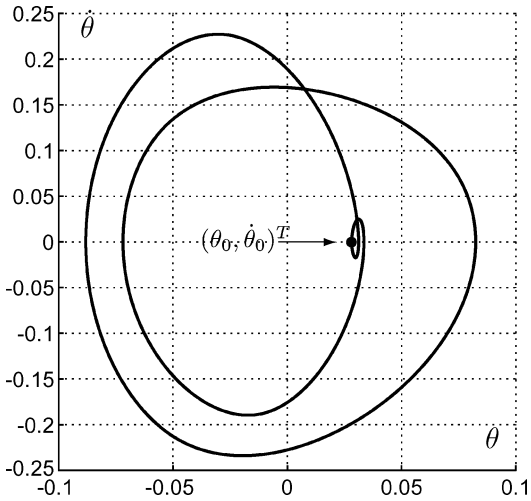


Fig. 7. Internal dynamics for spline (38).

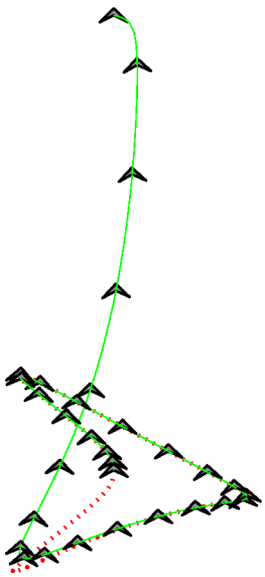


Fig. 8. Stabilization for spline (38).

and $d^i\gamma(5)/dt^i = d^i\gamma(0)/dt^i$, for $i = 0, \ldots, 3$. The maximum acceleration is given by 2.2 m/s. Fig. 7 shows the internal dynamics trajectories obtained using the fixed-point approach outlined in Section III. It is $|\theta(t)| \leq 0.085, |\dot{\theta}(t)| \leq 0.24$. The corresponding bounds obtained from (8) are $|\theta(t)| \leq 0.4861, |\dot{\theta}(t)| \leq 1.1727$. In this example, these bounds appear very conservative. This is justified by the fact that they must apply to any trajectory whose acceleration is bounded by 2.2 m/s. Fig. 8 shows the behavior of the closed-loop system obtained with a flatness-based stabilizing controller, as considered in [2].

V. Conclusion

Using a precise geometric characterization, the method based on the Poincaré map used in Section III allows one to find a set of trajectories that the VTOL can exactly track with bounded internal dynamics, whose estimates depend on the trajectory acceleration.

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On the Minimization of Maximum Transient Energy Growth

James F. Whidborne and John McKernan

***Abstract*—The problem of minimizing the maximum transient energy growth is considered. This problem has importance in some fluid flow control problems and other classes of nonlinear systems. Conditions for the existence of static controllers that ensure strict dissipativity of the transient energy are established and an explicit parametrization of all such controllers is provided. It also is shown that by means of a Q-parametrization, the problem of minimizing the maximum transient energy growth can be posed as a convex optimization problem that can be solved by means of a Ritz approximation of the free parameter. By considering the transient energy growth at an appropriate sequence of discrete time points, the minimal maximum transient energy growth problem can be posed as a semidefinite program. The theoretical developments are demonstrated on a numerical example.**

***Index Terms*—Fluid flow control, linear matrix inequalities, linear systems, optimization, transient response.**

I. Introduction

Following some initial perturbation to the state of a stable linear system, it is possible for the magnitude of the system state trajectory to grow to a large value before decreasing and converging to the origin. This can occur even though all the eigenvalues may have very negative real parts and no imaginary parts. This behavior is highly undesirable,

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particularly for certain nonlinear systems, where although linear eigenvalue analysis at an equilibrium point indicates very good stability, very small initial perturbations in the state variables may cause them to leave the domain of attraction resulting in instability.

This phenomenon is known to occur in fluid dynamic systems. For example, laminar flow can become turbulent even for Reynolds numbers for which linear stability analysis predicts stable eigenvalues. The reason for this was not widely recognized by fluid dynamicists until fairly recently [1]–[3]. The energy of the velocity perturbations is the square of the (appropriately weighted) Euclidian distance of the state from the origin, and the maximum energy following an initial unit energy state perturbation is sometimes used as a stability measure for fluid dynamics systems [2], [4]. Hence for fluid flow control systems, a useful control objective is the minimization of the maximum transient energy growth of the flow perturbations [5], [6]. Reducing the maximum transient energy growth is also important for a class of partially linear cascade systems initially investigated by Sussmann and Kokotovic [7]; see [8] for a review.

