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FINITE ELEMENT AND BOUNDARY ELEMENT METHODS
FOR ELASTO-PLASTIC STRESS ANALYSIS OF
TWO-DIMENSIONAL AND AXISYMMETRIC PROBLEMS

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In the name of God, Most Gracious, Most Merciful.

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SYNOPSIS

This work dealt with elasto-plastic analysis of 2-dimensional and axisymmetric problems using the finite-element and boundary-element techniques. The analysis is based upon a wide range of yield criteria and hardening rules so as to be suitable for most engineering materials.

Many accuracy measures have been employed with the boundary-integral equation derivations, such as isoparametric boundary elements, singular quadrature, corner models, finite-difference schemes for internal stress evaluation, high-order integration cells, and the use of boundary-integral expressions for domain loading terms.

A programming package has been developed with its own mesh generator and results plotter. The package has been validated using several case studies including a benchmark test case recommended by NAFEMS.

The finite-element analysis of elasto-plastic problems proved to be more stable and reliable than the corresponding boundary-element analysis.

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General Notation

<u>Symbol</u>	<u>Definition</u>
A	The total strain hardening parameter
$A^{(i)}$	The strain hardening parameter for isotropic hardening rule
$A^{(k)}$	The strain hardening parameter for kinematic hardening rule
$\underline{B}_S, \underline{B}_L$	The small and large deflection strain-displacement matrices
C_f, θ_f	The cohesion force and angle of internal friction
$\underline{D}$	The elastic stress/strain matrix
$\underline{D}_{ep}$	The elsto-plastic stress/strain matrix
$\underline{D}_p$	The plastic stress/strain matrix
$d\lambda$	Proportionality factor
$\overline{d\varepsilon}_e$	The uniaxial elastic (effective) strain increment
$\overline{d\varepsilon}_p$	The uniaxial plastic (effective) strain increment
$\overline{d\varepsilon}_T$	The uniaxial total (effective) strain increment
E	Young's modulus
E_t	The tangential modulus
$d\underline{\alpha}^Z, d\underline{\alpha}^P$	The Ziegler's and Prager's shift vectors respectively
$\underline{H}, \underline{G}$	The global matrices defined in terms of $\underline{g}^e$ and $\underline{h}^e$
H'	The strain-hardening parameter

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<u>Symbol</u>	<u>Definition</u>
I_1, I_2, I_3	The stress invariants
$\hat{i}, \hat{j}, \hat{k}$	Unit vectors along x, y, z axes, respectively
J_1, J_2, J_3	The invariants of the deviatoric stress tensor
$J(\xi, \eta)$	The Jacobian matrix in 2-dimensional plane
$\underline{K}_e$	The element stiffness matrix
$\underline{K}_{ep}$	The elasto-plastic small deflection stiffness matrix
$\underline{K}_L$	The elasto-plastic large deflection matrix
$\underline{K}_G$	The global stiffness matrix
$K(m_e), E(m_e)$	The first and second complete elliptic integrals
l, m, n	The directional cosines of the outward normal w.r.t x, y, z axes, respectively
$\hat{n}$	A unit vector in the outward normal direction to the boundary.
N_i	A Lagrangian shape function
$P_{\gamma\beta}$	The spherical stress tensor
R, Z	The body force intensities in r and z directions, respectively
$S_{\gamma\beta}$	The stress deviator tensor
S_1, S_2, S_3	The principal stress deviator components
$\underline{T}$	The global surface traction vector
u, v, w	The displacement components in the x, y, z or r, θ, z directions, respectively
U	The strain energy of the body

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<u>Symbol</u>	<u>Definition</u>
$\underline{U}$	The global body force vector
W	The work done by external loads
X, Y	The body force intensities in x and y directions, respectively
Y	The instantaneous yield stress
Y_0	Initial yield stress ($Y=Y_0$ for the case of kinematic hardening rule)
Y_t, Y_c	The absolute tensile and compressive yield stress
Y_r	The reduced yield stress in case of mixed hardening rule
$\underline{\alpha}$	The shift vector
Γ	The boundary of the domain Ω
Γ_e	The boundary of the e^{th} element (BEM)
Γ_c	An infinitesimal boundary of a corner
$\underline{\delta}$	The global displacement vector
$\epsilon_{\gamma\beta}$	The strain tensor
$\overline{\epsilon}$	The effective strain
$\underline{\epsilon}_e$	The elastic strain vector
$\underline{\epsilon}_p$	The plastic strain vector
$\underline{\epsilon}^L, \underline{\epsilon}^S$	The large and small deflection strain vectors, respectively

<u>Symbol</u>	<u>Definition</u>
$\bar{\epsilon}_p^i$	The total "accumulated" effective plastic strain for isotropic part
λ, μ	Lame's coefficients (μ is the modulus of rigidity)
ν	Possion's ratio
ξ, η	The Intrinsic coordinates
$\sigma_{\gamma\beta}$	The stress tensor
σ_m	The mean or hydrostatic stress
$\bar{\sigma}$	The effective stress
$\bar{\sigma}_r$	The reduced stress vector
$\bar{\sigma}^R, \bar{\sigma}_{re}$	The residul stress vector
$\bar{\sigma}_c$	The corrected stress vector
$\bar{\sigma}_m$	The equivelent stress vector for modified von Mises and modified Tresca polynomial yield criteria
χ	The total potential energy of the body
Ω	The domain of a specified problem
Ω_e	The domain of the e^{th} element (FEM)

Other local notations are defined in the text

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CHAPTER ONE

INTRODUCTION

1.0 General Introduction

In the last four decades approximate numerical methods have become more attractive, accurate, and reliable for solving Boundary Value Problems (BVP), as a result of rapid development of the high-speed digital computer technology.

The most commonly used numerical methods are:-

- the finite difference method (FDM)
- the finite element method (FEM)
- the boundary element method (BEM)

The field variables of the BVP can be governed by one or more ordinary or partial differential equation(s). According to the finite difference solution procedure the governing differential equation(s) within the domain is modelled by approximating the differential operators in the equation by simpler, localized algebraic ones valid at a series of nodes within the domain of the body. A pointwise discretization concept should be employed in order to define the domain of the given problem in terms of a grid of nodes and the unknown parameters can there be considered to be the values of the field function at those nodes. Therefore, in this method the governing differential equation(s) is transformed into a set of simultaneous algebraic equations, with a narrow band matrix, which may be solved directly, or iteratively, depending upon whether the problem is elliptic, parabolic, or hyperbolic, after employing the given boundary conditions.

The finite difference solution may be improved as more grid points are used. The FDM is difficult to use for problems with irregular or complex geometries or for problems with unusual types of boundary conditions. However, the FDM is still a reliable technique for computational fluid mechanics.

By using a suitable variational approach or weighted-residual method an integral formulation for the governing equation of a BVP may be

obtained.

The essence of the FEM involves division of the domain into a suitable number of sub-domains, or finite elements, by employing the piecewise discretization concept. Therefore the solution of a complex problem can be obtained by approximating the solution over each sub-domain by a simple function, i.e the field variables are interpolated over the finite element by polynomials. The FEM also produces an algebraic system of equations for the global domain, which may have a narrow band. At the present time, the FEM represents the most widely used numerical tool for many engineering applications. There are many FE packages available, such as, NASTRAN, PAFEC, FINEL, ASAS, ABAQUS, IDEAS,.. etc, which cover a wide range of engineering applications.

The previous domain methods, i.e FDM and FEM are based upon direct implementation of the given governing, domain, differential equations (FDM) or integral equations (FEM) without any need for further modification.

Using particular mathematical techniques, the volume integral in the governing integral equation can be transformed into a surface integral on the boundary of the domain, i.e into a Boundary Integral Equation (BIE). The essence of the BEM is to obtain an approximate numerical solution of the governing BIE, by means of piecewise discretization, whereby the boundary of the domain is divided into sub-boundaries, or boundary elements. The global algebraic boundary element equation can be assembled from the equation of the boundary, pieces, or elements.

The global boundary equation can be solved, after employing the boundary conditions, in order to find the values of the field variables over the boundary of the domain, and hence their values at any internal point can be obtained.

The advantages of the BEM compared with other numerical (domain)

approaches can be summarized as follows:-

- The dimensionality of the problem is reduced and the number of algebraic equations which need to be solved, are generally less than those generated with the FEM or the FDM for comparable accuracy.
- The reduced complexity and quantity of the input data makes the BEM model a quicker and easier process.
- The BEM is suitable for finite and infinite domain problems whilst the domain methods are limited to problems with defined boundaries.
- The method requires no internal modelling, and the values of the field variables can be calculated at any selected interior point.

There are two major categories in which the boundary integral method can be formulated. There are the direct formulation and indirect formulation. In the direct formulation the unknown functions are the actual physical variables of the problem which are either tractions and/or displacements at a point on the boundary, depending upon the boundary conditions. When the traction is specified in some region of the boundary, then the displacement becomes unknown in that region and vice versa.

The direct BIE is usually obtained by using the Maxwell-Betti reciprocal theorem, Somigliana's identity, or by integrating the weighted-residual expression of the governing differential equations.

On the other hand, the indirect methods formulate the problem in terms of unknown density functions, which have no physical significance. However, once these density functions are determined, the displacement

and/or stresses can be directly computed.

Due to the lack of physical interpretation in the indirect method, the direct method is generally more popular and will be employed in this work.

1.2 Literature Review

1.2.1 Elasto-plastic Stress Analysis

An early study in the plastic behaviour of materials was contributed by James Clerk Maxwell in 1856 when he made the early proposal of the elastic failure theory, i.e Distortion energy theory. In 1864 the first real experimental study of plasticity was carried out by Tresca [1] when he published a preliminary account of experiments into punching and extrusion, followed by the formulation of his well-known yield criterion.

The first milestone in the mathematical theory of plasticity was published in 1871 by Saint-Venant [3,4] who set up a system of five equations governing the stresses and strains in two-dimensional flow. In that year Levy proposed three-dimensional relationships between stress and the rate of plastic strain by adopting Saint-Venant's conception of an ideal plastic material [5]. Equations similar to the Levy equations were introduced independently by von Mises at the beginning of this century, therefore the equations are known as the Levy-Mises equations.

The next important contribution in the field was made in the twenties and thirties of this century, in Germany, by such researchers as Prandtl, von Karman, Nadai, Hencky, Geiringer, Lode, and others [2].

The Maxwell theory was independently reformulated by Huber 1904, and von Mises [6] 1913. In 1926 experimental results showed that the Levy-Mises stress-strain relations are valid only for perfectly-plastic material, which was also confirmed by early experiments dealing with the shape of yield loci by Taylor and Quinney in 1931 [7]. According to Lode's results the theory was generalized in two important directions. A first extension of the Levy-Mises equations was carried out by Prandtl 1924 [8] for planar problems and the generality was completed by Reuss [9] 1930. The generalized equations are known as the Prandtl-Reuss equations. A second generalization was introduced by Odquist (1930) and Schmidt (1932) [2].

Thus, by 1932, a theory had been constructed reproducing the main plastic and elastic properties of an isotropic metal at ordinary temperatures.

Around 1950, strain-hardening materials began to attract attention, D.C. Drucker [10] pointed out the severe restriction imposed on possible stress-strain relations by a mathematical formulation of the concepts of work hardening. In the next year he published his postulate [11], which led to the important and required constitutive relation between the plastic strain increment and stresses and stress increment, which was called the flow rule.

Ziegler [12] indicated that the plastic behaviour of a material can be described by specifying an initial yield condition, a flow rule, and a hardening rule. There is much of experimental work, described in the literature dealing with the investigation of the shape and orientation of the subsequent yield surface under different conditions. In order to obtain the real behaviour of the material and produce a mathematical model for the hardening rule, which can explain the behaviour of the material, a comprehensive literature review was carried out in References [13-15].

The simplest hardening rule is the isotropic hardening rule which

leads to an overall expansion of the yield surface. This rule was introduced by R.Hill [2] and it has been used extensively in the formulation of isotropic plasticity theory. The increase in the size of the initial yield surface has been indicated and observed by many experimentalists [16-24].

The theory of plastic anisotropy proposed by R.Hill [25] is given for a metal which is in a certain state of anisotropy due to some specific cause, and is not a theory which brings the state of anisotropy into correlation with the previous history of deformation.

The earliest anisotropic hardening rule is known as the Kinematic hardening rule and was formulated by Prager in 1955 [26,27] in considering the Bauschinger effect. Zeigler [12] made a rigorous formulation of the kinematic rule idea, excluding the contradictions involved in Prager's theory. Experiments concerned with the determination of subsequent yield surfaces [22], indicated an approximate agreement with the translational behaviour of the yield surface as predicted by the kinematic hardening rule.

Kadashevick and Novozhilov [28], and Eisenberg and Phillips [29] extended the theory of kinematic hardening so that it might include a nonlinear hardening material, Mroz [30] proposed another procedure for this purpose.

Y.Yoshimura [31,32] formulated a theory which correlates plastic anisotropy, and the Bauschinger effect, with the previous strain history. The most successful mathematical formulation of the distortion hardening displayed by some materials, was proposed by Baltov and Sawczuk [33], followed by Backhaus [34,35]. Danilov [36] generalized the results of Baltov and Sawczuk for the case of an arbitrary hardening curve which can even be applied to initially anisotropic bodies.

It can be concluded from the experimental results obtained by References [21,22,37,38] that the definition of the uniaxial yield stress

is of crucial importance in relation to the shape and orientation of the subsequent yield surface.

From experiments reported in References [39-41], it was observed that the Bauschinger effect was small, and the distortion of the yield surface was significant, for definitions of yielding corresponding to large offset strain or Lode-extrapolation point. While for yield point definitions corresponding to small offset strains, the distortion disappears. From experiments done by Phillips and his collaborators [42-49], it can be concluded that the lack of distortion of the yield surface is a universal feature when utilizing the proportional limit as the uniaxial yield limit.

1.2.2 Finite Element Elasto-plastic Analysis

The roots of the finite element method began in the field of structural engineering. Probably the earliest attempts were by Hrennikoff(1941) and Mchenry (1943)[50] who developed analogies between actual discrete elements and the rest of the continuous solid. Numerical solutions based upon energy methods were developed so as to deal with complex problems, as indicated by Argyris and Kelsey [51]. In 1956, the first finite element paper was published, without being so named, when Turner et al [52] employed a three-noded triangular piecewise discretization for the stress analysis of two-dimensional problems.

The name "finite element" was coined by Clough [53] who presented the technique for plane stress analysis. Engineers have developed the finite element method as a powerful tool for solving their linear and non-linear problems.

The first use of the finite element technique for solving plasticity problems, involving non-linear material behaviour, was given

by Gallagher et al. [54], the plasticity analysis implementing the concept of linearization in order to obtain a solution iteratively.

Two basic procedures may be employed for elasto-plastic analysis which are based upon the estimation of an initial strain, or an initial stress, as a result of linearization of non-linear, plasticity, problems. The initial strain approach has been discussed by Gallagher et al. [54], Argyris [55], Pope [56], and Swedlow et al. [57], whilst the initial stress approach which may be called "the Residual Force method" has been used by Yamada et al. [58], Zienkiewicz et al. [59], and Nayak and Zienkiewicz [60].

There are three common iterative schemes in the literature. An early scheme was the pure iterative solution scheme (Modified Newton-Raphson method) where an initial elastic stiffness matrix is used throughout the loading. This scheme was used by Padlog et al. [61], Gallagher et al. [54], Lansing et al. [62], Argyris [55], Zienkiewicz [59], and Nayak et al. [60]. The alternative of updating the stiffness matrix at every step of loading without iteration, corresponds to the so called variable stiffness technique, or Newton-Raphson method, which was employed by Goldberg, et al. [63], Pope [64], Marcal et al. [65], Yamada et al. [58], and Wilson [66]. The combined technique of updating with iteration is used by Balmer [67], Owen and Salonen [68], and Runesson [69].

In this work the initial stress approach or residual force approach has been used because it is more convenient for most of the relevant engineering materials than the initial strain approach which is not suitable for perfectly-plastic or low strain-hardening materials.

1.2.3 Boundary Element for Two-Dimensional Elasto-plastic Analysis

The boundary element method (BEM) is based upon classical integral equation formulation of boundary value problems. The important contributions of the formal understanding which established the foundation of the BIE has been made by many Russian researchers such as, Muskhelishvili [70], Mikhlin [71], Kupradze [72], and Smirnov [73].

Early attempts at employing a general numerical solution procedure in order to solve the integral boundary equation of linear problems have been introduced by Jaswon [74], Symm [75], Massonet [76], and others. More information concerning the historical and literature review of the application of BEM to linear problems can be taken from many References such as [77-80].

The BEM has been successfully applied for the solution of non-linear problems and in particular, elasto-plastic problems. The first formulation of the BEM for plasticity was introduced in 1971 by Swedlow and Cruse [81], and was the theoretical foundation for all subsequent work. They presented an extension of Somigliana's identity by incorporating a volume integral involving plastic strains. They employed neither the initial-stress approach nor the initial-strain approach, which should have been used in order to obtain the expression for stepwise plasticity analysis, although the basic ideas were well explored earlier by Lin in 1967[82].

Ricardella [83] implemented an algorithm based on Swedlow and Cruse's formulation for planar problems (two-dimensional) of elasticity and elasto-plasticity using von Mises yield criterion, and the isotropic hardening rule. He used piecewise constant interpolation of plastic strain rates. The complete integral expression for stress at internal points was not presented because of the singularity due to the partial

derivations of the Kernel functions.

Mendelson [84] formulated a boundary integral equation for elasto-plastic two- and three-dimensional bodies. He introduced Boundary Integral formulations for elasto-plastic problems using Direct, Indirect, and a Direct biharmonic (i.e Semidirect) approaches. The direct formulation was presented, including the integral expressions for internal stresses (two-, and three-dimensional problems).

Rzasnicki and Mendelson [85] in 1973 and Mendelson and Albers [86] in 1975 implemented the same formulation and presented numerical solutions for some two-dimensional cases. These formulations were essentially based on an initial-strain approach and, hence, are restricted to strain hardening materials. Kumar and Mukherjee [87] used a similar initial strain formulation and presented a solution for simple problems of expanding cylinders and spheres. They simplified the formulation by using a special form of the constitutive relationships.

None of the above authors gave the correct expressions for strains and stresses at internal points. Bui [88], in 1978, published the correct expressions for strains and stresses at internal points by using the concept of Mikhlin [89] for the derivative of the singular integral of the inelastic term.

In 1979 Telles and Brebbia [90] introduced the complete boundary element formulation for two- and three-dimensional plasticity problems. The proper derivatives of the singular domain integrals were given. They did not show any solutions of engineering examples. In 1980 [91] Telles and Brebbia also employed the BEM to solve two-dimensional elasto-plastic problems, based upon the initial-strain approach and using the isotropic, von Mises yield criterion with strain hardening, ideal plasticity and strain softening behaviour.

Banerjee, et al [92] described the first application of the method to three-dimensional elasto-plastic problems. The direct BEM for

elasto-plastic problems based upon initial-stress approach was formulated by Brebbia and Telles [93]. Banerjee and Cathei [94], derived the same formula and also calculated the internal stresses by computed internal displacements. They also applied a numerical differentiation scheme of the type used in finite difference techniques, instead of using the integral expression. They followed this using the elasto-plastic stress-strain relation to evaluate the corresponding stresses. Cathie [95] compared the two methods of evaluation of the strain and he concluded that the finite difference method required less computation than that required for the evaluation of strain using the integral expressions, his study was based upon constant or linear elements only.

Almost all of the further publications deal with constant or linear boundary elements and volume cells. Lee and Fenner [96], introduced an isoparametric quadratic formulation of the boundary integral method for two-dimensional elasto-plastic problems. Their formulation was based upon an initial-strain approach using von Mises yield criterion. They employed the two approaches for the calculation of the internal strains and stresses (i.e the finite difference method and the Integral expression). The use of such isoparametric representations not only improves the accuracy of the BEM analysis but also leads to a faster rate of convergence in the iterative algorithm.

1.2.4 Boundary Element For Axisymmetric Elasto-plastic Analysis

One of the most attractive advantages of the BIE method is the reduction-in-dimension of the problem. If the domain of the problem is symmetric in geometry and in all boundary data and loading with respect to an axis, then there is a further reduction possible because of the axisymmetry. Arising from axisymmetric conditions, a meridional section of an axisymmetric body needs only be considered, and hence reduces the problem to a quasi two-dimensional problem, i.e the surface integrals

may be reduced to line integrals. This can result in a considerable reduction in modelling effort and computer time.

The earliest indirect BIE formulation for axisymmetric elasticity problems was introduced by Kermanidis in 1975 [97], who also derived the fundamental ring solution by employing a three-dimensional point load solution in cylindrical coordinates, and integrating around the ring.

Mayr et al [98] (1976) extended the axisymmetric BIE formulation to consideration of the non-axisymmetric boundary tractions and displacements by employing a Fourier series expansion. In 1977 Cruse et al [99] obtained the elastic fundamental ring load solution using a Galerkin vector and also formulated a direct BIE formulation for axisymmetric problems with thermal and centrifugal loading.

All of the above works were carried out by using constant or linear boundary elements. Bakr and Fenner [100,101] extended the previous work using "Isoparametric" quadratic elements. Details of linear axisymmetric boundary element analysis can be found in the notes of Bakr [102].

The first elasto-plastic solution of axisymmetric problems by the direct BIE based on an initial-stress algorithm was introduced by Cathie and Banerjee in 1980 [103]. The internal stresses were calculated using an approximate formulation based on a simple finite difference procedure.

In 1981 Doblare et al [104], calculated the stress at internal points after extending and correcting the integral equation presented by [105], however they did not go on to solve any examples. Sarihan and Mukherjee [106] (1982) employed an alternative formulation, which was previously referred to by Telles and Brebbia (1980), as the "fictitious tractions and body forces approach". Telles in 1982 [107] presented an initial-strain elasto-plastic axisymmetric formulation, which can deal with general bodies of revolution.

In 1987 Henry et al [108] introduced a multi-region substructure analysis for elasto-plastic and thermoplastic axisymmetric media in which curvilinear boundary elements and volume cells were used.

1.3 Objective of the work

In this work an isoparametric quadratic formulation of the direct BIE method for elasto-plastic two-dimensional and axisymmetric problems using an initial-stress approach will be introduced.

The internal strain and stress will be evaluated by using either finite difference methods or an Integral expression.

A basic objective of this research is to develop a Finite and Boundary element Package for elasto-plastic analysis with small deflection problems. The elasto-plastic analysis can be carried out by using any one of the following yield criteria:-

- von Mises yield criterion,
- Tresca or Tresca polynomial yield criterion,
- Modified von-Mises yield criterion,
- Modified Tresca polynomial yield criterion,
- Mohr-Coloumb yield criterion, and
- Drucker-Prager yield criterion.

The following Hardening rules have been considered for elasto-plastic analysis:-

- Isotropic hardening rule,
- Kinematic hardening rule, and
- Mixed hardening rule.

Also the formulation will be capable of solving perfectly-plastic

problems or problems involving linear or non-linear strain-hardening, or softening, materials.

1.4 Outline of Thesis

This thesis is divided into eight chapters. Chapter 2 contains a review of small and large strain components of the derivation for two-dimensional and axisymmetric problems. The basic concepts of the elasto-plastic theory with different yield criteria and hardening rules are explained. Also in chapter 2 the concepts of elasto-plastic analysis have been employed for defining the elasto-plastic stress/strain matrix, for an explicit form with certain and specified yield criteria and hardening rules.

Chapter 3 deals with the FEM theory for elasto-plastic small and quasi-large deflection problems with two-dimensional, or axisymmetric problems. The FEM elasto-plastic solution algorithm has been explained with the direct iterative scheme, the Newton-Raphson iterative scheme, or the Modified Newton-Raphson iterative scheme. The stress correction procedure, which can also be used for BEM solutions has been explained. The $\underline{D}_{ep}$ evaluation procedure is explained in detail.

Chapter 4 introduces the derivation of the BIE for small and large two-dimensional elasto-plastic problems from first principles using the integration-by-parts theorems. The formulation is based upon the initial-stress approach using the previously described $\underline{D}_{ep}$ matrix. Transformation of the volume integral of the domain loading terms into boundary integral terms has been shown. The boundary discretization of the BIE has been employed in order to obtain the discretized form of the BIE for numerical purposes. Many techniques, such as singular integration, the double-node technique, and the finite difference method have been used for improving the accuracy at points nearest to the

boundary.

Chapter 5 deals with the BEM for axisymmetric problems. The derivation of the BIE for elasto-plastic problems with small and large deflections has been explained in details with a transformation for the domain-loading integral terms due to centrifugal and translational accelerations, into simpler boundary-integral terms. The elasto-plastic solution procedure in this chapter is similar to that given in the previous chapter.

Chapter 6 is concerned with the structure and the main features of the programming package FBNSA (Finite Element and Boundary Element Non-linear Stress Analys_is package) developed in this work.

Chapter 7 describes the engineering case studies used for validation of the programming package, with appropriate results and the relevant discussion.

Chapter 8 contains the final conclusions of this study and recommendations for future work.

CHAPTER TWO

ELASTO-PLASTIC RELATIONS FOR SMALL AND LARGE DEFLECTION PROBLEMS

2.0 INTRODUCTION

The deformation of a continuous solid body depends on the properties of the material and the environmental conditions, such as temperature of the body, duration of the process, rate of the applied load, ... etc. Thus the deformation of the continuum can be small or large with linear or non-linear elastic, viscoelastic, viscoplastic, plastic or creep material behaviour.

In this Chapter small and large deflection problems have been considered with linear-elastic and plastic material behaviour. Also the material is assumed isotropic and homogeneous. The behaviour of the material after yielding, for the uniaxial test case, can be assumed to be either perfectly-plastic, or with linear strain-hardening, or non-linear strain hardening.

The objective of solving a stress-analysis problem of a given solid body is to define the values of the field variables (i.e. displacement and force vectors, and stress and strain tensors) inside the domain and at the boundary. Therefore direct relationships between the field variables are required. Examples of such relationships are the kinematic relation which is a geometrical expression and which is valid within the elastic or the plastic region, and the kinetic relation which can also be used for linear or non-linear material behaviour, and a suitable constitutive relation, as shown in Fig(2.1).

To obtain the elasto-plastic relation for a complex problem under combined loads, an experimental test should be carried out so as to observe the elasto-plastic behaviour. In this way it is possible to assume a reasonable mathematical model to correlate between the uniaxial stress-strain test and the multiaxial case. In this work mathematical models from the literature to be shown later will be used, and developed.

2.1 Analysis of Stress and Equations of Equilibrium

To define the state of stress at a point inside a loaded body, the stress tensor can be used as follows [109 - 110].

$$\sigma_{\gamma\beta} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (2.1)$$

The principal stresses are the roots of the following equations:

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} = 0$$

Expanding the above determinant, the following cubic equation is obtained:

$$\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0 \quad (2.2)$$

where

$$\left. \begin{aligned} I_1 &= \sigma_{\gamma\gamma} = \sigma_x + \sigma_y + \sigma_z \\ I_2 &= (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) - (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) \\ I_3 &= \sigma_x \sigma_y \sigma_z + 2 \tau_{xy} \tau_{yz} \tau_{zx} - (\sigma_x \tau_{yz}^2 + \sigma_y \tau_{zx}^2 \\ &\quad + \sigma_z \tau_{xy}^2) \end{aligned} \right\} \quad (2.3)$$

The parameters I_1 , I_2 and I_3 are known as the stress invariants. The roots of the previous cubic equation are the principal stresses σ_1 , σ_2 and σ_3 .

2.1.1 Stress Deviator Tensor and Spherical Stress Tensor

It is convenient for plasticity theory to split the stress tensor as follows:

$$\sigma_{\gamma\beta} = P_{\gamma\beta} + S_{\gamma\beta}$$

where

$$P_{\gamma\beta} = \sigma_m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.4)$$

$\sigma_m = \frac{1}{3} I_1$ is the hydrostatic stress

$$S_{\gamma\beta} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix} \quad (2.5)$$

The tensor $P_{\gamma\beta}$ is called the spherical stress tensor, while $S_{\gamma\beta}$ is known as the stress deviator tensor.

Equation 2.2 can be reduced to a cubic equation, without a second-degree term, by using the following substitution.

$$\sigma_i = S_i + \frac{I_1}{3}, \quad i = 1, 2, 3$$

where $S_1, S_2, S_3 \dots$ are the principal stress deviator components.

Hence

$$\left(s + \frac{I_1}{3}\right)^3 - I_1 \left(s + \frac{I_1}{3}\right)^2 - I_2 \left(s + \frac{I_1}{3}\right) - I_3 = 0$$

$$\begin{aligned}
 \text{i.e.} \quad & \left(S^3 + I_1 S^2 + \frac{1}{3} I_1^2 S + \frac{I_1^3}{27} \right) \\
 & - \left(I_1 S^2 + \frac{2}{3} I_1^2 S + \frac{1}{9} I_1^3 \right) \\
 & - \left(I_2 S + \frac{1}{3} I_1 I_2 + I_3 \right) = 0
 \end{aligned}$$

which can be reduced to the following cubic equation:

$$S^3 - J_1 S^2 - J_2 S - J_3 = 0$$

where

$$J_1 = I_1 - I_1 = 0$$

$$J_2 = I_2 + \frac{1}{3} I_1^2$$

$$J_3 = I_3 + \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3$$

Finally, the above equation can be rewritten as follows:-

$$S^3 - J_2 S - J_3 = 0 \quad (2.6)$$

J_2, J_3 are called the invariants of the
Deviatoric stress tensor.

It can be proved that

$$\begin{aligned}
 J_2 = \frac{1}{3} \left[I_1^2 + 3I_2 \right] &= \frac{1}{3} \left[(\sigma_x + \sigma_y + \sigma_z)^2 - 3(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) \right. \\
 &\quad \left. + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]
 \end{aligned}$$

i.e.

$$J_2 = \frac{1}{6} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] \geq 0 \text{ always} \quad (2.7)$$

But

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 = (S_x - S_y)^2 + (S_y - S_z)^2 + (S_z - S_x)^2$$

i.e.

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 = 3(S_x^2 + S_y^2 + S_z^2)$$

Then

$$J_2 = \frac{1}{2} (S_x^2 + S_y^2 + S_z^2) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \quad (2.8)$$

which can be rewritten in tensorial terms as follows:-

$$J_2 = \frac{1}{2} \sum_{\gamma} \sum_{\beta} S_{\gamma\beta} S_{\gamma\beta} \quad (2.9)$$

An alternative form of J_2 , which is known as the effective stress $\bar{\sigma}$, can be defined as follows:-

$$\bar{\sigma} = C_{\sigma} \cdot \sqrt{J_2}$$

where

$\bar{\sigma}$ is a measure of the distortion energy in the multiaxial problem

C_{σ} is a numerical parameter such that $\bar{\sigma} = \sigma_1$, for the uniaxial stress case.

For such a case

$\sigma_x = \sigma_1$, while the other stress tensor components equal zero.

Then

$$J_2 = \frac{1}{6} [\sigma_1^2 + \sigma_1^2] = \frac{1}{3} \sigma_1^2$$

Hence $C_\sigma = \sqrt{3}$

i.e.

$$\bar{\sigma} = \sqrt{3 J_2} = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{\frac{1}{2}} \quad (2.10)$$

While for the J_3 invariant it can be proved that:

$$J_3 = S_x S_y S_z + 2 \tau_{xy} \tau_{yz} \tau_{zx} - (S_x \tau_{yz}^2 + S_y \tau_{zx}^2 + S_z \tau_{xy}^2) \quad (2.11)$$

or

$$J_3 = \frac{1}{3} \sum_{\gamma} \sum_{\beta} \sum_k S_{\gamma\beta} S_{\beta k} S_{k\gamma} \quad (2.12)$$

2.1.2 Methods for Finding the Principal Stress Components

In order to find the principal stresses, it is perhaps best to solve the cubic equation 2.6. The solution can be carried out by employing either the cosine identity method or sine identity method. For the second method it can be assumed that:-

$$S = \beta \cdot Z$$

Then the cubic equation 2.6, can be expressed as follows:-

$$z^3 - (J_2/\beta^2) \cdot z - (J_3/\beta^3) = 0$$

By comparing the above equation with the following sine identity:-

$$\sin^3 \phi - \frac{3}{4} \sin \phi + \frac{1}{4} \sin 3\phi = 0$$

Hence

$$\beta = \frac{2}{3} \bar{\sigma}$$

and

$$\left. \begin{aligned} \phi &= \frac{1}{3} \sin^{-1} \left(-\frac{3\sqrt{3} J_3}{2 J_2^{1.5}} \right) \\ \text{or} \quad \phi &= \frac{1}{3} \sin^{-1} \left(-\frac{27 J_3}{2 \bar{\sigma}^3} \right) \end{aligned} \right\} \quad (2.13)$$

where

$$-\frac{\pi}{6} \leq \phi \leq \frac{\pi}{6}$$

and it can be shown that:-

$$\left. \begin{aligned} S_1 &= \frac{2}{3} \bar{\sigma} \sin \left(\phi + \frac{2\pi}{3} \right) \\ S_2 &= \frac{2}{3} \bar{\sigma} \sin \phi \\ S_3 &= \frac{2}{3} \bar{\sigma} \sin \left(\phi - \frac{2\pi}{3} \right) \end{aligned} \right\} \quad (2.14)$$

It can also be proved that $S_1 \geq S_2 \geq S_3$,

i.e. $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and hence:-

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\bar{\sigma}}{\sqrt{3}} \cos \phi \quad (2.15)$$

2.1.3 Equilibrium Equations

The equilibrium equations are used for relating the stresses to the applied loads, and they may be written at a point inside a loaded body as follows:-

$$\frac{\partial \sigma_{y\beta}}{\partial x_\beta} + X_y = 0 \quad (2.16)$$

While the surface equilibrium equation for an infinitesimal area "dA" with normal vector $\hat{n}(l,m,n)$ can be obtained after defining the traction vector over dA, as follows:-

$$T = \frac{dF}{dA} = T_x \hat{i} + T_y \hat{j} + T_z \hat{k}$$

where

$\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x,y,z axes,
dF is the surface load vector over dA, as shown in Fig(2.2.a).

By using the equilibrium condition of the forces in the x-direction, it can be deduced that:-

$$\Sigma F_x = 0$$

$$\text{i.e. } dA T_x = \sigma_x dA_x + \tau_{yx} dA_y + \tau_{zx} dA_z$$

where $dA_x, dA_y, dA_z \dots$ are the projections of dA on the $yz, zx,$ and xy planes respectively, as shown in Fig(2.2.b).

The area vector can be defined as follows:-

$$\begin{aligned} dA &= dA \hat{n} = dA(l \hat{i} + m \hat{j} + n \hat{k}) \\ &= dA l \hat{i} + dA m \hat{j} + dA n \hat{k} \end{aligned}$$

Hence

$$\begin{aligned} dA_x &= \hat{i} \cdot d\vec{A} = dA l \\ dA_y &= \hat{j} \cdot d\vec{A} = dA m \\ dA_z &= \hat{k} \cdot d\vec{A} = dA n \end{aligned}$$

Finally, it can be proved that:-

$$\begin{aligned} T_x &= l \sigma_x + m \tau_{yx} + n \tau_{zx} \\ \text{Similarly } T_y &= l \tau_{xy} + m \sigma_y + n \tau_{zy} \\ T_z &= l \tau_{xz} + m \tau_{yz} + n \sigma_z \end{aligned} \quad \left. \vphantom{\begin{aligned} T_x \\ T_y \\ T_z \end{aligned}} \right\} \quad (2.17)$$

or in tensorial form as:-

$$T_\gamma = \sigma_{\gamma\beta} \cdot l_\beta \quad \gamma, \beta = 1, 2, 3 \quad (2.18)$$

when T_γ is the traction vector on the surface area,

$\sigma_{\gamma\beta}$ is the stress tensor at a loaded point,

l_β the directional cosines of the outside normal vector to the surface area at the considered point.

2.2 Analysis of Strain

The state of strain is considered as a relative measure of deformation, i.e. the strain is a geometrical property of deformation. There are many different methods for the definition of the strain at a point inside a deformed body, such as, Cauchy's engineering strain, Green's strain, Al-Mānsi's strain, ... etc. The above definitions of strain are based upon either Lagrangian coordinates or Eulerian coordinates. Both are credited to Euler [111]. The analysis of small deflection strains is similar in Lagrangian or Eulerian coordinates.

In Lagrangian description the frame of the axes is assumed to be fixed, or stationary, in space and everything is to be referred to the fixed axes, i.e. the relative deformation, is measured with respect to the original length. The Lagrangian description is also called "Stationary Lagrangian" or "Total Lagrangian".

In the "updated Lagrangian", or Euler, approach, it is assumed that the reference frame deforms with the structure. The equations governing the behaviour are modified according to the axes but at least the displacements are small with respect to the new or converted axes. This approach is rarely used in structures, but it is common in aerodynamics and fluid mechanics where the absolute movements of the deformed material may be vast [112].

Suppose points "a" and "b" are the end points of an infinitesimal line element in an undeformed body with coordinates (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$ respectively, as shown in Fig(2.3.a). Then the vector $\vec{ab}$ can be defined as follows:-

$$\vec{ab} = \vec{\Delta X_0} = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}$$

After subjecting the body to external forces, the points "a" and "b" are deformed into "a'" and "b'" with new coordinates as follows:-

$$a' \equiv (x+u, y+v, z+w) ,$$

$$b' \equiv (x + u + \Delta x + \Delta u, y + v + \Delta y + \Delta v, z + w + \Delta z + \Delta w),$$

and therefore the infinitesimal vector $\overrightarrow{\Delta X}_0$ changes to $\overrightarrow{\Delta X}$, such that:-

$$\overrightarrow{\Delta X} = \overrightarrow{\Delta X}_0 + \overrightarrow{\Delta q},$$

as shown in Fig(2.3.b), where

$$\overrightarrow{\Delta q} = \Delta u \hat{i} + \Delta v \hat{j} + \Delta w \hat{k} \dots\dots \text{is the deformation or displacement vector,}$$

u, v, w are the displacement components in the x, y and z directions respectively.

By using matrix notation, vectors $\Delta \underline{X}_0$ and $\Delta \underline{q}$ can be defined as:-

$$\Delta \underline{X}_0 = \{\Delta x \ \Delta y \ \Delta z\}, \text{ and}$$

$$\Delta \underline{q} = \{\Delta u \ \Delta v \ \Delta w\}.$$

Hence, it can be deduced that:-

$$\Delta \underline{X} = \Delta \underline{X}_0 + \Delta \underline{q}$$

Then the difference between the square of the final length a'b' and the square of the original length ab, can be deduced as:-

$$\begin{aligned} (a'b')^2 - (ab)^2 &= \overrightarrow{\Delta X} \cdot \overrightarrow{\Delta X} - \overrightarrow{\Delta X}_0 \cdot \overrightarrow{\Delta X}_0 \\ &= \Delta \underline{X}^t \cdot \Delta \underline{X} - \Delta \underline{X}_0^t \cdot \Delta \underline{X}_0 \end{aligned}$$

$$= (\underline{\Delta x}_0 + \underline{\Delta q})^t \cdot (\underline{\Delta x}_0 + \underline{\Delta q}) - \underline{\Delta x}_0^t \cdot \underline{\Delta x}_0$$

i.e.

$$(a'b')^2 - (a\ b)^2 = \Delta \underline{X}_o^t \cdot \Delta \underline{g} + \Delta \underline{g}^t \cdot \Delta \underline{X}_o + \Delta \underline{g}^t \Delta \underline{g} \quad (2.19)$$

The vector Δg can be expressed by using the chain rule of partial differentiation, as follows:-

$$\Delta \underline{g} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z + \frac{\partial^2 u}{\partial x^2} \frac{(\Delta x)^2}{2!} + \dots \\ \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \frac{\partial v}{\partial z} \Delta z + \text{higher order terms} \\ \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \frac{\partial w}{\partial z} \Delta z + \quad " \quad " \quad " \end{pmatrix}$$

Neglecting the high order terms, it can be deduced that

$$\Delta g = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$\text{or } \Delta \underline{q} = \underline{Q} \Delta \underline{X}_0 \quad (2.20)$$

where

$$\underline{Q} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Substituting the above equation into equation 2.19, it can be proved that:-

$$(a'b')^2 - (ab)^2 = \Delta \underline{X}_0^t [(\underline{Q} + \underline{Q}^t) + \underline{Q}^t \underline{Q}] \cdot \Delta \underline{X}_0 \quad (2.21)$$

By using Green's definition of the engineering strain tensor [111], it can be shown that

$$\begin{aligned} (a'b')^2 - (ab)^2 &= 2 \Delta \underline{X}_0^t \cdot \epsilon_{\gamma\beta} \cdot \Delta \underline{X}_0 \\ &= 2 \sum_{\gamma} \sum_{\beta} \epsilon_{\gamma\beta} dx_{\gamma} dx_{\beta} \end{aligned}$$

where $\epsilon_{\gamma\beta}$ is the Green's strain tensor.

Comparing the above equation with 2.21, it can be deduced that:-

$$\epsilon_{\gamma\beta} = \frac{1}{2} (\underline{Q}^t + \underline{Q}) + \frac{1}{2} \underline{Q}^t \cdot \underline{Q} \quad (2.22)$$

Hence, it can be proved that:-

$$\begin{aligned} (\epsilon_{\gamma\beta})^t &= \frac{1}{2} (\underline{Q}^t + \underline{Q})^t + \frac{1}{2} (\underline{Q}^t \cdot \underline{Q})^t \\ &= \frac{1}{2} (\underline{Q} + \underline{Q}^t) + \frac{1}{2} \underline{Q}^t \cdot \underline{Q} = \epsilon_{\gamma\beta} \end{aligned}$$

i.e. $\epsilon_{\gamma\beta} = \epsilon_{\beta\gamma}$ and Green's strain matrix is a symmetric matrix.

The $\underline{Q}$ matrix can be written in tensorial notation as follows:-

$$Q_{\gamma\beta} = \frac{\partial u_{\gamma}}{\partial x_{\beta}} \quad \text{and} \quad Q^t_{\gamma\beta} = \frac{\partial u_{\beta}}{\partial x_{\gamma}}$$

By substituting the above equation into equation 2.22, it can be deduced that:-

$$\epsilon_{\gamma\beta} = \frac{1}{2} \left(\frac{\partial u_{\gamma}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\gamma}} \right) + \frac{1}{2} \sum_k \left(\frac{\partial u_k}{\partial x_{\gamma}} \cdot \frac{\partial u_k}{\partial x_{\beta}} \right) \quad (2.23)$$

where $\gamma, \beta, k = 1, 2, 3$ are the degrees of freedom (either 2 for 2-dimensional or 3 for 3-dimensional problems).

The strain tensor components can be written for axisymmetric problems as follows:-

$$\left. \begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial w}{\partial r} \right)^2 \right] \\ \epsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] \\ \epsilon_{\theta} &= \frac{u}{r} + \frac{1}{2} \left(\frac{u}{r} \right)^2 \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} + \left[\frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right] \end{aligned} \right\} \quad \begin{matrix} 24 \\ (2.29) \end{matrix}$$

as explained in appendix 2.1 where u, w are the displacement components in the r and z directions respectively.

The artificial strain vector can be defined as follows:-

$$\underline{\varepsilon} = \{\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx}\}$$

The strain invariants, in terms of tensor components, have similar expressions to those for the stress invariants. An equivalent, or effective, strain $\bar{\varepsilon}$ can be defined as follows:-

$$\bar{\varepsilon} = C_\varepsilon [(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + 6(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2)]^{1/2}$$

or

$$\bar{\varepsilon} = C_\varepsilon [(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)]^{1/2}$$

where

$$\left. \begin{aligned} \gamma_{xy} &= 2 \varepsilon_{xy} \\ \gamma_{yz} &= 2 \varepsilon_{yz} \\ \gamma_{zx} &= 2 \varepsilon_{zx} \end{aligned} \right\} \begin{array}{l} \text{are the shear strain components} \\ \text{in x-y, y-z and z-x planes} \\ \text{respectively.} \end{array}$$

C_ε is a numerical coefficient which should be chosen such that $\bar{\sigma}$ and $\bar{\varepsilon}$ match the uniaxial stress state.

For the uniaxial stress-state, and an elastic material, $\sigma_x = \sigma$ and the other components are equal to zero.

$$\varepsilon_x = \frac{\sigma}{E}, \quad \varepsilon_y = \varepsilon_z = -\frac{\nu\sigma}{E}, \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$$

Then

$$\bar{\epsilon} = C_e \cdot \sqrt{\frac{2(1+\nu)^2 \sigma^2}{E^2}} = C_e \cdot \sqrt{2} (1+\nu)$$

$$C_e = \frac{1}{\sqrt{2} (1+\nu)}$$

i.e. the effective strain can be defined as follows:-

$$\begin{aligned} \bar{\epsilon} = \frac{1}{\sqrt{2} (1+\nu)} & \left[(\epsilon_x - \epsilon_y)^2 + (\epsilon_y - \epsilon_z)^2 + (\epsilon_z - \epsilon_x)^2 \right. \\ & \left. + \frac{3}{2} (\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \right]^{\frac{1}{2}} \end{aligned} \quad (2.25)$$

2.3 The Constitutive Equation for Elasticity

The concept of linear elasticity was proposed by Robert Hooke in 1678, Hooke's experiments related only force to extension. The law which currently bears his name is that proposing a linear relationship between stress and strain tensors. Cauchy developed this relation into what is usually known as the generalized Hooke's Law, which may be written as follows:-

$$\sigma_{\gamma\beta} = \lambda \cdot e \cdot \delta_{\gamma\beta} + 2 \cdot \mu \epsilon_{\gamma\beta} \quad (2.26)$$

or

$$\sigma_{\gamma\beta} = 2\mu \left(\epsilon_{\gamma\beta} + \frac{\nu}{1-2\nu} \cdot e \cdot \delta_{\gamma\beta} \right)$$

where

$\sigma_{\gamma\beta}$ is the stress tensor at a given point,

$\epsilon_{\gamma\beta}$ is the strain tensor at the same point,

e is the trace of the strain tensor

$$e = \epsilon_x + \epsilon_y + \epsilon_z = \epsilon_1 + \epsilon_2 + \epsilon_3 ,$$

$$\left. \begin{aligned} \lambda &= \frac{Ev}{(1-2v)(1+v)} \\ \mu &= \frac{E}{2(1+v)} \end{aligned} \right\} \text{ are Lamé's coefficients} \quad (2.27)$$

E is Young's modulus,

v is Poisson's ratio

μ is the modulus of Rigidity

Or in matrix form as follows:-

$$\underline{\sigma} = \underline{D} \cdot \underline{\epsilon} \quad (2.28)$$

where $\underline{D}$ matrix is the Elastic matrix, which for the 3-dimensional problems can be written as follows:-

$$\underline{D} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

For the case of plane strain problems the $\underline{D}$ matrix can be obtained by eliminating the last two rows and columns and with $\epsilon_z = 0$, whilst for the plane stress case the $\underline{D}$ matrix is defined in a similar way to that used for plane strain and in order to achieve $\sigma_z = 0$, ϵ_z is defined as follows:-

$$\epsilon_z = -\frac{\nu}{1-\nu} (\epsilon_x + \epsilon_y) \quad (2.29)$$

Also for the axisymmetric case the $\underline{D}$ matrix is similar to that employed in the plane strain case.

In the plastic region, the elastic part of the strain increment can be directly related to the stress increments by means of Hooke's Law as follows:-

$$\Delta \underline{\epsilon}_e = \underline{D}^{-1} \cdot \Delta \underline{\sigma}$$

where

$$\underline{D}^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

2.4 General elasto-plastic Stress/Strain Relations

The formulation of elasto-plastic behaviour of a given material under multiaxial stresses can be achieved by generalizing the uniaxial test results of the same material.

Generally the uniaxial stress-strain test can be written as follows:-

$$\sigma = f(\epsilon, \dot{\epsilon}, T, t)$$

where

- $\dot{\epsilon}$ is the strain rate due to the applied load,
- T is the temperature of the material,
- t is the time.

To simplify the multi-axial stress-strain equations in this work, the following assumptions have been considered:-

- (i) The material is homogeneous and initially isotropic although it may be anisotropic during plastic deformation.
- (ii) The material properties are temperature-independent.
- (iii) Strain-rate effects are eliminated by assuming that the applied loads are quasi-static, i.e. with strain rates between $(10^{-4} \text{ to } 10^{-2} \text{ sec}^{-1})$.
- (iv) Creep deformation is not considered and the analysis is assumed to be carried out instantaneously or at temperatures well below the recrystallisation temperature.

Plastic constitutive relations can be obtained in terms of the uniaxial behaviour, by specifying the following rules and conditions [116]:-

- (i) An initial yield condition which defines the elastic limit of the material (i.e. yield criterion), under any possible combination of stresses.
- (ii) A flow rule which relates the plastic strain increments to the stresses and the stress increments.

- (iii) A hardening rule for establishing the conditions for subsequent yield from a plastic state.

2.4.1 Yield Criteria, Yield Surface

The yield criteria can be classified according to the material behaviour as follows:-

a) Yield criteria for a metallic material

From experimental observation it has been found that the hydrostatic stress has no effect on yielding of metallic materials, i.e. the only effective component is the deviatoric stress.

The most widely used yield criteria are:-

(i) Von Mises Criterion

Using this criterion it can be shown that the yield surface in 3-dimensional principal stress space is represented by:-

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 \quad (2.30)$$

where

Y is the uniaxial yield stress whether it is the initial or the instantaneous value.

The above equation can be rewritten as follows:-

$$6J_2 = 2Y^2$$

or

$$\bar{\sigma} = Y \quad (2.31)$$

The geometrical representation of this yield criterion in principal stress space is a circular cylinder of radius $\sqrt{\frac{2}{3}} Y$, whose axis is the line through the origin equally inclined to the axes of coordinates, as shown in Fig. (2.4). This criterion is suitable for ductile materials [113].

(ii) Tresca Yield Criterion

Using this criterion it can be shown that the yield surface in 3-dimensional stress space is represented by:-

$$(\sigma_1 - \sigma_2 - Y)(\sigma_2 - \sigma_3 - Y)(\sigma_3 - \sigma_1 - Y)(\sigma_1 - \sigma_3 - Y) \\ (\sigma_2 - \sigma_1 - Y)(\sigma_3 - \sigma_2 - Y) = 0 \quad (2.32)$$

The above equation can be broken down into six separate equations, and can also be reduced to the following equation:-

$$\tau_{\max} = \frac{Y}{2}$$

From equation 2.15 it can be shown that:-

$$\frac{\bar{\sigma}}{\sqrt{3}} \cos \phi = \frac{Y}{2}$$

or

$$2\sqrt{J_2} \cdot \cos \phi = Y \quad (2.33)$$

where

$$\phi = \frac{1}{3} \sin^{-1} \left(-\frac{27 J_3}{3 \bar{\sigma}^3} \right)$$

The Tresca yield criterion may be represented in principal stress space by a regular hexagonal cylinder with its central axis on the line $\sigma_1 = \sigma_2 = \sigma_3$, as shown in Fig (2.4).

(iii) Tresca Polynomial Yield Criterion

From equation 2.33, it can be shown that:-

$$\cos \phi = \frac{Y}{2\sqrt{J_2}}$$

Since

$$\sin 3\phi = - \frac{3\sqrt{3} J_3}{2 J_2^{1.5}}, \text{ from equation 2.13,}$$

then from Trigonometry, it can be shown that:

$$\cos 3\phi = \pm \sqrt{1 - \frac{27 J_3^2}{4 J_2^3}}$$

And it can be proved that

$$(\cos 3\phi)^2 = (4 \cos^3 \phi - 3 \cos \phi)^2$$

$$1 - \frac{27 J_3^2}{4 J_2^3} = \left[4 \left(\frac{Y}{2\sqrt{J_2}} \right)^3 - 3 \frac{Y}{2\sqrt{J_2}} \right]^2,$$

i.e.

$$\frac{4 J_2^3 - 27 J_3^2}{4 J_2^3} = \frac{Y^2}{4 J_2} \left[\frac{4Y^2 - 12 J_2}{4 J_2} \right]^2$$

Finally, the yield surface equation in polynomial form, can be expressed as follows:

$$F = (4 J_2^3 - 27 J_3^2) - Y^2 (3J_2 - Y^2)^2 = 0 \quad (2.34)$$

(b) Yield Criteria for Soil and Concrete Materials

Hydrostatic stresses have a considerable effect on the yield criteria for soil and concrete materials.

The internal resistance of the material is generated by the internal friction force between the material particles. The friction force can be evaluated by using the angle of internal friction and the cohesion of the specified material. The common yield criteria are:-

(i) Mohr-Coulomb Criterion

By using the Coulomb failure law (1773), it can be shown that:-

$$\tau_n \leq C_f - \sigma_n \tan \theta_f$$

where

- τ_n is the applied shear-stress,
- σ_n is the normal stress (tensile stress, positive)
- C_f is the cohesion force (internal resistance)
- θ_f is the angle of internal friction,

as shown in Fig(2.5).

A graphical representation was first demonstrated by Mohr (1882), as shown in Fig(2.6), where the above equation represents a straight line, which is a tangent to the largest principal stress circle. Hence, it can be shown that:-

$$\tau_n = \frac{\sigma_1 - \sigma_3}{2} \cos \theta_f$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \sin \theta_f$$

At the onset of yield,

$$\tau_n = C_f - \sigma_n \tan \theta_f$$

i.e.

$$\frac{\sigma_1 - \sigma_3}{2} \cos \theta_f = C_f - \left[\frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \sin \theta_f \right] \tan \theta_f$$

By multiplying both sides by $\cos \theta_f$, and simplifying the above equation, it can be deduced that:-

$$\frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2} \sin \theta_f - C_f \cdot \cos \theta_f = 0$$

which can be employed as an equation for the yield surface as follows:-

$$F = \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2} \cdot \sin \theta_f - C_f \cdot \cos \theta_f = 0$$

Using equations 2.14, it can be shown that:-

$$\sigma_1 - \sigma_3 = \frac{2 \bar{\sigma}}{\sqrt{3}} \cos \phi$$

$$\sigma_1 + \sigma_3 = 2\sigma_m + \frac{2}{3} \bar{\sigma} \left\{ \sin \left(\phi + \frac{2\pi}{3} \right) + \sin \left(\phi - \frac{2\pi}{3} \right) \right\}$$

$$\sigma_1 + \sigma_3 = 2\sigma_m + \frac{2}{3} \bar{\sigma} \sin\phi$$

Then the yield surface can be rewritten as follows:-

$$F = \sigma_m \sin\theta_f + \frac{\bar{\sigma}}{\sqrt{3}} \cos\phi - \frac{\bar{\sigma}}{\sqrt{3}} \sin\phi \sin\theta_f - C_f \cos\theta_f \quad (2.35)$$

The first term indicates the effect of the hydrostatic stresses on the yield surface. The second and third terms are the effect of the deviatoric stress components only, whilst the fourth term shows the material resistance, which is equivalent to the yield stress.

Using equations 2.10 and 2.13, the equation for the yield surface can be rewritten in the form as follows:-

$$F = \frac{I_1}{3} \sin\theta_f + \sqrt{J_2} \cos\phi - \sqrt{\frac{J_2}{3}} \sin\phi \sin\theta_f - C_f \cos\theta_f \quad (2.36)$$

(ii) Drucker-Prager Criterion

An approximation to the Mohr-Coulomb Law was introduced by Drucker and Prager (1952) as a modification of the von Mises yield criterion. The influence of the hydrostatic stress components on the yield surface was represented by an additional term in the von Mises expression, to give:

$$\alpha I_1 + \sqrt{J_2} - k = 0 \quad (2.37)$$

where

αI_1 represents the influence of hydrostatic stress components on yielding,

k is the new equivalent yield stress level.

The Drucker-Prager criterion and the Mohr-Coulomb criterion are coincident with each other at $\phi = \pi/6$, as shown in Fig(2.7). Comparing equations 2.37 and 2.36 at $\phi = \pi/6$, it can be seen that:-

$$\left. \begin{aligned} \alpha &= \frac{2/\sqrt{3} \sin\theta_f}{3 - \sin\theta_f} \\ k &= \frac{6/\sqrt{3} \cos\theta_f}{3 - \sin\theta_f} C_f \end{aligned} \right\} \quad (2.38)$$

(c) Yield Criteria for Polymers

When comparing the yield criteria for polymeric materials with those for metals, we discover that the following additional points must be taken into account.

(i) The hydrostatic effect is considered.

(ii) Tensile and compressive yield stresses are not necessarily equal.

Such yield criteria can be summarised as follows:-

(i) Pressure Modified Von Mises Criterion

The equation for the von Mises criterion may be written as follows:-

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2$$

By substituting Y_c Y_t in place of Y^2 in the above equation, it can be deduced that:-

$$\begin{aligned} (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + C_1(\sigma_1 + \sigma_2 + \sigma_3) \\ = 2 Y_t Y_c \end{aligned} \quad (2.39)$$

where

C_1 is a constant to be determined
 Y_t, Y_c are the absolute tensile and compressive yield stresses respectively.

To find the value of C_1 , consider the uniaxial case as follows:-

$$\sigma_1 = \sigma_x, \quad \sigma_2 = \sigma_3 = 0$$

Then equation 2.39 becomes:-

$$\begin{aligned} 2\sigma_1^2 + C_1 \sigma_1 &= 2 Y_t Y_c \\ \text{or} \\ \sigma_1 C_1 &= 2(Y_t Y_c - \sigma_1^2) \end{aligned}$$

In the above equation the right side is equal to the left side, if σ_1 is equal to the yield stress (Y_t or Y_c), then it can be shown that:-

$$C_1 = 2(Y_c - Y_t), \quad \text{if } \sigma_1 = Y_t$$

or

$$C_1 = 2(Y_t - Y_c), \text{ if } \sigma_1 = Y_c$$

Substituting C_1 into equation 2.39, it can be deduced that:-

$$\begin{aligned} & (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \\ & + 2(Y_c - Y_t)(\sigma_1 + \sigma_2 + \sigma_3) = 2 Y_c Y_t \quad (2.40) \end{aligned}$$

The above equation is similar to the equation of modified von Mises mentioned in Ref.(114-115).

The locus of the yield criterion can be obtained by considering the 3-dimensional principal stress coordinate system $(0\sigma_1, 0\sigma_2, 0\sigma_3)$, as shown in Fig(2.8.a).

Point $P(\sigma_1, \sigma_2, \sigma_3)$ represents the state of stress at a point inside a stressed body. The state of stress at point P may be written as the sum of the three vectors $\overline{OP}_1 (= \sigma_1)$, $\overline{P}_1M (= \sigma_2)$ and $\overline{MP} (= \sigma_3)$.

Consider line $\overline{OH}$ (ℓ, m, n) equally inclined to each of the three principal stress axes, i.e.

$$\ell = m = n = \frac{1}{\sqrt{3}}$$

Then the projection of OP on line $\overline{OH}$ is:-

$$\overline{ON} = \sigma_1 \ell + \sigma_2 m + \sigma_3 n$$

$$\overline{ON} = \frac{1}{\sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3)$$

Then

$$\begin{aligned}\overline{PN}^2 &= \overline{OP}^2 - \overline{ON}^2 \\ &= (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)^2 \\ \overline{PN}^2 &= \frac{1}{3} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \}\end{aligned}$$

Substituting the above equation into equation 2.40, it can be proved that:-

$$3(\overline{PN})^2 + 2(Y_c - Y_t)(\sigma_1 + \sigma_2 + \sigma_3) = 2 Y_t Y_c$$

i.e.

$$(\overline{PN})^2 = \frac{2}{3} Y_t Y_c - 2(Y_c - Y_t) \sigma_m$$

Defining

$$\Delta = Y_c - Y_t$$

Hence

$$\overline{PN}^2 = \frac{2}{3} Y_t^2 + \frac{2}{3} \Delta Y_t - 2 \Delta \sigma_m$$

Finally, the radius of the locus of the yield surface about the axis $\overline{OH}$ is:-

$$\overline{PN} = \sqrt{\frac{2}{3} Y_t^2 + 2\Delta \left(\frac{Y_t}{3} - \sigma_m \right)} \quad (2.41)$$

If $\Delta = 0$, then $\overline{PN} = \sqrt{\frac{2}{3}} \cdot Y_t$, the radius of the Von Mises

about the $\overline{OH}$ axis.

Equation 2.41, can be rewritten as follows:-

$$\overline{PN} = \sqrt{C_t - C_y \sigma_m}$$

where

$$C_t = \frac{2}{3} Y_t Y_c$$

$$C_y = 2(Y_c - Y_t)$$

From the above equation, it can be deduced that $\overline{PN}$, the radius of the yield surface, is a function of σ_m and Δ . At the π - plane, where $\sigma_m = \text{zero}$, the radius $\overline{PN}$ is given by:-

$$\overline{PN} = \sqrt{C_t} = \sqrt{\frac{2}{3} Y_t Y_c}$$

Therefore, the yield surface of the modified Von Mises criterion has the form of a circular cone. The apex of the cone lies on the space diagonal at the point $\sigma_1 = \sigma_2 = \sigma_3 = C_t/C_y$, i.e. at $\overline{PN} = \text{radius} = \text{zero}$ and $\sigma_m = C_t/C_y$, as shown in Fig(2.8.b.).

The uniaxial, or equivalent, yield stress is given by:-

$$Y = \sqrt{Y_c Y_t} \quad (2.42)$$

and the modified effective stress is given by:-

$$\sqrt{\sigma^2 + 3 \Delta \sigma_m} \quad (2.43)$$

(ii) Pressure Modified Tresca Polynomial Criterion

The equation for the Tresca polynomial criterion is:-

$$(4J_2^3 - 27J_3^2) - Y^2(3J_2 - Y^2)^2 = 0$$

To consider the difference between the yield stress in tension and that in compression, and to take into account the influence of the hydrostatic stress σ_m , the above equation can be rewritten as follows:-

$$(4J_2^3 - 27J_3^2) - Y_t Y_c (3J_2 - Y_c Y_t)^2 + C_1 (\sigma_1 + \sigma_2 + \sigma_3) = 0$$

where C_1 is a constant which must be determined.

To obtain the value of C_1 , the uniaxial tension case is used, such that

$$\sigma_1 = \sigma_x = \text{yield stress, and}$$

$$\sigma_2 = \sigma_3 = 0.$$

Hence,

$$J_2 = \frac{1}{3} \sigma_x^2,$$

$$J_3 = \frac{2}{27} \sigma_x^3$$

Then the yield surface equation can be written as follows:-

$$\sigma_1 C_1 = Y_t Y_c (\sigma_1^2 - Y_t Y_c)^2$$

But

$$\sigma_x = Y_t, \text{ then}$$

$$C_1 = Y_t^2 Y_c (Y_c - Y_t)^2$$

Hence, the final yield equation for the pressure modified Tresca Polynomial Criterion may be written as follows:-

$$(4J_2^3 - 27J_3^2) - Y_c Y_t (3J_2 - Y_c Y_t)^2 + Y_c Y_t^2 (Y_c - Y_t)^2 (\sigma_1 + \sigma_2 + \sigma_3) = 0 \quad (2.44)$$

If $Y_t = Y_c = Y$, the above equation is reduced to the Tresca polynomial yield equation.

Defining

$$\Delta = Y_c - Y_t$$

Then equation 2.44 can be rewritten as follows:-

$$\frac{(4J_2^3 - 27J_3^2)^{\frac{1}{2}} + \sqrt{3Y_c \sigma_m} \Delta Y_t}{(3J_2 - Y_t Y_c)} = \sqrt{Y_c Y_t}$$

Therefore the uniaxial (or equivalent) yield stress is:-

$$Y = \sqrt{Y_c Y_t} \quad (2.45)$$

and the stress level is

$$\frac{(4J_2^3 - 27J_3^2)^{\frac{1}{2}} + \sqrt{3Y_c \sigma_m} \cdot \Delta \cdot Y_t}{(3J_2 - Y_t Y_c)} \quad (2.46)$$

2.4.2 Yield Surface and Loading Yield Surface

After yielding the initial yield surface will be modified for a stage of plastic deformation as a result of the effect of strain, or

work. hardening (or softening). The subsequent yield loading surface can be expressed as follows:-

$$F(\underline{\sigma}, \underline{\kappa}) = 0$$

where

$\underline{\kappa}$ is a measure of the degree of work hardening,
 $\underline{\sigma}$ is the stress vector.

The above function can be expressed in terms of the stress invariants as follows:-

$$F(I_1, I_2, I_3, \underline{\kappa}) = 0$$

or

$$F(\sigma_m, J_2, J_3, \underline{\kappa}) = 0$$

The hardening vector $\underline{\kappa}$ is generally a function of the plastic strain $\underline{\varepsilon}_p$ and a history parameter κ

$$\underline{\kappa} = f(\underline{\varepsilon}_p, \kappa)$$

Therefore, the equation of the yield surface is:

$$F(\sigma_m, \underline{\sigma}, \underline{\varepsilon}_p, \kappa) = 0$$

For the case of ductile materials, σ_m can be neglected, therefore:-

$$F(\underline{\sigma}, \underline{\varepsilon}, \kappa) = 0 \quad (2.47)$$

There are two methods in the literature for obtaining κ and these are as follows:-

(i) The Work-hardening Hypothesis

This hypothesis assumes that the amount of hardening depends only upon the total plastic work, and is independent of the strain path.

$$dk_1 = dW_p = \underline{\sigma}^t \cdot d\underline{\varepsilon}_p \quad (2.48)$$

For the case of a perfectly plastic material the amount of plastic work, after yielding, increases while $dk_1 =$ zero, i.e the above measure is not accurate for the perfect plasticity case, as shown in Fig(2.9).

(ii) The Strain-hardening Hypothesis

This hypothesis uses an expression, known as the effective plastic strain, as a measure of the work-hardening occurring.

$$dk_2 = \overline{d\varepsilon}_p$$

where

$$\overline{d\varepsilon}_p = \sqrt{\frac{2}{3}} \left\{ \sum_{\gamma \beta} \left(d\varepsilon_{\gamma\beta} \right)_p \left(d\varepsilon_{\gamma\beta} \right)_p \right\}^{\frac{1}{2}}$$

i.e. it is assumed that the work-hardening depends on the strain path.

(iii) The Elastic-work Hypothesis

For this hypothesis it is assumed that the amount of hardening depends only upon the total elastic work:-

$$dk_3 = dW_e = \underline{\sigma}^t \cdot d\underline{\varepsilon}_e, \quad (2.49)$$

as shown in Fig(2.10).

2.4.3 Correlation Between Uniaxial Test and Multiaxial Problems

An acceptable correlation between the uniaxial test and multiaxial cases is found if the following assumption is made:-

$$\underline{\sigma}^t \cdot d\underline{\epsilon}_p = Y \cdot (d\underline{\epsilon}_p)_u$$

as shown in Fig(2.11). By using a suitable yield criterion, it can be shown that:-

$$F(\underline{\sigma}, Y) = f(\underline{\sigma}) - Y = 0$$

where

$$Y = f(\underline{\sigma}) = \text{uniaxial stress}$$

Hence,

$$\overline{d\underline{\epsilon}_p} = \frac{\underline{\sigma}^t \cdot d\underline{\epsilon}_p}{f(\underline{\sigma})} = \text{uniaxial plastic strain increment} \quad (2.50)$$

Generally, $\underline{\sigma}$ is the average stress during the increment, $d\underline{\epsilon}_p$ and the total uniaxial plastic strain can be obtained as follows:-

$$\bar{\epsilon}_p = \int \overline{d\underline{\epsilon}_p}$$

There is no other valid rule for obtaining $\bar{\epsilon}_p$ and the equation for $\bar{\epsilon}$ (2.25), should be limited to the elastic part of the total strain vector, i.e. the total strain vector in the plastic region is:-

$$\bar{\epsilon} = \bar{\epsilon}_e + \bar{\epsilon}_p$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2} (1 + \nu)} [(\epsilon_x - \epsilon_y)^2 + (\epsilon_y - \epsilon_z)^2 + (\epsilon_z - \epsilon_x)^2 \\
&\quad + \frac{3}{2} (\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)]^{1/2} + \int d\epsilon_p \\
\int \frac{\sigma^t \cdot d\epsilon_p}{f(\sigma)} &= \int \frac{\sigma^t \cdot d\epsilon_p}{f(\sigma)}
\end{aligned}$$

From

$$\underline{\sigma} = \underline{P} + \underline{S}$$

where

$$\underline{P} = \{\sigma_m \ \sigma_m \ \sigma_m \ 0 \ 0 \ 0\} \text{ the hydrostatic stress vector}$$

then it can be shown that

$$\underline{\sigma}^t \cdot d\underline{\epsilon}_p = \underline{S}^t \cdot d\underline{\epsilon}_p + \underline{P}^t \cdot d\underline{\epsilon}_p$$

But, for the case of metallic materials,

$$\underline{P}^t \cdot d\underline{\epsilon}_p = \sigma_m (d\epsilon_x + d\epsilon_y + d\epsilon_z)_p = \text{zero}$$

Hence, it can be deduced that

$$\underline{\sigma}^t \cdot d\underline{\epsilon}_p = \underline{S}^t \cdot d\underline{\epsilon}_p \quad (2.51)$$

2.4.4 Flow Rule

The flow rule is the constitutive relationship between the plastic strain increment and the stress, and stress increment, during plastic deformation.

Drucker's postulate [11] states that the work done by an external agency during a complete cycle of loading and unloading must be non-negative. Hence, the unloading surface has continuously turning tangents, and it can be concluded that [114]:-

- (i) The instantaneous yield surface is convex with respect to the origin in stress space.
- (ii) The plastic strain increment vector is in the outward normal to the loading surface for all stress increments, as shown in Fig(2.12).

The second condition, which is known as the normality principle can be written for associated plasticity as follows [59,60].

$$d\epsilon_p = d\lambda \cdot \frac{\partial F}{\partial \sigma} = d\lambda \cdot \underline{a} \quad (2.52)$$

where

$$\underline{a} = \frac{\partial F}{\partial \underline{\sigma}} = \left\{ \frac{\partial F}{\partial \sigma_x} \quad \frac{\partial F}{\partial \sigma_y} \quad \dots \quad \frac{\partial F}{\partial \tau_{zx}} \right\}$$

$d\lambda$ is the proportionality factor which may depend upon $\underline{\sigma}$ (state of stress), $d\sigma$ (stress increment vector), $d\epsilon_p$ (strain plastic increment), and the history of plastic deformation.

For a non-associated plastic material the normality principle is not valid, but in general it will be assumed that the increment ($d\epsilon_p$), is proportional to the stress gradient of a plastic potential "Q", where Q can be expressed as follows:-

$$Q(\underline{\sigma}, \underline{\epsilon}_p, \kappa) = 0$$

This allows non-associated plasticity to be dealt with, and associated plasticity rules to be obtained as a special case

$$F \equiv Q$$

Hence, the flow rule can be written as follows:-

$$d\epsilon_p = d\lambda \cdot \frac{\partial Q}{\partial \sigma} = d\lambda \underline{b}$$

where

$$\underline{b} = \frac{\partial Q}{\partial \sigma} = \left\{ \frac{\partial Q}{\partial \sigma_x} \quad \frac{\partial Q}{\partial \sigma_y} \quad \dots \dots \frac{\partial Q}{\partial \tau_{zx}} \right\}$$

as shown in Fig(2.13).

The flow rule leads to an incremental, or flow, theory of plasticity in which there is path dependence of a final state of stress and strain as reached from some previous state. Flow theory is, in general, distinct from the deformation theory of plasticity, in which the total plastic strains are related to the final stress state.

Assuming Drucker's postulate to constitute a criterion which is physically correct, the deformation theory can be consistent with this postulate only for loading paths in the vicinity of proportional loading.

2.4.5 Hardening Rules

In a uniaxial test the value of the new yield stress after yielding can be obtained directly from the stress-strain diagram. While for the multiaxial case, hardening rules can be used to establish conditions for the subsequent yield surface by correlation with the uniaxial material behaviour.

The loading, or subsequent, yield surface should be a convenient mathematical idealization of some macroscopically observed behaviour. It should have the ability to describe, quantitatively, hardening behaviour in multiaxial problems, as determined from experimental results for a particular material.

From experimental observations, it can be deduced that there are three important parameters which must be considered in describing the subsequent yield surface. These parameters are:-

(a) Work or Strain Hardening (or softening) Behaviour

After yielding the level of the yield stress increases (or decreases) by an amount depending on the total plastic strain deformation. This behaviour known as strain hardening, (or softening).

A material which exhibits this behaviour is known as a strain-hardening, (or softening), material but for some materials (or under certain conditions) a material may not behave in this way and may instead, deform in a perfectly plastic manner.

(b) Bauschinger Effect (residual stress effect)

Consider a specimen subjected to an increasing uniaxial tensile load until yielding, and considerable plastic deformation, occurs. If the tensile load is decreased until it becomes a compressive load, then the magnitude of the initial compressive yield stress will be considerably lower than the initial yield in tension.

This anisotropic behaviour has been explained as being the result of the residual stress left in the specimen due to the tensile plastic deformation, and the developed anisotropy during plastic deformation. This phenomenon is known as the Bauschinger effect.

If the magnitude of the increase of yield stress in one direction results in a decrease of yield stress of the same magnitude in the reverse direction, then

$$Y_t + Y_c = 2(Y_t)_{\text{initial}} = 2(Y_c)_{\text{initial}} = (Y_c + Y_t)_{\text{initial}}$$

This condition is described as the "Ideal Bauschinger effect", as shown in Fig(2.14) and it is this form of the Bauschinger effect which has been considered in this work.

While in the Real Bauschinger effect, which is shown the real material behaviour, the magnitude of the increase or decrease in yield stress may depend on the number of cycles of applied load, as shown in Fig(2.14).

(c) Distortion of the Yield Surface

Experimental observation shows that there is a decrease in the width of the loading surface in the direction of normal to the applied stresses during the plastic deformation, as shown in Fig(2.15).

The degree of distortion depends on the definition of the defined yield stress which is used in the yield surface equation.

If the load-interpolation point is used for the definition of the yield stress, then the distortion is negligible. Whilst, if the other uniaxial points are used to define the yield stress, such as the proportional limit or the 0.1%, 0.2% proof stress, then the distortion could be significant [20].

The second and third of the above parameters lead to anisotropic behaviour, whilst the first parameter does not change the material isotropic property. The application of any of the above parameters, to mathematical models of the loading surface, depends on the type of applied load and the definition of the yield stress.

The main hardening rules which are used to simulate the influence of the above parameters on the feature of the loading surface are:-

(i) Isotropic Hardening Rule

In this rule strain-hardening (or softening) is accounted for by assuming that the loading surfaces, or subsequent yield surfaces, expand (or contract) uniformly about the origin in stress space during plastic flow, maintaining the same shape, centre, and orientation, as the initial yield surface.

This rule was introduced by Hill [2], and others, and ignores the Bauschinger effect and the distortion in the yield surface. According to the isotropic hardening rule, the material will exhibit an increase in the compressive yield stress equal to the increase in the tensile yield stress, i.e. the elastic region increases, as shown in Figs(2.16, 2.17, 2.18).

This rule is suitable for engineering applications with a monotonic load, or with a radial load [117].

(ii) Kinematic Hardening Rule

This hardening rule was proposed by Prager [26,27] to account for the ideal Bauschinger effect by assuming that, during plastic deformation, the loading surface translates as a rigid body in stress space, maintaining the size, shape and orientation of the initial yield surface.

In this rule the elastic region remains the same during the plastic deformation. Strain-hardening is not

simulated in this rule, as shown in Figs (2.19, 2.20). Kinematic hardening rules can be classified, as shown in [116], into

(a) Direct Kinematic Hardening Rule

In this version of the Kinematic Hardening Rule the zero stress components are assumed to have no influence on translation within the 9-dimensional stress space. The resulting equation for the initial yield surface is

$$F(\underline{\sigma}) = F(\underline{\sigma}', \underline{\sigma}'')$$

and the equation of loading surface is:-

$$F(\underline{\sigma} - \underline{\alpha}) = F(\underline{\sigma}' - \underline{\alpha}', \underline{\alpha}'')$$

$$= F(\underline{\sigma}' - \underline{\alpha}')$$

where

- $\underline{\sigma}'$ is the stress vector for non-zero components
- $\underline{\sigma}''$ is the stress vector for zero components
- $\underline{\alpha}$ is the shift vector
- $\underline{\alpha}'$ is the shift vector for non-zero stress components
- $\underline{\alpha}''$ is the shift vector for zero stress components.

But $\underline{\alpha}''$ is not necessarily equal to zero when $\underline{\sigma}'' = 0$, in which case the direct kinematic hardening rule is approximately the same as the Prager's Rule (or complete kinematic hardening rule).

(b) Complete Kinematic Hardening Rule

This rule considers all components of translation of the yield surface even for those stress components which are zero.

$$F(\underline{\sigma} - \underline{\alpha}) = F(\underline{\sigma}' - \underline{\alpha}' , \underline{\alpha}'')$$

(iii) Mixed Hardening Rule

Although the isotropic hardening rule does not simulate the Bauschinger effect it is capable of representing monotonic loading and has been widely used in engineering applications because of its simplicity.

The kinematic hardening rule is broadly suitable for modelling problems with cyclic loading but because it does not allow for strain-hardening it falls short of representing the behaviour of real materials.

By combining the isotropic and kinematic hardening rules, one can achieve a new hardening rule with the capability of simulating the influence of strain-hardening and the Bauschinger effect on the loading surface, this combination is known as the "Mixed Hardening Rule".

In this rule, the loading surface during plastic flow expands uniformly in all directions as a result of strain-hardening behaviour and translates at the same time due to the Bauschinger effect, as shown in Fig(2.21, 2.22).

The combination of the isotropic and kinematic hardening rules can be controlled by using a mixing hardening parameter as will be explained later.

(iv) Distortion Hardening Rule

This rule was introduced by Yoshimura [31-32], and Baltov and Sawczuk [33] proposed theories to consider the decrease in width of the yield surface in the direction of prestressing, as shown in Fig(2.15).

There are other hardening rules in the literature as shown in Fig(2.23).

The yield stress in this work is defined at the Lode-interpolation point since most of the materials used have a yield stress close to that at the Lode-interpolation point. Hence, the distortion hardening rule will not be considered in this work neither will any combination of the distortion hardening rule and other rules, as justified by Reference 117.

2.4.6 Strain-Hardening (or Softening) Parameter

From the uniaxial stress-strain diagram for a strain hardening material, one can define the strain-hardening parameter, as a measure of the strain hardening (or softening) as follows:-

$$H' = \frac{d \sigma}{d \epsilon_p} \quad (2.53)$$

as shown in Fig(2.24.a).

The total strain can be expressed as follows:-

$$\epsilon = \epsilon_e + \epsilon_p$$

where ϵ_e is the elastic strain which is equal to

$$\epsilon_e = \frac{\sigma}{E}$$

Hence

$$\epsilon_p = \epsilon - \frac{\sigma}{E}$$

Differentiating the above equation with respect to ϵ , it can be shown that:-

$$\frac{d\epsilon_p}{d\epsilon} = 1 - \frac{1}{E} \frac{d\sigma}{d\epsilon}$$

Hence

$$\begin{aligned} H' &= \frac{d\sigma}{d\epsilon} \frac{d\epsilon}{d\epsilon_p} = \frac{\frac{d\sigma}{d\epsilon}}{\frac{d\epsilon_p}{d\epsilon}} \\ &= \frac{\frac{d\sigma}{d\epsilon}}{1 - \frac{1}{E} \frac{d\sigma}{d\epsilon}} \end{aligned}$$

where $\frac{d\sigma}{d\epsilon}$ is measured after the yield point, therefore it can be assumed equal to the tangent elastic modulus

Then,

$$E_t = \frac{d\sigma}{d\epsilon}$$

Finally

$$H' = \frac{E_t}{1 - \frac{E_t}{E}} \quad (2.54)$$

H' can be constant if the tangent elastic modulus is a constant, as shown in Fig(2.24.b.) or equal to zero for a perfectly plastic material. It can be concluded that:-

H' = positive for strain hardening material (stable)

H' = negative for strain softening material (unstable)

H' = zero for ideal plastic material (translation).

2.5 Derivation of the Elasto-plastic Stress/Strain Matrix $\underline{D}_{ep}$

The subsequent yield function can be written from equation 2.47 as follows:-

$$F(\underline{\sigma}, \underline{\varepsilon}, \kappa) = 0$$

Hence,

$$dF = \left(\frac{\partial F}{\partial \underline{\sigma}} \right)^t \cdot d\underline{\sigma} + \left(\frac{\partial F}{\partial \underline{\varepsilon}_p} \right)^t \cdot d\underline{\varepsilon}_p + \left(\frac{\partial F}{\partial \kappa} \right) d\kappa = 0$$

It is convenient to separate the hardening terms by defining a parameter A such that:-

$$A = - \left[\left(\frac{\partial F}{\partial \underline{\varepsilon}_p} \right)^t \cdot d\underline{\varepsilon}_p + \left(\frac{\partial F}{\partial \kappa} \right) d\kappa \right] / d\lambda \quad (2.55)$$

Hence

$$dF = \left(\frac{\partial F}{\partial \underline{\sigma}} \right)^t \cdot d\underline{\sigma} - A d\lambda = 0$$

$$dF = \underline{a}^t d\underline{\sigma} - A d\lambda = 0 \quad (2.56)$$

The total strain increment can be expressed in terms of elastic and plastic strain increments as follows:-

$$d\bar{\epsilon} = d\bar{\epsilon}_e + d\bar{\epsilon}_p$$

where

$$d\bar{\epsilon}_e = \underline{D}^{-1} d\bar{\sigma},$$

and from the flow rate it can be shown that:-

$$d\bar{\epsilon}_p = d\lambda \underline{b} = d\lambda \frac{\partial Q}{\partial \bar{\sigma}}$$

$\underline{D}$ is the elastic stress-strain matrix

Hence

$$d\bar{\epsilon} = \underline{D}^{-1} d\bar{\sigma} + d\lambda \underline{b}$$

i.e.

$$d\bar{\sigma} = \underline{D} d\bar{\epsilon} - d\lambda (\underline{D} \underline{b}) \quad (2.57)$$

Multiplying equation 2.57 by $\underline{a}^t$ and substituting $A \cdot d\lambda$ for $\underline{a}^t d\bar{\sigma}$ from equation 2.56, then

$$A d\lambda = \underline{a}^t \underline{D} d\bar{\epsilon} - d\lambda (\underline{a}^t \underline{D} \underline{b})$$

Hence

$$d\lambda = \frac{\underline{a}^t \underline{D}}{A + \underline{a}^t \underline{D} \underline{b}} d\bar{\epsilon} \quad (2.58)$$

By substituting equation 2.58 into 2.57 it can be shown that

$$d\sigma = \underline{D}_{ep} d\varepsilon$$

where

$$\underline{D}_{ep} = \underline{D} - \underline{D}_p$$

$$\underline{D}_p = \frac{(\underline{D} \underline{b}) (\underline{D} \underline{a})^t}{A + B}$$

$$B = \underline{a}^t \underline{D} \underline{b}$$

The matrix $\underline{D}_{ep}$ is known as the elasto-plastic stress/strain matrix. Throughout this work, associated plasticity will be assumed, then

$$Q \equiv F, \quad \frac{\partial Q}{\partial \sigma} \equiv \frac{\partial F}{\partial \sigma}, \quad \underline{b} \equiv \underline{a}$$

Hence

$$\underline{D}_p = \frac{(\underline{D} \underline{a}) (\underline{D} \underline{a})^t}{A + B}, \quad (2.59)$$

$$B = \underline{a}^t \underline{D} \underline{a}, \quad (2.60)$$

$$d\varepsilon_p = d\lambda \underline{a}, \quad \text{and}$$

$$d\lambda = \frac{(\underline{D} \underline{a})^t}{A + B} d\varepsilon \quad (2.61)$$

It is clear that in order to evaluate the $\underline{D}_{ep}$ matrix, the following parameters will be required:-

$$A, \underline{a}, d\lambda$$

which can be obtained by employing a suitable hardening rule and a suitable yield criterion, as shown in the next section.

2.6 Evaluation of D_{ep} Parameters

To evaluate the D_{ep} for a certain problem a suitable hardening rule, which can be one of the following rules, should be selected.

2.6.1 Employing the Isotropic Hardening Rule

The subsequent yield surface for the case of the isotropic hardening rule may be written as follows:-

$$F(\underline{\sigma}, Y(\kappa)) = 0$$

where

κ depends on the plastic strain history,
 Y is the uniaxial instantaneous yield stress.

To obtain the parameter κ , one can apply the following hypotheses:-

(a) Work Hardening Hypothesis

According to this hypothesis, and for associated plasticity, it can be deduced that:-

$$d\kappa = Y \overline{d\varepsilon}_p = d\varepsilon_p^t \underline{\sigma} = d\lambda \underline{a}^t \underline{\sigma} \quad (2.62)$$

where $d\varepsilon_p = d\lambda \underline{a}$, flow rule.

Employing Euler's theorem, (Appendix 2.2), in the yield surface, it can be proved that:-

$$\underline{a}^t \underline{\sigma} + \frac{\partial F}{\partial Y} Y = n \quad F(\underline{\sigma}, Y(\kappa)) = 0 \quad (2.63)$$

Substituting equation 2.63 into equation 2.62 it can be shown that

$$d\kappa = -Y \left(\frac{\partial F}{\partial Y} \right) \cdot d\lambda \quad (2.64)$$

Therefore, from equations 2.64 and 2.63, it can be deduced that:-

$$d\lambda = \frac{-\overline{d\varepsilon_p}}{\frac{\partial F}{\partial Y}} \quad (2.65)$$

For an ideal plastic material, it can be shown that:-

$$F(\underline{\sigma}, Y) = F(\underline{\sigma}) - Y = 0,$$

$$\frac{\partial F}{\partial Y} = -1,$$

Therefore $d\kappa = \overline{d\varepsilon_p}$

The parameter A has been defined, (equation 2.55), as follows:-

$$A = - \left[\left(\frac{\partial F}{\partial \underline{\varepsilon}_p} \right)^t d\underline{\varepsilon}_p + \frac{\partial F}{\partial \kappa} d\kappa \right] / d\lambda$$

But

$$F(\underline{\sigma}, Y(\kappa)) = 0$$

Hence

$$\frac{\partial F}{\partial \underline{\varepsilon}_p} = \text{zero}$$

$$\frac{\partial F}{\partial \kappa} = \left(\frac{\partial F}{\partial \underline{\sigma}} \right)^t \cdot \frac{d\underline{\sigma}}{d\kappa} + \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial \kappa},$$

$$\frac{\partial \sigma}{\partial k} = \text{zero}$$

Finally

$$\frac{\partial F}{\partial k} = \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial k} = \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial \epsilon_p} \frac{\overline{d\epsilon_p}}{dk}$$

i.e.

$$A = - \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial \epsilon_p} \frac{\overline{d\epsilon_p}}{dk} \frac{dk}{d\lambda} \quad (2.66)$$

But from equations 2.62 and 2.64, it can be shown that:-

$$\frac{\overline{d\epsilon_p}}{dk} = \frac{1}{Y}, \text{ and}$$

$$\frac{dk}{d\lambda} = -Y \left(\frac{\partial F}{\partial Y} \right)$$

Then, equation 2.66 can be rewritten as follows:-

$$A = \left(\frac{\partial F}{\partial Y} \right)^2 \frac{dY}{\overline{d\epsilon_p}}$$

or

$$A = \left(\frac{\partial F}{\partial Y} \right)^2 H' \quad (2.67)$$

where

$$H' = \frac{d\sigma}{\overline{d\epsilon_p}} = \frac{dY}{\overline{d\epsilon_p}}$$

(b) Elastic Work Hypothesis

According to this hypothesis, the total increment in work is:-

$$dW_T = dW_e + dW_p$$

where

dW_e is the elastic work increment,

dW_p is the plastic work increment,

as shown in Fig(2.25).

If the total strain increment $d\epsilon_T$ is very small, then it can be assumed that:-

$$Y_e = Y_p = Y$$

where

Y_e is the average yield stress throughout the elastic strain increment

Y_p is the average yield stress throughout the plastic strain increment.

Therefore

$$d\kappa_3 = Y \cdot d\epsilon_e \quad \dots \text{for the uniaxial case,}$$

$$d\kappa_3 = \underline{\sigma}^t \cdot d\epsilon_e \quad \dots \text{for the multiaxial case}$$

$$\begin{aligned} d\kappa_3 &= \underline{\sigma}^t (d\epsilon - d\epsilon_p) \\ &= \underline{\sigma}^t d\epsilon - \underline{\sigma}^t a d\lambda \end{aligned} \quad (2.68)$$

Employing Euler's theorem to the yield surface, it can be shown that:-

$$\left(\frac{\partial F}{\partial \sigma} \right)^t \sigma + \frac{\partial F}{\partial Y} Y = 0$$

$$\bar{a}^t \sigma = - \frac{\partial F}{\partial Y} Y$$

Substituting the above equation into equation 2.68 it can be shown that:-

$$d\kappa_3 = Y \overline{d\varepsilon}_e = \sigma^t d\varepsilon + \frac{\partial F}{\partial Y} Y d\lambda$$

Hence

$$d\lambda = \left[\overline{d\varepsilon}_e - \frac{\sigma^t d\varepsilon}{Y} \right] / \left(\frac{\partial F}{\partial Y} \right)$$

After correlation between the uniaxial and multiaxial problem, it can be deduced that:

$$\sigma^t d\varepsilon = Y \overline{d\varepsilon}_T$$

Therefore

$$\frac{d\varepsilon^t \sigma}{Y} = \frac{\sigma^t d\varepsilon}{Y} = \overline{d\varepsilon}_T$$

where

$$\overline{d\varepsilon}_T = \overline{d\varepsilon}_e + \overline{d\varepsilon}_p \quad \dots \text{the total strain increment for the uniaxial case}$$

Hence

$$d\lambda = (\overline{d\varepsilon}_e - \overline{d\varepsilon}_T) / \left(\frac{\partial F}{\partial Y} \right)$$

$$d\lambda = - \overline{d\varepsilon_p} / \left(\frac{\partial F}{\partial Y} \right) \quad (2.69)$$

which is similar to equation 2.65, which was obtained by using the work hardening hypothesis.

The parameter A can be defined as

$$A = - \left(\frac{\partial F}{\partial K_3} \right) \left(\frac{dK_3}{d\lambda} \right)$$

$$\frac{\partial F}{\partial K_3} = \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial K_3}$$

Then

$$A = - \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial K_3} \frac{\partial K_3}{\partial \lambda}$$

or

$$A = - \frac{\partial F}{\partial Y} \frac{\partial Y}{\overline{\partial \varepsilon_p}} \frac{\overline{d\varepsilon_p}}{dK_3} \frac{dK_3}{d\lambda} \quad (2.70)$$

Substituting equation 2.69 into equation 2.68, it can be proved that:-

$$dK_3 = Y \left(\overline{d\varepsilon_T} + d\lambda \left(\frac{\partial F}{\partial Y} \right) \right)$$

i.e.

$$\frac{dK_3}{d\lambda} = Y \left(\frac{\overline{d\varepsilon_T}}{d\lambda} + \frac{\partial F}{\partial Y} \right) = Y \frac{\partial F}{\partial Y} \left(1 - \frac{\overline{d\varepsilon_T}}{\overline{d\varepsilon_p}} \right) \quad (2.71)$$

Substituting the above equation and equation 2.69 into equation 2.70, it can be deduced that:-

$$A = - \left(\frac{\partial F}{\partial Y} \right)^2 H' Y \frac{\overline{d\varepsilon}_F}{d\kappa_3} \left(1 - \frac{\overline{d\varepsilon}_T}{\overline{d\varepsilon}_P} \right),$$

$$A = - \left(\frac{\partial F}{\partial Y} \right)^2 H' Y \frac{\overline{d\varepsilon}_P}{d\kappa_3} \left(- \frac{\overline{d\varepsilon}_e}{\overline{d\varepsilon}_P} \right),$$

i.e.

$$A = - \left(\frac{\partial F}{\partial Y} \right)^2 H' \frac{Y \overline{d\varepsilon}_e}{d\kappa_3}$$

where

$$H' = \frac{\partial Y}{\overline{d\varepsilon}_P}$$

But $d\kappa_3 = Y \overline{d\varepsilon}_e$, then the above expression can be reduced into

$$A = \left(\frac{\partial F}{\partial Y} \right)^2 \cdot H'$$

which is similar to the previous expression which is obtained by using work hardening hypothesis.

With the yield surface $F(\underline{\sigma}, Y(k)) = F(\underline{\sigma}) - Y$, then

$$\frac{\partial F}{\partial Y} = \frac{\partial F(\underline{\sigma})}{\partial Y} - \frac{\partial Y}{\partial Y} = -1$$

$$\text{Hence } A = H' \quad (2.72)$$

which can be related to the uniaxial stress-strain diagram for the material as follows:-

(a) $\overline{\sigma} = Y = F(\underline{\sigma})$... from the stress-strain diagram

(b) $\overline{\varepsilon}_P = \int \overline{d\varepsilon}_P$

(c) $\overline{\varepsilon}_T = \overline{\varepsilon}_P + \overline{\varepsilon}_e = \overline{\varepsilon}_P + \overline{\sigma}/E$

(d) $E_t = \frac{d\sigma}{d\epsilon}$.. the tangential modulus, which can be obtained from the stress-strain diagram at point (σ, E)

For a linear strain hardening material, it can be deduced that:-

$$A = H' = \frac{E_t}{1 - \frac{E_t}{E}} = \text{constant}$$

and for a perfectly-plastic material:-

$$A = H' = \text{zero.}$$

2.6.2 Employing the Kinematic Hardening Rule

From the kinematic hardening rule the equation for the yield surface can be expressed as follows:-

$$F(\underline{\sigma} - \underline{\alpha}, Y_0) = 0$$

where

$\underline{\alpha}$ is the shift vector which represents the translation of the initial yield surface,

Y_0 is the uniaxial initial yield stress.

