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THE COLLEGE OF AERONAUTICS
CRANFIELD

THE FLUTTER OF THIN, PLANE, ELLIPTIC PANELS

by

D. J. Johns, R. J. Davies and G. H. F. Nayler



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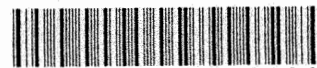
CORRIGENDA



Page 4. Equation (4.5). For $(80 \sqrt{3}/9)$ read $\left(\frac{80 \sqrt{3}}{9}\right)$

Page 9. Section 4.5.1. Equation (4.21) should read:

$$Z_{(x,y)} = a_1 \frac{Z_1}{H_1} + a_2 \frac{Z_2}{H_2} \cos \lambda + a_3 \frac{Z_3}{H_3} \sin \lambda$$



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The Flutter of Thin, Plane, Elliptic Panels

- by -

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SUMMARY

The effect of various parameters on the flutter of elliptic panels is studied by simple theoretical analyses. For clamped edge panels, compressive membrane stresses are shown to have destabilising effects. Structural damping is, in general, destabilising, except for very low values of the ratio of structural and aerodynamic damping coefficients.

The addition of a concentrated mass may also have a destabilising effect, depending upon the position of the added mass. The effect of sweepback is stabilising or destabilising, depending on whether the panel aspect ratio is less than, or greater than, unity. The effect of variations in the edge conditions is shown for the case of circular panels where the difference between simply supported and fully clamped edges corresponds to a difference of 20% in critical panel thickness.

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NOTATION

a_∞	speed of sound in undisturbed medium
$a_n, \bar{a}_n$	modal coefficients defined in equation 3.2.
a, b	semi-axes of elliptic panel in x and y directions
B_1, B_2	edge restraint parameters defined in equation 5.2.
c	panel chord
D	local flexural stiffness, $Eh^3/12(1 - \nu^2)$
E	Young's Modulus of elasticity of panel material
F	external force per unit length acting in the midplane of the panel, tensile force positive
F_c	critical compressive external force per unit length
F_x, F_y	values of F in x and y directions
g	structural damping coefficient
G_{mn}	matrix elements defined in equation 3.4.
h	thickness of panel
H_n	coefficients in assumed modes for elliptic panel (equations 4.1, 4.21)
H	value of H_n for circular panel, $\frac{8}{a}$
i	$\sqrt{-1}$
k	reduced frequency, $c\omega/2U$
K	flutter parameter, $\frac{2qc^3}{MD} \times \frac{8D}{3m\omega_1^2 c^4}$
L	edge-fixity parameter (Section 5)
m	panel mass per unit surface area
$\bar{M}$	magnitude of concentrated mass on panel
M	Mach number
$P_{(x,t)}$	net perturbation pressure, positive downwards
p	pressure in undisturbed supersonic flow
p_i	pressure on lower panel surface
q	dynamic pressure, $\frac{1}{2} \rho U^2$
r, s	coefficients defined after equation 4.4

Notation (Continued)

R, θ	polar co-ordinates in circular panel
t	time
S, T	coefficients defined after equation 4.7
S', T'	coefficients defined after equation 5.6
U	velocity of supersonic flow
v	coefficient defined after equation 4.15
x, y, z	co-ordinates defined in Fig. 1.
x_1, y_1	co-ordinates of position of concentrated mass $\bar{M}$
Z	vertical displacement of panel = $Z_{(xy)} e^{i\omega t}$
$Z_{(x,y)}$	flutter mode shape for elliptic panel
$Z_{n(x,y)}$	nth mode shape for elliptic panel
$\left. \begin{matrix} \alpha \\ \gamma \\ \delta \\ x \\ \psi \end{matrix} \right\}$	edge restraint coefficients defined in equation 5.4
β	concentrated mass parameter, $\bar{M}/\pi abm$
η, ξ	non-dimensional co-ordinates of concentrated mass $\bar{M}$ $\eta = \frac{x}{a}, \quad \xi = \frac{y}{b}$
κ	edge restraint parameter
θ	frequency ratio, ω_1/ω ; angular co-ordinate in circular panel relative to flow direction
λ	angle of sweepback
μ	panel-air mass ratio, $m/\rho c$
ν	Poisson's ratio
ρ	density of undisturbed supersonic stream
ϕ	$[\eta^2 + \xi^2 - 1]^4$
ω_n	in vacuo frequency associated with modal shape Z_n
ω	flutter frequency

1. Introduction

Up to the present time, most plane panel flutter analyses have dealt with rectangular panels having various edge conditions. In Ref. 1, Johns applied linear piston theory in an analysis of a clamped edge elliptic panel. This investigation has been extended in the present study to determine the effects of the following parameters on the flutter of unbuckled elliptic panels :-

- (a) Structural damping
- (b) Membrane stresses
- (c) Concentrated masses
- (d) Sweepback
- (e) Edge restraint

This Note is based, in part, on unpublished theses submitted by the two junior authors in partial fulfillment of the requirements for a Diploma of the College of Aeronautics. The work was supervised by the senior author, Mr. D. J. Johns, who prepared this Note.

2. Statement of the Problem

A flat panel of uniform thickness, h , is mounted in a rigid wall, with its upper surface exposed to a supersonic flow of velocity U , pressure p_∞ and density ρ (Fig. 1). The lower surface of the panel is exposed to still air, pressure p_i , whose acoustic action is ignored. The panel is assumed to be executing simple harmonic motion in the direction normal to the plane of the wall, with frequency ω and infinitesimally small amplitude.