The problem of constraining transient trajectory norms has been recently considered elsewhere [9]–[14]. In fact, an upper bound on the maximum transient energy growth can be obtained by a simple Lyapunov inequality, and so this upper bound can be minimized by a linear matrix inequality (LMI) approach [15]. However, in this paper, the minimization of the actual maximum transient energy growth is considered rather than the upper bound. The upper bound problem is presented for completeness and comparison.

Large transient energy growth behavior is often associated with nonnormality of the system state matrix [3]. While it can be shown that if the system state matrix is normal, there will be no transient energy growth, the converse is not true. That is, normality of the system state matrix is a sufficient but not a necessary condition for no transient energy growth [16]. The role of normality in affecting the transient behavior is explored in more depth in [17] and [18].

This paper is organized as follows. In Section II, following definitions of transient energy and maximum transient energy growth, conditions for unity maximum transient energy growth are established. Conditions that ensure strict dissipativity of the transient energy are provided. An upper bound on the maximum transient energy growth is given, and methods for evaluating the maximum transient energy growth and the upper bound are proposed. Section III considers the problem of determining static gain controllers that minimize the maximum transient energy growth. An explicit parametrization of all linear controllers that ensure strict dissipativity is provided. For systems where such controllers do not exist, a state feedback static controller that minimizes the upper bound may be determined. In Section IV, dynamic feedback controllers are considered. First, it is shown that if no static gain controller that restricts the maximum transient energy growth to unity exists, then no dynamic controller exists either. It is then shown that the problem of determining a controller to minimize the actual maximum transient energy growth (rather than the upper bound) may be solved by convex optimization over the free parameter in a Q-parametrization of the problem. Furthermore, by considering the response at a finite set of time points, an approximation of the problem can posed as a semidefinite program that can be solved using standard methods. In Section V, the theory is illustrated with a numerical example. Concluding remarks are given in the final section.

In this paper, the following notation is used.

$\|x\| := \sqrt{x^T x}$	Euclidian 2-norm of a vector x.
M^T	Transpose of a matrix M.
$M^\perp$	Left null space of a matrix M, that is, $M^\perp = U_2^T$, where $[\,U_1 \quad U_2\,]\begin{bmatrix}\Sigma_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix} = M$ is the singular value decomposition of M.
$\mathrm{vec}(M)$	Vector formed by stacking the columns of matrix M.
$\lambda_{\max}(M)$ and $\lambda_{\min}(M)$	Largest and smallest eigenvalues, respectively, of the symmetric matrix $M = M^T$.
$M > 0\,(M \geq 0)$	Symmetric matrix M is positive definite (semidefinite).
I_n	Identity matrix of dimension $n \times n$.
0_n	$n \times n$ matrix of zeros.

Also $\|\!|M|\!\| := \max\left\{\sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } M^T M\right\}$ is the spectral norm of a real matrix M.

II. Maximum Transient Energy Growth

Consider the asymptotically stable linear time-invariant system described by the initial value problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $x_0 \in \mathbb{R}^n$, which has the continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^n, t \mapsto \Phi(t)x_0$, where $\Phi(t)$ is the state transition matrix given by $\Phi(t) = e^{At} = \sum_{i=0}^{\infty} A^i t^i / i!$.

Definition 1: The transient energy $\mathcal{E}(t)$ is defined as $\mathcal{E}(t) := \max\left\{\|x(t)\|^2 : \|x_0\| = 1\right\}$.

Definition 2: The maximum transient energy growth $\widehat{\mathcal{E}}$ is defined as $\widehat{\mathcal{E}} := \max\{\mathcal{E}(t) : t \geq 0\}$.

Lemma 1: The maximum transient energy growth $\widehat{\mathcal{E}}$ of the system described by (1) is lower bounded by unity.

Proof: At $t = 0$, then $x(0) = x_0$ and $\|x_0\| = 1$. Hence $\max\{\mathcal{E}(t) : t \geq 0\} \geq 1$. ∎

The following lemma gives the conditions on the state matrix A for there to be unity transient energy growth.

Lemma 2: The maximum transient energy growth $\widehat{\mathcal{E}}$ of the system described by (1) is unity if and only if $A + A^T \leq 0$.

Proof:

Sufficiency: If $\mathrm{d}\|x(t)\|^2/\mathrm{d}t \leq 0$ for all $t \geq 0$ and all $\{x(t) : \|x_0\| = 1\}$, then $\max\{\|x(t)\| : t \geq 0\} = \|x(0)\| = 1$. Differentiating $\|x(t)\|^2$ gives $\mathrm{d}x^T(t)x(t)/\mathrm{d}t = x^T(t)(A + A^T)x(t)$, which, if $A + A^T \leq 0$, is nonpositive for all $x(t), t \geq 0$.