Prager [26,27] proposed that the shift vector in full stress space, or the increment of translation of the yield surface, should be assumed to act in the direction of the outward normal to the surface.

$$d\underline{\alpha}^p = C d\underline{\epsilon}_p$$

where

C is a parameter known as the modulus of kinematic hardening which characterises the hardening behaviour of the material.

In order to avoid the difficulties associated with the implementation of a complete kinematic hardening rule model in subspace, Ziegler [12] proposed a modification which can be used for the direct kinematic hardening rule (subspace), as well as for the complete kinematic hardening rule (fullspace). The Ziegler modification is:-

$$d\alpha^z = d\mu(\underline{\sigma} - \underline{\alpha}) \quad (2.73)$$

$$d\mu > 0$$

Prager's shift vector, $d\alpha^p = C d\varepsilon_p$ is interpreted as the projection of $d\alpha^z$ on the normal to the yield surface, at the instantaneous stress state, as shown in Fig(2.26). The equation for the loading surface can be rewritten as follows:-

$$F(\underline{\sigma}_r, Y_o) = 0 \quad (2.74)$$

where

$$\underline{\sigma}_r = \underline{\sigma} - \underline{\alpha} \dots \text{ is a reduced stress vector.}$$

The variation of the above yield surface is given by:-

$$dF = \left(\frac{\partial F(\underline{\sigma}_r, Y_o)}{\partial \underline{\sigma}_r} \right)^t \cdot d\underline{\sigma}_r = 0 \quad (2.75)$$

Suppose that

$$\underline{\sigma} = \beta \underline{\sigma}_r$$

where

β is an arbitrary parameter

Then

$$\frac{\partial \underline{\sigma}}{\partial \underline{\sigma}_r} = \beta, \quad \underline{\alpha} = (\beta - 1) \underline{\sigma}_r, \quad \frac{\partial \underline{\alpha}}{\partial \underline{\sigma}_r} = (\beta - 1)$$

Hence

$$\frac{\partial F}{\partial \underline{\sigma}_r} = \left(\frac{\partial F}{\partial \underline{\sigma}} \right)^t \frac{\partial \underline{\sigma}}{\partial \underline{\sigma}_r} + \left(\frac{\partial F}{\partial \underline{\alpha}} \right)^t \frac{\partial \underline{\alpha}}{\partial \underline{\sigma}_r}$$

$$\frac{\partial F}{\partial \underline{\sigma}_r} = \beta \frac{\partial F}{\partial \underline{\sigma}} + (\beta - 1) \frac{\partial F}{\partial \underline{\alpha}}$$

Finally, the above equation can be reduced to:-

$$\beta \left(\frac{\partial F}{\partial \underline{\sigma}} + \frac{\partial F}{\partial \underline{\alpha}} \right) - \left(\frac{\partial F}{\partial \underline{\sigma}_r} + \frac{\partial F}{\partial \underline{\alpha}} \right) = 0$$

Then, it can be shown that

$$\frac{\partial F}{\partial \underline{\sigma}_r} = \frac{\partial F}{\partial \underline{\sigma}} = - \frac{\partial F}{\partial \underline{\alpha}} = \underline{a} \quad (2.76)$$

As a result of the above relationship, equation 2.75 can be rewritten as follows:-

$$\left(\frac{\partial F}{\partial \underline{\sigma}} \right)^t d\underline{\sigma}_r = 0$$

But

$$d\sigma_r = d\sigma - d\alpha,$$

then

$$dF = \underline{a}^t (d\sigma - d\alpha) = 0$$

If $d\alpha$ is the Prager's shift then

$$dF = \underline{a}^t (d\sigma - C \, d\varepsilon_p) = 0 \quad (2.77)$$

To evaluate the parameter $d\lambda$, the above equation can be rewritten as follows:-

$$\underline{a}^t \, d\sigma - C \, \underline{a}^t \, d\varepsilon_p = 0 \quad (2.78)$$

From the flow rule, $d\varepsilon_p = d\lambda \cdot \underline{a}$, the above equation can be reduced to:-

$$\underline{a}^t \, d\sigma - C \, d\lambda \, \underline{a}^t \, \underline{a}$$

Hence

$$d\lambda = \frac{1}{C} \frac{\underline{a}^t \, d\sigma}{\underline{a}^t \, \underline{a}} \quad (2.79)$$

The variation of the yield surface, $F(\sigma - \alpha, Y_0) = F(\sigma, \varepsilon_p, Y_0)$, can be written as follows:-

$$dF = \left(\frac{\partial F}{\partial \sigma} \right)^t d\sigma + \left(\frac{\partial F}{\partial \varepsilon_p} \right)^t d\varepsilon_p$$

By comparing the above equation with equation 2.77, it can be deduced that:-

$$\left(\frac{\partial F}{\partial \underline{\varepsilon}_p} \right) = - C \underline{a} \quad (2.80)$$

Substituting the above equation into equation 2.55, and using the flow rule, (equation 2.52), it can be shown that:-

$$A = C \underline{a}^t \cdot \underline{a} \quad (2.81)$$

The Ziegler's shift vector can be written as:-

$$d \underline{\alpha}^z = d\mu (\underline{\sigma} - \underline{\alpha})$$

while the equivalent Prager's shift vector is:-

$$d \underline{\alpha}^p = C d \underline{\varepsilon}_p$$

Then, it can be shown that

$$\underline{a}^t d \underline{\alpha}^z = \underline{a}^t d \underline{\alpha}^p$$

or

$$d\mu \underline{a}^t (\underline{\sigma} - \underline{\alpha}) = C \underline{a}^t d \underline{\varepsilon}_p$$

By using the flow rule, it can be proved that:-

$$d\mu = \frac{C d\lambda \underline{a}^t \underline{a}}{\underline{a}^t (\underline{\sigma} - \underline{\alpha})}$$

Substituting equation 2.79 into the above equation, it can be shown that:-

$$d\mu = \frac{\underline{a}^t d \underline{\sigma}}{\underline{a}^t (\underline{\sigma} - \underline{\alpha})} = \frac{\underline{a}^t d \underline{\sigma}}{\underline{a}^t \underline{\sigma}_r} \quad (2.82)$$

Applying Euler's theorem to the equation for the yield surface (equation 2.74), it can be shown that:-

$$\left(\frac{\partial F}{\partial \underline{\sigma}_r} \right)^t \cdot \underline{\sigma}_r + \frac{\partial F}{\partial Y_o} Y_o = 0$$

But

$$\frac{\partial F}{\partial \underline{\sigma}_r} = \frac{\partial F}{\partial \underline{\sigma}} = \underline{a}, \text{ and}$$

$$\frac{\partial F}{\partial Y_o} = \frac{\partial}{\partial Y_o} (f(\underline{\sigma} - \underline{\alpha}) - Y_o) = -1$$

Then, the above equation can be reduced into:-

$$\underline{a}^t \cdot \underline{\sigma}_r = Y_o$$

Substituting the above equation into equation 2.82, it can be shown that

$$d\mu = \frac{\underline{a}^t d\underline{\sigma}}{Y_o}$$

From equation 2.78, it can be deduced that:-

$$\underline{a}^t d\underline{\sigma} = C \underline{a}^t d\underline{\varepsilon}_p$$

then

$$d\mu = C \frac{\underline{a}^t d\underline{\varepsilon}_p}{Y_o} \quad (2.83)$$

Parameter C is an important one and in order to evaluate it for the uniaxial case, the yield surface is represented by the following equation:-

$$F = f(\sigma_x - \alpha_x) - Y_0$$

which in differential form is as follows:-

$$dF = \frac{\partial f(\sigma_x - \alpha_x)}{\partial \sigma_x} \cdot (d\sigma_x - d\alpha_x) = 0$$

Hence

$$d\sigma_x = d\alpha_x$$

where

σ_x , α_x are the stress and shift values respectively for a uniaxial case in the x-direction,

$$d\alpha_x = C_x (d\epsilon_x)_p \text{ is the Prager's shift}$$

Then

$$C_x = \frac{d\alpha_x}{(d\epsilon_x)_p} = \frac{d\sigma_x}{(d\epsilon_x)_p} = \frac{2}{3} H'$$

where H' is obtained from the uniaxial test at point (σ_x, ϵ_x) , and for a linear strain-hardening material is constant and equal to:-

$$H' = E_t / \left(1 - \frac{E_t}{E} \right)$$

For the multiaxial stress case, it is required to evaluate the vector $\underline{C}$, which is based upon uniaxial test data and, it can be defined as follows:-

$$\underline{C} = \{C_x \ C_y \ C_z \ C_{xy} \ C_{yz} \ C_{zx}\}$$

where each component from vector $\underline{C}$ can be evaluated from the uniaxial case as follows:-

$$C_i = H'(\sigma_i, \epsilon_i) \quad i = 1, 2, \dots, 6$$

where

$$C_1 \equiv C_x, \quad C_2 \equiv C_y, \quad \dots \quad C_6 \equiv C_{zx},$$

That is, each stress component is treated as though it alone were present, and a corresponding component for $\underline{C}$ is determined on the basis of the uniaxial normal, or shear, stress-strain curve.

The single value of C is computed as a weighted average of $\underline{C}$ vector components [116] as follows:-

$$\frac{1}{C} = \sum_{i=1}^6 W_i \left(\frac{1}{C_i} \right)$$

where

$$W_i = \frac{\sigma_i^2}{\left(\sum_{j=1}^6 \sigma_j^2 \right)}$$

For a perfectly-plastic material, H' is equal to zero, and therefore, the $\underline{C}$ vector will be zero. Hence, the shift vector and all other parameters, will equal zero also. While for a material with linear strain hardening, $H' = \text{constant}$, it can be proved that

$$C = H'$$

It is clear that, from this rule, the state of stress and strain throughout plastic flow is characterised by the vector $\underline{C}$, whilst from the isotropic hardening rule this state can be defined from single scalar quantities referred to as "effective stress", $\bar{\sigma}$, and "effective strain", $\bar{\epsilon}$.

2.6.3 Employing the Mixed Hardening Rule

The equation of the yield surface according to the mixed hardening rule can be expressed as follows:-

$$F(\underline{\sigma} - \underline{\alpha}, Y_r) = 0 \quad (2.84)$$

where

Y_r is the current reduced yield surface from the simple uniaxial test, which is a measure of the "radius" of the expanded yield surface.

The increment of plastic strain can be split into two components [118], as follows:-

$$d\underline{\varepsilon}_p = d\underline{\varepsilon}_p^i + d\underline{\varepsilon}_p^k$$

where

$d\underline{\varepsilon}_p^i$ is associated with the expansion of the yield surface, and

$d\underline{\varepsilon}_p^k$ is associated with the translation of the yield surface.

These two strain components can be expressed as follows:-

$$\left. \begin{aligned} d\underline{\varepsilon}_p^i &= C_M d\underline{\varepsilon}_p \\ d\underline{\varepsilon}_p^k &= (1 - C_M) d\underline{\varepsilon}_p \end{aligned} \right\} \quad (2.85)$$

where

C_M is a numerical parameter, known as the "mixed hardening parameter" in the range $-1 \leq C_M \leq 1$, which defines the amount of isotropy in the total hardening. Since C_M can also be given negative

values, then isotropic softening can also be considered (i.e. $d\varepsilon_p^i$ can be negative with the yield surface contracting), as shown in Fig.(2.21b).

Since the shift vector $\underline{\alpha}$ is a function of ε_p only, and Y_r is a function of κ only, then

$$dF = \left(\frac{\partial F}{\partial(\underline{\sigma} - \underline{\alpha})} \right)^t \cdot d(\underline{\sigma} - \underline{\alpha}) + \frac{\partial F}{\partial Y_r} dY_r$$

Using equation 2.76, it can be shown that:-

$$dF = \underline{a}^t \cdot (d\underline{\sigma} - d\underline{\alpha}) + \frac{\partial F}{\partial Y_r} dY_r = 0 \quad (2.86)$$

where

$$d\underline{\alpha} = d\underline{\alpha}^P = C d\varepsilon_p^k \quad \text{is Prager's shift vector increment}$$

or

$$d\underline{\alpha} = d\underline{\alpha}^Z = d\mu(\underline{\sigma} - \underline{\alpha}) \quad \text{is Ziegler's shift vector increment}$$

From equation 2.55 the parameter A can be expressed, as follows:-

$$A = - \frac{1}{d\lambda} \left[\left(\frac{\partial F}{\partial \varepsilon_p^k} \right)^t \cdot d\varepsilon_p^k + \frac{\partial F}{\partial Y_r} dY_r \right]$$

From the flow rule (Eq.52), it can be deduced that

$$d\varepsilon_p^k = (1 - C_M) d\lambda \underline{a} \quad (2.88)$$

$$d\varepsilon_p^i = C_M d\lambda \underline{a} \quad (2.89)$$

From equation 2.80, it can be shown that:-

$$\left(\frac{\partial F}{\partial \varepsilon_p^k} \right) = - C \underline{a}$$

By using the above equation and equation 2.88, it can be shown that

$$\left(\frac{\partial F}{\partial \underline{\varepsilon}_p^k} \right)^t \cdot d\underline{\varepsilon}_p^k = - (1 - C_M) d\lambda \quad C_M \quad \underline{a}^t \quad \underline{a} \quad (2.90)$$

Using the work-hardening hypothesis, a correlation between the uniaxial and multiaxial cases can be expressed as follows:-

$$Y_r \quad \overline{d\varepsilon}_p^i = (\underline{\sigma} - \underline{\alpha})^t \quad d\underline{\varepsilon}_p^i$$

By using the flow rule, it can be shown that:-

$$Y_r \quad \overline{d\varepsilon}_p^i = C_M \quad d\lambda \quad (\underline{\sigma} - \underline{\alpha})^t \quad \underline{a} \quad (2.91)$$

Applying Euler's theorem to the yield surface equation (equation 2.84), it can be deduced that:-

$$\underline{a}^t \quad (\underline{\sigma} - \underline{\alpha}) + \frac{\partial F}{\partial Y_r} \quad Y_r = 0$$

That is

$$(\underline{\sigma} - \underline{\alpha})^t \quad \underline{a} = - \frac{\partial F}{\partial Y_r} \quad Y_r \quad (2.92)$$

Hence,

$$Y_r \quad \overline{d\varepsilon}_p^i = - C_M \quad d\lambda \quad \frac{\partial F}{\partial Y_r} \quad Y_r$$

and,

$$d\lambda = \frac{- \overline{d\varepsilon}_p^i}{C_M \quad \frac{\partial F}{\partial Y_r}} \quad (2.93)$$

The above equation could also be derived by using the elastic-work hypothesis as shown below. The elastic-work hypothesis states that:-

$$Y_r \quad \overline{d\varepsilon_e^i} = (\underline{\sigma} - \underline{\alpha})^t \quad d\underline{\varepsilon_e^i} = \underline{\sigma_r}^t \quad (d\underline{\varepsilon^i} - d\underline{\varepsilon_p^i})$$

where

$\overline{d\varepsilon_e^i}$ is the effective elastic strain increment for the isotropic part of the uniaxial test.

$d\underline{\varepsilon^i}$ is the total strain increment vector for the isotropic part of the multiaxial case.

$d\underline{\varepsilon_e^i}$, $d\underline{\varepsilon_p^i}$ are the isotropic elastic and plastic strain increment vectors, respectively.

The last equation can be rewritten as follows:-

$$Y_r \quad \overline{d\varepsilon_e^i} = \underline{\sigma_r}^t \quad d\underline{\varepsilon^i} - \underline{\sigma_r}^t \quad d\underline{\varepsilon_p^i}$$

Substituting equation 2.84 into the above equation, it can be shown that:-

$$Y_r \quad \overline{d\varepsilon_e^i} = \underline{\sigma_r}^t \quad d\underline{\varepsilon^i} - C_M \quad d\lambda \quad \underline{\sigma_r}^t \quad \underline{a} \quad (2.94)$$

Correlating between the uniaxial and multiaxial cases, it can be shown that:-

$$\underline{\sigma_r}^t \quad d\underline{\varepsilon^i} = Y_r \quad \overline{d\varepsilon^i}$$

or

$$\overline{d\varepsilon^i} = \frac{\underline{\sigma_r}^t \quad d\underline{\varepsilon^i}}{Y_r}$$

Equation 2.94 can then be rewritten as follows:-

$$\overline{d\epsilon}_e^i = \overline{d\epsilon}^i - \frac{C_M}{Y_r} \frac{d\lambda}{\sigma_r^t \cdot \underline{a}}$$

or

$$(\overline{d\epsilon}^i - \overline{d\epsilon}_e^i) = \frac{C_M}{Y_r} \frac{d\lambda}{\sigma_r^t \cdot \underline{a}}$$

Substituting equation 2.92 into the above equation it can be shown that:-

$$\overline{d\epsilon}_p^i = -C_M \, d\lambda \left(\frac{\partial F}{\partial Y_r} \right)$$

Finally,

$$d\lambda = - \frac{\overline{d\epsilon}_p^i}{C_M \left(\frac{\partial F}{\partial Y_r} \right)}$$

which is similar to equation 2.93 which was based upon the work-hardening hypothesis.

Substituting equations 2.92 and 2.90, into equation 2.89, the parameter A can be expressed as follows

$$A = (1 - C_M) \, C \, \underline{a}^t \, \underline{a} + C_M \left(\frac{\partial F}{\partial Y_r} \right)^2 \frac{d Y_r}{\overline{d\epsilon}_p^i}$$

or

$$A = (1 - C_M) A^k + C_M A^i \quad (2.95)$$

where

$$A^k = C \, \underline{a}^t \, \underline{a} \quad (2.96)$$

and

$$A^i = \left(\frac{\partial F}{\partial Y_r} \right)^2 \frac{dY_r}{d\varepsilon_p^i} = \left(\frac{\partial F}{\partial Y_r} \right)^2 H' \quad (2.97)$$

Defining the total kinematic strain vector as:-

$$\underline{\varepsilon}_T^k = \underline{\varepsilon}_e^k + \underline{\varepsilon}_p^k$$

$$\underline{\varepsilon}_T^k = (1 - C_M) \cdot (\underline{\varepsilon}_e + \underline{\varepsilon}_p)$$

where

$$\underline{\varepsilon}_e = \underline{D}^{-1} \underline{\sigma} ,$$

and

$$\underline{\varepsilon}_p = \underline{\Sigma} d\underline{\varepsilon}_p$$

then the parameter C can be obtained, as previously shown , for the kinematic hardening rule, at the stress and strain vectors $(\underline{\sigma}_r, \underline{\varepsilon}_T^k)$.

For the special case of a material with an idealised stress-strain diagram

$$C = \frac{dY_r}{d\varepsilon_p^i} = H' ,$$

and with a yield surface $F(\underline{\sigma} - \underline{\alpha}) - Y_r = 0$, it can be shown that:-

$$A = H' [C_M + (1 - C_M) \underline{a}^t \underline{a}]$$

Using Ziegler's modification to Prager's rule

$$d\alpha = d\mu(\sigma - \alpha) ,$$

it can be shown that:

$$\underline{a}^t d\alpha = C \underline{a}^t d\underline{\epsilon}_p^k = C(1 - C_M) \underline{a}^t \cdot d\underline{\epsilon}_p$$

Hence

$$d\mu = \frac{C(1 - C_M) \underline{a}^t d\underline{\epsilon}_p}{\underline{a}^t (\sigma - \alpha)} ,$$

and from equation 2.86, the above equation can be simplified to:-

$$d\mu = \frac{C(1 - C_M) \underline{a}^t d\underline{\epsilon}_p}{- Y_r \frac{\partial F}{\partial Y_r}} \quad (2.98)$$

2.7 ANALYSIS OF ELASTOPLASTIC CONTINUUM PROBLEMS USING A SPECIFIC YIELD CRITERION

The first step in the FEM or BEM elasto-plastic analysis is to evaluate the $\underline{D}_{ep}$ matrix at any specified point inside the domain of the problem. In section 2.6 the procedure for the evaluation of the $\underline{D}_{ep}$ matrix has been described for a general yield criterion, whilst in this section the $\underline{D}_{ep}$ matrix will be derived for given suitable yield criteria.

As an example the modified Von Mises yield criterion will be explained in detail here, whilst explanations involving the Von Mises and Tresca yield criteria will be found in appendices 2.3 and 2.4 respectively.

2.7.1 Analysis of the Elasto-plastic Problem Utilising the Modified Von Mises Yield Criterion

The analysis can be carried out by using one of the following hardening rules:-

(i) Isotropic Hardening Rule

The equation of the yield surface in principal stress space can be obtained from equations (2.42) and (2.43), as follows:-

$$F = \sqrt{\bar{\sigma}^2 + 3\Delta\sigma_m} - \sqrt{Y_c Y_t} = 0 \quad (2.99)$$

Hence, it can be concluded that:-

$$F = \text{Function} (J_2, \sigma_m, Y_c, Y_t)$$

Then the gradient vector $\underline{a}$ can be evaluated as follows:-

$$\underline{a} = \frac{\partial F}{\partial \underline{\sigma}} = \frac{\partial F}{\partial J_2} \frac{\partial J_2}{\partial \underline{\sigma}} + \frac{\partial F}{\partial \sigma_m} \frac{\partial \sigma_m}{\partial \underline{\sigma}}$$

$$\text{or } \underline{a} = C_1 \underline{a}_1 + C_2 \underline{a}_2 \quad (2.100)$$

$$\text{where } C_1 = \frac{\partial F}{\partial \sigma_m} = \frac{1}{2} (\bar{\sigma}^2 + 3\Delta\sigma_m)^{-\frac{1}{2}} 3\Delta$$

Defining, the modified equivalent stress as:-

$$\bar{\sigma}_m = (\bar{\sigma}^2 + 3\Delta\sigma_m)^{\frac{1}{2}}$$

Then
$$C_1 = \frac{3 \Delta}{2\bar{\sigma}_m} \quad (2.101)$$

While,
$$C_2 = \frac{\partial F}{\partial J_2} = \frac{1}{2} (\bar{\sigma}^2 + 3 \Delta \sigma_m)^{-\frac{1}{2}} \frac{\partial \bar{\sigma}^2}{\partial J_2}$$

But from equation (2.10), it can be shown that:-

$$\frac{\partial \bar{\sigma}^2}{\partial J_2} = 3$$

Then
$$C_2 = \frac{3}{2\bar{\sigma}_m} \quad (2.102)$$

i.e.
$$C_1 = \Delta \cdot C_2$$

The gradient vector $\underline{a}_1$ can be expressed as follows:-

$$\underline{a}_1 = \frac{\partial \sigma_m}{\partial \underline{\sigma}} = \left\{ \frac{\partial \sigma_m}{\partial \sigma_x} \quad \frac{\partial \sigma_m}{\partial \sigma_y} \quad \dots \quad \frac{\partial \sigma_m}{\partial \tau_{zx}} \right\}$$

But
$$\frac{\partial \sigma_m}{\partial \sigma_x} = \frac{\partial \sigma_m}{\partial \sigma_y} = \frac{\partial \sigma_m}{\partial \sigma_z} = \frac{1}{3}$$

$$\frac{\partial \sigma_m}{\partial \tau_{xy}} = \frac{\partial \sigma_m}{\partial \tau_{yz}} = \frac{\partial \sigma_m}{\partial \tau_{zx}} = 0$$

Then
$$\underline{a}_1 = \frac{1}{3} \{ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \} = \frac{1}{3} \underline{I} \quad (2.103)$$

where
$$\underline{I} = \{ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \}$$

The gradient vector $\underline{a}_2$ can be written as follows:-

$$\underline{a}_2 = \frac{\partial J_2}{\partial \underline{\sigma}} = \left\{ \frac{\partial J_2}{\partial \sigma_x} \quad \frac{\partial J_2}{\partial \sigma_y} \quad \dots\dots\dots \frac{\partial J_2}{\partial \tau_{zx}} \right\}$$

Using the definition of J_2 in equation (2.7), it can be shown that:-

$$\frac{\partial J_2}{\partial \sigma_x} = \frac{1}{6} \left[2(\sigma_x - \sigma_y) - 2(\sigma_z - \sigma_x) \right] = S_x$$

Similarly

$$\frac{\partial J_2}{\partial \sigma_y} = S_y, \quad \text{and} \quad \frac{\partial J_2}{\partial \sigma_z} = S_z$$

While

$$\frac{\partial J_2}{\partial \tau_{xy}} = 2 \tau_{xy}, \quad \frac{\partial J_2}{\partial \tau_{yz}} = 2 \tau_{yz}, \quad \text{and} \quad \frac{\partial J_2}{\partial \tau_{zx}} = 2 \tau_{zx}$$

$$\text{Then } \underline{a}_2 = \{ S_x \ S_y \ S_z \ 2 \tau_{xy} \ 2 \tau_{yz} \ 2 \tau_{zx} \}$$

The stress deviatoric vector can be defined as follows:-

$$\underline{S} = \{ S_x \ S_y \ S_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx} \}$$

which can be rewritten as follows:-

$$\underline{S} = \frac{2}{3} \underline{M}_s \cdot \underline{\sigma} \quad (2.104)$$

where

$$\underline{M}_s = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \end{bmatrix},$$

as shown in appendix 2.5.

Similarly, it can be shown that:-

$$\underline{a}_2 = \frac{2}{3} \underline{M}_a \underline{\sigma} \quad (2.105)$$

where

$$\underline{M}_a = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

as shown in appendix 2.5.

The elasto-plastic stress/strain matrix can be written as follows:-

$$\underline{D}_{ep} = \underline{D} - \underline{D}_p$$

$$\text{where } \underline{D}_p = \frac{(\underline{D} \underline{a})(\underline{D} \underline{a})^t}{A+B} = \frac{\underline{d} \underline{d}^t}{A+B}$$

For the Modified Von Mises criterion, it can be shown that:-

$$\underline{d} = \underline{D} \underline{a} = \underline{D} (C_2 \underline{a}_2 + C_1 \underline{a}_1)$$

From equations (2.102) and (2.105), it can be shown that:-

$$C_2 \underline{D} \underline{a}_2 = \frac{3}{2\bar{\sigma}_m} \underline{D} \frac{2}{3} \underline{M}_a \underline{\sigma}$$

or

$$C_2 \underline{D} \underline{a}_2 = \frac{\underline{D} \underline{M}_a \underline{\sigma}}{\bar{\sigma}_m}$$

Using an analysis similar to that explained in appendix 2.3, it can be shown that:-

$$C_2 \underline{D} \underline{a}_2 = \frac{3 G S}{\bar{\sigma}_m} \quad (2.106)$$

whilst the term $C_1 \underline{D} \underline{a}_1$ can be simplified into:-

$$C_1 \underline{D} \underline{a}_1 = \frac{\Delta}{2 \bar{\sigma}_m} \underline{D} \cdot \underline{I}$$

It can be shown that:-

$$\underline{D} \cdot \underline{I} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 - \nu + \nu + \nu \\ 1 - \nu + \nu + \nu \\ 1 - \nu + \nu + \nu \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{E}{(1-2\nu)} \underline{I}$$

Then

$$C_1 \underline{D} \underline{a}_1 = \frac{E \Delta}{2(1-2\nu) \bar{\sigma}_m} \underline{I} \quad (2.107)$$

Similarly, by using equations (2.106) and (2.107) the vector $\underline{d}$ can be expressed as follows:-

$$\begin{aligned}\underline{d} &= \underline{D} \underline{a} = \frac{3G\underline{S}}{\bar{\sigma}_m} + \frac{E \Delta}{2(1-2\nu) \bar{\sigma}_m} \underline{I} \\ &= \frac{1}{\bar{\sigma}_m} \left[3G\underline{S} + \frac{E}{2(1-2\nu)} \Delta \underline{I} \right] \quad (2.108)\end{aligned}$$

The parameter B is given by:-

$$\begin{aligned}B &= \underline{a}^t \underline{d} = (C_2 \underline{a}_2^t + C_1 \underline{a}_1^t) \frac{1}{\bar{\sigma}_m} \left[3G\underline{S} + \frac{E}{2(1-2\nu)} \Delta \underline{I} \right] \\ B &= \frac{1}{\bar{\sigma}_m} \left[3 C_2 \quad G \underline{a}_2^t \quad \underline{S} + \frac{E}{2(1-2\nu)} \Delta \quad C_2 \underline{a}_2^t \quad \underline{I} \right. \\ &\quad \left. + 3 C_1 \quad G \underline{a}_1^t \quad \underline{S} + \frac{E}{2(1-2\nu)} \Delta \quad C_1 \underline{a}_1^t \quad \underline{I} \right]\end{aligned}$$

But

$$\underline{a}_2^t \underline{I} = [S_x \quad S_y \quad S_z \quad 2\tau_{xy} \quad 2\tau_{yz} \quad 2\tau_{zx}] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = S_x + S_y + S_z = 0$$

Similarly

$$\underline{a}_1^t \underline{S} = \frac{1}{3} (S_x + S_y + S_z) = 0$$

While

$$\underline{a}_1^t \cdot \underline{I} = \frac{1}{3} [1 \ 1 \ 1 \ 0 \ 0 \ 0] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$$

Then

$$\begin{aligned} B &= \frac{1}{\bar{\sigma}_m} \left[3 G C_2 \underline{a}_2^t \underline{S} + \frac{E}{2(1-2\nu)} \Delta C_1 \right] \\ &= \frac{C_2}{\bar{\sigma}_m} \left[3 G \underline{a}_2^t \underline{S} + \frac{E}{2(1-2\nu)} \Delta^2 \right] \end{aligned}$$

Substituting equations (2.107) and (2.102) into the above equation, it can be deduced that:-

$$B = \frac{3}{2(\bar{\sigma}_m)^2} \left[3 \frac{2}{3} \bar{\sigma}^2 G + \frac{E}{2(1-2\nu)} \Delta^2 \right]$$

or

$$B = 3G \left(\frac{\bar{\sigma}}{\bar{\sigma}_m} \right)^2 + \frac{3 E}{4(1-2\nu)} \left(\frac{\Delta}{\bar{\sigma}_m} \right)^2 \quad (2.109)$$

The general form for parameter A is as follows:-

$$A = \frac{-1}{d\lambda} \left[\left(\frac{\partial F}{\partial \underline{\varepsilon}_p} \right)^t \cdot d\underline{\varepsilon}_p + \frac{\partial F}{dk} dk \right]$$

In which the term $\partial F / \partial k$ must be determined for the modified Von Mises yielding criterion. This can be accomplished as follows:-

$$\frac{\partial F}{\partial k} = \frac{\partial F}{\partial Y_c} \frac{\partial Y_c}{\partial k} + \frac{\partial F}{\partial Y_t} \frac{\partial Y_t}{\partial k}$$

$$\text{or } \frac{\partial F}{\partial k} = \frac{\partial F}{\partial Y_c} \frac{\partial Y_c}{(\partial \bar{\epsilon}_p)_c} \frac{(d\bar{\epsilon}_p)_c}{dk} + \frac{\partial F}{\partial Y_t} \frac{\partial Y_t}{(\partial \bar{\epsilon}_p)_t} \frac{(d\bar{\epsilon}_p)_t}{dk} \quad (110)$$

From the work hardening hypothesis it can be shown that:-

$$\text{and } \left. \begin{aligned} dk &= Y_c \frac{(d\bar{\epsilon}_p)_c}{(\partial \bar{\epsilon}_p)_c} = d\epsilon_p^t \frac{\sigma}{\sigma_m} = (d\lambda)_c \frac{a^t}{a} \frac{\sigma}{\sigma_m} \\ dk &= Y_t \frac{(d\bar{\epsilon}_p)_t}{(\partial \bar{\epsilon}_p)_t} = d\epsilon_p^t \frac{\sigma}{\sigma_m} = (d\lambda)_t \frac{a^t}{a} \frac{\sigma}{\sigma_m} \end{aligned} \right\} \quad (2.111)$$

Applying Euler's theorem to the yield surface expression given in equation (2.99), it can be shown that:-

$$\left(\frac{\partial F}{\partial \sigma} \right)^t \frac{\sigma}{\sigma_m} + \frac{\partial F}{\partial Y_c} Y_c + \frac{\partial F}{\partial Y_t} Y_t + \frac{\partial F}{\partial \sigma_m} \sigma_m = 0 \quad (2.112)$$

$$\text{where } \frac{\partial F}{\partial \sigma_m} \sigma_m = \frac{1}{2} \frac{3}{\bar{\sigma}_m} \Delta \sigma_m = \frac{3 \Delta \sigma_m}{2 \bar{\sigma}_m} = C_1 \sigma_m$$

$$\left(\frac{\partial F}{\partial \sigma} \right)^t \frac{\sigma}{\sigma_m} = \frac{a^t}{a} \frac{\sigma}{\sigma_m}$$

$$\left. \begin{aligned} \frac{\partial F}{\partial Y_c} Y_c &= \left(\frac{3 \sigma_m}{2 \bar{\sigma}_m} - \frac{Y_t}{2 \sqrt{Y_c Y_t}} \right) Y_c = \frac{3 \sigma_m Y_c}{2 \bar{\sigma}_m} - \frac{\sqrt{Y_t Y_c}}{2} \\ \frac{\partial F}{\partial Y_t} Y_t &= - \left(\frac{3 \sigma_m}{2 \bar{\sigma}_m} - \frac{Y_c}{2 \sqrt{Y_c Y_t}} \right) Y_t = \frac{-3 \sigma_m Y_t}{2 \bar{\sigma}_m} - \frac{\sqrt{Y_t Y_c}}{2} \end{aligned} \right\} \quad (2.113)$$

Substituting the above equations into equation (2.112), it can be deduced that:-

$$\begin{aligned}
 \frac{a^t}{a^c} \sigma + \frac{3 \sigma_m}{2 \bar{\sigma}_m} Y_c - \frac{\sqrt{Y_t Y_c}}{2} - \frac{3 \sigma_m Y_t}{2 \bar{\sigma}_m} - \frac{\sqrt{Y_t Y_c}}{2} + \frac{3 \Delta \sigma_m}{2 \bar{\sigma}_m} &= 0 \\
 \frac{a^t}{a^c} \sigma - \sqrt{Y_t Y_c} + \frac{3 \sigma_m}{2 \bar{\sigma}_m} (Y_c - Y_t) + \frac{3 \Delta \sigma_m}{2 \bar{\sigma}_m} & \\
 \frac{a^t}{a^c} \sigma = \sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\bar{\sigma}_m} \Delta & \quad (2.114)
 \end{aligned}$$

Substituting the above equation into equation (2.111), it can be seen that:-

$$dk = (d\lambda)_c \left[\sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\bar{\sigma}_m} \Delta \right]$$

or

$$dk = (d\lambda)_t \left[\sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\bar{\sigma}_m} \Delta \right]$$

$$\text{i.e.} \quad \frac{dk}{(d\lambda)_c} = \frac{dk}{(d\lambda)_t} = \sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\bar{\sigma}_m} \Delta \quad (2.115)$$

Also, from equation (2.111), it can be shown that:-

$$\frac{(\overline{d\varepsilon_p})_c}{dk} = \frac{1}{Y_c} \quad \text{and} \quad \frac{(\overline{d\varepsilon_p})_t}{dk} = \frac{1}{Y_t} \quad (2.116)$$

Hence, it can be deduced that:-

$$\left. \begin{aligned} (d\lambda)_c &= \frac{(\overline{d\varepsilon})_c \cdot Y_c}{\overline{a}^t \cdot \overline{\sigma}} = \frac{(\overline{d\varepsilon}_p)_c \cdot Y_c}{(\sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\overline{\sigma}_m} \Delta)} \\ (d\lambda)_t &= \frac{(\overline{d\varepsilon}_p)_t \cdot Y_t}{(\sqrt{Y_t Y_c} - \frac{3 \sigma_m}{\overline{\sigma}_m} \Delta)} \end{aligned} \right\} \quad (2.117)$$

Substituting equations (2.116) into (2.110), it can be seen that:-

$$\frac{\partial F}{\partial k} = \frac{\partial F}{\partial Y_c} H'_c \frac{1}{Y_c} + \frac{\partial F}{\partial Y_t} H'_t \frac{1}{Y_t}$$

where

$$H'_c = \frac{\partial Y_c}{\partial \varepsilon_p}, \quad \text{and} \quad H'_t = \frac{\partial Y_t}{\partial \varepsilon_p}$$

Therefore the expression for A is given by:-

$$A = - \frac{dk}{d\lambda} \left[\frac{\partial F}{\partial Y_c} \frac{H'_c}{Y_c} + \frac{\partial F}{\partial Y_t} \frac{H'_t}{Y_t} \right]$$

$$\text{where} \quad \frac{dk}{d\lambda} = \frac{dk}{(d\lambda)_c} = \frac{dk}{(d\lambda)_t}, \quad \text{i.e.} \quad (d\lambda)_c = (d\lambda)_t$$

$$\text{Then} \quad A = \left(\frac{3 \sigma_m \Delta}{\overline{\sigma}_m} - \sqrt{Y_c Y_t} \right) \left[\frac{\partial F}{\partial Y_c} \frac{H'_c}{Y_c} + \frac{\partial F}{\partial Y_t} \frac{H'_t}{Y_t} \right]$$

$$A = (2 C_1 \sigma_m - \sqrt{Y_c Y_t}) \left[\frac{\partial F}{\partial Y_c} \frac{H'_c}{Y_c} + \frac{\partial F}{\partial Y_t} \frac{H'_t}{Y_t} \right] \quad (2.118)$$

Substituting equations (2.113) into the above expression, it can be shown that;-

$$A = (2 C_1 \sigma_m - \sqrt{Y_c Y_t}) \left[\frac{3 \sigma_m}{\bar{\sigma}_m} \left(\frac{H'_c}{Y_c} - \frac{H'_t}{Y_t} \right) - \frac{1}{2\sqrt{Y_c Y_t}} \left(\frac{Y_t H'_c}{Y_c} + \frac{Y_c H'_t}{Y_t} \right) \right] \quad (2.119)$$

From equation (2.108) it can be shown that:-

$$\begin{aligned} \underline{d} \cdot \underline{d}^t &= \frac{1}{(\bar{\sigma}_m)^2} \left[3 G \underline{S} + \frac{E \Delta}{2(1-2\nu)} \underline{I} \right] \left[3 G \underline{S}^t + \frac{E}{2(1-2\nu)} \Delta \underline{I}^t \right] \\ \underline{d} \cdot \underline{d}^t &= \frac{1}{(\bar{\sigma}_m)^2} \left[9G^2 \underline{S} \underline{S}^t + \frac{3 G E \Delta}{2(1-2\nu)} (\underline{S} \underline{I}^t + \underline{I} \underline{S}^t) \right. \\ &\quad \left. + \left(\frac{E \Delta}{2(1-2\nu)} \right)^2 \underline{I} \underline{I}^t \right] \end{aligned}$$

But

$$\underline{I} \underline{S}^t = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [S_x \ S_y \ S_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}] ,$$

or

$$\underline{I} \quad \underline{S}^t = \begin{bmatrix} S_x & S_y & S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ S_x & S_y & S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ S_x & S_y & S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \underline{S}_1, \quad (2.120)$$

it follows that:-

$$\underline{S} \quad \underline{I}^t = (\underline{S}_1)^t, \quad \text{and}$$

$$\underline{S} \quad \underline{I}^t + \underline{I} \quad \underline{S}^t = \underline{S}_1 + \underline{S}_1^t = \underline{S}_2 = \begin{bmatrix} 2S_x & S_x S_y & S_x S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ S_y S_x & 2S_y & S_y S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ S_z S_x & S_y S_z & 2S_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \\ \tau_{xy} & \tau_{xy} & \tau_{xy} & 0 & 0 & 0 \\ \tau_{yz} & \tau_{yz} & \tau_{yz} & 0 & 0 & 0 \\ \tau_{zx} & \tau_{zx} & \tau_{zx} & 0 & 0 & 0 \end{bmatrix}$$

Also it can be shown that

$$\underline{I}_1 = \underline{I} \quad \underline{I}^t = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\underline{d} \quad \underline{d}^t = \frac{1}{(\bar{\sigma}_m)^2} \left[9 G^2 \underline{S} \quad \underline{S}^t + \frac{3GE}{2(1-2\nu)} \underline{S}_2 + \left(\frac{E \Delta}{2(1-2\nu)} \right)^2 \underline{I}_1 \right] \quad (2.122)$$

If $Y_c = Y_t$, i.e. $\Delta = 0$, and $\bar{\sigma}_m = \bar{\sigma}$ then the above equation is reduced to:-

$$\underline{d} \underline{d}^t = \frac{9G^2}{\bar{\sigma}^2} \underline{S} \underline{S}^t, \quad \text{which is similar to what has been}$$

obtained for the case of the Von Mises yield criterion.

By using equations (2.109), (2.119), and (2.122), $\underline{D}_p$ and $\underline{D}_{ep}$ can be evaluated.

The total effective plastic strain increment can be defined, by using the flow rule as follows:-

$$d\underline{\varepsilon}_p = d\lambda \underline{a} = d\lambda C_2 \underline{a}_2 + d\lambda C_1 \underline{a}_1 \quad (2.123)$$

or in explicit form as follows:-

$$\left. \begin{aligned} (d\varepsilon_x)_p &= d\lambda_2 S_x + d\lambda_1 \\ (d\varepsilon_y)_p &= d\lambda_2 S_y + d\lambda_1 \\ (d\varepsilon_z)_p &= d\lambda_2 S_z + d\lambda_1 \\ (d\gamma_{xy})_p &= 2 d\lambda_2 \tau_{xy} \\ (d\gamma_{yz})_p &= 2 d\lambda_2 \tau_{yz} \\ (d\gamma_{zx})_p &= 2 d\lambda_2 \tau_{zx} \end{aligned} \right\} \quad (2.124)$$

$$\text{where} \quad d\lambda_2 = C_2 \quad d\lambda = \frac{3}{2\bar{\sigma}_m} d\lambda, \quad (2.125)$$

$$\text{and} \quad d\lambda_1 = C_1 \quad d\lambda = \frac{3 \Delta}{2\bar{\sigma}_m} d\lambda, \quad (2.126)$$

$$\text{i.e.} \quad d\lambda_1 = \Delta d\lambda_2$$

Therefore it can be shown that:-

$$\left. \begin{aligned}
 S_x &= \frac{1}{d\lambda_2} [(d\varepsilon_x)_p - \Delta] , \\
 S_y &= \frac{1}{d\lambda_2} [(d\varepsilon_y)_p - \Delta] , \\
 S_z &= \frac{1}{d\lambda_2} [(d\varepsilon_z)_p - \Delta] , \\
 \tau_{xy} &= \frac{1}{2.d\lambda_2} (d\gamma_{xy})_p , \\
 \tau_{yz} &= \frac{1}{2.d\lambda_2} (d\gamma_{yz})_p , \text{ and} \\
 \tau_{zx} &= \frac{1}{2.d\lambda_2} (d\gamma_{zx})_p .
 \end{aligned} \right\} \quad (2.127)$$

Substituting the above equations into equation (2.10), in order to obtain $\bar{\sigma}$, then it can be deduced that:-

$$\bar{\sigma} = \frac{1}{\sqrt{2} . d\lambda_2} \left[(d\varepsilon_x - d\varepsilon_y)_p^2 + (d\varepsilon_y - d\varepsilon_z)_p^2 + (d\varepsilon_z - d\varepsilon_x)_p^2 + \frac{3}{2} (d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2)_p \right]^{\frac{1}{2}}$$

or

$$\bar{\sigma} = \frac{\sqrt{2} \bar{\sigma}_m}{d\lambda . 3} \left[(d\varepsilon_x - d\varepsilon_y)_p^2 + (d\varepsilon_y - d\varepsilon_z)_p^2 + (d\varepsilon_z - d\varepsilon_x)_p^2 + \frac{3}{2} (d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2)_p \right]^{\frac{1}{2}}$$

Finally

$$d\lambda = \frac{\sqrt{2}}{3} \cdot \left(\frac{\sigma_m}{\bar{\sigma}} \right) \left[(d\epsilon_x - d\epsilon_y)_P^2 + (d\epsilon_y - d\epsilon_z)_P^2 + (d\epsilon_z - d\epsilon_x)_P^2 \right. \\ \left. + \frac{3}{2} (d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2)_P \right]^{\frac{1}{2}} \quad (2.128)$$

From equations (2.111), (2.114) and (2.128), it can be shown that:-

$$(d\epsilon_p)_c = \left(\sqrt{Y_c Y_t} - \frac{3 \sigma_m}{\bar{\sigma}_m} \Delta \right) \frac{\sqrt{3}}{2} \left(\frac{\sigma_m}{\bar{\sigma}} \right) \frac{1}{Y_c} \\ \left[(d\epsilon_x - d\epsilon_y)_P^2 + (d\epsilon_y - d\epsilon_z)_P^2 + (d\epsilon_z - d\epsilon_x)_P^2 \right. \\ \left. + \frac{3}{2} (d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2) \right]^{\frac{1}{2}} \quad (2.129)$$

While, it can be deduced that:-

$$(\bar{d\epsilon}_p)_t = \left(\frac{Y_c}{Y_t} \right) (\bar{d\epsilon}_p)_c \quad (2.130)$$

If the $Y_c = Y_t = Y$, i.e. $\Delta = 0$, $H'_c = H'_t = H'$, $C_1 = 0$, and $\bar{\sigma}_m = \bar{\sigma}$, then, the following can be concluded:-

(i) From equation (2.109) the parameter B is reduced into:-

$$B = 3G \left(\frac{\sigma}{\sigma_m} \right)^2 + \frac{3E}{4(1-2\nu)} \left(\frac{\Delta}{\sigma_m} \right)^2 = 3G$$

which is similar to that obtained for the Von Mises yield criterion as explained in appendix 2.3.

(ii) From equation (2.119), the parameter A is reduced such that:-

$$A = (\text{Zero} - Y) \left[\text{Zero} - \frac{1}{2Y} \left(2 \frac{YH'}{Y} \right) \right]$$

$$= H'$$

similar to the A parameter obtained for the in Von Mises yield criterion.

(iii) Similarly equations (2.124) will be reduced to the Prandtl-Reuss form.

(ii) Kinematic Hardening Rule

The equation of the yield surface for the case involving the kinematic hardening rule is:-

$$F = \sqrt{(\sigma^K)^2 + 3 \Delta \sigma_m^K} - \sqrt{Y_c Y_t} = 0 \quad (2.131)$$

The gradient vector can be written as follows:-

$$\underline{a}^k = \frac{\partial F}{\partial \underline{\sigma}} = \frac{\partial F}{\partial J_2^K} \cdot \frac{\partial J_2^K}{\partial \underline{\sigma}} + \frac{\partial F}{\partial \sigma_m^K} \cdot \frac{\partial \sigma_m^K}{\partial \underline{\sigma}}$$

$$= C_2^K \underline{a}_2^K + C_1^K \underline{a}_1^K$$

which was the value for the case involving the isotropic hardening rule. It can also be shown that:-

$$\underline{a}_1^k = \frac{1}{3} \{ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \} = \underline{a}_1 = \frac{1}{3} \underline{I} \quad (2.132)$$

$$\text{Also } \underline{a}_2^k = \{ S_x^r \ S_y^r \ S_z^r \ 2\tau_{xy}^r \ 2\tau_{yz}^r \ 2\tau_{zx}^r \} \quad (2.133)$$

$$C_1^k = \frac{3}{2 \sigma_m^k}, \quad \text{and} \quad C_2^k = \frac{3 \Delta}{2 \sigma_m^k} \quad (2.134)$$

$$\text{where } \sigma_m^k = (\sigma_x^r + \sigma_y^r + \sigma_z^r) / 3$$

$$\sigma^k = \sqrt{3J_2^k},$$

$$J_2^k = \frac{1}{\sqrt{2}} \left\{ (\sigma_x^r - \sigma_y^r)^2 + (\sigma_y^r - \sigma_z^r)^2 + (\sigma_z^r - \sigma_x^r)^2 \right. \\ \left. + 6 \left[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2 \right] \right\}^{\frac{1}{2}}$$

and

$$\sigma_x^r = \sigma_x - \alpha_x, \quad \tau_{xz}^r = \tau_{xy} - \alpha_{xy}$$

$$\sigma_y^r = \sigma_y - \alpha_y, \quad \tau_{yz}^r = \tau_{yz} - \alpha_{yz}$$

$$\sigma_z^r = \sigma_z - \alpha_z, \quad \tau_{zx}^r = \tau_{zx} - \alpha_{zx}, \quad \text{and}$$

$$S_i^r = \sigma_i^r - \alpha_i$$

The plastic stress-strain matrix can be written as follows:-

$$\underline{D}_p = \frac{\underline{d}^k (\underline{d}^k)^t}{A + B},$$

where $\underline{d}^t = \underline{D} \underline{a} = D (C_2^K \underline{a}_2^K + C_1^K \underline{a}_1^K)$

which can be simplified, as shown previously, to:-

$$\underline{d}^t = \frac{1}{\underline{\sigma}_m^K} \left[3G \underline{s}^K + \frac{E}{2(1-2\nu)} \Delta \underline{I} \right] \quad (2.135)$$

where

$$\underline{\sigma}_m^K = \sqrt{\underline{\sigma}^K + 3 \Delta \underline{\sigma}_m^K}$$

In a manner similar to that used in previous analysis, it can be deduced that the parameter B for the case which includes the kinematic hardening rule is:

$$B = 3 G \left(\frac{\underline{\sigma}^K}{\underline{\sigma}_m^K} \right)^2 + \frac{3 E}{4(1-2\nu)} \left(\frac{\Delta}{\underline{\sigma}_m^K} \right)^2 \quad (2.136)$$

While the parameter A for the same kinematic hardening rule is:-

$$A = C (\underline{a}^K)^t \underline{a}^K$$

$$\begin{aligned} \text{where } (\underline{a}^K)^t \cdot \underline{a}^K &= (C_2^K \underline{a}_2^K + C_1^K \underline{a}_1^K)^t (C_2^K \underline{a}_2^K + C_1^K \underline{a}_1^K) \\ (\underline{a}^K)^t \cdot \underline{a}^K &= (C_2^K)^2 (\underline{a}_2^K)^t \underline{a}_2^K + (C_1^K)^2 (\underline{a}_1^K)^t \underline{a}_1^K \\ &\quad + (C_1^K + C_2^K) [(\underline{a}_2^K)^t \underline{a}_1^K + (\underline{a}_1^K)^t \underline{a}_2^K] \end{aligned}$$

But

$$(\underline{a}_2^K)^t \underline{a}_2^K = \begin{bmatrix} S_x^r & S_y^r & S_z^r & 2 \tau_{xy}^r & 2 \tau_{yz}^r & 2 \tau_{zx}^r \end{bmatrix} \begin{bmatrix} S_x^r \\ S_y^r \\ S_z^r \\ 2 \tau_{xy}^r \\ 2 \tau_{yz}^r \\ 2 \tau_{zx}^r \end{bmatrix}$$

$$(\underline{a}_2^k)^t \underline{a}_2^k = [(S_x^r)^2 + (S_y^r)^2 + (S_z^r)^2 + 4[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2]]$$

$$= \frac{2}{3} (\bar{\sigma}^k)^2 + 2[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2]$$

while

$$(\underline{a}_2^k)^t \underline{a}_1^k = \begin{bmatrix} S_x^r & S_y^r & S_z^r & 2 \tau_{xy}^r & 2 \tau_{yz}^r & 2 \tau_{zx}^r \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Similarly ,

$$(\underline{a}_1^k)^t \cdot (\underline{a}_2^k) = 0$$

and

$$(\underline{a}_1^k)^t \underline{a}_1^k = \left(\frac{1}{3}\right)^2 \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}$$

Then

$$\begin{aligned} (\underline{a}^k)^t \underline{a}^k &= \left(\frac{3}{2 \bar{\sigma}_m^k}\right)^2 \left[\frac{2}{3} (\bar{\sigma}^k)^2 + 2[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2] \right] \\ &+ 3 \left(\frac{\Delta}{2 \bar{\sigma}_m^k} \right)^2 \end{aligned}$$

which can be simplified as follows:-

$$\begin{aligned} (\underline{a}^k)^t \underline{a}^k &= \frac{3}{2} \left[\left(\frac{\bar{\sigma}^k}{\bar{\sigma}_m^k} \right)^2 + \frac{3[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2]}{(\bar{\sigma}_m^k)^2} \right. \\ &\left. + \frac{1}{2} \frac{\Delta^2}{(\bar{\sigma}^k)^4} \right] \end{aligned}$$

Therefore it can be shown that

$$A = \frac{3C}{2} \left[\left(\frac{\bar{\sigma}^K}{\bar{\sigma}_m^K} \right)^2 + \frac{3[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2]}{(\bar{\sigma}_m^K)^2} + \frac{1}{2} \frac{\Delta^2}{(\bar{\sigma}_m^K)^4} \right] \quad (2.136)$$

If $Y_c = Y_t = Y$, i.e. $\Delta = 0$. and $\bar{\sigma}^K = \bar{\sigma}_m^K$, then the above A parameter will be reduced to:-

$$A = \frac{3C}{2} \left[1 + \frac{3[(\tau_{xy}^r)^2 + (\tau_{yz}^r)^2 + (\tau_{zx}^r)^2]}{(\bar{\sigma}^K)^2} \right]$$

which is similar to the parameter A produced for the case involving the Von Mises's yield criterion. The parameter C can be obtained in a similar way to that which has been explained in section 2.6.2.

The flow rule equation can be written as follows:-

$$d\varepsilon_p = d\lambda \cdot \underline{a}^K = d\lambda (C_1^K \underline{a}_1 + C_2^K \underline{a}_2)$$

$$d\varepsilon_p = d\lambda_2^K \underline{\bar{a}}_2^K + d\lambda_1^K \underline{a}_1$$

$$\text{or } d\varepsilon_p = d\lambda_2^K [\underline{\bar{a}}_2^K + \Delta \underline{a}_1] \quad (2.137)$$

$$\text{where } d\lambda_2^K = \frac{3}{2\bar{\sigma}_m^K} \cdot d\lambda, \quad d\lambda_1^K = \frac{3 \Delta}{2\bar{\sigma}_m^K} d\lambda, \quad \text{and}$$

$$d\lambda = \frac{1}{C} \frac{(\underline{\bar{a}}^K)^t d\sigma}{(\underline{\bar{a}}^K)^t (\underline{\bar{a}}^K)}$$

The parameter $d\lambda$ can be rewritten in another form, as explained in the previous section, as follows:-

$$d\lambda = \frac{\sqrt{2}}{3} \left(\frac{\sigma_m^k}{\sigma^k} \right) \left[(d\epsilon_x - d\epsilon_y)_p^2 + (d\epsilon_y - d\epsilon_z)_p^2 + (d\epsilon_z - d\epsilon_x)_p^2 + \frac{3}{2} (d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2)_p \right]^{\frac{1}{2}} \quad (2.138)$$

The parameter $d\mu$ can be written from equation (2.82), as follows:-

$$d\mu = \frac{(\underline{a}^k)^t d\underline{\sigma}}{(\underline{a}^k)^t \underline{\sigma}_r}$$

where $\underline{\sigma}_r = \underline{\sigma} - \underline{\alpha}$ reduced stress vector.

From equation (2.111), it can be shown that:-

$$(\underline{a}^k)^t \underline{\sigma}_r = \frac{Y_c (\overline{d\epsilon}_p)_c}{d\lambda} = \frac{Y_t (\overline{d\epsilon}_p)_t}{d\lambda}$$

Substituting the above equation into the expression for $d\mu$, it can be shown that:-

$$(d\mu)_c = \frac{\underline{a}^t d\underline{\sigma}}{Y_c (\overline{d\epsilon}_p)_c} d\lambda \quad \dots \text{ in the compressive stress region.}$$

$$(d\mu)_t = \frac{\underline{a}^t d\underline{\sigma}}{Y_t (\overline{d\epsilon}_p)_t} d\lambda \quad \dots \text{ in the tensile stress region.}$$

But, from equation (2.78), it can be deduced that:-

$$(\underline{a}^k)^t \underline{d\sigma}_r = C(\underline{a}^k)^t \underline{d\varepsilon}_p$$

Then

$$(d\mu)_t = d\lambda \frac{C (\underline{a}^k)^t \underline{d\varepsilon}_p}{Y_t (\overline{d\varepsilon}_p)_t}$$

Using the flow rule, it can be shown that:-

$$(d\mu)_t = (d\lambda)^2 \frac{C (\underline{a}^k)^t \underline{a}^k}{Y_t (\overline{d\varepsilon}_p)_t} \quad (2.139)$$

Similarly,

$$(d\mu)_c = (d\lambda)^2 \frac{C (\underline{a}^k)^t \underline{a}^k}{Y_c (\overline{d\varepsilon}_p)_c} \quad (2.140)$$

If $Y_c = Y_t = Y$, i.e. $\Delta = 0$, and $(\overline{d\varepsilon}_p)_c = (\overline{d\varepsilon}_p)_t$, then the above equations are reduced to:-

$$d\mu = (d\lambda)^2 \frac{C (\underline{a}^k)^t \underline{a}^k}{Y \overline{d\varepsilon}_p}$$

also, it can be proved that

$$d\lambda = \overline{d\varepsilon}_p.$$

Then

$$d\mu = d\lambda \frac{C (\underline{a}^k)^t \underline{a}^k}{Y},$$

which is similar to the value obtained for the analysis involving the Von Mises yield criterion.

(iii) Mixed Hardening Rule

The mixed hardening, as explained in section (2.6.3), is a combination of isotropic and kinematic hardening rules according to the following equation:-

$$(\text{Parameter})_{\text{mixed}} = (1 - C_M) (\text{Parameter})_{\text{Kinematic}} + C_M (\text{Parameter})_{\text{Isotropic}}$$

$$\text{i.e. } A^M = (1 - C_M) A^K + C_M A^i, \text{ and}$$

$$B^M = (1 - C_M) B^K + C_M B^i,$$

$$\text{while } d\lambda = C_M (d\lambda)^i, \text{ and}$$

$$d\mu = (1 - C_M) (d\mu)^K.$$

2.7.2 Analysis of the Elasto-plastic Problem Utilising Other Yield Criteria

Analysis of the elasto-plastic stress problem by using Tresca, Tresca polynomial, Drucker-Prager, or Mohr-Coulomb criteria is similar to the analysis arising from the use of the Von Mises criterion.

While the analysis utilising the Modified Tresca polynomial criterion is similar to that arising from the use of the Modified Von Mises yield criterion. The gradient vector of the yield function for those yield criteria can be written as follows:-

$$\underline{a} = C_1 \underline{a}_1 + C_2 \underline{a}_2 + C_3 \underline{a}_3$$

$$\text{where } \underline{a}_1 = \frac{1}{3} \{1 \ 1 \ 1 \ 0 \ 0 \ 0\},$$

$$\underline{a}_2 = \frac{\partial J_2}{\partial \underline{\sigma}} = \{S_x \ S_y \ S_z \ 2 \ \tau_{xy} \ 2 \ \tau_{yz} \ 2 \ \tau_{zx}\}, \text{ and}$$

$$a_3 = \frac{\partial J_3}{\partial \sigma} = \begin{bmatrix} S_y S_z - \tau_{yz}^2 + \left(\frac{\sigma}{3}\right)^2 \\ S_z S_x - \tau_{zx}^2 + \left(\frac{\sigma}{3}\right)^2 \\ S_x S_y - \tau_{xy}^2 + \left(\frac{\sigma}{3}\right)^2 \\ 2 (\tau_{yz} \tau_{zx} - S_z \tau_{xy}) \\ 2 (\tau_{zx} \tau_{xy} - S_x \tau_{yz}) \\ 2 (\tau_{xy} \tau_{yz} - S_y \tau_{zx}) \end{bmatrix}$$

as proved in appendix 2.6.

The coefficients of the above vector for the different yield criteria are shown in Table (2.1), and derived in appendix 2.7.

In order to simplify the vector $\underline{a}_3$, the following equation can be employed:-

$$\underline{a}_3 = \underline{M}_3 \cdot \underline{\sigma}$$

Where

$$\underline{M}_3 = \frac{1}{3} \begin{bmatrix} S_x & S_z & S_y & \tau_{xy} & -2 \tau_{yz} & \tau_{zx} \\ S_z & S_y & S_x & \tau_{xy} & \tau_{yz} & -2 \tau_{zx} \\ S_y & S_x & S_z & -2 \tau_{xy} & \tau_{yz} & \tau_{zx} \\ 0 & 0 & 0 & -6 S_z & 6 \tau_{zx} & 0 \\ 0 & 0 & 0 & 0 & -6 S_x & 6 \tau_{xy} \\ 0 & 0 & 0 & 6 \tau_{yz} & 0 & -6 S_y \end{bmatrix}$$

as proved in appendix 2.8.

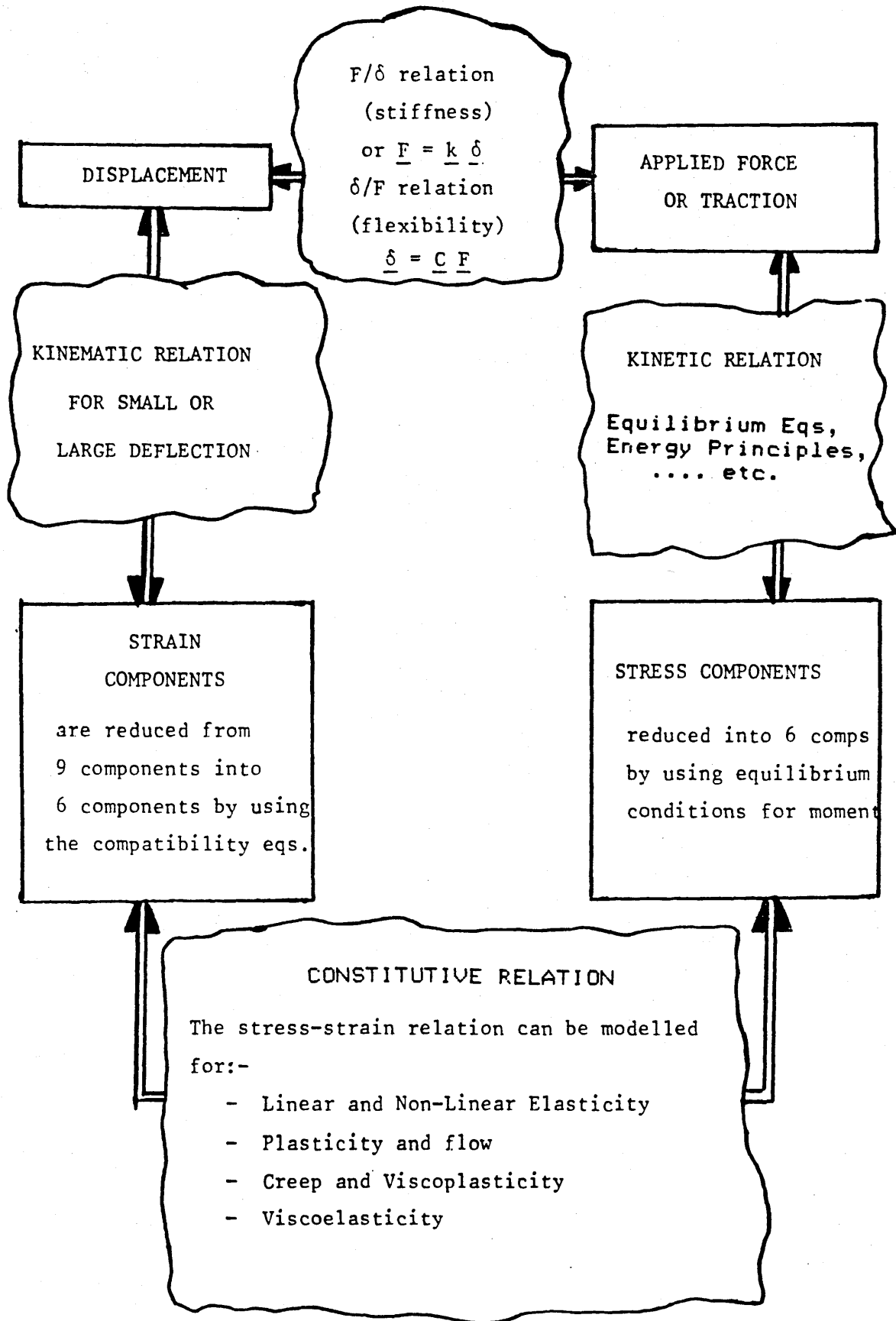
Hence, it can be deduced that:-

$$\underline{a} = C_1 \underline{a}_1 + (C_2 \underline{M}_2 + C_3 \underline{M}_3) \underline{\sigma} \quad (2.141)$$

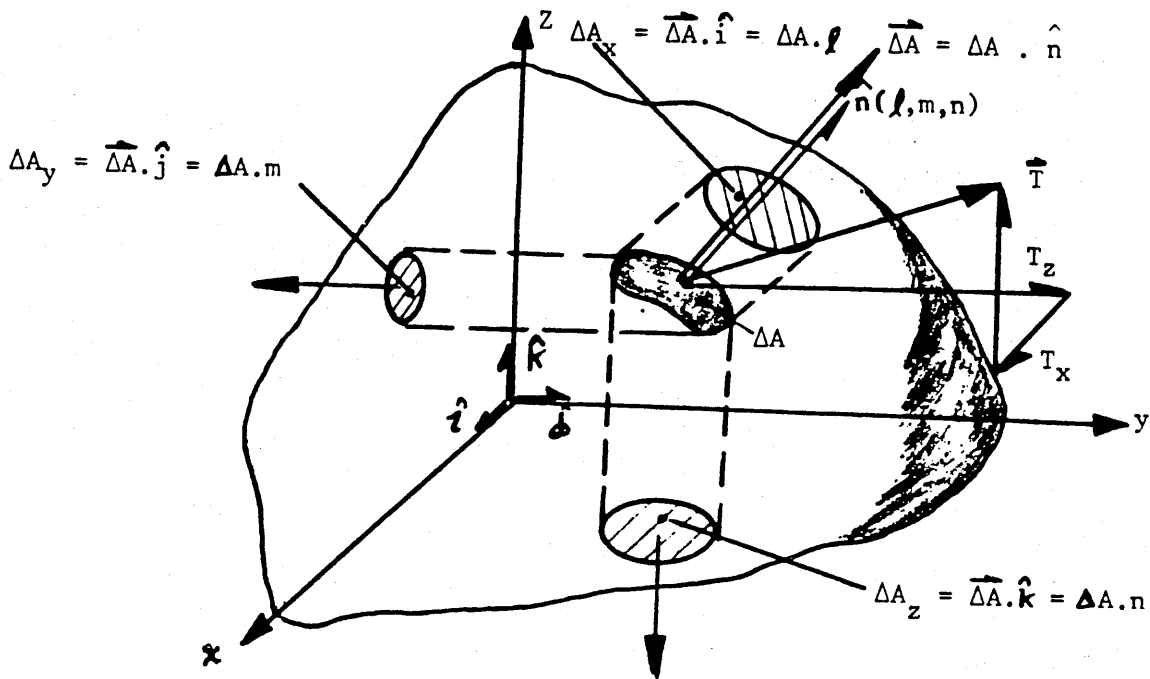
A singularity in the evaluation of the gradient vector (a) for yield criteria with non-smooth functions, such as the Tresca, Tresca polynomial, Modified Tresca, and Mohr-Coulomb yield criteria, can occur when the angle " ϕ " approaches $\pm \frac{\pi}{6}$, i.e. at the corner points of the Tresca polygon, or the Mohr-Coulomb Hexagonal Pyramid, where the direction of the normal vector is indeterminate.

If the stress states satisfy one of the singularity conditions then the vector a should be evaluated on the sides nearest to the corner and an average value assumed, as shown in Fig(2.27.a), and as suggested in [60].

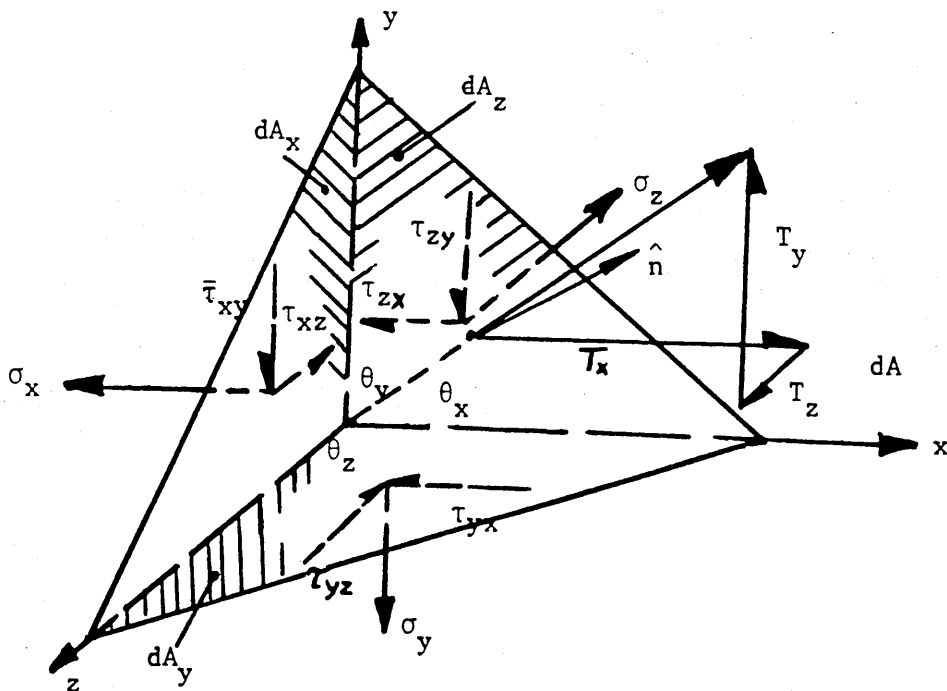
Another approach for avoiding the singularity in the vector a is by using the Von Mises Criteria instead of the Tresca, or Tresca polynomial yield criteria, since there is coincidence of the Von Mises and the Tresca at $\phi = \pm \frac{\pi}{6}$, as shown in Fig(2.27(b)). An alternative approach which is also applicable to Mohr-Coulomb criterion is given by Reference 119.



Fig(2.1) The Basic Relations between the Main Field Variables in a Continuum

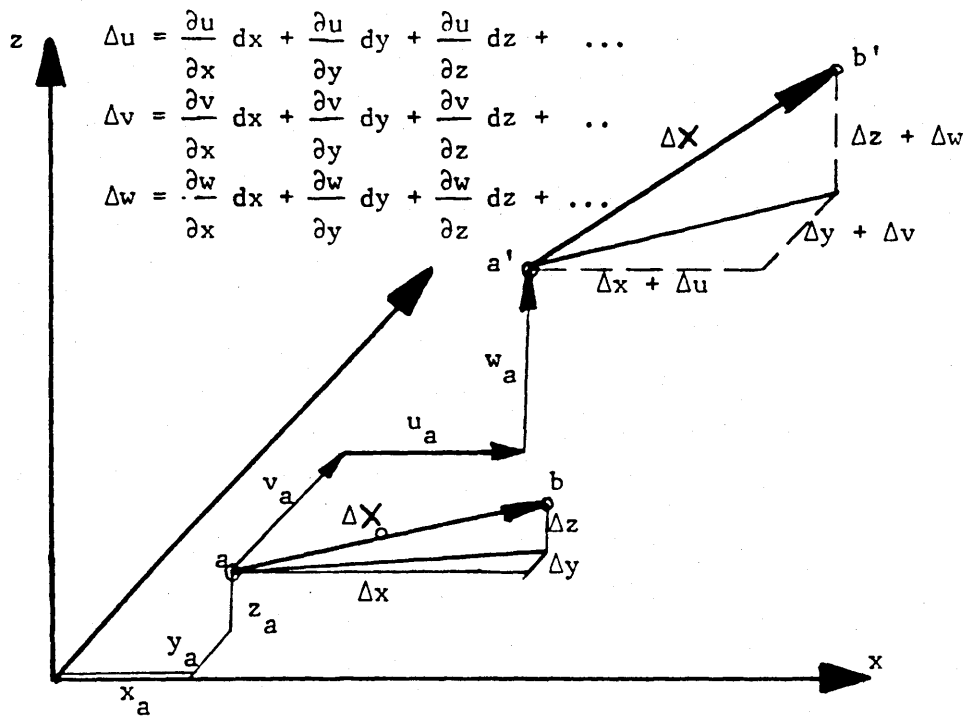


a) The Traction and Area Vectors and their Components

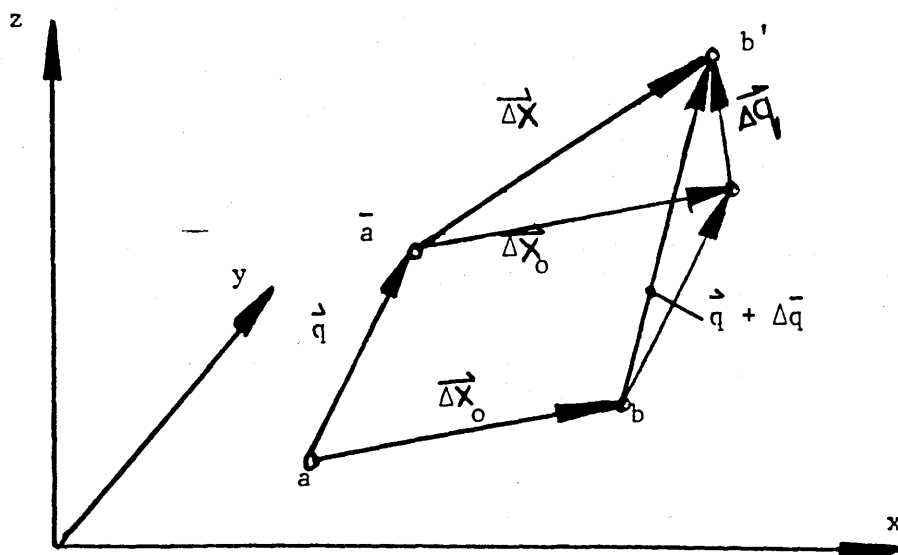


b) Stress Components and Surface Tractions on an Infinitesimal Element of Surface

Fig(2.2) Traction Components

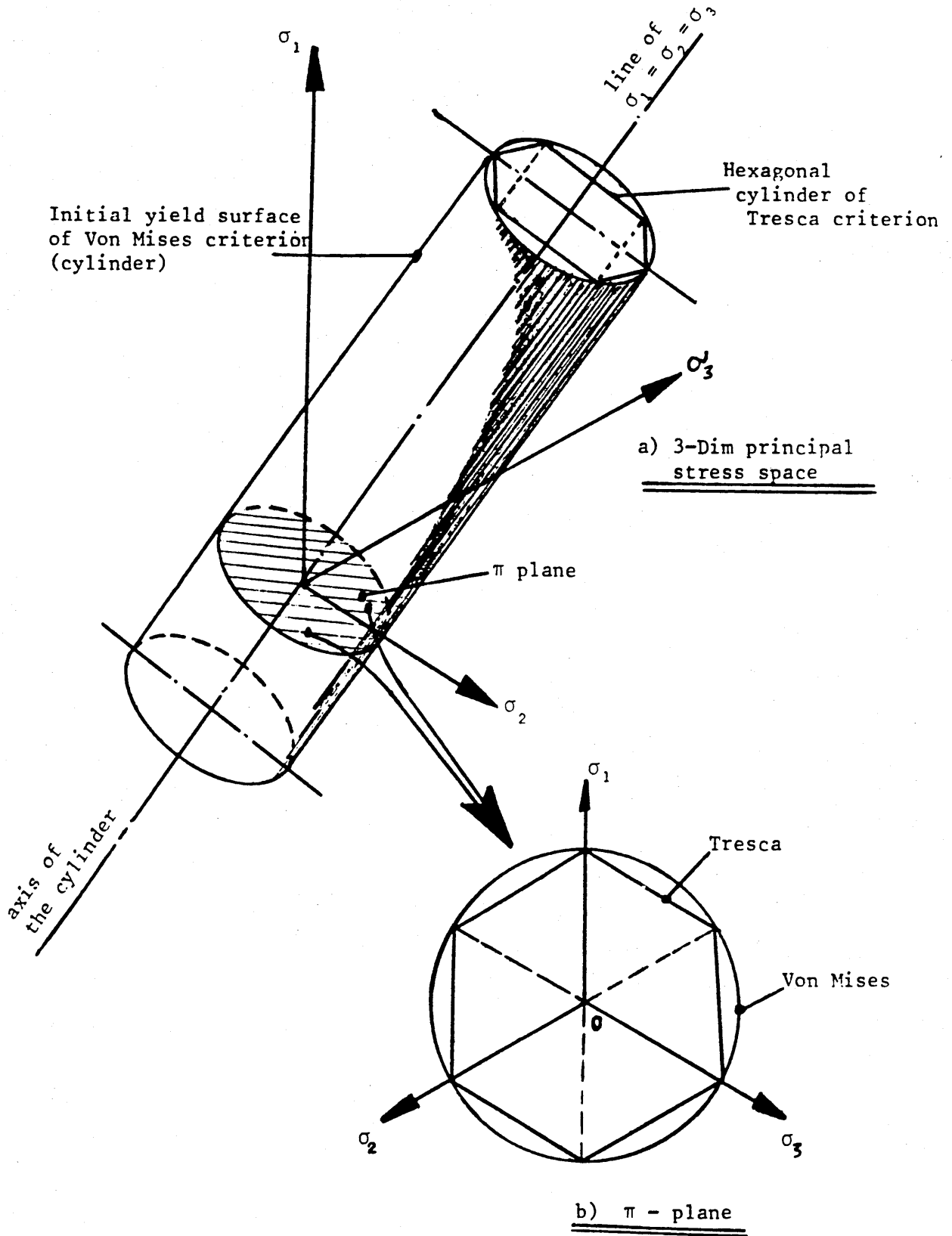


a) Lagrangian Description

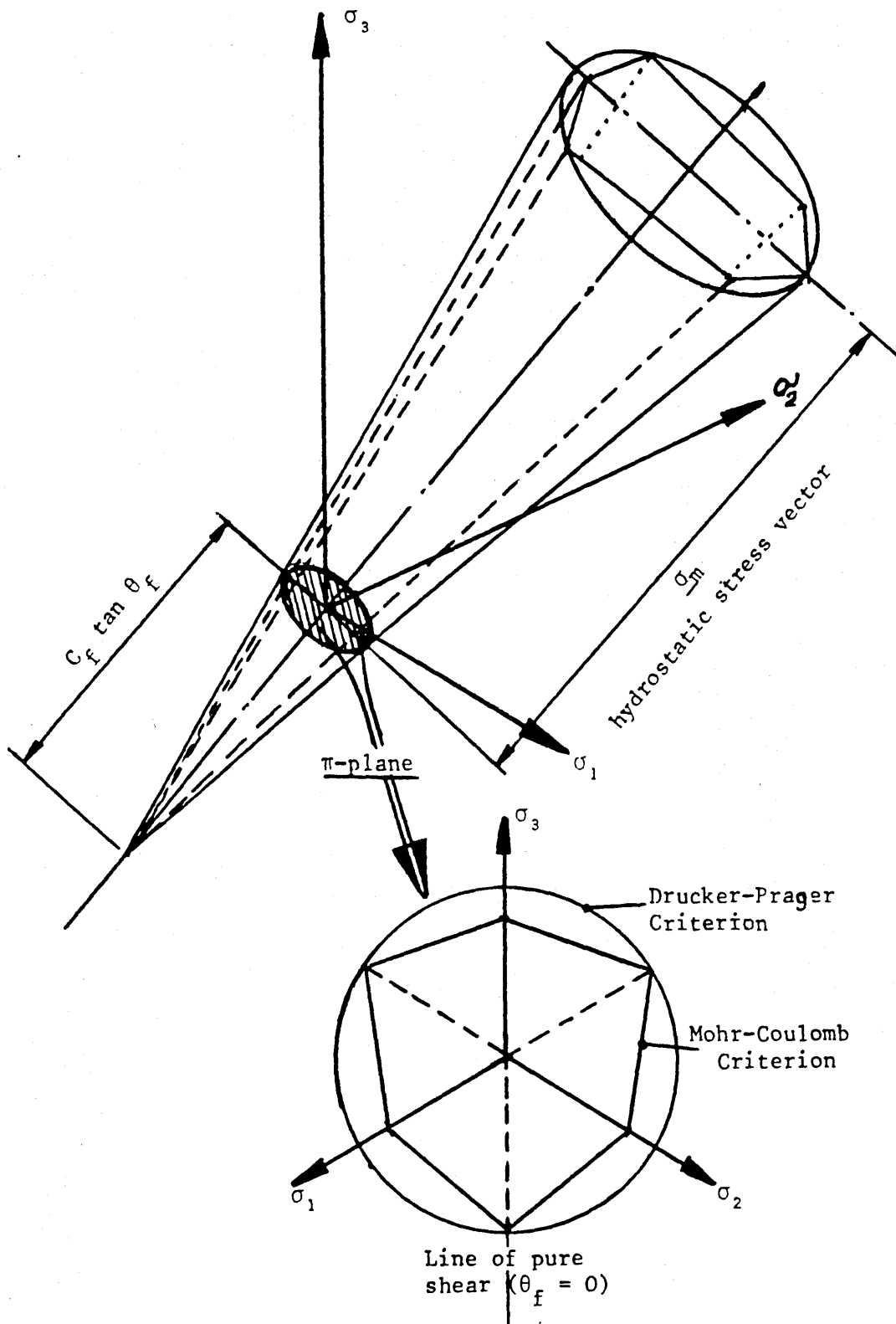


b) Vectorial Representation

Fig(2.3) Coordinates for an Infinitesimal Line in Deformed and Undeformed States in a Continuum



Fig(2.4) Geometrical Representation of Von Mises and Tresca Yield Criteria

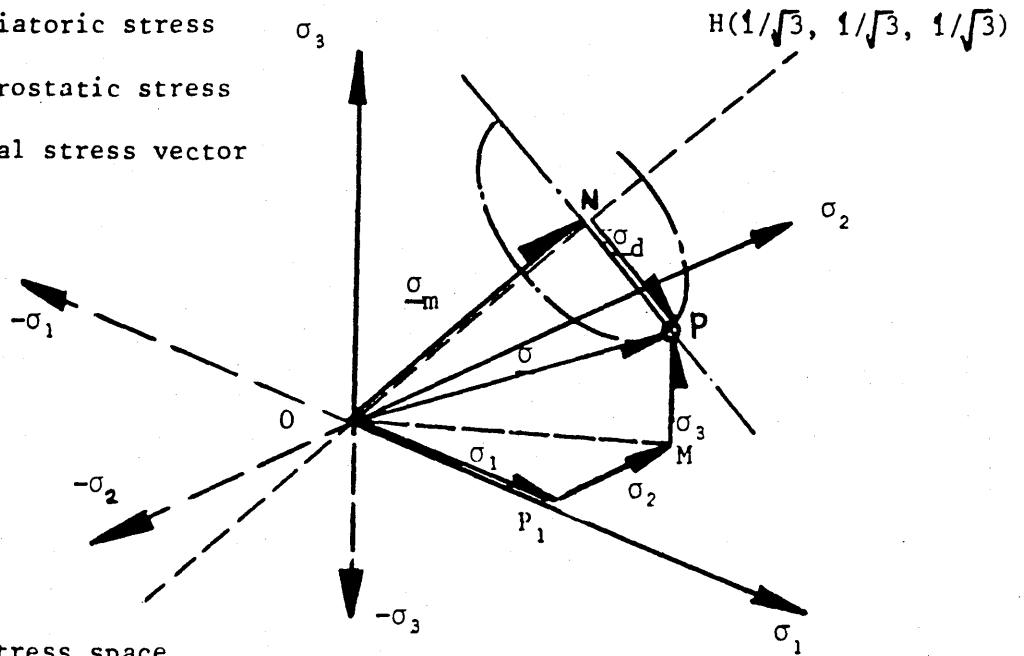


Fig(2.7) Representation of Mohr-Coulomb and Drucker-Prager Criteria in 3-Dim. Principal Stress Space and at the π -Plane

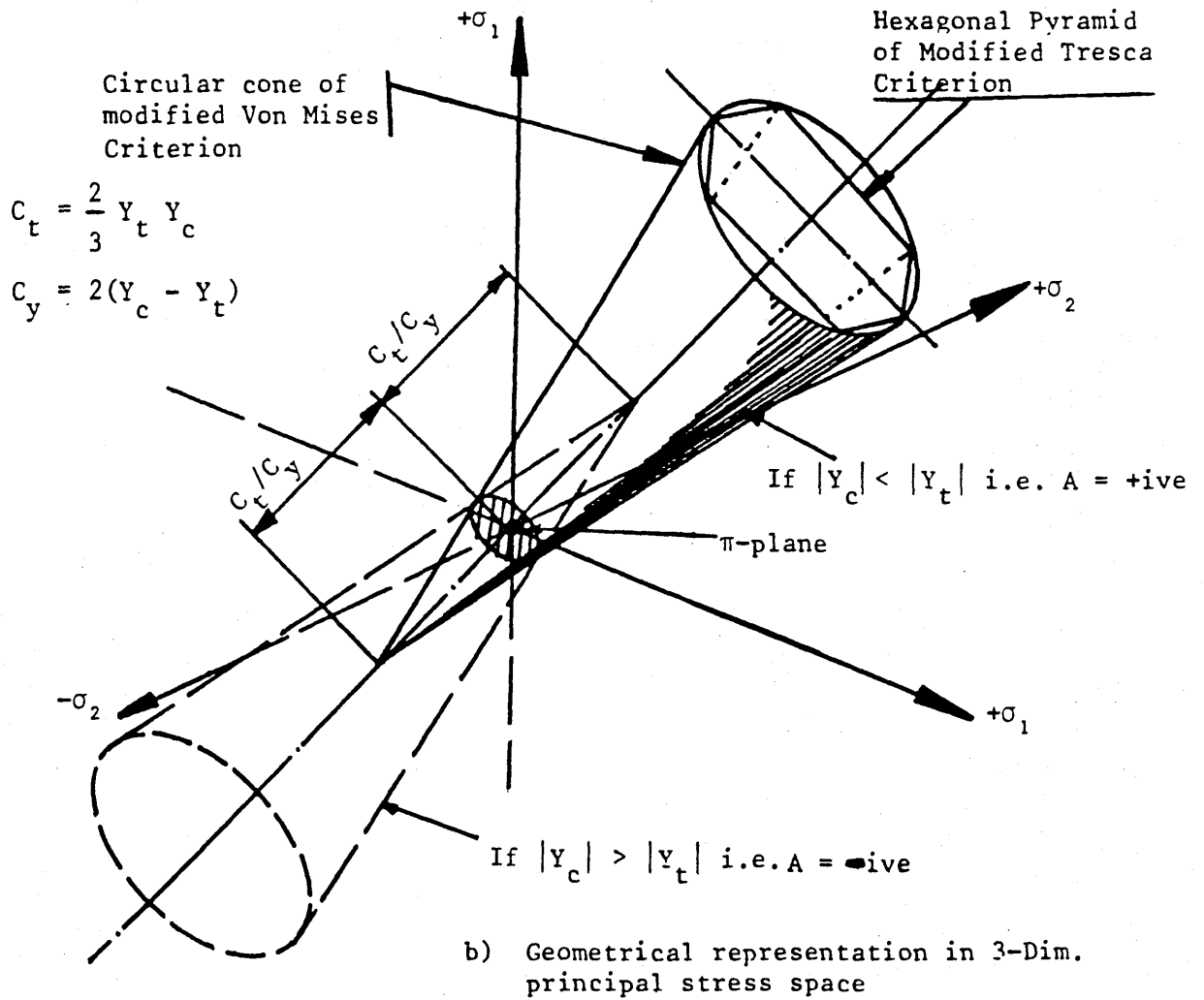
```

σd .. the deviatoric stress
      vector
σm .. the hydrostatic stress
      vector
σ    .. the total stress vector

```

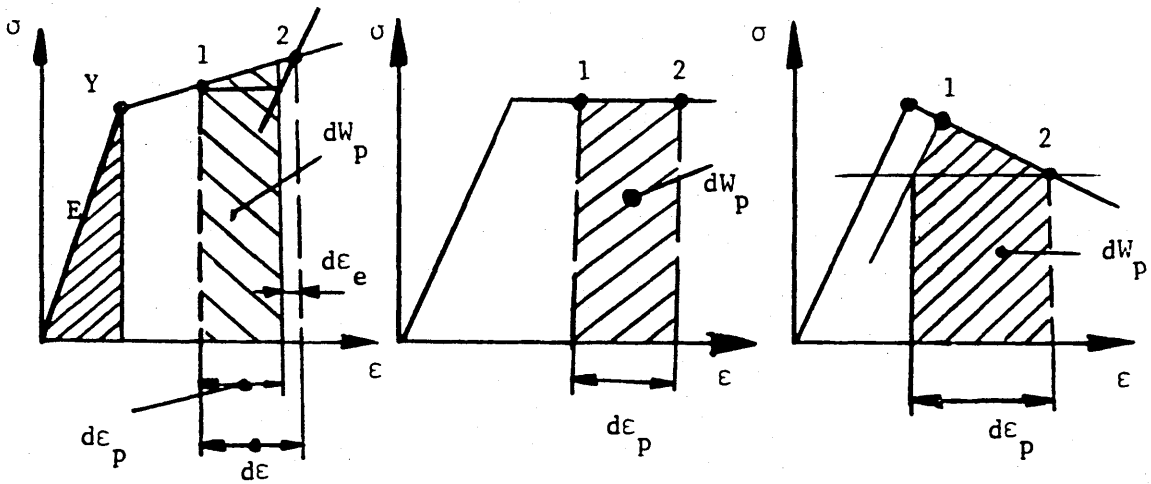


a) Principal stress space



b) Geometrical representation in 3-Dim.
principal stress space

Fig(2.8) Geometrical Representation of Pressure Modified Von Mises Criterion



$d\epsilon_p$ and dW_p increase
therefore $H' > 0$

$d\epsilon_p$ and dW_p both
increase but H' should
be equal to zero?!
(i.e. dW_p should
be = 0)