Assuming small deflection thin-plate theory, the differential equation of motion for such a panel is, in general,

$$D \nabla^4 Z + \frac{m \partial^2 Z}{\partial t^2} - F_x \frac{\partial^2 Z}{\partial x^2} - F_y \frac{\partial^2 Z}{\partial y^2} + (p_\infty - p_i) + p_{(x,y,t)} = 0 \quad (2.1)$$

where the displacement of the panel $Z = Z_{(x,y)} e^{i\omega t}$.

It will be assumed that there is no static pressure differential across the panel; i.e. $p_\infty - p_i = 0$.

First order piston theory gives,

$$p_{(x,y,t)} = \rho a_\infty \left(U \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial t} \right) = \rho a_\infty \left(U \frac{\partial Z}{\partial x} + i\omega Z \right) e^{i\omega t} \quad (2.2)$$

In Ref. 2 it was suggested that the aerodynamic damping term should be multiplied by two to allow for the effects of the quiescent air beneath the panel. This assumption will not be made here. It should, of course, be realised that piston theory does not permit the possibility of negative aerodynamic damping.

3. Method of Analysis

We may rewrite equation 2.1 as

$$DV^4 Z - m\omega^2 Z - F_x Z'' - F_y \ddot{Z} + \frac{\rho U}{M} (UZ' + i\omega Z) = 0 \quad (3.1)$$

where primes indicate differentiation with respect to x and dots denote differentiation with respect to y, and

$$\nabla^4 Z = Z'''' + 2\ddot{Z}'' + \ddot{\ddot{Z}}$$

The flutter mode shape $Z_{(x,y)}$ can be approximated by a linear combination of the form

$$Z_{(x,y)} = \sum_{r=1}^N a_r Z_r(xy) \quad (3.2)$$

where the coefficients a_r may represent complex amplitudes. Substitution of equation 3.2 into equation 3.1 yields

$$\sum_{r=1}^N a_r \left\{ D(Z_r'''' + 2\ddot{Z}_r'' + \ddot{\ddot{Z}}_r) - m\omega^2 Z_r - F_x Z_r'' - F_y \ddot{Z}_r + \frac{\rho U}{M} (UZ_r' + i\omega Z_r) \right\} = 0. \quad (3.3)$$

From equation 3.3, N simultaneous equations are obtained by the Galerkin process. The nth equation is obtained by multiplying equation 3.3 by Z_n and integrating over the area of the panel. The N equations may be written in matrix form

$$[G_{mn}] \begin{Bmatrix} a_n \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}, \quad (3.4)$$

and the flutter conditions are determined from the characteristic equation

$$|G_{mn}| = 0. \quad (3.5)$$

It is usual in panel flutter analyses to assume that the modes of displacement Z_r are the mode shapes for the unstressed panel vibrating in vacuo. The computation of these shapes for elliptic plates is rather tedious and, in the following analyses, modes are assumed which correspond to a uniformly distributed load and to a linearly varying load proportional to x. The accuracy of this assumption will be discussed later.

4. Analyses for Elliptic Clamped Edge Panels

4.1. Simple Two-Mode Solution

The flutter mode shape is approximated by

$$Z(x,y) = a_1 \frac{Z_1}{H_1} + a_2 \frac{Z_2}{H_2} \quad (4.1)$$

where

$$Z_1 = \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right]^2,$$

$$Z_2 = x \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right]^2,$$

$$H_1 = \left[\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2 b^2} \right],$$

$$H_2 = \left[\frac{5}{a^4} + \frac{1}{b^4} + \frac{2}{a^2 b^2} \right], \text{ and}$$

the ranges of the integrations in applying the Galerkin process are

$$-b \sqrt{1 - \frac{x^2}{a^2}} < y < +b \sqrt{1 - \frac{x^2}{a^2}}$$

$$-a < x < +a.$$

The resultant flutter determinant (equation 3.5) has the terms

$$G_{11} = \frac{8D}{3} + (\rho U i \omega - M m \omega^2) / 5 M H_1,$$

$$G_{22} = a^2 D + a^2 (\rho U i \omega - M m \omega^2) / 60 M H_2,$$

$$G_{12} = \rho U^2 / 10 M H_2,$$

$$G_{21} = -\rho U^2 / 10 M H_1,$$
(4.2)

To include hysteretic structural damping it is only necessary to factor the stiffness terms by $(1 + ig)$, where g is the structural damping coefficient. If g is assumed constant for the modes considered, D is replaced by $D(1 + ig)$.

The solution of the flutter determinant for zero structural damping yields the stability boundary (Ref. 1)

$$\frac{1}{\mu^2} = \left[1 - r/75s^2 \right] \left[4M^2 k^4 / (3 - k^2) \right], \quad (4.3)$$

and the flutter frequency,

$$\omega = \sqrt{\frac{150sD}{m}}, \quad (4.4)$$

where $r = 8H_1 H_2 / 3$ and $s = (2H_1 + 9H_2) / 45$ are functions of the panel geometry, and μ is the panel-air mass ratio, $m/\rho c$.

If aerodynamic damping effects are now ignored, which may be justified for zero structural damping as in the case of rectangular panels (Ref. 1), equations 4.3 and 4.4 reduce to

$$2qc^3 / MD = (80\sqrt{3}/9) \times \left[39 + 14 \frac{a^2}{b^2} + 3 \frac{a^4}{b^4} \right]. \quad (4.5)$$

Replacing M by $\sqrt{M^2 - 1}$ corresponds to using Ackeret-loading and should give better results for a wider range of supersonic Mach numbers.