Necessity: If $A + A^T \not\leq 0$, then there exists an x such that $x^T(A + A^T)x > 0$; hence there exists an x_0 such that $x^T(A + A^T)x > 0$ at $t = 0$. Thus $\|x(t)\|^2 > 1$ for some $t > 0$, so it is necessary that $A + A^T \leq 0$. ∎

If $A + A^T > 0$, then $\widehat{\mathcal{E}}$ can be evaluated by means of a line search over time of the spectral norm of $\Phi(t)$, since it is well known that $\|\!|M|\!\| = \max\{\|Mx\| : \|x\| = 1\}$. Monotonically decreasing and increasing upper bounds on the $\mathcal{E}(t)$ are available, e.g., [19, p. 138], and these can be used to provide the search interval of t and a maximum step size. However, the bounds can be very conservative and hence the search can be inefficient. The search procedure can be improved by establishing the necessary conditions for a maximum point of $\|\!|\Phi(t)|\!\|$. Clearly, at a turning point, the state derivative vector $\dot{x}(t)$ should be orthogonal to the state vector $x(t)$. Thus a local search can be made over t to obtain the inner product of $\dot{x}(t)$ and $x(t)$ to be zero, that is, $x^T(t)Ax(t) = 0$. Additional investigation and bounds are provided in [14].

Lemma 3: Consider the system described by (1). Then there exists $\beta < 0$ such that $\mathcal{E}(t) \leq e^{\beta t}$ for all $t \geq 0$ if and only if $A + A^T < 0$. This condition is known as *strict dissipativity* [12], [20, p. 660].

Proof: Similarly to Lemma 2, $A + A^T < 0$ is equivalent to $\mathrm{d}\|x(t)\|^2/\mathrm{d}t \leq \beta < 0$ for all $t \geq 0$ and all $\{x(t) : \|x_0\| = 1\}$. At $t = 0$, $\mathcal{E}(t) = e^{\beta t} = 1$ and $\mathrm{d}e^{\beta t}/\mathrm{d}t = \beta$. Furthermore, at $t = 0$, $\mathrm{d}e^{\beta t}/\mathrm{d}t$ is minimal for $t \geq 0$. Hence $\mathcal{E}(t) \leq e^{\beta t}$ for all $t \geq 0$. ■

Remark 1: It is clear that strict dissipativity is a sufficient condition for unity transient energy growth with asymptotic stability.

An upper bound on the maximum transient energy growth can be obtained by means of a Lyapunov function that describes an ellipsoid that bounds the trajectory.

Lemma 4: $\widehat{\mathcal{E}}_u \geq \widehat{\mathcal{E}}$ is an upper bound on the maximum transient energy growth $\widehat{\mathcal{E}}$ for a system described by (1), where

$$\widehat{\mathcal{E}}_u := \lambda_{\max}(P)\lambda_{\max}(P^{-1}) \tag{2}$$

where $P = P^T > 0$ satisfies

$$PA + A^T P \leq 0. \tag{3}$$

Proof: The function $L(x) = x^T P x$ is a (nonstrict) Lyapunov function if $\mathrm{d}L/\mathrm{d}t \leq 0$, that is, if P satisfies $PA + A^T P \leq 0$. If $L(x)$ is a Lyapunov function, then if $x(0)$ is in the ellipsoidal set $\{\xi^T P \xi \leq 1\}$, then $x(t)$ will remain in set $\{\xi^T P \xi \leq 1\}$ for all $t \geq 0$. Since $\lambda_{\min}(P)\|\xi\|^2 \leq \xi^T P \xi \leq \lambda_{\max}(P)\|\xi\|^2$ [21, Th. 8.18]; the identity $\lambda_{\min}(P) = 1/\lambda_{\max}(P^{-1})$ gives (2). ■

Note that $\lambda_{\max}(P) = \|\!|P|\!\|$ and that $\lambda_{\max}(P)\lambda_{\max}(P^{-1}) = \|\!|P|\!\|\,\|\!|P^{-1}|\!\|$, the well-known condition number of P.

A minimal upper bound can be obtained by solving the following LMI generalized eigenvalue problem (GEVP) [14, p. 75]:

$$\begin{aligned} &\min \gamma \\ &\text{subject to} \quad I \leq P \leq \gamma I, \quad PA + A^T P \leq 0 \end{aligned} \tag{4}$$

where $P > 0$ is real and symmetric. The inequality $I \leq P \leq \gamma I$ ensures that $\gamma \geq \lambda_{\max}(P) \geq \lambda_{\min}(P) \geq 1$; thus $\lambda_{\max}(P)/\lambda_{\min}(P) \leq \gamma$ and so $\widehat{\mathcal{E}} \leq \widehat{\mathcal{E}}_u \leq \gamma$.