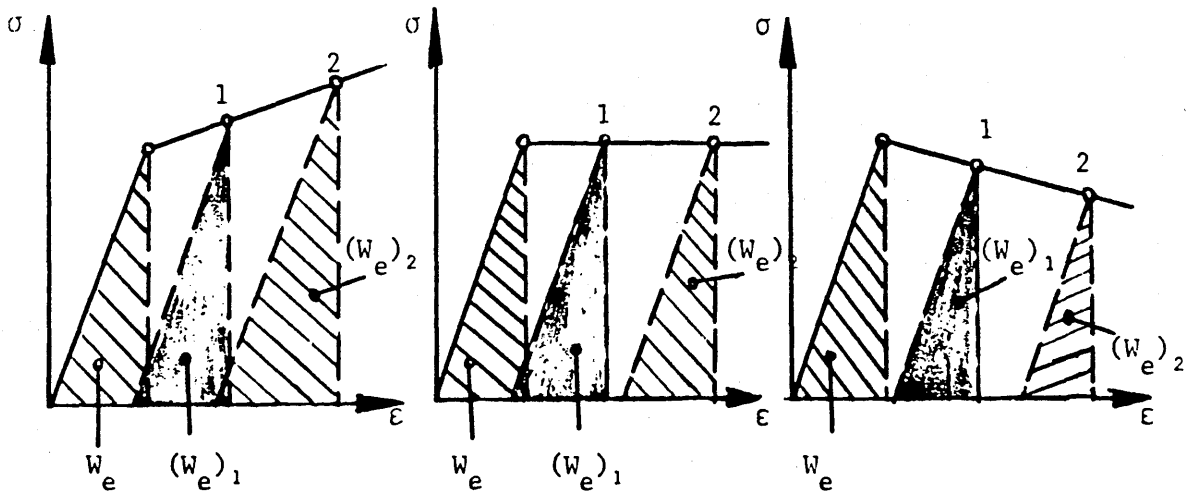
$d\epsilon_p$ and dW_p both increase
but H' should be negative?!
(i.e. dW_p should be
 $< \text{zero}$)

a) Strain-hardening
material

b) Perfect plastic
material

c) Strain-softening
material

Fig(2.9) The Work-Hardening Hypothesis



$dW_e > 0$
therefore $H' > 0$

$dW_e = (W_e)_2 - (W_e)_1 = 0$
therefore $H' = 0$

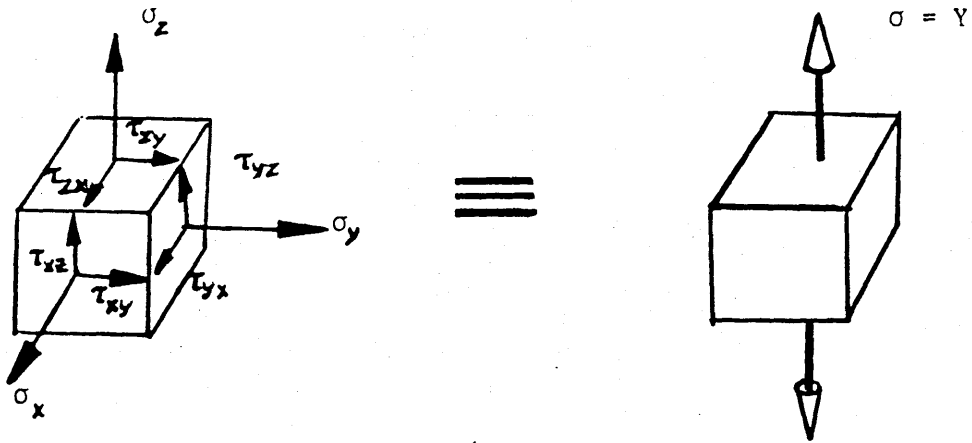
$dW_e = W_{e2} - W_{e1} < 0$
therefore $H' < 0$

a) Strain-hardening
material

b) Perfectly plastic
material

c) Strain-softening
material

Fig(2.10) The Elastic-Work Hypothesis



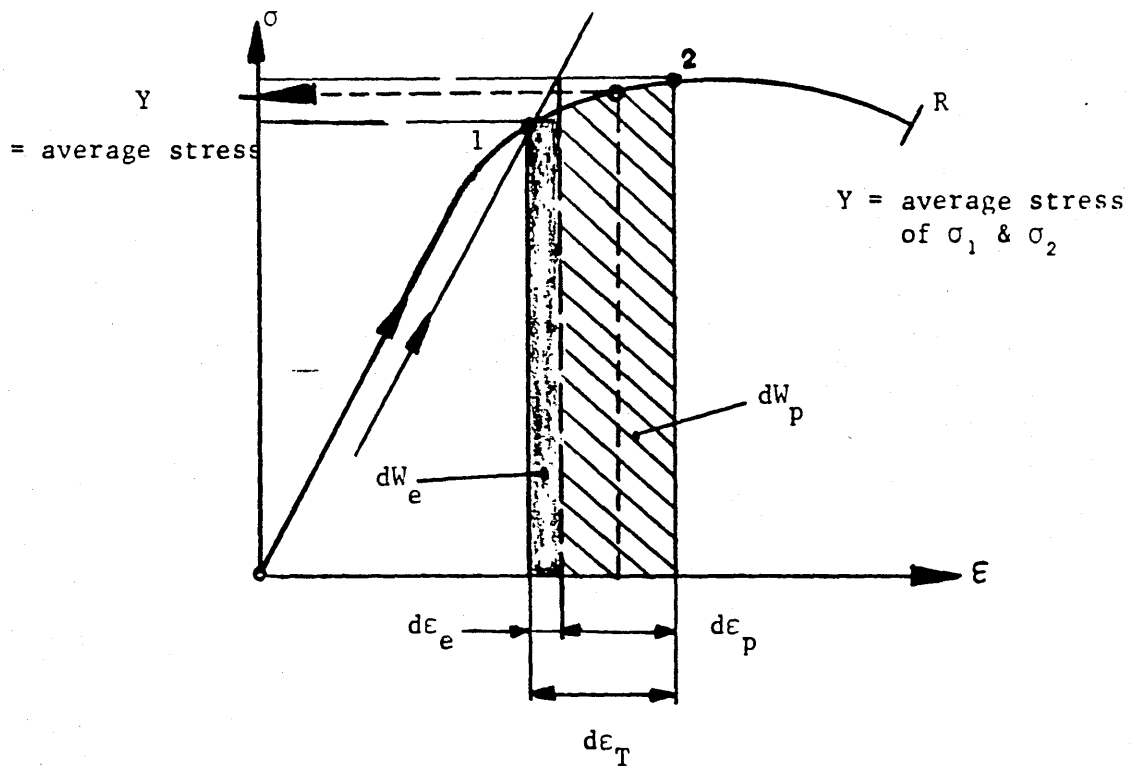
multi-axial stress state

$$dW_p = \underline{\sigma}^T \cdot d\underline{\epsilon}_p$$

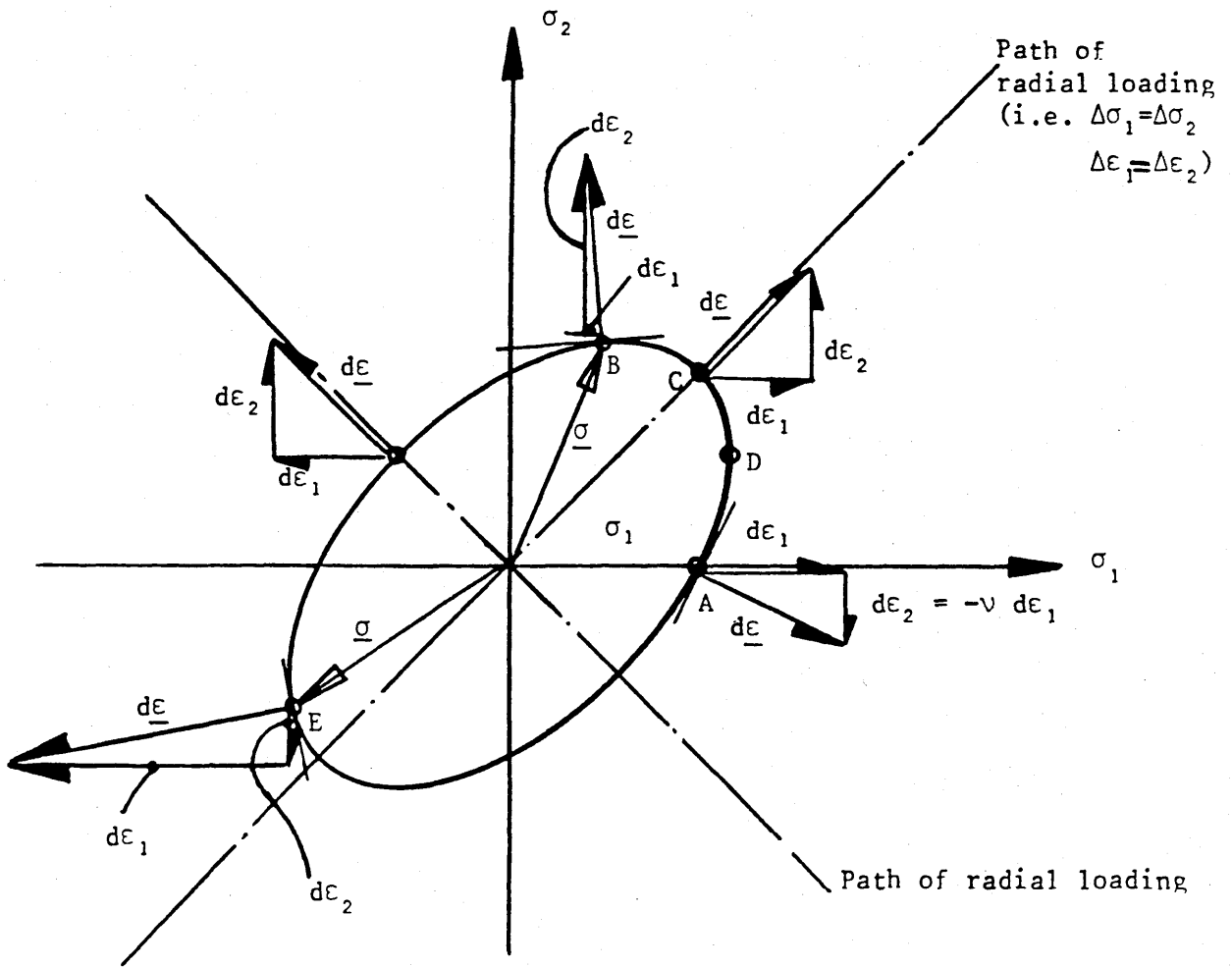
$$= \sigma_x (d\epsilon_x)_p + \sigma_y (d\epsilon_y)_p + \sigma_z (d\epsilon_z)_p + \tau_{xy} (d\gamma_{xy})_p + \tau_{yz} (d\gamma_{yz})_p + \tau_{xz} (d\gamma_{xz})_p$$

$$(dW_p)_u = \sigma \cdot d\epsilon$$

uniaxial stress state



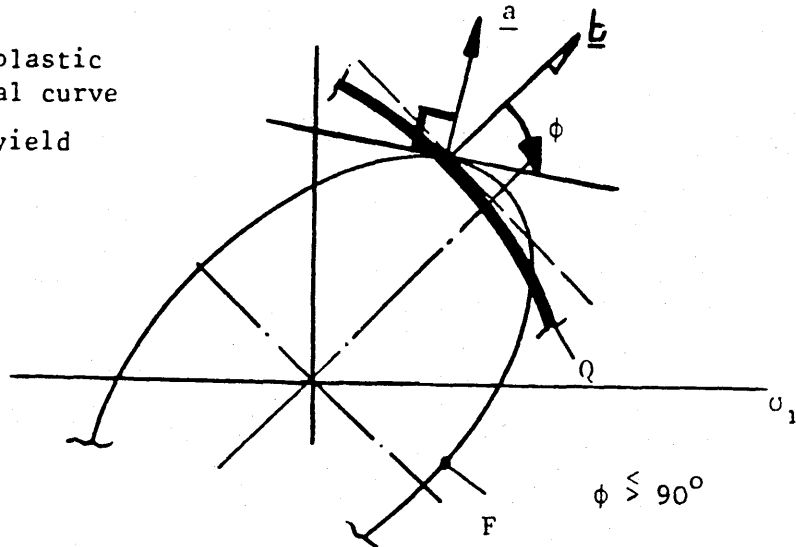
Fig(2.11) Correlation Between Uniaxial and Multi-axial Stress States



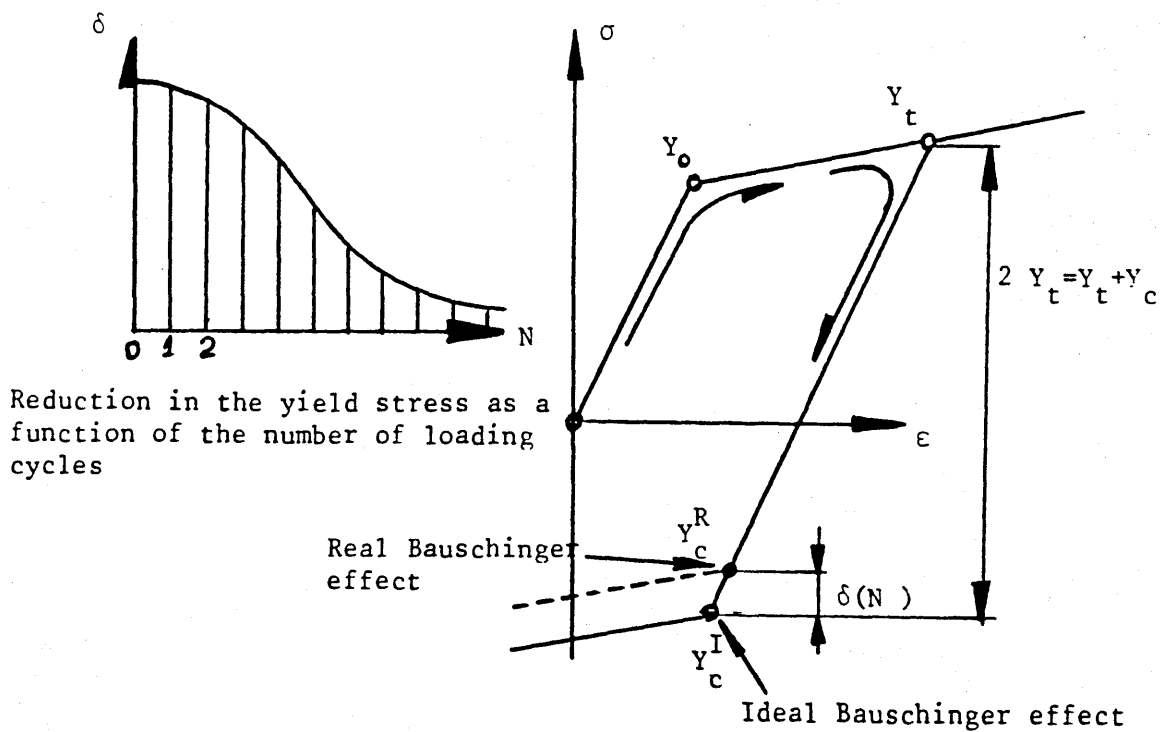
- A ... Represents uniaxial test condition (tension)
- B ... Represents biaxial condition (tension and compression)
- C ... Represents radial condition (load)
- D ... Represents biaxial condition (tension only)
- E ... Represents biaxial condition (compression only)

Fig(2.12) Normality Principle

Q ... is the plastic potential curve
 F ... is the yield surface



Fig(2.13) Flow Rules for Associated and Non-Associated Materials

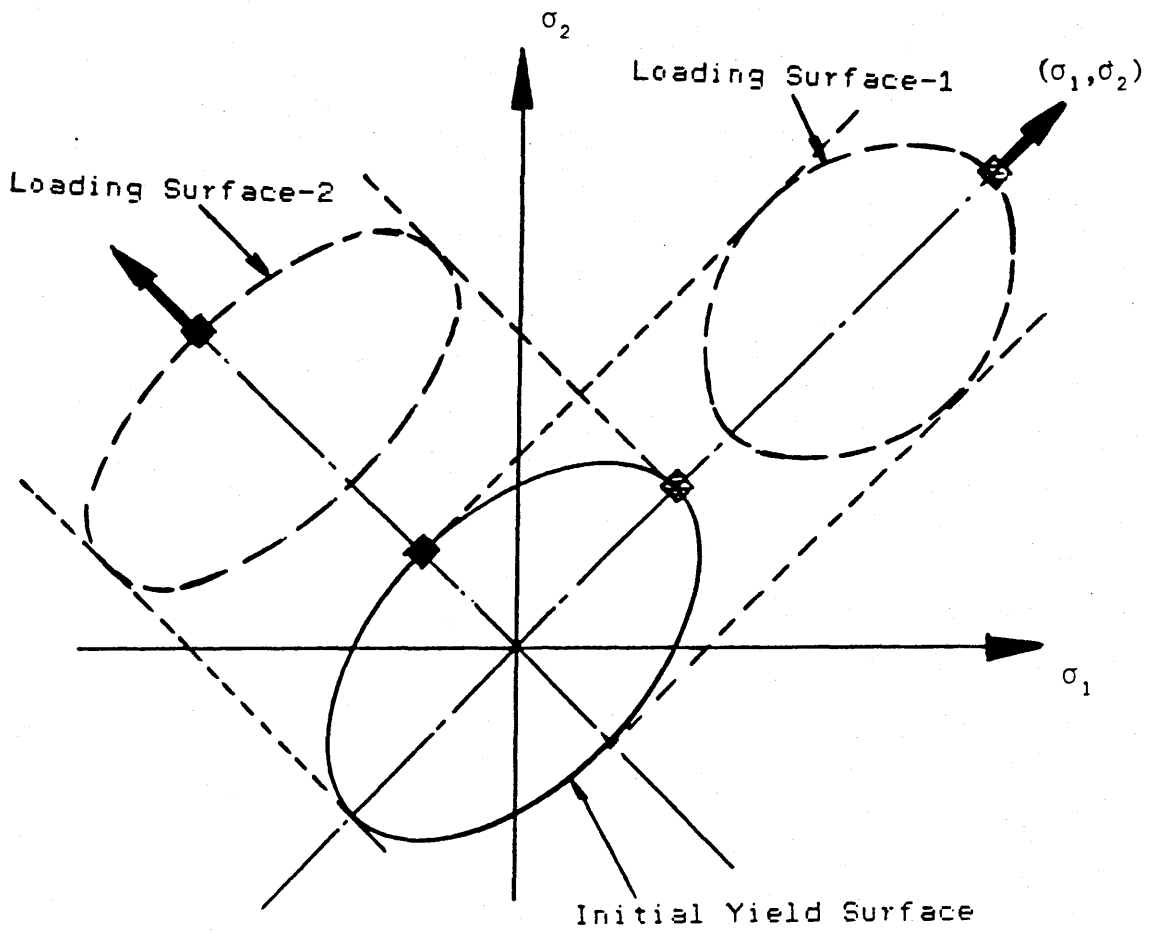


$$Y_t + Y_c^I = 2Y_0 \quad \text{for the ideal Bauschinger effect}$$

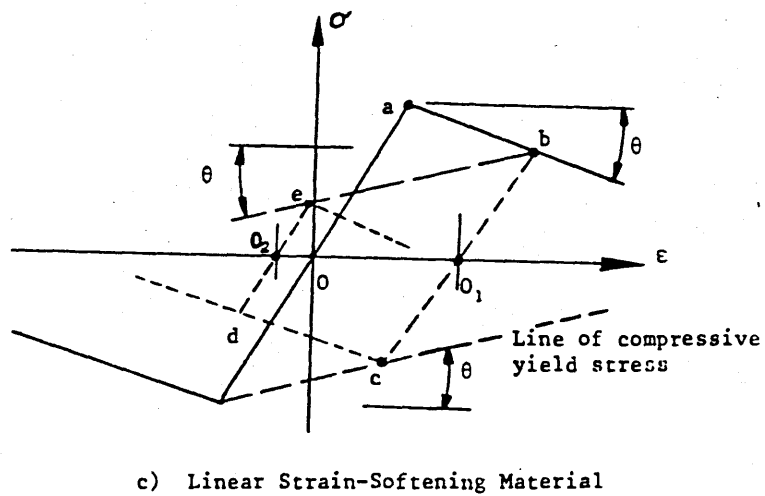
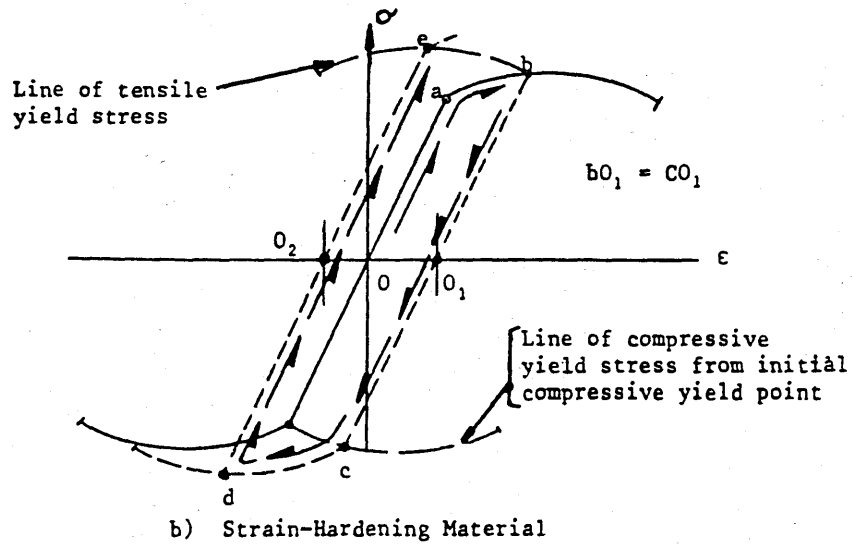
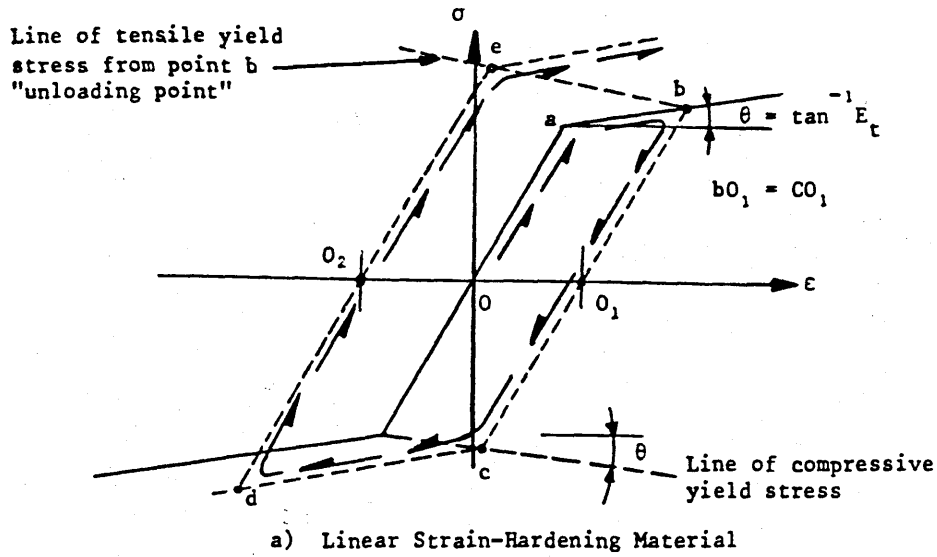
$$Y_t + Y_c^R < 2Y_0 \quad \text{for the real Bauschinger effect}$$

$$Y_t + Y_c^R = 2Y_0 - \delta$$

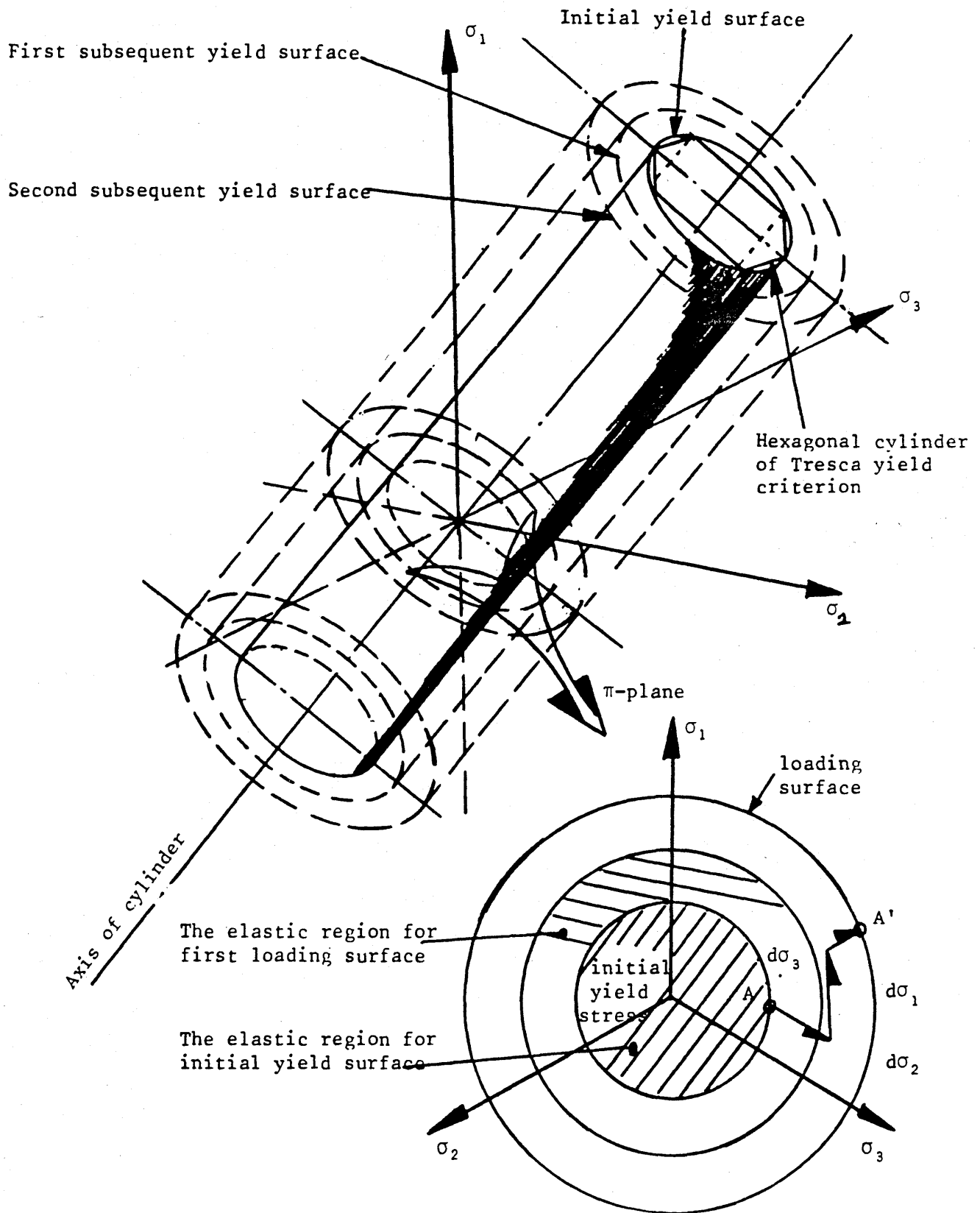
Fig(2.14) Bauschinger Effect



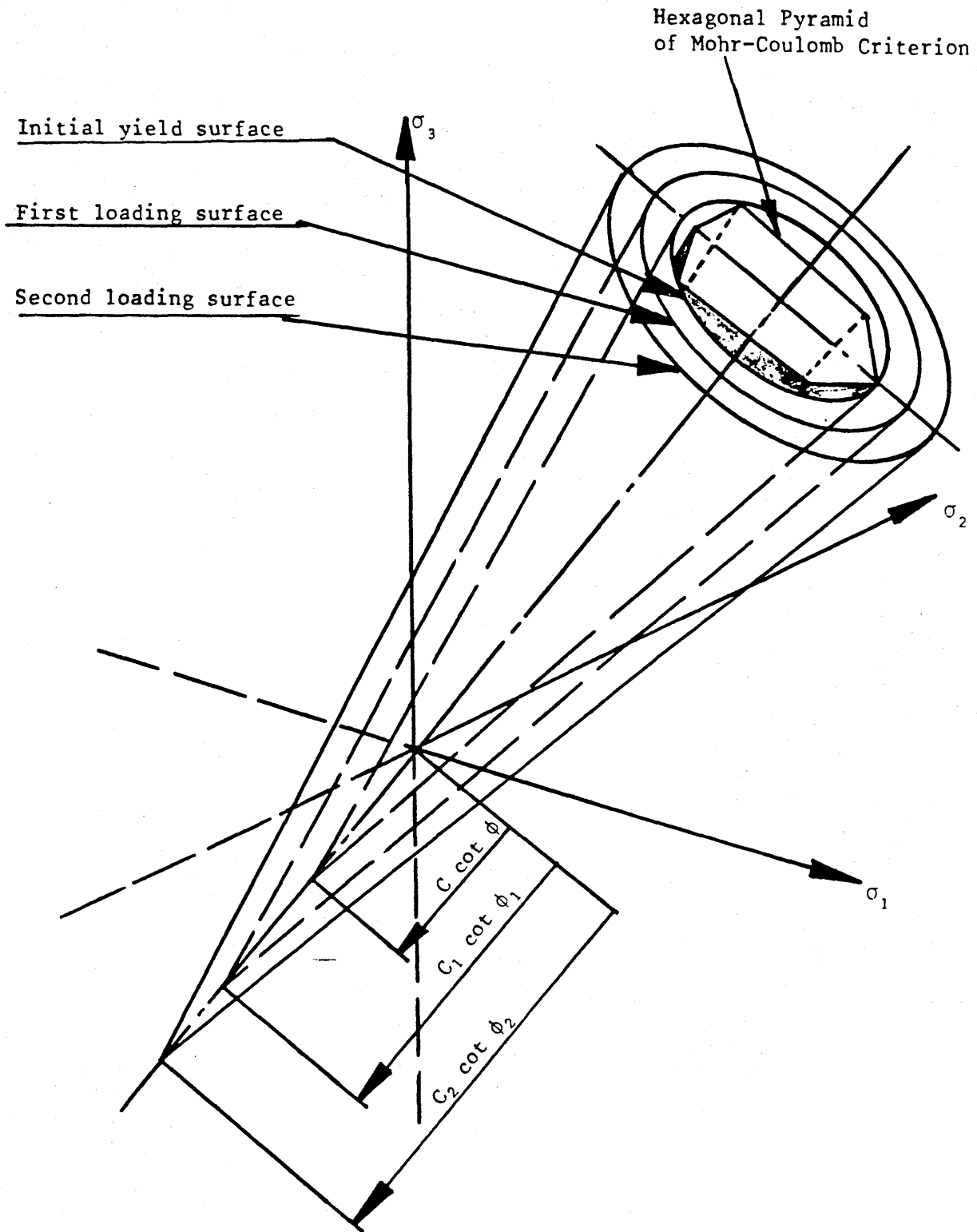
Fig(2.15) Distortion Hardening Rule
(distortion of the yield surface)



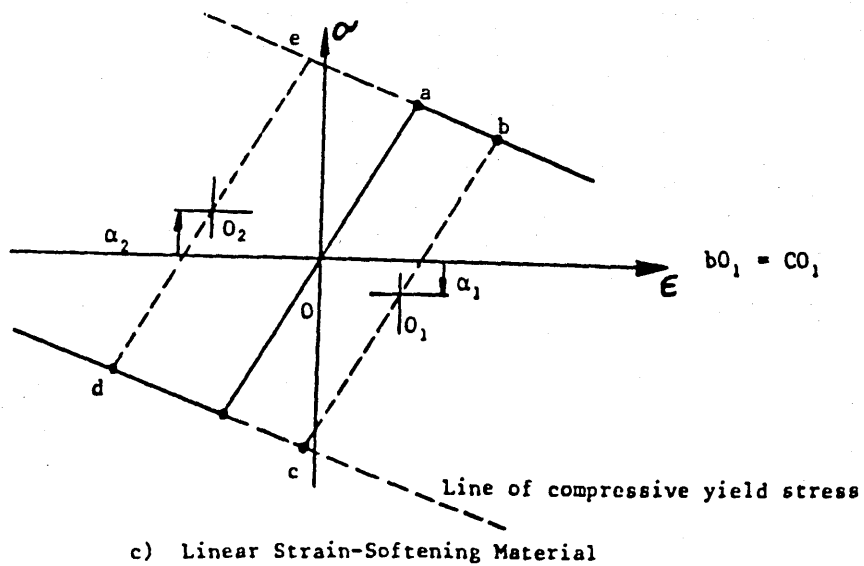
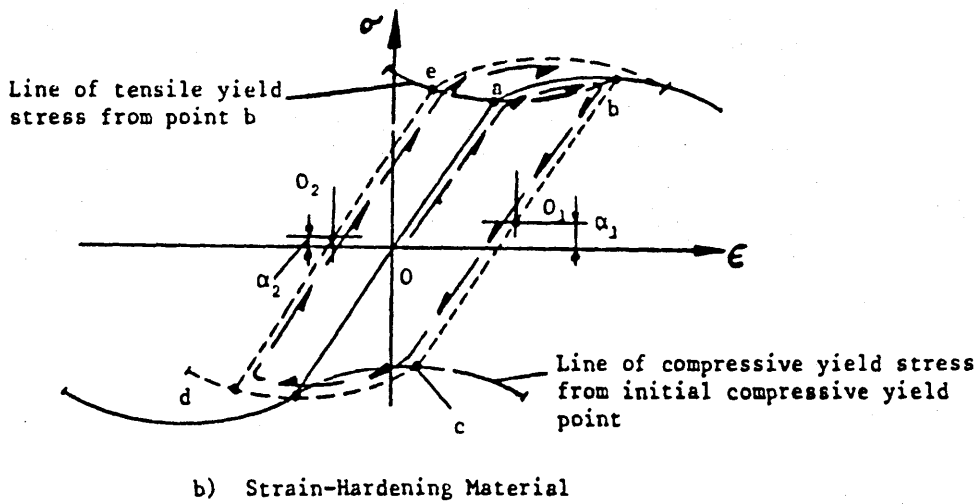
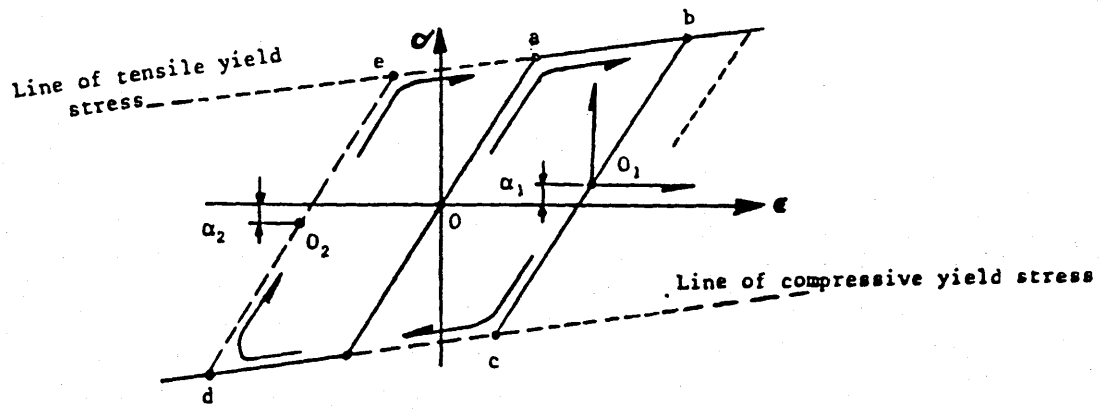
Fig(2.16) Isotropic Hardening Rule for One-Dimensional Problem



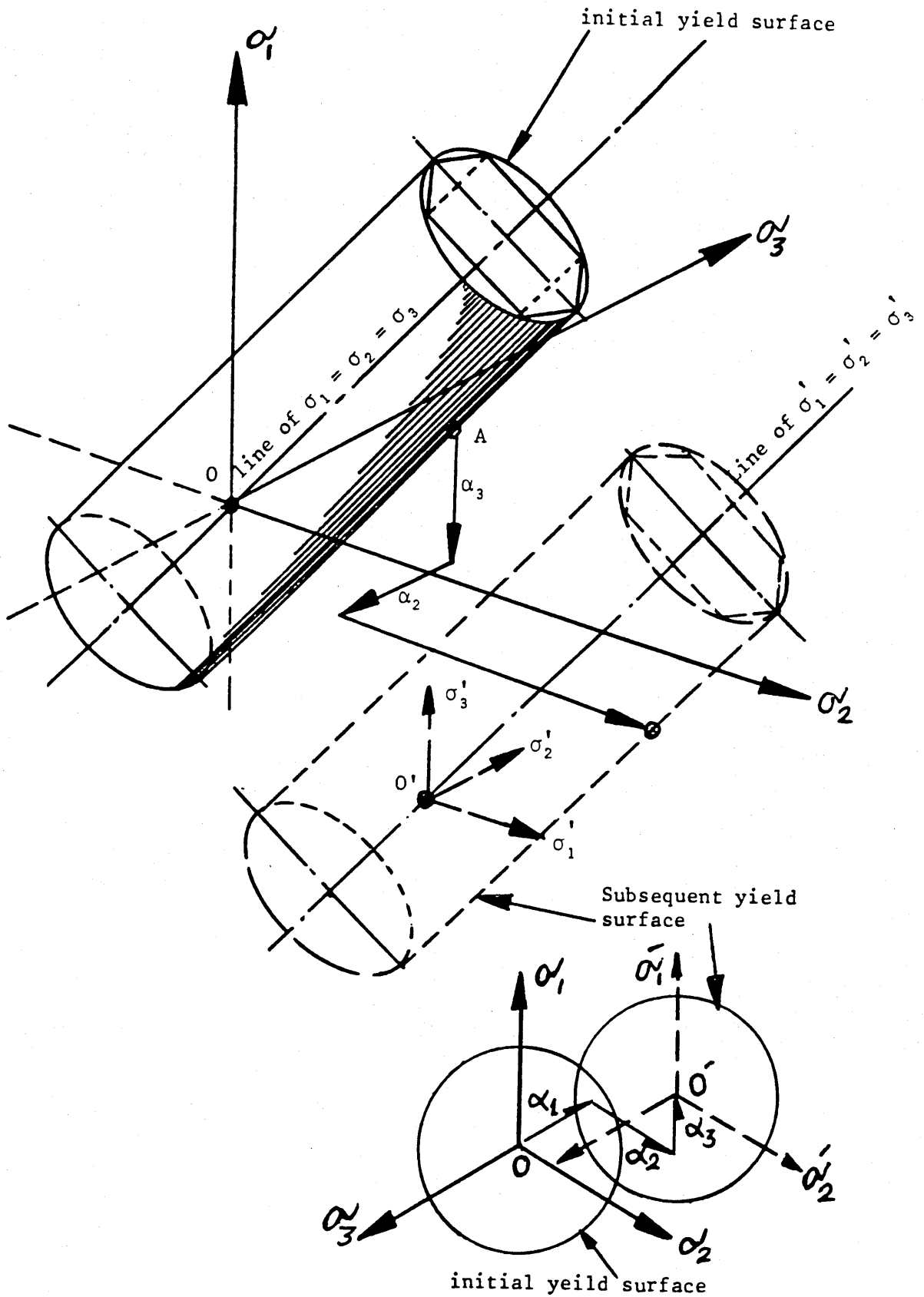
Fig(2.17) Isotropic Hardening Rule for Von Mises in 3-Dim Principal Stress Space and π -Plane



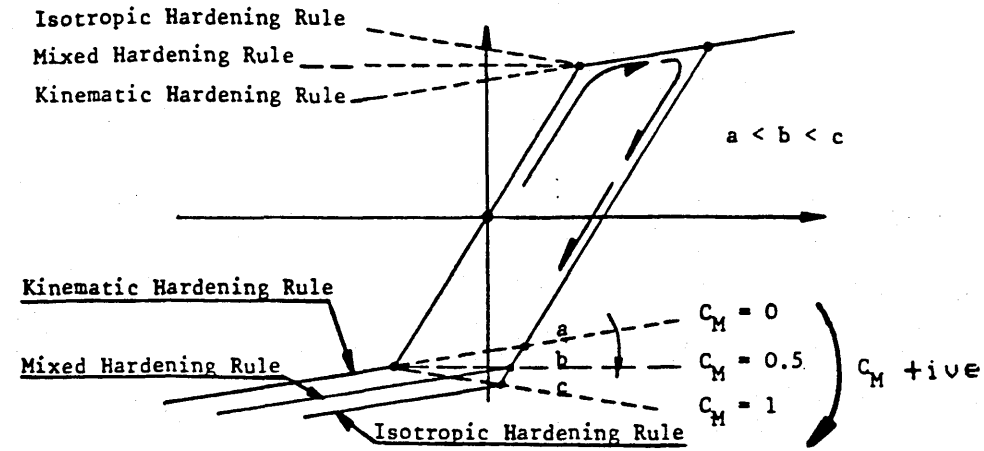
Fig(2.18) Geometrical Representation of Drucker -Prager Criterion by Using Isotropic Hardening Rule in 3-Dim Principal Stress Space



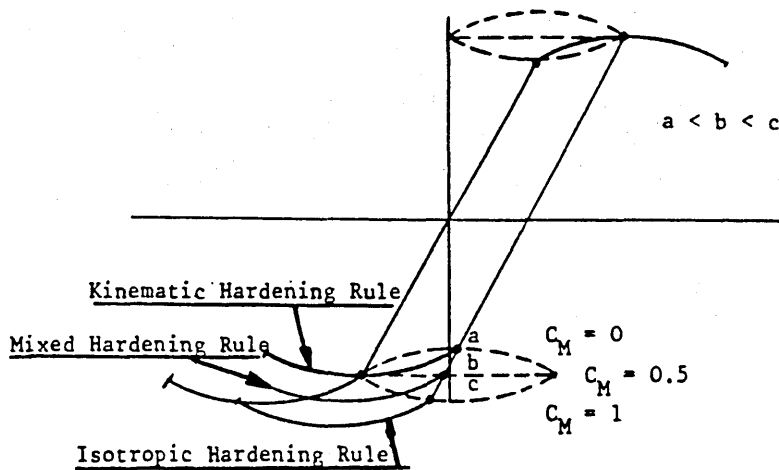
Fig(2.19) Kinematic Hardening Rule for Uniaxial Problem



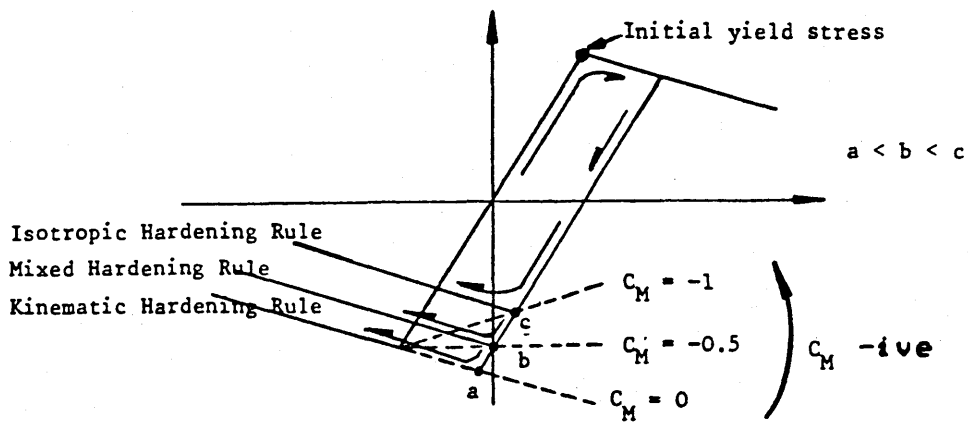
Fig(2.20) Geometrical Representation of Kinematic Hardening Rule for the Von Mises Criterion in 3-Dimensional Principal Stress Space and at the π -Plane



a) Linear Strain-Hardening Material



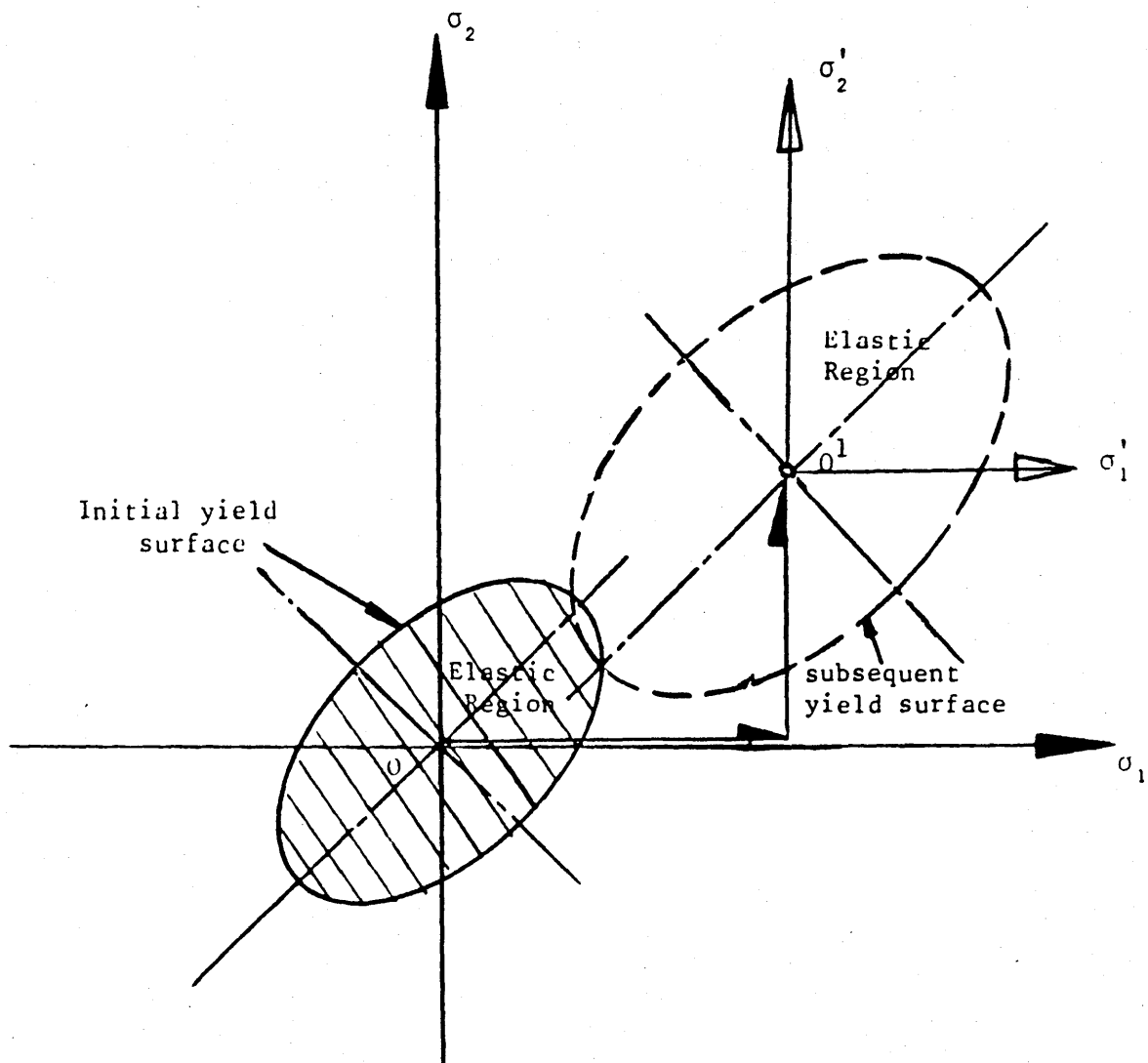
b) Non-Linear Strain-Hardening Material



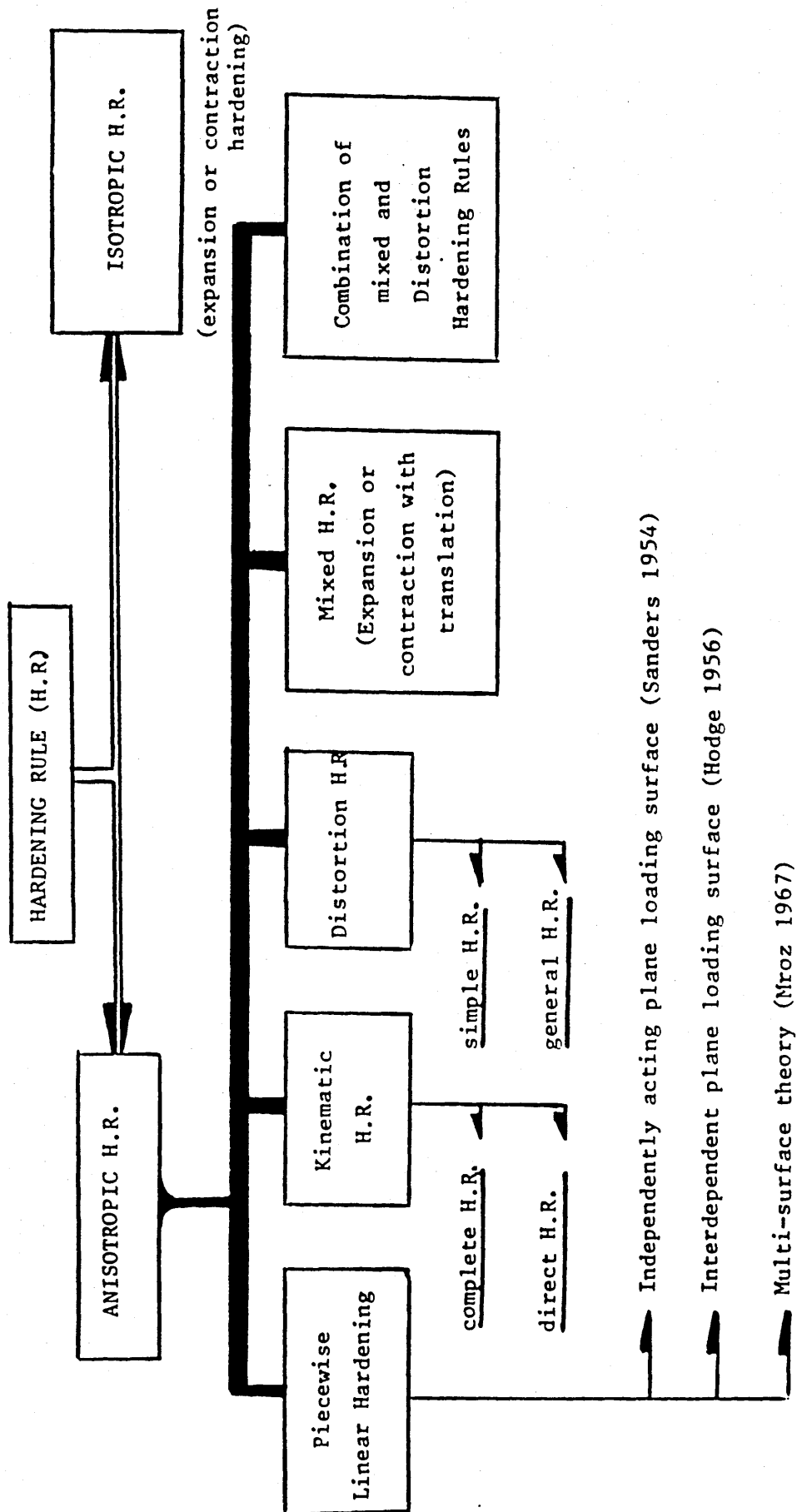
c) Linear Strain-Softening Material

a, b, c ... are the yield stresses for the Kinematic Hardening Rule, Mixed Hardening Rule and Isotropic Hardening Rule respectively

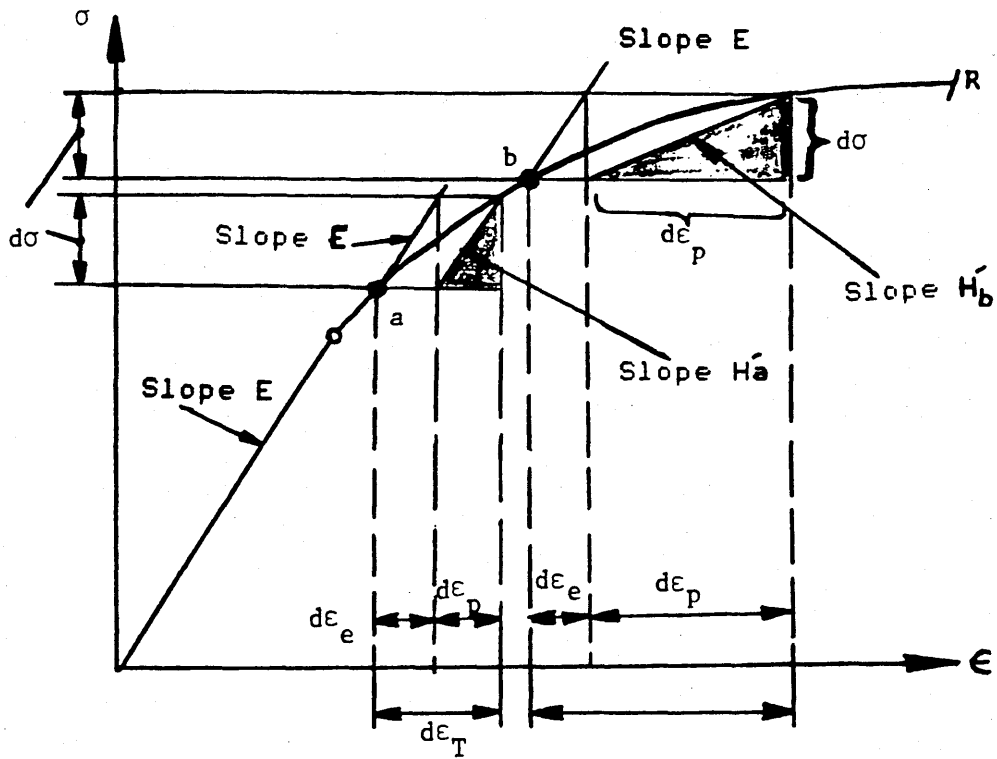
Fig(2.21) The Mixed Hardening Rule for the Uniaxial Problem



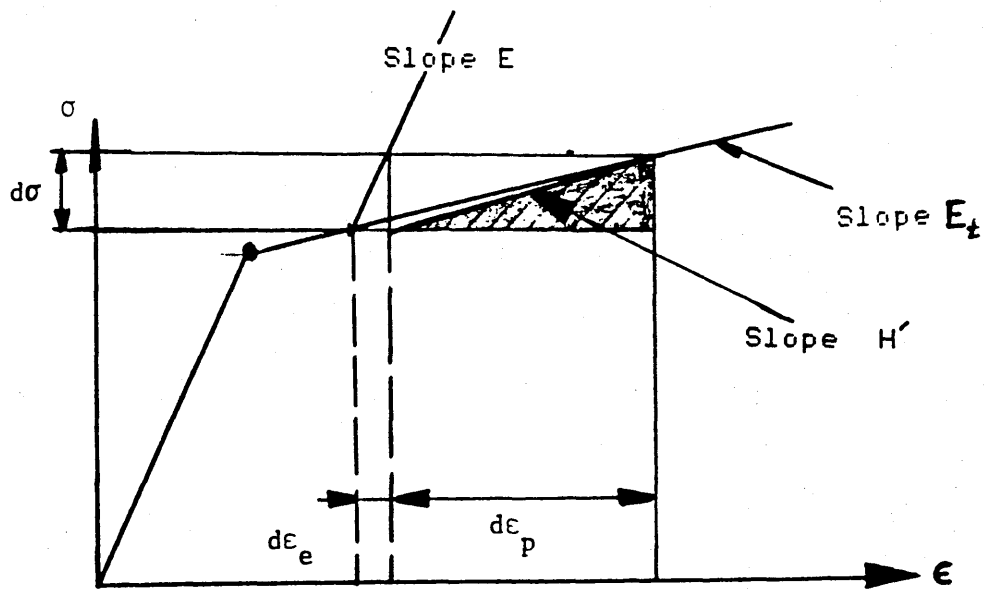
Fig(2.22) The Mixed Hardening Rule Applied to the Von Mises Yield Criterion shown in Two-Dimensional Principal Stress Space



Fig(2.23) Main Hardening Rules

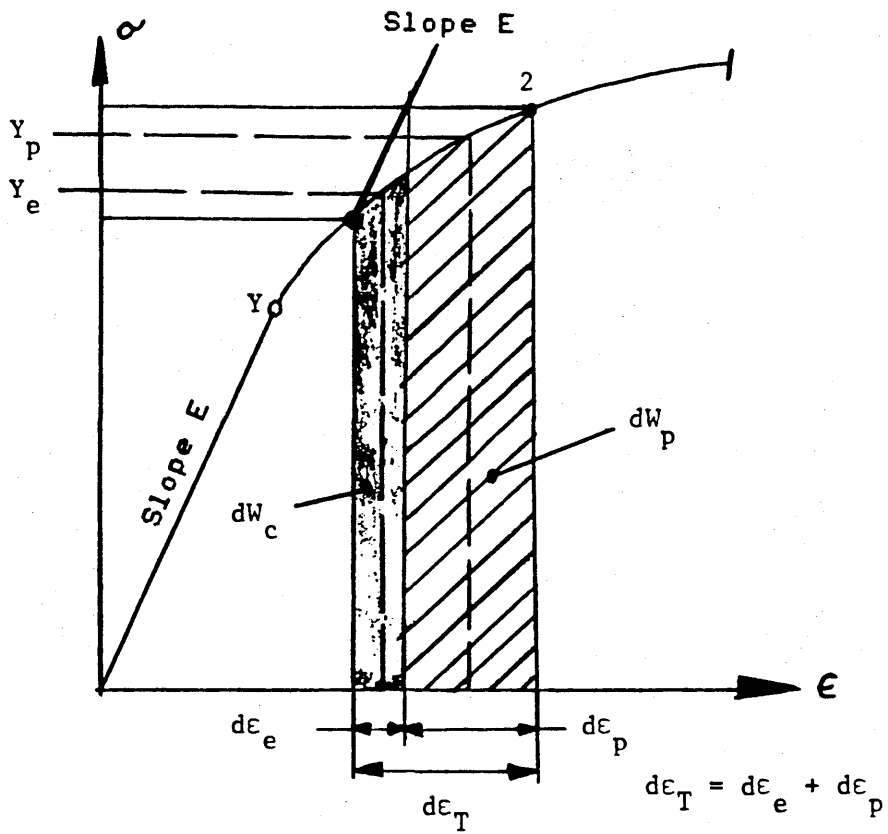


(a) For a Practical Material (non-linear strain-hardening)

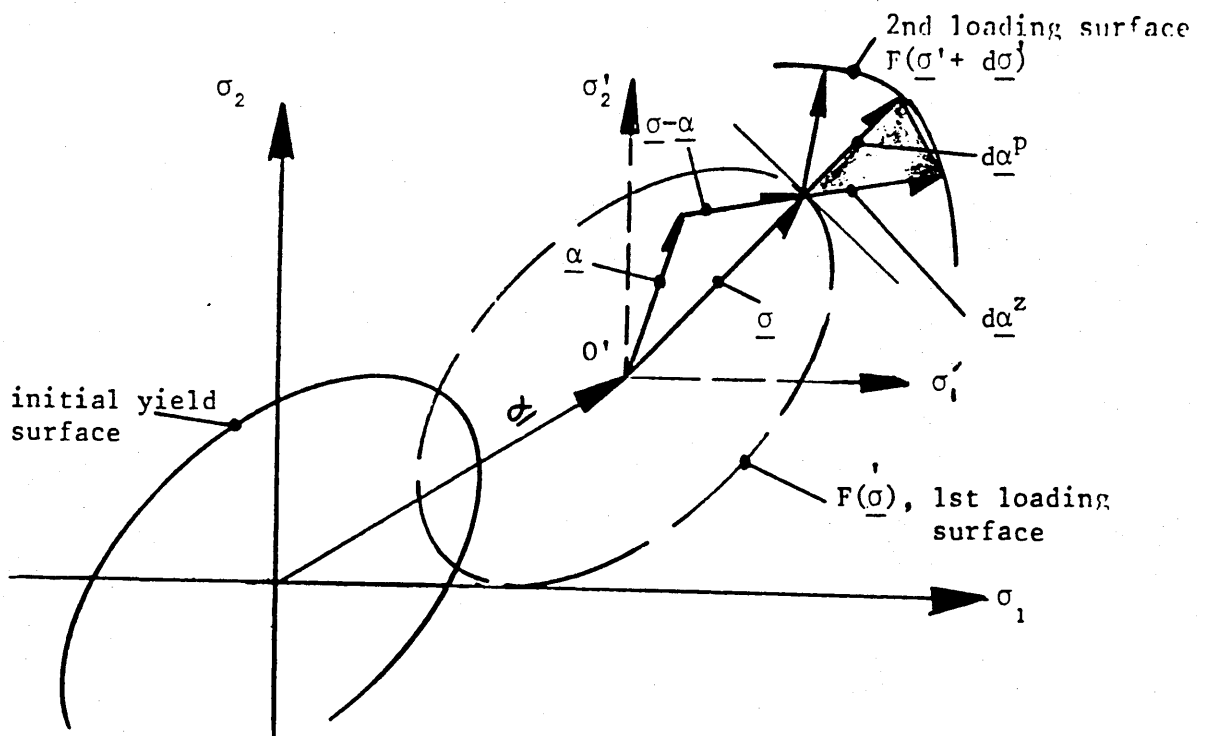


(b) For a Linear Strain-Hardening Material

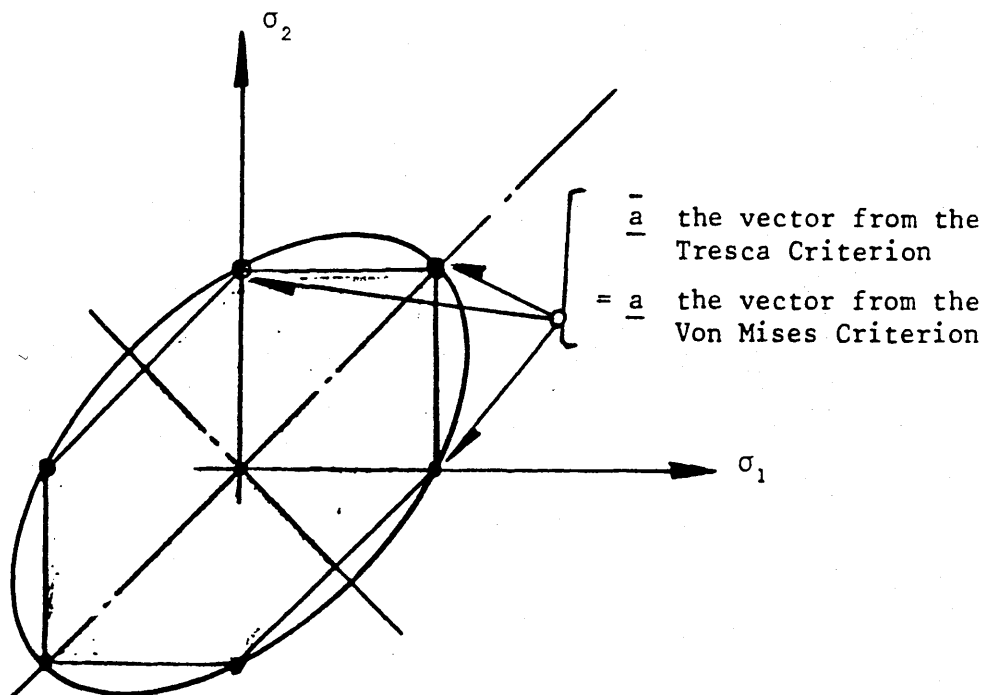
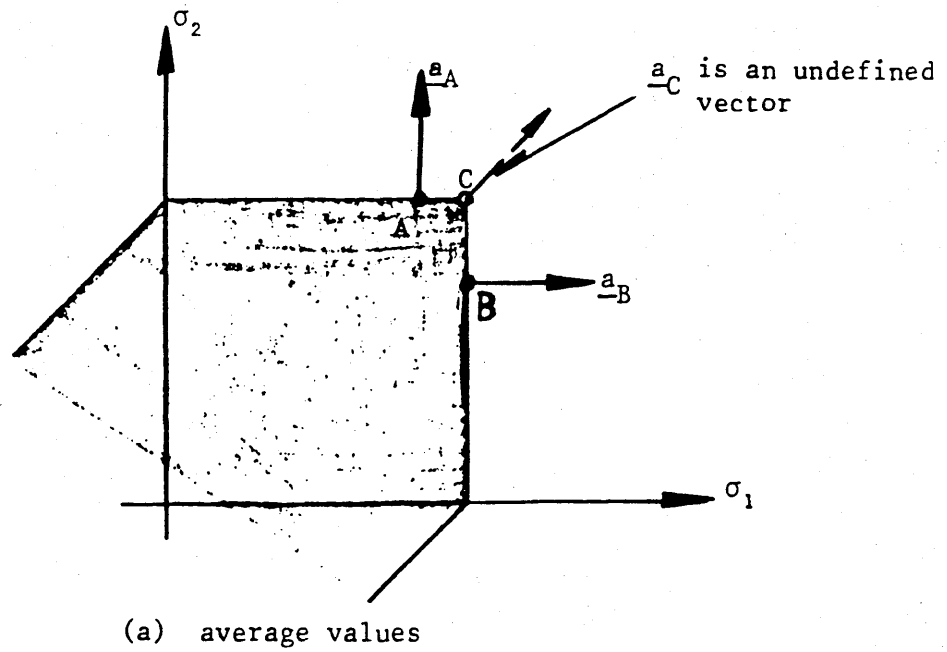
Fig(2,24) Strain-Hardening Parameter



Fig(2.25) Elastic Work Hypothesis



Fig(2.26) Mathematical Notation for Kinematic H.R.



Fig(2.27) $\underline{a}$ Vector for Tresca Yield Criterion

TABLE 2.1 Coefficients of Gradient Vector a

Yield Criterion	C ₁	C ₂	C ₃
Von Mises	0	$\frac{3}{2} \frac{\bar{\sigma}}{\bar{\sigma}}$	0
Tresca	0	$\frac{\sqrt{3}}{\bar{\sigma}} \cos \phi [1 + \tan \phi \tan 3\phi]$	$\frac{3\sqrt{3}}{\bar{\sigma}^2} \frac{\sin \phi}{\cos 3\phi}$
Modified Von Mises	$\frac{3}{2} \frac{\Delta}{\bar{\sigma}_m}$	$\frac{3}{2} \frac{\bar{\sigma}}{\bar{\sigma}_m}$	0
Modified Tresca Polynomial	$2 Y_c Y_t^2 \Delta^2$	$6(2J_2 - Y_c Y_t)(J_2 - Y_c Y_t)$	$-54 J_3$
Tresca Polynomial	0	$6(2J_2 - Y)(J_2 - Y^2)$	$-54 J_3$
Drucker-Prager	$3\alpha'$	$\frac{\sqrt{3}}{2\bar{\sigma}}$	0
Mohr-Coulomb	$\sin \theta_p$	$\frac{\sqrt{3}}{2\bar{\sigma}} \cos \phi [(1 + \tan \phi \tan 3\phi) + \frac{\sin \theta_p}{\sqrt{3}} (\tan 3\phi - \tan \phi)]$	$\frac{\sqrt{3} 3 (\sin \phi + \frac{1}{3} \sin \theta_p \cos \phi)}{2 \bar{\sigma}^2 \cdot \cos 3\phi}$

where $\bar{\sigma} = \sqrt{3 J_2}$ $\phi = \frac{1}{3} \sin^{-1} \frac{-3 \sqrt{3} J_3}{2 J_2^{1.5}}$, and

$\bar{\sigma}_m = (\bar{\sigma}^2 + 3 \cdot \Delta \cdot \sigma_m)^{\frac{1}{2}}$ $\alpha' = \frac{2 / \sqrt{3} \sin \theta_p}{3 - \sin \theta_p}$

$\Delta = Y_c - Y_t$

CHAPTER THREE

FINITE ELEMENT THEORY FOR TOW-DIMENSIONAL AND AXISYMETRIC ELASTO-PLASTIC PROBLEMS

3.0 Introduction

In this chapter finite element derivation, for small, and large deflection problems of elasto-plasticity have been carried out. 2-dimensional (plane-stress and plane-strain) problems and axisymmetric problems have been considered. Algorithms for stress correction, stress-strain matrix, and solution procedure have been explained. Efficient methods for predicting the initial yielding point for multiaxial stress based upon the Newton-Raphson scheme and the false-position rule are explained. The solution algorithm is valid for isotropic, kinematic and mixed hardening rules with any of the yield criteria explained in Chapter 2.

3.1 Two-Dimensional Finite Element Analysis

A 3-dimensional problem can be reduced to a 2-dimensional plane-stress case, if one dimension is small relative to the other dimensions, and the applied load is in-plane.

i.e.
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Hence,
$$\epsilon_{xz} = \epsilon_{yz} = 0,$$

while
$$\epsilon_z \neq 0$$

Also a 3-dimensional problem can be treated as 2-dimensional plane-strain problem, if the 3-dimensional problem has the same x-y plane section at any z as well as the same x-y plane loading, and there is no force component along the z-axis.

i.e.
$$\epsilon_z = \epsilon_{zx} = \epsilon_{yz} = 0, \text{ and } \sigma_z \neq 0$$

3.1.1 Piecewise and Pointwise Discretization

In the Finite Element process of analysis, simplification of the assumed solution is effected by discretizing the continuum domain into a suitable number of subdomains, called finite elements, and

connected by a finite number of geometrical nodes.

For each finite element a pointwise discretization is carried out in order to simplify the satisfaction of the boundary conditions, i.e the field functions over an element can be represented by their values at a suitable number of nodes within the element.

The field variables over the element can be approximated by using a suitable interpolation model. Lagrangian polynomial interpolation functions have been used in this work, as follows:-

$$\phi(x,y) = \sum_{i=1}^n \phi_i N_i(x,y) \quad (3.1)$$

where $\phi(x,y)$ is the field variable at x,y ,
 ϕ_i is the field variable at node i ,
 n is the total number of element nodes, and
 $N_i(x,y)$ is a Lagrangian shape function.

Defining the displacement vector for the e^{th} element as

$$\underline{\delta} = \{ u_1 \ v_1 \ u_2 \ v_2 \ \dots \ u_n \ v_n \}$$

then, from equation (3.1), the displacement components inside any element can be expressed in terms of the nodal displacement components and Lagrangian shape functions as follows:-

$$\left. \begin{aligned} u(x,y) &= \sum_{i=1}^n u_i N_i(x,y) \\ v(x,y) &= \sum_{i=1}^n v_i N_i(x,y) \end{aligned} \right\} \quad (3.2)$$

The shape functions in the above equations depend upon the nodal coordinates and are required to be evaluated for every subdomain. It is useful to employ a local system which is simple, unique and independent of the global system. Such a system is known

as an intrinsic system and its coordinates are the intrinsic coordinates ξ and η . Therefore equation(3.2) can be rewritten as follows:-

$$\left. \begin{aligned} u(\xi, \eta) &= \sum_{i=1}^n u_i N_i(\xi, \eta) \\ v(\xi, \eta) &= \sum_{i=1}^n v_i N_i(\xi, \eta) \end{aligned} \right\} \quad (3.3)$$

To obtain the equations of the transformation, i.e

$$x = x(\xi, \eta), \quad y = y(\xi, \eta),$$

the x and y are assumed to be field functions defined within the element domain in terms of their nodal values and obeying the same interpolation formula, i.e:-

$$\left. \begin{aligned} x(\xi, \eta) &= \sum_{i=1}^n x_i N_i(\xi, \eta) \\ y(\xi, \eta) &= \sum_{i=1}^n y_i N_i(\xi, \eta) \end{aligned} \right\} \quad (3.4)$$

Such transformation is known as "Isoparametric Transformation" and the above equations are called the isoparametric equations.

Finally, the displacement components, at any point (x, y) can be expressed, as follows:-

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix} \cdot \underline{\delta}$$

3.1.2 Strain-Displacement Matrix (B Matrix)

The strain components at point (x, y) inside a 2-dimensional solid body can be obtained by using equation(2.23), as follows:-

$$\underline{\epsilon} = \underline{\epsilon}_S + \underline{\epsilon}_L$$

where

$$\underline{\varepsilon} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad \text{is the total strain vector}$$

$$\underline{\varepsilon}_s = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad \text{is the small deflection strain vector, and}$$

$$\underline{\varepsilon}_L = \frac{1}{2} \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ 2\left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y}\right] \end{pmatrix} \quad \text{The large deflection strain vector.}$$

Defining vector $\underline{\theta}$, such that

$$\underline{\theta} = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial y} \right\},$$

which can be related to nodal values by using equation(3.2), as follows:-

$$\underline{\theta} = \underline{\Psi} \cdot \underline{\delta}$$

where

$$\underline{\Psi} = [\underline{\Psi}_1 \quad \underline{\Psi}_2 \quad \underline{\Psi}_3 \dots \underline{\Psi}_i \dots \underline{\Psi}_n],$$

$$\underline{\Psi}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ \frac{\partial N_i}{\partial y} & 0 \\ 0 & \frac{\partial N_i}{\partial x} \\ 0 & \frac{\partial N_i}{\partial y} \end{bmatrix}$$

Then $\underline{\varepsilon}_L$ at point (x,y) can be deduced as follows:-

$$\underline{\varepsilon}_L = \frac{1}{2} \underline{E} \cdot \underline{\theta}$$

Where

$$\underline{E} = \begin{bmatrix} \frac{\partial u}{\partial x} & 0 & \frac{\partial v}{\partial x} & 0 \\ 0 & \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{bmatrix}$$

The above matrix can be evaluated in terms of the nodal displacement vector and shape function as follows:-

$$\underline{E} = [\underline{e}_1 \quad \underline{e}_2 \cdots \underline{e}_i \quad \underline{e}_n] \cdot \underline{\delta}$$

where

$$\underline{e}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

Finally, it can be shown that:-

$$\underline{\varepsilon}_L = \frac{1}{2} \underline{E} \cdot \underline{\theta} = \frac{1}{2} \underline{E} \cdot \underline{\Psi} \cdot \underline{\delta} = \frac{1}{2} \underline{B}_L \cdot \underline{\delta} \quad (3.6)$$

where

$$\underline{B}_L = \underline{E} \cdot \underline{\Psi}$$

is the strain displacement matrix
for large displacements.

Also, it can be proved that:-

$$\underline{\varepsilon}_S = \underline{B}_S \cdot \underline{\delta} \quad , \quad (3.7)$$

where

$$\underline{B}_S = [\underline{b}_1 \quad \underline{b}_2 \cdots \underline{b}_n] \quad \text{is the strain-displacement matrix for small deflection}$$

$$\underline{b}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

Finally, it can be shown that:-

$$\underline{\varepsilon} = \underline{\varepsilon}_s + \underline{\varepsilon}_L = (\underline{B}_s + \frac{1}{2} \underline{B}_L) \cdot \underline{\delta}$$

In the solution procedure for plasticity problems, the increment evaluation will be employed, hence all the variables are to be in incremental form, therefore the variation of the total strain vector can be deduced as follows:-

$$\Delta \underline{\varepsilon} = \Delta \underline{\varepsilon}_s + \Delta \underline{\varepsilon}_L$$

where

$$\Delta \underline{\varepsilon}_s = \underline{B}_s \Delta \underline{\delta}$$

$$\Delta \underline{\varepsilon}_L = \Delta \left(\frac{1}{2} \underline{B}_L \cdot \underline{\delta} \right)$$

$$\Delta \underline{\varepsilon}_L = \Delta \left(\frac{1}{2} [\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_n] \cdot \underline{\delta}^t \cdot \underline{\theta} \cdot \underline{\delta} \right)$$

which can be proved equal to :

$$\Delta \underline{\varepsilon}_L = [\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_n] \cdot \underline{\delta} \cdot \underline{\theta} \cdot \Delta \underline{\delta}$$

$$= \underline{E} \cdot \underline{\theta} \cdot \Delta \underline{\delta}$$

$$\Delta \underline{\varepsilon}_L = \underline{B}_L \cdot \Delta \underline{\delta}$$

Finally

$$\Delta \underline{\varepsilon} = (\underline{B}_s + \underline{B}_L) \cdot \Delta \underline{\delta} \quad (3.8)$$

3.1.3 Strain-Displacement Matrix for Two-Dimensional Elasto-plastic Problems

For elasto-plastic problems the $\underline{D}$ matrix for plane-stress, and plane-strain cases, will be similar, Hence the stress and strain vector is to be modified as follows:-

$$\underline{\sigma} = \{ \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \}$$

$$\underline{\epsilon} = \{ \epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{xy} \}$$

Where

$$\epsilon_z = 0 \quad \text{for plane-strain, and}$$

$$\epsilon_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y) \quad \text{for plane-stress, as shown in equation(2.29)}$$

Therefore, the $\underline{B}_S$ and $\underline{B}_L$ matrices are to be modified as follows:-

$$\underline{B}_S = [\underline{b}_1 \quad \underline{b}_2 \quad \dots \quad \underline{b}_n]$$

where

$$\underline{b}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ C_v \frac{\partial N_i}{\partial x} & C_v \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (3.9)$$

$$\underline{B}_L = \underline{E} \cdot \underline{\Psi}$$

where

$$\underline{E} = [\underline{e}_1 \quad \underline{e}_2 \quad \underline{e}_3 \quad \dots \quad \underline{e}_n] \cdot \underline{\delta}$$

$$\underline{\epsilon}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_i}{\partial y} \\ C_v \frac{\partial N_i}{\partial x} & C_v \frac{\partial N_i}{\partial y} & C_v \frac{\partial N_i}{\partial x} & C_v \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (3.10)$$

$C_v = 0$ for plane-strain, and

$C_v = -\frac{\nu}{1-\nu}$ for plane-stress.

3.2 Axisymmetric Finite Element Analysis

An axisymmetric Problem of a solid body is a 3-dimensional problem in geometry but treated mathematically similar to a plane-stress 2-dimensional problem. The 3-dimensional problem can be reduced to an axisymmetric case, if it satisfies the following conditions, with respect to cylindrical axes(r, θ, z):

- (i) The cross section at any θ is the same (axisymmetric geometry),
- (ii) There is no loading component normal to the r - z plane,
- (iii) The load distribution in the r - z plane should be the same at any θ (axisymmetric loading).

3.2.1 Strain-Displacement Matrix ($\underline{B}$ Matrix)

The displacement vector for the e^{th} element can be defined for an axisymmetric problem as follows:

$$\underline{\delta} = \{u_1 \quad w_1 \quad u_2 \quad w_2 \quad \dots \quad u_n \quad w_n\}$$

Isoparametric equations similar to equation(3.2) can be employed to evaluate the displacement components at point (r, z) as follows:

$$\left. \begin{aligned} u(r,z) &= \sum_{i=1}^n u_i N_i(r,z) \\ w(r,z) &= \sum_{i=1}^n w_i N_i(r,z) \end{aligned} \right\} \quad (3.11)$$

The strain components for point (r,z) in an axisymmetric solid body can be obtained by using equations(2.24), as follows:-

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial w}{\partial r} \right)^2 \right]$$

$$\epsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right]$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{2} \left(\frac{u}{r} \right)^2$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} + \left[\frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right]$$

which can be rewritten as follows:-

$$\underline{\epsilon} = \underline{\epsilon}_S + \underline{\epsilon}_L \quad \text{is the total strain vector,}$$

Where

$$\underline{\epsilon} = \{ \epsilon_r \quad \epsilon_z \quad \epsilon_\theta \quad \gamma_{rz} \},$$

$$\underline{\epsilon}_S = \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{pmatrix} = \underline{B}_S \cdot \underline{\delta} \quad \text{is the small deflection strain vector,}$$

$$\underline{B}_S = [\underline{b}_1 \quad \underline{b}_2 \quad \underline{b}_3 \quad \dots \quad \underline{b}_n] ,$$

$$\underline{b}_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix},$$

and the large deflection strain vector is:-

$$\underline{\varepsilon}_L = \frac{1}{2} \begin{pmatrix} \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 \\ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \\ \left(\frac{u}{r}\right)^2 \\ 2 \left[\frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right] \end{pmatrix}$$

which can be simplified after defining the following vector:-

$$\underline{\theta} = \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \end{pmatrix}$$

Then,

$$\underline{\epsilon}_L = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{\partial w}{\partial r} & 0 & 0 \\ 0 & \frac{\partial u}{\partial z} & 0 & \frac{\partial w}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & \frac{u}{r} \\ \frac{\partial u}{\partial z} & \frac{\partial u}{\partial r} & \frac{\partial w}{\partial z} & \frac{\partial w}{\partial r} & 0 \end{bmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \end{pmatrix}$$

i.e. $\underline{\epsilon}_L = \frac{1}{2} \underline{E} \cdot \underline{\theta}$

The $\underline{\theta}$ vector and $\underline{E}$ matrix can be evaluated in terms of the nodal displacement vector by using equations(3.11), as follows:

$$\underline{\theta} = \underline{\Psi} \cdot \underline{\delta}$$

Where

$$\underline{\Psi} = [\underline{\Psi}_1 \quad \underline{\Psi}_2 \quad \underline{\Psi}_3 \quad \dots \quad \underline{\Psi}_n] ,$$

$$\underline{\Psi}_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ \frac{\partial N_i}{\partial z} & 0 \\ 0 & \frac{\partial N_i}{\partial r} \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \end{bmatrix} , \quad (3.12)$$

and

$$\underline{E} = [\underline{e}_1 \quad \underline{e}_2 \quad \underline{e}_3 \quad \dots \quad \underline{e}_i \quad \dots \quad \underline{e}_n] \cdot \underline{\delta} , \text{ and}$$

$$\underline{\epsilon}_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & \frac{\partial N_i}{\partial r} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & \frac{N_i}{r} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & 0 \end{bmatrix} \quad (3.13)$$

Finally, the total strain vector can be written as follows:-

$$\underline{\epsilon} = (\underline{B}_S + \frac{1}{2} \underline{B}_L (\underline{\delta})) \cdot \underline{\delta}$$

where

$$\underline{B}_L = \underline{E} \cdot \underline{\psi} \quad (3.14)$$

The strain increment vector due to $\Delta \underline{\delta}$, can be obtained by employing the variation approach as follows:

$$\Delta \underline{\epsilon} = \Delta \underline{\epsilon}_S + \Delta \underline{\epsilon}_L$$

Where

$$\Delta \underline{\epsilon}_S = \Delta (\underline{B}_S \cdot \underline{\delta})$$

$$\underline{\epsilon}_S = \underline{B}_S \cdot \Delta \underline{\delta}$$

While

$$\Delta \underline{\epsilon}_L = \Delta \left(\frac{1}{2} \underline{E}(\underline{\delta}) \cdot \underline{\psi} \cdot \underline{\delta} \right)$$

and it can be shown that

$$\Delta \underline{\epsilon}_L = \underline{B}_L \cdot \Delta \underline{\delta}$$

$$\Delta \underline{\epsilon}_L = \Delta \left(\frac{1}{2} [\underline{e}_1 \quad \underline{e}_2 \quad \dots \quad \underline{e}_n] \cdot \underline{\delta} \cdot \underline{\psi} \cdot \underline{\delta} \right)$$

$$\Delta \underline{\epsilon}_L = [\underline{e}_1 \quad \underline{e}_2 \quad \dots \quad \underline{e}_n] \cdot \underline{\delta} \cdot \underline{\psi} \cdot \Delta \underline{\delta}$$

$$\Delta \underline{\epsilon}_L = \underline{E}(\underline{\delta}) \cdot \underline{\psi} \cdot \Delta \underline{\delta} = \underline{B}_L \cdot \Delta \underline{\delta}$$

Finally, it can be proved that :-

$$\Delta \underline{\epsilon} = (\underline{B}_S + \underline{B}_L) \cdot \Delta \underline{\delta} \quad (3.15)$$

3.3 Derivation Of the FEM Equilibrium Equation

The state of stress and deformation of a solid body subjected to a set of external forces can be defined by applying an approximate energy principle such as the minimum total potential energy theorem which states that " for a stressed body to be in a stable equilibrium, the total potential energy of the body is minimum with respect to the variables of the problem ". This principle which is valid for linear and non-linear structures can be expressed as follows:

$$\chi = U - W = \text{Min}$$

or in terms of the first variation :

$$d\chi = dU - dW = 0$$

Where χ is the total potential energy of the body
 U is the strain energy of the body
 W is the work done by external loads

The strain energy absorbed by the body is :

$$U = \iiint_{\Omega} \left[\int_{\underline{\varepsilon}_0}^{\underline{\varepsilon}_t} \underline{\sigma}^t \cdot d\underline{\varepsilon} \right] d(\text{Vol})$$

The variation of the strain energy (due to displacement increment $d\underline{\delta}$) can be expressed as follows :

$$dU = \iiint_{\Omega} d\underline{\varepsilon}^t \cdot \underline{\sigma} \cdot d(\text{Vol})$$

Similarly, the total external work and its variation can be written as:-

$$W = \underline{F}^t \cdot \left[\int_{\underline{\delta}_0}^{\underline{\delta}_f} d\underline{\delta} \right]$$

Then

$$dW = \underline{F}^t \cdot d\underline{\delta} \quad ,$$

or

$$dW = d\underline{\delta}^t \cdot \underline{F}$$

Where $\underline{F}$ is the applied force vector

$d\underline{\delta}$ is the displacement increment vector

Hence, the variation of the total potential energy is:

$$d\chi = \iiint d\underline{\varepsilon}^t \cdot \underline{\sigma} \cdot d(\text{Vol}) - d\underline{\delta}^t \cdot \underline{F} = 0$$

By using the strain-displacement relationships, it can be shown that

$$d\underline{\varepsilon} = \underline{B}(\underline{\delta}) \cdot d\underline{\delta}$$

Where

$$\underline{B}(\underline{\delta}) = \underline{B}_s + \underline{B}_L(\underline{\delta})$$

Hence

$$d\chi = \iiint d\underline{\delta}^t \cdot \underline{B}^t(\underline{\delta}) \cdot \underline{\sigma} \cdot d(\text{Vol}) - d\underline{\delta}^t \cdot \underline{F}$$

Finally

$$d\chi = d\underline{\delta}^t \cdot \left[\iiint \underline{B}^t(\underline{\delta}) \cdot \underline{\sigma} \cdot d(\text{Vol}) - \underline{F} \right] = 0$$

Then

$$\underline{F} = \iiint_{\Omega} \underline{B}^t(\underline{\delta}) \cdot \underline{\sigma} \cdot d(\text{Vol})$$

Which is the general "Equilibrium Equation " for Linear and Non-linear structures.

3.4 Linearization Of Equilibrium Equation

It is necessary to linearize the equilibrium equation for non-linear structures. If the initial state is defined in terms of $(\underline{\sigma}_0, \underline{\varepsilon}_0, \underline{\delta}_0)$ an error or residual vector is defined such that:-

$$\underline{R}_0(\underline{\delta}_0) = \underline{F} - \iiint_{\Omega} \underline{B}^t(\underline{\delta}_0) \cdot \underline{\sigma}_0 \cdot d(\text{Vol})$$

To restore the equilibrium of the structure, the residual force vector will be applied as a complementary force vector and an incremental state may be obtained $(\Delta\underline{\sigma}, \Delta\underline{\varepsilon}, \Delta\underline{\delta})$ such that:-

$$\underline{F} - \iiint_{\Omega} (\underline{B}^t + \Delta\underline{B}^t)(\underline{\sigma} + \Delta\underline{\sigma}) d(\text{Vol}) = \underline{0}$$

or
$$\underline{R} - \iiint_{\Omega} \underline{B}^t \cdot \underline{\sigma} \cdot d(\text{Vol}) - \iiint_{\Omega} [\underline{\Delta B}^t \cdot \underline{\sigma} + \underline{B}^t \cdot \underline{\Delta \sigma} + \underline{\Delta B}^t \cdot \underline{\Delta \sigma}] d(\text{Vol}) = 0$$

After neglecting the small quantities, it can be shown that:-

$$\underline{R}_0(\underline{\delta}) = \iiint_{\Omega} \underline{B}^t \cdot \underline{\Delta \sigma} \cdot d(\text{Vol}) + \iiint_{\Omega} \underline{\Delta B}^t \cdot \underline{\sigma} \cdot d(\text{Vol})$$

From the elasto-plastic analysis, it can be proved that:-

$$\underline{\Delta \sigma} = \underline{D}_{ep} \cdot \underline{\Delta \epsilon}$$

$$\underline{\Delta \sigma} = \underline{D}_{ep} \cdot \underline{B} \cdot \underline{\Delta \delta}$$

Therefore

$$\begin{aligned} \iiint_{\Omega} \underline{B}^t \cdot \underline{\Delta \sigma} \cdot d(\text{Vol}) &= [\iiint_{\Omega} (\underline{B}_s^t + \underline{B}_L^t) \cdot \underline{D}_{ep} \cdot (\underline{B}_s + \underline{B}_L) d(\text{Vol})] \cdot \underline{\Delta \delta} \\ &= [\underline{K}_{ep} + \underline{K}_L] \cdot \underline{\Delta \delta} \end{aligned}$$

Where

$$\underline{K}_{ep} = \iiint_{\Omega} \underline{B}_s^t \cdot \underline{D}_{ep} \cdot \underline{B}_s \cdot d(\text{Vol}) \quad \text{the elasto-plastic small deflection stiffness matrix}$$

$$\underline{K}_L = \underline{K}_{sL} + \underline{K}_{Ls} + \underline{K}_{LL} \quad \text{the elasto-plastic large deflection stiffness matrix}$$

Where

$$\underline{K}_{sL} = \iiint_{\Omega} \underline{B}_s^t \cdot \underline{D}_{ep} \cdot \underline{B}_L \cdot d(\text{Vol})$$

$$\underline{K}_{Ls} = \underline{K}_{sL}^t$$

$$\underline{K}_{LL} = \iiint_{\Omega} \underline{B}_L^t \cdot \underline{D}_{ep} \cdot \underline{B}_L \cdot d(\text{Vol})$$

Then the variation of the $\underline{B}$ matrix is :

$$\underline{\Delta B}(\underline{\delta}) = \underline{\Delta B}_L(\underline{\delta}) \quad \text{where only } \underline{B}_s \text{ is constant}$$

Therefore the second part of the integration in the residual force

expression can be written as follows :

$$\iiint_{\Omega} \Delta \underline{B}^t \cdot \underline{\sigma} \, d\Omega = \iiint_{\Omega} \Delta \underline{B}_L^t \cdot \underline{\sigma} \, d\Omega$$

but

$$\underline{B}_L = \underline{E} \cdot \underline{\Psi}$$

i.e

$$\underline{B}_L^t = \underline{\Psi}^t \cdot \underline{E}^t$$

Then

$$\Delta \underline{B}_L^t \cdot \underline{\sigma} = \underline{\Psi}^t \cdot \underline{E}^t \cdot \underline{\sigma}$$

Hence , it can be proved, for the 2-dimensional case, that:-

$$\underline{E}^t(\underline{\delta}) \cdot \underline{\sigma} = \begin{bmatrix} \frac{\partial u}{\partial x} & 0 & \frac{\partial v}{\partial x} & 0 \\ 0 & \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial y} \\ C_v \frac{\partial u}{\partial x} & C_v \frac{\partial u}{\partial y} & C_v \frac{\partial v}{\partial x} & C_v \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{bmatrix}^t \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{pmatrix}$$

$$\underline{E}^t(\underline{\delta}) \cdot \underline{\sigma} = \underline{\Sigma} \cdot \underline{\Theta}$$

Where

$$\underline{\Sigma} = \begin{bmatrix} \sigma_x + C_v \sigma_z & \tau_{xy} & 0 & 0 \\ \tau_{xy} & \sigma_y + C_v \sigma_z & 0 & 0 \\ 0 & 0 & \sigma_x + C_v \sigma_z & \tau_{xy} \\ 0 & 0 & \tau_{xy} & \sigma_y + C_v \sigma_z \end{bmatrix}, \text{ and}$$

$$\underline{\theta} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$$

Similarly, for the axisymmetric case, it can be shown that the $\underline{\Sigma}$ is defined as follows:-

$$\underline{\Sigma} = \begin{bmatrix} \sigma_r & \tau_{rz} & 0 & 0 & 0 \\ \tau_{rz} & \sigma_z & 0 & 0 & 0 \\ 0 & 0 & \sigma_r & \tau_{rz} & 0 \\ 0 & 0 & \tau_{rz} & \sigma_z & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta \end{bmatrix}, \text{ and}$$

$$\underline{\theta} = \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial z} \\ \frac{u}{r} \end{pmatrix}$$

But

$$\underline{\theta} = \underline{\Psi} \cdot \underline{\delta}$$

then its variation is

$$\Delta \underline{\theta} = \underline{\Psi} \cdot \Delta \underline{\delta}$$

i.e

$$\Delta \underline{B}^t \cdot \underline{\sigma} = \underline{\Psi}^t \cdot \underline{\Sigma} \cdot \underline{\Psi} \cdot \Delta \underline{\delta}$$

The second part of the integration of the residual force can be written as follows:

$$\begin{aligned} \iiint_{\Omega} \Delta \underline{B}^t \cdot \underline{\sigma} \, d\Omega &= \left[\iiint_{\Omega} \underline{\Psi}^t \cdot \underline{\Sigma} \cdot \underline{\Psi} \, d(\text{Vol}) \right] \cdot \Delta \underline{\delta} \\ &= \underline{K}_{\sigma} \cdot \Delta \underline{\delta} \end{aligned}$$

Where

$$\underline{K}_{\sigma} = \iiint_{\Omega} \underline{\Psi}^t \cdot \underline{\Sigma} \cdot \underline{\Psi} \, d\Omega$$

Finally the residual force vector required to keep the equilibrium condition for the structure can be expressed as:

$$\begin{aligned} \underline{R}(\underline{\delta}) &= \left[\underline{K}_{\sigma} + \underline{K}_{LL} + \underline{K}_{SL} + \underline{K}_{LS} + \underline{K}_{ep} \right] \cdot \Delta \underline{\delta} \\ \underline{R}(\underline{\delta}) &= \underline{K} \cdot \Delta \underline{\delta} \end{aligned}$$

For the case of elasto-plastic analysis of small deflection problems , the following approximations can be employed :

$$\begin{aligned} \underline{B} &\approx \underline{B}_s \\ \underline{K} &\approx \underline{K}_{ep} \end{aligned}$$

For the finite element mesh, it can be proved that:

$$\underline{(K)}_{\text{Global}} = \sum_{e=1}^{Ne} \underline{K}_e$$

Where Ne is the number of elements ,
 $\underline{K}_e$ is the element stiffness matrix .

Then

$$\underline{R}_0 = \underline{F} - \sum_{e=1}^{Ne} \left[\iiint_{\Omega_e} \underline{B}^t \cdot \underline{\sigma} \, d\Omega_e \right]$$

Where the summation will be carried out using ordinary finite element assembly rules .

3.5 Evaluation Of Element Stiffness Matrix ($\underline{K}_e$)

The evaluation of $\underline{K}_e$ for a specified finite element with n-nodes can be achieved by employing a suitable numerical integration scheme

such as the Gauss quadrature method.