Using equation 4.5, Fig. 2 presents the variation of the parameter $2qc^3 / MD$ with panel eccentricity $\frac{a}{b}$. It can be concluded that the most critical value of $\frac{h}{c}$ is for the two-dimensional panel ($\frac{a}{b} = 0$) and it is interesting to note that the ratio of the critical thickness ratios for infinite and unit aspect ratio is 1.13, which agrees well with the ratio obtained by Hedgepeth (Ref. 3) for rectangular simply supported panels. These results show that at high Mach numbers the effect of aspect ratio, in the range unity to infinity, on the critical thickness ratio is relatively small, both for rectangular and elliptic panels.

In Ref. 4, Fung has pointed out that the results shown in Fig. 2 become increasingly inaccurate as the aspect ratio decreases, because the aerodynamic tip effect ignored in the simple analysis becomes increasingly more important. Hence, the right hand side of the curve in Fig. 2 must be viewed with caution.

Fig. 3 gives the variation of the critical panel thickness ratio with Mach number for the two dimensional ($a/b = 0$) and circular ($a/b = 1$) clamped edge panels, at sea level. Fig. 3 has been derived from equation 4.5 with M replaced by $\sqrt{M^2 - 1}$.

4.2. Effect of Structural Damping

The effect of structural damping on panel flutter has received very little treatment in the literature and there has been some doubt as to how such effects should be included. In Ref. 5 it was decided to incorporate into the analysis viscous damping, which is proportional to, and in phase with, the velocity of the panel oscillations. The alternative approach would have been to use hysteretic damping involving a complex stiffness term, which is proportional to the panel displacement and in phase with the velocity of the panel motion. It was reasoned that since the latter approach is only strictly applicable to simple harmonic motion it could lead to difficulties when complex frequencies are considered. Even so, it was shown that the addition of viscous structural damping could be destabilising. In another paper, (Ref. 6), it was shown that the effect of hysteretic structural damping could be stabilising or destabilising depending upon whether the Mach number of the flow was less than, or greater than, $\sqrt{2}$. This dependence on $M = \sqrt{2}$ is apparently a consequence of the aerodynamic assumptions which in Ref. 6 included damping terms based on linearised unsteady aerodynamic theory. Johns and Parks (Ref. 7), showed for a two-dimensional panel with simply supported edges that hysteretic structural damping could combine with aerodynamic damping to be destabilising. The analysis of Ref. 7 assumed linear piston theory for the aerodynamic forces.

In this section, the approach of Ref. 7 will be applied to the problem of the clamped edge elliptic panel.

The inclusion of hysteretic structural damping modifies the stiffness terms (D) by a factor $(1 + ig)$, where g is the structural damping coefficient. For the two mode analysis, the flutter determinant yields eventually the following two equations,

$$\frac{27}{16} K^2 \theta^4 = - \left[\theta^4 S - \theta^2 T + 1 \right] + g^2 \theta^2 \left[\theta^2 S + \theta \frac{R}{g} T + \left(\frac{R}{g} \right)^2 \right], \quad (4.6)$$

$$\frac{R}{g} = \left[\frac{T - 2S\theta^2}{T\theta^2 - 2} \right] \theta \quad (4.7)$$

$$\text{where } K = 8Ma_\infty^2 / 3\mu\omega_1^2 c^2 = \frac{2qc^3}{MD} \times \frac{8D}{3m\omega_1^2 c^4}$$

$$S = 9H_2/2H_1$$

$$T = 1 + S$$

$$R = \rho a_\infty / m\omega_1$$

$$\theta = \text{frequency ratio} = \omega_1 / \omega$$

$$\text{and } \omega_1 = \text{fundamental frequency} = \sqrt{\frac{40H_1 D}{3m}}$$

The parameter R/g is, effectively, the ratio of aerodynamic and structural damping coefficients and from equation 4.7 the value of the frequency ratio, θ , corresponding to this parameter may be determined for any elliptic panel. The critical Mach number, as given by K , may then be found from equation 4.6.

If, initially, g is assumed zero, solution of equations 4.7 and 4.6 gives respectively

$$\theta = \sqrt{2/T} \quad (4.8)$$

$$\frac{27}{4} K^2 = T^2 - 4S + 2T \cdot R^2 \quad (4.9)$$

These equations yield the results previously obtained in Section 4.1 viz. equations 4.4 and 4.3 respectively. Since R is always small one can conclude that for negligible aerodynamic damping, i.e. $R = 0$, equation 4.9 gives

$$\frac{27}{4} K^2 = T^2 - 4S \quad (4.10)$$

If, however, R is initially assumed to be zero in equations 4.7 and 4.6, one obtains, respectively,

$$\theta = \sqrt{T/2S} \quad (4.11)$$

$$\text{and } \frac{27}{4} K^2 = \frac{4S}{T^2} \left[T^2 - 4S \right] + 4g^2 S \quad (4.12)$$

Hence if g is now assumed to be zero, equation 4.12 reduces to give the result,

$$\frac{27}{4} K^2 = \frac{4S}{T^2} [T^2 - 4S], \quad (4.13)$$

which is completely dissimilar to equation 4.10.

Similar results to this were derived in Ref. 7 for the simply supported, two dimensional panel and a fuller discussion of the implications of such results is given there. For the purposes of this section it suffices only to say that equation 4.13 represents a lower critical Mach number than given by equation 4.10, the extent of the effect being indicated by the quantity $\sqrt{\frac{4S}{T^2}}$. The value of

this ratio for various elliptic planforms is given in Table 1, where the suffixes gR and Rg indicate the order in which these quantities are allowed to tend to zero.