Remark 2: Solving (4) has considerable numerical difficulties [14, p. 75]; however, suboptimal solutions can be obtained by tightening (3) to be a strict inequality as in [15, p. 65]

$$PA + A^T P < 0. \tag{5}$$

In addition, using the strict Lyapunov inequality ensures that solutions to the control problem (13) are asymptotically stable.

III. Optimal Static Gain Feedback Controllers

Now consider the linear time-invariant plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $C \in \mathbb{R}^{m\times n}$, $u : \mathbb{R}_+ \to \mathbb{R}^\ell$ is a piecewise continuous function and $y : \mathbb{R}_+ \to \mathbb{R}^m$ is a piecewise continuous function. Furthermore, it is assumed that $B^T B > 0$, that is, B has full column rank; and $CC^T > 0$, that is, C has full row rank, (i.e., all actuators and sensors are independent).

The next theorem provides conditions for all static output feedback controllers that have strict dissipativity.

Theorem 1: For the system of (6), the following are equivalent.

1) There exists a control $u = Ky$, where K is a constant matrix such that the closed-loop system has strict dissipativity.
2) The following two conditions hold:

$$B^{\perp}\left(A + A^T\right)B^{\perp T} < 0 \text{ or } BB^T > 0 \tag{7}$$

$$C^{T\perp}\left(A + A^T\right)C^{T\perp T} < 0 \text{ or } C^T C > 0. \tag{8}$$

Furthermore, if the above statements hold, all controller matrices K that ensure strict dissipativity are given by $K = -R^{-1}B^T\Psi C^T(C\Psi C^T)^{-1} + S^{1/2}L(C\Psi C^T)^{-1/2}$, where $S := R^{-1} - R^{-1}B^T\left[\Psi - \Psi C^T(C\Psi C^T)^{-1}C\Psi\right]BR^{-1}$, where L is an arbitrary matrix such that $\|\!|L|\!\| < 1$ and $R > 0$ is an arbitrary matrix such that

$$\Psi := \left(BR^{-1}B^T - A - A^T\right)^{-1} > 0. \tag{9}$$

Proof: For the closed-loop system transient energy to satisfy Lemma 3 and have strict dissipativity, it is required that

$$(A + BKC) + (A + BKC)^T < 0. \tag{10}$$

The remainder follows directly by application of [22, Th. 2.3.12.], with the condition that B is full column rank and C has full row rank. ■

Remark 3: A matrix R that satisfies (9) can be obtained by $R = I/\rho$. For the case where $BB^T > 0$ (i.e., B is full rank n), ρ is obtained simply by rearranging $BRB^T - A - A^T > 0$ giving the inequality

$$\rho > \lambda_{\max}(B^{-1}(A + A^T)(B^T)^{-1}). \tag{11}$$

For the case where $B^{\perp}(A + A^T)B^{\perp T} < 0$, ρ is obtained by an application of [22, Th. 2.3.10], this being an extension to Finsler's theorem.

If (3) in Lemma 4 is replaced by the strict Lyapunov inequality (5) in order to ensure asymptotic stability, then Theorem 1 can be extended to characterize all controllers that satisfy this inequality [23]. However, the problem of determining a P is nonconvex for most cases. If $C = I$, that is, the state feedback case, the problem can be solved [15], [23, p. 100]. Expanding (5) for $u = Kx$ gives $PA + A^T P + PBK + K^T B^T P < 0$. By the change of variable, $Q = P^{-1}$ and $Y = KQ$, the LMI

$$AQ + QA^T + BY + Y^T B^T < 0 \tag{12}$$

is obtained. Now since $\lambda_{\max}(P)\lambda_{\max}(P^{-1}) = \lambda_{\max}(Q)\lambda_{\max}(Q^{-1})$, a controller that minimizes the upper bound on the maximum transient energy growth can be obtained by solving the following LMI GEVP:

$$\begin{aligned} &\min \gamma \\ &\text{subject to } I \leq Q \leq \gamma I \\ &\qquad AQ + QA^T + BY + Y^T B^T < 0,\ Q = Q^T \end{aligned} \tag{13}$$

and the upper bound minimizing controller is $K = YQ^{-1}$.