The $\underline{K}_e$ matrix for 2-Dimensional problems can be written as follows:

$$\begin{aligned}\underline{K}_e &= [\underline{K}_\sigma + \underline{K}_{LL} + \underline{K}_{LS} + \underline{K}_{SL} + \underline{K}_{ep}]_e \\ \underline{K}_e &= \iint_{\Omega_e} [\underline{\Psi}^t(x,y) \cdot \underline{\Sigma} \cdot \underline{\Psi}(x,y) + \underline{B}_L^t(x,y) \cdot \underline{D}_{ep} \cdot \underline{B}_L(x,y) \\ &\quad + \underline{B}_L^t(x,y) \cdot \underline{D}_{ep} \cdot \underline{B}_S(x,y) + \underline{B}_S(x,y) \cdot \underline{D}_{ep} \cdot \underline{B}_L(x,y) \\ &\quad + \underline{B}_S^t(x,y) \cdot \underline{D}_{ep} \cdot \underline{B}_S(x,y)] d(\text{Vol})\end{aligned}\quad (3.16)$$

The $\underline{B}_S$, $\underline{B}_L$, and $\underline{\Psi}$ matrices are functions of the first cartesian derivatives of the shape functions. Since the shape functions are expressed in terms of the intrinsic coordinates ξ , η , it is useful to deduce the cartesian derivatives in terms of intrinsic derivatives. Consider the i^{th} shape function, and applying the chain rule of partial differentiation.

$$\begin{aligned}\frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta}\end{aligned}$$

In matrix form:-

$$\begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \underline{J} \left(\begin{matrix} x, y \\ \xi, \eta \end{matrix} \right) \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix}$$

where

$$\underline{J} \left(\begin{matrix} x, y \\ \xi, \eta \end{matrix} \right) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix},$$

which is known as the Jacobian matrix. Hence,

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = \underline{J}^{-1} \left(\frac{x, y}{\xi, \eta} \right) \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix}$$

and it can be shown that:-

$$\underline{J}^{-1} \left(\frac{x, y}{\xi, \eta} \right) = \frac{1}{|\underline{J}(\xi, \eta)|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix},$$

$$|\underline{J}(\xi, \eta)| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}$$

In a similar way, it can be shown that :-

$$d(\text{Vol}) = |\underline{J}(\xi, \eta)| t(\xi, \eta) d\xi d\eta$$

where $t(\xi, \eta)$ is the thickness of the element at the specified ξ and η

Hence, equation(3.16) can be rewritten for a typical quadrilateral element as follows:

$$\begin{aligned} K_e = \int_0^1 \int_0^1 [& \underline{\Psi}^t(\xi, \eta) \cdot \underline{\Sigma} \cdot \underline{\Psi}(\xi, \eta) + \underline{B}_I^t(\xi, \eta) \cdot \underline{D}_{ep} \cdot \underline{B}_L(\xi, \eta) \\ & + \underline{B}_S^t(\xi, \eta) \cdot \underline{D}_{ep} \cdot \underline{B}_L(\xi, \eta) + \underline{B}_L^t(\xi, \eta) \cdot \underline{D}_{ep} \cdot \underline{B}_S(\xi, \eta) \\ & + \underline{B}_S^t(\xi, \eta) \cdot \underline{D}_{ep} \cdot \underline{B}_S(\xi, \eta)] |\underline{J}(\xi, \eta)| t d\xi d\eta \end{aligned}$$

Employing the Gauss quadrature method for the above equation, it can

be deduced that:

$$\begin{aligned}
 \underline{K}_e = & \sum_{i=1}^{N_q} \sum_{j=1}^{N_q} [\underline{\Psi}^t(\xi_i, \eta_j) \cdot \underline{\Sigma} \cdot \underline{\Psi}(\xi_i, \eta_j) \\
 & + \underline{B}_L^t(\xi_i, \eta_j) \cdot \underline{D}_{ep} \cdot \underline{B}_L(\xi_i, \eta_j) \\
 & + \underline{B}_S^t(\xi_i, \eta_j) \cdot \underline{D}_{ep} \cdot \underline{B}_L(\xi_i, \eta_j) \\
 & + \underline{B}_L^t(\xi_i, \eta_j) \cdot \underline{D}_{ep} \cdot \underline{B}_S(\xi_i, \eta_j) \\
 & + \underline{B}_S^t(\xi_i, \eta_j) \cdot \underline{D}_{ep} \cdot \underline{B}_S(\xi_i, \eta_j)] \\
 & \cdot |J(\xi_i, \eta_j)| w_i \cdot w_j \cdot t(\xi_i, \eta_i)
 \end{aligned}$$

Where

ξ_i, η_j are the intrinsic coordinates of the integration points,

w_i, w_j are the corresponding weight values,

N_q is the number of integration points for each element.

For the axisymmetric case a similar numerical evaluation scheme can be employed but the $2\pi r(\xi_i, \eta_i)$ is used instead of the thickness $t(\xi_i, \eta_i)$ where $r(\xi_i, \eta_i)$ is the radius at the specified ξ_i and η_i . Similar expression can be written for triangular elements.

3.6 Solution Procedure

Due to non-linearity of the problem, the loading should be applied in a suitable number of increments. In each increment the non-linear equations can be solved by using an iterative scheme in order to improve the accuracy of the non-linear equation solution.

The algorithm solution starts in a manner similar to the linear static analysis. Fig(3.1) shows the main steps in the solution procedure. The algorithm can be summarized as follows:

Step-1 Initiation

(i) Before applying the load increment, the following parameters are

assumed to be known from the previous increment (or zero) as shown in Fig(3.2).

$\underline{\sigma}_0, \underline{\varepsilon}_0$ are the total initial stress and strain vectors respectively at every Gaussian stress point,

$\underline{\delta}_0$ is the total nodal displacement vector

(ii) The load increment is defined in terms of the followings:

$\Delta \underline{F}_1$ is the increment of the equivalent nodal loading vector,

$\Delta \underline{\eta}_1$ is the vector of initial and thermal strains during the given load increment at every Gaussian point.

Step-2 Increment Solution

i) Calculate the elasto-plastic stress-strain matrix at all of the Gaussian points.

a) for the iterative scheme

$$\underline{D}_{ep} = \underline{D}_e \quad \dots \quad \text{always}$$

b) for the modified Newton-Raphson scheme

$$\underline{D}_{ep} = \underline{D}_{ep}(\underline{\sigma}_0, \underline{\varepsilon}_0) \quad \text{at the first load increment trial.}$$

The same matrix will be set for the next trials of that load increment. $\underline{D}_{ep}$ is to be updated once for every new load increment.

c) for the Newton-Raphson scheme :

$$\underline{D}_{ep} = \underline{D}_{ep}(\underline{\sigma}_0, \underline{\varepsilon}_0)$$

and which will be updated for every trial and each new increment of load.

ii) Calculate the $\underline{B}_L(\underline{\delta}_0)$ to obtain the total $\underline{B}$ matrix

$$\underline{B}(\underline{\delta}_0) = \underline{B}_L(\underline{\delta}_0) + \underline{B}_s$$

iii) Calculate the Global stiffness matrix

$$\underline{K}_G(\underline{D}_{ep}(\underline{\sigma}_0, \underline{\varepsilon}_0), \underline{B}(\underline{\delta}_0)) = \sum_{e=1}^{Ne} (\underline{K})_{\text{element}}$$

Where N_e is the number of elements in the finite element mesh.

$$(\underline{K})_{\text{element}} = (\underline{K}_{LL} + \underline{K}_{sL} + \underline{K}_{Ls} + \underline{K}_{\sigma} + \underline{K}_{ep}) \dots \text{ for each element}$$

Fig(3.3) shows that for the iterative scheme, the Global stiffness matrix is similar to $\underline{K}_G$ derived for linear-elastic, small-deflection problems.

$$\text{i.e } \underline{K}_G = \underline{K}_G (\underline{D}_e, \underline{B}_s)$$

While for the modified Newton-Raphson scheme the stiffness matrix is updated once for every new load increment.

$$(\underline{K}_G)_{\beta+1} = \underline{K}_G [\underline{D}_{ep}(\underline{\sigma}_i, \underline{\varepsilon}_i), \underline{B}(\underline{\delta}_i)]$$

where i is the number of load increments, and for Newton-Raphson scheme the stiffness matrix is updated for every trial,

$$(\underline{K}_G)_{\beta+1} = \underline{K}_G [\underline{D}_{ep}(\underline{\sigma}_{\beta}, \underline{\varepsilon}_{\beta}), \underline{B}(\underline{\delta}_{\beta})]$$

where β is the number of the trial and $\underline{\sigma}_{\beta}$, $\underline{\varepsilon}_{\beta}$, $\underline{\delta}_{\beta}$ are values at trial β , and

$(\underline{K}_G)_{\beta+1}$ is the Global stiffness matrix at trial $\beta+1$.

iv) Solve the following increment equation

$$\underline{K}_G \cdot \Delta \underline{\delta}_1 = \Delta \underline{F}_1$$

to obtain $(\Delta \underline{\delta}_1)$, the new increment displacement, a banding or frontal technique can be employed.

v) Calculate the resulting strain increments and the corresponding stress increments at each Gaussian stress point.

$$\Delta \underline{\varepsilon}_1 = \underline{B} \cdot \Delta \underline{\delta}_1 - \Delta \underline{\eta}_1$$

$$\Delta \underline{\sigma}'_1 = D_{ep} \cdot \Delta \underline{\epsilon}_1$$

Step-3 Total Solution

Calculate the total values as follows:

$$\underline{\delta}_1 = \underline{\delta}_0 + \Delta \underline{\delta}_1$$

$$\underline{\epsilon}_1 = \underline{\epsilon}_0 + \Delta \underline{\epsilon}_1$$

$$\underline{\sigma}'_1 = \underline{\sigma}_0 + \underline{\sigma}'_1$$

Step-4 Plasticity Check And Updating The Variables

The equation of the initial yield surface is:

$$f(\underline{\sigma}) - Y = 0$$

The following procedure can be employed at each stress point inside the domain to check the level of the stress and the plastic flow.

- i) If $f(\underline{\sigma}_1) \leq Y$ then it is an elastic deformation and there is no plastic flow.

$$\text{i.e. } \underline{\sigma}'_1 = \underline{\sigma}_1$$

If all the stress points satisfy this condition then the iteration procedure is to be terminated and the new load increment can be applied (step-1).

- ii) If $f(\underline{\sigma}'_1) > Y$ and $f(\underline{\sigma}_0) \geq Y$ then it is a case of plastic deformation. If any of the Gaussain stress points satisfy this condition, then the stress point is in the plastic region, and the deformation increment is totally in the plastic zone.

Hence $\underline{\sigma}'_1$ does not represent the actual stress vector at the point and a stress-correction procedure is required to define the corrected stress $\underline{\sigma}'_1$, as explained in section 3.8.

iii) Critical case : if $f(\underline{\sigma}_0) < Y$ and $f(\underline{\sigma}_1) > Y$

For such a case the yielding will be initiated within the increment $\Delta\underline{\sigma}'_1$, i.e part of it does not undergo any plasticity whilst the remaining part must be corrected . A parameter ζ can be defined such that

$$f(\underline{\sigma}_0 + \zeta\Delta\underline{\sigma}'_1) = Y$$

i.e the elastic part of $\Delta\underline{\sigma}'_1$ is $\zeta\Delta\underline{\sigma}'_1$

Two different methods are introduced in this work to evaluate the parameter ζ , as follows:

a) Newton-Raphson scheme

Let the equation of the yield surface be

$$F(\underline{\sigma}, Y) = f(\underline{\sigma}) - Y = 0$$

where $\underline{\sigma} = \underline{\sigma}_0 + \zeta \cdot \Delta\underline{\sigma}'_1$

Then

$$\frac{\partial F}{\partial \zeta} = \left(\frac{\partial f}{\partial \underline{\sigma}} \right)^t \cdot \frac{\partial \underline{\sigma}}{\partial \zeta} + \text{zero}$$

where

$$\frac{\partial \underline{\sigma}}{\partial \zeta} = \frac{\partial \underline{\sigma}_0}{\partial \zeta} + \frac{\partial \zeta}{\partial \zeta} \Delta\underline{\sigma}'_1 + \zeta \frac{\partial \Delta\underline{\sigma}'_1}{\partial \zeta} = \Delta\underline{\sigma}'_1$$

Finally

$$\frac{\partial F}{\partial \zeta} = \underline{a}^t \cdot \Delta\underline{\sigma}'_1$$

The algorithm can be summarized as follows :

- 1) Assume an initial value ζ_0 , and $\zeta = \zeta_0$
- 2) $\underline{\sigma}(\zeta) = \underline{\sigma}_0 + \zeta \cdot \Delta\underline{\sigma}'_1$
- 3) $F(\zeta) = f(\underline{\sigma}) - Y$

$$4) \quad \underline{a}(\zeta) = \frac{\partial F}{\partial \underline{\sigma}}$$

$$5) \quad \frac{\partial F}{\partial \zeta} = \underline{a}^t \cdot \Delta \underline{\sigma}'_1$$

$$6) \quad \zeta_{\text{new}} = \zeta - F(\zeta) / \left(\frac{\partial F}{\partial \zeta} \right)$$

7) If $|\zeta_{\text{new}} - \zeta| > \text{a permissible error}$ go to (2) with $\zeta = \zeta_{\text{new}}$

A one dimensional illustration is shown in Fig (3.4) .

b) False-Position Rule

According to the given conditions:

$$f_0 = f(\underline{\sigma}_0) < Y$$

$$f_1 = f(\underline{\sigma}_0 + \Delta \underline{\sigma}') > Y$$

using linear interpolation

$$\frac{\zeta_1 - \zeta_0}{f_1 - f_0} = \frac{\zeta_2 - \zeta_0}{Y - f_0}$$

But

$$\zeta_0 = 0 \quad ; \quad \zeta_1 = 1$$

Then

$$\zeta_2 = \frac{Y - f_0}{f_1 - f_0}$$

The following algorithm can be employed for the r^{th} iteration , $r > 1$.

$$1) \quad \underline{\sigma}_r = \underline{\sigma}_0 + \zeta \cdot \Delta \underline{\sigma}'_1$$

$$2) \quad f_r = f(\underline{\sigma}_r)$$

$$3) \quad \zeta_{r+1} = \zeta_r \cdot \left(\frac{Y - f_0}{f_r - f_0} \right)$$

4) If $|\zeta_{+1} - \zeta_r| > \text{a permissible error go to (1)}$

A one-Dimensional illustration is shown in Fig(3.5).

Having obtained ζ , the following vectors are redefined as follow:

$$\underline{\varepsilon}_0 \equiv \underline{\varepsilon}_0 + \zeta \cdot \Delta \underline{\varepsilon}_1$$

$$\underline{\sigma}_0 \equiv \underline{\sigma}_0 + \zeta \cdot \Delta \underline{\sigma}'_1$$

$$\Delta \underline{\varepsilon}_1 \equiv (1 - \zeta) \cdot \Delta \underline{\varepsilon}_1$$

$$\Delta \underline{\sigma}'_1 \equiv (1 - \zeta) \cdot \Delta \underline{\sigma}'_1$$

Where the new $\Delta \underline{\sigma}'_1$ is the elasto-plastic stress to be corrected .

Step-5 Updating The Uniaxial Yield stress

For materials which exhibit strain hardening (or softening) an instantaneous yield stress should be defined as follows:

i) the reduced stress vector is

$$\underline{\sigma}_r = \underline{\sigma}_0 + \Delta \underline{\sigma}' - \underline{\sigma}_0$$

And the corresponding equivalent stress $\bar{\sigma}_r$ can therefore be defined.

ii) The new uniaxial yield stress for a perfectly-plastic material or a material which follows the kinematic hardening rule is:

$$Y_r = Y$$

whilst for isotropic or mixed hardening materials :

$$\bar{\varepsilon}_r = \frac{\bar{\sigma}}{E} + \bar{\varepsilon}_p^i$$

Then from the material uniaxial stress-strain diagram:

$$Y_r = Y(\bar{\varepsilon}_r)$$

The above updating should be carried out at all of the Gaussian points.

Step-6 Stress Correction

The stress correction is required in order to correlate between the uniaxial and multiaxial problems. In this procedure, the strain increment is assumed to be correct and a correction of the stress is to be carried out so as to match the stress-strain diagram of the material. The corrected stress increment should be calculated through a number of sub-increments (to improve the accuracy of the iteration), without solving the finite element equations of the system , as explained in section 3.8.

After the stress correction procedure the stress vector will be reduced to $\underline{\sigma}_c$.

Step-7 Calculate The Nodal Residual Force

The applied force increment produces $\underline{\sigma}'$ stress vectors at the Gaussian stress points , while the correct value should be $\underline{\sigma}_c$. Therefore the error in the stress vector calculation at the Gaussian point is:

$$\underline{\sigma}_{re} = \underline{\sigma}_c - \underline{\sigma}'$$

Hence, the required equivalent nodal residual force to restore the equilibrium of the body is:

$$\Delta \underline{F}_R = \sum_{e=1}^{Ne} \left[- \iiint_{\Omega_e} \underline{B}^t \cdot \underline{\sigma}_{re} d\Omega_e \right]$$

The Global Residual force vector is obtained by using the ordinary assembler.

Step-8 Convergence Check

The Global residual force vector is to be applied as a load

increment and the previous procedure is to be repeated from step 1 until at least one of the following conditions is satisfied .

- 1) An acceptable error in the displacement increment is reached. Such an error may be measured by the following ratio:

$$\underline{E}_{r1} = \frac{(\underline{\Delta\delta}^t \cdot \underline{\Delta\delta})}{(\underline{\delta}^t \cdot \underline{\delta})}$$

- 2) The value of the correcting load vector becomes too small.

This can be measured by the following scalar quantity:

$$\underline{E}_{r2} = \frac{(\underline{\Delta F}^t \cdot \underline{\Delta F})}{(\underline{F}^t \cdot \underline{F})}$$

Divergence may occur (i.e the number of FE iterations may exceed a given maximum limit while the permissible error is still not achieved) if:

- i) The applied load increment is too large with respect to the Global stiffness matrix. This will cause the solution of the non-linear system of equations to be inaccurate. Similarly the stress correction can not be achieved.
- ii) Vibratory divergence which may happen when the specified permissible error is too small.
- iii) Unstable structure. If enough stress points exceed the ultimate stress level, the stiffness matrix becomes unstable and the procedure will fail due to structural collapse.

3.7 Elasto-Plastic Stress-Strain Matrix Algorithm

The $\underline{D}_{ep}$ matrix is a function of total stress and strain vectors. The following steps can be employed to obtain the $\underline{D}_{ep}$ matrix at any given Gaussian stress point.

(1) - Prepare required data

$\underline{\sigma}, \underline{\varepsilon}$ are the current stress and strain vectors at the given Gaussian stress point.

$\underline{\varepsilon}_p^i$ is the total accumulated (isotropic) effective plastic strain [$\underline{\varepsilon}_p^i = 0$, for pure kinematic hardening].

$\underline{\alpha}$ is the current stress shift vector. [$\underline{\alpha} = \underline{0}$, for pure isotropic hardening].

C_M is the value of the mixed hardening parameter.

(2) - Calculate the gradient vector

The equation for the yield surface is:

$$F((\underline{\sigma} - \underline{\alpha}), Y_r) = 0$$

and the gradient vector is defined as follows:

$$\underline{a} = \frac{\partial F}{\partial \underline{\sigma}} = \frac{\partial F}{\partial (\underline{\sigma} - \underline{\alpha})}$$

Defining the reduced stress vector

$$\underline{\sigma}_r = \underline{\sigma} - \underline{\alpha}$$

Then

$$\underline{a} = \underline{a}(\underline{\sigma}_r) = \underline{a}_r$$

(3) - Calculate the hardening parameter (A)

A is given by:

$$A = (1 - C_M) A^{(k)} + C_M A^{(i)}$$

Where

$A^{(k)}$ is the kinematic hardening parameter,

and

$A^{(i)}$ is the isotropic hardening parameter.

(i) Isotropic hardening parameter

The following steps can be applied to evaluate $A^{(i)}$:

a) Obtain the equivalent stress and strain quantities :

$$\bar{\sigma} = f(\underline{\sigma}) = f(\underline{\sigma} - \underline{\alpha})$$

The expression of $f(\underline{\sigma})$ for different yield criteria of different materials are listed in Table(3.1)

where the parameters are as defined in Chapter 2.

$$\bar{\epsilon} = \epsilon_p + \frac{\bar{\sigma}}{E}$$

b) Calculate the tangential elastic modulus

$$E_t = \frac{d\sigma}{d\epsilon} \quad \text{which is the slope of the material tensile stress-strain diagram at } (\bar{\epsilon}, \bar{\sigma}).$$

$$c) H' = \frac{E_t}{\left(1 - \frac{E_t}{E}\right)}$$

$$d) A^{(i)} = \left(\frac{\partial F}{\partial Y_r}\right)^2 \cdot H'$$

Where the values of $(\partial F / \partial Y_r)$ for different yield criteria, are as given in Table (3.1), and Y_r is the instantaneous yield stress as defined in step 5 of section 3.6.

ii) Kinematic hardening parameter $A^{(K)}$

$$A^{(K)} = C \cdot \underline{a}^t \cdot \underline{a}$$

Where

$$\underline{a} = \underline{a}(\underline{\sigma}_r) ; \underline{\sigma}_r = \underline{\sigma} - \underline{\alpha}$$

$$C = H' \quad \text{for a linear stress hardening material,}$$

$$C = 0 \quad \text{for a perfectly-plastic material,}$$

$$C = W_T / C_T \quad \text{for a non-linear strain hardening or softening material.}$$

Defining

$$\underline{\varepsilon}' = \underline{\varepsilon} - C_M \cdot \underline{\varepsilon}_p$$

$$\underline{\varepsilon}_p = \underline{\varepsilon} - \underline{D}^{-1} \cdot \underline{\sigma}$$

and

$$\varepsilon'_1 = \varepsilon'_x ; \quad \varepsilon'_2 = \varepsilon'_y \quad \dots\dots\dots ; \quad \varepsilon'_6 = \varepsilon'_{zx}$$

$$\sigma_1 = \sigma_x ; \quad \sigma_2 = \sigma_y \quad \dots\dots\dots ; \quad \sigma_6 = \tau_{zx}$$

then

$$W_T = \sum_{i=1}^6 (\sigma_i)^2$$

$$C_T = \sum_{i=1}^6 \frac{(\sigma_i)^2}{C_i}$$

where

$$C_i = \frac{E_i}{1 - \left(\frac{E_i}{E} \right)^2}$$

For each component of the stress vector, and

$$E_i = \frac{d\sigma}{d\varepsilon} \quad \text{at } \varepsilon'_i \quad \text{from the material diagram.}$$

4) Final Calculation

Defining:

$$\underline{d} = \frac{(\underline{D} \underline{a})}{(A + \underline{a}^t \underline{D} \underline{a})}$$

Then

$$\underline{D}_{ep} = \underline{D} - (\underline{D} \underline{a}) \underline{d}^t$$

3.8 Stress Correction Algorithm

To achieve a reasonable accuracy in the stress correction procedure, the total stress vector is to be divided into N divisions, where

$$N = 1 + \frac{(\text{deg})}{(\text{deg})_{\max}}$$

$$\text{deg} = \left| \frac{\bar{\sigma}_r - Y_r}{Y} \right| \quad \text{and}$$

$(\text{deg})_{\max}$ is a given maximum permissible degree for a single subdivision.

Considering the r^{th} increment, where the following new subscripts are employed:

$()_r \equiv$ parameter at the end of the r^{th} subincrement

$()_m \equiv$ mean value of the parameter at the r^{th} subincrement

The following steps can be used for correcting the stress in the i^{th} subincrement:

$$\left. \begin{array}{l} \text{i) } \Delta \underline{\sigma}_r = \frac{\Delta \underline{\sigma}'}{N} \\ \Delta \underline{\alpha}_r = \underline{0} \\ (\Delta \underline{\epsilon}_p^{-i})_r = 0 \end{array} \right\} \quad \dots \quad \text{For } r = 1$$

$$\left. \begin{aligned} \Delta \underline{\sigma}_r &= (\Delta \underline{\sigma})_{r-1} \\ \Delta \underline{\alpha}_r &= (\Delta \underline{\alpha})_{r-1} \\ (\Delta \underline{\epsilon}_p^{-i})_r &= (\Delta \underline{\epsilon}_p^{-i})_{r-1} \end{aligned} \right\} \dots \text{ For } r > 1$$

$$\Delta \underline{\epsilon}_r = \frac{(\Delta \underline{\epsilon})_{\text{total}}}{N}$$

ii) Calculate the following mean values:

$$\underline{\epsilon}_m = \underline{\epsilon}_{r-1} + \frac{\Delta \underline{\epsilon}_r}{2}$$

$$\underline{\sigma}_m = \underline{\sigma}_{r-1} + \frac{\Delta \underline{\sigma}_r}{2}$$

$$\underline{\alpha}_m = \underline{\alpha}_{r-1} + \frac{\Delta \underline{\alpha}_r}{2}$$

$$(\underline{\epsilon}_p^{-i})_m = (\underline{\epsilon}_p^{-i})_{r-1} + \frac{(\Delta \underline{\epsilon}_p^{-i})}{2}$$

iii) Calculate $(\Delta \underline{\epsilon}_p^{-i})_r$ [for isotropic and mixed hardening]

a) call the $\underline{D}_{ep}$ algorithm using the mean conditions

$(\underline{\sigma}_m, \underline{\epsilon}_m, (\underline{\epsilon}_p^{-i})_m, \underline{\alpha}_m)$ to obtain: $C, \underline{a}, \underline{d}, \underline{D}_{ep}$

b) $d\lambda = \underline{d}^t \cdot \Delta \underline{\epsilon}_r$

c) $(\Delta \underline{\epsilon}_p)_r = -d\lambda \cdot \underline{a} \cdot \left(\frac{\partial F}{\partial Y} \right)$

d) $(\Delta \underline{\epsilon}_p^{-i})_r = -C_M \cdot d\lambda \cdot \left(\frac{\partial F}{\partial Y} \right)$

iv) Calculate the shift increment [for kinematic and mixed hardening]

$$\Delta \underline{\alpha}_r = (1 - C_M) \cdot C \cdot (\Delta \underline{\varepsilon}_p)_r \quad \text{Prager's shift}$$

$$\Delta \underline{\alpha}_r = d\mu \cdot (\underline{\sigma}_m - \underline{\alpha}_m) \quad \text{Ziegler's shift}$$

where

$$d\mu = C \cdot (1 - C_M) \cdot \underline{a}^t \cdot \frac{(\Delta \underline{\varepsilon}_p)_r}{Y_m}$$

$Y_m = Y$ for a perfectly-plastic material with kinematic hardening, while for mixed hardening the uniaxial yield stress is:

$$Y_m = Y(\varepsilon_m) \quad \dots \text{from the material digram}$$

where

$$\varepsilon_m = \frac{\bar{\sigma}_m}{E} + (\bar{\varepsilon}_p^i)_m$$

and

$$\bar{\sigma}_m = f(\underline{\sigma}_m - \underline{\alpha}_m) \quad \dots \text{according to the yield function (table 3.1).}$$

v) Calculate the corrected stress increment

$$\Delta \underline{\sigma}_r = \underline{D} \cdot \Delta \underline{\varepsilon}_r - d\lambda (\underline{D} \underline{a})$$

or

$$\Delta \underline{\sigma}_r = \underline{D}_{ep}(\underline{\sigma}_m, \underline{\varepsilon}_m) \cdot \Delta \underline{\varepsilon}_r$$

Although these two formulae are equivalent, the first one may result in less rounding-off error than the second.

vi) Calculate the total value

$$\underline{\varepsilon}_r = \underline{\varepsilon}_{r-1} + \Delta \underline{\varepsilon}_r$$

$$\underline{\sigma}_r = \underline{\sigma}_{r-1} + \Delta \underline{\sigma}_r$$

$$\underline{\alpha}_r = \underline{\alpha}_{r-1} + \Delta \underline{\alpha}_r$$

$$(\bar{\varepsilon}_p^i)_r = (\bar{\varepsilon}_p^i)_{r-1} + (\Delta \bar{\varepsilon}_p^i)_r$$

vii) Calculate the uniaxial yield stress

$$Y_r = Y \quad \dots \text{ for a perfectly-plastic material or for the kinematic hardening condition.}$$

while for the isotropic or mixed hardening condition:-

$$Y_r = Y(\bar{\epsilon}_r) \quad \dots \text{ from material diagram}$$

where

$\bar{\epsilon}_r$ is the uniaxial strain equivalent to the multiaxial stress vector($\underline{\epsilon}_r$).

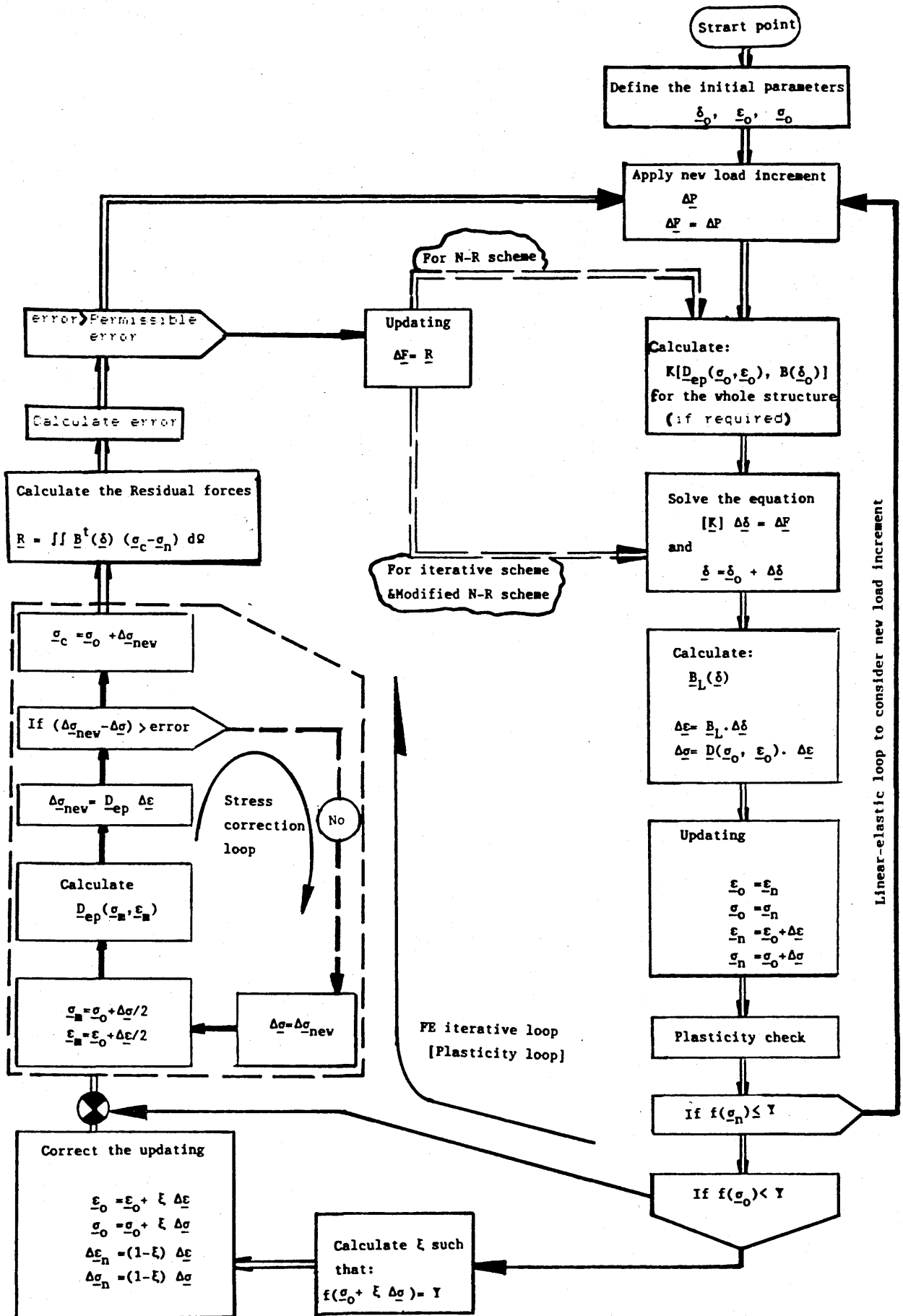
$$\bar{\epsilon}_r = \frac{\bar{\sigma}_r}{E} + (\bar{\epsilon}_p^i)_r$$

$$\bar{\sigma}_r = f(\underline{\sigma}_r - \underline{\alpha}_r) \quad \text{as obtained from the yield function}$$

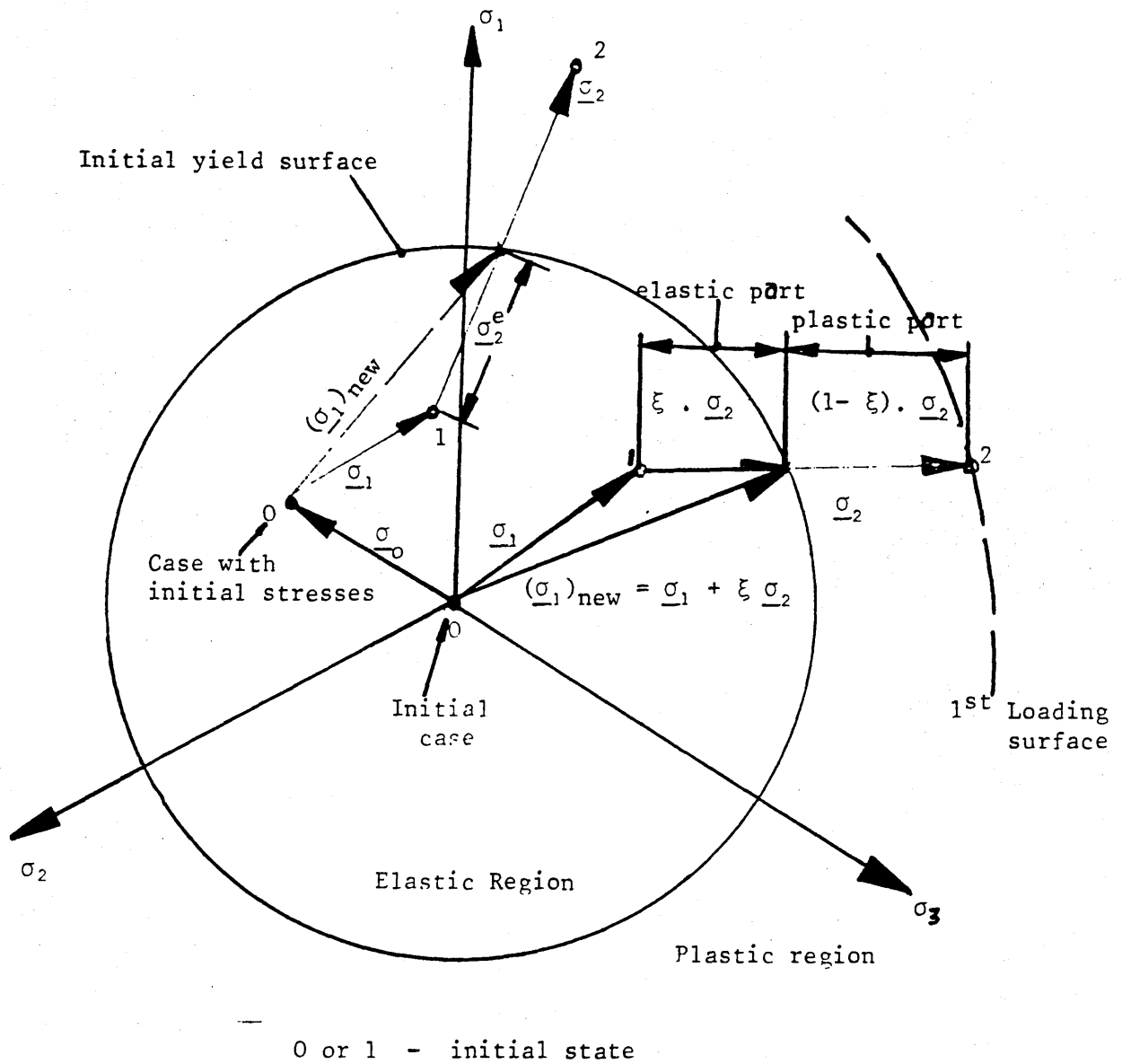
viii) The total value of the stress vector should be adjusted to match the uniaxial condition, as follows

$$(\underline{\sigma}_r)_{\text{corrected}} = \frac{Y_r}{\bar{\sigma}_r} (\underline{\sigma}_r - \underline{\alpha}_r) + \underline{\alpha}_r$$

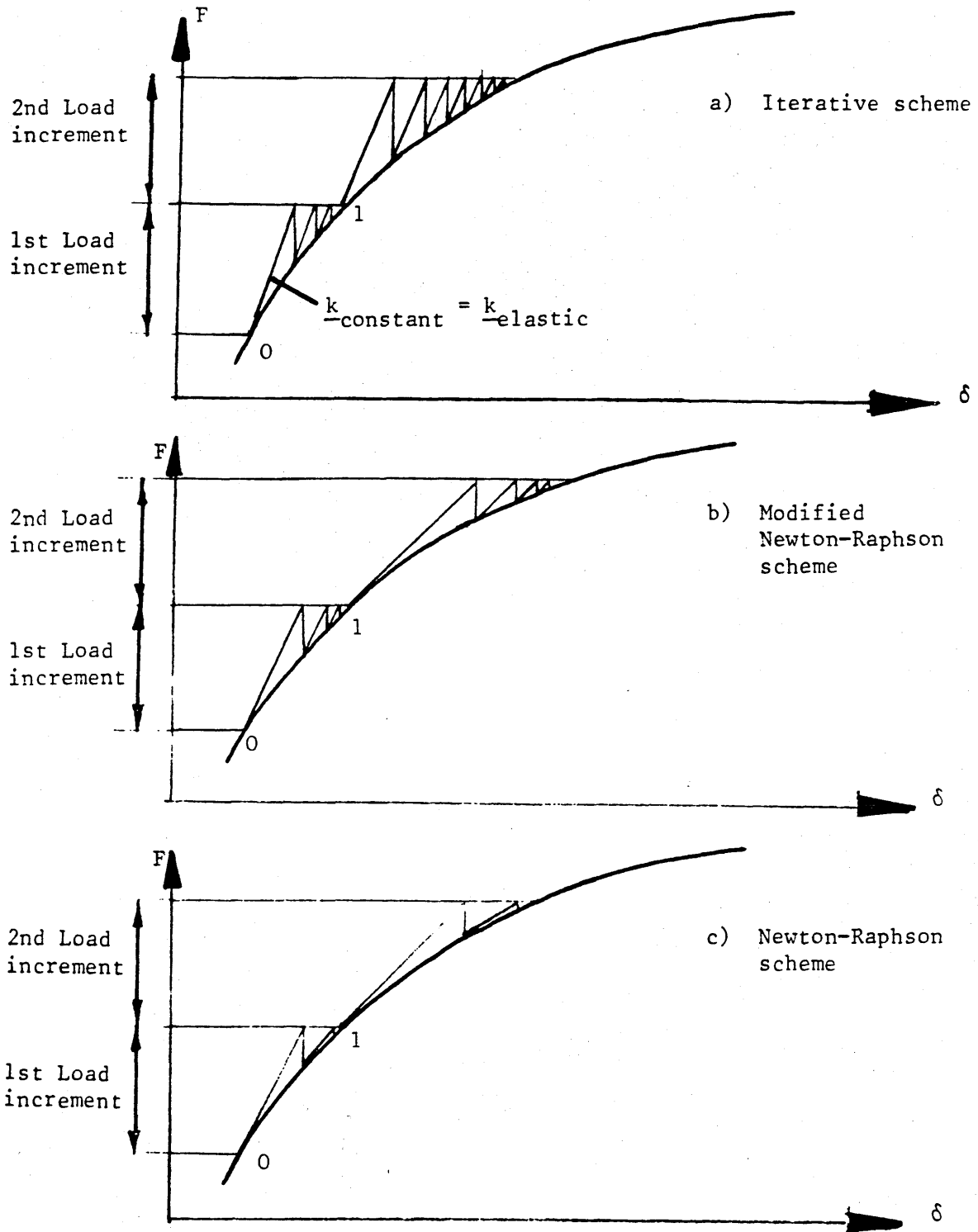
The calculations should be repeated from step (ii) until a reasonable convergence to an accurate result is obtained.



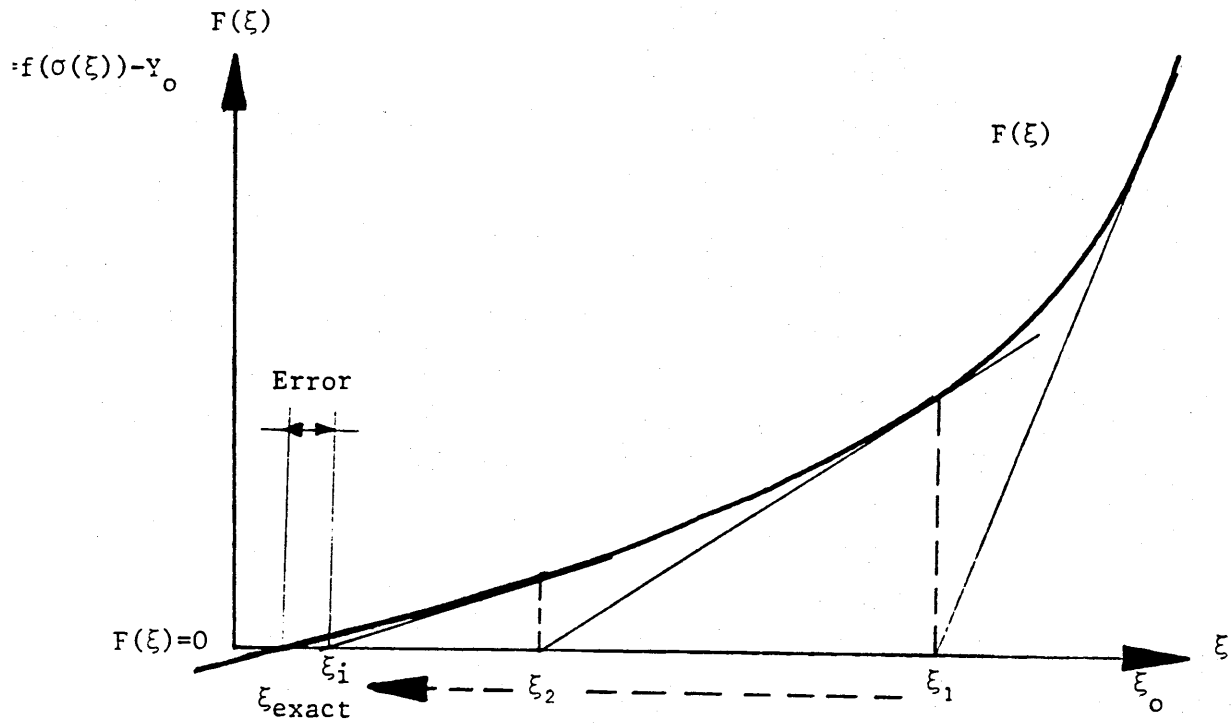
Fig(3.1) Solution procedure for Elastoplastic problems



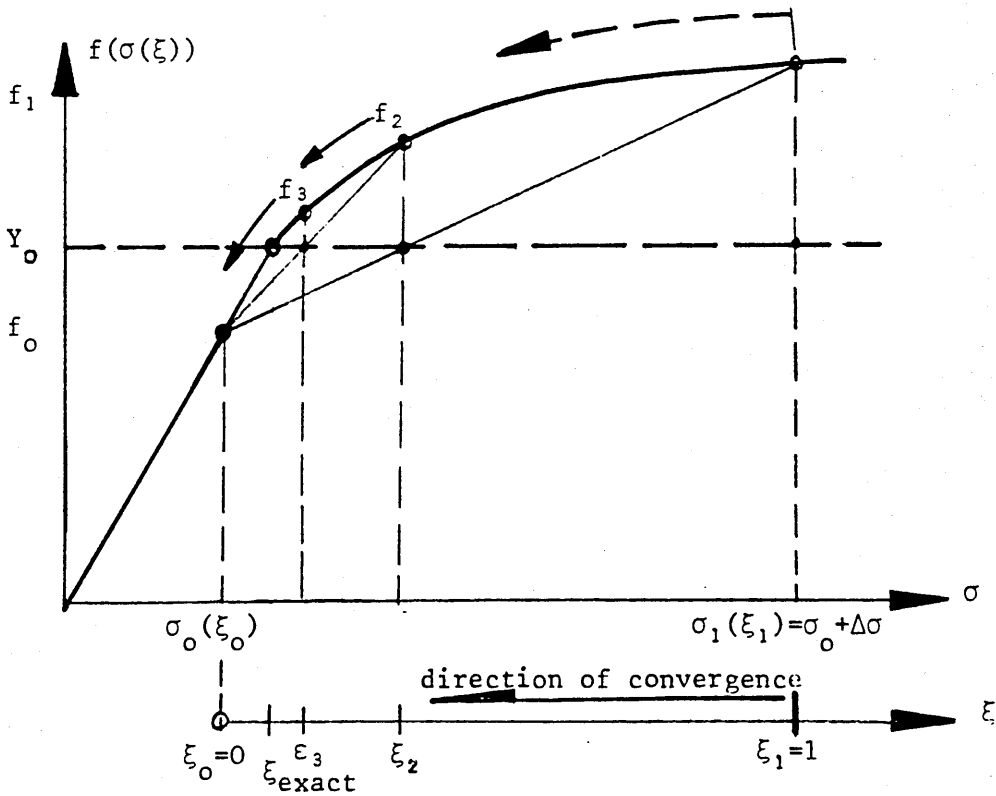
Fig(3.2) The Critical Condition of an Elastoplastic Load Increment in 3-Dim. Principal Stress Space



Fig(3.3) One-Dimensional Illustration of Solution Schemes



Fig(3.4) Illustration of Newton-Raphson Scheme for Parameter ξ



Fig(3.5) Illustration of False-Position Rule for Parameter ξ

NO	Yield Criterion	Uniaxial (or equivalent yield stress), Y	Stress level (effective stress)	$\frac{\partial F}{\partial Y}$
	Von Mises	Y_o	$\bar{\sigma}$	-1
	Tresca	Y_o	$\frac{2 \bar{\sigma} \cos \phi}{3}$	-1
	Tresca Polynomial	Y_o	$\frac{2 \bar{\sigma} \cos \phi}{\sqrt{3}}$	$-\frac{6Y_r (3J_2 - Y_r^2)}{(J_2 - Y_r^2)}$
	Modified Von Mises	$\sqrt{Y_c Y_t}$	$\bar{\sigma}^2 + 3 \cdot \sigma_m \cdot \Delta$	-1
	Modified Tresca Polynomial	$\sqrt{Y_c Y_t}$	$\frac{\sqrt{4J_2^3 - 27J_3^2 + Y_t^2 \Delta^2} \cdot 3 \cdot Y_c \cdot \sigma_m}{(3J_2 - Y_t Y_c)}$	-1
	Mohr-Coulomb	$\varphi \cos(\theta)$	$\sigma_m \sin \theta + \frac{\bar{\sigma}}{\sqrt{3}} \cos \phi - \frac{\bar{\sigma}}{3} \sin \phi$	-1
	Drucker-Prager	$\frac{6 \cdot \varphi \cdot \cos \theta}{\sqrt{3} \cdot (3 - \sin \theta)}$	$\frac{2/\sqrt{3} \sin \theta}{3 - \sin \theta} \cdot J_1 + \sqrt{J_2}$	-1

Table 3.1 Parameters for Different Yield Criteria

CHAPTER FOUR

BOUNDARY ELEMENT THEORY FOR TWO-DIMENSIONAL ELASTO-PLASTIC PROBLEMS

4.1 Derivation of Inverse Expression for Elasto-Plastic Small and Large Deflection Problems

The boundary element method (BEM) is based upon the numerical evaluation of governing boundary integral equations (BIE). the first step for the solution of an engineering problem by the BEM is therefore, the derivation of such governing boundary integral equations.

There are many approaches for such derivations available in the literature[81,84,91,92,94,120]. The technique employed in this work is based upon the point-collocation weighted-residual method. Having formulated the differential equation(s) for the specified engineering problem then an equivalent weighted-residual expression can be obtained and integrated-by-parts enough times so as to deduce what is known as the "Inverse Weighted-Residual Expression" (IWRE) as explained in this section.

a) Governing Differential Equations For Elasto-Plastic Problems

Using the strain-displacement relationships the components of the strain vector at any point within the specified domain can be expressed, in terms of displacement components as follows:

$$\underline{\varepsilon} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\ \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] \end{pmatrix}$$

The strain vector can be split into small and large deflection contributions as follows:

$$\underline{\varepsilon} = \underline{\varepsilon}^S + \underline{\varepsilon}^L$$

where

$$\underline{\varepsilon}^S = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad \text{Small deflection strain vector, and}$$

$$\underline{\varepsilon}^L = \begin{pmatrix} \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\ \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \end{pmatrix} \quad \text{Large deflection strain vector.}$$

It is clear that a part of the total strain vector ($\underline{\varepsilon}_e$) is recoverable (elastic), and the remaining part ($\underline{\varepsilon}_p$) is irrecoverable (plastic), i.e

$$\underline{\varepsilon} = \underline{\varepsilon}_e + \underline{\varepsilon}_p$$

Using the generalised Hooke's law, it can be shown that the stress vector at the given point can be expressed as:-

$$\underline{\sigma} = \underline{D} \underline{\varepsilon}_e$$

Hence, it can be deduced that:-

$$\underline{\sigma} = \underline{D} (\underline{\varepsilon} - \underline{\varepsilon}_p)$$

$$\underline{\sigma} = \underline{D} (\underline{\varepsilon}^S + \underline{\varepsilon}^L - \underline{\varepsilon}_p)$$

i.e $\underline{\sigma} = \underline{D} (\underline{\varepsilon}^S - \underline{\varepsilon}^{Lp})$

where $\underline{\varepsilon}^{Lp} = \underline{\varepsilon}_p - \underline{\varepsilon}^L \quad (4.1)$

From the elastic stress-strain relationships the above equation can be written explicitly as follows:

$$\left. \begin{aligned} \sigma_x &= d_{11} \left(\frac{\partial u}{\partial x} \right) + d_{12} \left(\frac{\partial v}{\partial y} \right) - \sigma_x^{Lp} \\ \sigma_y &= d_{21} \left(\frac{\partial u}{\partial x} \right) + d_{22} \left(\frac{\partial v}{\partial y} \right) - \sigma_y^{Lp} \\ \tau_{xy} &= d_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \tau_{xy}^{Lp} \end{aligned} \right\} \quad (4.2)$$

Where

$$\sigma_x^{Lp} = d_{11} \epsilon_x^{Lp} + d_{12} \epsilon_y^{Lp}$$

$$\sigma_y^{Lp} = d_{21} \epsilon_x^{Lp} + d_{22} \epsilon_y^{Lp}$$

$$\tau_{xy}^{Lp} = d_{33} \gamma_{xy}^{Lp}$$

d_{11} , d_{12} , d_{21} , d_{22} , d_{33} ... are the elements of the $\underline{D}$ matrix.

The equilibrium equations of an infinitesimal elemnet at the given point inside a body subjeti to a domain load are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

which are valid for linear and non-linear problems. Substituting equation (4.2) into equilibrium equations, the general differential equations can be deduced as follows:

$$\left. \begin{aligned} d_{11} \frac{\partial^2 u}{\partial x^2} + d_{12} \frac{\partial^2 v}{\partial x \partial y} + d_{33} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + f_x &= 0 \\ d_{21} \frac{\partial^2 u}{\partial y \partial x} + d_{22} \frac{\partial^2 v}{\partial y^2} + d_{33} \left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial x^2} \right) + f_y &= 0 \end{aligned} \right\} \quad (4.3)$$

Where

$$f_x = X - d_{11} \left(\frac{\partial \epsilon_x^{Lp}}{\partial x} \right) - d_{12} \left(\frac{\partial \epsilon_y^{Lp}}{\partial x} \right) - d_{33} \left(\frac{\partial \gamma_{xy}^{Lp}}{\partial y} \right)$$

$$f_y = Y - d_{21} \left(\frac{\partial \epsilon_x^{Lp}}{\partial y} \right) - d_{22} \left(\frac{\partial \epsilon_y^{Lp}}{\partial y} \right) - d_{33} \left(\frac{\partial \gamma_{xy}^{Lp}}{\partial x} \right)$$

b) Inverse Weighted-Residual Expression

If an approximate solution in terms of 'u' and 'v' exists and satisfies the given boundary conditions, then Residual functions can be obtained by substituting the approximate solution in the governing differential equations. Using appropriate weighting functions $\bar{u}$ and $\bar{v}$, a weighted-residual expression for such differential equations can be written as follows:-

$$\begin{aligned}
 & \iint_{\Omega} \bar{u} \left[d_{11} \frac{\partial^2 u}{\partial x^2} + d_{12} \frac{\partial^2 v}{\partial x \partial y} + d_{33} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \right] dx dy \\
 & + \iint_{\Omega} \bar{v} \left[d_{21} \frac{\partial^2 u}{\partial x \partial y} + d_{22} \frac{\partial^2 v}{\partial y^2} + d_{33} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) \right] dx dy \\
 & + \iint_{\Omega} \bar{u} \left[X - d_{11} \frac{\partial \epsilon_x^{Lp}}{\partial x} - d_{12} \frac{\partial \epsilon_y^{Lp}}{\partial x} - d_{33} \frac{\partial \gamma_{xy}^{Lp}}{\partial y} \right] dx dy \\
 & + \iint_{\Omega} \bar{v} \left[Y - d_{21} \frac{\partial \epsilon_x^{Lp}}{\partial y} - d_{22} \frac{\partial \epsilon_y^{Lp}}{\partial y} - d_{33} \frac{\partial \gamma_{xy}^{Lp}}{\partial x} \right] dx dy = 0
 \end{aligned}$$

Using integration-by-parts theorems, as indicated in appendix 4.1, the above weighted residual expression can be integrated twice and simplified, as shown in appendix 4.2, leading to the following inverse weighted-residual expression:

$$\begin{aligned}
 & \oint_{\Gamma} \left[\bar{u} \bar{T}_x + \bar{v} \bar{T}_y \right] ds + \\
 & \iint_{\Omega} \left[\bar{u} \bar{X} + \bar{v} \bar{Y} \right] dx dy - \iint_{\Omega} (\bar{\sigma})^t \cdot \underline{\epsilon}^{Lp} dx dy = \\
 & \oint_{\Gamma} \left[\bar{u} T_x + \bar{v} T_y \right] ds + \iint_{\Omega} \left[\bar{u} X + \bar{v} Y \right] dx dy \quad (4.4)
 \end{aligned}$$

Which is a generalised form for the Maxwell-Betti reciprocal theorem, in which the starred parameters denote an alternative elastic solution for the same structure with the same boundary conditions but under a starred system of loading, i.e.

$$\underline{\underline{\sigma}}^* = \underline{\underline{D}} \underline{\underline{\varepsilon}}^*$$

$$\underline{\underline{T}}_x^* = \ell \underline{\underline{\sigma}}_x^* + m \underline{\underline{\tau}}_{xy}^*$$

$$\underline{\underline{T}}_y^* = \ell \underline{\underline{\tau}}_{xy}^* + m \underline{\underline{\sigma}}_y^*$$

$$\underline{\underline{X}}^* = - \left(\frac{\partial \underline{\underline{\sigma}}_x^*}{\partial x} + \frac{\partial \underline{\underline{\tau}}_{xy}^*}{\partial y} \right)$$

$$\underline{\underline{Y}}^* = - \left(\frac{\partial \underline{\underline{\tau}}_{xy}^*}{\partial x} + \frac{\partial \underline{\underline{\sigma}}_y^*}{\partial y} \right), \text{ etc.}$$

4.2 Fundamental Solution

The domain integral term:-

$$\iint_{\Omega} (u \underline{\underline{X}}^* + v \underline{\underline{Y}}^*) dx dy$$

in equation(4.4), can be reduced in a way similar to that employed in the point-collocation, weight-residual, method by using weighting functions $\underline{\underline{u}}^*$ and $\underline{\underline{v}}^*$ such that $\underline{\underline{X}}^*$ and $\underline{\underline{Y}}^*$ are proportional to Dirac delta functions. This is the case which defines $\underline{\underline{u}}^*$ and $\underline{\underline{v}}^*$ as the displacement components due to a concentrated load (with components e_x and e_y) acting at (x_i, y_i) inside the given body. Since this problem is difficult to solve generally, an alternative approach has been employed by boundary element workers[120,121,122]. In this approach the two-dimensional domain Ω is considered to be part of an infinite domain for which the concentrated-load problem is solved. The two-dimensional domain Ω with, its associated boundary tractions is cut out of the infinite domain, and will be found to have the correct stress distribution. The differential equations

of the concentrated-load problem in the infinite domain are usually singular (i.e. with terms being proportional to Dirac delta functions) and their solution is usually known as "the fundamental solution".

4.2.1 Fundamental Equation Of Equilibrium

Consider a two-dimensional elasticity problem in the infinite x - y domain, with a state of loading due to body forces distributed uniformly within a circle of radius " ϵ " centred at (x_i, y_i) , as shown in Fig(4.1).

Let e_x, e_y be the resultant force components (per unit thickness) in the x and y directions respectively. The domain load intensities can, therefore, be defined as follows:

$$\left. \begin{aligned} X &= \frac{e_x}{\pi \epsilon^2} \\ Y &= \frac{e_y}{\pi \epsilon^2} \end{aligned} \right\} \quad \dots \text{ for } r < \epsilon$$

$$\left. \begin{aligned} X &= 0 \\ Y &= 0 \end{aligned} \right\} \quad \dots \text{ for } r > \epsilon$$

where
$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

Considering the limiting case when $\epsilon \rightarrow 0$ (concentrated load at x_i, y_i) a two-dimensional Dirac delta function $\delta(x-x_i, y-y_i)$ can be defined such that:-

$$\delta(x-x_i, y-y_i) = \begin{cases} \lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon^2} & \dots \text{ for } r \rightarrow 0 \\ 0 & \dots \text{ for } r > 0 \end{cases}$$

Hence, the corresponding load intensities can be defined as follows:

$$\left. \begin{aligned} \hat{X}(x-x_i, y-y_i) &= e_x \delta(x-x_i, y-y_i) \\ \hat{Y}(x-x_i, y-y_i) &= e_y \delta(x-x_i, y-y_i) \end{aligned} \right\} \quad (4.5)$$

Substituting in the equation of equilibrium (2.16), then the governing equations of equilibrium for the case of the fundamental solution can be written as follows:-

$$\left. \begin{aligned} \frac{\partial \hat{\sigma}_x}{\partial x} + \frac{\partial \hat{\tau}_{xy}}{\partial y} + e_x \delta(x-x_i, y-y_i) &= 0 \\ \frac{\partial \hat{\tau}_{xy}}{\partial x} + \frac{\partial \hat{\sigma}_y}{\partial y} + e_y \delta(x-x_i, y-y_i) &= 0 \end{aligned} \right\} \quad (4.6)$$

The properties of the two-dimensional Dirac delta function are summarised in appendix 4.4 from which it can be shown that:

$$\iint_{\Omega} f(x, y) \delta(x-x_i, y-y_i) dx dy = C_i f(x_i, y_i)$$

Where C_i is a geometric coefficient,

$$C_i = 1 \quad \text{if } (x_i, y_i) \text{ is inside } \Omega$$

$$C_i = 0 \quad \text{if } (x_i, y_i) \text{ is outside } \Omega$$

$$C_i = 0.5 \quad \text{if } (x_i, y_i) \text{ is on a smooth boundary } \Gamma$$

4.2.2 Fundamental Displacements

There are many some methods in the literature for obtaining a solution for equation(4.6), that is, the Fundamental solution [123,124,125]. The method employed in this work is based upon the "Galerkin vector" approach [111,126], because it has been found to be suitable for cases with domain loading, as explained later.

Defining the Galerkin vector $\underline{\hat{G}}$ such that:

$$\underline{\hat{G}} = \{ \hat{G}_x \hat{G}_y \}$$

and the displacement components, as explained in appendix 4.3 being:

$$\left. \begin{aligned} \bar{u}(x,y) &= \nabla^2 \bar{G}_x - \frac{1}{2(1-p)} \frac{\partial}{\partial x} \left(\frac{\partial \bar{G}_x}{\partial x} + \frac{\partial \bar{G}_y}{\partial y} \right) \\ \bar{v}(x,y) &= \nabla^2 \bar{G}_y - \frac{1}{2(1-p)} \frac{\partial}{\partial y} \left(\frac{\partial \bar{G}_x}{\partial x} + \frac{\partial \bar{G}_y}{\partial y} \right) \end{aligned} \right\} \quad (4.7)$$

Where $p = \nu$ (Poisson's ratio) for plane-strain problems, and

$p = \frac{\nu}{1+\nu}$ for plane-stress problems.

Then the governing differential equations for the concentrated-load problem, described above, can be reduced to the following bi-harmonic equations:

$$\left. \begin{aligned} \nabla^2 (\nabla^2 \bar{G}_x) + \frac{e_x}{\mu} \delta(x-x_i, y-y_i) &= 0 \\ \nabla^2 (\nabla^2 \bar{G}_y) + \frac{e_y}{\mu} \delta(x-x_i, y-y_i) &= 0 \end{aligned} \right\} \quad (4.8)$$

as shown in appendix 4.3.

Usually the point (x_i, y_i) , at which the concentrated load is applied (or the solution is infinite), is known as the source point, whilst any other point (x, y) at which the solution is required, is called the field point, as illustrated in Fig(4.2).

A general solution to equation (4.8) has been obtained as explained in appendix 4.3, as follows:

$$\bar{G}_x = \frac{e_x}{8\pi\mu} g(r)$$

$$\bar{G}_y = \frac{e_y}{8\pi\mu} g(r)$$

where

$$g(r) = -r^2 (\text{Log}(r) - C_1) + C_2 \text{Log}(r) + C_3$$

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

C_1 , C_2 , and C_3 are arbitrary constants

From which, it can be deduced, as shown in appendix 4.3, that:

$$\left. \begin{aligned} \bar{u}^* &= G_{11} e_x + G_{12} e_y \\ \bar{v}^* &= G_{21} e_x + G_{22} e_y \end{aligned} \right\} \quad (4.9)$$

where

$$G_{\gamma\beta} = \frac{1}{16\pi\mu(1-p)} \left[2(1-p) \nabla^2 g \delta_{\gamma\beta} - \frac{\partial}{\partial x_\gamma} \left(\frac{\partial g}{\partial x_\beta} \right) \right]$$

For the special case of $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$,
i.e $g(r) = r^2 (1 - \text{Log}(r))$, it can be deduced that:

$$\begin{aligned} G_{\gamma\beta}(x-x_i, y-y_i) &= \frac{1}{C_a} \left[[C_1 \text{Log}\left(\frac{1}{r}\right) - 0.5] \delta_{\gamma\beta} \right. \\ &\quad \left. + \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \right] \end{aligned} \quad (4.10)$$

Where

$$C_a = 8\pi\mu(1-p) ,$$

$$C_1 = 3-4p , \text{ and}$$

$$\delta_{\gamma\beta} = \text{Kronecker delta.}$$

Explicit expression for $G_{\gamma\beta}$ are given in Table(4.1).

4.2.3 Fundamental Strain and Stress Components

The total strain components at a field point (x,y) due to the concentrated point-load acting at the source point (x_i, y_i) , are usually called fundamental strains and can be defined in terms of the fundamental displacements $\bar{u}^*$ and $\bar{v}^*$ as follows:-

$$\epsilon_x^* = \frac{\partial \bar{u}^*}{\partial x} = \frac{\partial G_{11}}{\partial x} e_x + \frac{\partial G_{12}}{\partial x} e_y$$

$$\epsilon_y^* = \frac{\partial v^*}{\partial y} = \frac{\partial G_{21}}{\partial y} e_x + \frac{\partial G_{22}}{\partial y} e_y$$

$$\gamma_{xy}^* = \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} = \left(\frac{\partial G_{11}}{\partial y} + \frac{\partial G_{12}}{\partial x} \right) e_x + \left(\frac{\partial G_{21}}{\partial y} + \frac{\partial G_{22}}{\partial x} \right) e_y$$

which can be rewritten in the following tensorial form:-

$$\epsilon_{\gamma\beta}^*(x-x_i, y-y_i) = \sum_{k=1}^2 A_{\gamma\beta k} e_k$$

where

$$\begin{aligned} A_{\gamma\beta k}(x-x_i, y-y_i) = & \frac{-1}{C_{a \cdot r}} \left[2 \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_k} \right. \\ & + (1-2p) \cdot \left(\frac{\partial r}{\partial x_\gamma} \delta_{\beta k} + \frac{\partial r}{\partial x_\beta} \delta_{\gamma k} \right) \\ & \left. - \frac{\partial r}{\partial x_k} \delta_{\gamma\beta} \right] \end{aligned}$$

,as proved in appendix 4.6.

The above equations can be written explicitly as follows:

$$\left. \begin{aligned} \epsilon_x^* &= -(R_{11} e_x + R_{12} e_y) \\ \epsilon_y^* &= -(R_{12} e_x + R_{22} e_y) \\ \gamma_{xy}^* &= -(R_{13} e_x + R_{23} e_y) \end{aligned} \right] \quad (4.11)$$

Where R_{ij} are as defined in Table(4.2).

The corresponding stress components, the fundamental stress, at the field point (x,y) can be evaluated by using the generalised Hooke's law as follows:

$$\underline{\sigma}^* = \underline{D} \underline{\epsilon}^*$$

Where

$$\underline{\sigma}^* = \begin{pmatrix} \sigma_{-x}^* \\ \sigma_{-y}^* \\ \tau_{-xy}^* \end{pmatrix} \quad \begin{array}{l} \text{Fundamental stress components} \\ \text{vector} \end{array}$$

$$\underline{\varepsilon}^* = \begin{pmatrix} \varepsilon_{-x}^* \\ \varepsilon_{-y}^* \\ \gamma_{-xy}^* \end{pmatrix} \quad \begin{array}{l} \text{Fundamental strain components} \\ \text{vector} \end{array}$$

It can be shown, as explained in appendix 4.7 that the tensorial form for the above equation is as follows:

$$\sigma_{\gamma\beta}^* = \sum_{k=1}^2 D_{\gamma\beta k} e_k \quad (4.12)$$

Where

$$D_{\gamma\beta k} = \frac{-1}{C_b \cdot r} \left[2 \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_k} + (1-2p) \cdot \left(\frac{\partial r}{\partial x_\beta} \delta_{k\gamma} + \frac{\partial r}{\partial x_\gamma} \delta_{k\beta} + \frac{\partial r}{\partial x_k} \delta_{\gamma\beta} \right) \right]$$

$$C_b = 4\pi(1-p)$$

4.2.4 Fundamental Traction

From The physical interpretation of the generalised Maxwell-Betti reciprocal theorem, equation (4.4), it is clear that $\underline{u}^*$, $\underline{v}^*$ are solutions for the same domain Ω . The infinite-domain solution can be assumed to be still valid, if the domain Ω is cut with all acting internal forces, from the infinite domain, as shown in Fig(4.3). The internal forces will be substituted at the boundary of the domain Ω by mechanically equivalent traction

components $\hat{T}_x^*$, $\hat{T}_y^*$, which are known as fundamental tractions, i.e.

$$\left. \begin{aligned} \hat{T}_x^* &= l \sigma_x^* + m \tau_{xy}^* \\ \hat{T}_y^* &= l \tau_{xy}^* + m \sigma_y^* \end{aligned} \right\} \quad (4.13)$$

, as described before.

Hence, substituting from equations (4.12), into (4.13), it can be deduced, as explained in appendix 4.8 that:

$$\left. \begin{aligned} \hat{T}_x^* &= F_{11} e_x + F_{12} e_y \\ \hat{T}_y^* &= F_{21} e_x + F_{22} e_y \end{aligned} \right\} \quad (4.14)$$

Where

$$\begin{aligned} F_{\gamma\beta} &= \frac{-1}{C_b r} \left[2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \right. \\ &\quad + (1-2p) \left(l_\beta \frac{\partial r}{\partial x_\gamma} - l_\gamma \frac{\partial r}{\partial x_\beta} \right) \\ &\quad \left. + (1-2p) \frac{\partial r}{\partial n} \delta_{\gamma\beta} \right] \end{aligned} \quad (4.15)$$

$$\frac{\partial r}{\partial n} = l \cdot \frac{\partial r}{\partial x} + m \cdot \frac{\partial r}{\partial y}$$

4.3 Boundary Integral Equations for Elasto-Plastic Problems

Substituting from equations (4.5), (4.9), and (4.13), the domain integral equation given by equation (4.4) can be rewritten as follows:-

$$\begin{aligned} &\iint_Q (u e_x + v e_y) \delta(x-x_i, y-y_i) dx dy \\ &+ \oint_\Gamma u (F_{11} e_x + F_{12} e_y) ds + \oint_\Gamma v (F_{21} e_x + F_{22} e_y) ds \\ &\text{.....continue} \end{aligned}$$

$$\begin{aligned}
& - \iint_{\Omega} (\underline{\sigma}^*)^t \cdot \underline{\varepsilon}^{Lp} \, dx dy \\
& = \oint_{\Gamma} T_x (G_{11} e_x + G_{12} e_y) \, ds \\
& \quad + \oint_{\Gamma} T_y (G_{21} e_x + G_{22} e_y) \, ds \\
& \quad + \iint_{\Omega} X (G_{11} e_x + G_{12} e_y) \, dx dy \\
& \quad + \iint_{\Omega} Y (G_{21} e_x + G_{22} e_y) \, dx dy
\end{aligned} \tag{4.16}$$

From the properties of Dirac delta functions, it can be deduced that:

$$\left. \begin{aligned}
\iint_{\Omega} u(x,y) \cdot \delta(x-x_i, y-y_i) \, dx dy &= C_i \cdot u(x_i, y_i) \\
&\equiv C_i \cdot u_i \\
\text{and} \\
\iint_{\Omega} v(x,y) \cdot \delta(x-x_i, y-y_i) \, dx dy &= C_i \cdot v(x_i, y_i) \\
&\equiv C_i \cdot v_i
\end{aligned} \right\} \tag{4.17}$$

From the definition of $\underline{\sigma}^*$, it can be shown that:

$$(\underline{\sigma}^*)^t \cdot \underline{\varepsilon}^{Lp} = (\underline{\varepsilon}^*)^t \cdot \underline{D} \cdot \underline{\varepsilon}^{Lp} \tag{4.18}$$

Using equation (4.1), then it can be deduced that:

$$\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}^S - \underline{D} \cdot \underline{\varepsilon}^{Lp}$$

i.e.

$$\underline{D} \cdot \underline{\varepsilon}^{Lp} \equiv \underline{\sigma}' - \underline{\sigma} = \underline{\sigma}^R \tag{4.19}$$

where

$$\underline{\sigma}' = \underline{D} \cdot \underline{\varepsilon}^S$$

$\underline{\sigma}^R$ represents the difference between the stress vector for an elastic small-deflection solution and the actual stress vector.

Substituting equations (4.11) and (4.19) into equation (4.18), it can be proved that:

$$-(\underline{\sigma}^*)^t \cdot \underline{\varepsilon}^{Lp} = \left. \begin{aligned} & (R_{11} e_x + R_{12} e_y) \sigma_x^R \\ & + (R_{12} e_x + R_{22} e_y) \sigma_y^R \\ & + (R_{13} e_x + R_{23} e_y) \tau_{xy}^R \end{aligned} \right\} \quad (4.20)$$

From equations (4.17) and (4.20) the integral equation (4.16) can be rearranged as follows:

$$\begin{aligned} & e_x \left[C_i u_i + \oint_{\Gamma} (u F_{11} + v F_{21}) ds \right. \\ & \left. + \iint_{\Omega} (\sigma_x^R R_{11} + \sigma_y^R R_{12} + \tau_{xy}^R R_{13}) dx dy \right] \\ & + e_y \left[C_i v_i + \oint_{\Gamma} (u F_{12} + v F_{22}) ds \right. \\ & \left. + \iint_{\Omega} (\sigma_x^R R_{12} + \sigma_y^R R_{22} + \tau_{xy}^R R_{23}) dx dy \right] \\ & = e_x \left[\oint_{\Gamma} (T_x G_{11} + T_y G_{21}) ds + \iint_{\Omega} (X G_{11} + Y G_{21}) dx dy \right] \\ & + e_y \left[\oint_{\Gamma} (T_x G_{12} + T_y G_{22}) ds + \iint_{\Omega} (X G_{12} + Y G_{22}) dx dy \right] \end{aligned}$$

Due to the arbitrariness of (e_x, e_y) and (x_i, y_i) , the following

two integral equations can be deduced for every source point (x_i, y_i) :

$$\begin{aligned}
 & C_i u_i + \oint_{\Gamma} (u F_{11} + v F_{21}) ds = \oint_{\Gamma} (T_x G_{11} + T_y G_{21}) ds \\
 & + U(x_i, y_i) - \iint_{\Omega} (\sigma_x^R R_{11} + \sigma_y^R R_{12} + \tau_{xy}^R R_{13}) dx dy \\
 \text{and} \\
 & C_i v_i + \oint_{\Gamma} (u F_{12} + v F_{22}) ds = \oint_{\Gamma} (T_x G_{12} + T_y G_{22}) ds \\
 & + V(x_i, y_i) - \iint_{\Omega} (\sigma_x^R R_{21} + \sigma_y^R R_{22} + \tau_{xy}^R R_{23}) dx dy
 \end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
 U(x_i, y_i) &= \iint_{\Omega} \left(X(x, y) \cdot G_{11}(x-x_i, y-y_i) \right. \\
 &\quad \left. + Y(x, y) G_{21}(x-x_i, y-y_i) \right) dx dy \\
 V(x_i, y_i) &= \iint_{\Omega} \left(X(x, y) \cdot G_{12}(x-x_i, y-y_i) \right. \\
 &\quad \left. + Y(x, y) \cdot G_{22}(x-x_i, y-y_i) \right) dx dy
 \end{aligned} \tag{4.22}$$

Although the terms U, V defined by equation (4.22) can be evaluated at any source point (x_i, y_i) , equations (4.21) cannot be considered as boundary integral equations due to the presence of non-linear terms containing $\underline{\sigma}^R$. A linearised approach will be employed, as explained in section (4.6), in which $\underline{\sigma}^R$ will be assumed from a previous iteration, leading to an approximate evaluation of its domain integral terms in equations (4.21). This approach produces approximate boundary integral equations which can be analysed by using the standard algorithm of the boundary element method discussed in section 4.5.

4.4 Boundary Integral Expressions for Domain Loading Terms

The domain-integral terms in equations (4.22) can be evaluated numerically by dividing the domain under consideration into a number of simple cells, known as integration cells, and approximating the integral over each cell, as follows:

$$\begin{aligned} \iint_{\Omega} f(x,y) dx dy &= \sum_{c=1}^{N_c} \iint_{\Omega_c} f(x,y) dx dy \\ &\approx \sum_{c=1}^{N_c} f(x_c, y_c) \cdot A_c \end{aligned} \quad (4.23)$$

Where

N_c is the number of integral cells ,
 (x_c, y_c) are the Coordinates of the cell centroid ,
 A_c is the area of the c^{th} cell .

There are some types of domain loading, for which it is possible to express their domain integral in terms of equivalent boundary integrals. Amongst such types are inertial loading, centrifugal loading and steady-state thermal loading. In this work only inertial loading (due to constant linear acceleration) and centrifugal loading (due to uniform rotational speed) will be considered. Substituting the expression of $G_{\gamma\beta}$ in terms of the Galerkin vector (as defined by equation (4.10)) into equations (4.22), then it can be deduced that:

$$\begin{aligned} U_{\gamma} &= \sum_{\beta=1}^2 \iint_{\Omega} X_{\beta} G_{\gamma\beta} dx dy \\ &= \frac{1}{2C_a} \left[2(1-p) \iint_{\Omega} X_{\gamma} v^2 g dx dy - \sum_{\beta=1}^2 \iint_{\Omega} X_{\beta} \frac{\partial^2 g}{\partial x_{\gamma} \partial x_{\beta}} dx dy \right] \end{aligned}$$

For the domain-loading cases for which :

$$\nabla^2 X_Y = 0 \quad , \quad \frac{\partial^2 X_\beta}{\partial x_Y \partial x_\beta} = 0$$

The above equation can be integrated by parts (twice) and reduced to the following boundary integral expression:-

$$U_Y = \frac{1}{2C_a} \left(2(1-p) \int_{\Gamma} \left[-X_Y \frac{\partial g}{\partial n} + \frac{\partial X_Y}{\partial n} g \right] ds \right. \\ \left. + \sum_{\beta=1}^2 \int_{\Gamma} \left[\ell_\beta \cdot X_\beta \cdot \frac{\partial g}{\partial x_Y} - \ell_Y \frac{\partial X_\beta}{\partial x_\beta} g \right] ds \right) \quad (4.24)$$

,as proved in appendix 4.9.

There are two special cases of domain loading which can be summarised as follows:-

a) Uniform Domain Loading

Considering X and Y to be constants over the whole domain, then equation (4.24), can be simplified and expressed explicitly as follows :

$$U = \oint_{\Gamma} (f_{xx} X + f_{xy} Y) ds$$

$$V = \oint_{\Gamma} (f_{yx} X + f_{yy} Y) ds$$

where

$$f_{Y\beta} = \frac{1}{2C_a} \left[-2(1-p) \frac{\partial g}{\partial n} \delta_{Y\beta} + \ell_\beta \frac{\partial g}{\partial x_Y} \right]$$

Explicit expressions for $f_{Y\beta}$ are given in Table(4.3). The corresponding engineering application for this type of domain loading would be a structure moving with a constant acceleration $\vec{a}$,

since it can be shown that :

$$\vec{X} = - \rho \vec{a}$$

i.e $X = -\rho a_x$, $Y = -\rho a_y$

b) Linear Domain Loading

Consider the special case, for which:-

$$X(x,y) = a_0 + a_1 x + a_2 y$$

$$Y(x,y) = b_0 + b_1 x + b_2 y$$

where $a_0, a_1, a_2, b_0, b_1, b_2$ are given constant parameters .

Hence, equation (4.24) can be reduced to:-

$$U = \oint_{\Gamma} (f_{xx} X + f_{xy} Y) ds - \oint_{\Gamma} C_x \bar{Q} ds$$

$$V = \oint_{\Gamma} (f_{yx} X + f_{yy} Y) ds - \oint_{\Gamma} C_y \bar{Q} ds$$

Where

$$\bar{Q} = \frac{1}{2C_a} g(r) ,$$

$$C_x = 2(1-p)(\oint a_1 + m a_2) - (\oint a_1 + m b_1) ,$$

$$C_y = 2(1-p)(\oint b_1 + m b_2) - (\oint a_2 + m b_2)$$

,as explained in appendix 4.9.

The practical engineering application for this type of domain loading would be that of a structure rotating with a constant angular velocity, as shown in Fig (4.4.a).

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Defining, the linear acceleration vector as:

$$\vec{a} = \vec{\omega} \wedge (\vec{\omega} \wedge \vec{R}) = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

where

$$\vec{R} = (x-x_0) \hat{i} + (y-y_0) \hat{j}, \text{ and}$$

(x_0, y_0) is a point on the axis of rotation ,

it can be deduced that:

$$X = \rho \left[(\omega_y^2 + \omega_z^2) (x-x_0) - \omega_x \omega_y (y-y_0) \right]$$

$$Y = \rho \left[(\omega_x^2 + \omega_z^2) (y-y_0) - \omega_x \omega_y (x-x_0) \right]$$

$$Z = \rho \left[-\omega_x \omega_z (x-x_0) - \omega_y \omega_z (y-y_0) \right]$$

In eliminating the loading component in the z-direction (a necessary condition for 2-dimensional analysis) only the two following cases can be considered:

i) Rotation About The z-axis:

$$\text{i.e. } \omega_x = \omega_y = 0, \omega_z \neq 0$$

$$\text{Hence: } X = \rho (x-x_0) \omega_z^2$$

$$Y = \rho (y-y_0) \omega_z^2$$

as shown in Fig (4.4.b).

ii) Rotation about an axis in the x-y Plane:-

$$\text{i.e. } \omega_z = 0$$

$$\text{Hence } X = \rho \left[(x-x_0) \omega_y^2 - (y-y_0) \omega_x \omega_y \right],$$

$$Y = \rho \left[(y-y_0) \omega_x^2 - (x-x_0) \omega_x \omega_y \right].$$

,as shown in Fig (4.4.c).

4.5 Solution of the Boundary Integral Equations (B.I.E)

In order to solve the governing B.I.E's the discretization concept is employed, this concept is based upon division of the boundary of the domain into a suitable number of sub-boundaries connected by boundary nodes, as shown in Fig(4.5.a).

Hence, if $f(x,y)$ is a continuous function defined over Γ , then it can be deduced that:

$$\oint_{\Gamma} f(x,y) ds = \sum_{e=1}^{N_e} \int_{\Gamma_e} f(x,y) ds$$

Where

N_e is the number of sub-boundaries, and

Γ_e is the e^{th} sub-boundary.

Applying the above concept to equations(4.21), without considering the elasto-plastic terms, which will be dealt with later, then these equations can be reduced to:

$$\begin{aligned} C_i u_i + \sum_{e=1}^{N_e} & \left[\int_{\Gamma_e} F_{11}(x-x_i, y-y_i) \cdot u(\Gamma_e) ds \right. \\ & \left. + \int_{\Gamma_e} F_{21}(x-x_i, y-y_i) \cdot v(\Gamma_e) ds \right] \\ = \sum_{e=1}^{N_e} & \left[\int_{\Gamma_e} G_{11}(x-x_i, y-y_i) \cdot T_x(\Gamma_e) ds \right. \\ & \left. + \int_{\Gamma_e} G_{21}(x-x_i, y-y_i) \cdot T_y(\Gamma_e) ds + U(x_i, y_i) \right] \quad (4.25) \end{aligned}$$

and

$$\begin{aligned} C_i v_i + \sum_{e=1}^{N_e} & \left[\int_{\Gamma_e} F_{12}(x-x_i, y-y_i) \cdot u(\Gamma_e) ds \right. \\ & \left. + \int_{\Gamma_e} F_{22}(x-x_i, y-y_i) \cdot v(\Gamma_e) ds \right] \dots \text{continue} \end{aligned}$$

$$= \sum_{e=1}^N \left[\int_{\Gamma_e} G_{12}(x-x_i, y-y_i) \cdot T_x(\Gamma_e) ds + \int_{\Gamma_e} G_{22}(x-x_i, y-y_i) \cdot T_y(\Gamma_e) ds + V(x_i, y_i) \right] \quad (4.25)$$

Where $u(\Gamma_e), v(\Gamma_e)$ are the displacement components over Γ_e
 $T_x(\Gamma_e), T_y(\Gamma_e)$ are the traction components over Γ_e .

A sub-boundary is usually defined in terms of a number of boundary nodes and is known as "a boundary element", as shown in Fig(4.5.b). If the boundary nodes are also used as the field-function nodes, then the boundary element may be called an isoparametric element. A suitable transformation can be used to transform the element from the x-y plane into a uniform element in the intrinsic, ξ -axis, as shown in Fig (4.6).

The intrinsic parameter ξ can be defined, such that the nodes are at equal distances in the ξ -line, i.e at the j^{th} local node:

$$\xi_j = \frac{j-1}{n-1}$$

where n is the number of boundary nodes for the element.

A field point (x, y) , which is any point moving on Γ_e , can be expressed in terms of ξ by means of the following Lagrangian interpolation equations:

$$\left. \begin{aligned} x(\xi) &= \sum_{j=1}^n N_j(\xi) \cdot x_j \\ y(\xi) &= \sum_{j=1}^n N_j(\xi) \cdot y_j \end{aligned} \right\} \quad (4.26)$$

where $N_j(\xi) = \prod_{\substack{r=1 \\ r \neq j}}^n \frac{(n-1)\xi - (r-1)}{(j-r)}$ is the Lagrangian shape function

Equations(4.26) can therefore be shown to be a parametric representation of the element equation in the x-y plane in terms of the parameter ξ .

The field variables can be approximated in terms of their nodal values, at the nodes used for the geometrical definition of the element. Hence the following isoparametric equations could, therefore, be employed for such approximations:

$$u(\Gamma_e) = \sum_{j=1}^n N_j(\xi) \cdot u_j ,$$

$$v(\Gamma_e) = \sum_{j=1}^n N_j(\xi) \cdot v_j ,$$

$$T_x(\Gamma_e) = \sum_{j=1}^n N_j(\xi) \cdot (T_x)_j , \text{ and}$$

$$T_y(\Gamma_e) = \sum_{j=1}^n N_j(\xi) \cdot (T_y)_j .$$

By substituting the above equations into equations (4.25), they can be rewritten in the following matrix equation at the source point (x_i, y_i) :

$$\begin{bmatrix} C_i & 0 \\ 0 & C_i \end{bmatrix} \underline{\delta}_i = \sum_{e=1}^N [\underline{g}^e \cdot \underline{T}_e - \underline{h}^e \cdot \underline{\delta}_e] + \underline{U}_i \quad (4.27)$$

where

$$\underline{T}_e = \{ (T_x)_1 (T_y)_1 (T_x)_2 (T_y)_2 \dots (T_x)_n (T_y)_n \}$$

= the vector of surface tractions for the e^{th} element, in terms of local subscripts,

$$\underline{\delta}_e = \{ u_1 v_1 u_2 v_2 \dots u_n v_n \}$$

= the vector of nodal displacements of the e^{th} element, in terms of local subscripts,

$\underline{h}^e(x_i, y_i)$, $\underline{g}^e(x_i, y_i)$ are the element matrices for the source point (x_i, y_i) , and defined as:

$$\underline{g}_{\gamma, 2(j-1)+\beta}^e = \int_{\Gamma_e} G_{\beta\gamma}(x-x_i, y-y_i) N_j(\xi) ds$$

$$\underline{h}_{\gamma, 2(j-1)+\beta}^e = \int_{\Gamma_e} F_{\beta\gamma}(x-x_i, y-y_i) N_j(\xi) ds$$

$$\underline{U}_i = \begin{bmatrix} U(x_i, y_i) \\ V(x_i, y_i) \end{bmatrix}$$

$$\gamma=1,2, \quad \beta=1,2$$

Using the parametric equations (4.26), it can be shown that:-

$$\frac{dx}{d\xi} = \sum_{j=1}^n x_j \cdot \frac{dN_j}{d\xi}$$

$$\frac{dy}{d\xi} = \sum_{j=1}^n y_j \cdot \frac{dN_j}{d\xi}$$

$$\frac{dN_j(\xi)}{d\xi} = \sum_{\substack{r=1 \\ r \neq j}}^n \binom{n-1}{j-r} \prod_{\substack{s=1 \\ s \neq r \\ s \neq j}}^n \frac{(n-1)\xi - (s-1)}{j-s}$$

Hence, an infinitesimal length "ds" on the element may be expressed as follows:

$$ds = J(\xi).d\xi$$

where

$$J(\xi) = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

Then, it can be shown that:

$$\left. \begin{aligned} g_{\gamma, 2(j-1)+\beta}^e &= \int_0^1 G_{\beta\gamma}(x-x_i, y-y_i) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi \\ h_{\gamma, 2(j-1)+\beta}^e &= \int_0^1 F_{\beta\gamma}(x-x_i, y-y_i) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi \end{aligned} \right\} \quad (4.28)$$

Equation (4.27) can be expressed at any source point. Hence, taking source point (x_i, y_i) at all boundary nodes (say M), $2M$ equations can be derived and assembled in the following matrix form.

$$\underline{C} \underline{\delta} + \underline{H} \underline{\delta} = \underline{G} \underline{T} + \underline{U} \quad (4.29)$$

where $\underline{G}, \underline{H}$ are global matrices defined in terms of g^e and h^e as follows

$$G[2(I-1)+\gamma, 2(J-1)+\beta] = \sum_{e=1}^{N_e} g_{\gamma, 2(j-1)+\beta}^e$$

$$H[2(I-1)+\gamma, 2(J-1)+\beta] = \sum_{e=1}^{N_e} h_{\gamma, 2(j-1)+\beta}^e$$

$\underline{C}$ is the global matrix of geometrical coefficient c_i

$$C[2(I-1)+\gamma, 2(j-1)+\beta] = \delta_{IJ} \delta_{\gamma\beta} C_i$$

I is the i^{th} boundary node,

$J=TA(e, j)$ is the global number of the j^{th} local node on the e^{th} boundary element,

$\gamma = 1, 2, \beta = 1, 2$ for the number of degrees of freedom.

The matrix equation (4.29) can be simplified as follows:-

$$\underline{\tilde{H}} \underline{\delta} = \underline{G} \underline{T} + \underline{U}$$

Where

$$\underline{\tilde{H}} = \underline{H} + \underline{C}$$

4.6 Elasto-plastic Solution Procedure

The BEM analysis of an elasto-plastic problem is based upon linearised boundary integral equations in terms of elastic kernels, and elasto-plastic effects are represented by means of some "initial" or "residual" stresses and /or strains distributed over the parts of the domain which undergo plastic deformation. The external load is applied incrementally and the stress and strain increments are calculated at the boundary nodes and at a considerable number of internal nodes employed for the definition of integration cells. Wherever a value of the total stress exceeds the yield level at any point, a stress-correction scheme is invoked to estimate the value of the stress which should correspond to the same total strain, according to the yield and hardening rules employed.

The difference between the non-corrected and corrected stresses defines a set of residual stresses which should be considered as an external form of loading. This external loading is assumed to maintain equilibrium and new iterations are carried out until an accurate elasto-plastic solution is obtained. This approach is known as the initial stress method and is similar to that previously been employed for finite-element elasto-plastic analysis.

The main steps of the above procedure as in Fig(4.7), can be summarised as follows:

Step 1 Initial Calculations

In this step the fixed matrices and vectors used in the iterative analysis are estimated.

- i) Calculate first the global $\underline{G}$ and $\underline{\tilde{H}}$ matrices for the whole boundary employed in the matrix equation:

$$\underline{\tilde{H}} \underline{\delta} = \underline{G} \underline{T} + \underline{U}$$

- ii) The boundary conditions are given in terms of prescribed displacements or tractions at boundary nodes:

Defining:

$\underline{\delta}_u$, $\underline{T}_t$ As the unknown parts of $\underline{\delta}$ and $\underline{T}$ vectors,

$\underline{\delta}_t$, $\underline{T}_u$ The prescribed parts of $\underline{\delta}$ and $\underline{T}$ vectors

Then the previous equation can be partitioned as follows:

$$\begin{bmatrix} \tilde{H}_{uu} & \tilde{H}_{ut} \\ \tilde{H}_{tu} & \tilde{H}_{tt} \end{bmatrix} \begin{pmatrix} \underline{\delta}_u \\ \underline{\delta}_t \end{pmatrix} = \begin{bmatrix} G_{uu} & G_{ut} \\ G_{tu} & G_{tt} \end{bmatrix} \begin{pmatrix} \underline{T}_u \\ \underline{T}_t \end{pmatrix} + \underline{U}$$

which can be rewritten as:

$$\begin{bmatrix} \tilde{H}_{uu} & -G_{ut} \\ \tilde{H}_{tu} & -G_{tt} \end{bmatrix} \begin{pmatrix} \underline{\delta}_u \\ \underline{T}_t \end{pmatrix} = \begin{bmatrix} G_{uu} & -\tilde{H}_{ut} \\ G_{tu} & -\tilde{H}_{tt} \end{bmatrix} \begin{pmatrix} \underline{T}_u \\ \underline{\delta}_t \end{pmatrix} + \underline{U}$$

i.e $\underline{A} \underline{X} = \underline{Y}$ (4.30)

where $\underline{X} = \{\underline{\delta}_u \quad \underline{T}_t\}$

Hence, the matrix of equation coefficients can be defined as :

$$\underline{A} = \begin{bmatrix} \tilde{H}_{uu} & -G_{ut} \\ \tilde{H}_{tu} & -G_{tt} \end{bmatrix}$$

and the vector equivalent to the applied domain loads, prescribed displacements and tractions is:

$$\underline{Y} = \begin{bmatrix} G_{uu} & -\tilde{H}_{ut} \\ G_{tu} & -\tilde{H}_{tt} \end{bmatrix} \begin{pmatrix} \underline{T}_u \\ \underline{\delta}_t \end{pmatrix} + \underline{U}$$

Matrix $\underline{A}$ and vector $\underline{Y}$, are calculated and used in the following steps.

Step 2 Definition Of Applied Load Increment

Since, in the BEM, the applied surface load is expressed in terms of prescribed tractions whilst unknown tractions represent reactions at nodes with prescribed displacements, then the vector $\underline{Y}$, defined previously, is a measure of the total acting forces and reactions. According to the approximation employed in the linearisation scheme, for example:

$$\Delta \delta \approx \underline{D}_{ep} \cdot \Delta \epsilon$$

If the total vector $\underline{Y}$ is used, an excessive number of iterations are required and even then these may result in divergence of the solution. Hence, a number of increments should be employed for each of which only a part of $\underline{Y}$ being used. Since, the elastic theory does not require such an approximation, it is common practice to take the first increment of $\underline{Y}$ as large as possible, just sufficient to make the structure start to yield, however, for large elasto-plastic deformation, the convergence of the algorithm becomes very sensitive and a very small $\Delta \underline{Y}$ should, therefore, be used.

If the structure collapses before absorbing the total amount of $\underline{Y}$ then the algorithm would definitely diverge when this happens. Hence it is important to define a set of increment coefficients $(\lambda_1, \lambda_2 \dots \lambda_n)$ with values taken according to circumstances, where:

$$\sum_{i=1}^n \lambda_i = 1$$

and

$$(\Delta \underline{Y})_i = \lambda_i \underline{Y} \dots \text{is the } i^{\text{th}} \text{ load increment.}$$

The Following Steps are Used for Each New Load Increment

Step 3 Elastic Solution

From equation (4.30) solve the following incremental equation

$$\underline{A} \cdot \Delta \underline{X} = (\Delta \underline{Y})_i$$

Then from $\Delta \underline{X}$ the values of $\Delta \underline{\delta}_{-u}$ and $\Delta \underline{T}_{-t}$ are defined, from which it is possible to calculate

$$\Delta \underline{\delta} = \{ \Delta \underline{\delta}_{-u} \quad \Delta \underline{\delta}_{-t} \}, \text{ and}$$

$$\Delta \underline{T} = \{ \Delta \underline{T}_{-u} \quad \Delta \underline{T}_{-t} \}$$

From the above result, elastic boundary integral equations can be employed for the calculation of stress and strain increments ($\Delta \underline{\sigma}'$, $\Delta \underline{\varepsilon}$) at boundary nodes. Hence using the techniques described in section 4.7, the value of $\Delta \underline{\sigma}'$ and $\Delta \underline{\varepsilon}$ are to be calculated at all internal nodes of the integration cells. Finally the total values are obtained as follows:

$$\underline{\delta} = \underline{\delta}_{old} + \Delta \underline{\delta}$$

$$\underline{\sigma}'_{new} = \underline{\sigma}_{old} + \Delta \underline{\sigma}'$$

$$\underline{\varepsilon}_{new} = \underline{\varepsilon}_{old} + \Delta \underline{\varepsilon}$$

Step 4 Plasticity Check

The level of stress should be checked, at each geometric point of the integration cells, by using a suitable yield criterion. If the stress at all of the points remains within the yield stress, then no correction of stresses is required and a new load increment (represented by $\Delta \underline{Y}$) is added and the elastic solution (step 3) is repeated. If the stress at any geometrical point of the cells exceeds the yield stress, then an updating procedure is required to distinguish between the totally elastic and elasto-plastic parts of the stress and strain vectors, as explained in section(3.6), step-4.