Equation 4.7 can be rearranged to read

$$\theta^2 = \frac{2}{T} + \frac{g}{R} \frac{\theta}{T} (T - 2S\theta^2) \quad (4.14)$$

which for very small values of the ratio g/R gives

$$\left. \begin{aligned} \theta^2 &\approx \frac{2}{T} + \frac{g}{R} \sqrt{\frac{2}{T}} \left(1 - \frac{4S}{T^2}\right) \\ \theta^4 &\approx \frac{4}{T^2} + \frac{4g}{TR} \sqrt{\frac{2}{T}} \left(1 - \frac{4S}{T^2}\right) \\ \theta &= \sqrt{\frac{2}{T}} \end{aligned} \right\} \quad (4.15)$$

Substituting these values into equation 4.6 yields, after some manipulation,

$$\frac{27}{4} K^2 = T^2 - 4S + 2TR^2 \left[1 + \frac{g}{R} \sqrt{\frac{T}{2}} \left(1 + \frac{4S}{T^2} \right) \right], \quad (4.16)$$

which should be compared with equation 4.9.

The square bracketed term indicates that the effect of structural damping for small values of g/R is indeed stabilising but since R is always small and the term in g/R is also small, the overall effect is probably of no practical importance.

It can thus be shown from equation 4.6 and 4.7 that, except for very small values of the ratio g/R , the effect of structural damping is destabilising. For a two dimensional panel and with $R = .088$, the corresponding results for $g = 0$ and $g = 0.05$ are $\theta = .485$ and $\theta = .6$ respectively and the ratio of critical Mach numbers is $\frac{M(g = 0.05)}{M(g = 0)} = 0.9$. The effect of structural damping is more destabilising as $R \rightarrow 0$ i. e. as altitude increases and the most extreme effect is evidenced by the results of equations 4.10 and 4.13, which were discussed above.

4.3. Effect of Membrane Stresses

The presence of mid-plane stresses in an actual panel is inevitable, since very little of an aircraft structure is not load-carrying. The inclusion of mid-plane stresses in two dimensional analyses is common, e.g. Refs. 1, 3 and 7, but there are few similar three-dimensional analyses. However, Hedgepeth (Ref. 3) has examined the flutter of a rectangular simply supported panel of finite size with mid-plane stresses in both directions and he concluded that only the mid-plane stresses in the direction of the airflow were important. The transverse stresses had no effect on the critical Mach number.

In this Section a three dimensional analysis is presented for the clamped edge elliptic plate assuming a constant load per unit length in both mutually perpendicular directions i.e. $F_x = F_y = F$ in equations 3.1 and 3.3.

The flutter determinant which results has additional terms in the expressions for G_{11} and G_{22} (see equation 4.2), viz. in G_{11} :

$$+ \frac{F}{H_1} \frac{2}{3} \left\{ \frac{1}{a^2} + \frac{1}{b^2} \right\}$$

and in G_{22} :

$$+ \frac{a^2}{15} \frac{F}{H_2} \left\{ \frac{3}{a^2} + \frac{1}{b^2} \right\}.$$

Solution of the flutter determinant for zero structural damping yields the following expressions when aerodynamic damping is ignored, which may be compared with equations 4.5 and 4.4,

$$2qc^3/MD = \left(\frac{80\sqrt{3}}{9}\right) \times \left[39 + 14 \frac{a^2}{b^2} + 3 \frac{a^4}{b^4} + \frac{Fa^2}{10D} \left(13 + \frac{a^2}{b^2}\right) \right] \quad (4.17)$$

$$\omega = \sqrt{\frac{150sD + Fv}{m}} \quad (4.18)$$

where $v = \frac{1}{3} \left(\frac{23}{a^2} + \frac{11}{b^2} \right)$ is a function of the panel geometry.

As a check on the accuracy of the solution, the buckling condition for the panel can be obtained from the expression for G_{11} , and compared with more exact solutions. Thus, in the absence of inertia and aerodynamic forces, the buckling force F_c is given by,

$$\frac{F_c a^2}{D} = 4 \left[3 + 2 \frac{a^2}{b^2} + 3 \frac{a^4}{b^4} \right] / \left[1 + \frac{a^2}{b^2} \right]$$

or
$$\frac{F_c b^2}{D} = 4 \left[3 + 2 \frac{a^2}{b^2} + 3 \frac{a^4}{b^4} \right] / \left[\frac{a^2}{b^2} \left(1 + \frac{a^2}{b^2} \right) \right]$$

In Table 2, the variation of the buckling parameter $\frac{F_c a^2}{D}$ with panel eccentricity, a/b , is shown and compared with the results of Refs. 8, 9. This shows that the energy method used in Ref. 8 is only 0.7% in error for a circular panel, compared with the exact solution of Ref. 9. The present analysis gives a corresponding error of 9%.

Inspection of equation 4.17 shows that the stabilising effect of the tensile membrane stresses is greatest for $a/b = 0$ i.e. the two-dimensional panel. Also, as $a/b \rightarrow \infty$, i.e. when the stresses are mainly transverse to the air flow, the stabilising effect is insignificant. The latter result agrees with the conclusions of Ref. 3 that forces in the y direction have no effect on the stability boundaries for rectangular simply supported panels. For $a/b = 0$, the reduction in critical panel thickness/chord ratio for $Fa^2/D = 10$ compared with $Fa^2/D = 0$ is 9.2%.

4.4. Effect of a Concentrated Mass

The addition of a concentrated mass to the panel is of interest analytically because

- (a) it may be necessary experimentally to mount a strain gauge or other transducer to the panel, and the effect of this must be known,
- (b) the concentrated mass may have a stabilising effect analogous to the mass balancing of control surfaces.