IV. Optimal Dynamic Feedback Controllers

Consider the linear time-invariant plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{14}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $u(t) \in \mathbb{R}^\ell$, $C \in \mathbb{R}^{m\times n}$, $y(t) \in \mathbb{R}^m$, with feedback controller

$$\begin{aligned} \dot{x_k}(t) &= A_k x_k(t) + B_k y(t), \quad x_k(0) = x_{k0} \\ u(t) &= C_k x_k(t) + D_k y(t) \end{aligned} \tag{15}$$

with $A_k \in \mathbb{R}^{n_k\times n_k}$, $B_k \in \mathbb{R}^{n_k\times m}$, $C \in \mathbb{R}^{\ell\times n_k}$, $D \in \mathbb{R}^{\ell\times m}$. The closed-loop system is given by

$$\dot{x}_c(t) = A_c x_c(t), \quad x_c(0) = x_{c0} \tag{16}$$

where

$$A_c := \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix}, \quad x_c(t) := \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix}. \tag{17}$$

A. Unity Maximum Transient Energy Growth

Lemma 5: A necessary condition for unity maximum transient energy growth $\widehat{\mathcal{E}} = 1$ of the plant (14) with a stabilizing feedback controller (15) is that $(A + BD_kC) + (A + BD_kC)^T \leq 0$.

Proof: From Definition 1, the transient energy of the plant (14) is given by $\mathcal{E}(t) := \max\left\{\|x(t)\|^2 : \|x_0\| = 1,\ x_{k0} = 0\right\}$. Let us replace $\mathcal{E}(t)$ by a modified energy function $\mathcal{E}_\epsilon(t)$, where $\mathcal{E}_\epsilon(t) := \max\left\{\|W_\epsilon x_c(t)\|^2 : \left\|W_\epsilon^{-1} x_{c0}\right\| = 1\right\}$, where $W_\epsilon := \mathrm{diag}(I_n, \epsilon I_{n_k})$ and $\epsilon \in \mathbb{R}_+$. Clearly as $\epsilon \to 0$, $\mathcal{E}_\epsilon \to \mathcal{E}$. Applying Lemma 2 to (17), $\max\{\mathcal{E}_\epsilon(t)\} = 1$ if and only if $W_\epsilon(A_c + A_c^T)W_\epsilon \leq 0$, that is

$$\begin{bmatrix} A_D + A_D^T & (BC_k + (BC_k)^T)\epsilon \\ (B_kC + C^TB_k^T)\epsilon & (A_k + A_k^T)\epsilon^2 \end{bmatrix} \leq 0 \qquad (18)$$

where $A_D = A + BD_kC$. Since all the principal submatrices of a negative semidefinite matrix are negative semidefinite [24, p. 397], $(A + BD_kC) + (A + BD_kC)^T \leq 0$ is a necessary condition for (18) to hold and for $\widehat{\mathcal{E}} = 1$. ∎

Remark 4: From the above lemma, it is clear that if no static controller that achieves unity maximum transient energy growth exists, then no dynamic controller exists either.

B. Minimal Transient Energy Growth by Convex Optimization

The transient energy of the plant is $\mathcal{E}(t) = \max\left\{\|x(t)\|^2 : \|x_0\| = 1,\ x_{k0} = 0\right\}$ and can be evaluated by $\|\Phi_c(t)\|$, where $\Phi_c(t) := [I_n \quad 0_{n_k}]e^{A_c t}[I_n \quad 0_{n_k}]^T$. The operation $\max\{\mathcal{E}(t) : t \geq 0\}$ represents a norm on the matrix function $\Phi_c(t)$. By means of a Q-parametrization, control system performance indexes that are norms can be minimized by exploiting the convex properties of norms [25]. For simplicity, here we just consider the case for an open-loop stable system; details on a parametrization for the unstable case are given in [25]. Assuming that the system given by (14) is stable, a convex realization of $\Phi_c(t)$ is given by $\Phi_c(t) = \mathcal{L}^{-1}[\Phi_c(s)] = 1/2\pi \int_{-j\infty}^{j\infty} \Phi_c(s)e^{st}ds$, where $\Phi_c(s) = U_1(s) + U_2(s)Q(s)U_3(s)$ with $U_1(s) = (sI - A)^{-1}$, $U_2(s) = (sI - A)^{-1}B$, $U_3(s) = C(sI - A)^{-1}$, and $Q(s)$ is the free parameter transfer function matrix with dimension $\ell \times m$ and is stable and proper.

The problem is then posed as follows:

$$\widehat{\mathcal{E}}_{\min} = \min_{\text{stable } Q} \max_{t \geq 0} \|\Phi_c(t)\|. \qquad (19)$$

Approximations of the set of all stable, proper $Q(s)$ can be parameterized by means of a Ritz approximation [25], [26] of $Q(s)$ given by $\tilde{Q}(s)$. The final optimal controller is given by $K_{\mathrm{opt}}(s) = (I + \tilde{Q}_{\mathrm{opt}}(s)C(sI - A)^{-1}B)^{-1}\tilde{Q}_{\mathrm{opt}}(s)$.