Step 5 Plastic Iterative Loop

If the updated stresses at some of the points of the structure, calculated by means of the elastic equations (step-3), exceeded, the instantaneous yield stress, then stress correction and

elasto-plastic iterations are required to obtain the correct solution. This can be carried out by means of the following steps:

- i) Calculate the correct stress vectors($\underline{\sigma}_c$) at the specified boundary and internal nodes, by using the stress correction algorithm which is explained in section (3.8).
- ii) Calculate the corresponding residual stress vectors as:

$$\underline{\sigma}^R = \underline{\sigma}' - \underline{\sigma}_c$$

- iii) Evaluate the equivalent residual force by employing the domain-numerical-integration scheme, section(4.7), over each integration cell in a way similar to that used with the Finite-Element Method.

Hence, at every source point (x_i, y_i) , it can be shown that:

$$\underline{F}_i = - \sum_{c=1}^{N_c} \iint_{\Omega_c} \underline{R} \cdot \underline{\sigma}^R d\Omega_c$$

Where

$$\underline{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \end{bmatrix}$$

The Fundamental parameters
as in Table (4.2),

$$\underline{\sigma} = \{\sigma_x^R \quad \sigma_y^R \quad \tau_{xy}^R\} \text{ Residual stress vector}$$

Hence, for all boundary nodes which are source points, a global vector can be calculated as follows:

$$\underline{F}^R = \{\underline{F}_1 \quad \underline{F}_2 \dots \quad \underline{F}_{N_b}\}$$

Where N_b Number of boundary nodes

- iv) Use the vector $\underline{F}^R$ to solve the following equation :

$$\underline{A} \cdot \underline{\Delta X} = \underline{F}^R$$

To obtain the incremental $\Delta\delta_u$ and ΔT_t , the following total vectors can be updated:

$$\underline{\delta} = \underline{\delta} + \Delta\underline{\delta}$$

$$\underline{T} = \underline{T} + \Delta\underline{T}$$

and used to calculate the boundary stress and strain vectors.

v) Calculate the corresponding increments of strain and stress vectors at the internal nodes, and hence update their total vectors.

vi) If the increment of $\underline{\delta}$ or $\underline{T}$ is not small enough i.e if:

$$\frac{\Delta\underline{\delta}^t \cdot \Delta\underline{\delta}}{\underline{\delta}^t \cdot \underline{\delta}} > \text{a permissible error}$$

or

$$\frac{\Delta\underline{T}^t \cdot \Delta\underline{T}}{\underline{T}^t \cdot \underline{T}} > \text{a permissible error}$$

then return to step (i) else consider the next load increment from step-3.

4.7 BEM New Accuracy Measures and Related Topics

It is clear from section 4.6 that the boundary element elasto-plastic stress analysis is based upon iterative stress-correction of an incremental elastic solution. Because of an accumulation of errors, an accurate plastic solution would never be achieved unless the elastic solution is itself extremely accurately determined. The elasto-plastic BEM analysis in this work would not have been possible without first improving the accuracy of the BEM solution. Several accuracy measures have been investigated and implemented in the developed program. Isoparametric elements are thought to be incapable of dealing with problems of edges and corners and for this reason non-conforming elements have been proposed[127-131].

In the present investigation, however, some corner models have

been derived with the aim of improving the accuracy of isoparametric elements and hence justifying their use in this work. One of the problems with the use of general elements is the difficulty of obtaining analytical solutions for the singular integrations involved. For this reason, a special singular quadrature scheme has been adopted and used for the evaluation of such integrations.

The elasto-plastic algorithm requires the evaluation also of some domain integrals which are usually carried out in terms of integrations over subdomains or integration cells. In this work a sophisticated library of finite elements has been employed for accurate representation of those integration cells and suitable Gaussian-quadrature schemes are used .

Since the elasto-plastic algorithm is based upon incremental stress evaluation and correction (from elastic to actual elasto-plastic stress), it was essential to obtain accurate values for the stress anywhere within the domain. Although boundary integral equations for strains (and stress) can be derived, they are only valid for points inside the domain due to the r^{-1} and r^{-2} singularities in the Kernel functions. For this reason, a special derivation has been carried out for boundary points. For the case of points very close to the boundary a finite-difference scheme has been employed. A review of the basic accuracy measures developed and employed in this work is given.

4.7.1 Corner Model

The geometries of most engineering structures contain corners. A point on a corner of a given domain can cause problems in the evaluation of element matrices due to the discontinuity of the normal to the boundary at that point. Some investigators ignore corners completely by using non-conforming elements which have no nodes at their ends. Since such elements do not satisfy a C^0 -continuity of the field functions at the boundary, it is thought that the isoparametric elements may prove to be more accurate than the non-conforming ones.

However, a special treatment for corner effect on the boundary integrals should be invoked with isoparametric elements so as to obtain reasonable accuracy as explained in this section.

4.7.1.1 Jump Functions

The Jump function can be defined as the value of the boundary integration of a fundamental solution parameter when crossing a corner.

Considering a closed boundary " Γ ", which contains a corner defined by two points "a" and "b" with a boundary " Γ_c ", as shown in Fig(4.8.a), then it can be shown that:

$$\oint_{\Gamma} f \cdot ds = \lim_{a \rightarrow b} \left[\int_{\Gamma'} f \cdot ds + \int_a^b f \cdot ds \right]$$

or

$$\oint_{\Gamma} f \cdot ds = \int_{\Gamma'} f \cdot ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} f \cdot ds$$

where

$$\Gamma = \Gamma' + \Gamma_c$$

Γ' is the part of the boundary " Γ " which does not contain the corner (i.e the smooth part of the boundary),

Γ_c is the infinitesimal boundary of the corner.

Hence, the jump function can be defined as follows:-

$$\begin{aligned} \text{Jump function of } f(x,y) \\ \text{integral at corner } c \end{aligned} = \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} f(x,y) \, ds$$

Usually this jump function does not exist unless $f(x,y)$ is singular at the specified corner.

Applying the above concept to the fundamental solution parameters defined by equations (4.10) and (4.15), it can be shown that:-

$$\oint_{\Gamma} F_{\gamma\beta} \cdot u_{\gamma} ds = \int_{\Gamma'} F_{\gamma\beta} \cdot u_{\gamma} ds + \lim_{a \rightarrow b} \int_a^b F_{\gamma\beta} \cdot u_{\gamma} ds$$

At the limit $a \rightarrow b$, the corner point (x_c, y_c) coincides with points "a" and "b", then:-

$$(x_c, y_c) \rightarrow (x_a, y_a) \rightarrow (x_b, y_b), \text{ also} \\ u_{\gamma}(x_c, y_c) \rightarrow u_{\gamma}(x_a, y_a) \rightarrow u_{\gamma}(x_b, y_b).$$

Finally

$$\oint_{\Gamma} F_{\gamma\beta} \cdot u_{\gamma} ds = \int_{\Gamma'} F_{\gamma\beta} \cdot u_{\gamma} ds + \mathcal{J}_{\gamma\beta}^{(x_i, y_i)} \cdot u_{\gamma}(x_c, y_c)$$

where

$$\mathcal{J}_{\gamma\beta}^{(x_i, y_i)} = \lim_{\Gamma_C \rightarrow 0} \int_{\Gamma_C} F_{\gamma\beta}(x-x_i, y-y_i) \cdot ds \quad (4.31)$$

Which is the jump function for $F_{\gamma\beta}$ at corner C.

Similarly:-

$$\oint_{\Gamma} G_{\gamma\beta} \cdot T_{\gamma} ds = \int_{\Gamma'} G_{\gamma\beta} T_{\gamma} ds + \mathcal{G}_{\gamma\beta}^{(x_i, y_i)} T_{\gamma}(x_c, y_c)$$

where

$$\mathcal{G}_{\gamma\beta}^{(x_i, y_i)} = \lim_{\Gamma_C \rightarrow 0} \int_{\Gamma_C} G_{\gamma\beta}(x-x_i, y-y_i) ds \quad (4.32)$$

which is the jump function for $G_{\gamma\beta}$ at corner c

4.7.1.2 Smooth Corner Model

In order to evaluate jump functions at a given corner point, a smooth corner model may be considered, where the corner point (x_c, y_c) is to be a centre of a circular arc which is tangent to the boundary at points "a" and "b", as shown in Fig(4.8.b).

Suppose that the source point (x_i, y_i) is not a corner point, i.e. is not the centre of the arc, then it can be proved that the jump functions, (x_i, y_i) and (x_i, y_i) are each equal to zero, because there will be no singularities in $F_{\gamma\beta}$ and $G_{\gamma\beta}$ under those conditions. If the source point (x_i, y_i) is on the corner point (x_c, y_c) , then singularities are expected in the fundamental solution parameters. Let (x, y) be a field point moving on the circular arc. The distance from this field point to (x_c, y_c) is the radius of the arc defined as:

$$r = \sqrt{(x-x_c)^2 + (y-y_c)^2}$$

By using polar coordinates, it can be proved that:-

$$x = x_c + r \cos \theta$$

$$y = y_c + r \sin \theta$$

Then

$$\frac{\partial r}{\partial x} = \frac{x-x_c}{r} = \cos \theta,$$

$$\frac{\partial r}{\partial y} = \frac{y-y_c}{r} = \sin \theta, \text{ and}$$

$$\frac{\partial r}{\partial n} = 1.$$

By substituting the above equations into the expressions of $G_{\gamma\beta}$, as shown in Table(4.1), it can be deduced that:-

$$G_{11} = \frac{1}{C_a} [C_1 \log \frac{1}{r} + \cos^2 \theta],$$

$$G_{12} = G_{21} = \frac{1}{C_a} \cos \theta \cdot \sin \theta, \text{ and}$$

$$G_{22} = \frac{1}{C_a} [C_1 \log \frac{1}{r} + \sin^2 \theta].$$

Similarly, it can be proved, using equation (4.15), that:

$$F_{11} = \frac{-1}{C_b \cdot r} [2(1-p) + \cos 2\theta],$$

$$F_{12} = F_{21} = \frac{-1}{C_b \cdot r} [2 \cos \theta \sin \theta], \text{ and}$$

$$F_{22} = \frac{-1}{C_b \cdot r} [2(1-p) + \sin 2\theta],$$

Then the jump functions for this smooth corner model can be obtained by using the above expressions for fundamental parameters, in the equation (4.32) as follows:-

$$g_{II}^{(x_c, y_c)} = \lim_{a \rightarrow b} \int_a^b G_{11} (x-x_c, y-y_c) r d\theta$$

or in polar coordinates as follows:-

$$g_{II}^{(x_c, y_c)} = \lim_{r \rightarrow 0} \int_{\theta_a}^{\theta_b} G_{11} (x-x_c, y-y_c) r d\theta$$

$$g_{II}^{(x_c, y_c)} = \lim_{r \rightarrow 0} \frac{-(3-4p) \log r \cdot (\theta_b - \theta_a)}{8 \pi \mu (1-p) r^{-1}}$$

By using L'Hopital's Rule, it can be deduced that:-

$$g_{II}^{(x_c, y_c)} = 0$$

Similarly, it can be proved that: $g_{12} = g_{21} = g_{22} = 0$

The jump functions for parameters $F_{\gamma\beta}(x-x_c, y-y_c)$ are:-

$$f_{II}^{(x_c, y_c)} = \lim_{a \rightarrow b} \int_{\theta_a}^{\theta_b} F_{11} (x-x_c, y-y_c) r d\theta$$

$$J_{11}^{(x_c, y_c)} = \lim_{r \rightarrow 0} \int_{\theta_a}^{\theta_b} \frac{-1}{C_b} [2(1-p) + \cos 2\theta] d\theta$$

i.e.

$$J_{11}^{(x_c, y_c)} = \frac{-1}{C_b} \left[2(1-p)(\theta_a - \theta_b) + \frac{\sin(2\theta_b) - \sin(2\theta_a)}{2} \right]$$

By using the following geometrical simplification,

$$\theta_b - \theta_a = \phi_0 = \pi - \phi_c$$

$$\phi_m = \frac{\theta_a + \theta_b}{2}$$

where

ϕ_c is the corner angle

ϕ_m is the mean value of θ_a and θ_b

, as illustrated in Fig(4.8.b).

Hence,

$$J_{11}^{(x_c, y_c)} = \frac{\phi_c}{2\pi} - \frac{1}{2} - \frac{1}{C_b} \cos(2\phi_m) \cdot \sin(\phi_c)$$

Similarly, it can be proved that:-

$$J_{12}^{(x_c, y_c)} = J_{21}^{(x_c, y_c)} = -\frac{1}{C_b} \sin(2\phi_m) \cdot \sin(\phi_c)$$

$$J_{22}^{(x_c, y_c)} = \frac{\phi_c}{\pi} - \frac{1}{2} + \frac{1}{C_b} \cos(2\phi_m) \cdot \sin(\phi_c)$$

4.7.1.3 Use of Rigid Translation Conditions

Suppose that the whole domain is moved rigidly in the x and y directions by amounts u_0 and v_0 respectively, then there is no boundary traction applied and therefore there are no internal stresses produced, i.e.

$$T_x = T_y = 0,$$

$$U_i = V_i = 0,$$

$$u(x,y) = u_o, \text{ and}$$

$$v(x,y) = v_o.$$

Therefore the B.I.E.'s given by equations (4.21) for any source point are:

$$\left. \begin{aligned} C_i u_o + u_o \oint_{\Gamma} F_{11} ds + v_o \oint_{\Gamma} F_{21} ds &= 0 \\ C_i v_o + u_o \oint_{\Gamma} F_{12} ds + v_o \oint_{\Gamma} F_{22} ds &= 0 \end{aligned} \right\} \quad (4.33)$$

For the special case:- $u_o = u_i$, and $v_o = v_i$, the above B.I.E.'s can be rewritten as follows:

$$\left. \begin{aligned} C_i u_i + u_i \oint_{\Gamma} F_{11} ds + v_i \oint_{\Gamma} F_{21} ds &= 0 \\ C_i v_i + u_i \oint_{\Gamma} F_{12} ds + v_i \oint_{\Gamma} F_{22} ds &= 0 \end{aligned} \right\} \quad (4.34)$$

By subtracting equations (4.34) from equations (4.21), it can be deduced that:

$$\begin{aligned} \oint_{\Gamma} F_{11} (u - u_i) ds + \oint_{\Gamma} F_{21} (v - v_i) ds &= \\ \oint_{\Gamma} G_{11} T_x ds + \oint_{\Gamma} G_{21} T_y ds + U_i & \\ \oint_{\Gamma} F_{12} (u - u_i) ds + \oint_{\Gamma} F_{22} (v - v_i) ds &= \\ \oint_{\Gamma} G_{12} T_x ds + \oint_{\Gamma} G_{22} T_y ds + V_i & \end{aligned}$$

If the boundary is not smooth, i.e it has a discontinuity or corner, then the above B.I.E.'s can be rewritten as follows:

$$\begin{aligned}
 & \int_{\Gamma'} F_{11} (u-u_i) ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{11} (u-u_i) ds \\
 & + \int_{\Gamma'} F_{21} (v-v_i) ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{21} (v-v_i) ds \\
 & = \oint_{\Gamma} G_{11} T_x ds + \oint_{\Gamma} G_{21} T_y ds + U_i
 \end{aligned}$$

And

$$\begin{aligned}
 & \int_{\Gamma'} F_{12} (u-u_i) ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{12} (u-u_i) ds \\
 & + \int_{\Gamma'} F_{22} (v-v_i) ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{22} (v-v_i) ds \\
 & = \oint_{\Gamma} G_{12} T_x ds + \oint_{\Gamma} G_{22} T_y ds + V_i
 \end{aligned}$$

The boundary integration over " Γ_c " can be simplified as follows:-

$$\begin{aligned}
 \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{\gamma\beta} (u-u_i) ds &= \lim_{a \rightarrow b} \int_a^b F_{\gamma\beta} (u-u_i) ds \\
 &= \int_{\gamma\beta}^{(x_i, y_i)} (u_c - u_i)
 \end{aligned}$$

Similarly:-

$$\lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{\gamma\beta} (v-v_i) ds = \lim_{a \rightarrow b} \int_a^b F_{\gamma\beta} (v-v_i) ds \quad \dots \text{continue}$$

$$= \int_{\gamma\beta} (x_i, y_i) \cdot (v_c - v_i)$$

Where

u_c, v_c are the displacement components at the corner point (x_c, y_c)

For the case of a source point (x_i, y_i) is not a corner, and from previous analysis:

$$\int_{\gamma\beta} (x_i, y_i) = 0$$

whilst, if the source point (x_i, y_i) is on the corner point (x_c, y_c) , then:

$$u_c - u_i = 0, \quad v_c - v_i = 0$$

Therefore, the terms of the boundary integration over " Γ_c " are always equal to zero, and the B.I.E.'s (4.21) can be therefore be reduced to the following equations:-

$$\int_{\Gamma'} F_{11} (u - u_i) ds + \int_{\Gamma'} F_{21} (v - v_i) ds = \int_{\Gamma'} G_{11} T_x ds + \int_{\Gamma'} G_{21} T_y ds + U_i$$

And

$$\int_{\Gamma'} F_{12} (u - u_i) ds + \int_{\Gamma'} F_{22} (v - v_i) ds = \int_{\Gamma'} G_{12} T_x ds + \int_{\Gamma'} G_{22} T_y ds + V_i$$

By defining

$$C_{\gamma\beta}(x_i, y_i) = - \int_{\Gamma'} F_{\beta\gamma}(x - x_i, y - y_i) ds ,$$

then the previous B.I.E.'s can be simplified as follows:

$$\left. \begin{aligned}
 C_{11}(x_i, y_i) \cdot u_i + C_{12}(x_i, y_i) \cdot v_i + \int_{\Gamma'} F_{11} u \, ds + \int_{\Gamma'} F_{21} v \, ds = \\
 \int_{\Gamma'} G_{11} T_x \, ds + \int_{\Gamma'} G_{21} T_y \, ds + U_i \\
 C_{21}(x_i, y_i) \cdot u_i + C_{22}(x_i, y_i) \cdot v_i + \int_{\Gamma'} F_{12} u \, ds + \int_{\Gamma'} F_{22} v \, ds = \\
 \int_{\Gamma'} G_{12} T_x \, ds + \int_{\Gamma'} G_{22} T_y \, ds + V_i
 \end{aligned} \right\} \quad (4.35)$$

4.7.2 Numerical Integration

The integration of the variable over a specified boundary or domain can be evaluated by employing a suitable numerical integration technique, such as the one-dimensional or the two-dimensional Gaussian scheme. The accuracy of the BEM results depends directly upon the accuracy of the integration technique, as explained in this section.

4.7.2.1 Integration over the Boundary

To obtain the element matrix, a boundary integration over the boundary of the elements is required. The one-dimensional Gaussian-quadrature scheme can be employed as follows:-

$$\begin{aligned}
 \int_{\Gamma_e} f(x-x_i, y-y_i) \, ds &= \int_0^1 f[x(\xi)-x_i, y(\xi)-y_i] \cdot J(\xi) \cdot d\xi \\
 &= \sum_{q=1}^{N_q} f[x(\xi_q)-x_i, y(\xi_q)-y_i] \cdot J(\xi_q) \cdot W_q
 \end{aligned}$$

Where f is a specified integrand,

ξ_q is the abscissa of the Gaussian quadrature,

w_q is the corresponding weight, and

N_q is the number of Gaussian points used.

If the source point (x_i, y_i) is too close to, or coinciding with, the field point $(x(\xi), y(\xi))$, which is a Gauss point, then a singularity may occur in the fundamental parameter terms which contain $\log r$ and $1/r$.

A mathematical approach can be employed for evaluating the boundary integration of the singular terms. The basic concept of this approach is based upon the division of the integral into two parts, where the first part contain the regular terms which can be evaluated by means of Gaussian quadrature schemes as usual, whilst the second part contains the singular terms which will be evaluated by special numerical techniques. The singular integration method employed in this work is based upon the following equation:

$$\int_0^1 \log\left(\frac{1}{x}\right) f(x) dx = \sum_{s=1}^{N_s} f(x_s) w_s \quad (4.36)$$

where x_s, w_s are as given by [132] and this technique will be —applied to integrands containing $G_{\gamma\beta}$ terms.

Considering a boundary element which has n -nodes and a source point (x_i, y_i) which is coincident with the j_{th} local node, equation (4.28) can be rewritten as follows:

$$g_{\gamma, 2(j-1)\beta}^e = \int_0^1 G_{\beta\gamma}(x(\xi) - x_i, y(\xi) - y_i) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

$$g_{\gamma, 2(j-1)\beta}^e = \frac{1}{C_a} \int_0^1 [C_1 \log\left(\frac{1}{r}\right) \delta_{\gamma\beta} + \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma}] \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

For the case of $\beta=\gamma$, the above integration can be rewritten as follows:

$$\frac{1}{C_a} \int_0^1 [C_1 \log\left(\frac{1}{r}\right) + \left(\frac{\partial r}{\partial x_\gamma}\right)^2] \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi = I^S + I^G \quad (4.37)$$

Where

$$I^G = \frac{1}{C_a} \int_0^1 \left(\frac{\partial r}{\partial x_\gamma}\right)^2 \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

which is the regular term, and

$$I^S = \frac{1}{C_a} \int_0^1 C_1 \log\left(\frac{1}{r}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

which is the singular term.

The boundary " Γ_e " can be divided into two parts, as shown in Fig(4.9), then the singular integration can be rewritten as follows:

$$I^S = \frac{C_1}{C_a} \left[\int_0^{\xi_j} \log\left(\frac{1}{r}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi + \int_{\xi_j}^1 \log\left(\frac{1}{r}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi \right]$$

$$I^S = \frac{C_1}{C_a} [I_1^S + I_2^S] \quad (4.38)$$

where

$$I_1^S = \int_0^{\xi_j} \log\left(\frac{1}{r}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi \quad \text{is the singular integration over part I}$$

and

$$I_2^S = \int_{\xi_j}^1 \log\left(\frac{1}{r}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi \quad \text{is the singular integration over part II}$$

$$\xi_j = \frac{j-1}{n-1}$$

To evaluate I_1^S , the following analysis can be applied:-

$$\log \frac{1}{r} = \log\left(\frac{\xi_j}{\xi_j - \xi}\right) - \log\left(\frac{\xi_j - \xi}{r \xi_j}\right) = \log\left(\frac{\xi_j}{\xi_j - \xi}\right) + \log\left(\frac{\xi_j - \xi}{r \xi_j}\right)$$

Taking the limits when $r \rightarrow 0$

$$\lim_{r \rightarrow 0} \log\left(\frac{1}{r}\right) = \lim_{\substack{r \rightarrow 0 \\ \xi \rightarrow \xi_j}} \log\left(\frac{\xi_j}{\xi_j - \xi}\right) + \lim_{\substack{r \rightarrow 0 \\ \xi \rightarrow \xi_j}} \log\left(\frac{\xi_j - \xi}{r \xi_j}\right)$$

from which it can be shown that the first part of the previous equation is singular, whilst the second part is regular.

Hence, the integral I_1^S can also be written as:-

$$I_1^S = I_1^{SG} + I_1^{SS} \quad (4.39)$$

$$I_1^{SG} = \int_0^{\xi_j} \log\left(\frac{\xi_j - \xi}{r \xi_j}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

which can be dealt with by using Gaussian quadrature, and

$$I_1^{SS} = \int_0^{\xi_j} \log\left(\frac{\xi_j}{\xi_j - \xi}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

Defining the variable ρ_1 , such that:-

$$\rho_1 = \frac{\xi_j - \xi}{\xi_j}$$

$$\begin{aligned} \text{then at } \xi=0, & \quad \rho_1=1, \\ \xi=\xi_j, & \quad \rho_1=0, \end{aligned}$$

$$\text{and} \quad d\rho_1 = - \frac{d\xi}{\xi_j}$$

$$\text{i.e} \quad d\xi = -\xi_j d\rho_1.$$

Hence, it can be proved that:-

$$I_1^{SS} = \int_0^1 \log\left(\frac{1}{\rho_1}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot \xi_j \cdot d\rho_1$$

which is in a suitable form for the evaluation by means of the singular quadrature given by [132]

Similarly, the term I_2^S can be rewritten as:

$$I_2^S = I_2^{SG} + I_2^{SS} \quad (4.40)$$

Where

$$I_2^{SG} = \int_{\xi_j}^1 \log\left(\frac{\xi - \xi_j}{r \cdot (1 - \xi_j)}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

and

$$I_2^{SS} = \int_{\xi_j}^1 \log\left(\frac{1-\xi_j}{\xi-\xi_j}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot d\xi$$

Defining a second variable ρ_2 , such that:-

$$\rho_2 = \frac{\xi - \xi_j}{1 - \xi_j}$$

$$\begin{array}{ll} \text{then at } \xi = \xi_j, & \rho_2 = 0, \\ \xi = 1, & \rho_2 = 1, \end{array}$$

$$\text{and } d\rho_2 = \frac{d\xi}{(1 - \xi_j)}$$

Hence, it can be shown that:-

$$I_2^{SS} = \int_0^1 \log\left(\frac{1}{\rho_2}\right) \cdot N_j(\xi) \cdot J(\xi) \cdot (1 - \xi_j) \cdot d\rho_2$$

Substituting from equations (4.39), and (4.40) into equation (4.38), it can be deduced that:-

$$I_S = \frac{C_1}{C_a} [I_1^{SS} + I_1^{SG} + I_2^{SS} + I_2^{SG}] \quad (4.41)$$

and from equation (4.41), it can be proved that equation (4.37) can be expressed as follows:

$$g_{\gamma, 2(j-1)\beta}^e = I_0 + I_1 + I_2$$

Where

$$I_0 = I_G + \frac{C_1}{C_a} [I_1^{SG} + I_2^{SG}]$$

$$I_1 = \frac{C_1}{C_a} I_1^{SS}$$

$$I_2 = \frac{C_1}{C_a} I_2^{SS}$$

It is clear from the above equation that:-

- (i) I_0 contains regular terms only, and that it can be evaluated accurately by means of the standard Gauss quadrature,
- (ii) Both I_1 and I_2 are singular but they are in a format compatible with the singular quadrature scheme given by equation (4.36)

It is also clear from above analysis that:

$$I_2 = 0 \quad \text{if } j=1$$

$$I_1 = 0 \quad \text{if } j=n, \text{ and}$$

both I_1 and I_2 should be evaluated if $1 < j < n$.

4.7.2.2 Integration over the Domain

Domain integration are required for the evaluation of body force expressions which arise from the residual stresses generated inside the domain after elasto-plastic deformation. These expressions are defined, for a given source point, as follows:

$$F_{\gamma 1} = \iint_{\Omega} (R_{\gamma 1} \sigma_x^R + R_{\gamma 2} \sigma_y^R + R_{\gamma 3} \tau_{xy}^R) d\Omega$$

where $\gamma=1,2$

Usually the above integration can be evaluated approximately by dividing the domain Ω into asuitable number of integration cells such that:

$$F_{\gamma} = \sum_{c=1}^{N_c} (F_{\gamma})_c$$

Where

$$(F_{\gamma})_c = \iint_{\Omega_c} (R_{\gamma 1} \sigma_x^R + R_{\gamma 2} \sigma_y^R + R_{\gamma 3} \tau_{xy}^R) d\Omega$$

Where

N_c is the number of integration cells, and
 Ω_c is the domain of the C_{th} cell.

A simplified approach based upon the multiplication of the average value of the integrand by the area of the cell, as shown in Fig(4.10), can be used for the evaluation of $(F_{\gamma})_c$. This approach when tested, proved to be inaccurate due to the singularities involved, and the crude approximation employed.

A more accurate technique has been developed in this work. In this approach the domain must be divided into a number of standard finite elements (in fact quadrilateral and triangular elements of first, second and third order may be used), as shown in Fig(4.11).

The nodes defining the topology of these elements can be considered as a set of internal points, the stresses at which are to be evaluated. Consequently the residual stresses at these points can be specified within the elasto-plastic algorithm. Using the standard isoparametric transformation, then:

$$(F_{\gamma 1})_c = \int_0^1 \int_0^1 (R_{\gamma 1} \sigma_x^R + R_{\gamma 2} \sigma_y^R + R_{\gamma 3} \tau_{xy}^R) \cdot |J(\xi, \eta)| \cdot d\xi \cdot d\eta$$

where

$f(\eta)=1$ for quadrilateral elements,

$f(\eta)=1-\eta$ for triangular elements, and

$|J(\xi, \eta)|$ is the Jacobian determinant, as defined in chapter 3.

The residual body force can be evaluated numerically by employing Gauss-quadrature scheme, such that:

$$(F_r)_c = \sum_{s=1}^N \sum_{r=1}^q f(\eta_s) \cdot I(x_{rs}, y_{rs}) \cdot W_r \cdot W_s \cdot |J|$$

where

$$I(x_{rs}, y_{rs}) = R_{\gamma 1} \sigma_x^R + R_{\gamma 2} \sigma_y^R + R_{\gamma 3} \tau_{xy}^R \quad (4.42)$$

and x_{rs} , y_{rs} are to be evaluated from the following isoparametric equations,

$$x_{rs} = \sum_{j=1}^n N_j(\xi_{rs}, \eta_s) \cdot x_j$$

$$y_{rs} = \sum_{j=1}^n N_j(\xi_{rs}, \eta_s) \cdot y_j$$

Where

n is the number of nodes for the integration cell,

$\xi_{rs} = f(\eta_s) \cdot a_r$,

(a_r, η_s) are abscissas of the Gaussian quadrature, and

(W_r, W_s) are the corresponding Weights.

The values of σ_x^R , σ_y^R , τ_{xy}^R in equation(4.42) are to be evaluated at corresponding Gaussian quadrature points by means of isoparametric interpolation equations.

This developed approach proved to be more efficient than the average-cell approach due to the followings:

- (1) Calculating the stress away from quadrature points eliminates singularities encountered in the B.I.E.'s.
- (2) Using high order elements as integration cells leads to accurate geometrical modelling of domains with curved boundaries.

(3) Since boundary nodes are also used with integration cells, the elasto-plastic algorithm is found to be capable of dealing with these nodes as efficient as with internal nodes, and will therefore lead to a more accurate analysis.

(4) The wide range of cell types, and quadrature schemes, is very useful for finding the optimum parameters for different domains.

4.7.3 Stress-Strain Calculation

Using the accuracy measures developed in this work, the values of the displacement components obtained from the B.I.E.'s are always independent of the position of the source point.

Unfortunately, the accuracy of the strain components as obtained from the appropriate boundary integral equations is a function of the position of the source point because of the r^{-1} and r^{-2} singularities in the Kernel functions. New measures have been considered so as to obtain accurate values for stresses and strains for different positions of source points, as explained in this section.

4.7.3.1 Stresses at Internal Points

The strain components at point (x_i, y_i) , inside the domain are the direct derivatives of the displacement components at the same point with respect to x_i, y_i . Hence by differentiating the B.I.E.'s of the displacements (equations (4.21)), with respect to x_i and y_i , it can be deduced that:

$$\left. \begin{aligned} C_i \frac{\partial u_i}{\partial x_i} &= E_{11}(x_i, y_i), \\ C_i \frac{\partial u_i}{\partial y_i} &= E_{12}(x_i, y_i), \end{aligned} \right\} \quad (4.43)$$

.....continue

$$C_i \frac{\partial v_i}{\partial x_i} = E_{21}(x_i, y_i), \text{ and}$$

$$C_i \frac{\partial v_i}{\partial y_i} = E_{22}(x_i, y_i).$$

(4.43)

Where $E_{\gamma k}(x_i, y_i)$ can be expressed as follows:

$$\begin{aligned} E_{\gamma k}(x_i, y_i) = & \oint_{\Gamma} \left(T_x \frac{\partial G_{1\gamma}}{\partial x_k} + T_y \frac{\partial G_{2\gamma}}{\partial x_k} \right) ds \\ & - \oint_{\Gamma} \left(u \frac{\partial F_{1\gamma}}{\partial x_k} + v \frac{\partial F_{2\gamma}}{\partial x_k} \right) ds \\ & - \int \int_{\Omega} \left[\sigma_x^R \frac{\partial R_{\gamma 1}}{\partial x_k} + \sigma_y^R \frac{\partial R_{\gamma 2}}{\partial x_k} + \tau_{xy}^R \frac{\partial R_{\gamma 3}}{\partial x_k} \right] dx dy \\ & + \frac{\partial U_{\gamma}}{\partial x_k} \end{aligned}$$

where

$$k=1,2$$

$$x_1 \equiv x_i, \text{ and } x_2 \equiv y_i$$

The derivatives of $F_{\gamma\beta}$, $R_{\gamma\beta}$ with respect to x_i and y_i are listed in Table(4.4) and Table(4.5) respectively. The derivatives of U_{γ} can be generally be written as follows:

$$\frac{\partial U_{\gamma}}{\partial x_k} = \oint \left[\frac{\partial f_{\gamma x}}{\partial x_k} x + \frac{\partial f_{\gamma y}}{\partial x_k} y - \frac{\partial Q^*}{\partial x_k} C_{\gamma} \right] ds$$

where the derivatives of the Kernel functions are:

$$\frac{\partial f_{\gamma\beta}}{\partial x_k} = \frac{1}{2C_a} \left(2(1-p) \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_k} \delta_{\gamma\beta} \right.$$

$$\left. -2 \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_\gamma} \ell_\beta - C_r (1-2p) \ell_k \delta_{k\gamma} \right),$$

$$\frac{\partial Q^*}{\partial x_k} = - \frac{r C_r}{2 C_a} \frac{\partial r}{\partial x_k},$$

$$C_r = 2 \log r + 1,$$

$$x_k \equiv x_i \text{ or } y_i,$$

, as proved in appendix 4.10, and are given in explicit form in Table(4.6)

Then the strain components are:

$$\epsilon_x(x_i, y_i) = \frac{\partial u_i}{\partial x_i}, \quad \epsilon_y(x_i, y_i) = \frac{\partial v_i}{\partial y_i}, \quad \text{and} \quad \gamma_{xy}(x_i, y_i) = \frac{\partial u_i}{\partial y_i} + \frac{\partial v_i}{\partial x_i}$$

and the corresponding elastic stress components can be obtained by means of the generalised Hooke's law.

4.7.3.2 Stresses at a Boundary Node

The evaluation of strain components at a boundary point, by means of the previous boundary integral equations (equation (4.43)), is impossible due to the singularities occurring in $E_{\gamma\beta}(x_i, y_i)$.

Another approach can be used which is based upon boundary tractions and displacements. At any point on an isoparametric boundary element, it can be shown, by using the chain rule of partial differentiation, that:-

$$\frac{du}{d\xi} = \frac{\partial u}{\partial x} \frac{dx}{d\xi} + \frac{\partial u}{\partial y} \frac{dy}{d\xi}, \quad \text{and}$$

$$\frac{du}{d\xi} = \frac{\partial u}{\partial x} \frac{dx}{d\xi} + \frac{\partial u}{\partial y} \frac{dy}{d\xi}$$

Dividing the above equations by $J(\xi)$, where

$$J(\xi) = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

they may be rewritten as follows:-

$$\left. \begin{aligned} -m \cdot \left(\frac{\partial u}{\partial x}\right) + \ell \cdot \left(\frac{\partial u}{\partial y}\right) &= \left(\frac{\partial u}{\partial \xi}\right) / J(\xi) \\ -m \cdot \left(\frac{\partial v}{\partial x}\right) + \ell \cdot \left(\frac{\partial v}{\partial y}\right) &= \left(\frac{\partial v}{\partial \xi}\right) / J(\xi) \end{aligned} \right\} \quad (4.44)$$

From equation (2.16), it can be shown that, the traction components are:

$$\begin{aligned} T_x &= \ell \cdot \sigma_x + m \cdot \tau_{xy} \\ T_y &= \ell \cdot \tau_{xy} + m \cdot \sigma_y \end{aligned}$$

By substituting the stress components in terms of displacements (equation (4.2)), it can be proved that:-

$$\left. \begin{aligned} T_x &= \ell \cdot [d_{11} \left(\frac{\partial u}{\partial x}\right) + d_{12} \left(\frac{\partial v}{\partial y}\right)] + m \cdot d_{33} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] + T_x^o \\ T_y &= \ell \cdot d_{33} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] + m \cdot [d_{21} \left(\frac{\partial u}{\partial x}\right) + d_{22} \left(\frac{\partial v}{\partial y}\right)] + T_y^o \end{aligned} \right\} \quad (4.45)$$

Where

$$\left. \begin{aligned} T_x^o &= \ell [d_{11} \epsilon_x^{Lp} + d_{12} \epsilon_y^{Lp}] + m d_{33} \gamma_{xy}^{Lp} \\ T_y^o &= \ell d_{33} \gamma_{xy}^{Lp} + m [d_{21} \epsilon_x^{Lp} + d_{22} \epsilon_y^{Lp}] \end{aligned} \right\} \quad (4.46)$$

Solving the four linear simultaneous equations (4.44) and

(4.45), the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ can be obtained.

Finally the strain components can be expressed as:-

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The elastic stress components are evaluated as follows:

$$\underline{\sigma} = \underline{D} \cdot \underline{\epsilon}$$

4.7.3.3 Stresses at Corner Points

If the source point is a corner point, then it is difficult to use the previous technique to evaluate the stress and strain components. This difficulty arises from the discontinuity in the field variables, such as traction, and the direction cosines of the normal vector.

An alternative approach based upon a double-node representation of corners is developed in this work for calculating the strain and stress components.

The corner source point (x_i, y_i) is a node in two different elements, element "1" and element "2" as shown in Fig(4.12). The surface traction equilibrium equations for element "1" can be written as follows:-

$$(T_x)_1 = l_1 \cdot \sigma_x + m_1 \cdot \tau_{xy}$$

$$(T_y)_1 = l_1 \cdot \tau_{xy} + m_1 \cdot \sigma_y$$

Where

l_1, m_1 are the directional cosines of the normal to element "1" at node (x_i, y_i)

By multiplying the first of the above equations by l_1 and the second equation by m_1 , it can be deduced that:-

$$l_1 \cdot (T_x)_1 - m_1 \cdot (T_y)_1 = l_1^2 \cdot \sigma_x - m_1^2 \cdot \sigma_y$$

Similarly, for element "2", it can be shown that:-

$$(T_x)_2 = l_2 \cdot \sigma_x + m_2 \cdot \tau_{xy}$$

$$(T_y)_2 = l_2 \cdot \tau_{xy} + m_2 \cdot \sigma_y$$

where

l_2, m_2 are the directional cosines of the normal to element "2" at node (x_1, y_1) ,

and

$$l_2 \cdot (T_x)_2 - m_2 \cdot (T_y)_2 = l_2^2 \cdot \sigma_x - m_2^2 \cdot \sigma_y$$

Since $(T_x)_1, (T_y)_1, (T_x)_2$, and $(T_y)_2$ are known parameters, then the following parameters can be evaluated:-

$$f_1 = l_1 \cdot (T_x)_1 - m_1 \cdot (T_y)_1$$

$$f_2 = l_2 \cdot (T_x)_2 - m_2 \cdot (T_y)_2$$

Hence, it can be proved that:-

$$f_1 = l_1^2 \cdot \sigma_x - m_1^2 \cdot \sigma_y$$

$$f_2 = l_2^2 \cdot \sigma_x - m_2^2 \cdot \sigma_y$$

or in matrix form:

$$\begin{bmatrix} l_1^2 & -m_1^2 \\ l_2^2 & -m_2^2 \end{bmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Defining

$$\text{Det.} = \begin{vmatrix} l_1^2 & -m_1^2 \\ l_2^2 & -m_2^2 \end{vmatrix} = (l_2 \cdot m_1 - l_1 \cdot m_2) \cdot (l_2 \cdot m_1 + l_1 \cdot m_2)$$

If

$$\text{Det.} = 0,$$

then $l_2 \cdot m_1 = \pm l_1 \cdot m_2$, i.e. the tangential to element "1" and element "2" are parallel and the point is not a corner point.

If $\text{Det.} \neq 0$, then the above matrix equation can be solved by using Cramer's Rule as follows:

$$\sigma_x = \frac{\begin{vmatrix} f_1 & -m_1^2 \\ f_2 & -m_2^2 \end{vmatrix}}{\text{Det.}} = \frac{m_1^2 \cdot f_2 - m_2^2 \cdot f_1}{\text{Det.}}$$

$$\sigma_y = \frac{\begin{vmatrix} l_1^2 & f_1 \\ l_2^2 & f_2 \end{vmatrix}}{\text{Det.}} = \frac{l_1^2 \cdot f_2 - l_2^2 \cdot f_1}{\text{Det.}}$$

The shear stress can be averaged as follows:-

$$\tau_{xy} = \frac{1}{2} [m_1 \cdot (T_x)_1 + l_1 \cdot (T_y)_1 + m_2 \cdot (T_x)_2 + l_2 \cdot (T_y)_2 - (l_1 \cdot m_1 + l_2 \cdot m_2) \cdot (\sigma_x + \sigma_y)]$$

4.7.3.4 Stresses at Points near to the Boundary

If the source point is very close to the boundary, then the strain and stress components, which are calculated by using equation (4.43), are found to be inaccurate. The component values are found to be more accurate, if the point can be assumed to lie on the boundary, as shown in Fig(4.13.a). Otherwise it is recommended

to use a finite-difference scheme for evaluating the strain components, since the displacements can be obtained very accurately from the B.I.E.'s. Four points "a,b,c,d" can be defined around the source point (x_i, y_i) as shown in Fig(4.13.b), where:

$$\begin{aligned} a &= (x_i - \Delta x, y_i), \\ b &= (x_i + \Delta x, y_i), \\ c &= (x_i, y_i - \Delta y), \text{ and} \\ d &= (x_i, y_i + \Delta y). \end{aligned}$$

The values of Δx and Δy should be selected such that "a,b,c,d" are still inside the domain, or on the boundary.

The strain components can therefore be obtained as follows:-

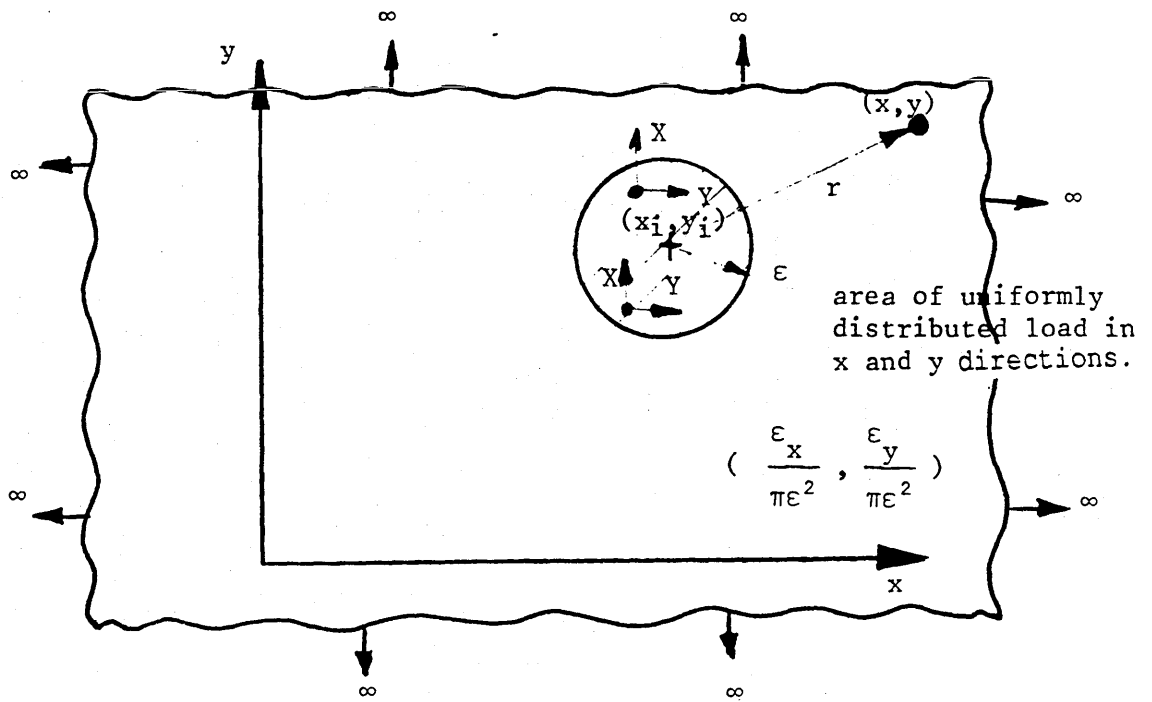
$$(\epsilon_x)_i = \left(\frac{\partial u}{\partial x} \right)_i = \frac{u_b - u_a}{2 \Delta x}$$

$$(\epsilon_y)_i = \left(\frac{\partial v}{\partial y} \right)_i = \frac{v_d - v_c}{2 \Delta y}$$

$$(\gamma_{xy})_i = \left(\frac{\partial u}{\partial y} \right)_i + \left(\frac{\partial v}{\partial x} \right)_i = \frac{v_b - v_a}{2 \Delta x} + \frac{u_d - u_c}{2 \Delta y}$$

and the corresponding stress vector can be expressed as follows:-

$$\underline{\sigma}_i = \underline{D} \quad \underline{\epsilon}_i$$



Fig(4.1) Infinite Domain with Uniformly Distributed Load over Area $\pi \epsilon^2$

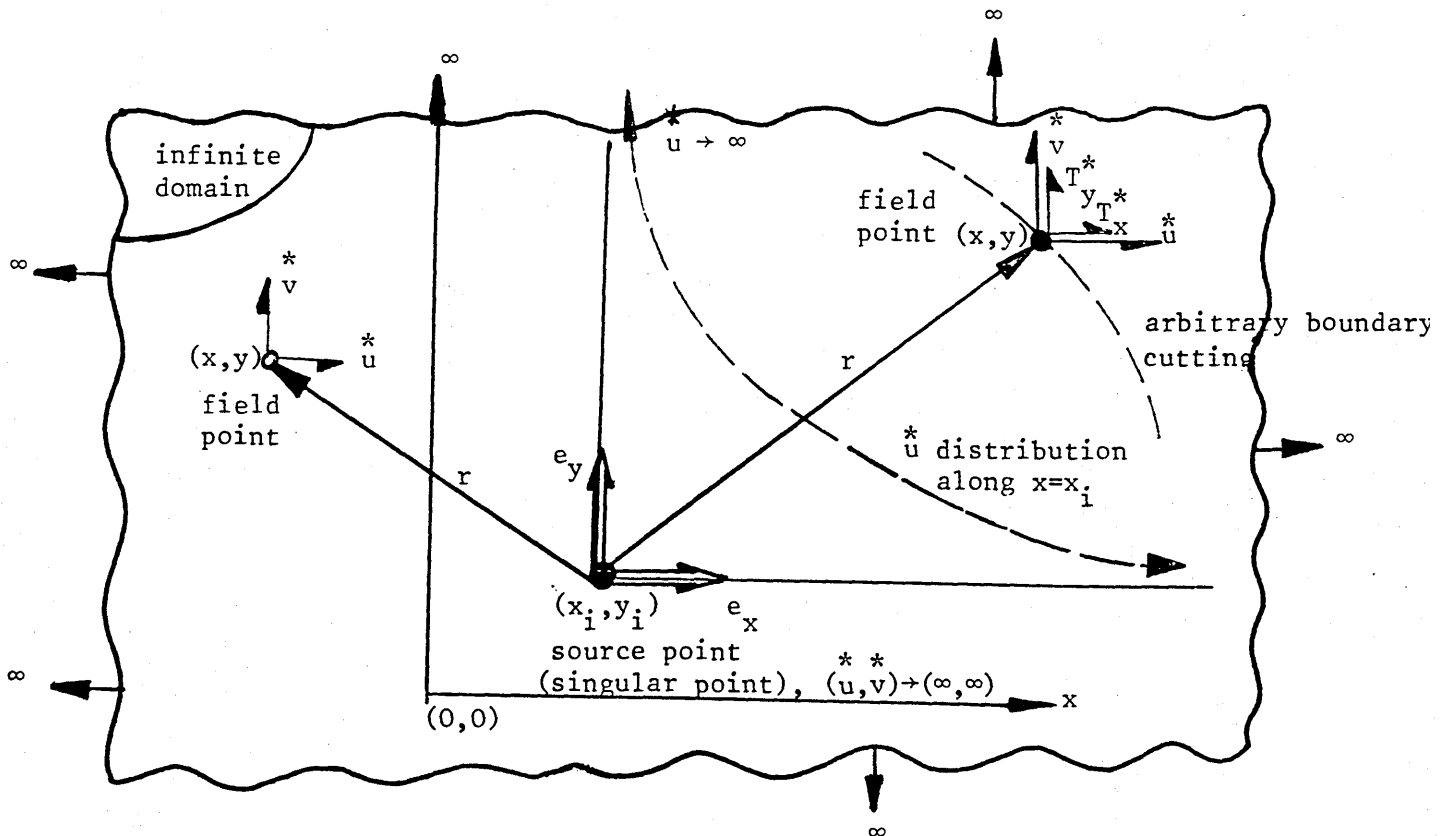
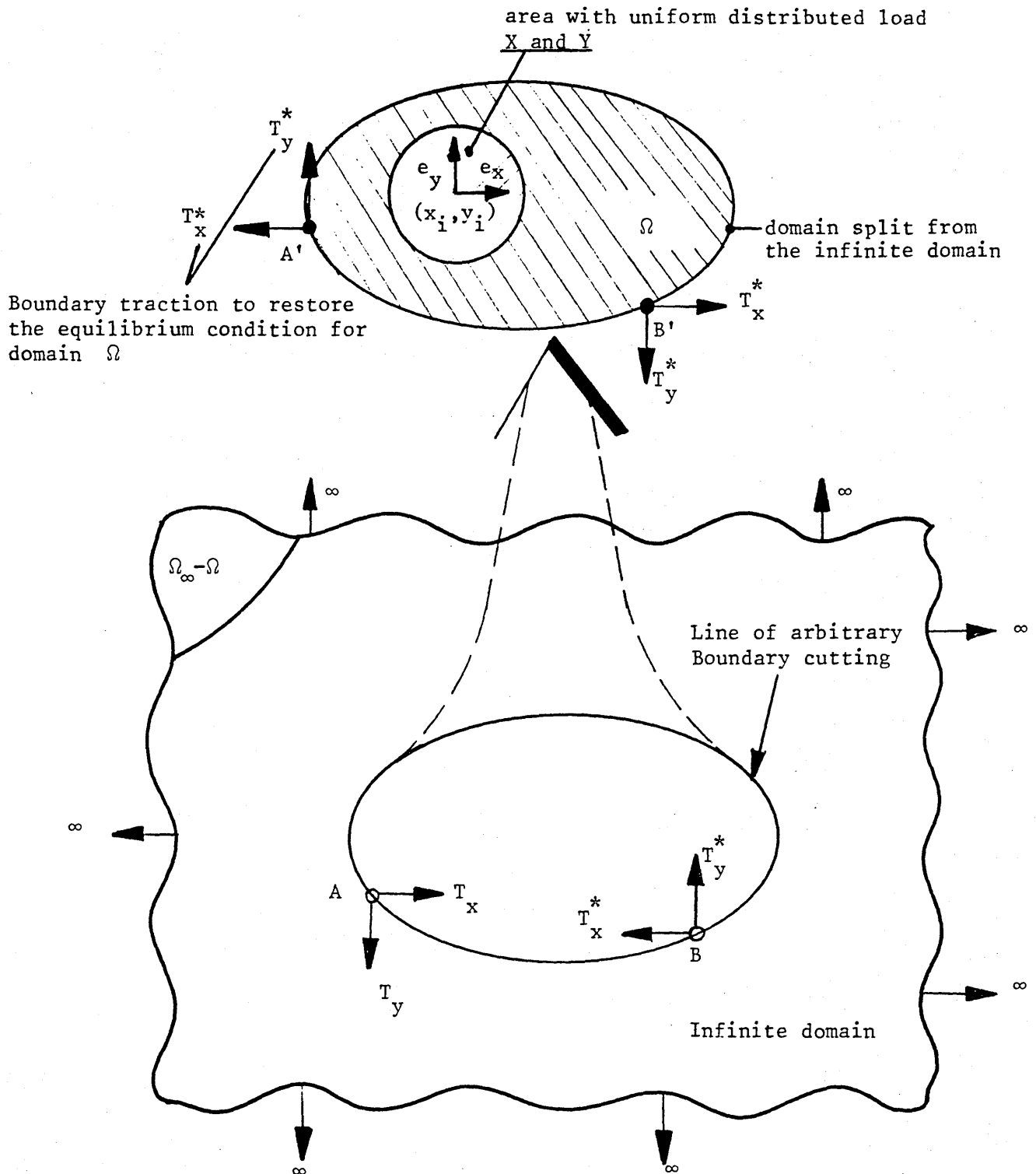
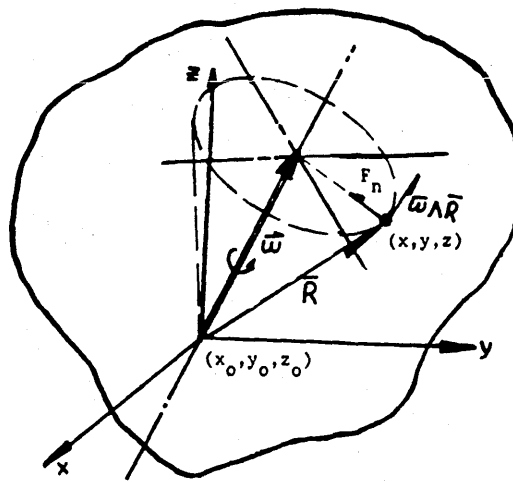


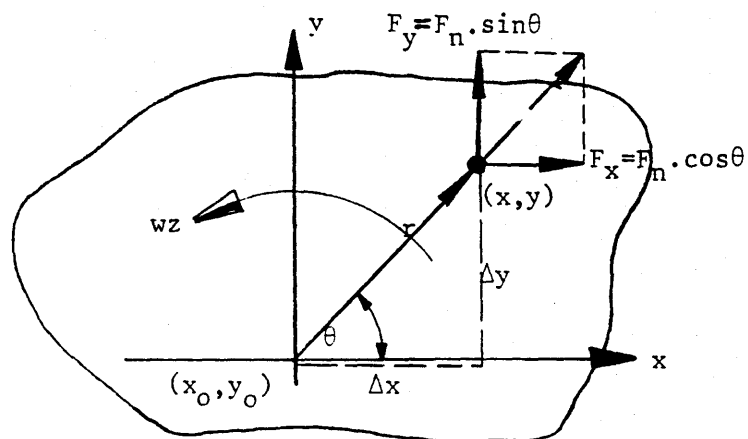
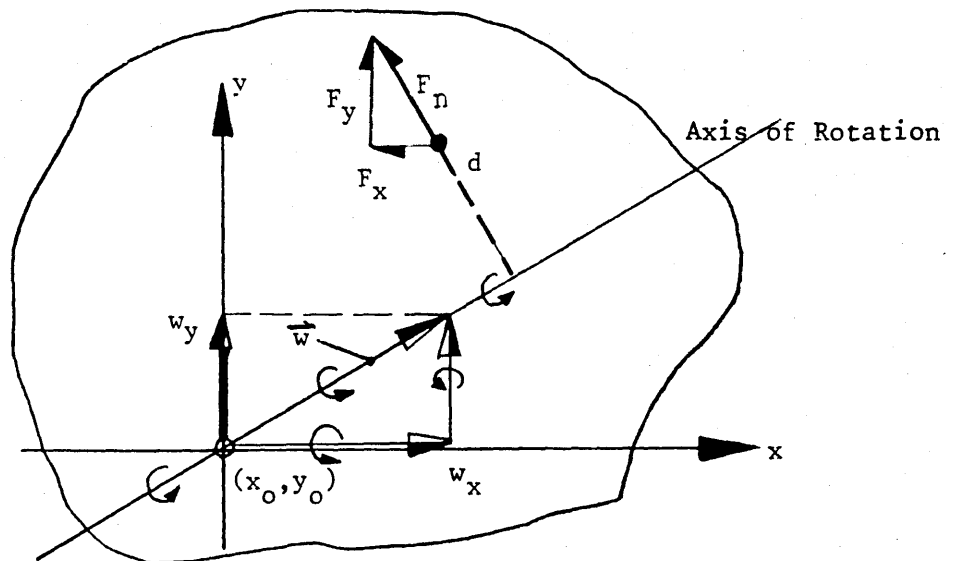
Fig.(4.2) Distribution of the Fundamental Displacement Solution on an Infinite Domain due to a Concentrated Load (e_x, e_y) at Source Point (x_i, y_i)



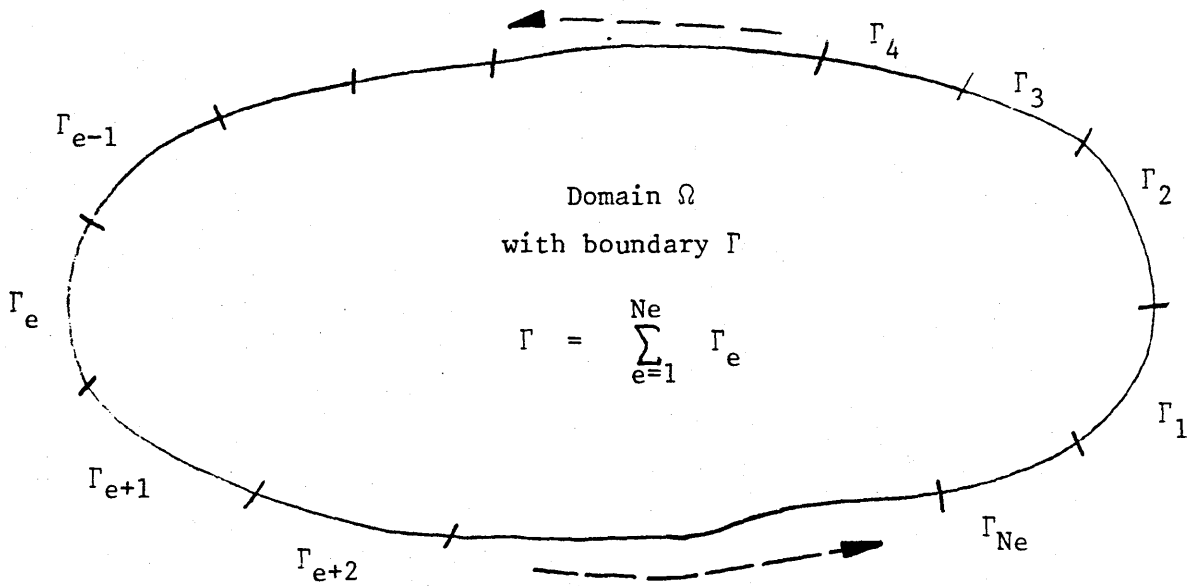
Fig(4.3) Infinite Domain Divided into Two Parts by an Arbitrary Boundary Condition cutting



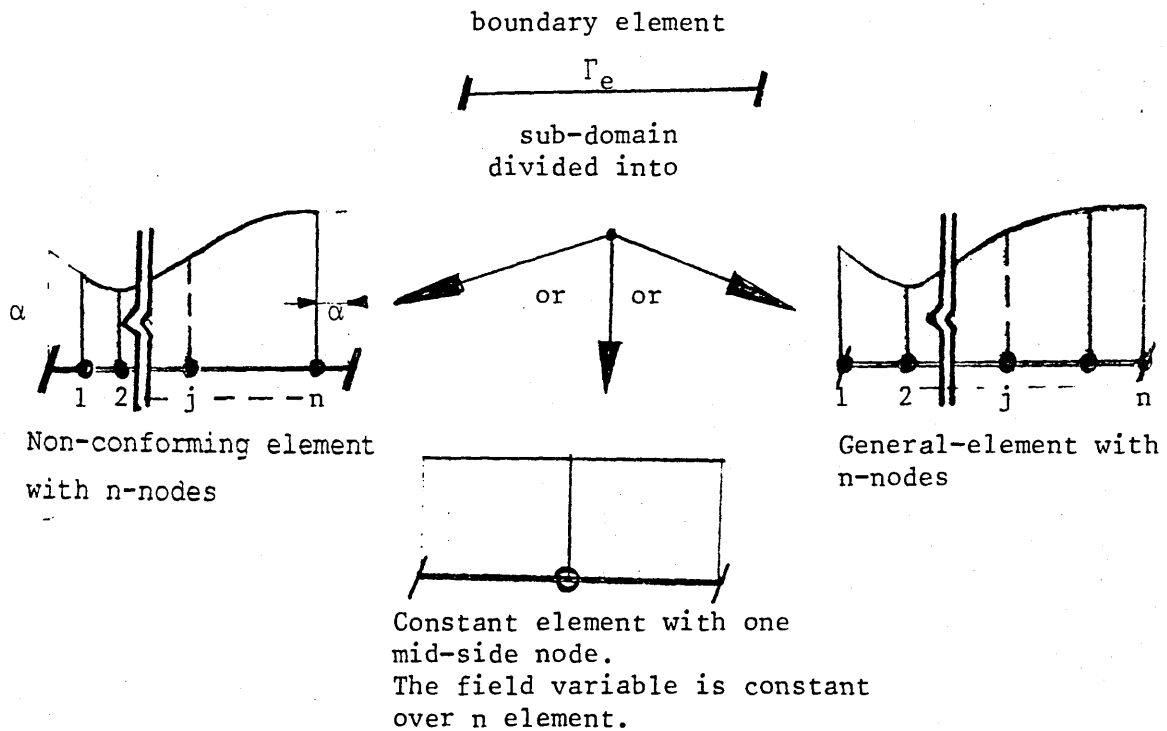
a) General Rotation in Space

b) Case of Rotation about the z -axisc) Case of Rotation about a Line in the x - y Plane

Fig(4.4) Cases of Rotational Inertia

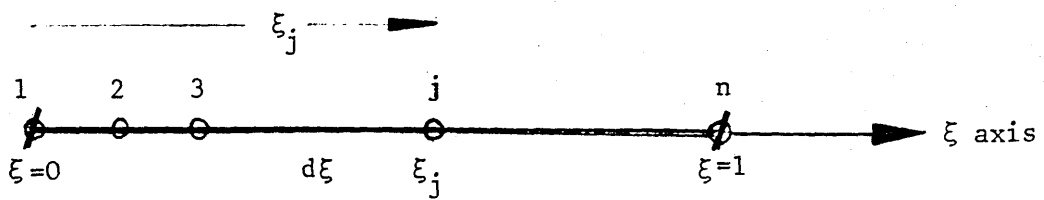
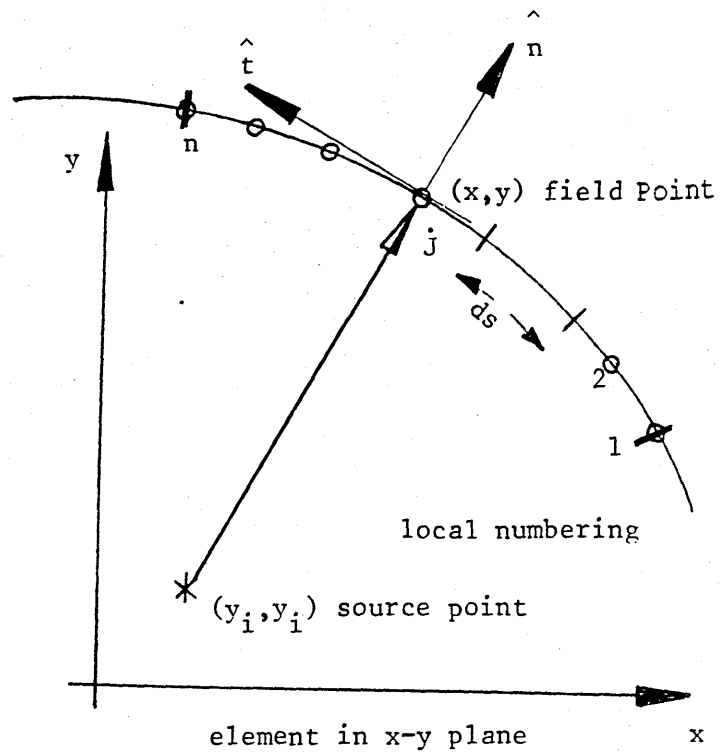


- a) Piece-Wise Discretisation of the Boundary Γ , into a Number of Sub-Boundaries



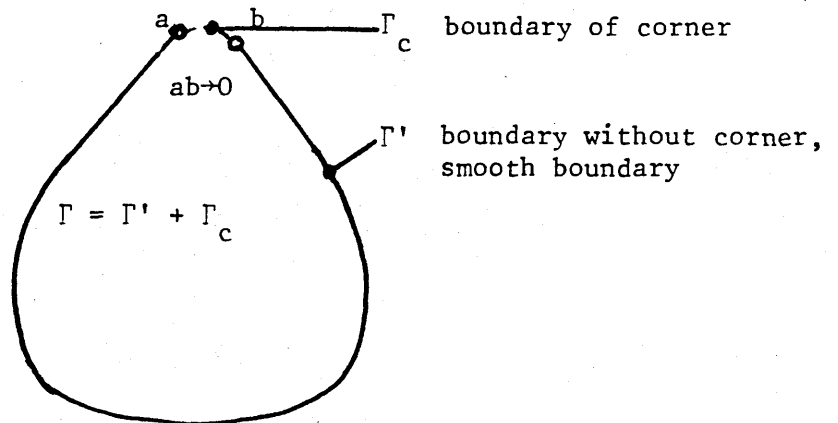
- b) Point-Wise Discretisation Dividing the Sub-Boundary into a Suitable Number of Nodes 'n'

Fig(4.5) Discretisation Concepts

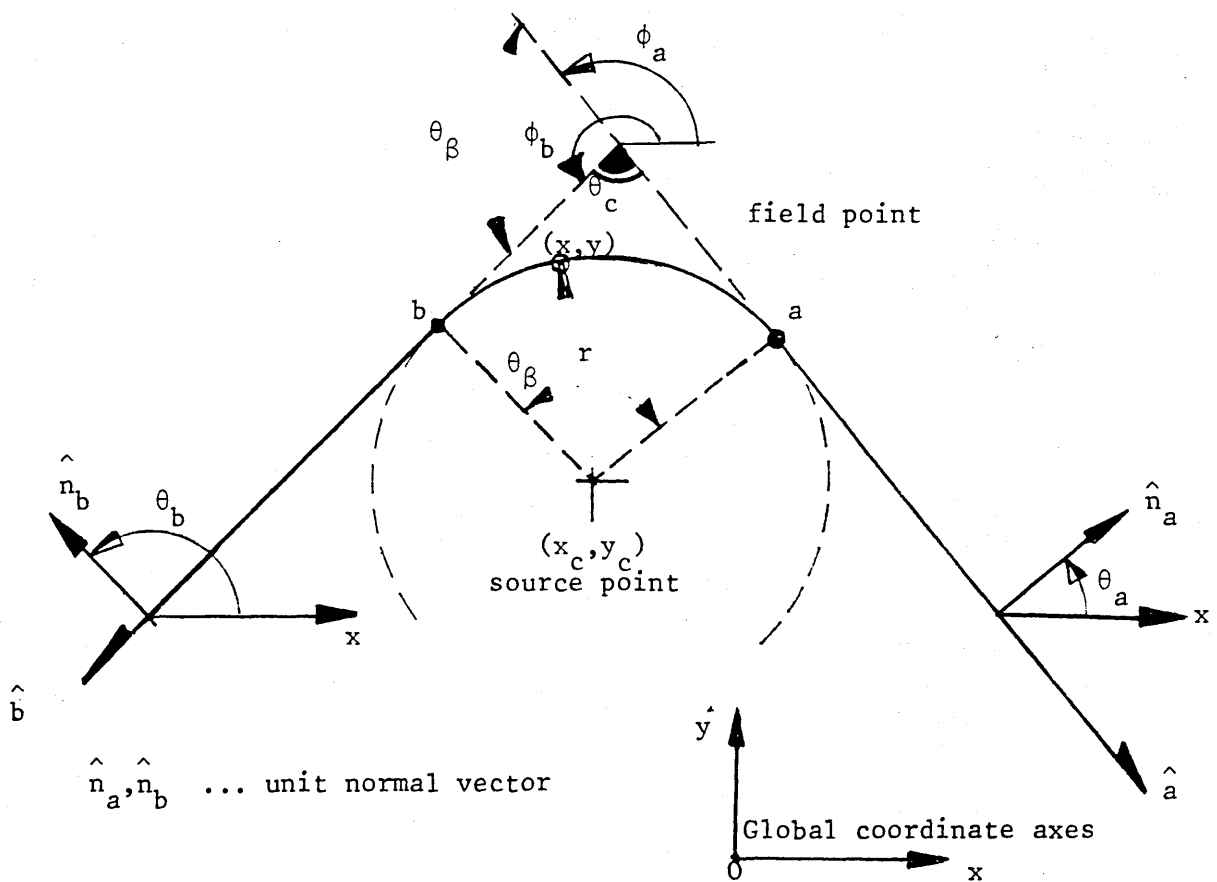


$$\xi_j = \frac{j-1}{n-1} \quad \dots \text{intrinsic coordinate at node } j$$

Fig(4.6) n -node Isoparametric Boundary Element and the Intrinsic Transformation

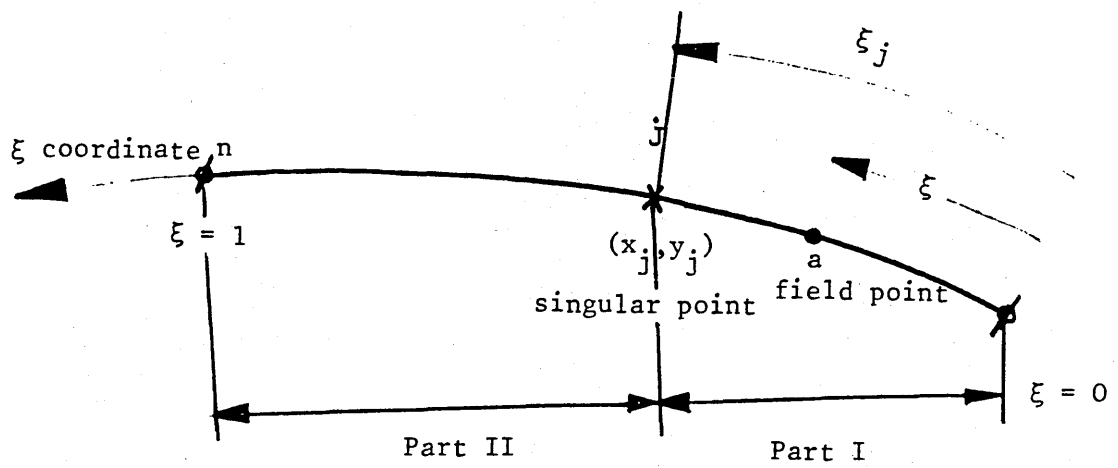


a) Smooth boundary and boundary of the corner point

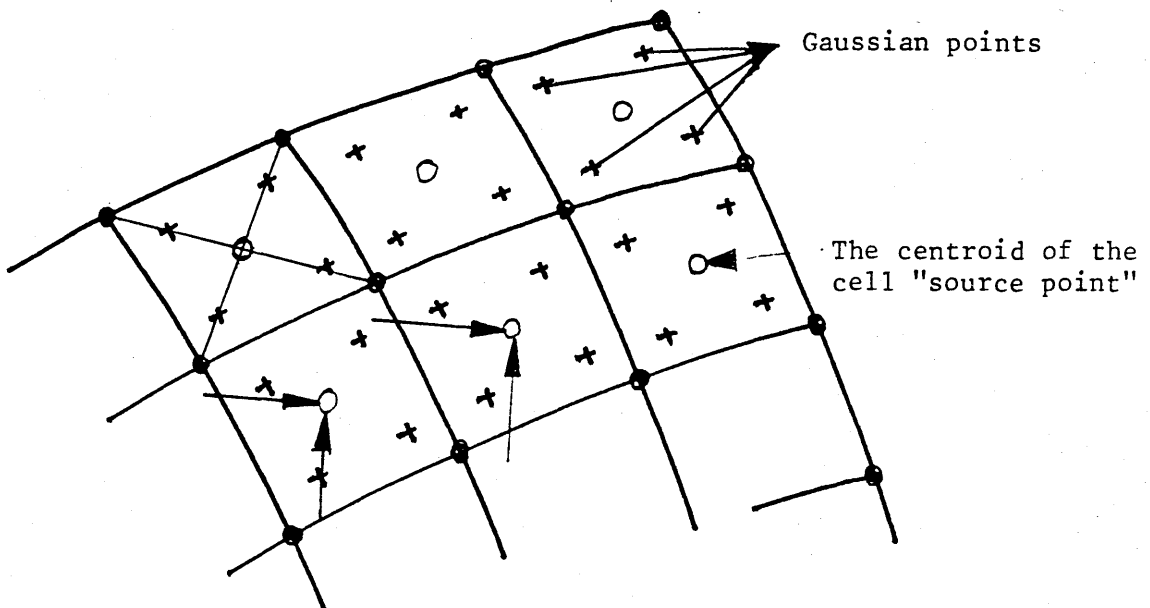


b) Smooth corner model

Fig(4.8) Boundary with Corner

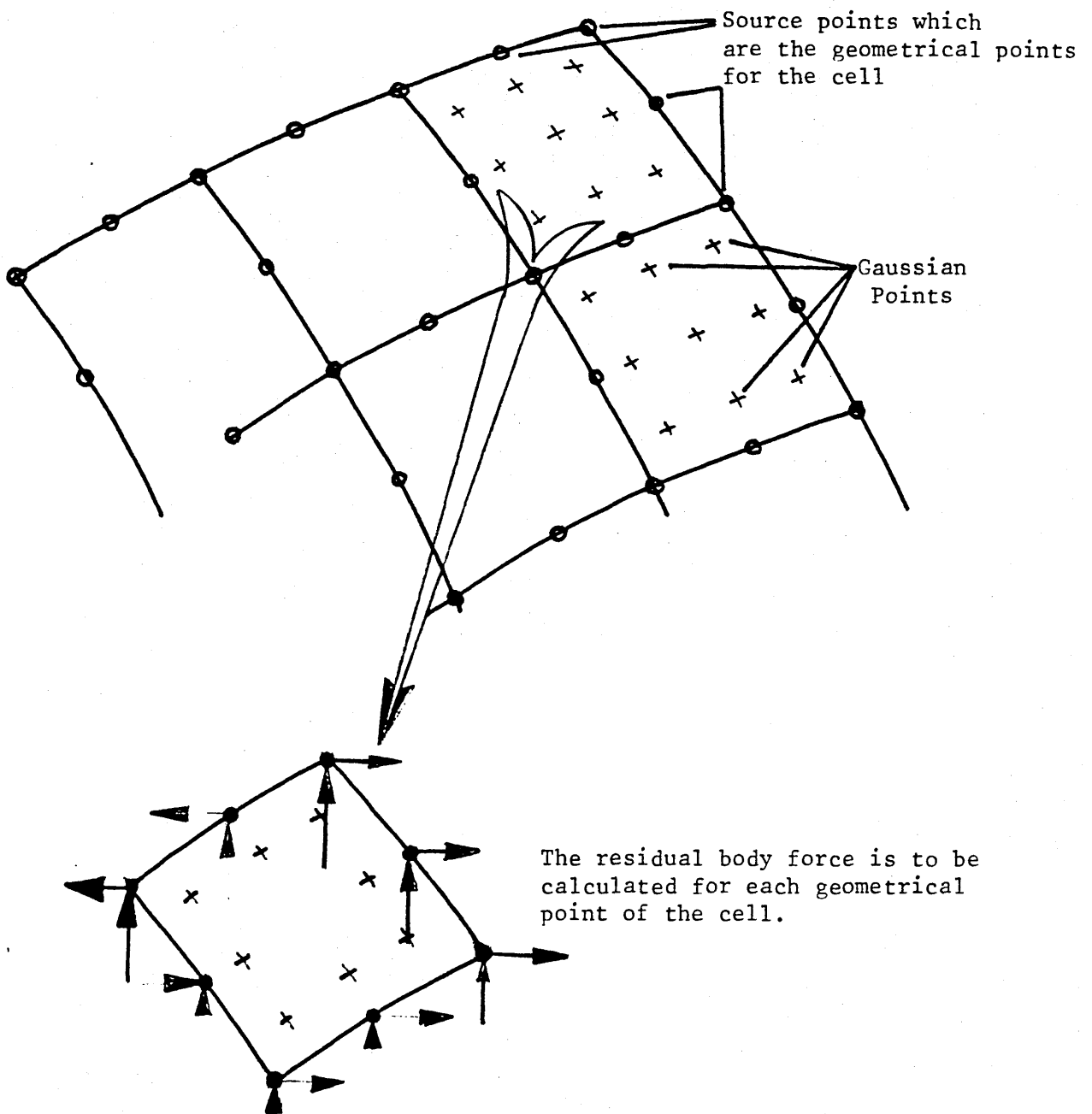


Fig(4.9) Isoparametric Element with Singularity Condition

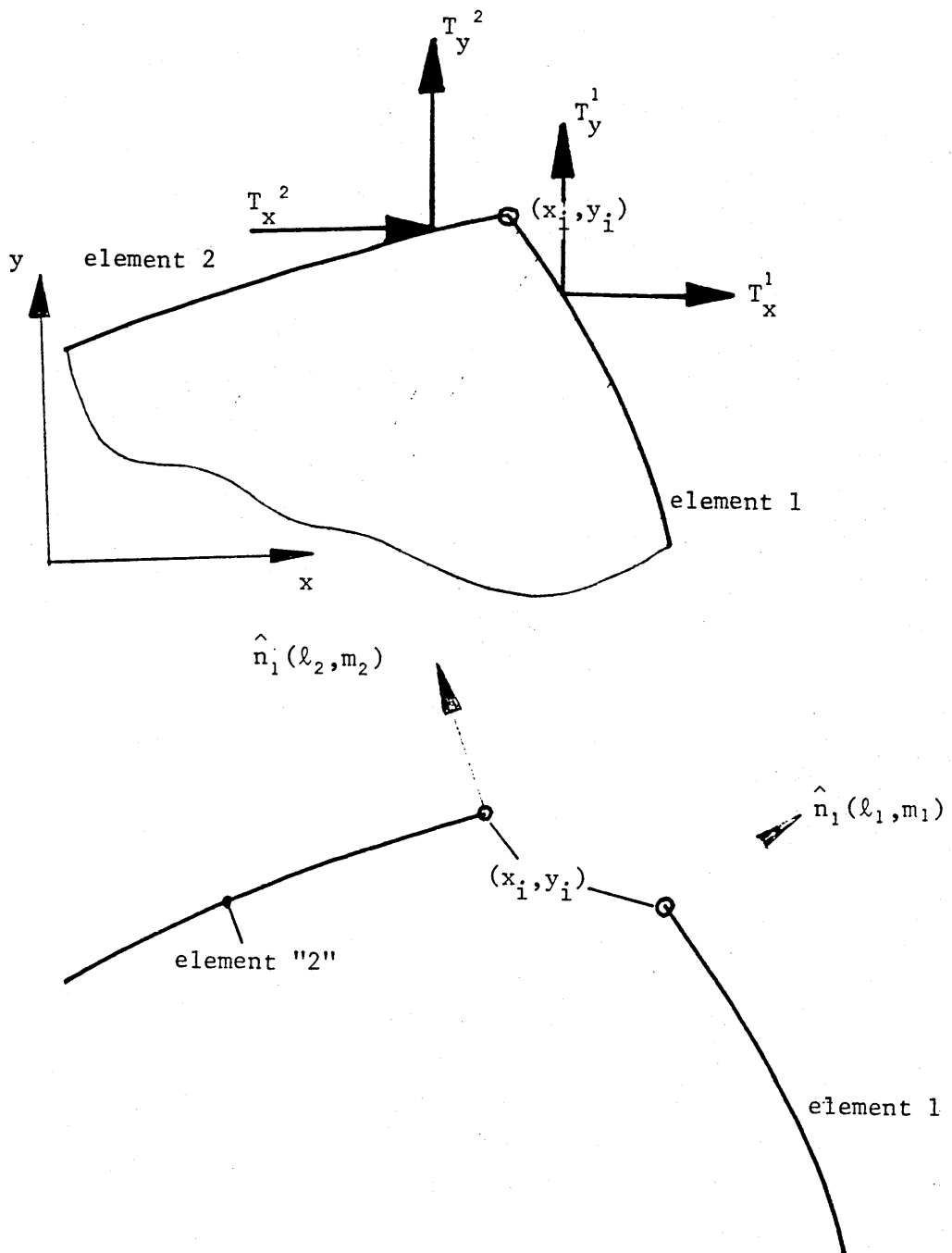


The residual body force is to be evaluated at the centroid of each cell

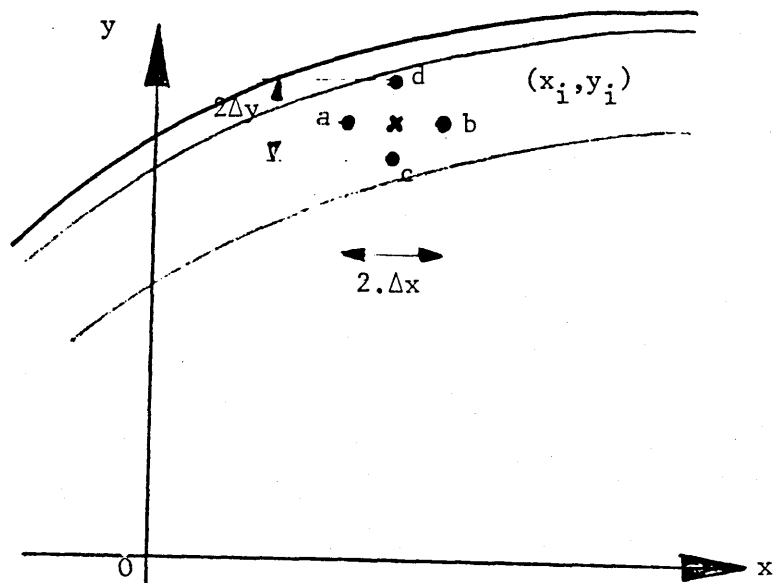
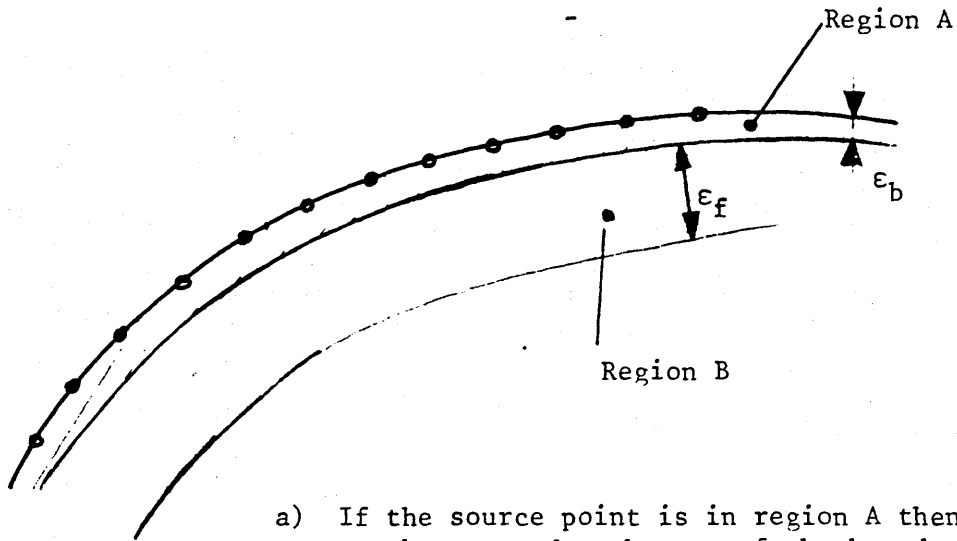
Fig(4.10) Average-Cell Approach



Fig(4.11) Domain Divided into a Number of Integration Cells, Each Cell with n -nodes.



Fig(4.12) Double-Node Representation for a Corner Point



Fig(4.13) Finite Difference Scheme

$$G_{11} = \frac{1}{C_a} [C_1 \text{Log}(1/r) + \left(\frac{\partial r}{\partial x}\right)^2]$$

$$G_{12} = \frac{1}{C_a} [\left(\frac{\partial r}{\partial x}\right) \left(\frac{\partial r}{\partial y}\right)] = G_{21}$$

$$G_{22} = \frac{1}{C_a} [C_1 \text{Log}(1/r) + \left(\frac{\partial r}{\partial y}\right)^2]$$

$$F_{11} = \frac{-1}{C_b r} [2 \left(\frac{\partial r}{\partial n}\right) \left(\frac{\partial r}{\partial x}\right)^2 + C_2 \left(\frac{\partial r}{\partial n}\right)]$$

$$F_{12} = \frac{-1}{C_b r} [2 \left(\frac{\partial r}{\partial n}\right) \left(\frac{\partial r}{\partial x}\right) \left(\frac{\partial r}{\partial y}\right) + C_2 \left(\left(\frac{\partial r}{\partial x}\right)^m - 1 \left(\frac{\partial r}{\partial y}\right) \right)]$$

$$F_{21} = \frac{-1}{C_b r} [2 \left(\frac{\partial r}{\partial n}\right) \left(\frac{\partial r}{\partial x}\right) \left(\frac{\partial r}{\partial y}\right) + C_2 \left(\left(\frac{\partial r}{\partial y}\right)^1 - m \left(\frac{\partial r}{\partial x}\right) \right)]$$

$$F_{22} = \frac{-1}{C_b r} [2 \left(\frac{\partial r}{\partial n}\right) \left(\frac{\partial r}{\partial y}\right)^2 + C_2 \left(\frac{\partial r}{\partial n}\right)]$$

Where

$$C_a = 8 \pi \mu (1-p)$$

$$C_b = 4 \pi (1-p)$$

$$C_1 = (3-4p)$$

$$C_2 = (1-2p)$$

$$p = \nu \quad \text{for plane-strain problems}$$

$$p = \frac{\nu}{1+\nu} \quad \text{for plane-stress problems}$$

Table (4.1) Explicit Expression for Displacement and Traction
Fundamental Solution Parameters ($G_{\gamma\beta}$ and $F_{\gamma\beta}$)

$$R_{11} = \frac{1}{C_a r} \left(\frac{\partial r}{\partial x} \right) \left[2 \left(\frac{\partial r}{\partial x} \right)^2 + (1-4p) \right]$$

$$R_{12} = \frac{1}{C_a r} \left(\frac{\partial r}{\partial x} \right) \left[2 \left(\frac{\partial r}{\partial y} \right)^2 - 1 \right]$$

$$R_{13} = \frac{2}{C_a r} \left(\frac{\partial r}{\partial y} \right) \left[2 \left(\frac{\partial r}{\partial x} \right)^2 + (1-2p) \right]$$

$$R_{21} = \frac{1}{C_a r} \left(\frac{\partial r}{\partial y} \right) \left[2 \left(\frac{\partial r}{\partial x} \right)^2 - 1 \right]$$

$$R_{22} = \frac{1}{C_a r} \left(\frac{\partial r}{\partial y} \right) \left[2 \left(\frac{\partial r}{\partial y} \right)^2 + (1-4p) \right]$$

$$R_{23} = \frac{2}{C_a r} \left(\frac{\partial r}{\partial x} \right) \left[2 \left(\frac{\partial r}{\partial y} \right)^2 + (1-2p) \right]$$

Where

$$C_a = 8 \pi \mu (1-p)$$

$p = \nu$ for plane-strain problems

$p = \frac{\nu}{1+\nu}$ for plane-stress problems

Table (4.2) Explicit Expression for Strain Fundamental Solution
Parameters ($R_{\gamma\beta}$)

$$f_{xx} = \frac{q^*}{2C_a} \left(1 \frac{\partial r}{\partial x} - 2(1-p) \frac{\partial r}{\partial n} \right)$$

$$f_{xy} = \frac{q^*}{2C_a} m \frac{\partial r}{\partial x}$$

$$f_{yx} = \frac{q^*}{2C_a} l \frac{\partial r}{\partial y}$$

$$f_{yy} = \frac{q^*}{2C_a} \left(m \frac{\partial r}{\partial y} - 2(1-p) \frac{\partial r}{\partial n} \right)$$

Where

$$q^* = r(2 \log r + 1)$$

$$C_a = 8 \pi \mu (1-p)$$

Table(4.3) Explicit Expression for Domain Terms and Kernel's Parameters

$$\frac{\partial F_{11}}{\partial x_i} = \frac{1}{C_b r^2} [2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x} (C_4 - 4 (\frac{\partial r}{\partial x})^2) + 1 (C_2 + 2 (\frac{\partial r}{\partial x})^2)]$$

$$\frac{\partial F_{21}}{\partial x_i} = \frac{1}{C_b r^2} [2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial y} (C_4 - 4 (\frac{\partial r}{\partial x})^2) + m (-C_4 + 2 (\frac{\partial r}{\partial x})^2)]$$

$$\frac{\partial F_{12}}{\partial x_i} = \frac{1}{C_b r^2} [-2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial y} (C_4 - 4 (\frac{\partial r}{\partial y})^2) - m (C_2 + 2 (\frac{\partial r}{\partial y})^2)]$$

$$\frac{\partial F_{22}}{\partial x_i} = \frac{1}{C_b r^2} [-2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x} (C_2 + 4 (\frac{\partial r}{\partial y})^2) + 1 (C_2 + 2 (\frac{\partial r}{\partial y})^2)]$$

$$\frac{\partial F_{11}}{\partial y_i} = \frac{1}{C_b r^2} [-2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial y} (C_2 + 4 (\frac{\partial r}{\partial x})^2) + m (C_2 + 2 (\frac{\partial r}{\partial x})^2)]$$

$$\frac{\partial F_{21}}{\partial y_i} = \frac{1}{C_b r^2} [-2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x} (C_4 - 4 (\frac{\partial r}{\partial x})^2) - 1 (C_2 + 2 (\frac{\partial r}{\partial x})^2)]$$

$$\frac{\partial F_{12}}{\partial y_i} = \frac{1}{C_b r^2} [2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x} (C_4 - 4 (\frac{\partial r}{\partial y})^2) + 1 (-C_4 + 2 (\frac{\partial r}{\partial y})^2)]$$

$$\frac{\partial F_{22}}{\partial y_i} = \frac{1}{C_b r^2} [2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial y} (C_4 - 4 (\frac{\partial r}{\partial y})^2) + m (C_2 + 2 (\frac{\partial r}{\partial y})^2)]$$

Where

$$C_a = 8 \pi \mu (1-p),$$

$$C_2 = (1-2p),$$

$$C_4 = (1+2p),$$

$p = \nu$ for plane-strain problems, and

$p = \frac{\nu}{1+\nu}$ for plane-stress problems.

Table(4.4) Derivation of Traction Fundamental Solution Parameters
($F_{\gamma\beta}$) With respect to (x_i, y_i) of the Source Point

$$\frac{\partial R_{11}}{\partial x_i} = \frac{1}{C_a r^2} [C_3 + 4 C_4 \left(\frac{\partial r}{\partial x} \right)^2 - 8 \left(\frac{\partial r}{\partial x} \right)^4]$$

$$\frac{\partial R_{21}}{\partial x_i} = \frac{1}{C_a r^2} [1 + 8 \left(\frac{\partial r}{\partial x} \right)^2 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{31}}{\partial x_i} = \frac{4}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [C_4 - 4 \left(\frac{\partial r}{\partial x} \right)^2]$$

$$\frac{\partial R_{12}}{\partial x_i} = \frac{2}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [3 - 4 \left(\frac{\partial r}{\partial x} \right)^2]$$

$$\frac{\partial R_{22}}{\partial x_i} = \frac{2}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [-C_3 - 4 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{32}}{\partial x_i} = \frac{2}{C_a r^2} [-C_2 + 4(1-p) \left(\frac{\partial r}{\partial y} \right)^2 - 8 \left(\frac{\partial r}{\partial x} \right)^2 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{11}}{\partial y_i} = \frac{2}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [-C_3 - 4 \left(\frac{\partial r}{\partial x} \right)^2]$$

$$\frac{\partial R_{21}}{\partial y_i} = \frac{2}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [3 - 4 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{31}}{\partial y_i} = \frac{2}{C_a r^2} [-C_2 + 4(1-p) \left(\frac{\partial r}{\partial x} \right)^2 - 8 \left(\frac{\partial r}{\partial x} \right)^2 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{12}}{\partial y_i} = \frac{1}{C_a r^2} [1 - 8 \left(\frac{\partial r}{\partial x} \right)^2 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\frac{\partial R_{22}}{\partial x_i} = \frac{1}{C_a r^2} [C_3 + 4 C_4 \left(\frac{\partial r}{\partial y} \right)^2 - 8 \left(\frac{\partial r}{\partial y} \right)^4]$$

$$\frac{\partial R_{32}}{\partial y_i} = \frac{4}{C_a r^2} \left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial r}{\partial y} \right) [C_4 - 4 \left(\frac{\partial r}{\partial y} \right)^2]$$

$$\text{where } \begin{array}{ll} C_2 = (1-2p), & C_3 = (1-4p), \\ C_4 = (1+2p), & C_a = 8\mu(1-p), \end{array}$$

$p = \nu$ for plane-strain problems, and

$p = \frac{\nu}{1+\nu}$ for plane-stress problems.