The analysis follows that of Section 3, except that equation 3.3 is modified by the addition of a concentrated mass of magnitude $\bar{M}$ at a point on the panel having co-ordinates (x_1, y_1) . The flutter determinant obtained in Section 4.1 (i.e. equation 4.2) has the following additional terms in each of the expressions G_{mn} , viz.

$$\begin{aligned} \text{in } G_{11} & - \omega^2 \beta m \phi / H_1, \\ \text{in } G_{22} & - \omega^2 \beta m \phi a^2 \eta^2 / H_2, \\ \text{in } G_{21} & - \omega^2 \beta m \phi a \eta / H_1, \\ \text{in } G_{12} & - \omega^2 \beta m \phi a \eta / H_2. \end{aligned}$$

In the above terms, $\bar{M} = \beta m \pi a b$ defines the term β ,

$$\begin{aligned} \eta &= \frac{x_1}{a} \quad \text{and} \quad \xi = \frac{y_1}{b} \quad \text{defines the co-ordinates of } \bar{M} \\ \text{and } \phi &= \left[\eta^2 + \xi^2 - 1 \right]^4. \end{aligned}$$

Solution of the flutter determinant for zero structural damping yields the following result for the flutter frequency

$$\omega^2 = 300sD / \left\{ 2m + 5\beta m \phi (1 + 12\eta^2) \right\}. \quad (4.19)$$

If aerodynamic damping is also neglected, the critical dynamic pressure parameter can be expressed by

$$\left(\frac{2qc^3}{MD} \right)^2 = \frac{6400}{12} a^8 \left[\frac{8H_1}{3} - \frac{\omega^2}{5D} (m + 5\beta m \phi) \right]^2 + 6400a^8 \left[\frac{\omega^2}{D} \beta m \phi \eta \right]^2 \quad (4.20)$$

The above results are strictly true only for an additional mass system which is symmetrical about the x-axis but not about the y axis, i.e. $\bar{M}/2$ should be concentrated at $\eta, \pm\xi$. For a mass system which is symmetrical about the y axis also, the additional terms in G_{11} and G_{22} would be the same as above but those in G_{12} and G_{21} would be zero, i.e. for $\bar{M}/4$ concentrated at $\pm\eta, \pm\xi$.

From the above results it can be shown that the addition of a concentrated mass on the y-axis is always stabilising and is most effective when added at the centre of the panel. For example, for $\beta = 1$ the result obtained from equation 4.20, and to be compared with equation 4.5, is

$$\frac{2qc^3}{MD} = \frac{80\sqrt{3}}{9} \times \frac{16}{7} \left[33 + 13 \frac{a^2}{b^2} + 6 \frac{a^4}{b^4} \right] \quad (4.21)$$

Table 3 presents the results for a circular panel and a two-dimensional panel with a concentrated mass, ($\beta = 1$), situated at various points along the x axis. It can be seen that a marked increase in stability occurs when $\eta = 0$ and this decreases as η increases until, for $\eta = 0.6$ and 0.8 , there is, in fact, a reduction in stability. However, if the added mass had been utilised to double the panel thickness, the value of the parameter $2qc^3/MD$, in terms of the original thickness, would be 6,880 for the circular panel and 4,800 for the two dimensional panel. Thus it can be inferred that the addition of a concentrated mass as a means of stabilisation is not as effective as increasing the panel thickness.

Table 4 presents corresponding results for a circular panel with a concentrated mass at the centre of the panel, showing the effects of variations in β .

4.5. Effect of Sweepback

4.5.1. Three Mode Solution

For the elliptic panel shown in Fig. 1, the air flow velocity, U , has components $U\cos\lambda$ and $U\sin\lambda$ in the x and y directions respectively, when the panel y axis is swept by an angle λ . The effect of the sweepback is also introduced by assuming that the flutter mode shape is now approximated by (cf. equation 4.1)

$$Z_{(xy)} = a_1 \frac{Z_1}{H_1} + a_2 \frac{Z_2}{2H_2} \cos\lambda + a_3 \frac{Z_3}{3H_3} \sin\lambda \quad (4.21)$$

where $Z_3 = y \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right]^2$

$$H_3 = \left[\frac{1}{a^4} + \frac{5}{b^4} + \frac{2}{a^2 b^2} \right]$$

The resultant flutter determinant has the terms

$$\begin{aligned}
 G_{11} &= \left[\frac{8D}{3} + \left(\frac{\rho U i \omega - M m \omega^2}{5 M H_1} \right) \right] \\
 G_{22} &= \left[a^2 D + a^2 \left(\frac{\rho U i \omega - M m \omega^2}{60 M H_2} \right) \right] \cos^2 \lambda \\
 G_{33} &= \left[b^2 D + b^2 \left(\frac{\rho U i \omega - M m \omega^2}{60 M H_3} \right) \right] \sin^2 \lambda \\
 G_{12} &= \rho U^2 \cos^2 \lambda / 10 M H_2 \\
 G_{21} &= -\rho U^2 \cos^2 \lambda / 10 M H_1 \\
 G_{13} &= \rho U^2 \sin^2 \lambda / 10 M H_3 \\
 G_{31} &= -\rho U^2 \sin^2 \lambda / 10 M H_1 \\
 G_{23} &= G_{32} = 0.
 \end{aligned} \tag{4.23}$$

A general solution of the flutter determinant to give a closed form result is not possible so a particular solution has been obtained for the following values of the various parameters (in lb. slug. ft. units),

$$\begin{aligned}
 D &= 67.5 \\
 m &= 0.0432 \\
 a &= 1 \\
 b &= 0.75 \\
 \rho a_\infty^2 &= 2967 \\
 \rho a_\infty &= 2.64
 \end{aligned}$$

These values correspond to an aluminium elliptic panel at sea level having a panel thickness/chord ratio $h/c = 4 \times 10^{-3}$. Table 5 presents the results of this analysis, giving values of flutter frequency, ω , and critical Mach number, M_c , for different angles of sweepback, compared with a two-mode analysis. Those results for $\lambda = 0^\circ$ and 90° could be obtained directly from equations 4.4 and 4.5: there would be slight differences in the values for M_c so obtained because of the neglect of aerodynamic damping in the equation 4.5.