Using a state-space basis for the Ritz approximation [26], some examples of the above problem appear in [27]. However, the search over time for the peak of the maximum transient energy growth, described in Section II, is computationally intensive, and so the method is very inefficient. However, the problem can be solved approximately by choosing an appropriate sequence of unique points in time, $\{t_i\}_{i=1}^N = \{t_1, t_2, \ldots, t_N\}$, $t_i > 0$, and minimizing the maximum energy growth over all t_i. This translates simply into an LMI, since it is well known that, for some real matrix M

$$\|M\| < \gamma \Leftrightarrow \begin{bmatrix} \gamma I & M \\ M^T & \gamma I \end{bmatrix} > 0. \qquad (20)$$

Thus if at each time point t_i the following LMI is satisfied:

$$\begin{bmatrix} \gamma I_n & \Phi_c(t_i) \\ \Phi_c^T(t_i) & \gamma I_n \end{bmatrix} > 0 \qquad (21)$$

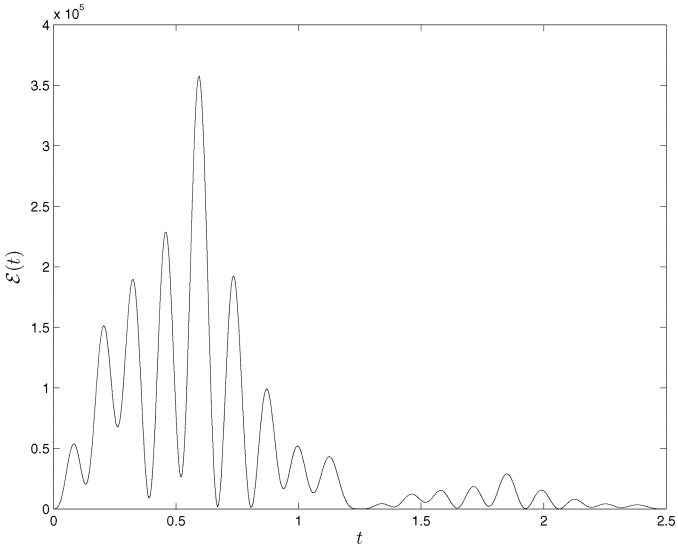


Fig. 1. Open-loop transient energy.

then, provided $\{t_i\}_{i=1}^N$ is appropriately chosen, the problem given by (19) will be approximately solved. Since $\Phi_c(t)$ is a continuous function, as N increases and the intervals between adjacent time-points decrease, the approximation accuracy will increase and $\widehat{\mathcal{E}} \lesssim \gamma^2$. Hence the problem given by (19) can be approximated by the semidefinite program (SDP)

$$\begin{aligned} &\min \gamma \\ &\text{subject to} \begin{bmatrix} \gamma I_n & \Phi_c(t_i, \tilde{q}) \\ \Phi_c^T(t_i, \tilde{q}) & \gamma I_n \end{bmatrix} > 0, \quad i = 1, \ldots, N \end{aligned} \qquad (22)$$

where $\tilde{q} = \mathrm{vec}(\tilde{Q})$ and $\Phi(t_i, \tilde{q})$ depends affinely on the decision vector $\tilde{q}$. The problem can be solved using standard SDP software.

The choice of $\{t_i\}_{i=1}^N$ can be made by observation of the transient energy response $\mathcal{E}(t)$ after the optimization. This requires some trial and error, but for the examples given in the next section, it is quite easy. The points are simply chosen to be sufficiently close to each other where $\mathcal{E}(t)$ is near its maximum and can be more widely spread elsewhere. Development of a rigorous method remains for future work.

V. Example

The linear system plant with

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -625 \\ 0 & -1 & -30 & 400 & 0 & 0 & 250 \\ -2 & 0 & -1 & 0 & 0 & 0 & 30 \\ 5 & -1 & 5 & -1 & 0 & 0 & 200 \\ 11 & 1 & 25 & -10 & -1 & 1 & -200 \\ 200 & 0 & 0 & -150 & -100 & -1 & -1000 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (23)$$

and

$$B = \begin{bmatrix} I_4 \\ 0_4 \end{bmatrix} \qquad (24)$$

is from [12]. The maximum transient energy growth for the open-loop system is calculated as $\widehat{\mathcal{E}} = 358148$ by means of a line search over time of the spectral norm of $\Phi(t)$. The transient energy $\mathcal{E}(t)$ is shown in Fig. 1. Solving the GEVP of (4), an upper bound on $\widehat{\mathcal{E}}$ is obtained as $\widehat{\mathcal{E}}_u = 439709$.