Table(4.5) Derivation of Strain Fundamental Solution Parameters $(R_{\gamma\beta})$ with respect to (x_i, y_i) of the Source point

$$\frac{\partial f_{xx}}{\partial x_i} = \frac{1}{2C_a} [-2(1-p)((\frac{\partial r}{\partial n})(\frac{\partial r}{\partial x}) + 1 C_r) + 1 (2 (\frac{\partial r}{\partial x})^2 + C_r)]$$

$$\frac{\partial f_{xx}}{\partial y_i} = \frac{1}{2C_a} [-2(1-p)((\frac{\partial r}{\partial n})(\frac{\partial r}{\partial y}) + m C_r) + 2 (1 (\frac{\partial r}{\partial x})(\frac{\partial r}{\partial y}))]$$

$$\frac{\partial f_{xy}}{\partial x_i} = \frac{1}{2C_a} m [2 (\frac{\partial r}{\partial x})^2 + C_r]$$

$$\frac{\partial f_{xy}}{\partial y_i} = \frac{1}{C_a} m (\frac{\partial r}{\partial x})(\frac{\partial r}{\partial y})$$

$$\frac{\partial f_{yx}}{\partial x_i} = \frac{1}{C_a} 1 (\frac{\partial r}{\partial x})(\frac{\partial r}{\partial y})$$

$$\frac{\partial f_{yx}}{\partial y_i} = \frac{1}{2C_a} 1 [2 (\frac{\partial r}{\partial y})^2 + C_r]$$

$$\frac{\partial f_{yy}}{\partial x_i} = \frac{1}{2C_a} [-2(1-p)((\frac{\partial r}{\partial n})(\frac{\partial r}{\partial x}) + 1 C_r) + 2 m (\frac{\partial r}{\partial x})(\frac{\partial r}{\partial y})]$$

$$\frac{\partial f_{yy}}{\partial y_i} = \frac{1}{2C_a} [-2(1-p)((\frac{\partial r}{\partial n})(\frac{\partial r}{\partial y}) + m C_r) + m (2 (\frac{\partial r}{\partial y})^2 + C_r)]$$

Where

$$C_a = 8 \pi \mu (1-p),$$

$$C_r = 2 \log(r) + 1,$$

$$p = \nu \quad \text{for plane-strain problems}$$

$$p = \frac{\nu}{1+\nu} \quad \text{for plane-stress problems}$$

Table(4.6) Derivation of the Domain Terms and the Kernel's Parameters With respect to (x_i, y_i) of the Source Point

CHAPTER FIVE

BOUNDARY ELEMENT THEORY FOR AXISYMMETRIC ELASTO-PLASTIC PROBLEMS

5.0 INTRODUCTION

In this chapter Boundary Integral Equations have been derived for small and large deflections of elastoplastic axisymmetric problems.

A fundamental solution of displacement has been obtained by using a three dimensional approach similar to that suggested by [97,133]. The Galerkin vector approach is also summarised. The traction, strain and stress fundamental components have been obtained from the displacement fundamental components by using the equilibrium and kinematic relations together with a generalised Hooke's Law.

The theory of the Boundary Element Method, as applied to axisymmetric problems, will be explained in detail. The procedures for the solution of the Boundary Integral Equation and the description of the elasto-plastic algorithms, will, however, not be mentioned here since are very similar to that explained in the previous chapter.

Some measures for the improvement of the accuracy of the solution, such as corner models, singular integration for the elliptic functions, axisymmetric domain loading expression, ... etc, have been developed and are explained in this chapter.

5.1 Derivation of the Inverse Expression for the Elastoplastic, Small and Large Deflection Problem

The derivation of the inverse weighted residual expression can be achieved by applying the weighted residual approach to the governing differential equations which define the field variable(s) distribution over the whole domain in a differential form.

(a) Governing Differential Equations for Elasto-Plastic Problems

The total strain vector can be defined as follows:-

$$\underline{\varepsilon} = \underline{\varepsilon}^s + \underline{\varepsilon}^L$$

where

$$\underline{\varepsilon} = \{\varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz}\} \text{ is the total strain vector,}$$

and

$$\underline{\varepsilon}^s = \left\{ \frac{\partial u}{\partial r} \quad \frac{\partial w}{\partial z} \quad \frac{u}{r} \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right\} \text{ is the small deflection, strain vector,}$$

$$\underline{\varepsilon}^L = \left[\begin{array}{c} \frac{1}{2} \left(\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial w}{\partial r} \right)^2 \right) \\ \frac{1}{2} \left(\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right) \\ \frac{1}{2} \left(\frac{u}{r} \right)^2 \\ \frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \end{array} \right] \text{ is the large deflection strain vector}$$

as explained by equations 2.24.

In a manner similar to that employed for 2-dimensional elasto-plastic, small and large deflection problems, it can be shown from equation 4.1, that:-

$$\underline{\sigma} = D(\underline{\varepsilon}^s - \underline{\varepsilon}^{LP})$$

where

$\underline{\varepsilon}^{LP} = \underline{\varepsilon}^P - \underline{\varepsilon}^L$ is the large deflection and plastic strain vector.

The above equation can be rewritten in explicit form as follows:

$$\left. \begin{aligned} \sigma_r &= d_{11} \left(\frac{\partial u}{\partial r} \right) + d_{12} \left(\frac{\partial w}{\partial z} \right) + d_{13} \frac{u}{r} - \sigma_r^{LP} \\ \sigma_z &= d_{21} \left(\frac{\partial u}{\partial r} \right) + d_{22} \left(\frac{\partial w}{\partial z} \right) + d_{23} \frac{u}{r} - \sigma_z^{LP} \\ \sigma_\theta &= d_{31} \left(\frac{\partial u}{\partial r} \right) + d_{32} \left(\frac{\partial w}{\partial z} \right) + d_{33} \frac{u}{r} - \sigma_\theta^{LP} \\ \tau_{rz} &= d_{44} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - \tau_{rz}^{LP} \end{aligned} \right\} \quad (5.1)$$

where

$$\left. \begin{aligned} \sigma_r^{LP} &= d_{11} \varepsilon_r^{LP} + d_{12} \varepsilon_z^{LP} + d_{13} \varepsilon_\theta^{LP} \\ \sigma_z^{LP} &= d_{21} \varepsilon_r^{LP} + d_{22} \varepsilon_z^{LP} + d_{23} \varepsilon_\theta^{LP} \\ \sigma_\theta^{LP} &= d_{31} \varepsilon_r^{LP} + d_{32} \varepsilon_z^{LP} + d_{33} \varepsilon_\theta^{LP} \\ \tau_{rz}^{LP} &= d_{44} \gamma_{rz}^{LP} \end{aligned} \right\} \quad (5.2)$$

$d_{11}, d_{12}, d_{13} \dots d_{44}$ are the elements of the axisymmetric $\underline{D}$ matrix.

The equilibrium equations of an infinitesimal element at a given point inside a body subjected to a domain load are:-

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0$$

Substituting equation (5.1) into the above equilibrium equation, the general differential equations can be deduced as follows:-

$$\left. \begin{aligned} & \frac{\partial}{\partial r} \left[d_{11} \frac{\partial u}{\partial r} + d_{12} \frac{\partial w}{\partial z} + d_{13} \frac{u}{r} - \sigma_r^{Lp} \right] \\ & + \frac{\partial}{\partial z} \left[d_{44} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - \tau_{rz}^{Lp} \right] \\ & + \frac{1}{r} \left[d_{11} \frac{\partial u}{\partial r} + d_{12} \frac{\partial w}{\partial z} + d_{13} \frac{u}{r} - \sigma_r^{Lp} \right] \\ & - \frac{1}{r} \left[d_{31} \frac{\partial u}{\partial r} + d_{32} \frac{\partial w}{\partial z} + d_{33} \frac{u}{r} - \sigma_\theta^{Lp} \right] + R = 0 \\ & \frac{\partial}{\partial r} \left[d_{44} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - \tau_{rz}^{Lp} \right] \\ & + \frac{\partial}{\partial z} \left[d_{21} \frac{\partial u}{\partial r} + d_{22} \frac{\partial w}{\partial z} + d_{23} \frac{u}{r} - \sigma_z^{Lp} \right] \\ & + \frac{1}{r} \left[d_{44} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - \tau_{rz}^{Lp} \right] + Z = 0 \end{aligned} \right\} \quad (5.3)$$

(b) Inverse Weighted-Residual Expression

A weighted-residual expression equivalent to equations (5.3), can be deduced as follows:-

$$\begin{aligned} & \iint (2\pi r \dot{u}) \left[\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} (\sigma_r - \sigma_\theta) + R \right] dr dz \\ + & \iint (2\pi r \dot{w}) \left[\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z \right] dr dz = 0 \end{aligned}$$

Applying the integration-by-parts theorems, as described in appendix 5.1, the above weighted residual expression can be written in terms of the following weak form:-

$$\begin{aligned} & \int \dot{u}^* \cdot \sigma_r \cdot \ell \, d\Gamma - \iint \left(\frac{\partial \dot{u}^*}{\partial r} + \frac{\dot{u}^*}{r} \right) \sigma_r \, d\Omega \\ + & \int \dot{u}^* \cdot \tau_{rz} \cdot m \, d\Gamma - \iint \frac{\partial \dot{u}^*}{\partial z} \cdot \tau_{rz} \, d\Omega \\ + & \int \dot{w}^* \cdot \tau_{rz} \cdot \ell \cdot d\Gamma - \iint \left(\frac{\partial \dot{w}^*}{\partial r} + \frac{\dot{w}^*}{r} \right) \tau_{rz} \, d\Omega \\ + & \int \dot{w}^* \cdot \sigma_z \cdot m \, d\Gamma - \iint \frac{\partial \dot{w}^*}{\partial z} \sigma_z \, d\Omega \\ + & \iint \left(\frac{\sigma_r - \sigma_\theta}{r} + R \right) \dot{u} \, d\Omega \\ + & \iint \left(\frac{\tau_{rz}}{r} + Z \right) \dot{w} \, d\Omega = 0 \end{aligned}$$

where

$$d\Omega = 2\pi r \, dr \, dz, \quad d\Gamma = 2\pi r \, ds$$

and

ℓ, m = are the directional cosines in r and z directions respectively.

The Boundary Integration terms can be written as follows:-

$$I_B = \oint [\dot{u}(\sigma_r \cdot \ell + \tau_{rz} \cdot m) + \dot{w}(\tau_{rz} \ell + \sigma_z m)] \, d\Gamma$$

But, from equations (2.18), it can be shown that the traction components in the r and z directions are respectively:-

$$T_r = \ell \cdot \sigma_r + m \cdot \tau_{rz}$$

$$T_z = \ell \cdot \tau_{rz} + m \cdot \sigma_z$$

Therefore

$$I_B = \oint (\dot{u} T_r + \dot{w} T_z) \, d\Gamma \quad (5.4)$$

The domain integration terms can be written, after simplification, as follows:-

$$\begin{aligned} I_D &= \iint (R\dot{u} + Z\dot{w}) \, d\Omega \\ &- \iint \left(\frac{\partial \dot{u}}{\partial r} \sigma_r + \frac{\partial \dot{u}}{\partial z} \tau_{rz} + \frac{\dot{u}}{r} \sigma_\theta \right) \, d\Omega \\ &- \iint \left(\frac{\partial \dot{w}}{\partial r} \tau_{rz} + \frac{\partial \dot{w}}{\partial z} \sigma_z \right) \, d\Omega \end{aligned}$$

The above equation can then be rewritten in terms of displacement components, using equations 5.1, as follows:-

$$\begin{aligned}
 I_D &= \iint (R\dot{u} + Z\dot{w}) d\Omega \\
 &- \iint \frac{\partial \dot{u}}{\partial r} \left(d_{11} \frac{\partial u}{\partial r} + d_{12} \frac{\partial w}{\partial z} + d_{13} \frac{u}{r} \right) d\Omega \\
 &- \iint \frac{\partial \dot{u}}{\partial z} \cdot d_{44} \cdot \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) d\Omega \\
 &- \iint \frac{\dot{u}}{r} \left(d_{31} \frac{\partial u}{\partial r} + d_{32} \frac{\partial w}{\partial z} + d_{33} \frac{u}{r} \right) d\Omega \\
 &- \iint \frac{\partial \dot{w}}{\partial r} \cdot d_{44} \cdot \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) d\Omega \\
 &- \iint \frac{\partial \dot{w}}{\partial z} \left(d_{21} \frac{\partial u}{\partial r} + d_{22} \frac{\partial w}{\partial z} + d_{23} \frac{u}{r} \right) d\Omega \\
 &+ \iint \left[\frac{\partial \dot{u}}{\partial r} \sigma_r^{LP} + \frac{\partial \dot{w}}{\partial z} \sigma_z^{LP} + \frac{\dot{u}}{r} \sigma_\theta^{LP} \right. \\
 &\quad \left. + \left(\frac{\partial \dot{w}}{\partial r} + \frac{\partial \dot{u}}{\partial z} \right) \tau_{rz}^{LP} \right] d\Omega
 \end{aligned}$$

Using integration-by-parts theorems, as described in appendix 5.1, the above expression can be integrated once more, and simplified, as shown in appendix 5.2, into the following expression:-

$$\begin{aligned}
 I_D &= \iint (R\dot{u} + Z\dot{w}) d\Omega - \oint (\dot{T}_r u + \dot{T}_z w) d\Gamma \\
 &+ \iint (\underline{\sigma}^{LP})^t \cdot \underline{\underline{\varepsilon}} d\Omega - \iint (\dot{R} u + \dot{Z} w) d\Omega
 \end{aligned}$$

Finally:-

$$I_D + I_B = 0$$

i.e.

$$\begin{aligned} \oint_{\Gamma} (T_r \dot{u} + T_z \dot{w}) d\Gamma + \iint_{\Omega} (Ru + Z \dot{w}) d\Omega \\ + \iint_{\Gamma} (\underline{\sigma}^{LP})^t \cdot \underline{\varepsilon} d\Omega = \oint_{\Omega} (\dot{T}_r u + \dot{T}_z w) d\Gamma \\ + \iint_{\Omega} (\dot{R}u + \dot{Z} w) d\Omega \end{aligned} \quad (5.5)$$

where

u, w represent the approximate solution which satisfies the given boundary conditions,

$\dot{u}, \dot{w}$ are the proper weighted residual functions in the r and z directions respectively, which are the elastic solution of the structure, with the same boundary conditions, and under point loads in the r and z directions,

$$\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon} ,$$

$$\dot{T}_r = \ell \dot{\sigma}_r + m \dot{\tau}_{rz} , \quad \dot{T}_z = \ell \dot{\tau}_{rz} + m \dot{\sigma}_z$$

$$\dot{R} = - \left(\frac{\partial \dot{\sigma}_r}{\partial r} + \frac{\partial \dot{\tau}_{rz}}{\partial z} + \frac{\dot{\sigma}_r - \dot{\sigma}_\theta}{r} \right) , \quad \text{and}$$

$$\dot{Z} = - \left(\frac{\partial \dot{\tau}_{rz}}{\partial r} + \frac{\partial \dot{\sigma}_z}{\partial z} + \frac{\dot{\tau}_{rz}}{r} \right) .$$

5.2 FUNDAMENTAL SOLUTION

A special form for the weighting functions $\dot{u}$ and $\dot{w}$ can be evaluated by employing the point collocation method. This can be achieved by applying a concentrated ring loading of intensities $\dot{R}$ and $\dot{Z}$ in the r and z directions respectively, inside an infinite, elastic, isotropic, and homogeneous domain, as shown in Fig(5.1).

5.2.1 Fundamental Equation of Equilibrium

Consider an axisymmetric elastic problem of infinite domain and with state of loading due to body forces distributed uniformly within a circular torus of radius r_i and with a circular cross-section in the r - z plane of radius " ϵ " centred at (r_i, z_i) , as shown in Fig(5.1).

Suppose e_r , e_z be the total force components with constant axisymmetric domain load intensities $\dot{R}$ and $\dot{Z}$ in the r and z directions, respectively, such that:-

$$e_r = \iiint_{\Omega} \dot{R} r \, dr \, d\theta \, dz = \dot{R} \cdot \iiint_{\Omega} r \, dr \, d\theta \, dz$$

and

$$e_z = \iiint_{\Omega} \dot{Z} r \, dr \, d\theta \, dz = \dot{Z} \cdot \iiint_{\Omega} r \, dr \, d\theta \, dz$$

For the limiting case with $\epsilon \rightarrow 0$, the volume integration can be evaluated as follows:-

$$e_r = \dot{R} \cdot (2\pi r_i) \cdot \lim_{\epsilon \rightarrow 0} (\pi \epsilon^2)$$

$$e_z = \dot{Z} \cdot (2\pi r_i) \cdot \lim_{\epsilon \rightarrow 0} (\pi \epsilon^2)$$

Hence, the domain load intensities, over the whole domain, can therefore be defined as follows:-

$$\left. \begin{aligned} \dot{R} &= \frac{e_r}{2\pi r_i} \frac{1}{\lim_{\epsilon \rightarrow 0} (\pi \epsilon^2)} \\ \dot{Z} &= \frac{e_z}{2\pi r_i} \frac{1}{\lim_{\epsilon \rightarrow 0} (\pi \epsilon^2)} \end{aligned} \right\} \begin{aligned} &\text{for } r = r_i, \\ &\text{and } z = z_i \end{aligned}$$

and

$$\left. \begin{aligned} \dot{R} &= 0 \\ \dot{Z} &= 0 \end{aligned} \right\} \begin{aligned} &\text{for } r \neq r_i \\ &\text{or } z \neq z_i \end{aligned}$$

Defining a two-dimensional Dirac delta function in the r - z plane in a way similar to that employed in section 4.2.1 such that:-

$$\delta(r - r_i, z - z_i) = \begin{cases} \frac{1}{\lim_{\epsilon \rightarrow 0} (\pi \epsilon^2)} & \text{for } r = r_i, \text{ and } z = z_i \\ \text{zero} & \text{for } r \neq r_i, \text{ or } z \neq z_i \end{cases}$$

hence, the domain load intensities for the limiting case, when $\epsilon \rightarrow 0$ (concentrated ring load at r_i, z_i), the singular problem, can be defined as follows:-

$$\left. \begin{aligned} \dot{R}(r - r_i, z - z_i) &= \frac{e_r}{2\pi r_i} \cdot \delta(r - r_i, z - z_i) \\ \dot{Z}(r - r_i, z - z_i) &= \frac{e_z}{2\pi r_i} \cdot \delta(r - r_i, z - z_i) \end{aligned} \right\} \quad (5.6)$$

Then the equations of equilibrium for the above concentrated ring load problem can be written as follows:-

$$\left. \begin{aligned} \frac{\partial \sigma_r^*}{\partial r} + \frac{\partial \tau_{rz}^*}{\partial z} + \frac{\sigma_r^* - \sigma_\theta^*}{r} + \frac{e_r}{2\pi r_i} \delta(r - r_i, z - z_i) &= 0 \\ \frac{\partial \tau_{rz}^*}{\partial r} + \frac{\partial \sigma_z^*}{\partial z} + \frac{\tau_{rz}^*}{r} + \frac{e_z}{2\pi r_i} \delta(r - r_i, z - z_i) &= 0 \end{aligned} \right\} \quad (5.7)$$

The domain integration terms for $\dot{R}$ and $\dot{Z}$, in equation 5.5, can be evaluated as follows:-

$$\begin{aligned} & \iint (u \dot{R} + w \dot{Z}) \cdot 2\pi r \, dr \, dz \\ &= \iint \left(\frac{u \cdot e_r + w \cdot e_z}{2\pi r_i} \right) \cdot 2\pi r \cdot \delta(r - r_i, z - z_i) \, dr \, dz \end{aligned}$$

Employing the property of the Dirac delta function, which states that:-

$$\iint f(r, z) \cdot \delta(r - r_i, z - z_i) \, dr \, dz = C_i f(r_i, z_i)$$

or

$$\begin{aligned} & \iint f(r, z) \cdot \delta(r - r_i, z - z_i) \cdot 2\pi r \, dr \, dz \\ &= \iint [2\pi r \cdot f(r, z)] \cdot \delta(r - r_i, z - z_i) \cdot dr \, dz \\ &= C_i \cdot [2\pi r_i \cdot f(r_i, z_i)] , \end{aligned}$$

as explained in Appendix 4.2, it can be deduced that

$$\begin{aligned}
 & \iint (u \hat{R} + w \hat{Z}) 2\pi r \, dr \, dz \\
 &= C_i \left[\frac{u_i \cdot e_r + w_i \cdot e_z}{2\pi r_i} \right] \cdot 2\pi r_i \\
 &= C_i u_i e_r + C_i w_i e_z \quad (5.8)
 \end{aligned}$$

where

$$\left. \begin{aligned} u_i &= u(r_i, z_i) \\ w_i &= w(r_i, z_i) \end{aligned} \right\} \begin{array}{l} \text{the displacement components at the} \\ \text{singular point, (load point)} \end{array}$$

5.2.2 Fundamental Displacement

The solution of the singular differential equation (5.6) is known as the fundamental solution, and the associated displacement components are called the fundamental displacements.

The fundamental displacement solution can be obtained by using one of the following approaches:-

(i) Three-Dimensional Approach

In this approach the axisymmetric fundamental displacement components can be obtained from the three dimensional fundamental displacement components which result from the concentrated load vector, per unit volume, $(\hat{X}, \hat{Y}, \hat{Z})$ at point x_i, y_i, z_i , as follows:-

$$\bar{u}_Y = \sum G_{Y\beta} \hat{X}_\beta \quad (5.9)$$

where

$\bar{u}_r$ represents displacement per unit volume

$$\gamma = 1, 2, 3, \quad \beta = 1, 2, 3$$

$$G_{\gamma\beta} = \frac{1}{16\pi\mu(1-\nu)} \frac{1}{r} \left[(3 - 4\nu)\delta_{\gamma\beta} + \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \right]$$

r is the distance between the source point (x_i, y_i, z_i) and any field point (x, y, z) in a given 3-dimensional domain [134].

Using cylindrical-polar coordinates, it can be shown that:-

$$x = r \cdot \cos\theta,$$

$$y = r \cdot \sin\theta,$$

$$z = z,$$

and the coordinates of the source point are (r_i, θ_i, z_i) with concentrated load vector $(\hat{R}, \hat{\theta}, \hat{Z})$, per unit volume as shown in Fig(5.2). Hence, it can be deduced, that, at the source point:-

$$\hat{X} = \hat{R} \cos\theta_i - \hat{\theta} \sin\theta_i,$$

$$\hat{Y} = \hat{R} \sin\theta_i + \hat{\theta} \cos\theta_i,$$

$$\hat{Z} = \hat{Z}.$$

While the displacement components occurring at a field point (x, y, z) , or (r, θ, z) , and due to the above concentrated load vector, can be expressed as follows:-

$$u_r = u_x \cos\theta + u_y \sin\theta,$$

$$u_\theta = u_y \cos\theta - u_x \sin\theta, \text{ and}$$

$$u_z = u_z.$$

Finally, it can be proved that:-

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \underline{R}^t(\theta_1) \begin{pmatrix} \dot{R} \\ \dot{\theta} \\ \dot{Z} \end{pmatrix} \quad (5.10)$$

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} = \underline{R}(\theta) \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \quad (5.11)$$

where

$$\underline{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting equation (5.9) into equation (5.11), it can be deduced that:-

$$\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{u}_z \end{bmatrix} = \underline{R}(\theta) \cdot \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \cdot \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}$$

Substituting equation (5.10), into the above equation, it can be shown that:-

$$\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{u}_z \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{pmatrix} \dot{R} \\ \dot{\theta} \\ \dot{Z} \end{pmatrix}$$

where

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \underline{R}(\theta) \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \cdot \underline{R}^t(\theta_1)$$

The above displacement components are due to the point load vector $(\dot{R}, \dot{\theta}, \dot{Z})$ per unit volume acting in an infinitesimal volume ($dvol = \pi \epsilon^2 r_i d\theta$) as shown in Fig(5.1). For axisymmetric loading problems, the loading in the θ direction should be zero, i.e. $\dot{\theta} = 0$. The total force e_r, e_z in the r and z direction respectively, can be expressed in terms of $\dot{R}$ and $\dot{Z}$ as follows:-

$$e_r = 2\pi^2 \epsilon^2 r_i \cdot \dot{R} ,$$

$$e_z = 2\pi^2 \epsilon^2 r_i \cdot \dot{Z}$$

The corresponding displacement components are:-

$$\begin{pmatrix} \dot{u}_r \\ \dot{u}_z \end{pmatrix} = \frac{1}{2\pi^2 \epsilon^2 r_i} \begin{bmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{bmatrix} \begin{pmatrix} e_r \\ e_z \end{pmatrix} dVol$$

where $dVol = \pi \epsilon^2 r_i d\theta$, hence,

$$\text{i.e.} \quad \begin{pmatrix} \dot{u}_r \\ \dot{u}_z \end{pmatrix} = \frac{1}{2\pi} \begin{bmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{bmatrix} \begin{pmatrix} e_r \\ e_z \end{pmatrix} d\theta$$

Integrating the above equations with respect to θ , the full "fundamental solution" due to the concentrated ring-force may be expressed as follows:-

$$\begin{pmatrix} \dot{u}_r \\ \dot{u}_z \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{bmatrix} G_{rr} & G_{rz} \\ G_{zr} & G_{zz} \end{bmatrix} \begin{pmatrix} e_r \\ e_z \end{pmatrix} \quad (5.12)$$

where

$$\begin{bmatrix} G_{rr} & G_{rz} \\ G_{zr} & G_{zz} \end{bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{bmatrix} d\theta$$

$$g_{11} = G_{11} \cos\theta_i \cdot \cos\theta + G_{12} \sin\theta_i \cdot \cos\theta + G_{21} \cos\theta_i \sin\theta + G_{22} \sin\theta_i \sin\theta ,$$

$$\begin{aligned}
g_{13} &= G_{13} \cos\theta + G_{23} \sin\theta, \\
g_{31} &= G_{31} \cos\theta_1 + G_{32} \sin\theta_1, \text{ and} \\
g_{33} &= G_{33}.
\end{aligned}$$

Defining the complete elliptic integrals of the first and second kind as follows:-

$$\begin{aligned}
K(m_e) &= \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m_e^2 \sin^2 \phi}} = \int_0^{\pi/2} \frac{R_i}{\rho_i} \frac{d\phi}{\sin\phi} \\
E(m_e) &= \int_0^{\pi/2} \sqrt{1 - m_e^2 \sin^2 \phi} d\phi = \int_0^{\pi/2} \frac{\rho_i}{R_i} \sin\phi d\phi
\end{aligned}$$

Then, it can be shown that:-

$$G_{\gamma\beta} = \frac{1}{C_c} \left[K_{\gamma\beta} \cdot K(m_e) + E_{\gamma\beta} E(m_e) \right] \quad (5.13)$$

where

$$\begin{aligned}
\gamma &= r, z, \quad \beta = r, z, \\
C_c &= 8\pi^2 r_i (1 - \nu) \cdot \mu, \\
\mu &= \text{modulus of rigidity}, \\
m_e^2 &= 1 - (\rho_i/R_i)^2 \\
\rho_i &= \sqrt{(r - r_i)^2 + (z - z_i)^2}, \\
R_i &= \sqrt{(r + r_i)^2 + (z - z_i)^2}, \text{ and}
\end{aligned}$$

the parameters $K_{\gamma\beta}$ and $E_{\gamma\beta}$ are defined in Table(5.1). The $G_{\gamma\beta}$ values are similar to those given in the literature [99, 100].

(ii) Galerkin-Vector Approach

In this approach, the displacement components are expressed in terms of a Galerkin vector as follows:-

$$u = \left(\nabla^2 - \frac{1}{r^2} \right) \dot{G}_r - \frac{1}{2(1-\nu)} \frac{\partial}{\partial r} (\nabla \cdot \vec{G})$$

$$w = \nabla^2 \dot{G}_z - \frac{1}{2(1-\nu)} \frac{\partial}{\partial z} (\nabla \cdot \vec{G})$$

where

$$\vec{G} = \{\dot{G}_r \quad \dot{G}_z\} \quad \text{is the Galerkin vector.}$$

And the equilibrium equations, can therefore, be expressed in terms of the following bi-harmonic equations:-

$$\left(\nabla^2 - \frac{1}{r^2} \right) \left(\nabla^2 - \frac{1}{r^2} \right) \dot{G}_r + F_r/\mu = 0$$

$$\nabla^2 \nabla^2 \dot{G}_z + F_z/\mu = 0$$

where F_r, F_z are the body forced in the r and z directions respectively.

A solution of the equilibrium equations can be achieved by obtaining the Galerkin vector in terms of associated Legendre functions [99] as follows:-

$$\dot{G}_r = \frac{\sqrt{rr_i}}{8\pi^2\mu} \sqrt{x^2 - 1} Q_k^{-1}(x) \cdot e_r$$

$$\dot{G}_z = \frac{\sqrt{rr_i}}{8\pi^2\mu} \sqrt{x^2 - 1} Q_k^{-1}(x) \cdot e_z$$

where

$$x = 1 + \frac{(r - r_i)^2 + (z - z_i)^2}{2rr_i}$$

which can also be expressed in terms of complete elliptic integrals as follows (100).

$$\left. \begin{aligned} \dot{G}_r &= \frac{\sqrt{rr_i}}{6\pi^2\mu} \frac{e_r}{m_e^3} \left[2(1 - m_e^3) K(m_e) - (2 - m_e^2) E(m_e) \right] \\ \dot{G}_z &= \frac{\sqrt{rr_i}}{2\pi^2\mu} \frac{e_z}{m_e} E(m_e) \end{aligned} \right\} (5.14)$$

$$m_e^2 = \frac{2}{1+x} = \frac{4rr_i}{(r+r_i)^2 + (z-z_i)^2}$$

Hence, the fundamental displacement solution parameters can be deduced as follows:-

$$\left. \begin{aligned} G_{rr} &= \frac{1}{2(1-\nu)} \left[(1-2\nu) \left(\nabla^2 - \frac{1}{r^2} \right) \dot{G}_r + \frac{\partial^2 \dot{G}_r}{\partial z^2} \right] \\ G_{rz} &= - \frac{1}{2(1-\nu)} \frac{\partial^2 \dot{G}_z}{\partial r \partial z} \\ G_{zr} &= - \frac{1}{2(1-\nu)} \left[\frac{\partial^2 \dot{G}_r}{\partial r \partial z} + \frac{1}{r} \frac{\partial \dot{G}_r}{\partial z} \right] \\ G_{zz} &= \frac{1}{2(1-\nu)} \left[(1-2\nu) \nabla^2 \dot{G}_z + \frac{1}{r} \frac{\partial \dot{G}_z}{\partial r} + \frac{\partial^2 \dot{G}_z}{\partial r^2} \right] \end{aligned} \right\} (5.15)$$

5.2.3 Evaluation of the First and Second Complete Elliptic Integrals, K, E

The evaluation of $K(m_e)$ and $E(m_e)$ can be carried out by using one of the following approaches.

(i) Approximate Series:-

$K(m_e)$ can be evaluated by using the following Chebyshev approximate form [135].

$$K(m_e) = \sum_{i=0}^n a_i \eta^i + \ln \left(\frac{1}{\eta} \right) \cdot \sum_{j=0}^m b_j \eta^j \quad (5.16)$$

where

$$a_0 = \ln 4, \quad b_0 = 0.5, \quad \eta = 1 - m_e^2, \quad \text{and}$$

n, m are the number of terms to be considered for achieving the required accuracy.

The values of coefficients a_i and b_j are listed in Table (5.2).

Similarly,

$$E(m_e) = \sum_{i=0}^n c_i \eta^i + \ln \left(\frac{1}{\eta} \right) \sum_{j=0}^m d_j \eta^j \quad (5.17)$$

where

$$c_0 = 1, \quad d_0 = 0$$

Table(5.3) shows the values of the coefficients c_j and d_j .

(ii) Gauss Quadrature Scheme

The elliptic integrals can be reduced to the following form by assuming that $\xi = \sin\phi$, as follows:-

$$K(m_e) = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-m_e^2\xi^2)}}$$

$$E(m_e) = \int_0^1 \sqrt{\frac{1-m_e^2\xi^2}{1-\xi^2}} d\xi$$

By employing the modified Gaussian quadrature with n_q points, it can be shown that:-

$$K(m_e) = \sum_{q=1}^{n_q} \frac{W_q}{\sqrt{(1-\xi_q^2)(1-m_e^2\xi_q^2)}}$$

$$E(m_e) = \sum_{q=1}^{n_q} W_q \cdot \sqrt{\frac{1-m_e^2\xi_q^2}{1-\xi_q^2}}$$

$K(m_e)$ and $E(m_e)$ can also be deduced by assuming

$$\xi = \frac{\phi}{\pi/2}$$

then, the complete elliptic integrals can be expressed as follows:-

$$K(m_e) = \frac{\pi}{2} \int_0^1 \frac{d\xi}{\sqrt{1-m_e^2 \sin^2(\pi\xi/2)}}$$

$$E(m_e) = \frac{\pi}{2} \int_0^1 \sqrt{1-m_e^2 \sin^2(\pi\xi/2)}$$

by using modified Gaussian quadrature, it can be shown that:-

$$K(m_e) = \frac{\pi}{2} \sum_{q=1}^{n_q} \frac{W_q}{\sqrt{1 - m_e^2 \sin^2(\pi \xi_q/2)}}$$

$$E(m_e) = \frac{\pi}{2} \sum_{q=1}^{n_q} W_q \cdot \sqrt{1 - m_e^2 \sin^2(\pi \xi_q/2)}$$

This approach proves to be more accurate than the approximate series procedure.

5.2.4 Fundamental Strain and Stress Components

The strain components at a field point (r, z) , due to the concentrated ring load at source point (r_i, z_i) , can be expressed from the fundamental displacement components, equation 5.12, as follows:-

$$\epsilon_r^* = \frac{\partial u^*}{\partial r} = \frac{\partial G_{rr}}{\partial r} e_r + \frac{\partial G_{rz}}{\partial r} e_z$$

$$\epsilon_z^* = \frac{\partial w^*}{\partial z} = \frac{\partial G_{zr}}{\partial z} e_r + \frac{\partial G_{zz}}{\partial z} e_z$$

$$\epsilon_\theta^* = \frac{u^*}{r} = \frac{G_{rr}}{r} e_r + \frac{G_{rz}}{r} e_z$$

$$\gamma_{rz}^* = \frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial r} = \left(\frac{\partial G_{rr}}{\partial z} + \frac{\partial G_{zr}}{\partial r} \right) e_r + \left(\frac{\partial G_{rz}}{\partial z} + \frac{\partial G_{zz}}{\partial r} \right) e_z$$

Defining the R matrix, such that:-

$$\underline{\epsilon}^* = \underline{R} \cdot \begin{pmatrix} e_r \\ e_z \end{pmatrix}$$

where

$$\underline{\epsilon} = \{\epsilon_r^* \quad \epsilon_z^* \quad \epsilon_\theta^* \quad \gamma_{rz}^*\}$$

and

$$\underline{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \\ R_{41} & R_{42} \end{bmatrix}$$

it follows that:-

$$\epsilon_Y^* = \sum \dot{R}_{Ys} \epsilon_s \quad (5.18)$$

where

$$s = 1, 2$$

$$\gamma = 1, 2, 3, 4, \quad , \quad e_1 = e_r \quad , \quad e_2 = e_z$$

$$R_{11} = \frac{\partial G_{rr}}{\partial r} \quad , \quad R_{12} = \frac{\partial G_{rz}}{\partial r} \quad , \quad R_{21} = \frac{\partial G_{zr}}{\partial z}$$

$$R_{22} = \frac{\partial G_{zz}}{\partial z} \quad , \quad R_{31} = \frac{G_{rr}}{r} \quad , \quad R_{32} = \frac{G_{rz}}{r}$$

$$R_{41} = \frac{\partial G_{rr}}{\partial z} + \frac{\partial G_{zr}}{\partial r} \quad , \quad R_{42} = \frac{\partial G_{rz}}{\partial z} + \frac{\partial G_{zz}}{\partial r}$$

and

$$\frac{\partial G_{Y\beta}}{\partial r_s} = \frac{1}{C_c} \left[\frac{\partial K_{Y\beta}}{\partial r_s} K + \frac{\partial E_{Y\beta}}{\partial r_s} \cdot E + \frac{\partial K}{\partial r_s} \cdot K_{Y\beta} + \frac{\partial E}{\partial r_s} E_{Y\beta} \right]$$

The $\underline{R}$ matrix can be rewritten in tensorial form as follows:-

$$R_{Ys} = \frac{1}{C_c} \left[\alpha_{Ys}^* K(m_e) + \beta_{Ys}^* \cdot E(m_e) \right] \quad (5.19)$$

where

$$\gamma = 1, 2, 3, 4$$

$$s = 1, 2$$

The parameters $\alpha_{\gamma s}^*$ and $\beta_{\gamma s}^*$ are as defined in Table(5.4), and as proved in appendix 5.3.

The corresponding fundamental stress components can be deduced by employing the elastic stress-strain relationships given by , equation 2.26, as follows:-

$$\left. \begin{aligned} \sigma_r^* &= 2\mu \left[\epsilon_r^* + \left(\frac{\nu}{1-2\nu} \right) \epsilon_v^* \right] \\ \sigma_z^* &= 2\mu \left[\epsilon_z^* + \left(\frac{\nu}{1-2\nu} \right) \epsilon_v^* \right] \\ \sigma_\theta^* &= 2\mu \left[\epsilon_\theta^* + \left(\frac{\nu}{1-2\nu} \right) \epsilon_v^* \right] \\ \tau_{rz}^* &= \mu \cdot \gamma_{rz}^* \end{aligned} \right\} \quad (5.20)$$

where $\epsilon_v^* = \epsilon_r^* + \epsilon_z^* + \epsilon_\theta^*$

5.2.5 Fundamental Traction

If the infinite domain, which is subjected to the load vector (e_r, e_z) at point (r_i, z_i) , is cut by any arbitrary boundary condition, then fundamental surface traction components $\dot{T}_r$ and $\dot{T}_z$, are required to restore the equilibrium condition for the domain.

The fundamental traction at point (r, z) can be defined in terms of the fundamental stresses acting , at the same point, as follows:-

$$\dot{T}_r = l \dot{\sigma}_r + m \dot{T}_{rz}$$

$$\dot{T}_z = m \dot{\sigma}_z + l \dot{T}_{rz}$$

Substituting equations 5.20 into the above equations, it can be shown that:

$$\dot{T}_r = 2\mu \cdot l \left[\dot{\epsilon}_r + \left(\frac{v}{1-2\nu} \right) \dot{\epsilon}_v \right] + m \cdot \mu \cdot \dot{\gamma}_{rz}$$

$$\dot{T}_z = 2\mu \cdot m \left[\dot{\epsilon}_z + \left(\frac{v}{1-2\nu} \right) \dot{\epsilon}_v \right] + l \cdot \mu \cdot \dot{\gamma}_{rz}$$

or

$$\dot{T}_Y = 2\mu l_Y \left[\dot{\epsilon}_Y + \left(\frac{v}{1-2\nu} \right) \dot{\epsilon}_v \right] + \left[l \delta_{zY} + m \delta_{rY} \right] \cdot \mu \dot{\gamma}_{rz} \quad (5.21)$$

where

$$Y = r, z \quad \text{i.e.} \quad l_r = l$$

$$l_z = m$$

Substituting equation 5.18 into the above equation, it can be shown that:-

$$\dot{T}_Y = 2\mu l_Y \left[\sum_{s=1}^2 R_{Ys} e_s + \left(\frac{v}{1-2\nu} \right) \sum_S (R_{1s} + R_{2s} + R_{3s}) e_s \right]$$

$$+ \mu (l \delta_{zY} + m \delta_{rY}) \cdot \sum_S R_{4s} e_s$$

Finally,

$$\dot{T}_Y = \sum_S F_{Ys} e_s \quad (5.22)$$

where

$$s = 1, 2 \equiv r, z, \quad \gamma = 1, 2 \equiv r, z, \quad \text{and}$$

$$F_{\gamma s} = 2\mu \int_{\gamma} \left[R_{\gamma s} + \left(\frac{\nu}{1-2\nu} \right) (R_{1s} + R_{2s} + R_{3s}) \right] \\ + \mu (\int \delta_{z\gamma} + m \delta_{r\gamma}) R_{4s}$$

$F_{\gamma s}$ can be simplified and written as follows:-

$$F_{\gamma s} = \frac{1}{8\pi^2 r_i (1-\nu)} [A_{\gamma\beta} \cdot K(m_e) + (B_{\gamma\beta}) E(m_e)] \quad (5.23)$$

where $A_{\gamma\beta}$, $B_{\gamma\beta}$ are as tabulated in Table (5.5) and as derived in appendix 5.4,

5.3 BOUNDARY INTEGRAL EQUATIONS FOR ELASTO-PLASTICITY PROBLEMS

The Boundary Integral Equations can be obtained by substituting equations (5.8), (5.12), (5.20) and (5.22), into the final equation of the Inverse Weighted-Residual expression (5.5), as follows:-

$$C_i u_i e_r + C_i w_i e_z \\ + \oint_{\Gamma} (F_{rr} e_r + F_{rz} e_z) \cdot u \cdot 2\pi r \, ds \\ + \oint_{\Gamma} (F_{zr} e_r + F_{zz} e_z) \cdot w \cdot 2\pi r \, ds$$

$$\begin{aligned}
&= \oint_{\Gamma} (G_{rr} e_r + G_{rz} e_z) \cdot T_r \cdot 2\pi r \, ds \\
&+ \oint_{\Gamma} (G_{zr} e_r + G_{zz} e_z) \cdot T_z \cdot 2\pi r \, ds \\
&+ \iint_{\Omega} (G_{rr} e_r + G_{rz} e_z) R \, 2\pi r \, dr \, dz \\
&+ \iint_{\Omega} (G_{zr} e_r + G_{zz} e_z) Z \, 2\pi r \, dr \, dz \\
&+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R} \cdot \begin{pmatrix} e_r \\ e_z \end{pmatrix} 2\pi r \, dr \, dz \quad (5.24)
\end{aligned}$$

Since e_r and e_z are arbitrary parameters the above equation can be divided into the following Boundary Integral Equations:-

$$\begin{aligned}
C_i u_i &+ \oint_{\Gamma} (u F_{rr} + w F_{zr}) 2\pi r \, ds \\
&= \oint_{\Gamma} (T_r G_{rr} + T_z G_{zr}) 2\pi r \, ds \\
&+ \iint_{\Omega} (R G_{rr} + Z G_{zr}) 2\pi r \, dr \, dz \\
&+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_r 2\pi r \, dr \, dz \quad (5.25)
\end{aligned}$$

and

$$C_i w_i + \oint_{\Gamma} (u F_{rz} + w F_{zz}) 2\pi r \, ds$$

$$\begin{aligned}
&= \oint_{\Gamma} (T_r G_{rz} + T_z G_{zz}) 2\pi r \, ds \\
&+ \iint_{\Omega} (R G_{rz} + Z G_{zz}) 2\pi r \, dr \, dz \\
&+ \iint (\underline{\sigma}^{LP})^t \cdot \underline{R}_z \cdot 2\pi r \, dr \, dz \quad (5.26)
\end{aligned}$$

where $\underline{R}_r = \{ R_{11}^* \quad R_{21}^* \quad R_{31}^* \quad R_{41}^* \}$, and
 $\underline{R}_z = \{ R_{12}^* \quad R_{22}^* \quad R_{32}^* \quad R_{42}^* \}$
 $\underline{\sigma}^{LP} = \{ \sigma_r^{LP} \quad \sigma_z^{LP} \quad \sigma_{\theta}^{LP} \quad \tau_{rz}^{LP} \}$

The domain integration terms in the above BIE's are due to the domain loading and the residual stress $\underline{\sigma}^{eP}$ is produced over the whole domain as a result of approximation errors in the solution of the non-linear problem.

5.4 EVALUATION OF THE DOMAIN LOADING TERMS

An axisymmetric body rotating with a uniform angular velocity about the z-axis can be considered to be subjected to an axisymmetric body force plus an inertia force, whilst domain loading may be generated in the z-direction due to gravitational force or translational acceleration in the z direction. Additional radial domain loading may be produced by some other source.

Therefore, the body force per unit volume, due to the centrifugal linear acceleration, or arising from other sources, at point (r,z) can be written as follows:-

$$R(r,z) = \rho \omega^2 r + F_r$$

$$Z(r,z) = F_z$$

Then the domain terms in the r and z directions can be deduced as follows:-

$$U_i(r, z) = \iint_{\Omega} \left[\left(\rho \omega^2 r + F_r \right) G_{rr} + F_z \cdot G_{zr} \right] 2\pi r \, dr \, dz$$

$$W_i(r, z) = \iint_{\Omega} \left[\left(\rho \omega^2 r + F_r \right) G_{rz} + F_z \cdot G_{zz} \right] 2\pi r \, dr \, dz$$

Using equations (5.15), it can be shown that:-

$$U_i = \iint R \cdot \frac{1}{2(1-\nu)} \left[(1-2\nu) \left(\nabla^2 - \frac{1}{r^2} \right) G_r^* + \frac{\partial^2 G_r^*}{\partial z^2} \right] 2\pi r \, dr \, dz$$

$$+ \iint z \cdot \frac{(-1)}{2(1-\nu)} \left[\frac{\partial^2 G_r^*}{\partial z \partial r} + \frac{1}{r} \frac{\partial G_r^*}{\partial z} \right] 2\pi r \, dr \, dz$$

and

$$W_i = \iint R \left[\frac{(-1)}{2(1-\nu)} \frac{\partial^2 G_z}{\partial r \partial z} \right] 2\pi r \, dr \, dz$$

$$+ \iint z \frac{1}{2(1-\nu)} \left[(1-2\nu) \nabla^2 G_z^* + \frac{1}{r} \frac{\partial G_z^*}{\partial r} + \frac{\partial^2 G_z^*}{\partial r^2} \right] 2\pi r \, dr \, dz$$

The above domain terms can be integrated by parts and reduced to the following boundary integration terms, as shown in appendix 5.5.

$$U_i(r, z) = Q_w \cdot \oint_{\Gamma} f_{rw} \, d\Gamma + Q_r \cdot \oint_{\Gamma} f_{rr} \, d\Gamma + Q_z \cdot \oint_{\Gamma} f_{rz} \, d\Gamma \quad (5.27)$$

$$W_i(r, z) = Q_w \cdot \oint_{\Gamma} f_{zw} \, d\Gamma + Q_r \cdot \oint_{\Gamma} f_{zr} \, d\Gamma + Q_z \cdot \oint_{\Gamma} f_{zz} \, d\Gamma$$

where

$$Q_w = \frac{\rho\omega^2}{2(1-\nu)} \quad , \quad Q_r = \frac{F_r}{2(1-\nu)} \quad , \quad Q_z = \frac{F_z}{2(1-\nu)}$$

$$f_{rw} = (1-2\nu) \left(r \frac{\partial G_r^*}{\partial n} - \ell G_r^* \right) + r \frac{\partial G_r^*}{\partial z} \cdot m \quad ,$$

$$f_{rr} = (1-2\nu) \frac{\partial G_r^*}{\partial n} + \frac{\partial G_r^*}{\partial z} m \quad ,$$

$$f_{rz} = - \left(\frac{\partial G_r^*}{\partial r} + \frac{G_r^*}{r} \right) \cdot m \quad ,$$

$$f_{zw} = -mr \frac{\partial G_z^*}{\partial r} \quad ,$$

$$f_{zr} = -m \frac{\partial G_z^*}{\partial r} \quad , \quad \text{and}$$

$$f_{zz} = 2(1-\nu) \frac{\partial G_z^*}{\partial n} - m \frac{\partial G_z^*}{\partial z} \quad .$$

Also

$$\begin{aligned} \frac{\partial G_r^*}{\partial r_\beta} = & \delta_{r\beta} \frac{G_r^*}{2r} - \frac{3G_r^*}{m_e} \frac{\partial m_e}{\partial r_\beta} + \frac{C_\nu}{m_e^3} \left[2(1-m_e^3) \frac{\partial K}{\partial r_\beta} \right. \\ & \left. - (2-m_e^2) \frac{\partial E}{\partial r_\beta} + 2m_e \frac{\partial m_e}{\partial r_\beta} (E-3m_e K) \right] \end{aligned}$$

$$\frac{\partial G_z^*}{\partial r_\beta} = \frac{G_z^*}{2r} \delta_{r\beta} - C_\nu \frac{K}{m_e^2} \frac{\partial m_e}{\partial r_\beta}$$

$$\frac{\partial G_r^*}{\partial n} = \ell \cdot \frac{\partial G_r^*}{\partial r} + m \frac{\partial G_r^*}{\partial z}$$

$$= \cancel{\ell} \frac{G_r^*}{2r} - \frac{3G_r^*}{m_e} \frac{\partial m_e}{\partial n} + \frac{C_\nu}{m_e^3} \left[2(1-m_e^3) \frac{\partial K}{\partial n} \right. \\ \left. - (2-m_e^2) \frac{\partial E}{\partial n} + 2m_e \frac{\partial m_e}{\partial n} (E-3m_e K) \right]$$

$$\frac{\partial G_z^*}{\partial n} = \frac{G_z^*}{2r} \cdot \cancel{\ell} - \frac{C_\nu K}{m_e^2} \frac{\partial m_e}{\partial n}$$

$$\frac{\partial E}{\partial r_\beta} = C_\beta [E-K]$$

$$\frac{\partial K}{\partial r_\beta} = C_\beta \left[\frac{E}{1-m_e^2} - K \right]$$

$$C_\beta = C_r \text{ or } C_z$$

$$C_r = \frac{m_e^2 [r_i^2 - r^2 + (z - z_i)^2]}{8r^2 r_i}$$

$$C_z = - \frac{m_e^2 (z - z_i)}{4rr_i}$$

$$C_\nu = \frac{\sqrt{rr_i}}{2\pi^2 \mu}$$

$$\frac{\partial m_e}{\partial r} = \frac{m_e}{2} \left[\frac{1}{r} - 2(r+r_i) \right], \text{ and}$$

$$\frac{\partial m_e}{\partial z} = \frac{m_e^3}{4rr_i} (z-z_i),$$

as shown in appendix 5.5.

5.5 BOUNDARY INTEGRAL EQUATION FOR A NON-SMOOTH BOUNDARY

The BIE's (5.23) and (5.24) are suitable for a smooth boundary without corners, whilst for a boundary with corners the previous BIE's can be used with some modification which allows for the effect of the corners, i.e. the boundary integration of the variables over the boundary of the corners.

The effect of corners can be considered by employing one of the following approaches:-

5.5.1 Use of a Corner Model

Suppose that the source point (r_i, z_i) is a corner point (r_c, z_c) . The BIE's for such a source point can be written, after separating the boundary integration over the whole boundary into integration over the corner boundary " Γ_c " and integration over the smooth boundary of the problem " Γ' ", as follows:-

$$\begin{aligned}
 C_i \ u(r_c, z_c) &+ \oint_{\Gamma'} (u F_{rr} + w F_{zr}) 2\pi r \, ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} (u F_{rr} + w F_{zr}) 2\pi r \, ds \\
 &= \oint_{\Gamma'} (T_r G_{rr} + T_z G_{zr}) 2\pi r \, ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} (T_r G_{rr} + T_z G_{zr}) 2\pi r \, ds \\
 &+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_r \cdot 2\pi r \, dr \, dz + U_i(r_c, z_c) \\
 C_i \ w(r_c, z_c) &+ \oint_{\Gamma'} (u F_{rz} + w F_{zz}) 2\pi r \, ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} (u F_{rz} + w F_{zz}) 2\pi r \, ds \\
 &= \oint_{\Gamma'} (T_r G_{rz} + T_z G_{zz}) 2\pi r \, ds + \lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} (T_r G_{rz} + T_z G_{zz}) 2\pi r \, ds \\
 &+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_z \cdot 2\pi r \, dr \, dz + W_i(r_c, z_c)
 \end{aligned}$$

The smooth corner model for a 2-Dimensional problem can be used for the axisymmetric case [135] to evaluate the boundary integration over the corner boundary " Γ_c ", as explained in section 4.7.1.2. Therefore it can be shown:-

$$\lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} F_{\gamma\beta} (r-r_c, z-z_c) \cdot U_\gamma d\Gamma = J_{\gamma\beta} (r_c, z_c) \cdot U_\gamma (r_c, z_c)$$

$$\lim_{\Gamma_c \rightarrow 0} \int_{\Gamma_c} G_{\gamma\beta} (r-r_c, z-z_c) \cdot T_\gamma d\Gamma = \text{Zero}$$

where $\gamma = r, z$, $\beta = r, z$

Then the previous BIE's can be rewritten as follows:-

$$\begin{aligned} C_{11} u_c + C_{12} w_c &+ \oint_{\Gamma'} (u F_{rr} + w F_{zr}) 2\pi r ds \\ &= \oint_{\Gamma'} (T_r G_{rr} + T_z G_{zr}) 2\pi r ds \\ &+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_r 2\pi r dr dz + U_i \end{aligned} \quad (5.28)$$

$$\begin{aligned} C_{21} u_c + C_{22} w_c &+ \oint_{\Gamma'} (u F_{rz} + w F_{zz}) 2\pi r ds \\ &= \oint_{\Gamma'} (T_r G_{rz} + T_z G_{zz}) 2\pi r ds \\ &+ \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_z 2\pi r dr dz + W_i \end{aligned} \quad (5.29)$$

where $C_{11} = C_i + J_{11}(r_c, z_c)$,

$$C_{12} = J_{12}(r_c, z_c) ,$$

$$C_{21} = J_{21}(r_c, z_c) , \text{ and}$$

$$C_{22} = C_i + J_{22}(r_c, z_c) .$$

For the case of the smooth corner model, it can be shown, as explained in section 4.7.1.2, that:-

$$C_{11} = \frac{\phi_c}{2\pi} - \frac{1}{C_b} \cos(2\phi_m) \cdot \sin\phi_c$$

$$C_{12} = C_{21} = -\frac{1}{C_b} \sin(2\phi_m) \cdot \sin\phi_c$$

$$C_{22} = \frac{\phi_c}{2\pi} + \frac{1}{C_b} \cos(2\phi_m) \cdot \sin\phi_c$$

The above coefficients can be written for a source point on the smooth boundary (i.e. $\phi_c = \pi$), as follows:-

$$C_{11} = C_{22} = 0.5 , \quad C_{12} = C_{21} = 0$$

While for a point inside the domain, it can be shown that:

$$C_{11} = C_{22} = 1, \quad C_{12} = C_{21} = 0$$

5.5.2 Use of Rigid Translation Condition

For axisymmetric problems a rigid translation of the whole domain in the z-direction can be achieved without applying boundary tractions and hence without generating internal stresses. However, a rigid translation in the r direction cannot be achieved without

generating internal stresses. If a rigid translation in the z direction (w_0) is considered, then the BIE's (5.25) and (5.26) can be written respectively as follows:-

$$\oint_{\Gamma} w_0 F_{zr} d\Gamma = 0, \text{ and}$$

$$C_i w_0 + \oint_{\Gamma} w_0 F_{zz} d\Gamma = 0$$

Subtracting the above equations from equations (5.25) and (5.26) respectively, and by considering a special case with $w_0 = w_i$, then it can be shown that:-

$$\begin{aligned} C_i u_i + \oint_{\Gamma} \left[u F_{rr} + (w - w_i) F_{zr} \right] d\Gamma \\ = \oint_{\Gamma} (T_r G_{rr} + T_z G_{zr}) d\Gamma \\ + \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_r d\Omega + U_i \end{aligned}$$

and

$$\begin{aligned} \oint_{\Gamma} \left[u F_{rz} + (w - w_i) F_{zz} \right] d\Gamma \\ = \oint_{\Gamma} (T_r G_{rz} + T_z G_{zz}) d\Gamma \\ + \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_z d\Omega + W_i \end{aligned}$$

Defining

$$C_{\gamma\beta}(r_i, z_i) = - \oint_{\Gamma'} F_{\beta\gamma}(r - r_i, z - z_i) 2\pi r \, ds$$

where

$\Gamma' = \Gamma - \Gamma_c$ represents a smooth boundary without corners

Then, the above BIE's can be written as follows:-

$$\begin{aligned} C_i u_i + C_{12}(r_i, z_i) w_i + \oint_{\Gamma} u F_{rr} \, d\Gamma + \oint_{\Gamma'} w F_{zr} \, d\Gamma \\ = \oint_{\Gamma} T_r G_{rr} \, d\Gamma + \oint_{\Gamma'} T_z G_{zr} \, d\Gamma + \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_r \, d\Omega + U_i \quad (5.30) \end{aligned}$$

and

$$\begin{aligned} C_{22}(r_i, z_i) w_i + \oint_{\Gamma} u F_{rz} \, d\Gamma + \oint_{\Gamma'} w F_{zz} \, d\Gamma \\ = \oint_{\Gamma} T_r G_{rz} \, d\Gamma + \oint_{\Gamma'} T_z G_{zz} \, d\Gamma \\ + \iint_{\Omega} (\underline{\sigma}^{LP})^t \cdot \underline{R}_z \, d\Omega + W_i \quad (5.31) \end{aligned}$$

as shown in section 4.7.1.3.

5.5.3 Combination of Rigid Translation and the Corner Model

From the 2-dimensional BEM studies, it can be concluded that

$$(C_{11} + C_{22})_{s.c.} = (C_{11} + C_{22})_{R.T.}, \quad \text{and}$$

$$(C_{12} + C_{21})_{s.c.} = (C_{12} + C_{21})_{R.T.}$$

where

$(\quad)_{s.c.}$ are coefficients evaluated by using the smooth corner model

$(\quad)_{R.T.}$ are coefficients evaluated by using rigid translation.

Similar combinations can be employed for evaluating the C_{11} and C_{21} for the axisymmetric case, as follows

$$\left. \begin{aligned} (C_{11})_{combined} &= (C_{11} + C_{22})_{s.c.} - (C_{22})_{R.T.} \\ (C_{21})_{combined} &= (2 C_{12})_{s.c.} - (C_{12})_{R.T.} \end{aligned} \right\} \quad (5.32)$$

5.6 INTEGRATION OF THE VARIABLE OVER THE SPECIFIED BOUNDARY

The element matrices can be evaluated by using a suitable numerical integration technique, such as the Gaussian-quadrature scheme, as follows:-

$$\begin{aligned} \int_{\Gamma_e} f(r - r_i, z - z_i) ds &= \int_0^1 f[r(\xi) - r_i, z(\xi) - z_i] \cdot |J(\xi)| \cdot 2\pi \\ &\quad r(\xi) \cdot d\xi \\ &= \sum_{q=1}^{N_q} f[r(\xi_q) - r_i, z(\xi_q) - z_i] \cdot |J(\xi_q)| \cdot 2\pi r(\xi_q) \cdot w_q \end{aligned}$$

where

f is a fundamental solution parameter at field point $[r(\xi_q), z(\xi_q)]$ for the source point (r_i, z_i)

ξ_q is the abscissa of the Gaussian quadrature

w_q is the corresponding weight

$r(\xi_q)$ is the radius at the Gaussian point

N_q is the number of Gaussian points used.

If the source point (r_i, z_i) coincides with the specified field point a singularity exists in the elliptic integral terms due to

$$\log(1/\rho_i), \quad \rho_i = \sqrt{(r - r_i)^2 + (z - z_i)^2}$$

A singular integration method, similar to that explained in section 4.7.2.1. can be employed to evaluate the boundary integration of the singular terms.

Considering a boundary element with nodes and a source point (r_i, z_i) being coincident with the j^{th} local node, the element matrices can be written as follows:-

$$g^e(\gamma, 2(j-1) + \beta) = \int_{\Gamma_e} G_{\beta\gamma}(r-r_i, z-z_i) N_j(\xi) ds \quad (5.33a)$$

$$h^e(\gamma, 2(j-1) + \beta) = \int_{\Gamma_e} F_{\beta\gamma}(r-r_i, z-z_i) N_j(\xi) ds \quad (5.33b)$$

Substituting equation(5.13) into equation(5.33a) it can be deduced that:

$$g^e(\gamma, 2(j-1) + \beta) = \frac{1}{C_c} \int_0^1 [K_{\beta\gamma} \cdot K(m_e) + E_{\beta\gamma} \cdot E(m_e)] \cdot N_j(\xi) \cdot |J(\xi)| \cdot 2\pi r(\xi) \cdot d\xi \quad (5.34)$$

However, from equations (5.16) and (5.17), it can be shown that

$$K(m_e) = K_1 + 2 \log\left(\frac{R_i}{\rho_i}\right) \cdot K_2 \quad (5.35)$$

$$E(m_e) = E_1 + 2 \log\left(\frac{R_i}{\rho_i}\right) \cdot E_2 \quad (5.36)$$

where

$$K_1 = \sum_{i=0}^{\infty} a_i \eta^i, \quad K_2 = \sum_{j=0}^{\infty} b_j \eta^j$$

$$E_1 = \sum_{i=0}^{\infty} c_i \eta^i, \quad E_2 = \sum_{j=0}^{\infty} d_j \eta^j$$

$$\eta = 1 - m_e^2 = \left(\frac{\rho_i}{R_i} \right)^2,$$

$$R_i^2 = (r + r_i)^2 + (z - z_i)^2, \quad \text{and}$$

$$\rho_i^2 = (r - r_i)^2 + (z - z_i)^2$$

Substituting equations (5.35) and (5.36) into equation (5.39) it can be deduced that

$$g^e(\gamma, 2(j-1) + \beta) = I^G + I^S \quad (5.37)$$

where

$$I^G = \frac{1}{C_c} \int_0^1 \left[K_{\beta\gamma} \cdot K_1 + E_{\beta\gamma} E_1 \right] N_j(\xi) \cdot |J(\xi)| \cdot 2\pi r(\xi) \cdot d\xi$$

which is the regular term that can be evaluated by using the Gaussian scheme, while

$$I^S = \frac{1}{C_c} \int_0^1 \left[K_{\beta\gamma} \cdot K_2 + E_{\beta\gamma} E_2 \right] \cdot N_j(\xi) \cdot |J(\xi)| \cdot 2 \log \left(\frac{R_i}{\rho_i} \right) \cdot 2\pi r(\xi) d\xi$$

is the singular term.

The boundary " Γ_e " can be divided into two parts, as shown in Fig(5.4). Then the singular integration can be rewritten as follows:-

$$I^s = \frac{4\pi}{C_c} \left[\int_0^{\xi_j} C_k \cdot \log\left(\frac{R_i}{\rho_i}\right) + \int_{\xi_j}^1 C_k \cdot \log\left(\frac{R_i}{\rho_i}\right) \right] \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot d\xi$$

$$I^s = \frac{4\pi}{C_c} \left[I_1^s + I_2^s \right] \quad (5.38)$$

where

$$C_k = K_{\beta\gamma} K_2 + E_{\beta\gamma} E_2 \quad (5.39)$$

$$I_1^s = \int_0^{\xi_j} C_k \cdot \log\left(\frac{R_i}{\rho_i}\right) \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot d\xi \dots$$

is the singular integration over Part I,

$$I_2^s = \int_{\xi_j}^1 C_k \cdot \log\left(\frac{R_i}{\rho_i}\right) \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot d\xi \dots$$

is the singular integration over Part II, and

$$\xi_j = \frac{j-1}{n-1}$$

The evaluation of I_1^s can be achieved after employing the following modifications:-

$$\log \frac{R_i}{\rho_i} = \log \frac{\xi_j}{\xi_j - \xi} \cdot \frac{\xi_j - \xi}{\xi_j} \cdot \frac{R_i}{\rho_i}$$

$$\log \frac{R_i}{\rho_i} = \log \frac{\xi_j}{\xi_j - \xi} + \log \frac{(\xi_j - \xi) R_i}{\xi_j \rho_i}$$

Taking the limits when $\rho_i \rightarrow 0$, i.e. $\xi \rightarrow \xi_j$, it is clear that the first term is singular whilst the second term is regular. The I_1^s can be rewritten as follows:-

$$I_1^s = I_1^{sg} + I_1^{ss} \quad (5.40)$$

where

$$I_1^{sg} = \int_0^{\xi_j} C_k \cdot \log \left(\frac{\xi_j - \xi}{\xi_j} \cdot \frac{R_i}{\rho_i} \right) N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) d\xi$$

which can be dealt with using Gaussian quadrature, and

$$I_1^{ss} = \int_0^{\xi_j} C_k \cdot \log \left(\frac{\xi_j}{\xi_j - \xi} \right) \cdot N_j(\xi) \cdot r(\xi) \cdot |J(\xi)| \cdot d\xi$$

Defining $\rho_1 = \frac{\xi_j - \xi}{\xi_j}$, then

$$d\xi = -\xi_j d\rho_1, \text{ and}$$

at

$$\xi = 0 \quad \rho_1 = 1$$

$$\xi = \xi_j \quad \rho_1 = 0$$

Therefore

$$I_1^{sg} = - \int_1^0 \log \left(\frac{1}{\rho_1} \right) \cdot N_j(\xi) \cdot r(\xi) \cdot |J(\xi)| \cdot C_k \cdot \xi_j d\rho_1$$

$$I_1^{ss} = \int_0^1 \log \left(\frac{1}{\rho_1} \right) \cdot N_j(\xi) \cdot r(\xi) \cdot |J(\xi)| \cdot C_k \cdot \xi_j \cdot d\rho_1$$

(5.91)

which is in a suitable form for evaluation by means of the singular quadrature given by equation(4.36).

Similarly the term I_2^s can be rewritten as:-

$$I_2^s = I_2^{ss} + I_2^{sg} \quad (5.42)$$

where

$$I_2^{sg} = \int_{\xi_j}^1 C_k \cdot \log \left(\frac{(\xi - \xi_j) R_i}{(1 - \xi_j) \rho_2} \right) \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) d\xi \quad (5.43)$$

and

$$I_2^{ss} = \int_{\xi_j}^1 C_k \cdot \log \left(\frac{(1 - \xi_j)}{(\xi - \xi_j)} \right) \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot d\xi$$

Defining

$$\rho_2 = \frac{\xi - \xi_j}{1 - \xi_j}, \quad \text{then}$$

at

$$\xi = \xi_j \quad \rho_2 = 0$$

$$\xi = 1 \quad \rho_2 = 1$$

and

$$d\rho_2 = \left(\frac{1}{1 - \xi_j} \right) d\xi$$

Hence, it can be shown that:-

$$I_2^{ss} = \int_0^1 C_k \cdot \log \frac{1}{\rho_2} N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot (1 - \xi_j) d\rho_2 \quad (5.44)$$

From equations (5.38), (5.40) and (5.43), it can be shown that

$$I_s = \frac{4\pi}{C_c} [I_1^{ss} + I_1^{sg} + I_2^{sg} + I_2^{ss}] \quad (5.45)$$

Using equation (5.37), and the above equation, it can be shown that

$$g^e(\gamma, 2(j-1) + \beta) = I_0 + I_1 + I_2$$

where

$$I_0 = I^G + \frac{4\pi}{C_c} [I_1^{sg} + I_2^{sg}]$$

$$I_1 = \frac{4\pi}{C_c} I_1^{ss}$$

$$I_2 = \frac{4\pi}{C_c} I_2^{ss}$$

Similarly, it can be shown that

$$h^e(\gamma, 2(j-1) + \beta) = I_0 + I_1 + I_2$$

$$I_0 = I^G + \frac{4\pi}{8\pi^2 r_i (1-\nu)} [I_1^{sg} + I_2^{sg}]$$

$$I_1 = \frac{4\pi}{8\pi^2 r_i (1-\nu)} I_1^{ss}$$

$$I_2 = \frac{4\pi}{8\pi^2 r_i (1-\nu)} I_2^{ss}$$

$$I^G = \frac{1}{8\pi^2 r_i (1-\nu)} \int_0^1 \left[A_{\gamma\beta} \cdot K_1 + B_{\gamma\beta} \cdot E_1 \right] N_j(\xi) |J(\xi)|$$

$$2\pi r(\xi) \cdot d\xi$$

$$I_1^{sg} = \int_0^{\xi_j} \left[A_{\gamma\beta} K_2 + B_{\gamma\beta} E_2 \right] \cdot \log \frac{\xi_j - \xi}{\xi_j} \cdot \left(\frac{R_i}{\rho_i} \right) \cdot N_j(\xi) \cdot |J(\xi)| r(\xi) \cdot d\xi$$

$$I_2^{sg} = \int_{\xi_j}^1 \left[A_{\gamma\beta} K_2 + B_{\gamma\beta} E_2 \right] \cdot \log \frac{(\xi - \xi_j) R_i}{(1 - \xi_j) \rho_i} \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot d\xi$$

$$I_1^{ss} = \int_0^1 \left[A_{\gamma\beta} K_2 + B_{\gamma\beta} E_2 \right] \cdot \log \left(\frac{1}{\rho_1} \right) \cdot N_j(\xi) \cdot r(\xi) \cdot |J(\xi)| \cdot \xi_j \cdot d\rho_1$$

$$I_2^{ss} = \int_0^1 \left[A_{\gamma\beta} K_2 + B_{\gamma\beta} E_2 \right] \cdot \log \frac{1}{\rho_2} \cdot N_j(\xi) \cdot |J(\xi)| \cdot r(\xi) \cdot (1 - \xi_j) \cdot d\rho_2$$

5.7 INTEGRATION OVER THE DOMAIN

Domain integration should be carried out in order to evaluate the nodal forces necessary to re-equilibrate the structure subjected to the residual stresses inside the domain.

The residual forces can be evaluated, in a way similar to that explained in section 4.7.2.2, as follows:-

$$(F_\gamma)_c = \sum_{s=1}^N \sum_{\beta=1}^N f(\eta_s) \cdot I_\gamma(r_{\beta s}, z_{\beta s}) \cdot W_\beta \cdot W_s \cdot |J| \cdot 2\pi \cdot r_{\beta s}$$

where

$$I_\gamma \equiv R_{\gamma 1} \sigma_r^R + R_{\gamma 2} \sigma_z^R + R_{\gamma 3} \sigma_\theta^R + R_{\gamma 4} \tau_{rz}^R$$

$$\left. \begin{aligned} r_{\beta s} &= \sum_{j=1}^n N_j(\xi_{\beta s}, \eta_s) \cdot r_i \\ z_{\beta s} &= \sum_{j=1}^n N_j(\xi_{\beta s}, \eta_s) \cdot z_i \end{aligned} \right\} \begin{array}{l} \text{are the co-ordinates of} \\ \text{the Gaussian points} \end{array}$$

n is the number of nodes for the integration cells

$$\xi_{\beta s} = f(\eta_s) \cdot a_{\beta}$$

(a_{β}, η_s) is the abscissa of the Gaussian quadrature

W_{β}, W_s are the corresponding weights.

The values of σ_r^R , σ_z^R , σ_{θ}^R , and τ_{rz}^R , in the above equations are the residual stresses at the Gaussian point which can be evaluated by means of isoparametric interpolation equations.

5.8 STRESS-STRAIN CALCULATION

The strain components can be calculated for a specified node by using a suitable approach according to the position of the node, as explained in the following sections.

5.8.1 Stresses at a Boundary Node

The approach employed in this section is similar to that explained in section 4.7.3.2 but with slight modifications.

Equation (4.44) can be rewritten for the axisymmetric case as follows:-

$$\left. \begin{aligned} -m \left(\frac{\partial u}{\partial r} \right) + \left(\frac{\partial u}{\partial z} \right) &= \left(\frac{\partial u}{\partial \xi} \right) / J(\xi) \\ -m \left(\frac{\partial w}{\partial r} \right) + \left(\frac{\partial w}{\partial z} \right) &= \left(\frac{\partial w}{\partial \xi} \right) / J(\xi) \end{aligned} \right\} \quad (5.46)$$

where

$$J(\xi) = \sqrt{\left(\frac{\partial r}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}$$

The surface-traction equilibrium equation can be written as follows:-

$$\begin{aligned} T_r &= \int \sigma_r + m \tau_{rz} \\ T_z &= \int \tau_{rz} + m \sigma_z \end{aligned}$$

Substituting equations (5.1) into the above equation, it can be proved that the values of

$$T_r = \int \left[d_{11} \left(\frac{\partial u}{\partial r} \right) + d_{12} \left(\frac{\partial w}{\partial z} \right) + d_{13} \frac{u}{r} \right] + m \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] \cdot d_{44} - T_r^o \quad (5.47)$$

$$T_z = \int d_{44} \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] + m \cdot \left[d_{21} \frac{\partial u}{\partial r} + d_{22} \frac{\partial w}{\partial z} + d_{23} \frac{u}{r} \right] - T_z^o$$

where

$$\begin{aligned} T_r^o &= \int [d_{11} \epsilon_r^{LP} + d_{12} \epsilon_z^{LP} + d_{13} \epsilon_\theta^{LP}] + m \cdot d_{44} \gamma_{rz}^{LP} \\ T_z^o &= \int d_{44} \cdot \gamma_{rz}^{LP} + m \cdot [d_{21} \epsilon_r^{LP} + d_{22} \epsilon_z^{LP} + d_{23} \epsilon_\theta^{LP}] \end{aligned}$$

From equations (5.46) and (5.47), it can be deduced that

$$\underline{A} \cdot \underline{U} = \underline{T}$$

where

$$\underline{A} = \begin{bmatrix} -m & \ell & 0 & 0 \\ 0 & 0 & -m & \ell \\ \int d_{11} & m d_{44} & m d_{44} & \int d_{12} \\ m d_{21} & \int d_{44} & \int d_{44} & m d_{22} \end{bmatrix}$$

$$\underline{U} = \left\{ \frac{\partial u}{\partial r} \quad \frac{\partial u}{\partial z} \quad \frac{\partial w}{\partial r} \quad \frac{\partial w}{\partial z} \right\} \text{ is the vector of the unknown variables}$$

$$\underline{T} = \begin{bmatrix} \frac{\partial u}{\partial \xi} / J \\ \frac{\partial w}{\partial \xi} / J \\ T_r - T_r^0 - \lambda d_{13} \cdot \frac{u}{r} \\ T_z - T_z^0 - m d_{23} \cdot \frac{u}{r} \end{bmatrix}$$

By solving the above equation

$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial z}, \frac{\partial w}{\partial r} \text{ and } \frac{\partial w}{\partial z}$$

can be evaluated. Therefore the strain components can be calculated, followed by the elastic stress components using the generalised Hooke's Law.

5.8.2 Stresses at a Corner Point

To deal with the discontinuity in the field variables at a corner point, an approach similar to that used in section 4.7.3.3. will be employed.

The corner point is a node in two different elements, as shown in Fig(5.5). From equations (4.47) it can be shown that:-

$$\sigma_r = \frac{m_1^2 f_2 - m_2^2 f_1}{\text{Det.}}$$

$$\sigma_z = \frac{l_1^2 f_2 - l_2^2 f_1}{\text{Det.}}$$

where

$$\text{Det.} = (l_2 m_1 - l_1 m_2)(l_2 m_1 + l_1 m_2)$$

$$f_1 = l_1 (T_r)_1 - m_1 (T_z)_1$$

$$f_2 = l_2 (T_r)_2 - m_2 (T_z)_2$$

${}_1 m_1$ are the directional cosines of the normal to element 1 at the corner node

${}_2 m_2$ are the directional cosines of the normal to element 2 at the corner node

$(T_r)_1, (T_z)_1$ are the surface tractions in the r and z directions respectively, for element 1

$(T_r)_2, (T_z)_2$ are the surface tractions in the r and z direction respectively, for element 2.

From Hooke's Law, the tangential stress can be obtained as follows:-

$$\epsilon_\theta = \frac{1}{E} \left[\sigma_\theta - \nu(\sigma_r + \sigma_z) \right]$$

i.e.

$$\sigma_\theta = E \cdot \epsilon_\theta + \nu(\sigma_r + \sigma_z)$$

where

$$\epsilon_\theta = \frac{u}{r}$$

The average value of the shear stress can be calculated from equation (4.48) as follows:-

$$\tau_{rz} = \frac{1}{2} \left[m_1 (T_r)_1 + l_1 (T_z)_1 + m_2 (T_r)_2 + l_2 (T_z)_2 - (l_1 m_1 + l_2 m_2)(\sigma_r + \sigma_z) \right]$$

5.8.3 Stresses at an Internal Node

The stresses at any internal source point can be evaluated by direct derivation of displacement components at the same point. Unfortunately the derivation of the fundamental solution parameters of equations (5.23) and (5.24), is very complex, therefore a finite difference method can be employed, as explained in section 4.7.3.4, in order to evaluate the strain components.

Consider four points "a,b,c", and "d" around the specified node (r_i, z_i) be defined as follows:-

$$\begin{aligned} a &= (r_i - \Delta r, z_i) , \\ b &= (r_i + \Delta r, z_i) , \\ c &= (r_i, z_i - \Delta z) , \text{ and} \\ d &= (r_i, z_i + \Delta z) . \end{aligned}$$

Therefore the strain components can be obtained as follows:-

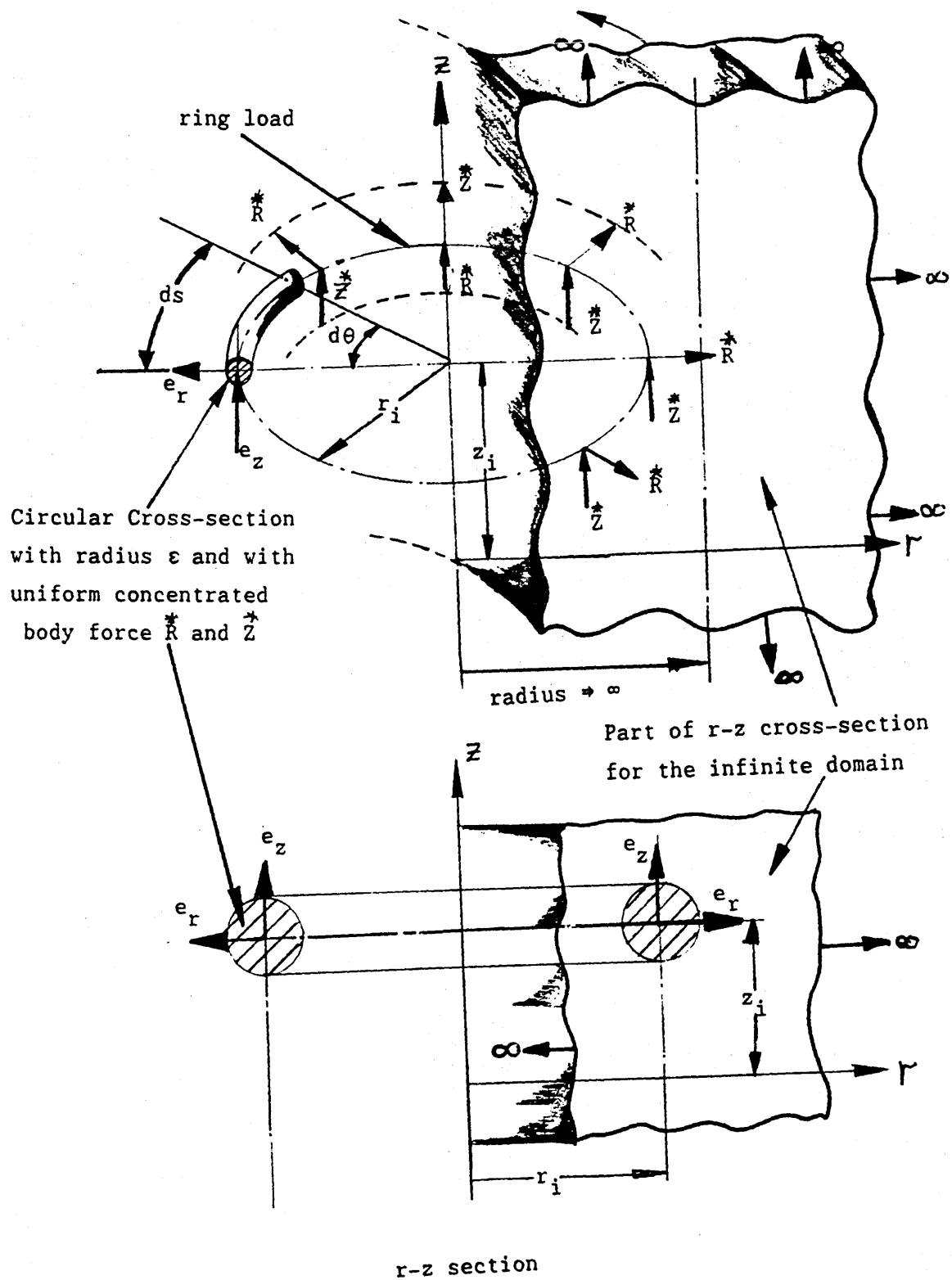
$$(\epsilon_r)_i = \left(\frac{\partial u}{\partial r} \right)_i = \frac{u_b - u_a}{2\Delta r}$$

$$(\epsilon_z)_i = \left(\frac{\partial w}{\partial z} \right)_i = \frac{w_d - w_c}{2\Delta z}$$

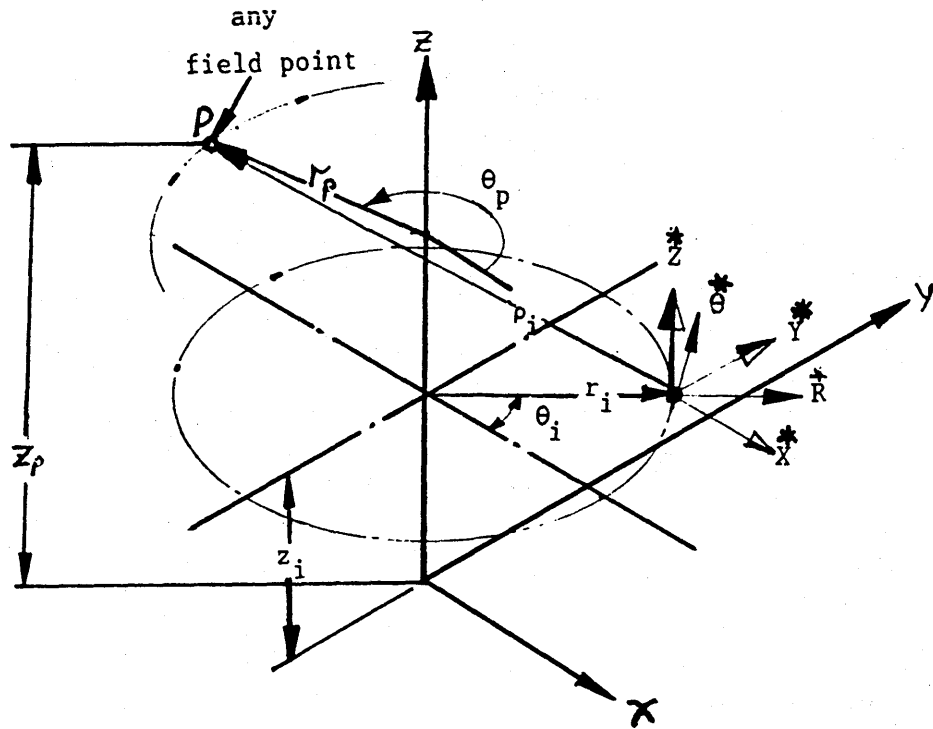
$$(\epsilon_\theta)_i = \frac{u_i}{r_i}$$

$$(\gamma_{rz})_i = \left(\frac{\partial w}{\partial r} \right)_i + \left(\frac{\partial u}{\partial z} \right)_i = \frac{w_b - w_a}{2\Delta r} + \frac{u_d - u_c}{2\Delta z}$$

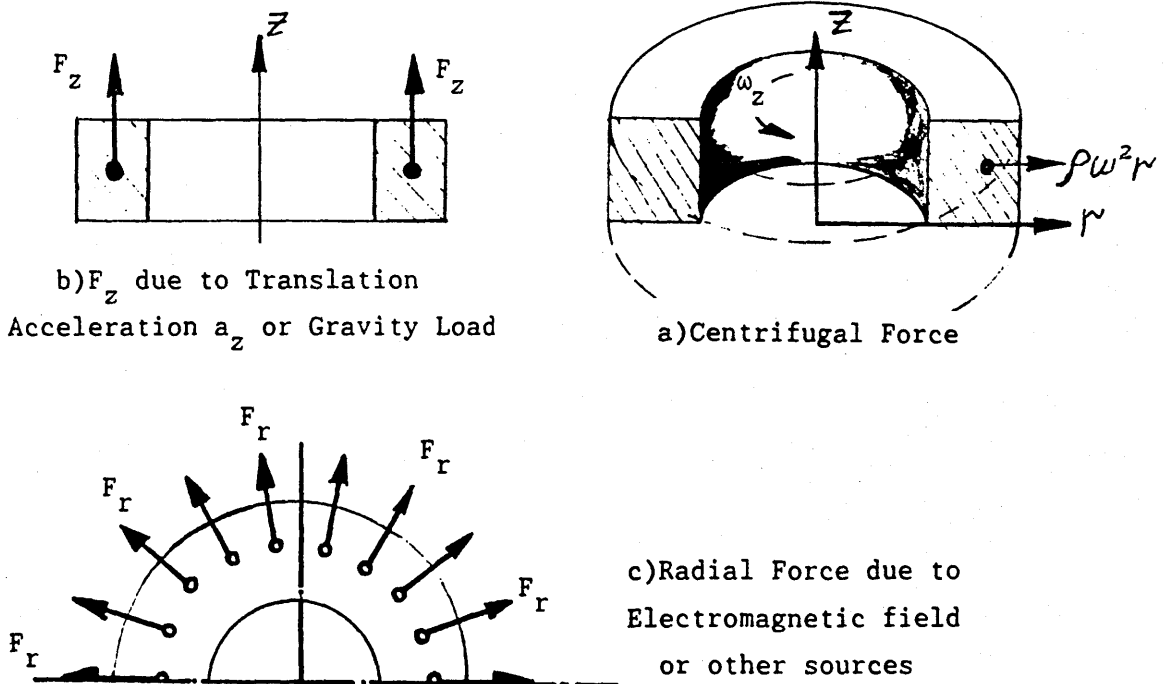
The corresponding stress components can be expressed by using the generalised Hooke's Law.



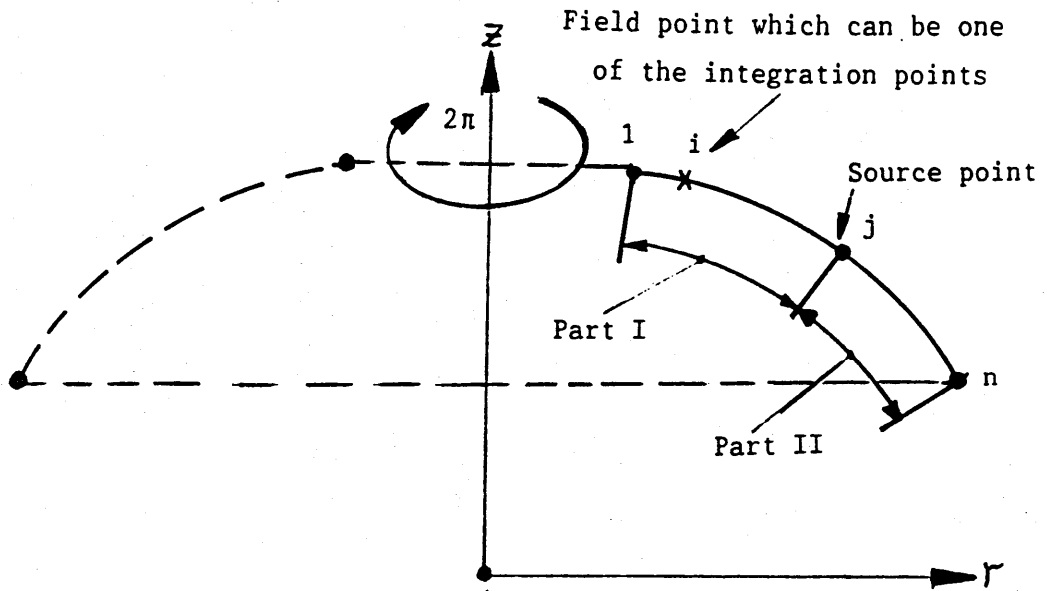
Fig(5.1) Infinite Three Dimensional Body under Axisymmetric Concentrated Ring Load



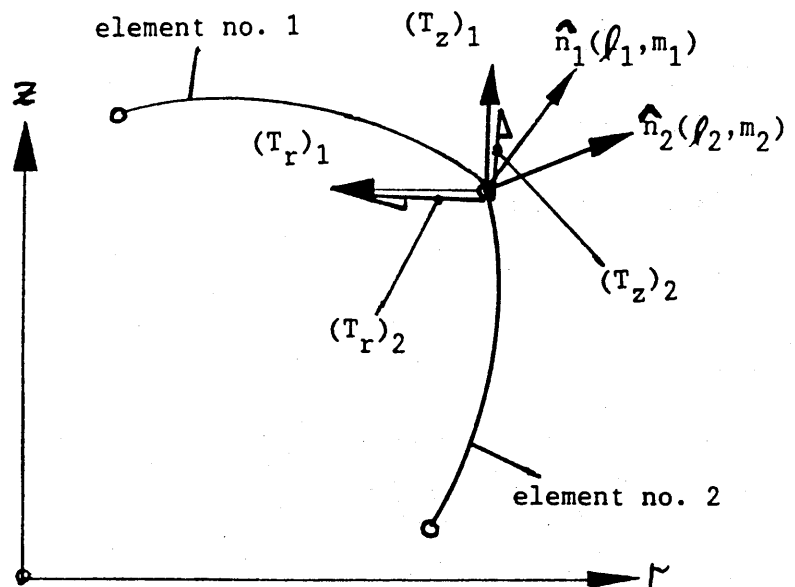
Fig(5.2) Axisymmetric Body with Concentrated Load in Cartesian and Polar Coordinates



Fig(5.3) Possibil Sources of Axisymmetric Load



Fig(5.4) Axisymmetric Boundary Element with Singularity.



Fig(5.5) Corner Point in the Boundary of an Axisymmetric Body

$$K_{rr} = [(3-4\nu)(r^2 + r_i^2) + 4(1-\nu)(z-z_i)^2] / 2rR_i$$

$$K_{rz} = r_i(z-z_i) / 2rR_i$$

$$K_{zr} = -(z-z_i) / 2R_i$$

$$K_{zz} = (3-4\nu)r_i / R_i$$

$$E_{rr} = \frac{(r^2 - r_i^2)^2 - (z-z_i)^4}{4rR_i\rho_i} - \left(\frac{7-8\nu}{4r}\right)R_i$$

$$E_{rz} = r_i(z-z_i)[(r^2 - r_i^2) - (z-z_i)^2] / (2rR_i\rho_i)$$

$$E_{zr} = (z-z_i)[(r^2 - r_i^2) - (z-z_i)^2] / (2R_i\rho_i)$$

$$E_{zz} = r_i(z-z_i)^2 / (R_i\rho_i^2)$$

Where

$$R_i = \sqrt{(r+r_i)^2 + (z-z_i)^2},$$

$$\rho_i = \sqrt{(r-r_i)^2 + (z-z_i)^2},$$

r_i, z_i coordinates of source point, and

r, z coordinates of field point.