4.5.2. Two Mode Solution

An alternative approach to the problem is to assume that the flutter mode shape is approximated by

$$Z_{(x,y)} = \bar{a}_1 \frac{Z_1}{H} + \bar{a}_2 \left(\frac{Z_2}{H_2} \cos \lambda + \frac{Z_3}{H_3} \sin \lambda \right) \quad (4.24)$$

The four terms in the flutter determinant can now be written directly from the expressions in equation 4.23, viz :

$$\begin{aligned} \bar{G}_{11} &= G_{11} \\ \bar{G}_{22} &= G_{22} + G_{33} \\ \bar{G}_{12} &= G_{12} + G_{13} \\ \bar{G}_{21} &= G_{21} + G_{31} \end{aligned} \quad (4.25)$$

Using the parameters given above in Section 4.5.1., Table 5 presents the results of this analysis, compared with those from the three mode solution. The agreement between the two solutions is good and in both cases the variation of M_c with sweepback follows a reasonable trend.

5. Analysis for Circular Panels with Variable Edge Restraint

The analyses in the previous section were for elliptic clamped edge panels and it is desirable that corresponding investigations be made for panels with other forms of edge restraint. However, simple analytical expressions for either the vibration modes, or the pressure modes as used in Section 4, are not available for general elliptic panels and the present analysis is restricted therefore to circular panels for which the required pressure modes are known.

Thus the assumed flutter mode shape is

$$Z_{(xy)} = a_1 \bar{Z}_1 + a_2 \bar{Z}_2 \quad (5.1)$$

$$\begin{aligned} \text{where } \bar{Z}_1 &= \left[1 - \frac{x^2 + y^2}{a^2} \right] \left[B_1 - \frac{x^2 + y^2}{a^2} \right] \\ \bar{Z}_2 &= \frac{x}{a} \left[1 - \frac{x^2 + y^2}{a^2} \right] \left[B_2 - \frac{x^2 + y^2}{a^2} \right] \end{aligned} \quad (5.2)$$

These modes correspond to uniform and linearly varying pressure loadings respectively, where the terms B_1 and B_2 are given by (Ref. 10) :-

$$B_1 = \frac{1 + \frac{5}{\nu}}{1 + \frac{1}{\nu}} \quad \text{and} \quad B_2 = \frac{1 + \frac{7}{\nu}}{1 + \frac{3}{\nu}} \quad \text{for simply supported edges,}$$

and $B_1 = B_2 = 1$ for fully clamped edges.

For values of edge restraint intermediate between simply supported and fully clamped, the appropriate boundary conditions to be satisfied at the edge of the plate are

$$Z = 0.$$

$$D \left(\frac{\partial^2 Z}{\partial R^2} + \frac{\nu}{R} \frac{\partial Z}{\partial R} + \frac{\nu}{R^2} \frac{\partial^2 Z}{\partial \theta^2} \right) = -L \frac{\partial Z}{\partial R},$$

where L is an elastic constant for the support at the edge, i.e.

$$L = 0 \text{ for simply supported edges}$$

$$L = \infty \text{ for fully clamped edges}$$

$$L > 0 \text{ for intermediate, elastic, edge restraint.}$$

Thus it can be inferred from similar analyses in Refs. 11 and 12 that more general expressions for B_1 and B_2 are

$$B_1 = \frac{1+5\kappa}{1+\kappa} \quad \text{and} \quad B_2 = \frac{1+7\kappa}{1+3\kappa}$$

$$\text{where } \kappa = 1 / \left(\nu + \frac{La}{D} \right).$$

Thus the values of L quoted above would correspond to

$$\kappa = 1/\nu \text{ for simply supported edges}$$

$$\kappa = 0 \text{ for fully clamped edges}$$

$$0 < \kappa < 1/\nu \text{ for elastically restrained edges.}$$

Using the above modes in equation 3.3, and applying the Galerkin process, yields the following terms in the flutter determinant (neglecting membrane stresses)

$$\left. \begin{aligned} G_{11} &= \frac{8D}{3} \cdot \alpha + \left(\frac{\rho U i \omega - M m \omega^2}{5MH} \right) \cdot \delta \\ G_{22} &= D \cdot \gamma + \left(\frac{\rho U i \omega - M m \omega^2}{60MH} \right) \cdot x \\ G_{12} &= -G_{21} = \frac{\rho U^2}{10MH a} \cdot \psi \end{aligned} \right\} \quad (5.3)$$

$$\left. \begin{aligned} \text{where } \alpha &= 1 + \frac{3}{2} (B_1 - 1) \\ \delta &= 1 + \frac{5}{2} (B_1 - 1) + \frac{5}{3} (B_1 - 1)^2 \\ \gamma &= 1 + 2(B_2 - 1) \\ x &= 1 + 3(B_2 - 1) + \frac{5}{2} (B_2 - 1)^2 \\ \psi &= 1 + \frac{5}{3} (B_2 - 1) + \frac{5}{6} (B_1 - 1) + \frac{5}{3} (B_1 - 1) (B_2 - 1) \\ \text{and } H &= \frac{8}{a^4} \end{aligned} \right\} \quad (5.4)$$