No state feedback controller providing asymptotic stability and unity maximum transient energy growth was found to exist. The GEVP of (13) is solved to obtain a controller that minimizes the upper bound on the maximum transient energy growth. The minimal upper bound is

TABLE I
SOLUTIONS TO SEQUENCE OF APPROXIMANT SDPS

n_q	0	1	2	3	4	5	6	7	8	9
$\gamma^2_{\min}$	508.39	148.30	65.25	57.37	54.32	53.04	52.27	51.82	51.54	51.29
$\widehat{\mathcal{E}}$	511.34	140.00	65.80	57.75	54.50	53.64	52.44	51.91	51.72	56.65

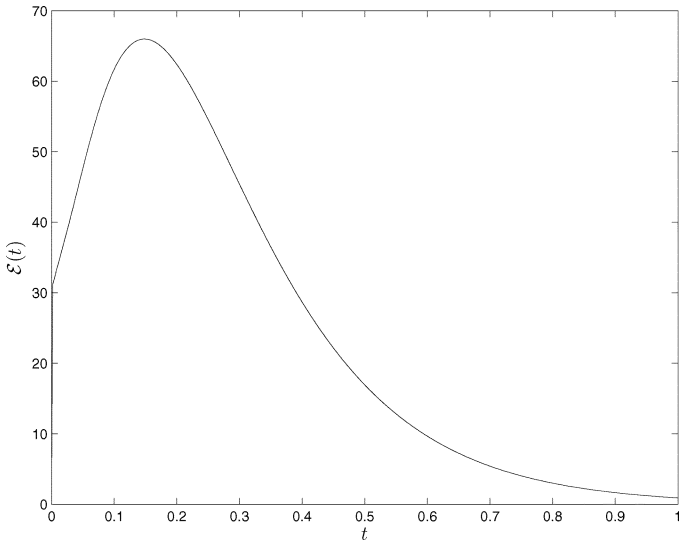


Fig. 2. Transient energy with an upper bound minimizing controller.

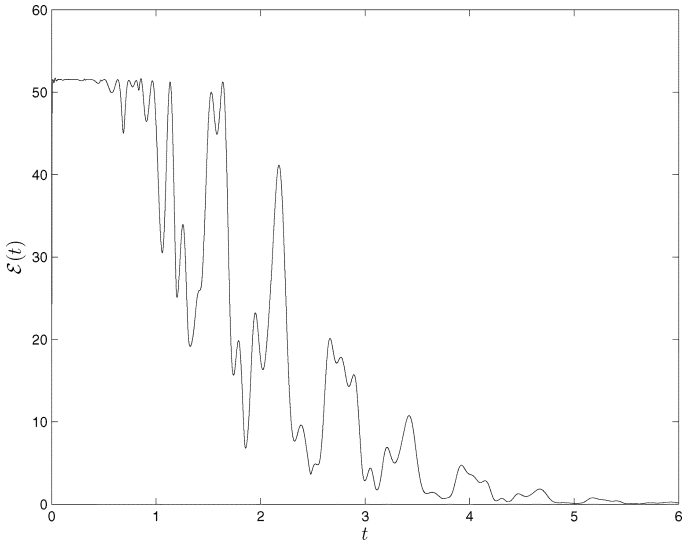


Fig. 3. Transient energy with an approximately minimizing state feedback controller with $\boldsymbol{n_q = 8}$.

$\widehat{\mathcal{E}}_u = 178.47$ with maximum transient energy growth of $\widehat{\mathcal{E}} = 65.99$. The transient energy of the closed-loop system $\mathcal{E}(t)$ is shown in Fig. 2. Note that initially $\mathcal{E}(t)$ rises very quickly (within 10 μs) from 1 to about 30.5. This is a consequence of the very large gains in K. An LMI that also constrains the control effort energy has been proposed in [17].

The dynamic controller problem with $C = I_7$ is now considered. A sequence of SDPs given by (22) was solved using Ritz approximations formed from eigenvalues located at -50, as described in [26]. The degree of the Ritz approximations is given by n_q. The time sequence $\{t_i\}_{i=1}^N$ was kept constant for the whole sequence of problems, and was chosen by trial and error. Table I shows the solutions to the SDPs along with the actual maximum transient energy growth resulting from the