Table(5.1) Displacement Fundamental Solution Parameters $K_{\gamma\beta}$, $E_{\gamma\beta}$

i	a_i	b_i
$n=2$		
1	1.119697E-01	1.213486E-01
2	7.253230E-02	2.887472E-02
$n=3$		
1	9.79324618E-02	1.24750843E-01
2	5.45433073E-02	6.01197058E-02
3	3.20261966E-02	1.09455763E-02
$n=4$		
1	9.666338350E-02	1.249859468E-01
2	3.589980090E-02	6.880295505E-02
3	3.742539571E-02	3.328521016E-02
4	1.451338556E-02	4.418398230E-03
$n=5$		
1	9.65786196226E-02	1.24999295975E-01
2	3.15594316275E-02	7.01487577829E-02
3	2.37612248576E-02	4.49838755399E-02
4	2.59628884526E-02	1.87516602769E-02
5	6.63980111468E-03	1.84723416323E-03
$n=6$		
1	9.6573843102223E-02	1.2499996748737E-01
2	3.0953955531153E-02	7.0298097586169E-02
3	1.6941959131641E-02	4.8159884398615E-02
4	1.9742905159930E-02	3.0724421769603E-02
5	1.7237188608291E-02	1.0504911494346E-02
6	3.0521141417676E-03	7.8967991858043E-04
$n=7$		
1	9.65736020516771E-02	1.24999998585309E-01
2	3.08909633861795E-02	7.03114105853296E-02
3	1.52618320622534E-02	4.87379510945218E-02
4	1.25565693543211E-02	3.57218443007327E-02
5	1.68695685967517E-02	2.09857677336790E-02
6	1.09423810688623E-02	5.81807961871996E-03
7	1.40704915496101E-03	3.42805719229748E-04
$n=8$		
1	9.6573590797589018E-02	1.2499999994117923E-01
2	3.0885573486752694E-02	7.0312426464627361E-02
3	1.4978988178704629E-02	4.8818058565403952E-02
4	9.6587579861753113E-03	3.7068398934155422E-02
5	1.1208918554644092E-02	2.7189861116788250E-02
6	1.3855601247156560E-02	1.4105380776158048E-02
7	6.6905509906897936E-03	3.1831309927862886E-03
8	6.4998443379390180E-04	1.5049181783601883E-04
$n=9$		
1	9.65735903017425285E-02	1.24999999997640658E-01
2	3.08851730018997099E-02	7.03124954595466082E-02
3	1.49420291422820783E-02	4.88271550481180099E-02
4	8.92664629455646620E-03	3.73355466822860296E-02
5	7.51938672180838102E-03	2.95037293486887130E-02
6	1.05899536209893585E-02	2.06902400051008404E-02
7	1.07959904905916349E-02	9.28116038296860419E-03
8	3.96847090209897819E-03	1.72161470979865212E-03
9	3.00725199036864838E-04	6.66317524646073151E-05
$n=10$		
1	9.6573590280856255384E-02	1.249999999990808051E-01
2	3.0885146271305189866E-02	7.0312499739038352054E-02
3	1.4938013532687165242E-02	4.8828041906862397978E-02
4	8.7898018745550646778E-03	3.7377739758623604144E-02
5	6.1796274460533176084E-03	3.0124849012898930266E-02
6	6.8479092826245051197E-03	2.3931913323110790077E-02
7	9.8489293221768937682E-03	1.5530941631977203877E-02
8	8.0030039806499853708E-03	5.9739042991554291551E-03
9	2.2966348983969586869E-03	9.2155463496324984638E-04
10	1.3930878570066467279E-04	2.9700280966555612066E-05

Table(5.2) Coefficients for Elliptic Integrals K

i	c_i	d_i
		n=2
1	4.630106E-01	2.452740E-01
2	1.077857E-01	4.125321E-02
		n=3
1	4.44789300E-01	2.49698607E-01
2	8.50922292E-02	8.15096894E-02
3	4.09147972E-02	1.38343651E-02
		n=4
1	4.432515145E-01	2.499836641E-01
2	6.260761942E-02	9.200109374E-02
3	4.757404429E-02	4.069468414E-02
4	1.736314854E-02	5.263789328E-03
		n=5
1	4.43152874726E-01	2.49999202736E-01
2	5.75669984841E-02	9.35649078307E-02
3	3.17611455247E-02	5.42605244487E-02
4	3.06623474572E-02	2.18360211693E-02
5	7.65296060328E-03	2.12479182845E-03
		n=6
1	4.4314746159513E-01	2.4999996385465E-01
2	5.6881588013808E-02	9.3734005947003E-02
3	2.4052363568173E-02	5.7852808337762E-02
4	2.3657946984506E-02	3.5359651640905E-02
5	1.9623272084535E-02	1.1885115619289E-02
6	3.4336945487476E-03	8.8741784464644E-04
		n=7
1	4.43147193467733E-01	2.49999998448655E-01
2	5.68115681053803E-02	9.37488062098189E-02
3	2.21862206993846E-02	5.84950297066166E-02
4	1.56847700239786E-02	4.09074821593164E-02
5	1.92284389022977E-02	2.35091602564984E-02
6	1.21819481486695E-02	6.45682247315060E-03
7	1.55618744745296E-03	3.78886487349367E-04
		n=8
1	4.4314718112155806E-01	2.499999993617622E-01
2	5.6805657874695858E-02	9.3749920249680113E-02
3	2.1876220647186198E-02	5.8582839536559024E-02
4	1.2510592410844644E-02	4.2382807456947900E-02
5	1.3034146073731432E-02	3.0302747728412848E-02
6	1.5377102528552019E-02	1.5525129948040721E-02
7	7.3356174974290365E-03	3.4838679435896492E-03
8	7.0980964089987229E-04	1.6427210797048025E-04
		n=9
1	4.43147180583368137E-01	2.49999999997461423E-01
2	5.68052233293082895E-02	9.37499951163670673E-02
3	2.18361314054868967E-02	5.85927071842652739E-02
4	1.17167669446577228E-02	4.26725101265917523E-02
5	9.03552773754088184E-03	3.28110691727210618E-02
6	1.18419259955012494E-02	2.26603098916041221E-02
7	1.17858410087339355E-02	1.00879584943751004E-02
8	4.30253777479311659E-03	1.86453791840633632E-03
9	3.25192015506390418E-04	7.20316963457154599E-05
		n=10
1	4.4314718056033952648E-01	2.499999999990177208E-01
2	5.6805194567559156648E-02	9.3749999721203140658E-02
3	2.1831811676130481568E-02	5.8593661255531491732E-02
4	1.1569595745295402175E-02	4.2717890547383095644E-02
5	7.5950934225594322802E-03	3.3478943665761626232E-02
6	7.8204040609595541727E-03	2.6145014700313878932E-02
7	1.0770635039866455473E-02	1.6804023346363384981E-02
8	8.6384421736040744302E-03	6.4321465864383017666E-03
9	2.4685033304607227339E-03	9.8983328462253847867E-04
10	1.4946621757181326771E-04	3.1859195655501571800E-05

Table(5.3) Coefficients for Elliptic Integrals E

$$\alpha_{11}^* = \frac{(3-4\nu)}{R_i} - \left(\frac{r+r_i}{R_i^2} + 1/r \right) K_{rr} - C_r (E_{rr} + K_{rr})$$

$$\alpha_{12}^* = - \left(\frac{r+r_i}{R_i^2} + 1/r \right) K_{rz} - C_r (E_{rz} + K_{rz})$$

$$\alpha_{21}^* = - \frac{1}{2R_i} - \left(\frac{z-z_i}{R_i^2} \right) K_{zr} - C_z (E_{zr} + K_{zr})$$

$$\alpha_{22}^* = - \left(\frac{z-z_i}{R_i^2} \right) K_{zz} - C_z (E_{zz} + K_{zz})$$

$$\alpha_{31}^* = K_{rr}/r$$

$$\alpha_{32}^* = K_{rz}/r$$

$$\alpha_{41}^* = \frac{4(1-\nu)(z-z_i)}{rR_i} - \left(\frac{z-z_i}{R_i^2} \right) - \left(\frac{r+r_i}{R_i^2} \right) K_{zr} - C_z (K_{rr} + E_{rr}) - C_r (K_{zr} + E_{zr})$$

$$\alpha_{42}^* = \frac{r_i}{2rR_i} - \left(\frac{z-z_i}{R_i^2} \right) K_{rz} - \left(\frac{r+r_i}{R_i^2} \right) K_{zz} - C_z (K_{rz} + E_{rz}) - C_z (K_{zz} + E_{zz})$$

$$\beta_{11}^* = \frac{r^2 - r_i^2}{R_i \rho_i^2} - \left(\frac{7-8\nu}{4r} \right) \left[\frac{R_i}{r} - \frac{r+r_i}{R_i} \right] - (\alpha + 1/r) DE_{rr} + Cr \left(\frac{K_{rr}}{1-m_e^2} + E_{rr} \right)$$

$$\beta_{12}^* = \frac{r_i(z-z_i)}{R_i \rho_i^2} - (\alpha + 1/r) E_{rz} - Cr \left(\frac{K_{rz}}{1-m_e^2} + E_{rz} \right)$$

$$\beta_{21}^* = \frac{(r^2 - r_i^2) + 3(z-z_i)^2}{R_i \rho_i^2} - \beta E_{zr} - C_z \left(\frac{K_{zr}}{1-m_e^2} + E_{zr} \right)$$

$$\beta_{22}^* = \frac{2r_i(z-z_i)}{R_i \rho_i^2} - \beta E_{zz} - C_z \left(\frac{K_{zz}}{1-m_e^2} + E_{zz} \right)$$

$$\beta_{31}^* = E_{rr}/r$$

$$\beta_{32}^* = E_{rz}/r$$

$$\beta_{41}^* = \frac{r(z-z_i)}{R_i \rho_i^2} - \alpha E_{zr} - \beta DE_{rr} - \gamma - C_z (K_{rr} + E_{rr}) - C_r (K_{zr} + E_{zr})$$

$$\beta_{42}^* = \frac{r_i[r^2 - r_i^2 - 3(z-z_i)^2]}{2rR_i \rho_i^2} - \beta E_{rz} - \alpha E_{zz} + C_z \left(\frac{K_{rr}}{1-m_e^2} + E_{rr} \right) - C_r \left(\frac{K_{zr}}{1-m_e^2} + E_{zr} \right)$$

Where

$$\alpha = \frac{r+r_i}{R_i^2} + \frac{2(r-r_i)}{\rho_i^2}$$

$$\gamma = \frac{(z-z_i)^3}{r R_i \rho_i^2} + \frac{(7-8\nu)(z-z_i)}{4 r R_i}$$

$$\beta = (z-z_i) (1/R_i^2 + 2/\rho_i^2)$$

$$DE_{rr} = \frac{(r^2 - r_i^2)^2 - (z-z_i)^4}{4 r R_i \rho_i^2}$$

Table(5.4) Expression of Parameters $\alpha_{\gamma s}^*$ and $\beta_{\gamma s}^*$ of Strain Fundamental Solution

$$A_{rr} = \ell \left[a \frac{\partial K_{rr}}{\partial r} + b \left(\frac{K_{rr}}{r} \frac{\partial K_{zz}}{\partial z} \right) \right] + m \left(\frac{\partial K_{rr}}{\partial z} + \frac{\partial K_{zz}}{\partial r} \right) + C_{rr} (K_{rr} + E_{rr}) + C_{rz} (K_{zz} + E_{zz})$$

$$B_{rr} = \ell \left[a \frac{\partial E_{rr}}{\partial r} + b \left(\frac{E_{rr}}{r} \frac{\partial E_{zz}}{\partial z} \right) \right] + m \left(\frac{\partial E_{rr}}{\partial z} + \frac{\partial E_{zz}}{\partial r} \right) - C_{rr} \left(\frac{K_{rr}}{1-m_e^2} E_{rr} \right) - C_{rz} \left(\frac{K_{zz}}{1-m_e^2} E_{zz} \right)$$

$$A_{rz} = \ell \left[a \frac{\partial K_{rz}}{\partial r} + b \left(\frac{K_{rz}}{r} \frac{\partial K_{zz}}{\partial z} \right) \right] + m \left(\frac{\partial K_{rz}}{\partial z} + \frac{\partial K_{zz}}{\partial r} \right) + C_{rr} (K_{rz} + E_{rz}) + C_{rz} (K_{zz} + E_{zz})$$

$$B_{rz} = \ell \left[a \frac{\partial E_{rz}}{\partial r} + b \left(\frac{E_{rz}}{r} \frac{\partial E_{zz}}{\partial z} \right) \right] + m \left(\frac{\partial E_{rz}}{\partial z} + \frac{\partial E_{zz}}{\partial r} \right) - C_{rr} \left(\frac{K_{rz}}{1-m_e^2} E_{rz} \right) - C_{rz} \left(\frac{K_{zz}}{1-m_e^2} E_{zz} \right)$$

$$A_{zz} = m \left[a \frac{\partial K_{zz}}{\partial r} + b \left(\frac{K_{zz}}{r} \frac{\partial K_{rr}}{\partial r} \right) \right] + \ell \left(\frac{\partial K_{rz}}{\partial z} + \frac{\partial K_{zz}}{\partial r} \right) + C_{zr} (K_{zz} + E_{zz}) + C_{zz} (K_{rr} + E_{rr})$$

$$B_{zz} = m \left[a \frac{\partial E_{zz}}{\partial r} + b \left(\frac{E_{zz}}{r} \frac{\partial E_{rr}}{\partial r} \right) \right] + \ell \left(\frac{\partial E_{rz}}{\partial z} + \frac{\partial E_{zz}}{\partial r} \right) - C_{zr} \left(\frac{K_{zz}}{1-m_e^2} E_{zz} \right) - C_{zz} \left(\frac{K_{rr}}{1-m_e^2} E_{rr} \right)$$

$$A_{zz} = m \left[a \frac{\partial K_{zz}}{\partial r} + b \left(\frac{K_{zz}}{r} \frac{\partial K_{rz}}{\partial r} \right) \right] + \ell \left(\frac{\partial K_{rz}}{\partial z} + \frac{\partial K_{zz}}{\partial r} \right) + C_{zr} (K_{zz} + E_{zz}) + C_{zz} (K_{rz} + E_{rz})$$

$$B_{zz} = m \left[a \frac{\partial E_{zz}}{\partial r} + b \left(\frac{E_{zz}}{r} \frac{\partial E_{rz}}{\partial r} \right) \right] + \ell \left(\frac{\partial E_{rz}}{\partial z} + \frac{\partial E_{zz}}{\partial r} \right) - C_{zr} \left(\frac{K_{zz}}{1-m_e^2} E_{zz} \right) - C_{zz} \left(\frac{K_{rz}}{1-m_e^2} E_{rz} \right)$$

Where

$$C_{rr} = \ell a C_r + m C_z$$

$$C_{zr} = \ell C_r + a m C_z$$

$$C_{rz} = m C_r + \ell b C_z$$

$$C_{zz} = b m C_r + \ell C_z$$

$$C_r = \frac{(r^2 - r_i^2) - (z - z_i)^2}{2r(r + r_i)^2 + (z - z_i)^2}$$

$$C_z = \frac{(z - z_i)}{(r + r_i)^2 + (z - z_i)^2}$$

$$a = 2 \left(\frac{1-v}{1-2v} \right)$$

$$b = 2 \left(\frac{v}{1-2v} \right)$$

Table(5.5) Expressions of Parameters $A_{\gamma\beta}$ and $B_{\gamma\beta}$ of the Fundamental Tracn Solution

CHAPTER SIX

FINITE ELEMENT AND BOUNDARY ELEMENT PROGRAMMING SYSTEM

6.0 Introduction

A Finite Element and Boundary Element package, which is capable of elastic or elasto-plastic stress analysis for Two-dimensional and axisymmetric problems, was designed according to the theory explained in the previous chapters. The pre-processor, with mesh generation programs for FEM and BEM is also explained in this chapter. This package has been designated FBNSA and its main segments, as shown in Fig(6.1), are described briefly below:

6.1 Pre-processor and Post-processor segment

The main objective of the pre-processor is to discretize the domain problem into a suitable number of finite elements or to divide the boundary of the problem into a specified number of boundary elements of a certain type, and to prepare the required data for the FEM or BEM programs. This discretization can be carried out by using the MESHFEBE mesh generation program which was modified from work carried out in a previous Ph.D programme [137], by considering transition finite elements and by providing software suitable for boundary discretization. The boundary element mesh generator can create meshes based upon constant, isoparametric or non-conforming, boundary elements. The program has the capability to divide the problem into an FEM region and/or a BEM region, according to the modelling requirements as shown in Figs(6.2-6.3). The MESHFEBE program requires a data file, which describes the block modelling of the structure and division ratios in the x and y directions as well as the type of the generated elements. The results of the MESHFEBE program are stored in files to be used directly for analysis and/or for mesh plotting.

The mesh plotting program PMESH was originally designed and based upon the GINO plotting package. When the Cranfield Computer Centre recently obtained the more advanced multi-purpose plotting

package "UNIRAS", it was necessary to write another program (UNIMESH2), based upon the UNIRAS package and using its new facilities for producing high quality mesh plotting. The two programs developed in this work can be used immediately after generating a mesh in order to check the topology of the generated mesh and to define the nodal numbers for boundary and loading conditions.

6.2 Stress Analysis Programs

The package is capable of carrying out an elastic or elasto-plastic analysis by using one of the following programs:

(i) FEMEP Program

This software can be used for solving 2-dimensional or axisymmetric elastic or elasto-plastic problems by employing the finite element theory described in Chapter 3. An advanced fast frontal solver was developed and employed for the solution of such problems. Under user control, elasto-plastic analysis may be carried out by Interpolative (Newton-Raphson), Iterative (Modified Newton-Raphson) or Combined schemes. Due to the expected large CPU time required for a full elasto-plastic analysis, the program has the facility for protection against computer crashes or from illogical selection of loading increment, by storing information at intermediate stages.

(ii) BEM2DEP Program

This software is based upon the 2-dimensional BEM technique which is explained in Chapter 4. After an initial phase where different types of boundary elements were tested, it was decided to employ isoparametric boundary elements only because they proved to be more accurate than either constant or non-conforming elements, particularly after the introduction of many accuracy measures. The stresses at any arbitrary boundary or internal node, are calculated by means of three different methods, as explained in Chapter 4,

depending upon the location of the node. The program has its own automatic device which defines, whether the node under consideration is at a corner, on the boundary, close to the boundary (within certain given tolerance), or well inside the domain, and accordingly the correct algorithm for stress-calculation will be employed without reference to the user.

(iii) BEMAXI Program

The function of this program is to analyse elastic and elasto-plastic axisymmetric problems. The theoretical foundation of this programme is explained in Chapter 5. Most of the facilities employed for the 2D program have been developed and used for the axisymmetric program also.

6.3 Common Segments

The previous analysis programs should be linked with the following segments in order to complete the procedure:

(a) Domain element library

The package contains a comprehensive finite element library thus giving the user a wide range of selection in order to achieve the required accuracy. Table (6.1) shows the available isoparametric finite elements. In this part of the package the derivation of the intrinsic shape functions, and the intrinsic coordinates can be obtained for any finite element listed in Table(6.1). For the BEM elasto-plastic solution procedure, the domain should be discretized into a suitable number of internal cells, which are finite elements, therefore the available library can be used with the BEM technique. This capability may give more flexibility in selecting appropriate integration cells with the added facility of adopting any Gaussian-quadrature scheme.

(b) Nodal equivalent force

The role of this section is to evaluate the equivalent nodal forces due to uniform or non-uniform pressure loading, gravity loading, or inertial loading. This part of the package is linked to the FE program only.

(c) Plasticity Algorithms

(i) D_{ep} Matrix Segments

This section was designed for 2-dimensional, axisymmetric, and 3-dimensional cases. In order to evaluate the D_{ep} matrix at a specified point in the domain the following data are required

- The elastic material properties(E, ν), and its uniaxial stress-strain diagram
- The stress state at the point
- The shift vector and/or the total isotropic plastic strain

The main subroutines which can be employed to obtain the D_{ep} matrix, as shown in Fig(6.4), are given below:

D matrix

The purpose of this subroutine is merely to determine the elastic stress/strain matrix based upon the material properties.

YIELD

The function of this subroutine is to evaluate the equivalent uniaxial stress at a specified point from the stress vector and to compare the uniaxial yield stress of the material with the calculated equivalent stress in order to check whether the point is in an elastic or elasto-plastic condition. This subroutine

requires the specification of the yield criterion employed and the instantaneous yield stress. The selection of a yield criterion will be carried out by setting a certain control number in the data file, whilst the instantaneous yield stress will be defined from the given uniaxial stress-strain diagram by means of subroutine TANMOD.

TANMOD

This subroutine evaluates the tangential elastic modulus and also updates the uniaxial yield stress at a specified point, and was designed for both linear and non-linear strain hardening. For the case of a material with non-linear hardening, the uniaxial stress-strain diagram will be fed from the initial yield point in terms of a suitable number of (σ_i, ϵ_i) points. A Lagrangian interpolation scheme is therefore employed for defining the value of σ and $d\sigma/d\epsilon$ at a given ϵ from such given numerical data.

GRADVE

The function of this subroutine is to determine the gradient vector ($\underline{a} = \partial F / \partial \underline{\sigma}$) as defined in chapter 2, as follows:

$$\underline{a} = C_1 \underline{a}_1 + C_2 \underline{a}_2 + C_3 \underline{a}_3$$

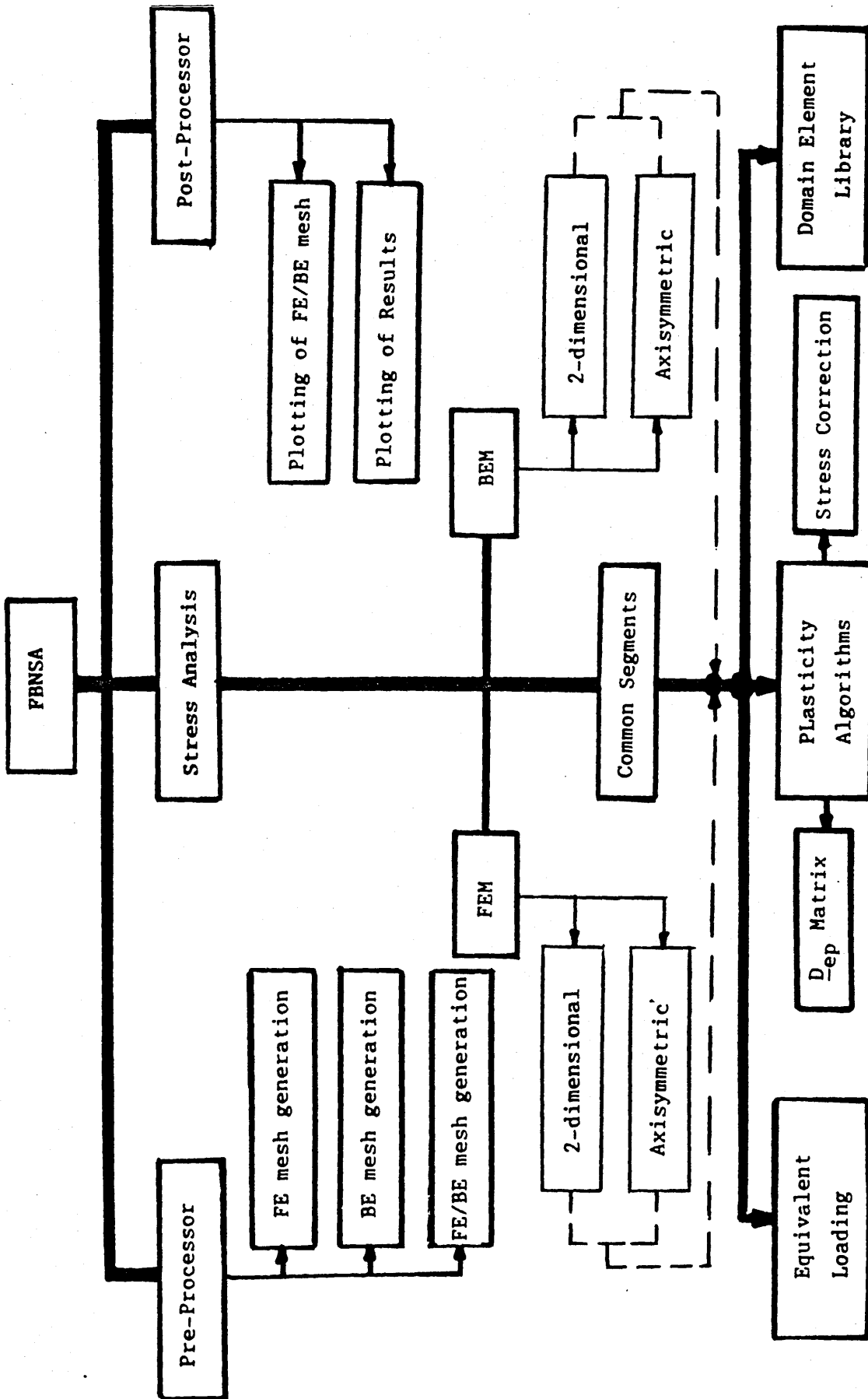
where C_1, C_2 , and C_3 are given in Table(2.1) for different yield criteria.

HARPAR

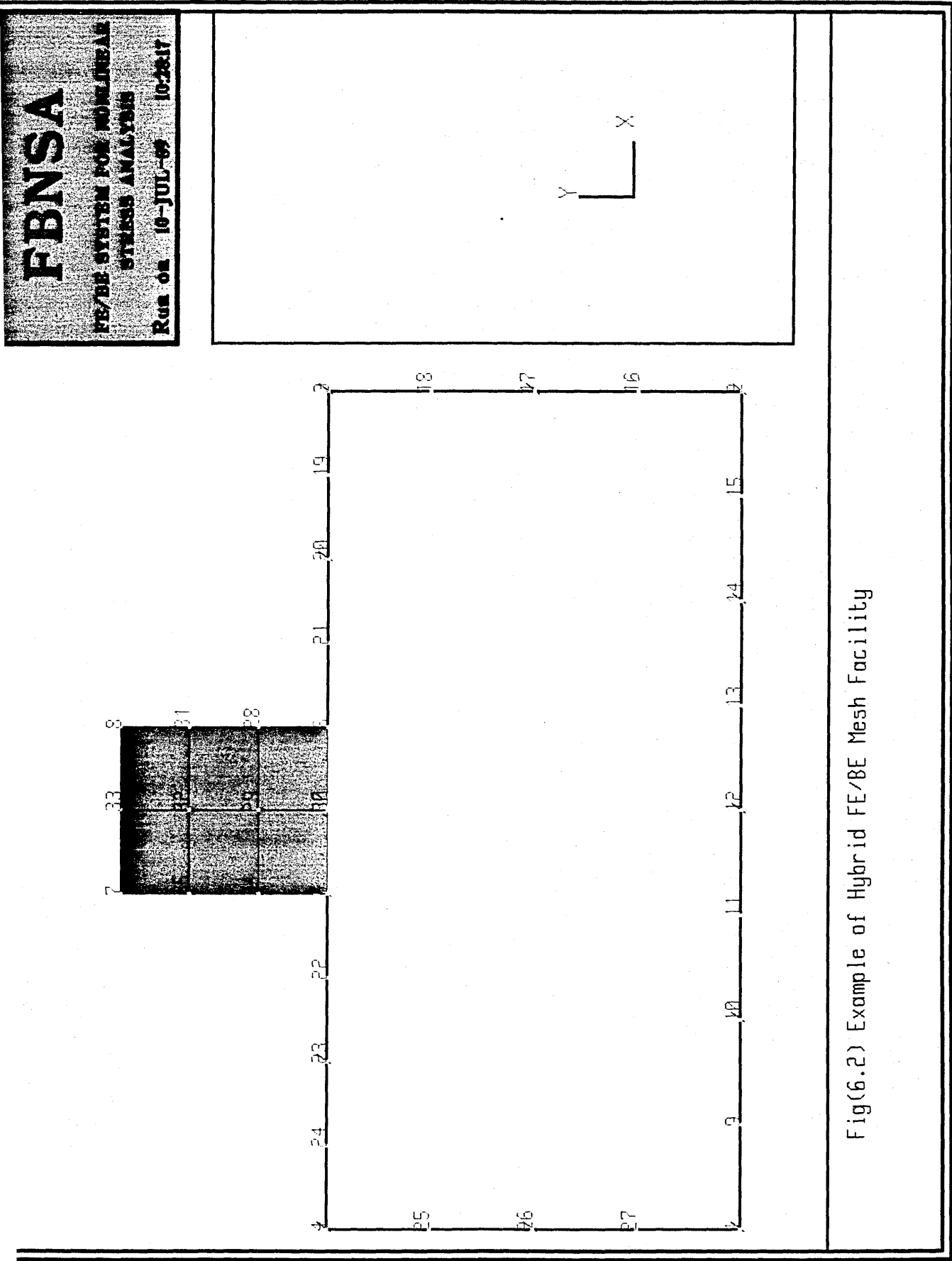
The main purpose of this subroutine is to determine the hardening parameter "A". This subroutine requires the definition of the hardening rule employed for the analysis which can be specified under user control by setting a value for another control parameter in the data file.

(ii) Stress Correction Subroutine

This segment is based upon a very accurate algorithm, as described in Chapter 3. The stress and strain increments at every point will be checked for first time yielding, and those occurring in the plastic range will automatically be divided into a large number of smaller increments, a procedure which improves the accuracy and convergence of iterations. For finite element analysis, stresses are calculated and corrected at both the nodal points and the Gauss-quadrature points. The latter are useful for the estimation of residual forces required for elasto-plastic iterations.



Fig(6.1) Main Outline of the FBNSA Package



FBNSA

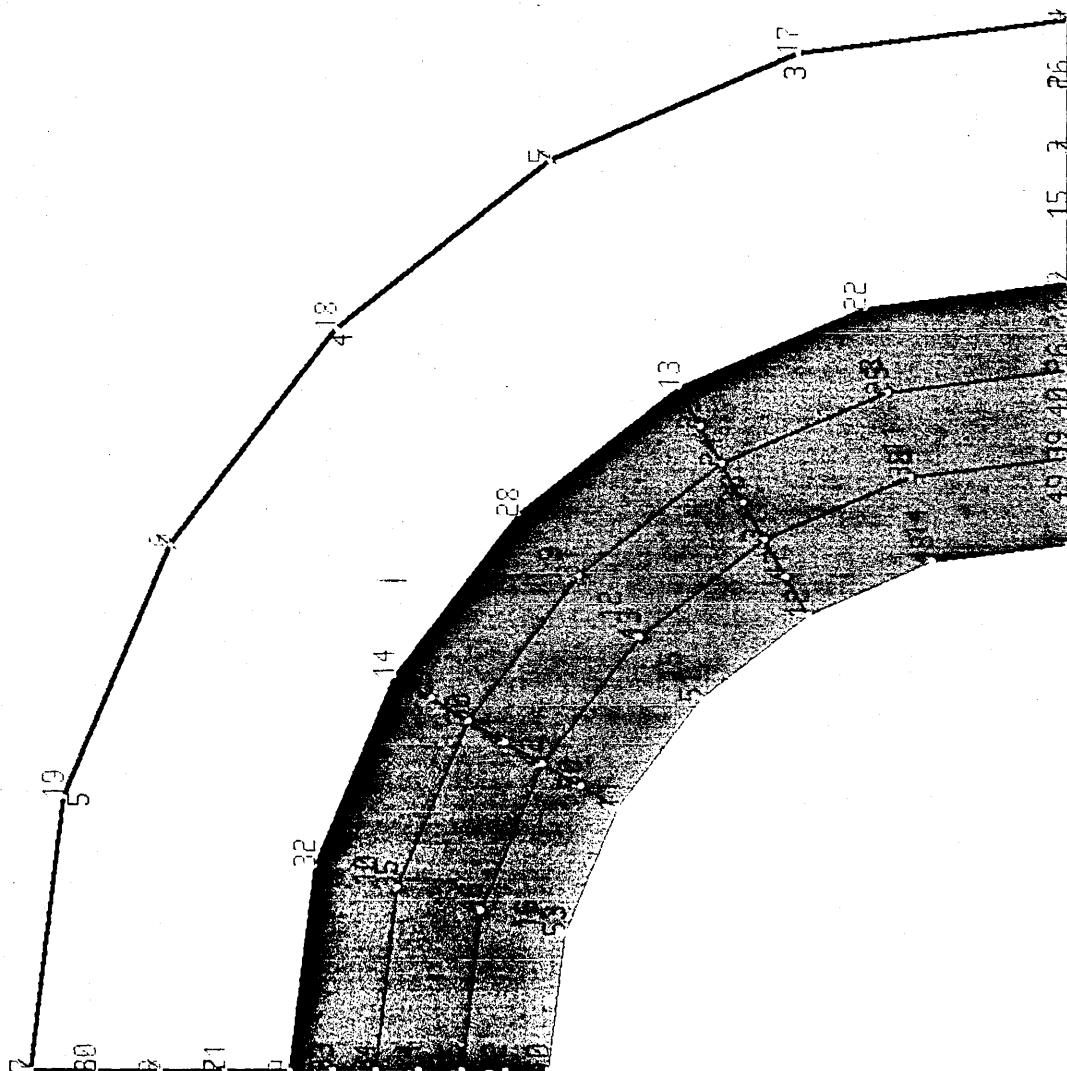
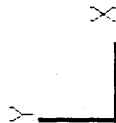
FE/BE SYSTEM FOR NONLINEAR
STRESS ANALYSIS

Run on 16-JUL-88 10:28:17

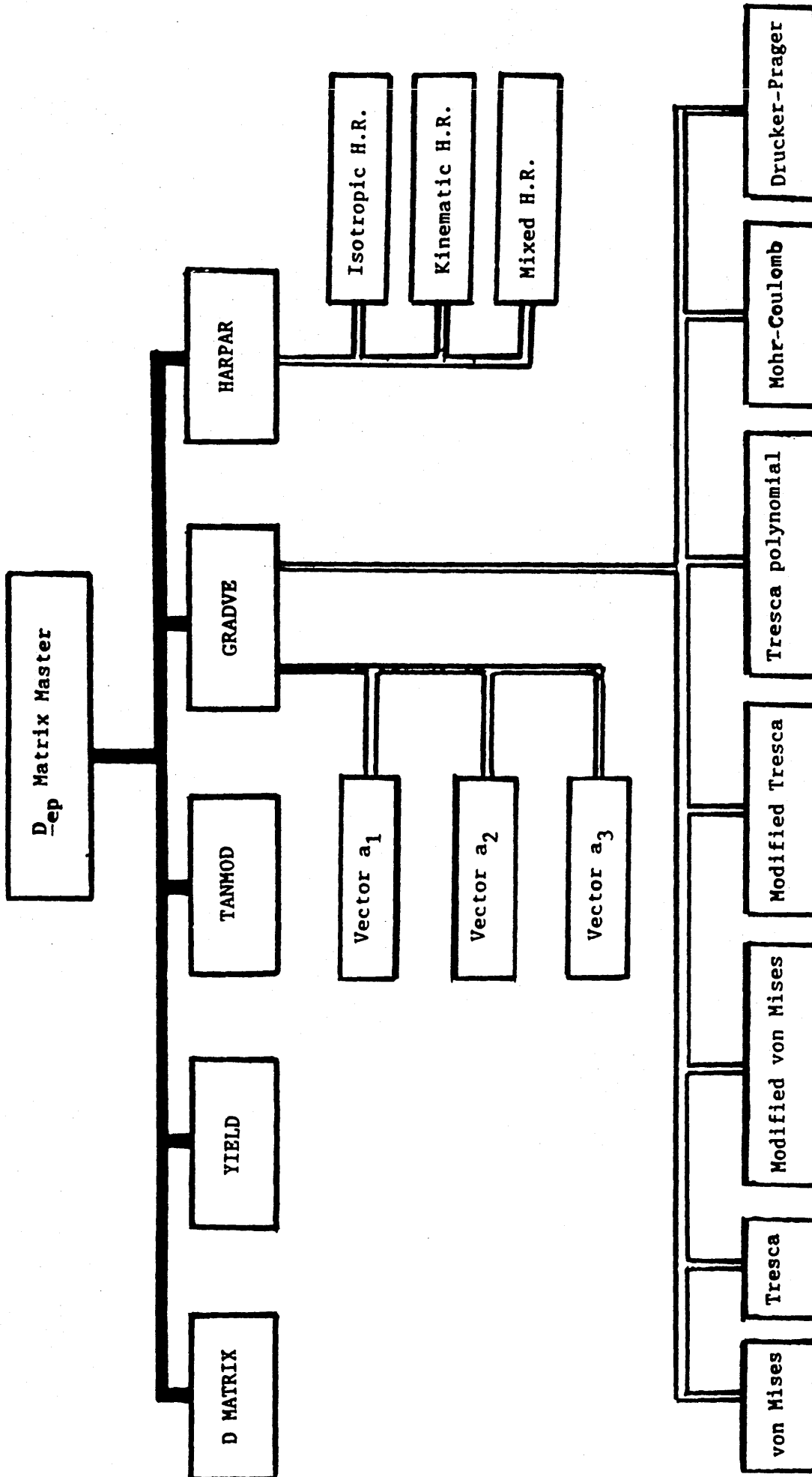
FBNSA

FE/BE SYSTEM FOR NONLINEAR
STRESS ANALYSIS

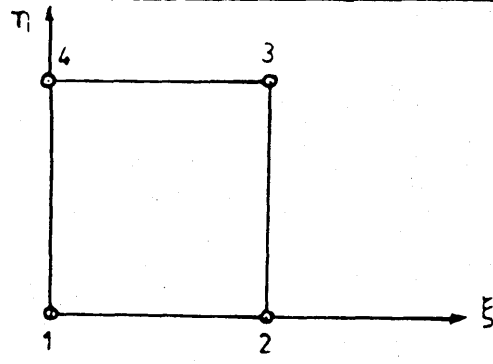
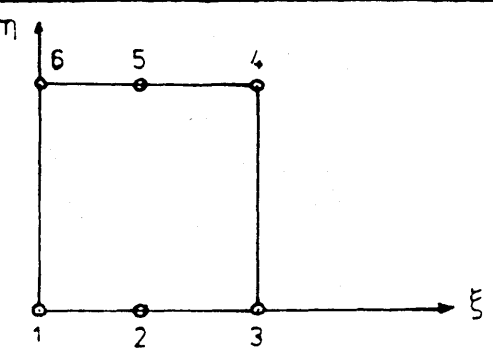
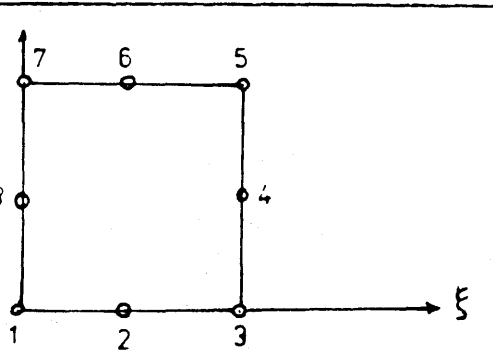
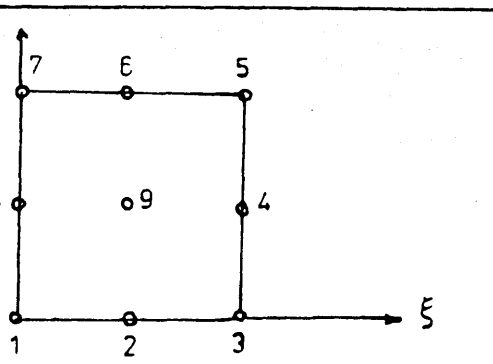
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fig(6.3) Thick Cylinder Problem Discretized by means of
Hybrid FE/BE Facility

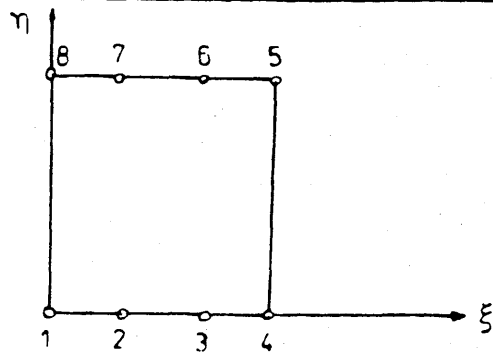
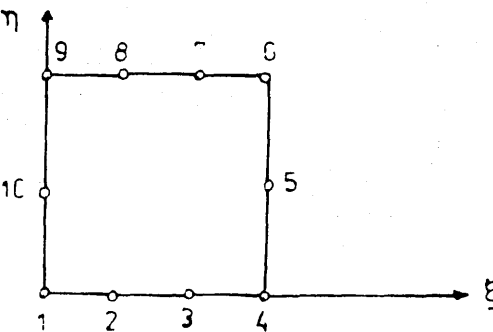
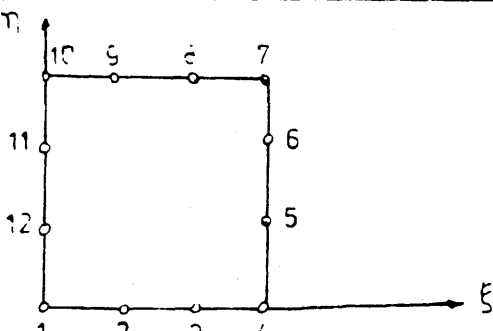


Fig(6.4) Main Subroutines for D_{ep} Matrix Calculation

No.	Element	Name
1		4-Node Quadrilateral
2		6-Node Serendipity
3		8-Node Serendipity
4		9-Node Lagrangian

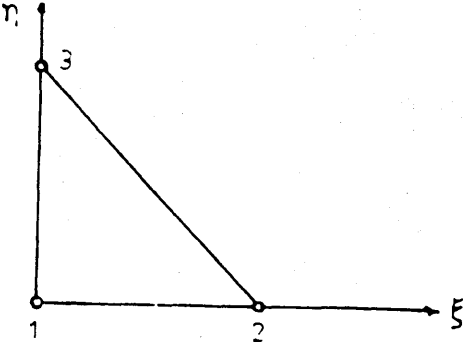
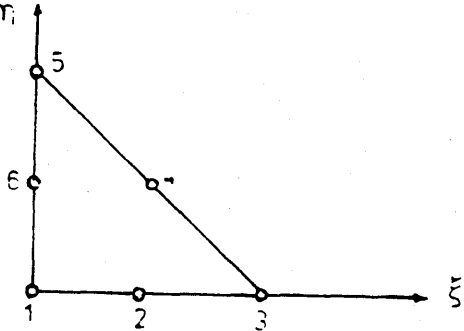
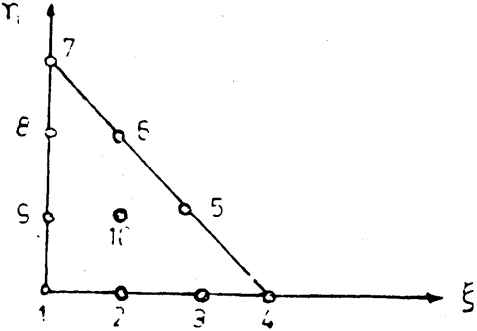
Quadrilateral Family(Part I)

Table(6.1) Domain Element Library

No.	Element	Name
5		8-Node Serendipity
6		10-Node Serendipity
7		12-Node Serendipity

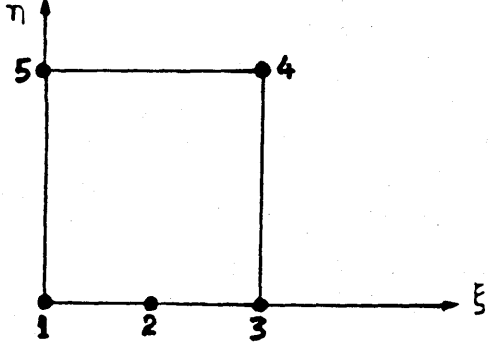
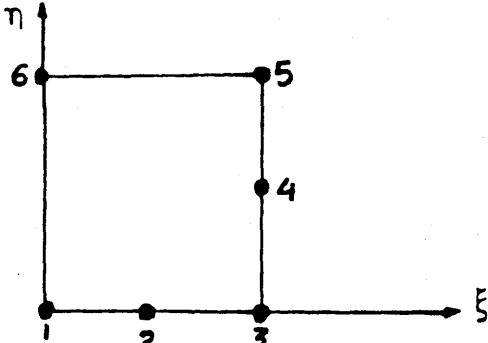
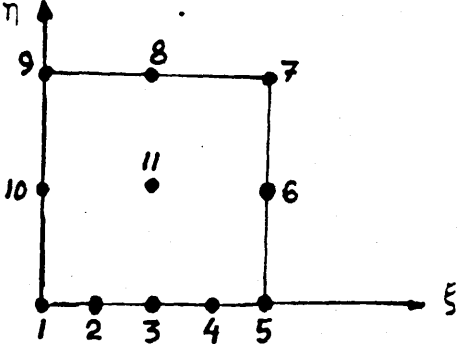
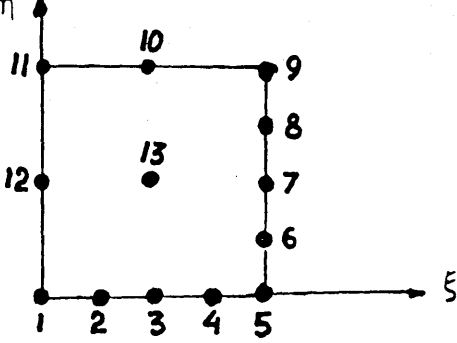
Quadrilateral Family(Part II)

Table(6.1) Domain Element Library

No.	Element	Name
1		3-Node Linear
2		6-Node Quadrilateral
3		10-Node Cubic

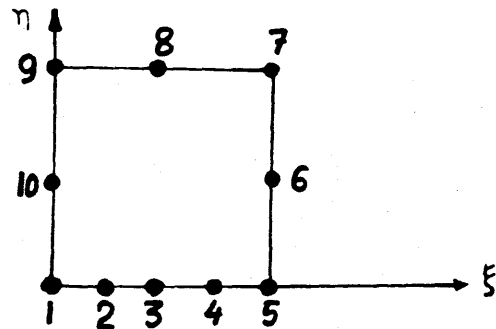
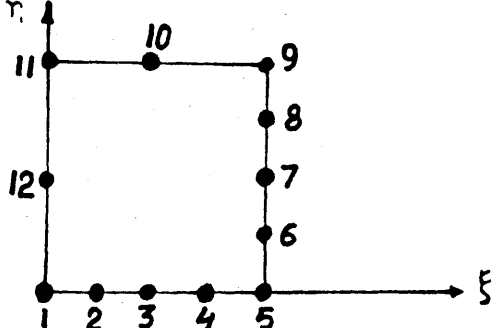
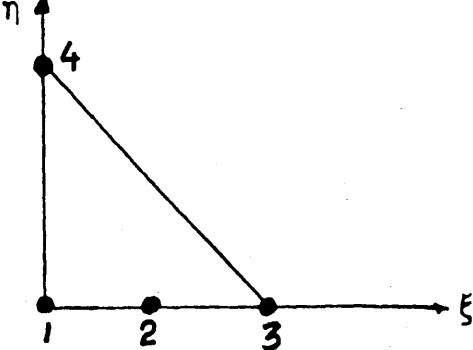
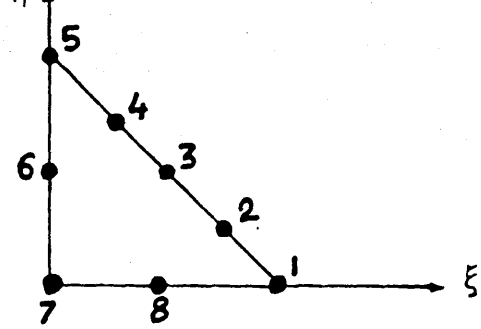
Triangular Family

Table(6.1) Domain Element Library

No.	Element	Name
1		One directional 1 st order Quadrilateral Transition element
2		Two directional 1 st order Quadrilateral Transition element
3		One directional 2 nd order Lagrangian Transition element
4		Two directional 2 nd order Lagrangian Transition element

Transition Family(Part I)

Table(6.1) Domain Element Library

No.	Element	Name
5		One directional 2 nd order Serendipity Transition element
6		Two directional 2 nd order Serendipity Transition element
7		One directional 1 st order Triangular Transition element
8		One directional 2 nd order Traingular Transition element

Transition Family(Part II)

Table(6.1) Domain Element Library

CHAPTER SEVEN

RESULTS AND DISCUSSION

7.0 Introduction

It is clear from the previous chapters that there is a range of facilities available in the package "FBNSA", which are capable of 2-dimensional and axisymmetric elasto-plastic stress analysis using a wide range of yield criteria, hardening rules, iterative schemes and different accuracy measures. The validation of these facilities is essential in order to prove the soundness and reliability of the programming package. A comparison between the results obtained from the FEM and BEM is useful for concluding which of these methods is most economical and accurate. It is obviously also beneficial to investigate the effect of the different available accuracy measures on the elasto-plastic solutions obtained. In this chapter standard 2-dimensional and axisymmetric benchmark cases have been used to test the validation of the package. A number of case studies have been selected for which comparisons have been made between FEM and BEM results, and with exact analytical solutions or published results. The problems include a pressurized thick cylinder solved both as a 2-dimensional and an axisymmetric case, a notched tensile specimen, a strip footing problem, and toroidal shape problem. The problems have been chosen so as to involve all of the available facilities, such as yield criteria and hardening rules, and to employ both the FEM and the BEM solution procedures.

7.1 Two-dimensional Problems

7.1.1 Benchmark Test

The first numerical example is a benchmark test where the FEM and BEM results are compared with a solution from the NAFEMS literature [138]. This example was thought to be a good method for checking the accuracy of the output results, and illustrating the ability of the software to converge to an accurate solution, according to the standard specified measures. In this example a

thick cylinder is subjected to a gradually increasing internal pressure and is analysed as a 2-dimensional plane strain case. The von Mises yield criterion is assumed together with the isotropic hardening rule. The main dimensions of the problem and the material properties are as follows:

a=inner radius = 0.1 m,
 b=outer radius = 0.2 m,
 E=Elastic modulus=21000 dN/mm² (210x10² MPa),
 Y=Yield stress =24 dN/mm² (24 MPa), and
 ν =Poisson's ratio=0.3

The FEM and BEM analyses were carried out for two types of material, firstly, a perfectly-plastic material subjected to internal pressures of (100,140,166, and 192 MPa), and secondly a linear-strain-hardening material (H=52500 MPa) subjected to internal pressures of (100,140,240, and 340 MPa). The integration over the domain of the problem for both the FEM and the BEM procedures was carried out using 2-point Gauss rule. Fig(7.1) shows the FEM mesh using 8-noded serendipity elements, whilst Fig(7.2) shows the BEM mesh using 2-noded isoparametric boundary elements. The internal cell discretization is shown in Fig(7.3) and employs 8-noded serendipity finite elements.

The displacement at the inner radius were evaluated with the aid of our FEM and BEM programs, at the pressure values suggested in the NAFEMS report[138], and the results are plotted together with those published in Ref[138], in Fig(7.4) for H=0, and Fig(7.5) for H=52500 MPa.

The FEM results show a close agreement with the results provided by NAFEMS. However, it is clear from Figs(7.4) and (7.5) that the BEM results were consistent with an over stiff model for H=0 and with an under stiff model for H=52500 MPa. It is also clear that the BEM results begin to diverge from the FEM solution at high plastic deformation.

7.1.2 Pressurized Thick-walled Cylinder

The thick-walled cylinder as used in the previous benchmark test has been employed for a thorough investigation of the following aspects:-

- A comparison between the FEM and the BEM.
- A comparison between the trigonometric Tresca and polynomial Tresca yield criteria.
- A test of different hardening rules.
- The use of different FEM solvers and iterative schemes.
- The effect of the finite-difference stress scheme, and its parameters, on the BEM results.
- The effect on the BEM results of the size of integration cell and the order of Gaussian quadrature used.

Two additional programs have been developed for the analysis of pressurized, thick-walled, cylinders with $H=0$. The first of these new programs uses the von Mises yield criterion, and the second uses the Tresca criterion, as described in appendix 7.1. The FEM and BEM results have been compared with the analytical results obtained from the analytical-solution programs, as described above. The pressure load for this analysis was applied in 8 increments, 100,120,140,150,160,170,180,190 MPa, respectively.

The FEM mesh, using 8-node serendipity elements is shown in Fig(7.6). While the BEM mesh was similar to that used in the previous example and as shown in Fig(7.2). The BEM solution was carried out using two different internal cell discretizations, mesh-1 as shown in Fig(7.3) and mesh-2 as shown in Fig(7.7).

The variation of the non-dimensional radial displacement ($u_r \mu / 4Yb$) at the outer surface, with the non-dimensional pressure (P/Y), is shown in Fig(7.8), and for the inner surface is shown in Fig(7.9). The results shown were obtained by using different FEM and BEM meshes as indicated, for a typical case employing the von Mises yield criterion and with $H=0$. It is clear from Fig(7.8) that the FEM results are very close to the

corresponding published results [60], whilst the BEM results show some divergence when the plastic deformation increases. It is clear from Figs(7.9) and (7.10) that different internal cell discretizations may lead to different BEM results. In order to validate the stress distributions obtained for the previous case from the package, the FEM and BEM results have been plotted versus the analytical solution, on a radial line from the inner to the outer surface, for a value of $P/Y=0.5833$, i.e at plastic front $c/a=1.4$, as shown in Figs(7.10) and (7.11). From these figures it can be seen that the BEM results are quite acceptable. As might be expected, at higher ratios of c/a the stresses obtained from the BEM begin to diverge in the same manner as did the displacements, showing again that, for high plastic deformations, the accumulation of errors in the BEM procedure can drastically reduce its accuracy.

Different finite element iterative schemes were tested and were all shown to converge to the same solution, for the same FEM model. The only difference observed between these different FEM schemes was in the CPU time needed by each scheme, and from Fig(7.12) it is clear that although the direct iterative scheme is the simplest, it requires the largest CPU time particularly for high plastic deformations. The difference between the CPU times needed for the Newton-Raphson and Modified Newton-Raphson schemes was not considerable. However, the author believes that when the number of FEM equations increases the direct iterative scheme will become more economical, since the elimination of the FEM equations is required only once with that scheme. This may justify the inclusion of different schemes in the programing package, so as to allow the user to employ the optimum scheme for his analysis.

Further analysis of the pressurized cylinder, has been carried out, still utilising the von Mises yield criterion, but with different hardening schemes. The non-dimensional inner radial displacement ($4\mu_a/Ya$) is shown plotted against the non-dimensional pressure (P/Y) in Fig(7.13), and the radial distribution of the hoop stress (for $P/Y=0.5833$) is shown in Fig(7.14). These figures indicate that for a monotonically increasing load, the different hardening rules, such as isotropic, kinematic and mixed rules give

similar results. From an economical point of view only, the isotropic hardening rule is recommended for cases with monotonically increasing loads.

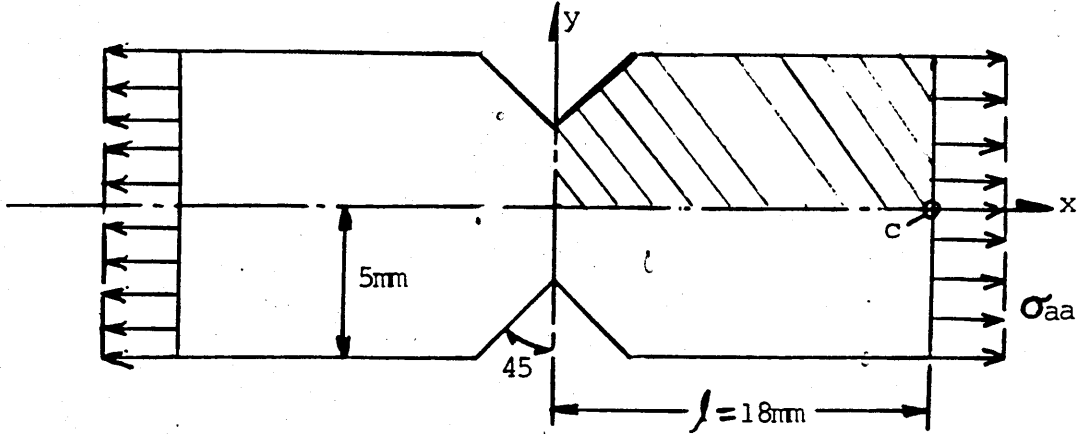
The use of the FDM for the calculation of the stress and strain values at points with certain distance from the boundary of the domain, has improved the BEM results, as shown in Fig(7.15). It is clear from Fig(7.16) that the use of the FDM instead of the BIE's will not increase the CPU time required for the solution of the problem, and does not appear to generate any additional error in the results. This interesting conclusion has justified the use of the FDM scheme alone throughout the domain, for the determination of the axisymmetric BEM stress distribution, thus avoiding the use of the very complex expressions encountered with axisymmetric strain BIE's.

The Tresca yield criterion is usually recommended for metals with low ductility and is, generally, more conservative as a design criterion than the von Mises. A new implementation of the Tresca criterion, based upon polynomial representation has been developed in this work. A comparison of this form of the Tresca criterion with that based upon trigonometric functions has also been incorporated into thick walled cylinder case study. The results for displacement, hoop stress and radial stress obtained from these two versions of the Tresca criterion are illustrated in Figs(7.17-7.19) respectively. It is clear from those figures that the developed implementation is as accurate as the old one, but as might be expected, it shows saving in CPU time required for the solution of each load increments (see Table 7.1).

7.1.3 Notched Tensile Specimen

In this example a notched specimen subjected to a tensile stress (σ_a) is considered as both a plane stress and a plane strain problem. This case is of interest in case some results for it have been published by Nayak and Zienkiewicz[60], and Telles[120]. Elasto-plastic analysis for this problem is based upon the von Mises yield criterion and the isotropic hardening rule. The main

dimensions of the problem are as shown below:-



with the following material properties:-

$$E = 7000 \text{ Kgf/mm}^2 \quad (6.86 \times 10^4 \text{ MPa})$$

$$Y = 24.3 \text{ Kgf/mm}^2 \quad (238.31 \text{ MPa})$$

$$\nu = 0.2, \text{ and } H = 0$$

The FEM and BEM analysis was based upon a quarter of the specimen as a result of the symmetry in geometry and applied load. The FEM mesh is shown in Fig(7.20), using 8-noded serendipity elements.

i) Plane stress case

Fig(7.21) shows the BEM mesh for this case using 2-noded isoparametric elements. For this case it is possible that before the specimen collapses it will be partly plastically deformed and partly elastically deformed. Hence, the internal cell discretization can be considered for the plastic zone only. Two different internal cell meshes, defined as a coarse and a fine mesh, are considered, as shown in Fig(7.22) and Fig(7.23) respectively using, 6-noded triangular finite elements.

The deflection of point c, which has the x,y coordinates (18,0), with applied stress σ_a is plotted in Fig(7.24). It is clear that the FEM results obtained here are the same as the FEM results published by Reference[60]. The BEM results obtained by employing the 3-point Gauss rule and a FDM step of 2mm(i.e $\Delta x = \Delta y = 2$) using the coarse internal cell mesh, are stiffer than the FEM results, whilst the BEM results with a FDM step of 2.5mm are closer

to the FEM results at high plastic deformation. Also the BEM results obtained with the aid of the fine mesh are stiffer than the FEM results and become softer at high plastic deformation. Although the maximum permissible error can be selected as small as 10^{-7} , or even less, for the FEM iterations, it was observed that such a small permissible error can cause a state of vibratory divergence in the BEM iterations, and that the permissible error level should be increased to 10^{-2} : 10^{-3} so as to restore the convergence.

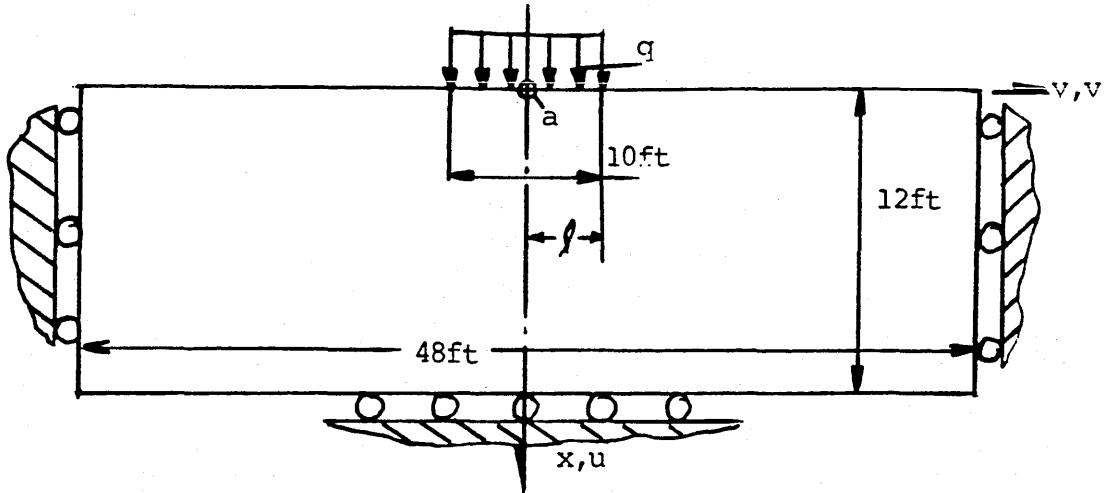
ii) Plane strain case

For this case plastic deformation prevails over the whole domain before the collapse of the specimen, and it is essential to use an internal cell mesh for the whole domain, as shown in Fig(7.25). The BEM mesh was formed in terms of 2-noded isoparametric elements as illustrated by Fig(7.26). The non-dimensional displacement at point c is plotted versus $2\sigma_a/Y$ in Fig(7.27), and shows that the present BEM results agree completely with those published by Telles[120], whilst the present FEM results can be seen to be closer to the expected analytical solution than those of Ref[60]. The maximum value of non-dimensional load ($2\sigma_a/y$) obtained analytically by Chen [139] was between (1.52:1.73). The present BEM results have a limiting load of 1.63 whilst the FEM results have a value less than 1.78.

7.1.4 Case Study Involving the Mohr-Coloumb

and Drucker-Prager Yield Criteria

The main objective of this particular case study was to check the validation of the Mohr-Coloumb and Drucker-Prager yield criteria. The application was of a flexible strip foundation, under a uniform pressure loading, over an infinite soil domain. The main dimensions and boundary conditions can be shown as follows:-



The material properties assumed for soil are

$$\begin{aligned}
 E &= 30\,000 \text{ psi (206.85 MPa),} \\
 C_f &= 10 \text{ psi (68.95} \times 10^3 \text{ Pa),} \\
 \nu &= 0.3, \quad \theta_f = 20^\circ, \text{ and } H = 0.0
 \end{aligned}$$

The material is assumed to be represented by an associated plasticity and isotropic hardening rule. Fig(7.28) shows the FEM mesh for one symmetric half of the problem, using 6-noded triangular elements, whilst Fig(7.29) shows the BEM mesh using 3-noded isoparametric elements, and the corresponding internal cells, required for the BEM solution, are illustrated in Fig(7.30). A complete version of the problem has also been considered for the BEM solution, where the BEM mesh and the integration cell mesh are as shown in Figs(7.31) and (7.32), respectively. The FEM results for the Mohr-Coloumb criterion are more stiff than the Drucker-Prager results, as shown in Fig(7.33). The BEM results show reasonable agreement with the FEM results at low plasticity, whilst for high plasticity they diverge until it becomes difficult to achieve any results near to the FEM solution, as shown in Fig(7.34).

7.2 Axisymmetric Problems

Due to the modular design of the package, all of the

subroutines used for elasto-plastic analysis, with different yield criteria and hardening rules, have already been tested in the previous 2-dimensional case studies.

The remaining task now is to check the axisymmetric FEM and BEM derivations, where the simplest possible elasto-plastic analysis can be used. In the following axisymmetric examples, only the von Mises yield criterion, together with the isotropic hardening rule, are employed.

7.2.1 Axisymmetric Benchmark Test

In this example the FEM and BEM results have been compared with the NAFEMS standard test results [138], which are for a case similar to the previous 2-dimensional benchmark test case, i.e. a thick walled cylinder subjected to internal pressure and solved as an axisymmetric body. The main dimensions and material properties are similar to those used for the previous 2-dimensional benchmark test. Fig(7.35) shows the FEM mesh, using two 8-noded serendipity elements, whilst Fig(7.36) shows the BEM mesh using 2-noded isoparametric elements and Fig(7.37) shows the internal cell discretization. The integration over the domain is achieved using the 2-point Gauss rule.

The inner radial displacement/load characteristic curve is shown in Fig(7.38) for the case of a perfectly plastic material. Also, Table(7.2) shows the stress values at the inner Gaussian point for the present FEM solution compared with the NAFEMS results. Fig(7.39) shows the inner radial displacement/load characteristic for a material with $H=52500$ MPa, and Table(7.3) shows the stress values at the inner Gaussian point for the FEM solution.

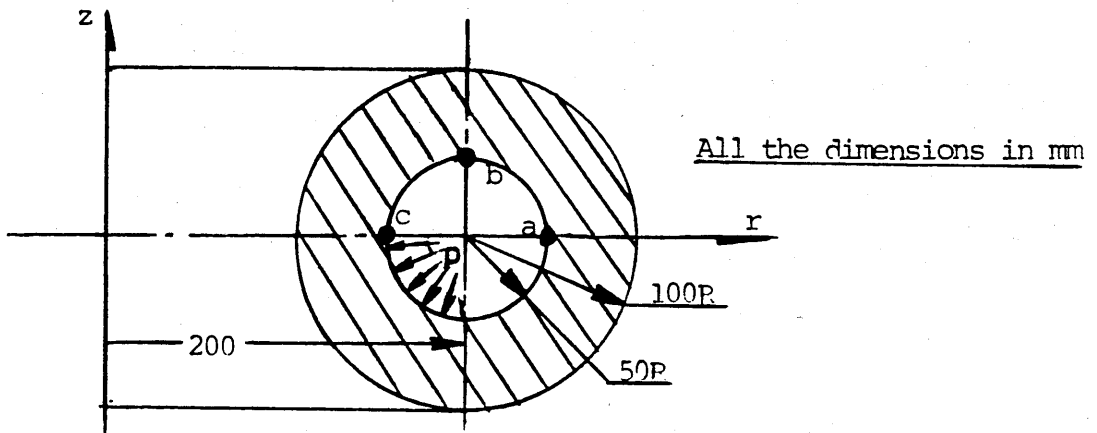
It is clear from Figs(7.38) and (7.39), and Tables(7.2) and (7.3), that the FEM results are very close to the published NAFEMS results for both $H=0$ and $H=52500$ MPa. The BEM results are also very accurate for the case where $H=0$. However, for $H=52500$ MPa, the BEM results begin to diverge at the last load increment.

The same case study was tested with more load increments and using a finer FEM mesh, as shown in Fig(7.37), and the

non-dimensional pressure-displacement plots are as shown in Fig(7.40). These results are similar to those observed for the case of 2-dimensional BEM analysis, in that the BEM results begin to diverge from the FEM results for at high plastic deformations.

7.2.2 Toroidal Shape Case Study

The problem considered for this case study is one of a toroidal shape under an internal pressure. The main dimensions of the problem are shown in the figure below:-



The material is assumed perfectly-plastic with the following properties:-

$$E=21000 \text{ dN/mm}^2 \text{ (} 210 \times 10^2 \text{ MPa),}$$

$$Y=4 \text{ dN/mm}^2 \text{ (} 24 \text{ MPa), and}$$

$$\nu=0.3, \text{ and } H=0$$

The problem is symmetric about the r -axis, therefore only one-half of the problem is considered. The FEM mesh is as shown in Fig(7.41) using 8-noded serendipity elements. Due to the effect of corners on the BEM results, a complete torus has been considered for the BEM solutions in order to improve their accuracy. Figs(7.42) and (7.43) show the BEM mesh and corresponding internal cell discretization for the whole domain. The internal pressure was applied gradually with increments $(8,10,12 \dots \text{etc}) \text{ dN/mm}^2$ until a full plastic state was reached. Fig(7.44) shows the radial displacement

at the inner surface for $P/Y=0.625$ and Fig(7.45) shows the radial displacement at point "a" for different values of P/Y , whilst Figs(7.46) and (7.47) show the radial displacements at point "b" and "c" respectively.

It is clear from these figures that the FEM results and BEM results are very close within the elastic limit and in the low elasto-plastic range. Again the results obtained from the BEM begin to diverge from those obtained from the FEM, at high plastic deformations.

7.3 General Discussion

From the previous examples some general comments can be made:-

7.3.1 Parameters for Elasto-plastic Analysis

The choice of yield criterion and an appropriate hardening rule for elasto-plastic analysis using the FBNSA package is left to the user. It is clear that the polynomial implementation of the Tresca yield criterion, developed in this work, is slightly more economical, in terms of computer CPU time, than the Tresca's trigonometric implementation. For cases with monotonic increasing loading, all of the hardening rules employed in this work will produce approximately the same answer. The present programming package does not have a facility for dealing with problems of unloading and re-loading, for which different hardening rules may produce different answers.

The stress-correction algorithm may converge quickly, if the number of strain increments within one loading step is increased. If the maximum permissible error is made too small, a vibratory divergence may occur, increasing the number of iterations and CPU time unnecessarily.

7.3.2 FEM Parameters

High order finite elements have, of course, proved to be more accurate than low order ones. The accuracy of the solution, and the rate of iteration convergence may be improved if the finite element mesh is made very fine within the expected plastic zone. For small-size meshes, it would appear that the direct-iterative scheme is less efficient than the Newton-Raphson, but that the later scheme may require enormous CPU time for a large mesh, because of the updating of the structure stiffness matrix required with every single Newton-Raphson iteration.

The large loading increment used with the benchmark tests proved that very accurate answers can be obtained with the aid of the present FEM package, irrespective of the magnitude of the load increment, as long as the total load is beyond the ultimate strength of the structure. The performance of the iterative schemes could be improved to give a lower probability of divergence and a smaller number of iterations to convergence, if the magnitude of the load increments is decreased particularly for large plastic deformations.

7.3.3 BEM Parameters

It is quite clear that users of the BEM still require considerable experience if they are to handle some of its critical parameters. Although stress can be evaluated very accurately at points on the boundary or far from it, it was not possible in the present package to calculate the stress, with the same level of accuracy, at points very near to the boundary. Hence, coarse integration-cell meshes may give more reasonable answers than fine meshes, but will of course, reduce the overall level of accuracy. As a results of this, BEM convergence is very sensitive to many parameters, such as, the magnitude of load increment, the maximum permissible error, etc.

The finite-difference scheme employed for the evaluation of stresses at internal points is as accurate as the B.I.E method, and even more accurate for points within a specified zone from the

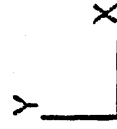
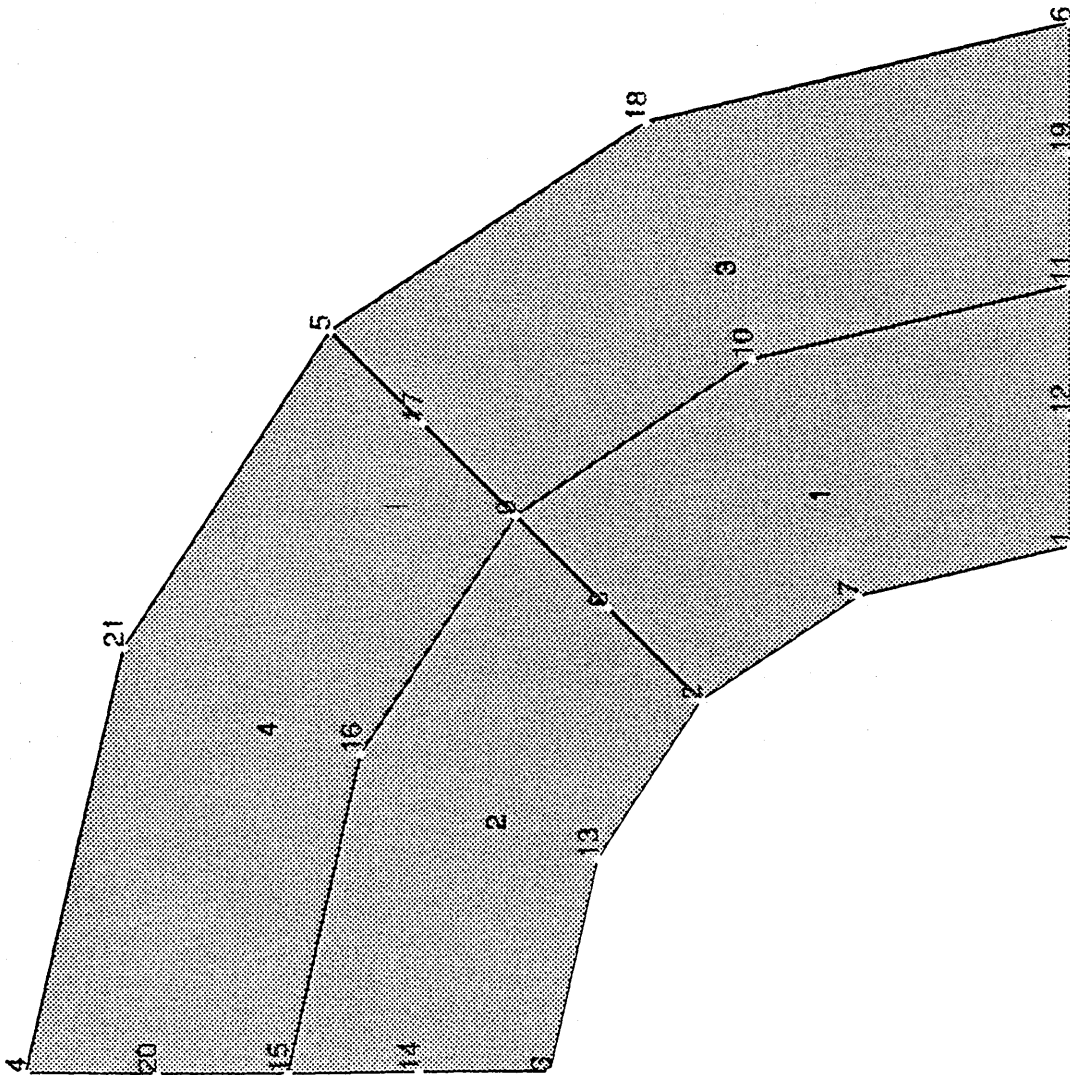
boundary. The results for 2-dimensional BEM analysis can be improved if the FDM zone is selected carefully, so as to use the B.I.E method at points well inside the domain. For axisymmetric BEM analysis, only the FDM was programmed for simplicity.

Different accuracy measures, such as singular integration, corner models, and different types of integration cell, have helped to provide a very accurate elastic solution but it has been observed that within the plastic range, the errors are cumulative and cause the BEM solution to diverge from an accurate answer, at large plastic deformation. Integration cells are only required within the plastic zone, and the CPU time for the BEM solution can be reduced by limiting the integration-cell discretization to the expected area of the plastic zone.

7.3.4 FEM versus BEM

It would appear from the previous examples that the FEM is still more efficient than the BEM for elasto-plastic analysis. One particular problem associated with the BEM is that it produces solutions which diverge from the correct values at high plastic deformations. For low plasticity, the CPU time required for a BEM solution may be smaller than that required for an FEM solution, but this is not always true, particularly when the number of equations increases, since the frontal solver used with the FEM will save much CPU time, but this facility does not apply for the BEM.

For problems with localised small plastic zones, the BEM may be more efficient than the FEM, whenever integration cells can be limited to such zones, provided that they can be predicted.

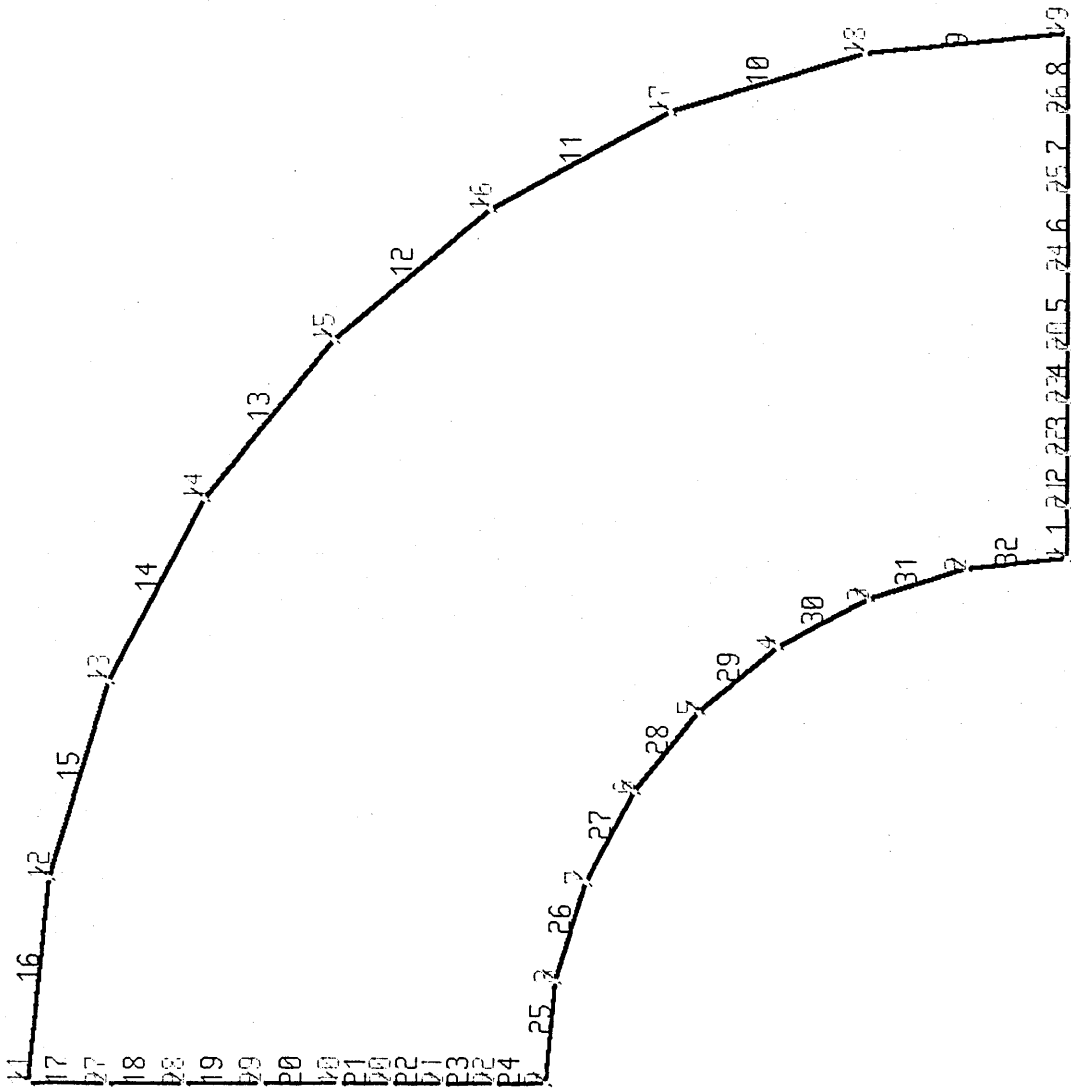
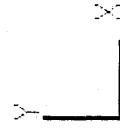


Fig(7.1) Finite Element Mesh for Thick Pressurized Cylinder
used in Plane Strain Analysis (Benchmark Test)

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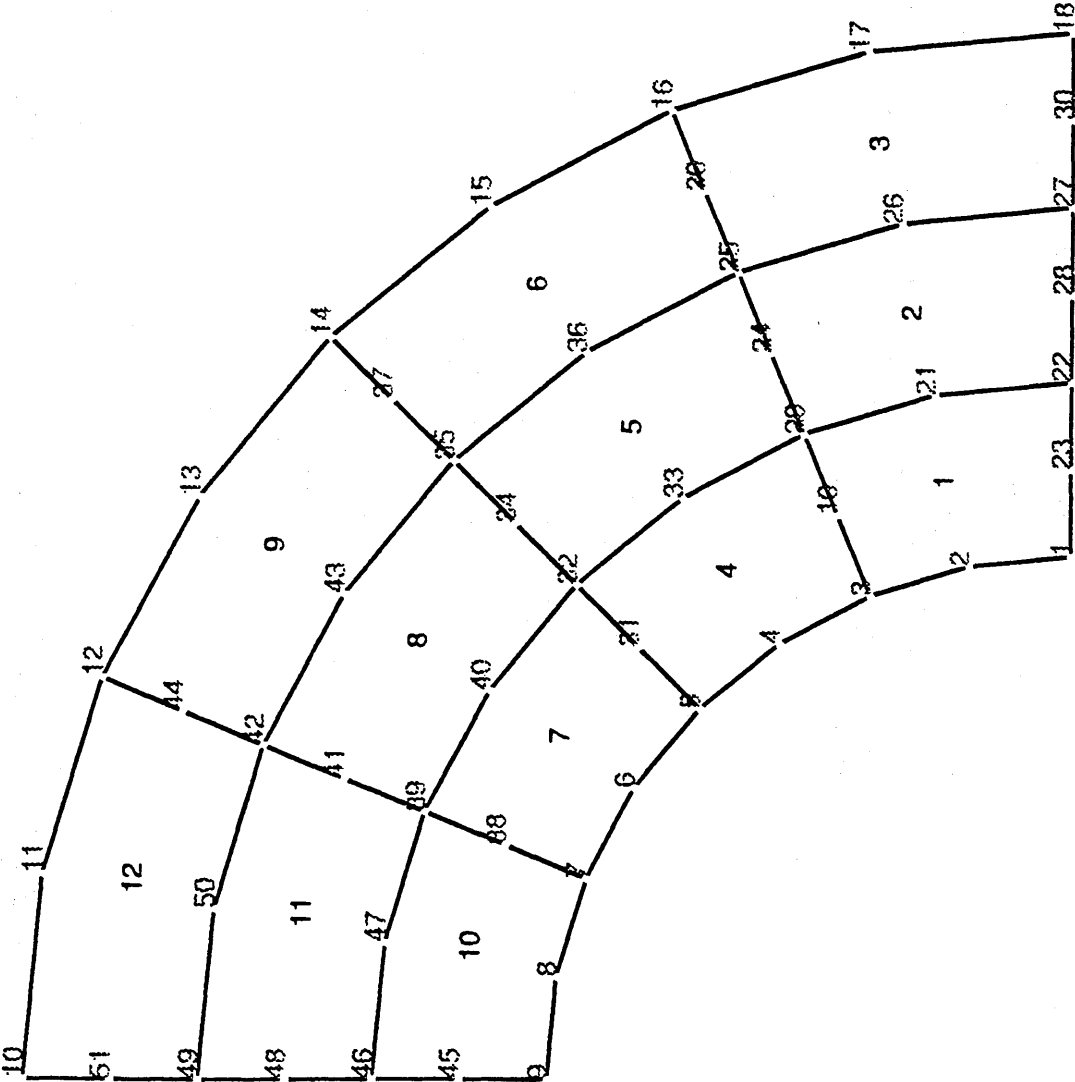
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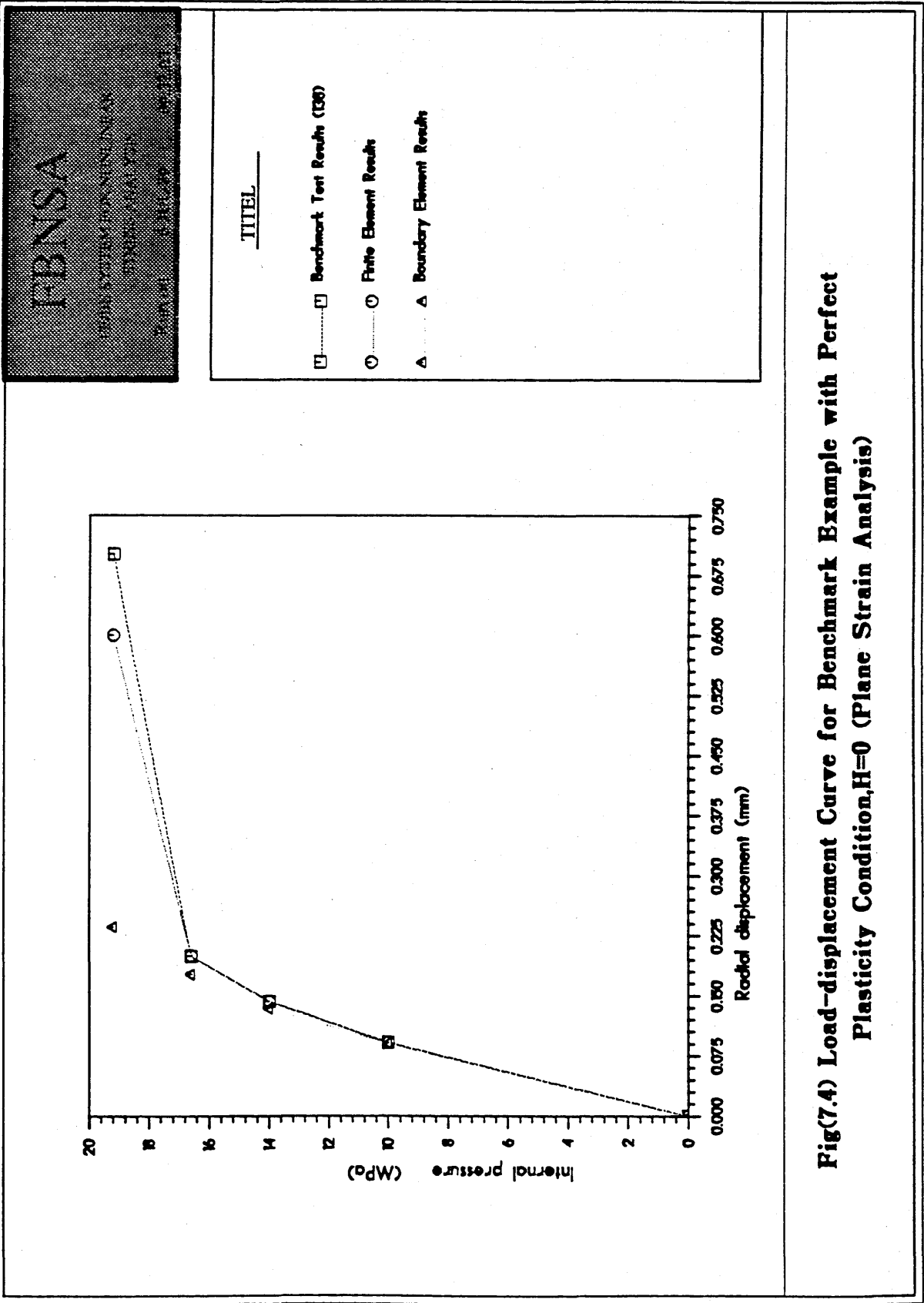


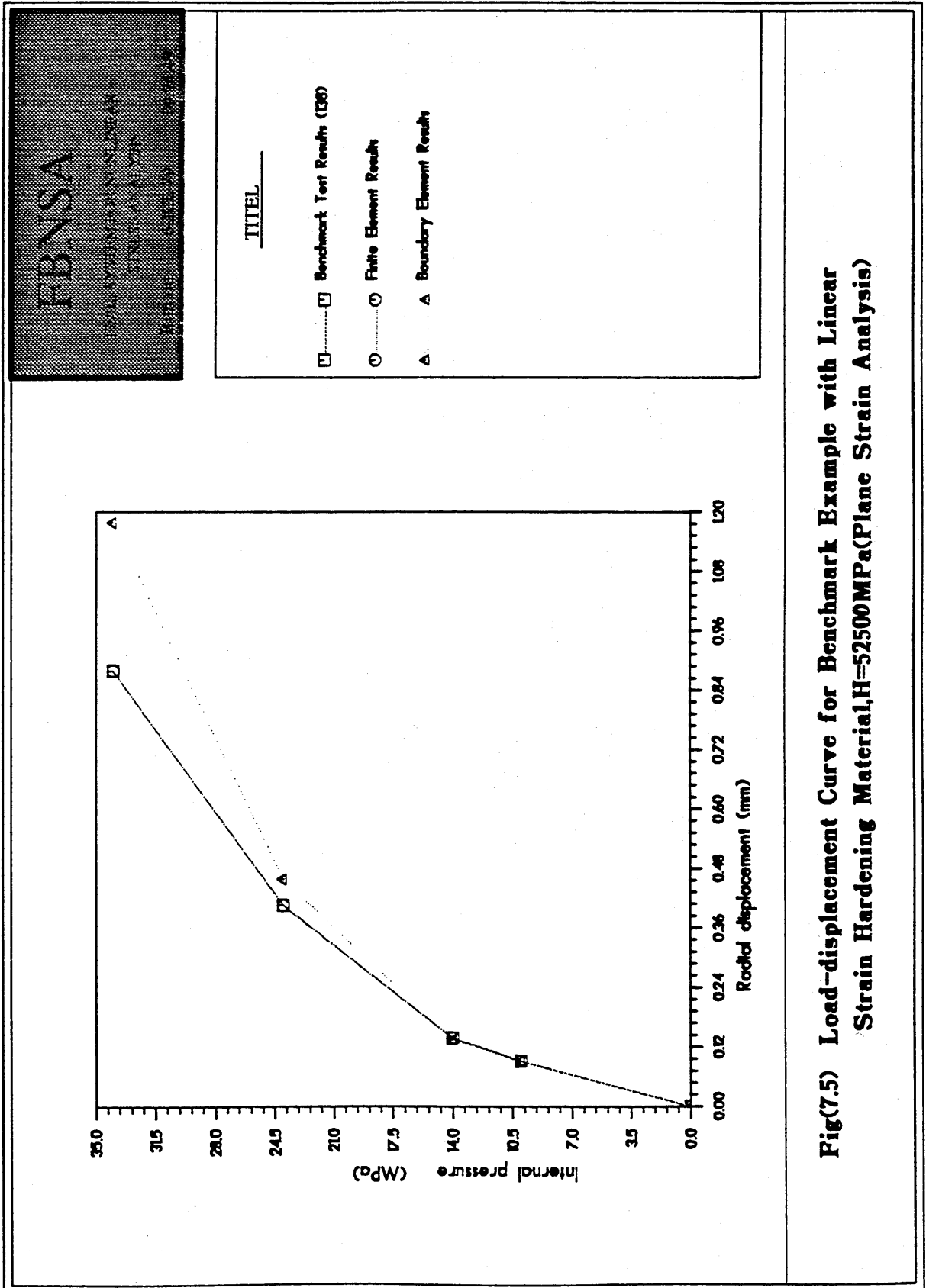
Fig(7.2) BEM Discretization for Thick Cylinder using
2-noded Isoparametric Elements

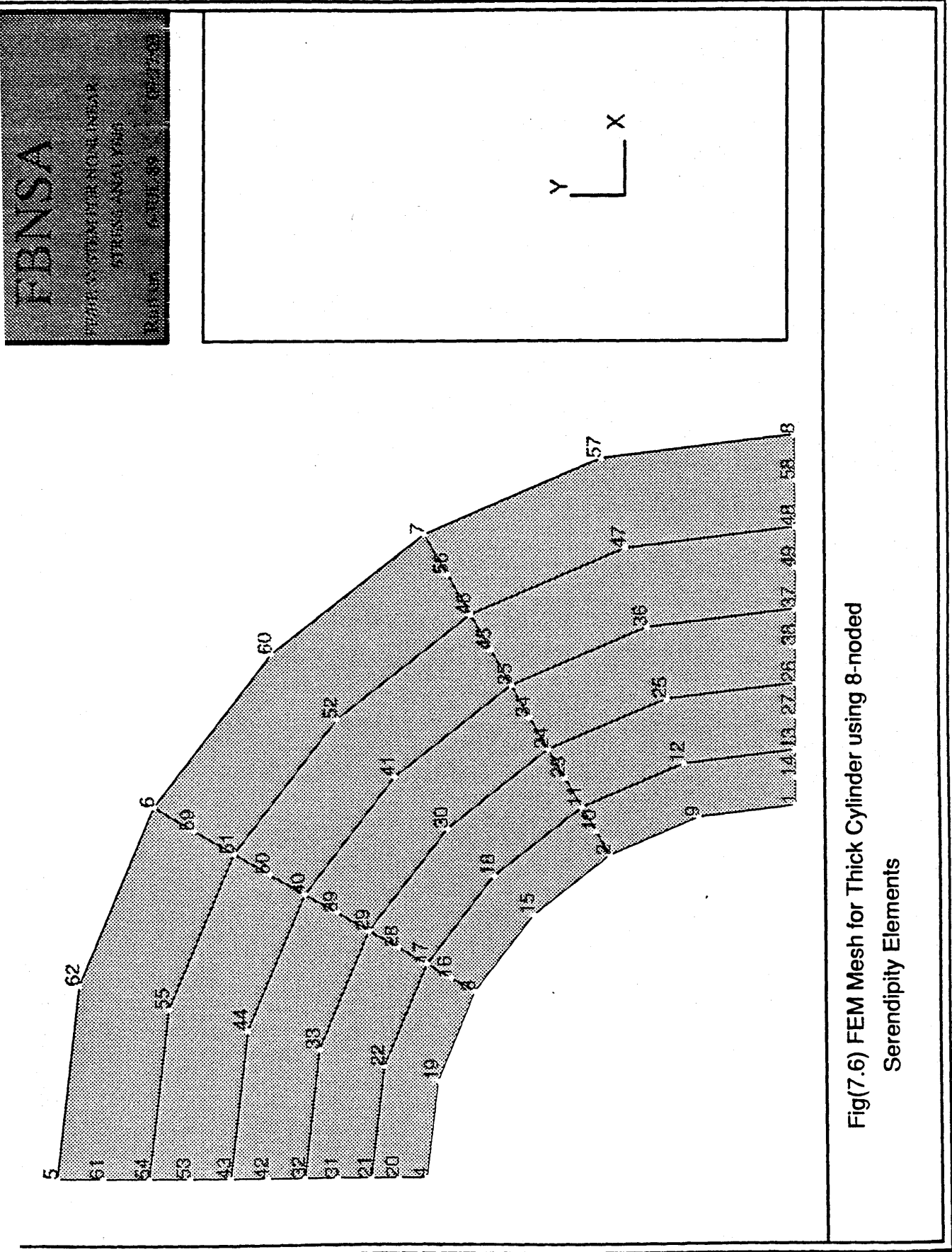
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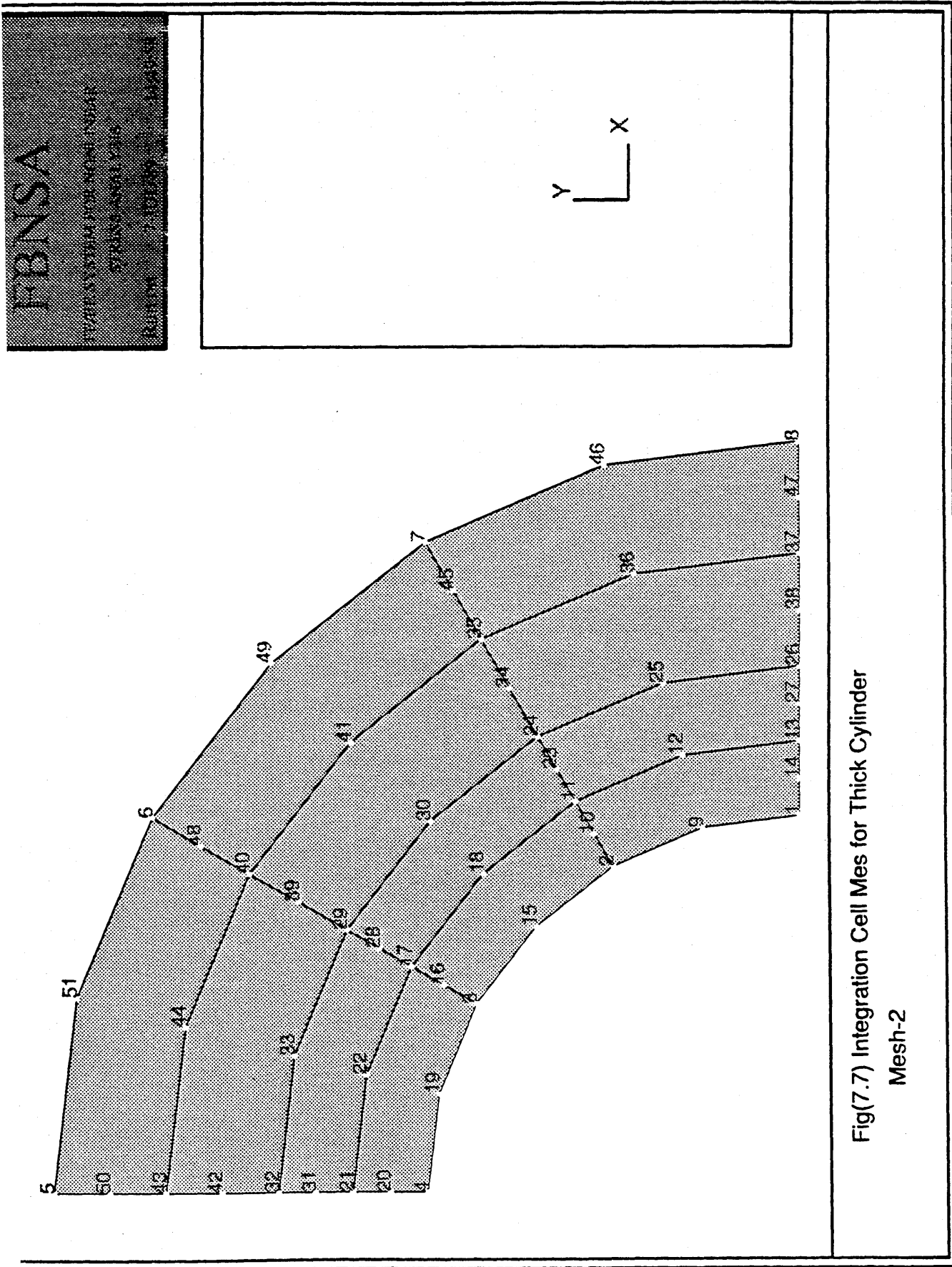