If hysteretic structural damping is assumed, as in Section 4.2, the following two equations are obtained from the flutter determinant

$$\frac{27}{16} \frac{\psi^2}{\delta x} K^2 \theta^4 = - \left[\theta^4 S' - \theta^2 T' + 1 \right] + g^2 \theta^2 \left[\theta^2 S' + \theta \frac{R}{g} T' + \left(\frac{R}{g} \right)^2 \right] \quad (5.5)$$

$$\frac{R}{g} = \left[\frac{T' - 2S' \theta^2}{T' \theta^2 - 2} \right] \theta, \quad (5.6)$$

where $S' = \frac{9}{2} \cdot \frac{\delta y}{\alpha x},$

$$T' = 1 + S'$$

$$\theta = \text{frequency ratio} = \frac{\omega_i}{\omega}$$

and $\omega_1 = \text{fundamental frequency} = \sqrt{\frac{40HD}{3m} \frac{\alpha}{\delta}}.$

Equations 5.5 and 5.6 are similar to equations 4.6 and 4.7, and hence a similar analysis can be made here of the effects of structural and aerodynamic damping on the flutter of clamped and simply supported circular panels. The principal findings of this section are summarised in Table 6, together with other comparable data from earlier sections and from Refs. 1, 11 and 12.

6. Limitations in the Previous Analyses

6.1. Use of Static Deflection Modes

In the previous analyses, static deflection mode shapes were used because the 'in vacuo' vibrational mode shapes for general elliptic panels could not be represented in a convenient analytic form. However, analyses in Ref. 11 for circular panels have indicated that very little difference in the results is obtained when vibrational mode shapes are used. When vibrational modes are used, the flutter determinant has to be determined separately for each chosen value of edge fixity coefficient κ , since κ does not enter into the deflection function directly. Thus it is suggested that static deflection mode shapes are more convenient to use and do not introduce any significant errors.

The flutter determinant obtained in Ref. 11 for the circular panel using vibrational mode shapes had terms G_{m_n} which, when compared to those in equation 4.2, were found to agree to within ± 2 per cent. The calculated flutter parameters are compared in Table 6 with those determined from static deflection mode analyses, and those from Ref. 1, assuming zero structural damping.

6.2. Assumption of Only Two Degrees of Freedom

It is of interest to determine how the inclusion of more degrees of freedom in the panel flutter problem affects the results obtained, i.e. whether or not the solution converges as the number of terms in the flutter mode increases, and, if it does converge, how realistic is the two mode approximation. Analyses of this type have been made in Ref. 11 for a clamped edge circular panel using a flutter mode of the form :

$$Z = (1 - R^2)^2 \left[A_0 + A_1 R \cos \theta + A_2 R^2 \cos^2 \theta \dots + A_n R^n \cos^n \theta \right]$$

The first and second terms represent the static deflection shapes assumed in equation 4.1 and the other terms represent mode shapes which have nodal lines perpendicular to the flow direction and situated symmetrically about the centre.

The results obtained indicate that the flutter parameter K was larger for the three-mode than for the two-mode approximation by about 25 per cent but that the use of further terms produced only a very small change in K. Further, the four term approximation gave somewhat lower, and the five term approximation somewhat higher, results than the preceding approximations. It was also noticed that for the range of $\frac{Fa^2}{D} > 7$, K no longer showed a linear dependence on $\frac{Fa^2}{D}$ - contrary

to the result shown in equation 4.14. However, in a more recent paper, Lee (Ref.13) has investigated the same convergence problem by using a slightly different approximation, viz.

$$Z = (1 - R^2)^2 \left[A_0 + A_1 R \cos \theta + A_2 R^2 \cos 2\theta \dots + A_n R^n \cos n\theta \right]$$

The only difference is in replacing $\cos^n \theta$ by $\cos n\theta$, which is the same for the vibration mode in vacuo. Calculations using this approximation were carried out and, for comparison, analyses using three vibration modes in vacuo were made. The same rapid convergence of the approximate modes approach was observed, as in Refs. 11 and 12, but it was found that K varied linearly with respect to the midplane load parameter.

To summarise, therefore, the two mode results presented earlier may be expected to be conservative for design purposes on the basis of the results of Refs. 11, 12 and 13.

7. Conclusions

Theoretical analyses have shown the destabilising effects of compressive membrane stresses and structural damping (in general) on the flutter of clamped edge elliptic panels. The addition of a concentrated mass to the panel has been investigated and the possibility of such an addition being destabilising shown. The effect of sweepback is stabilising or destabilising, depending on whether the panel aspect ratio $\frac{b}{a}$ is less than, or greater than, unity; the predicted variation of critical Mach number with sweepback has been shown to agree well from two and three mode analyses.

Simple analyses have also shown that the effect of edge restraint on the flutter of circular panels is equivalent to a difference of 20 per cent in critical panel thickness between simply supported and fully clamped panels.