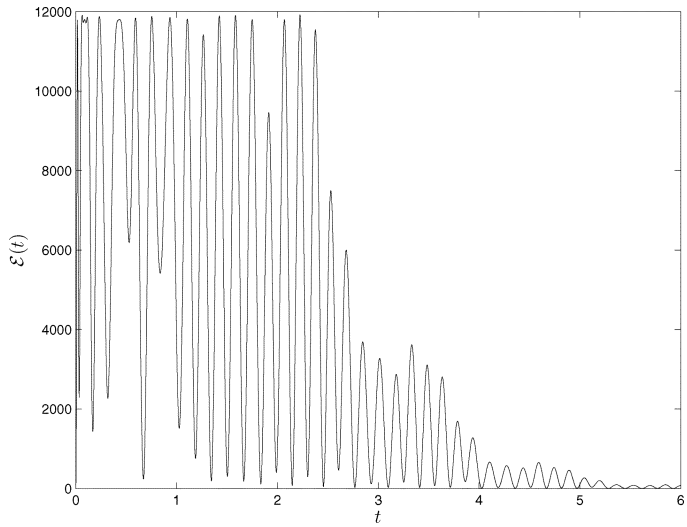


Fig. 4. Transient energy with an approximately minimizing output feedback controller with $\boldsymbol{n_q = 9}$.

controllers obtained from solving (22). It can be seen that the solution converges with increasing n_q. For $n_q \geq 9$, the chosen time sequence $\{t_i\}_{i=1}^N$ is no longer dense enough to give a good approximation to the $\widehat{\mathcal{E}}$ minimization problem. The transient energy of the closed-loop system $\mathcal{E}(t)$ for the controller for $n_q = 8$ is shown in Fig. 3. Again, $\mathcal{E}(t)$ initially rises very rapidly.

Finally, it is assumed that only the sixth state variable can be measured. Eigenvalues located at -50 are again chosen for the Ritz approximation. For a Ritz approximant of degree $n_q = 9$, the maximum transient energy gain is $\widehat{\mathcal{E}} = 11919$. The transient energy $\mathcal{E}(t)$ of the resulting closed-loop system is shown in Fig. 4.

VI. CONCLUDING REMARKS

Upper bound problem LMIs have been suggested previously [15], and some similar results have also appeared recently [11], [12], [14]. These have been extended to consider the robust problem in [13]. Other upper bounds have been proposed in [9].

The proposed method for minimizing the maximum transient energy growth approach can also be used for other norms of the transient response such as the $\mathcal{L}_\infty$-norm [18] that has been used to investigate bounded peaking for linear quadratic optimal control [28]. Investigation of other norms such as those used in [14] remains for further work. Also remaining is a methodical determination of an appropriate sequence of time points $\{t_i\}_{i=1}^N$ so that the problem is solved to a prespecified accuracy.

The controllers resulting from the Q-parametrization are high order and high gain, and although they lead to low maximum transient energy growth, it is clear that these controllers do not necessarily provide good control system designs. The intention is not necessarily to design controllers that meet all desired closed-loop requirements but to provide designers with a means of determining the minimum of the maximum transient energy gain so that the specifications for the controller design can be sensibly set. Alternatively, additional performance criteria can be included in the SDP to improve the design. Note that the results of

[11] and [12] allow for the inclusion of a decay rate constraint, i.e., $\left\|e^{At}\right\| < Me^{\alpha t}$. The inclusion of time-domain constraints to "shape" $\mathcal{E}(t)$ is straightforward within the proposed approach.

Another difficulty with the Q-parametrization approach is the choice of the eigenvalues for the Ritz approximation. This is discussed in more detail in [26]. The Q-parametrization allows for an observer structure that includes state and estimator gain matrices [29]. This is required if the plant is not open-loop stable. However, the suboptimal state feedback controller from Section III could be used in the observer structure, and it is envisaged that this may improve convergence of the Ritz approximation. Determining an appropriate suboptimal estimator gain matrix remains for future work, as does consideration of the robust problem.

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Consensus Seeking Over Random Weighted Directed Graphs

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***Abstract*—We examine the consensus problem for a group of agents that communicate via a stochastic information network. Communication among agents is modeled as a weighted directed random graph that switches periodically. The existence of any edge is probabilistic and independent from the existence of any other edge. We further allow each edge to be weighted differently. Sufficient conditions for asymptotic almost sure consensus are presented for the case of positive weights and for the case of arbitrary weights.**

***Index Terms*—Consensus problem, directed graphs, fast switching, random graphs, weighted graphs.**

I. INTRODUCTION

In a consensus problem, a set of dynamic agents seeks to agree upon certain quantities of interests based upon shared information. Consensus problems are used to model many different phenomena involving information flow among agents, including flocking, swarming, synchronization, distributed decision making, and schooling; see, e.g., the survey paper [1].

Algebraic graph theory [2] is a natural framework for analyzing consensus problems; see, e.g., [3]–[6] and [7]. Within this framework, each agent is modelled as a vertex of a graph, and communication among

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On the minimization of maximum transient energy growth

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