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$\frac{a}{b} =$	0	1/2	1	2	∞
$\frac{M_{Rg}}{M_{gR}} =$	.645	.668	.77	.925	.98

TABLE 1

Ratio of Critical Mach Numbers for Clamped Edge Elliptic Panels.

$$\text{i.e. } \frac{M_{Rg}}{M_{gR}} = \sqrt{\frac{4S}{T^2}}$$

$\frac{a}{b} =$	0	1/2	1	2	3	4	5
Ref. 8		-	14.79	44.08	99.09	178.40	282.50
Present Analysis	12.00	11.30	16.00	47.20	105.60	183.06	296.62
Ref. 9	9.87	-	14.68	-	-	-	-

TABLE 2

Buckling Coefficient for Clamped Edge Elliptic Panels. $F_c a^2/D$

Mass Position $\eta =$	0	.2	.6	.8	1.0
Flutter Parameter $\frac{2qc^3}{MD}$ (Circular Panel)	1830	1550	776	634	860
Flutter Parameter $\frac{2qc^3}{MD}$ (Two-Dimensional Panel)	1162	983	496	304	600

TABLE 3

Effect of Concentrated Mass Position on the Flutter of a Clamped Edge Panel

($\beta = 1$; Mass on x-axis)

Mass Parameter β	0	.2	.4	.6	.8	1.0
Flutter Parameter $\frac{2qc^3}{MD}$	860	1318	1542	1680	1768	1830

TABLE 4

Effect of Variations in Concentrated Mass on the Flutter of a Circular Clamped Edge Panel. (Mass on Centre of Panel)

λ	Three Mode Solution		Two Mode Solution	
	ω rad/sec	M_c	ω rad/sec	M_c
0	846	3.24	846	3.24
30	864	3.51	915	3.47
45	883	3.76	906	3.79
60	913	4.12	958	4.22
90	1059	4.98	1059	4.98

TABLE 5

Effect of Sweepback on Flutter Mach Number for a Clamped Elliptic Aluminium Panel at Sea Level. ($h/c = 4 \times 10^{-3}$, $a = 12$ in., $b = 9$ in.)

	Simply Supported		Clamped	
	Circular	Two-Dimensional	Circular	Two-Dimensional
$m\omega^2 c^4/D$	1775 (1648)	((832))	4693 (4480)	2020
$2qc^3/MD$	485 (468)	((274))	860 (804)	600

TABLE 6

Flutter Parameters for Circular and Two-Dimensional Panels

The figures not in brackets correspond to analyses using static deflection mode shapes in this Note. The figures in brackets correspond to analyses using vibrational mode shapes, i.e. (()) from Ref. 1 and () from Ref. 11.

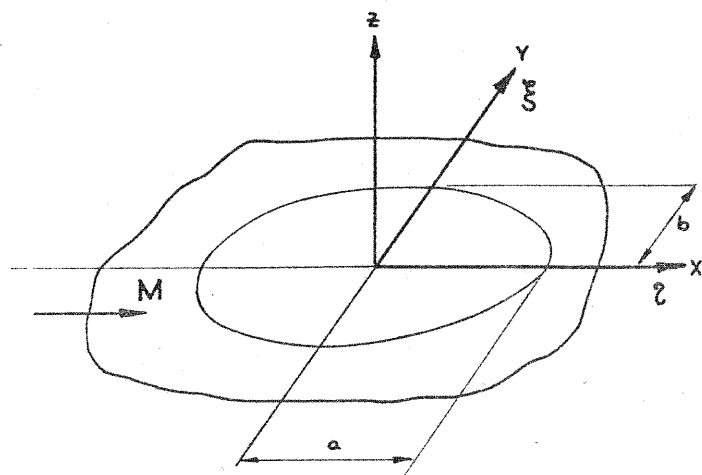


FIG.1. CO-ORDINATE SYSTEM FOR THE ELLIPTIC PANEL.

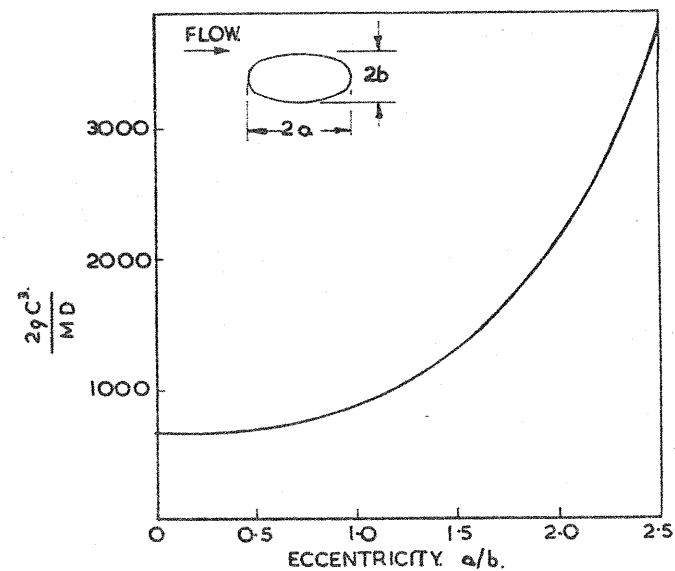


FIG.2. VARIATION OF CRITICAL FLUTTER PARAMETER WITH PANEL ECCENTRICITY.

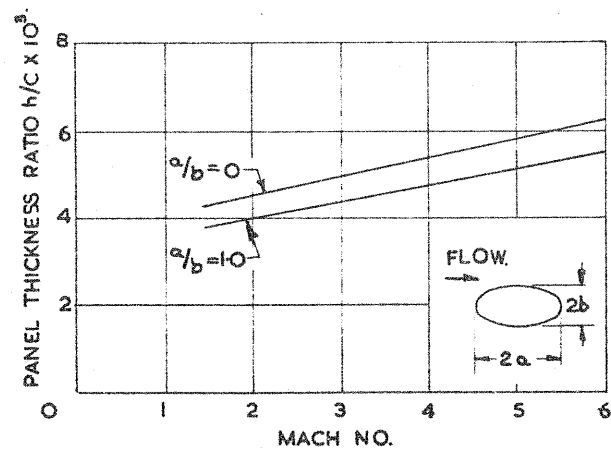


FIG.3. VARIATION OF CRITICAL PANEL THICKNESS WITH MACH NO. AT SEA LEVEL