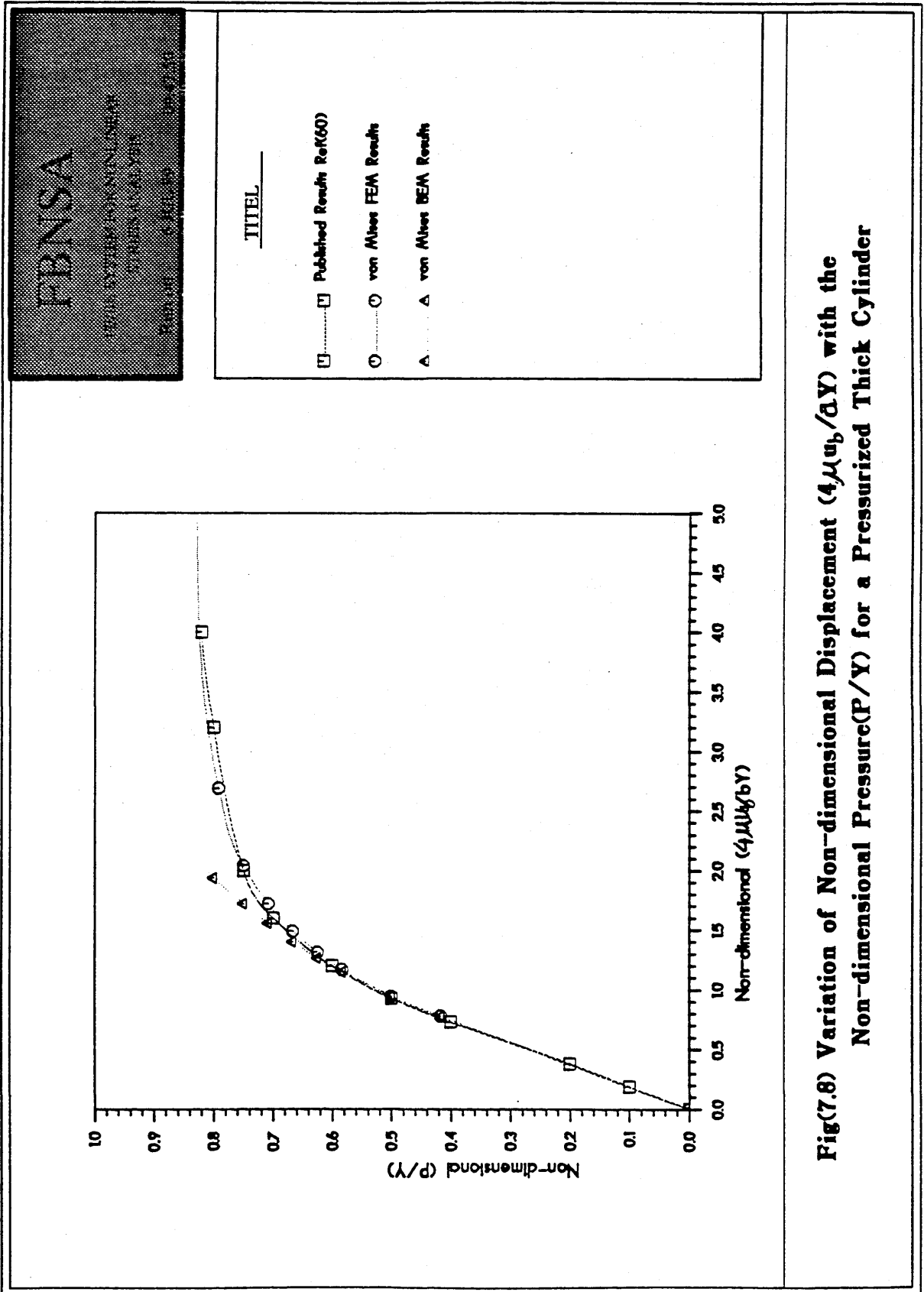
Fig(7.3) Integration Cell Discretization for Thick Cylinder Problem
using 8-noded Serendipity Finite Elements, Mesh-1

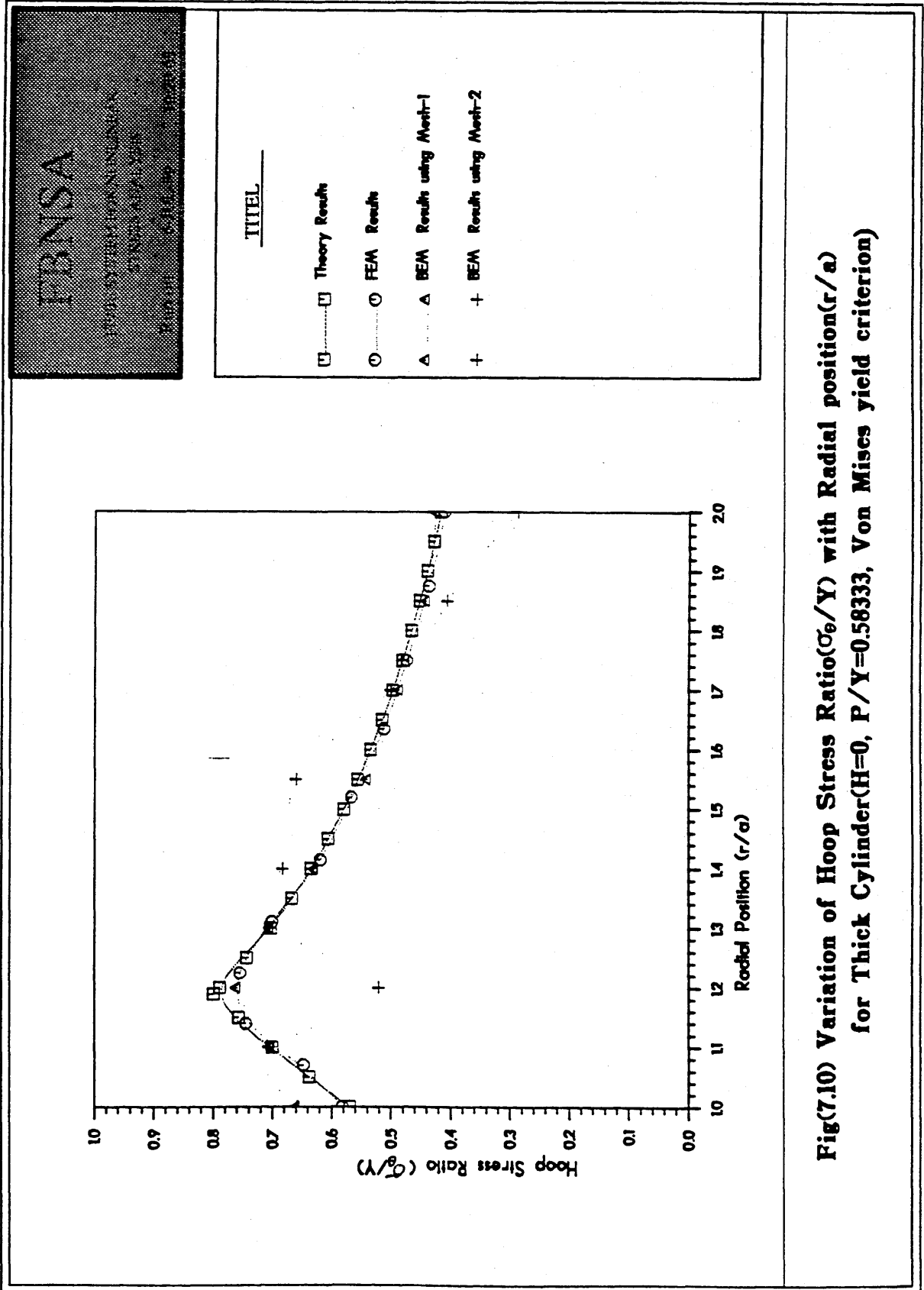


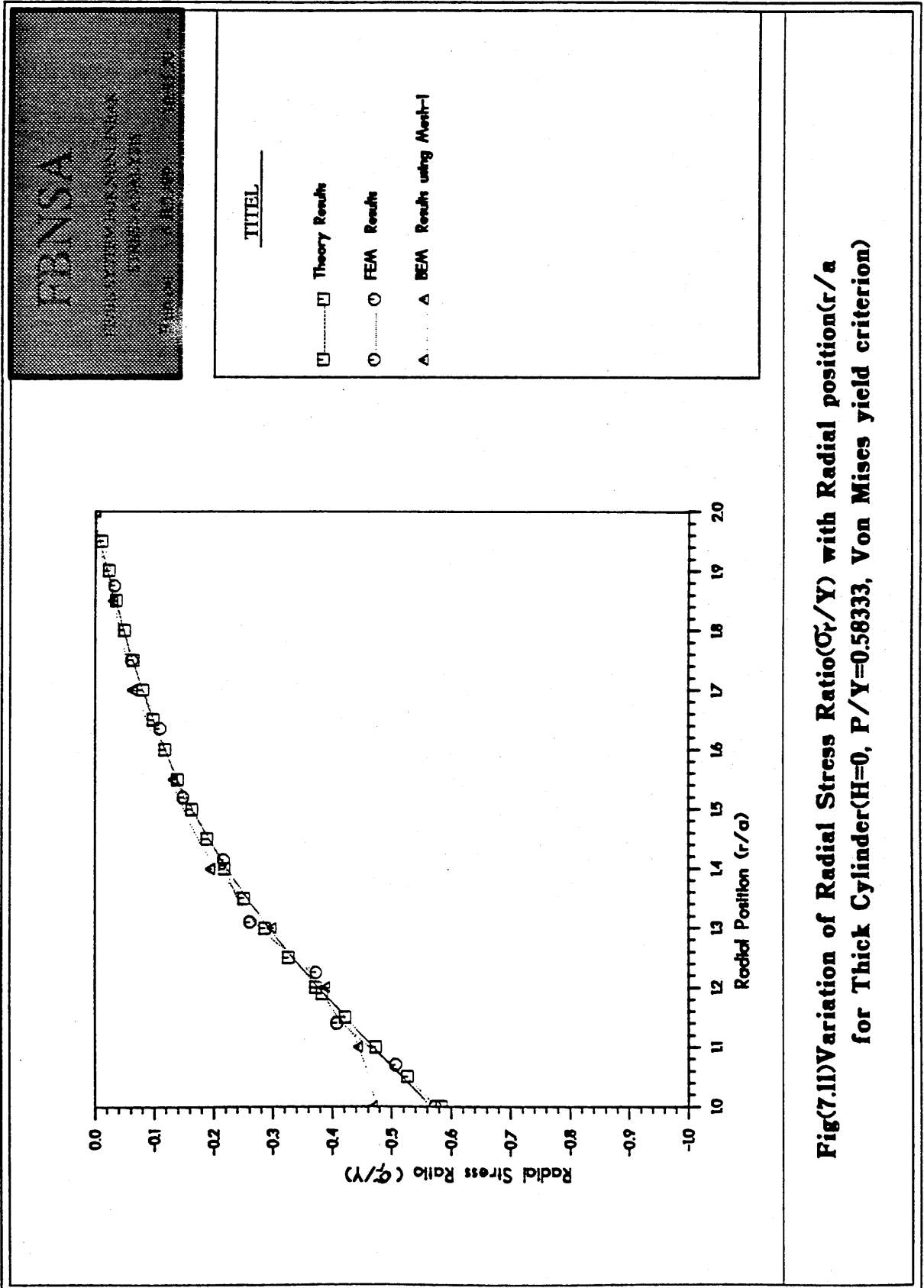


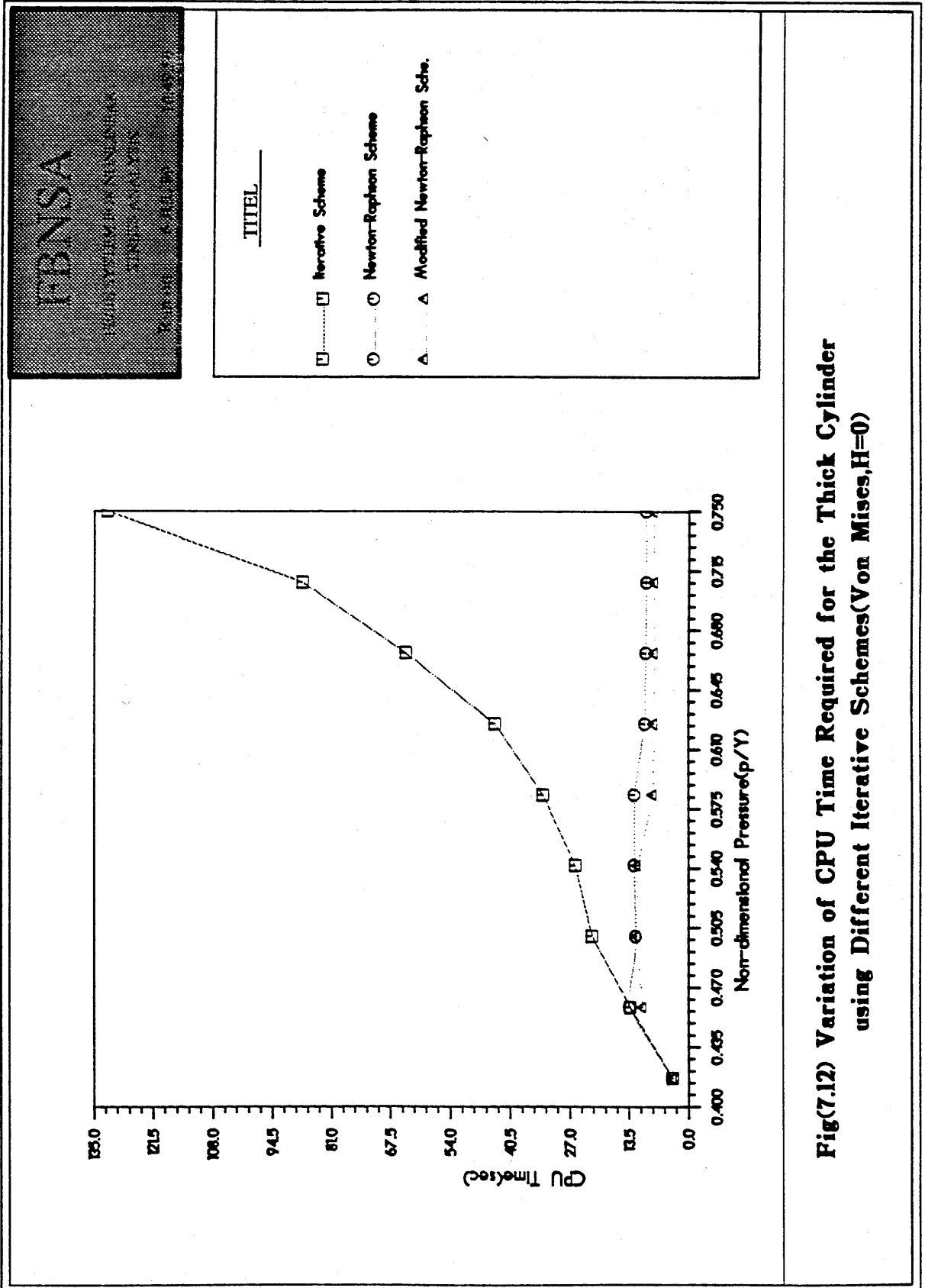


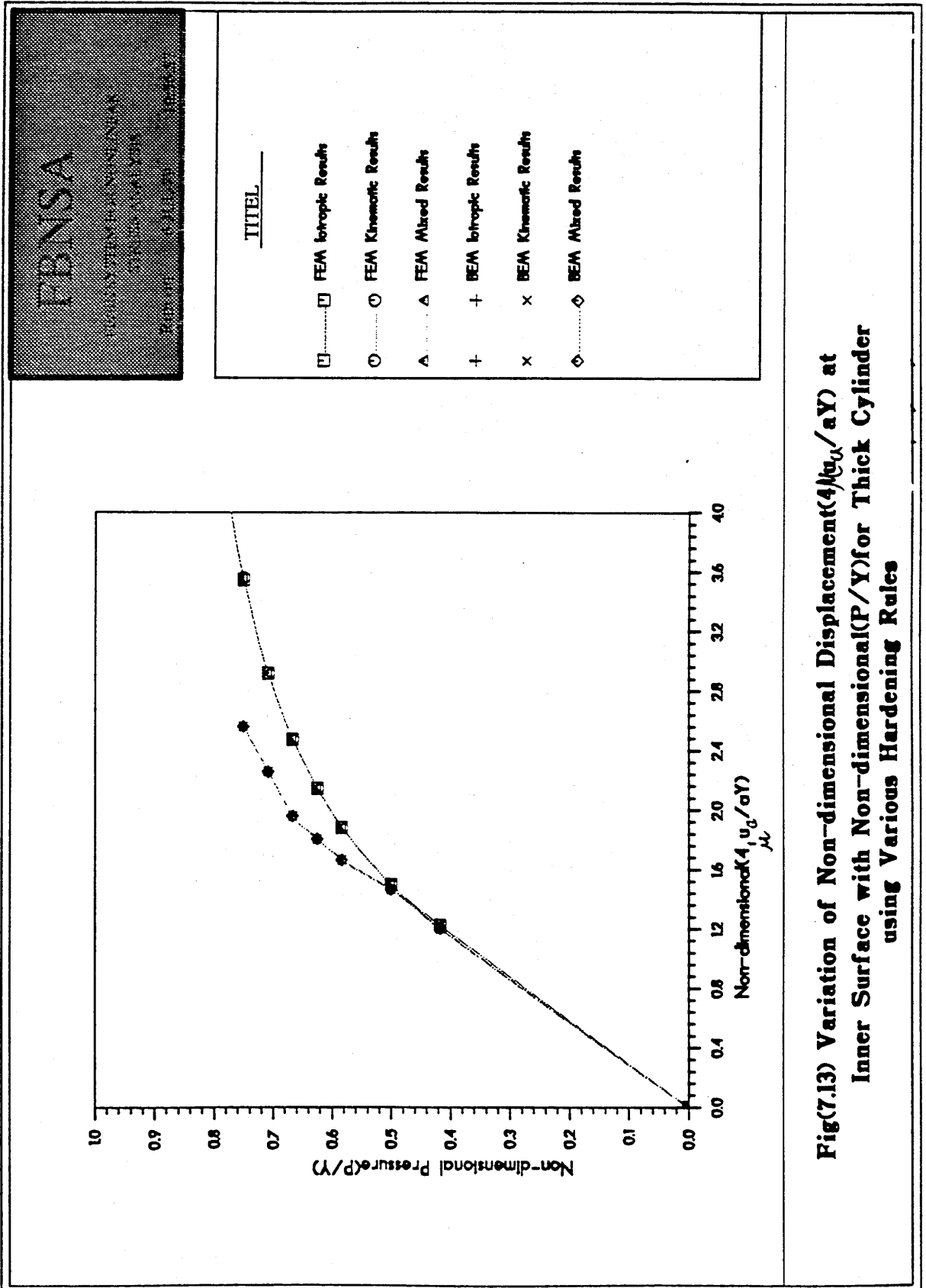


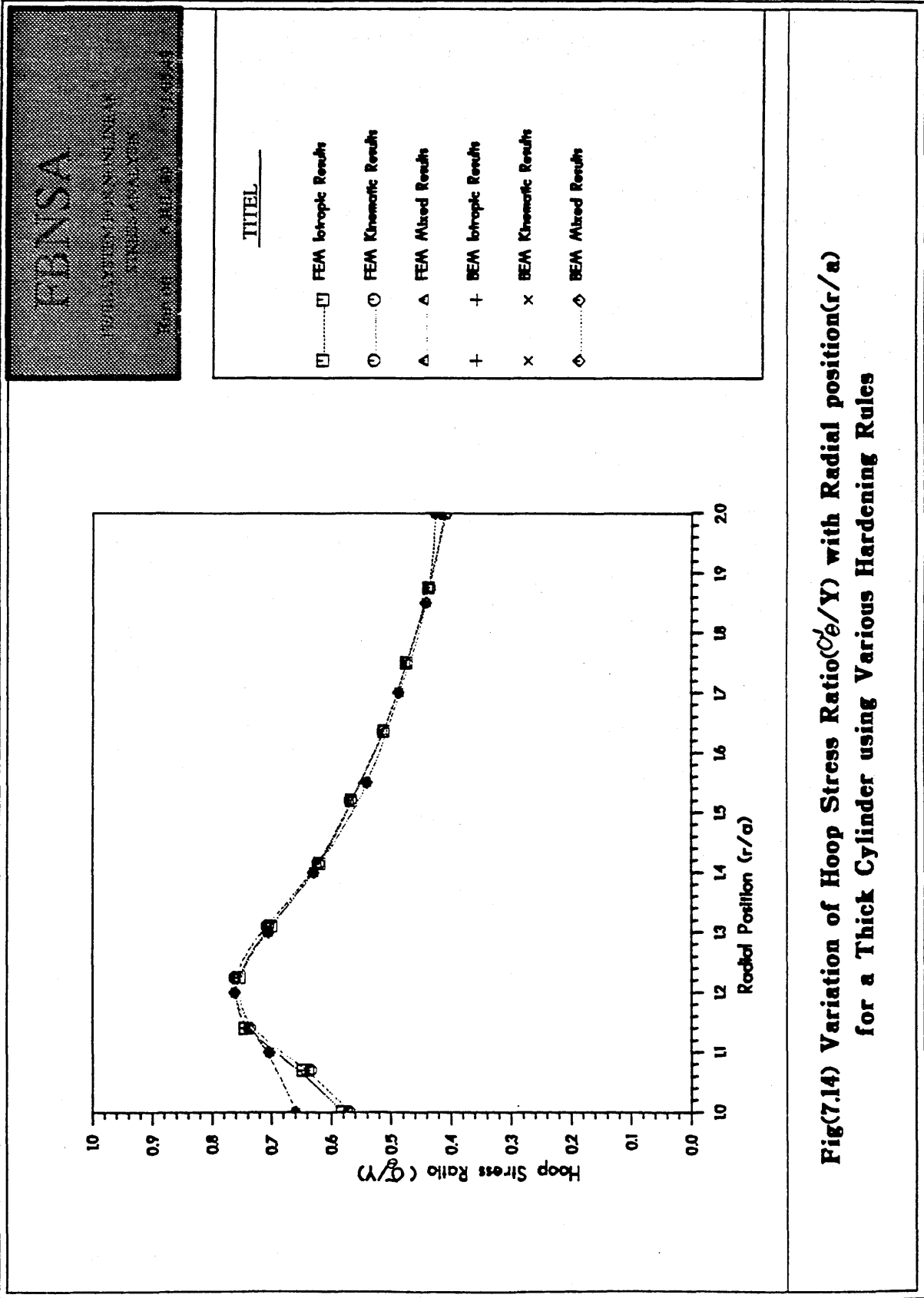


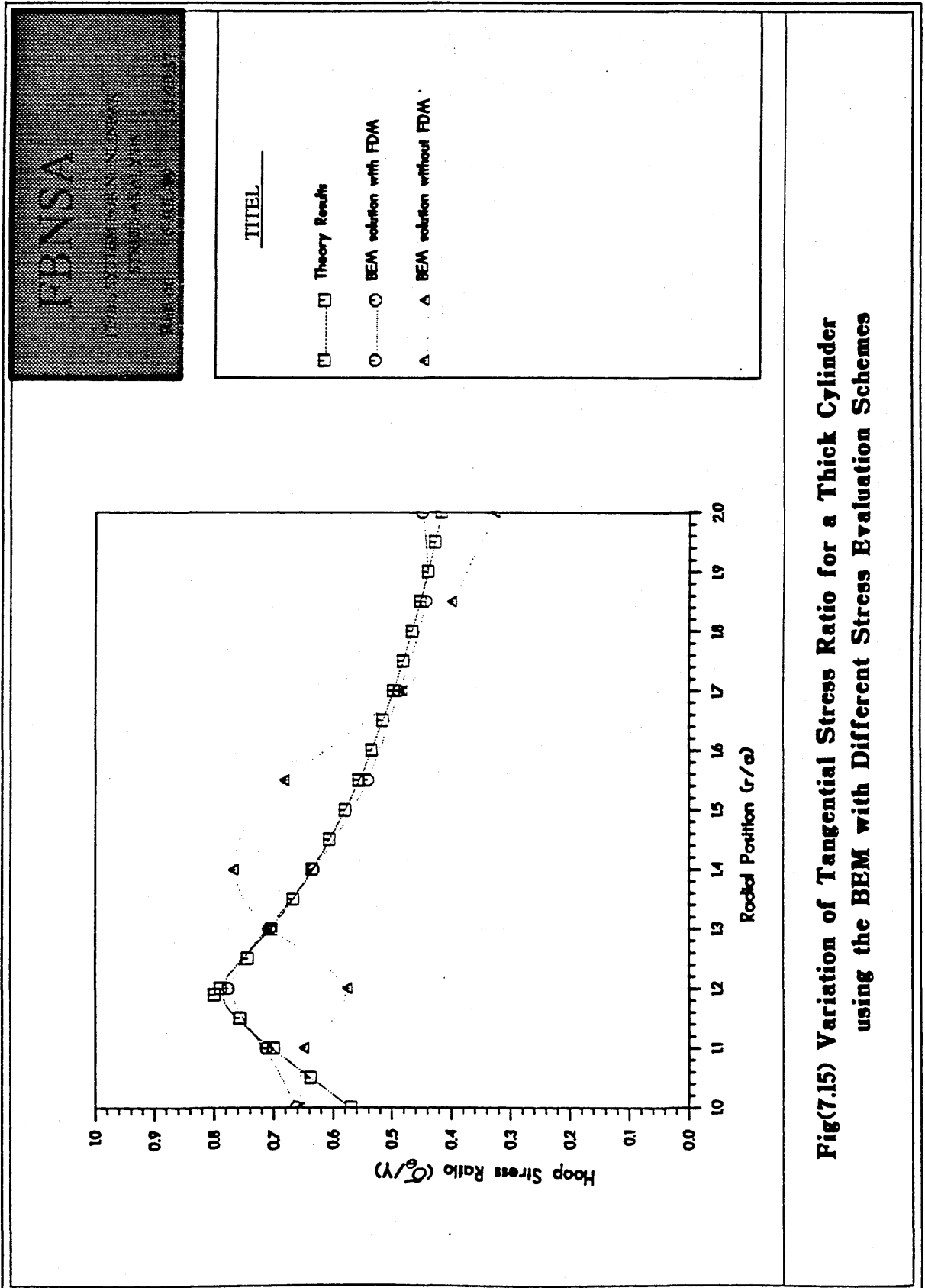


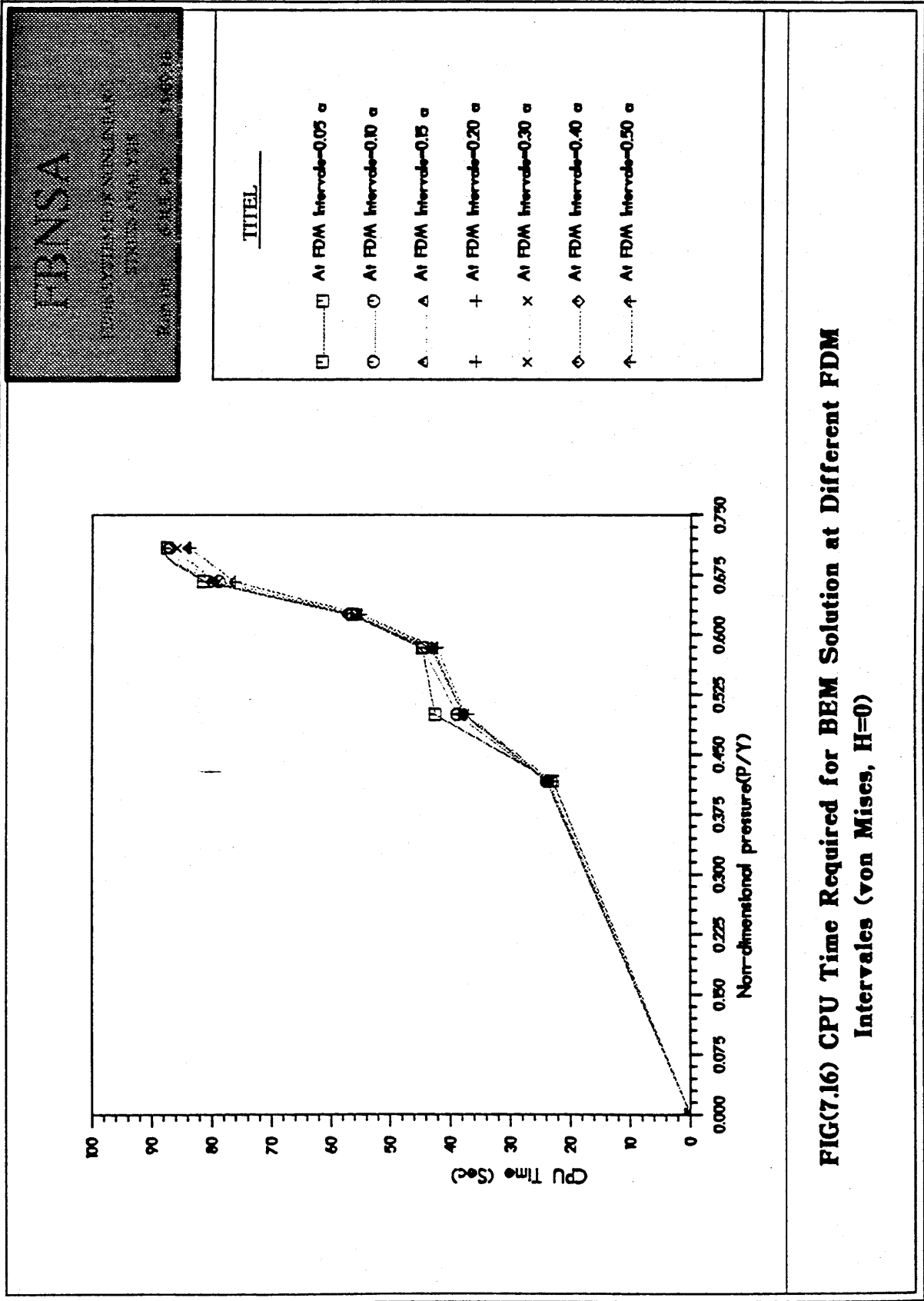


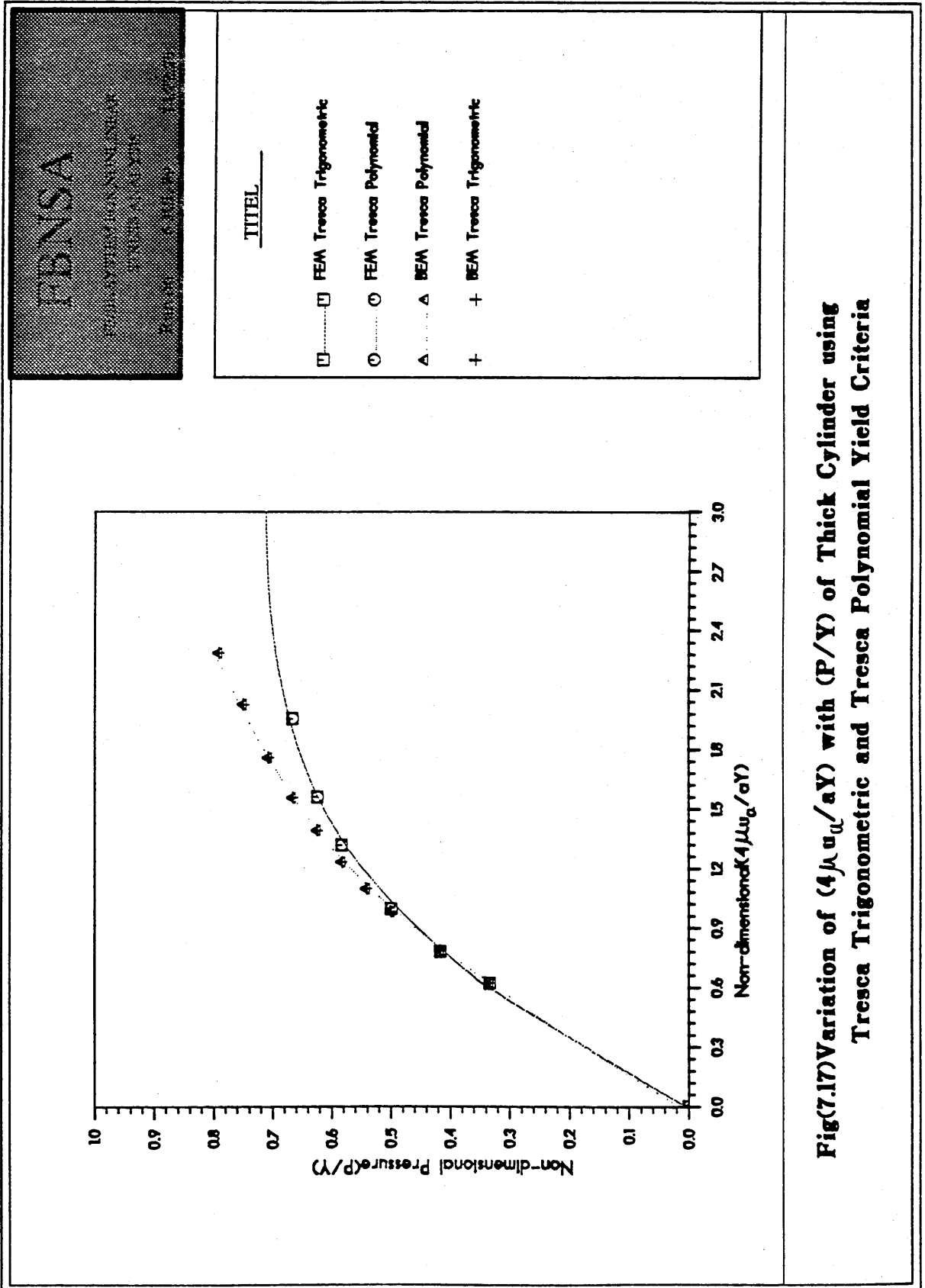


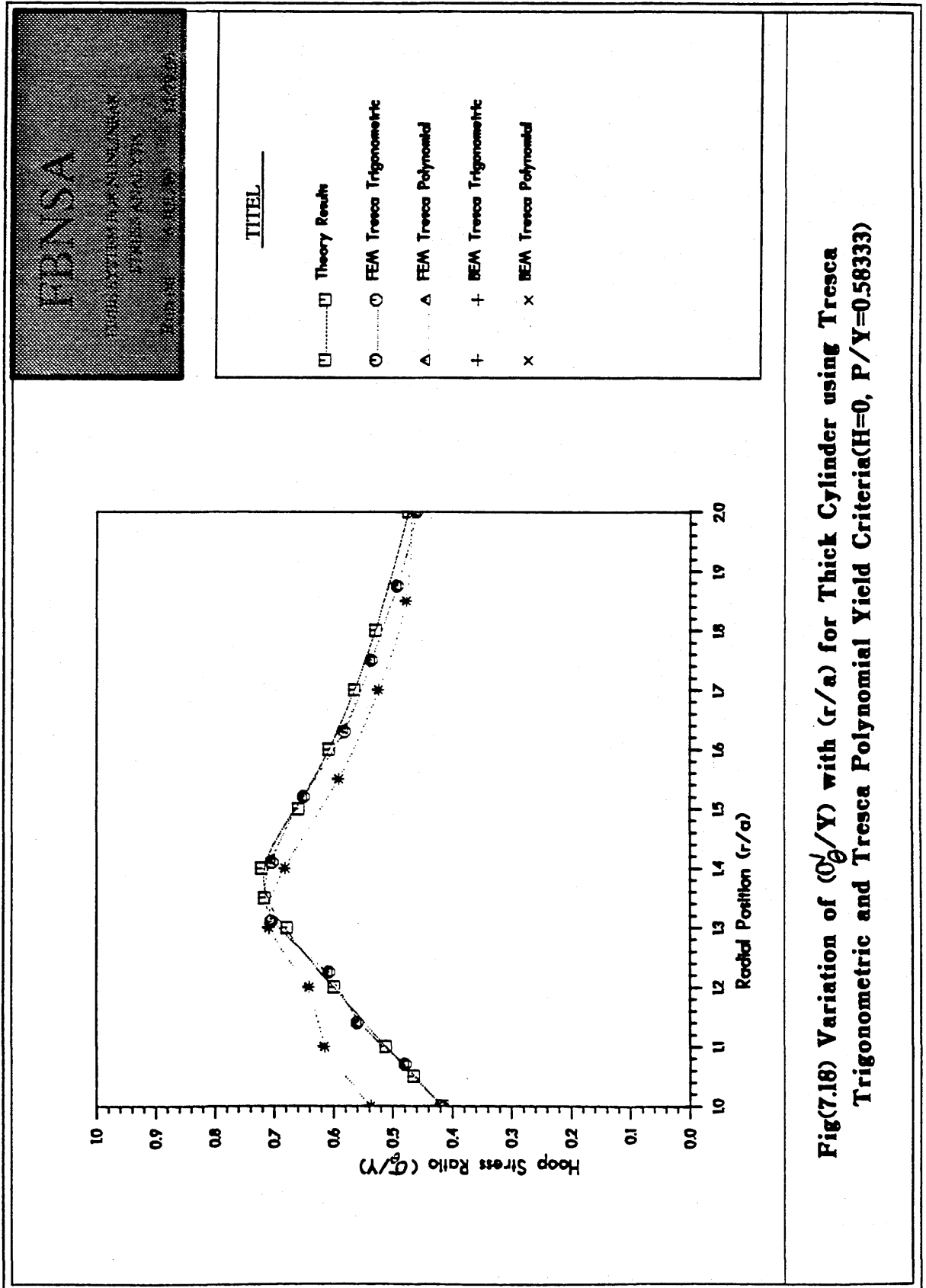


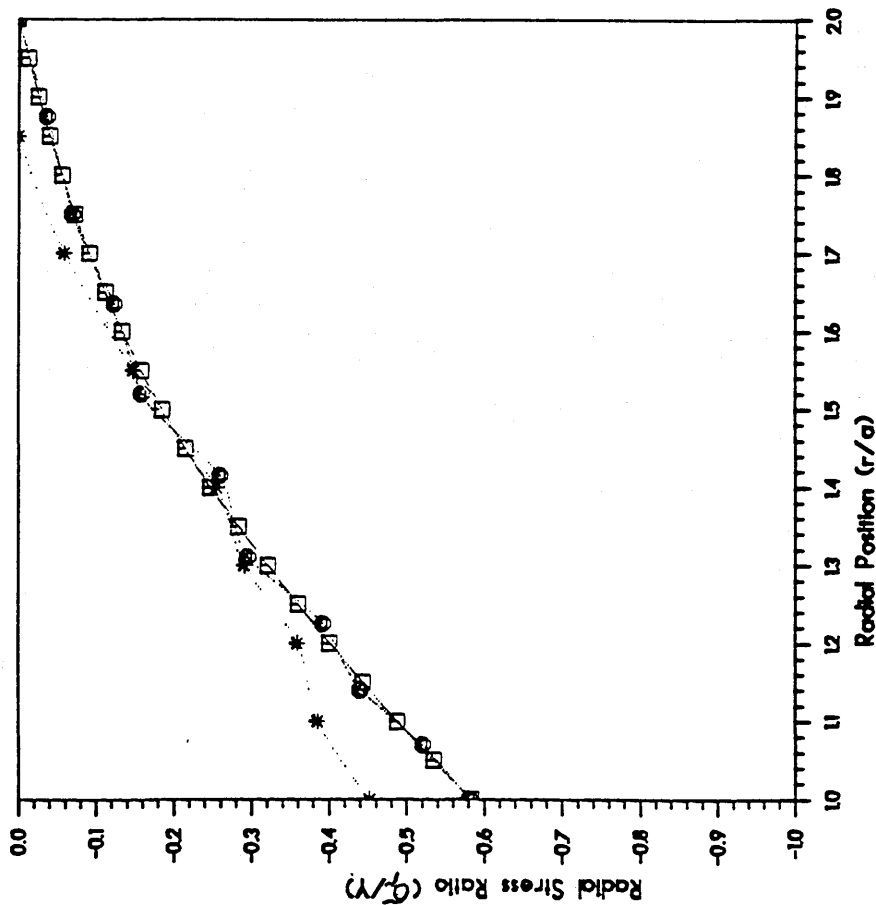










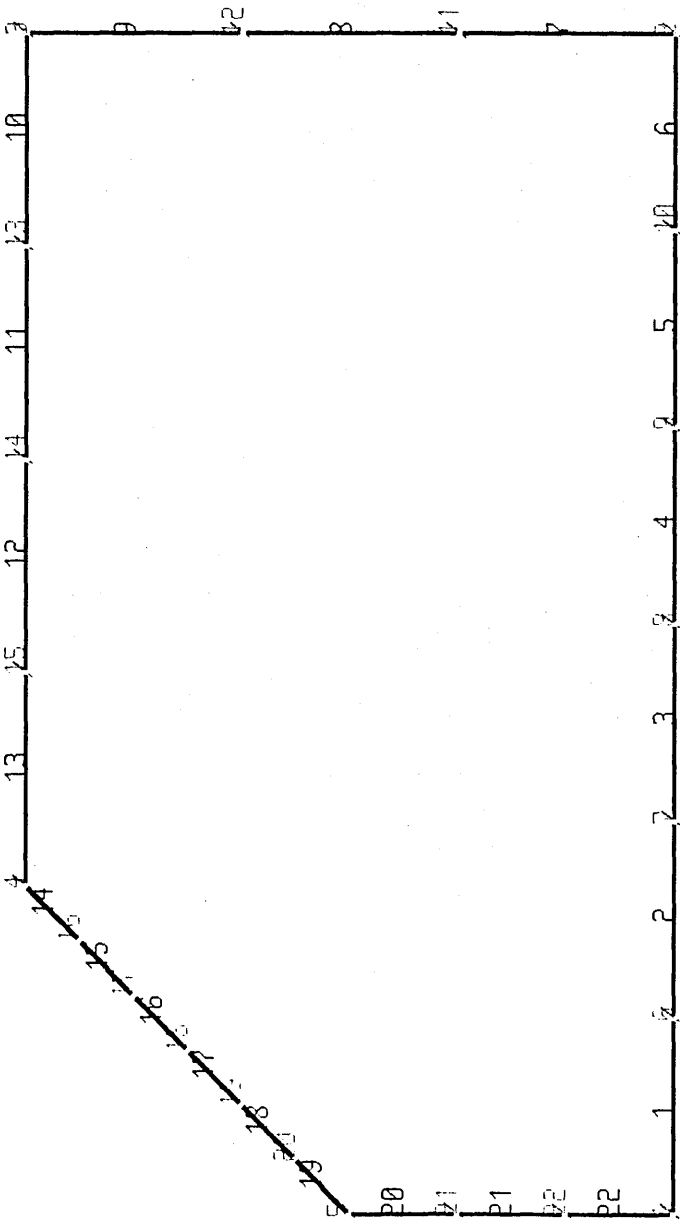


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- Theory Results
- FEM Tresca Polynomial
- △ FEM Tresca Trigonometric
- + BEA Tresca Polynomial
- x BEA Tresca Trigonometric

Fig(7.19) Variation of (σ_r/Y) with (r/a) for Thick Cylinder using Tresca Trigonometric and Tresca Polynomial Yield Criteria($H=0$, $P/Y=0.58333$)

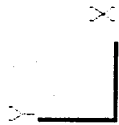
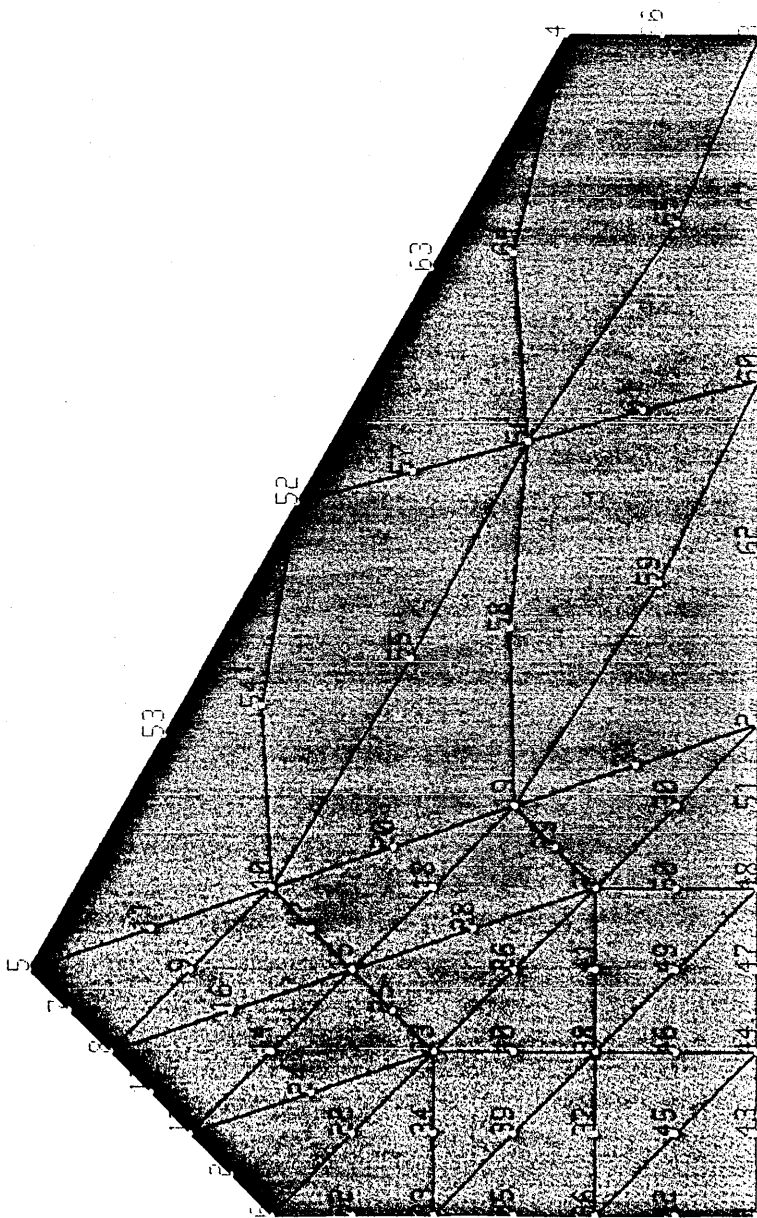
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Fig(7.21) BEM Mesh for Notched Tensile Example using 2-noded
Isoparametric Elements (Plane Stress Case)

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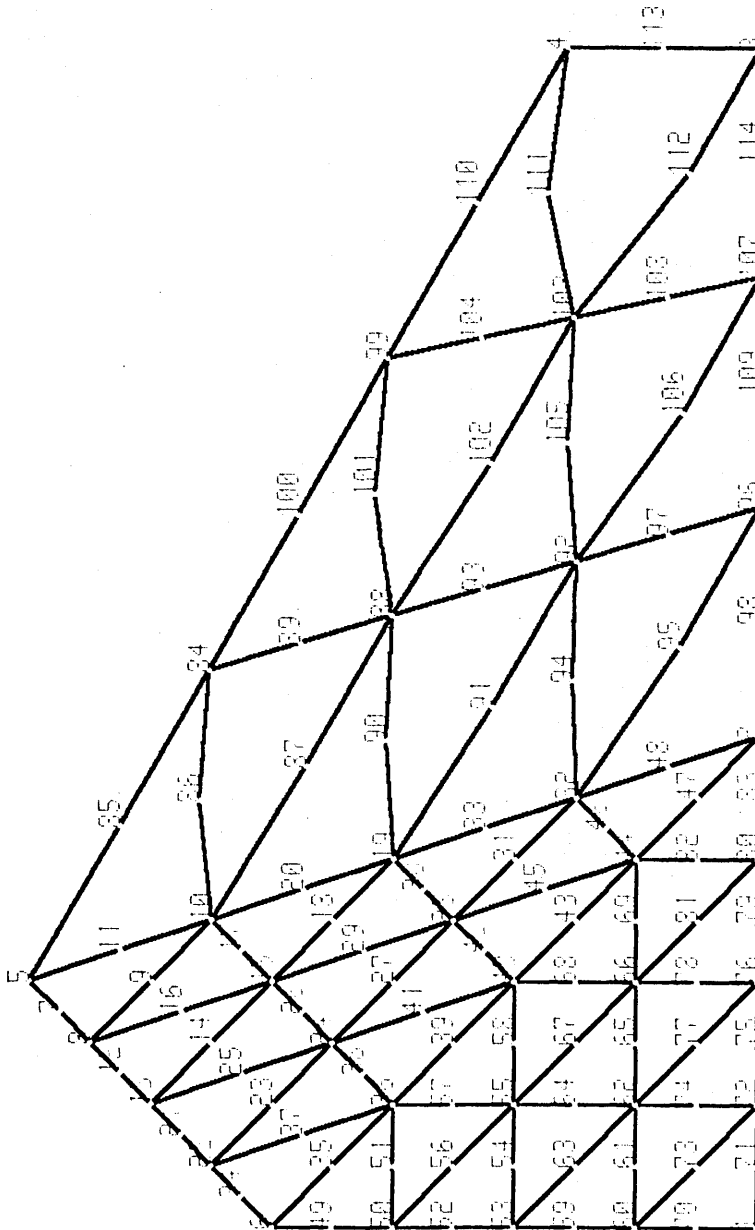


Fig(7.22) Integration Cell Discretization of Notched Tensile Example using
6-noded Triangular Finite Elements(Plane Stress Case), Coarse-Mesh

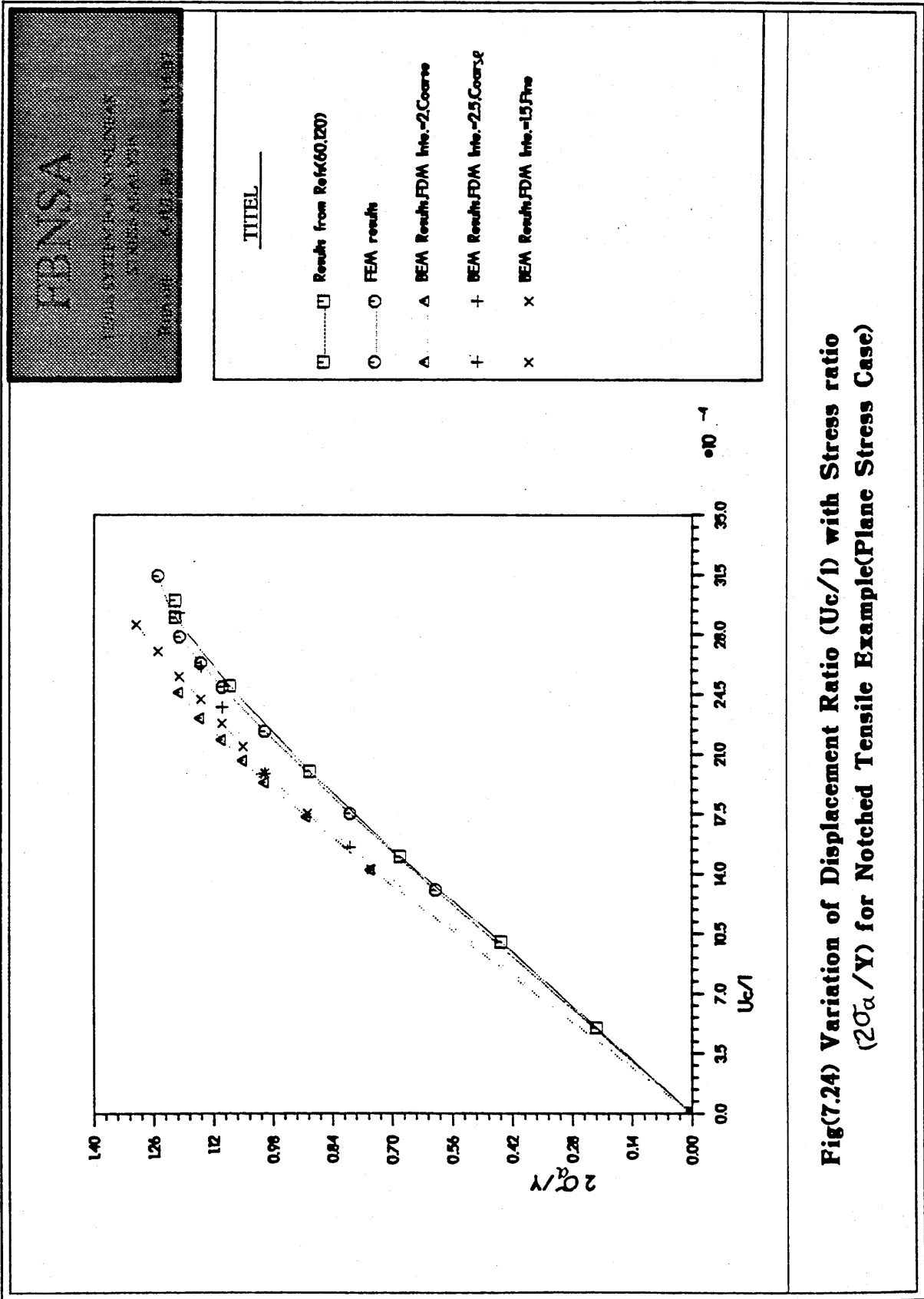
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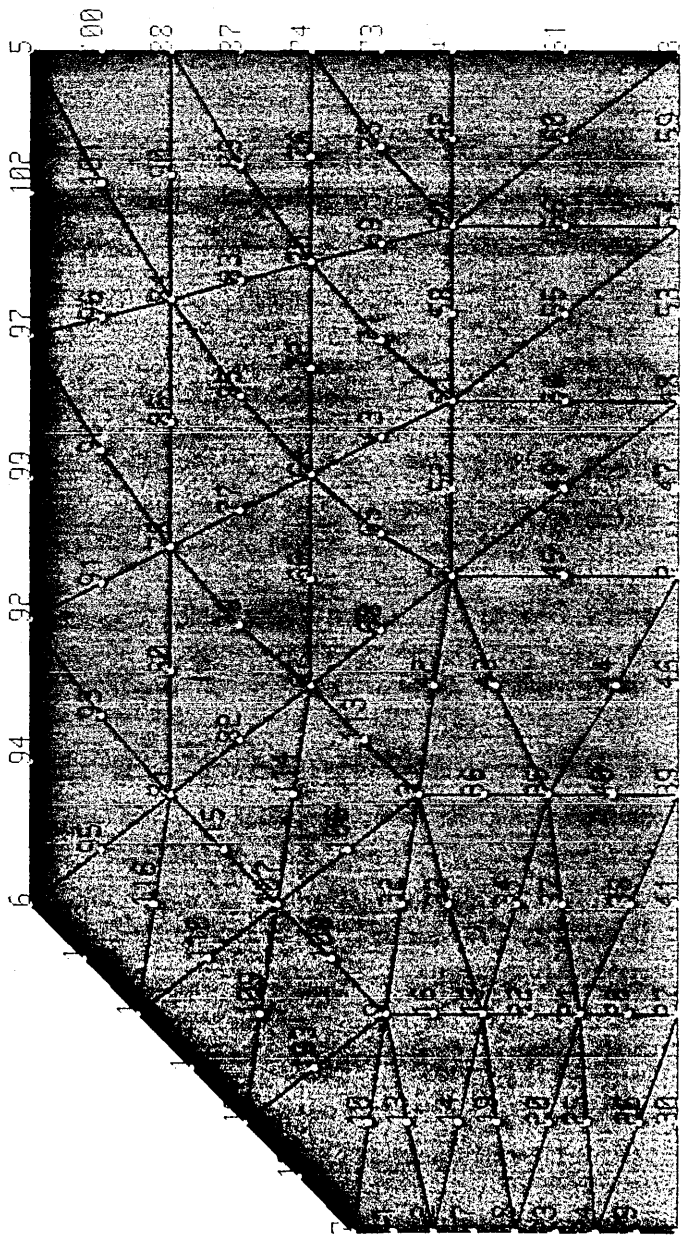
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Fig(7.23) Integration Cell Discretization of Notched Tensile Example
using 6-noded Triangular Finite Elements(Plane Stress Case), Fine Mesh



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Fig(2.25) Integration Cell Mesh for a Notched Tensile Specimen using
6-noded Triangular Finite Elements(Plane Strain Case)

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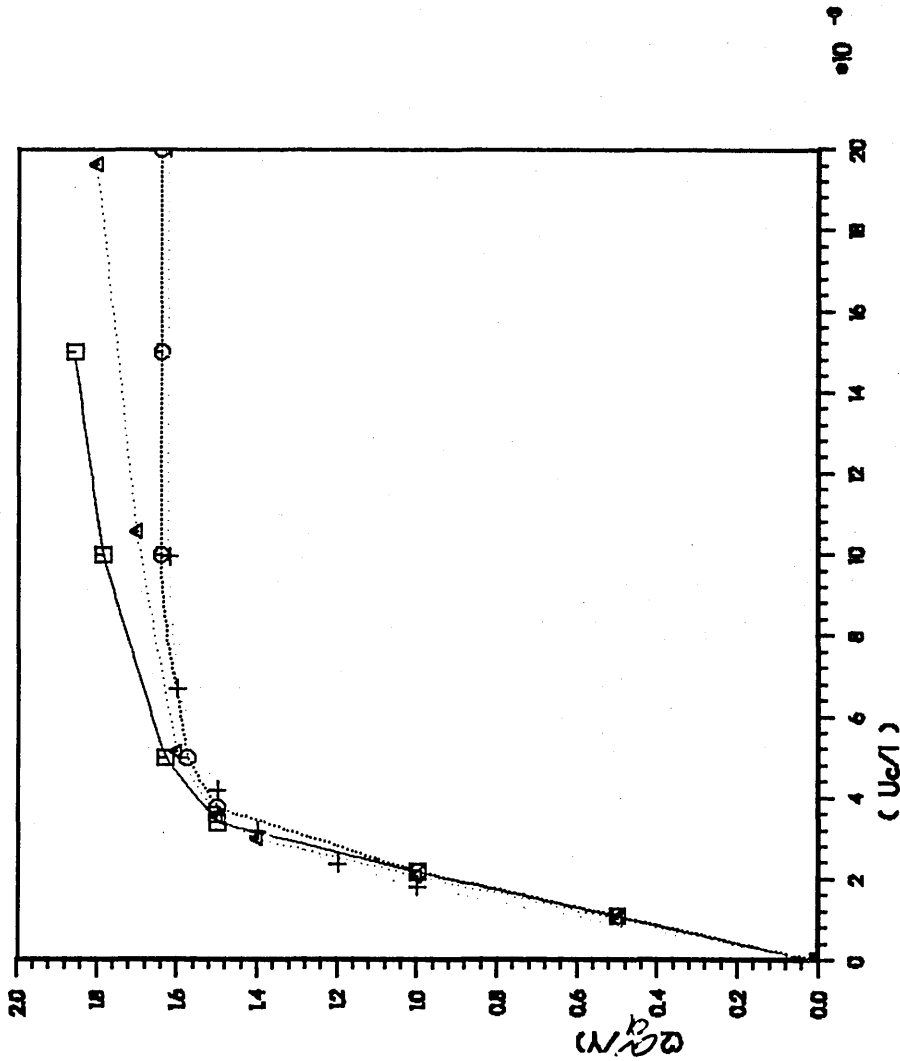
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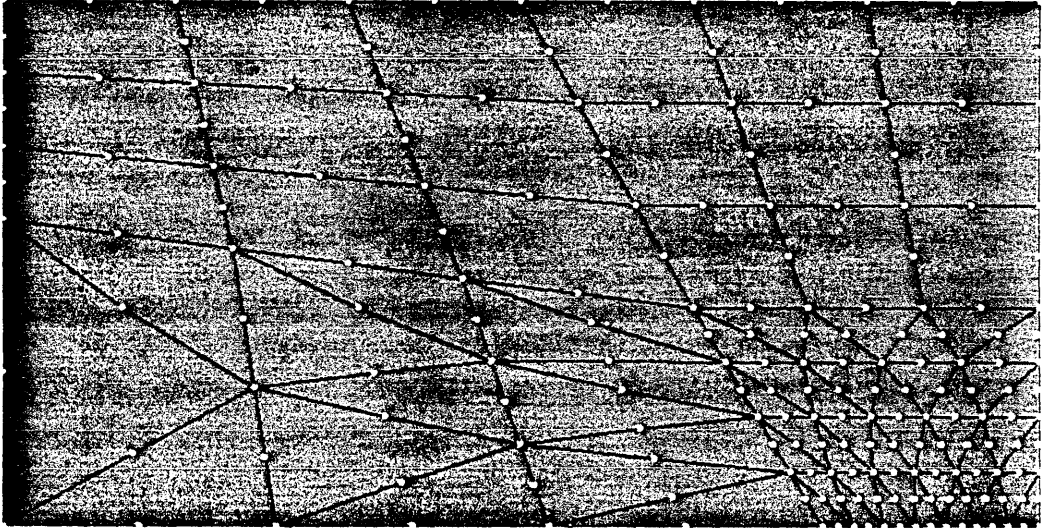


Fig(7.27) Variation of Displacement Ratio(U_c/l) with Stress Ratio (σ/σ_Y) for Notched Tensile Specimen (Plane Strain Case)

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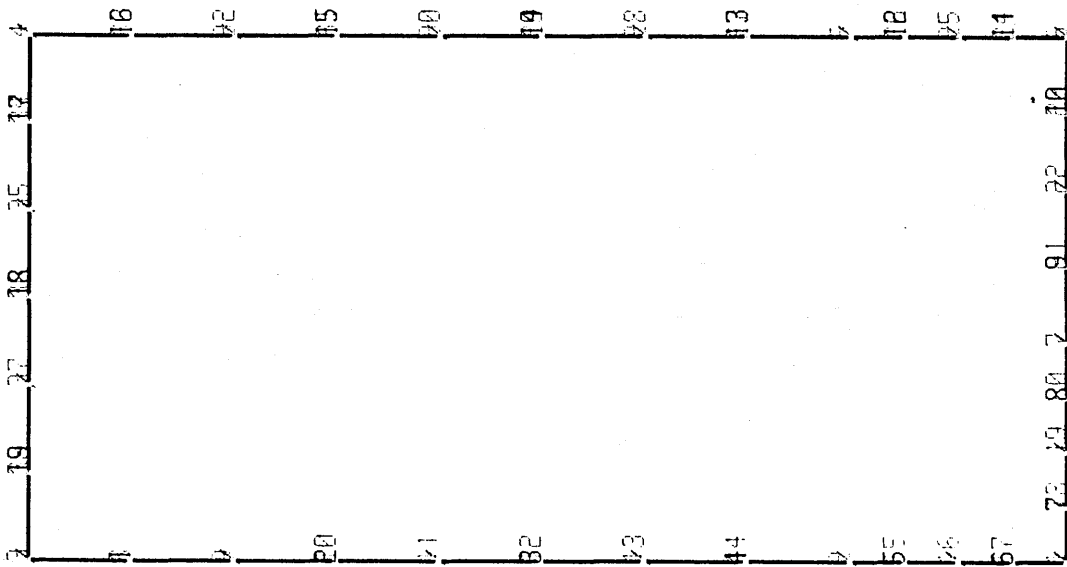
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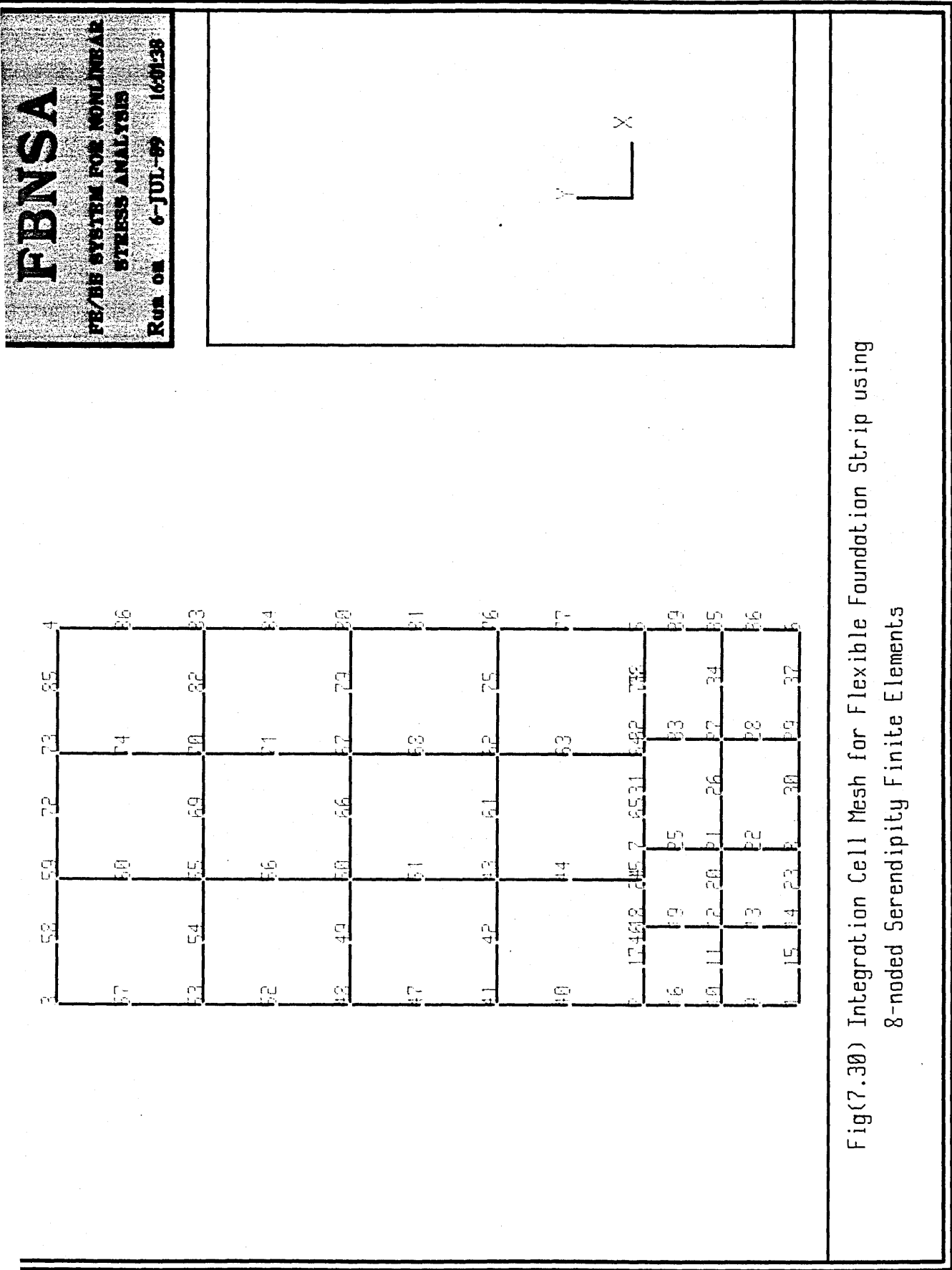


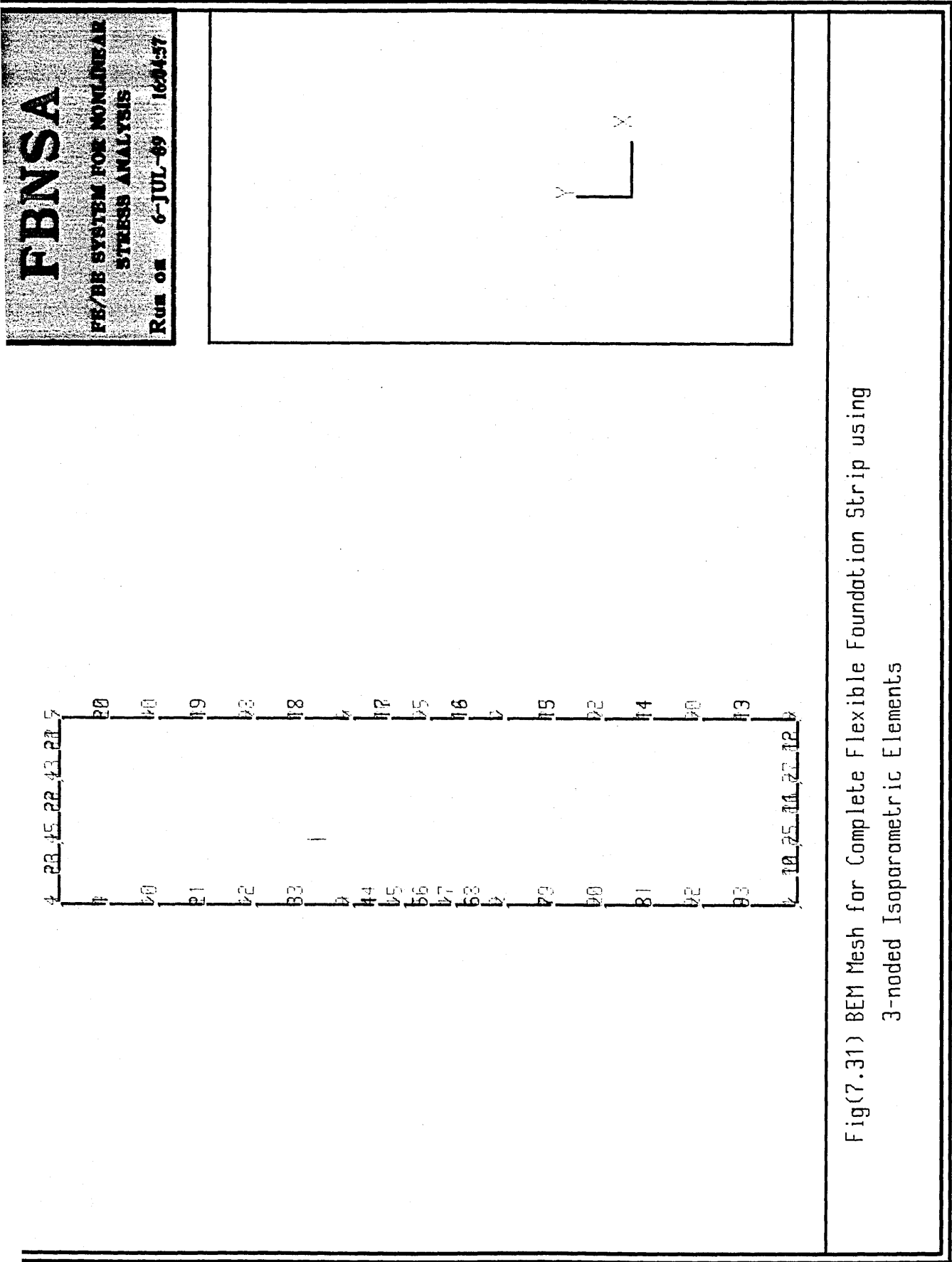
Fig(7.28) Finite Element Mesh for Flexible Foundation Strip using
6-noded Triangular Elements

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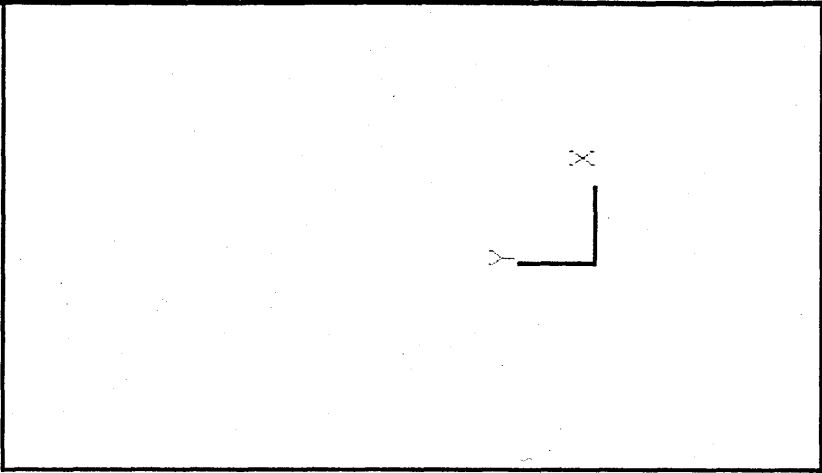
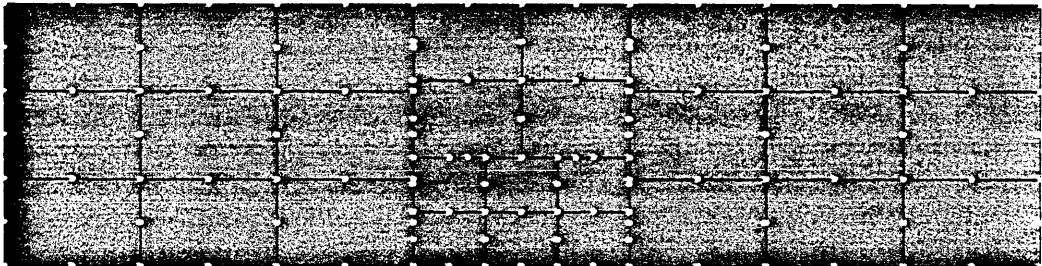


Fig(7.29) BEM Mesh for Flexible Foundation Strip using 3-noded
Isoparametric Elements





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Fig(7.32) Integration Cell Mesh for Complete Flexible Foundation Strip
using 8-noded Serendipity Elements

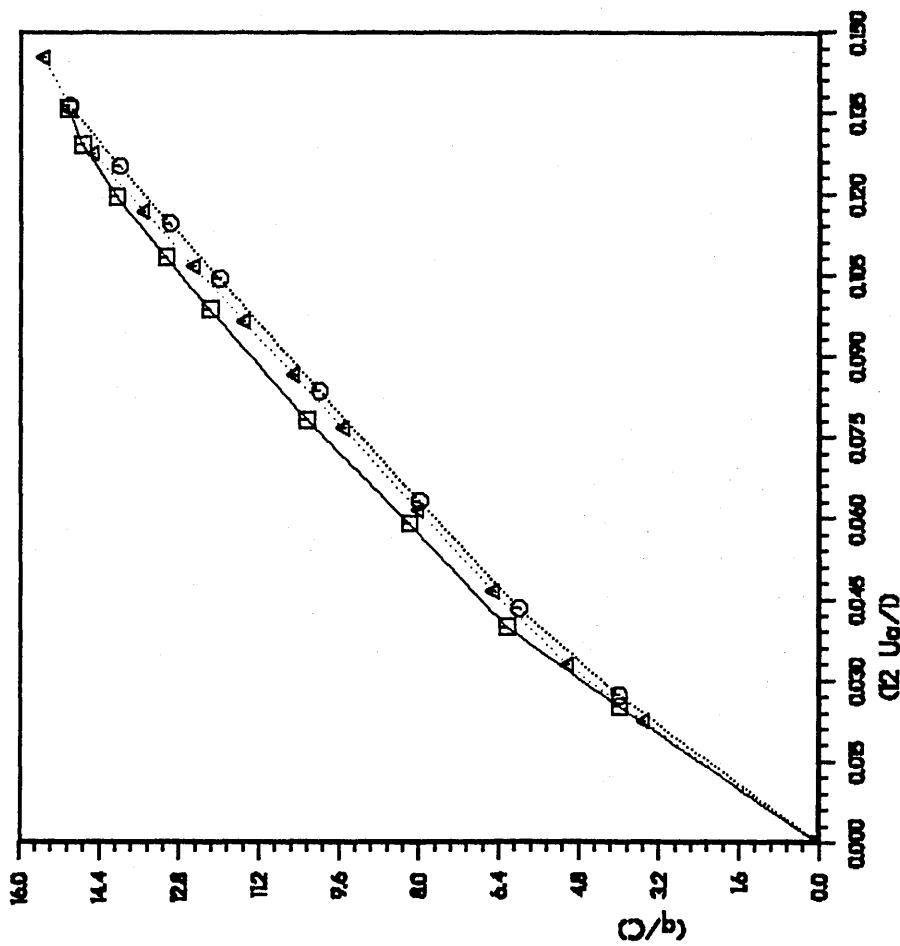
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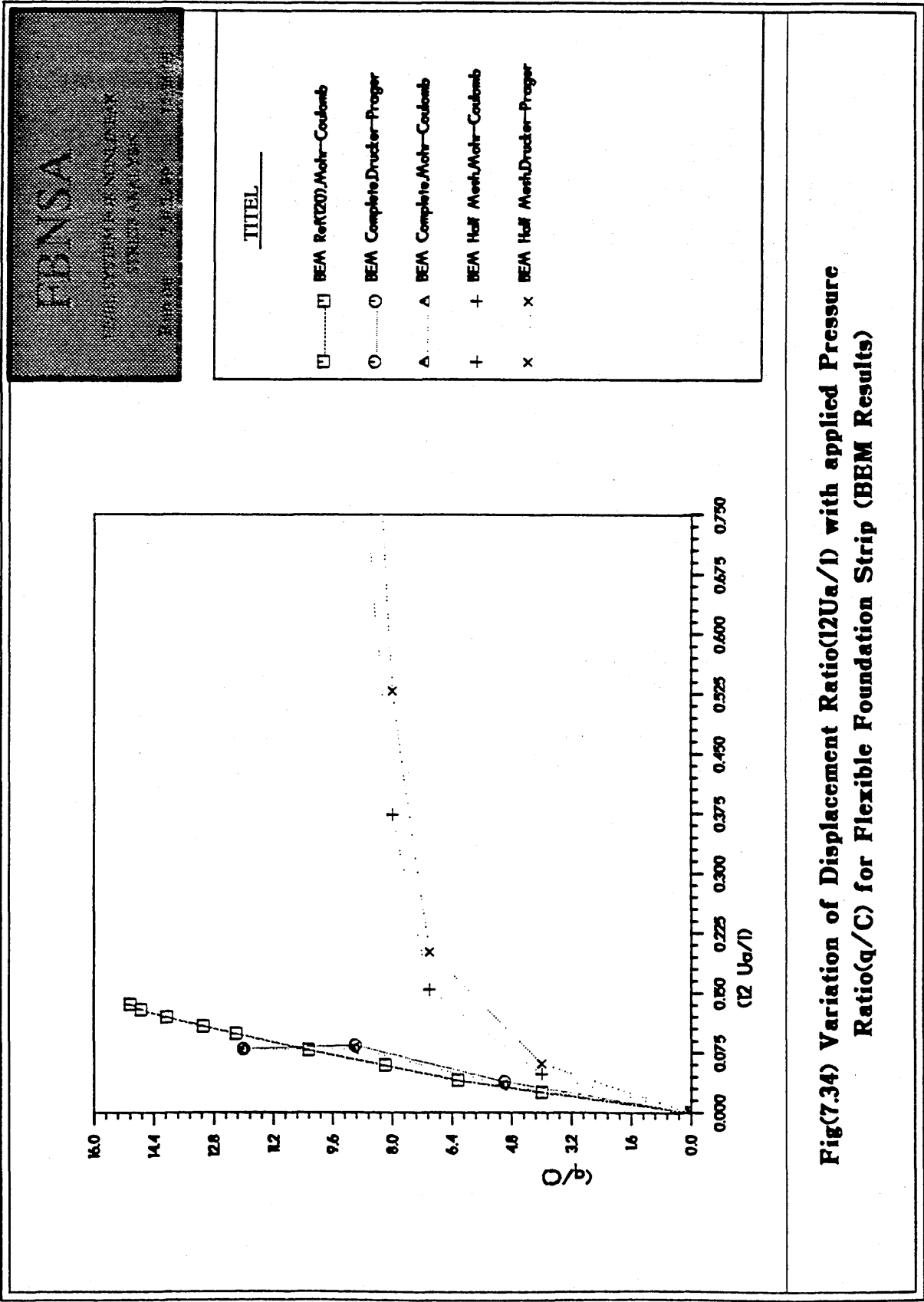
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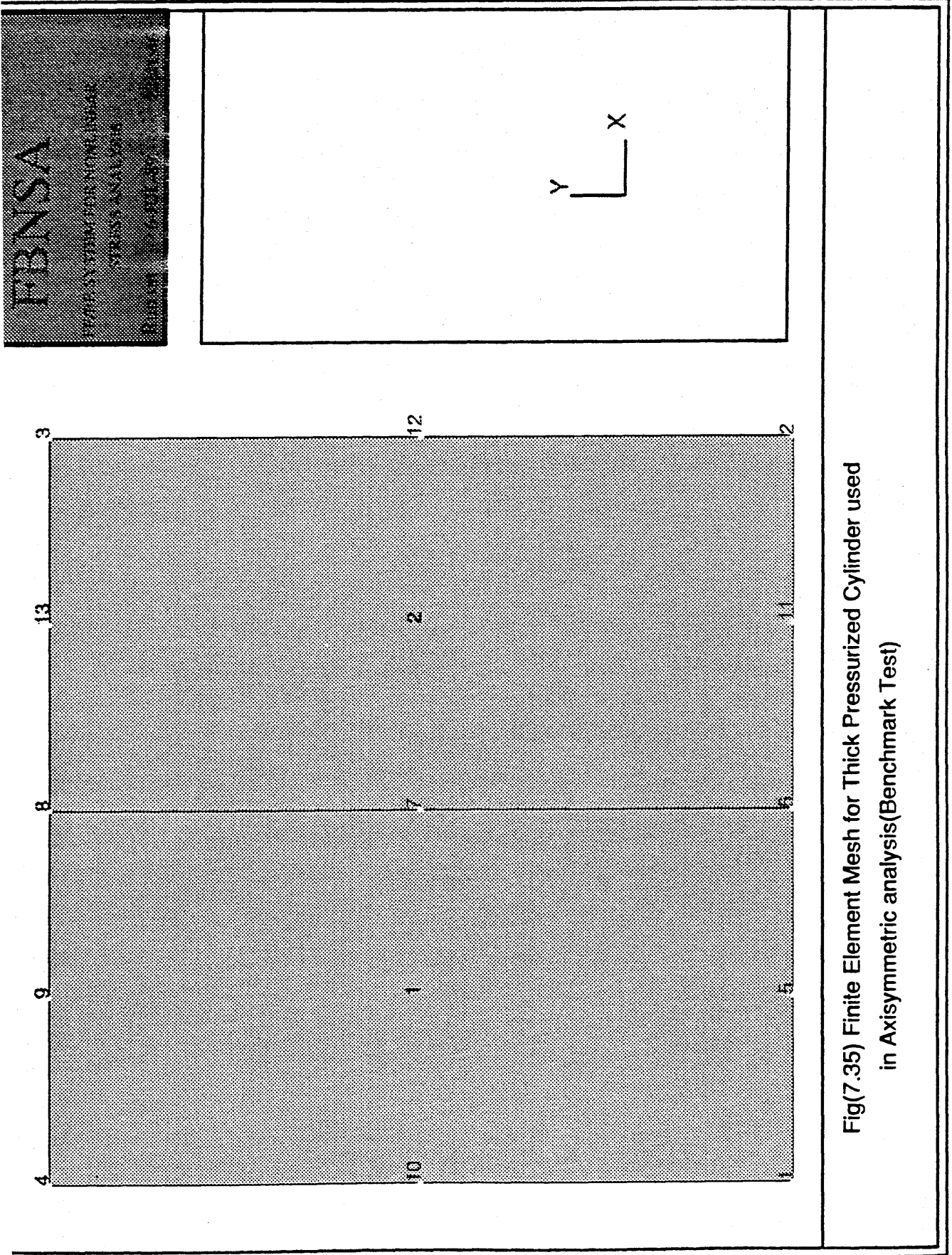
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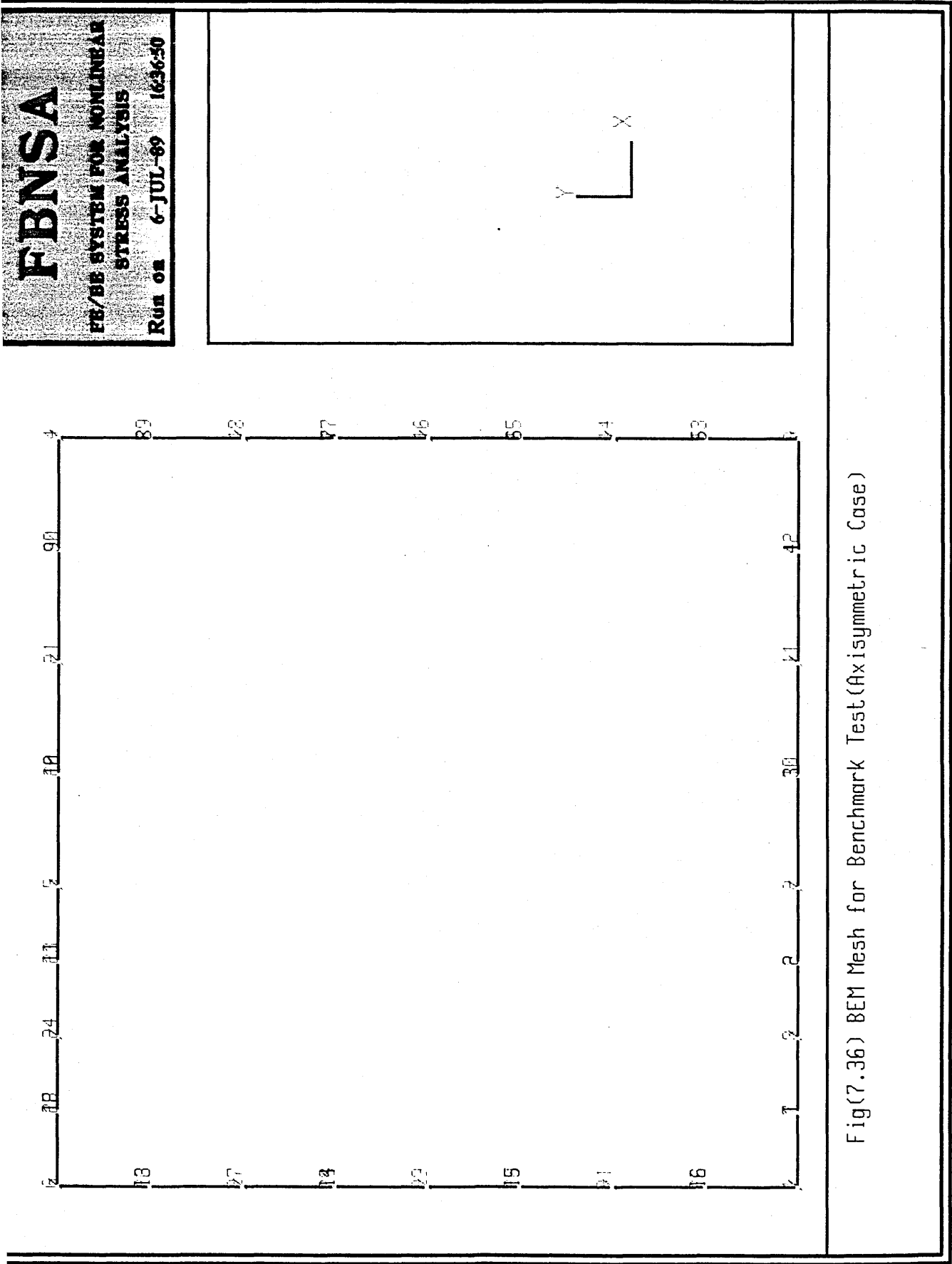
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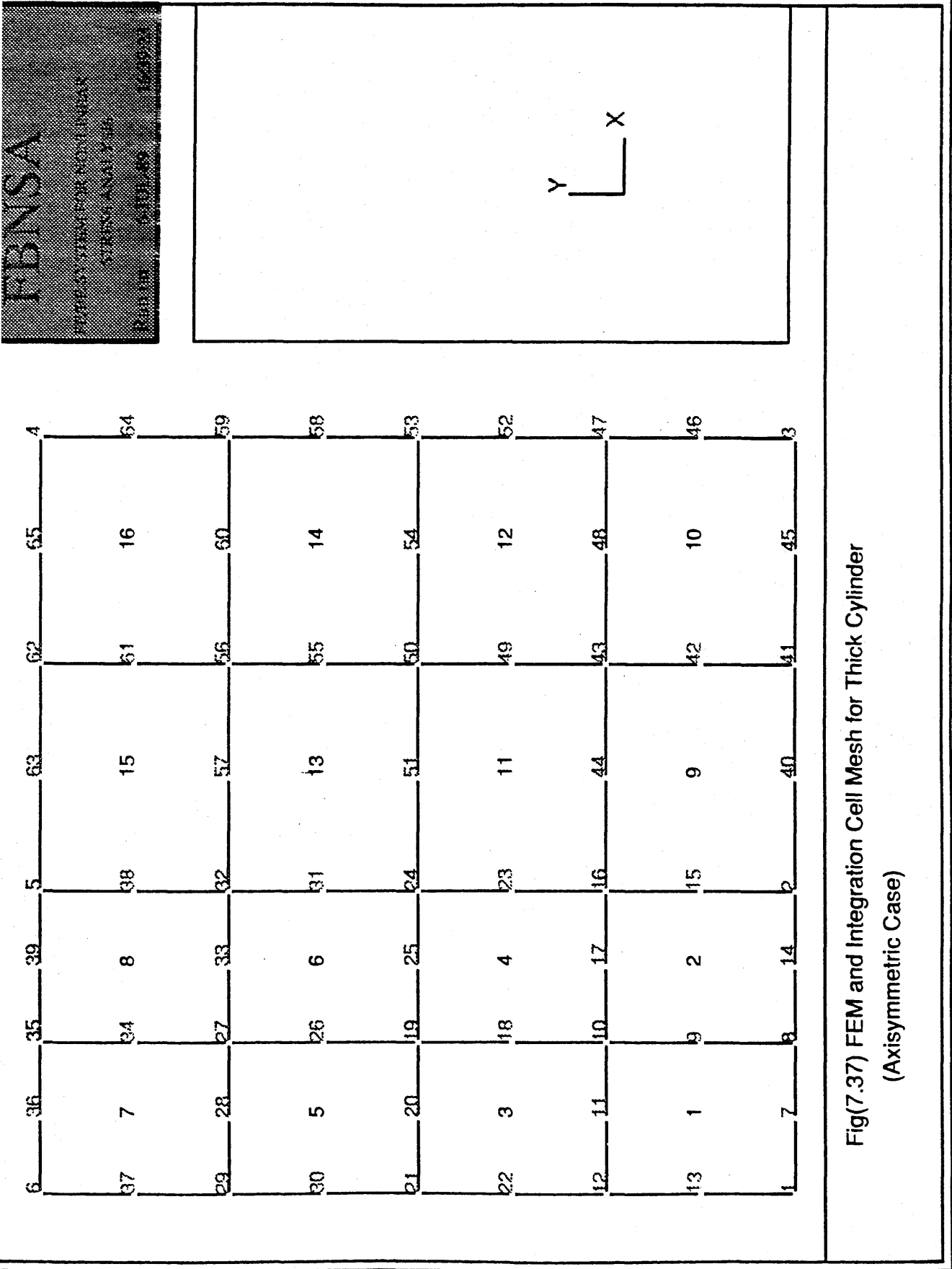


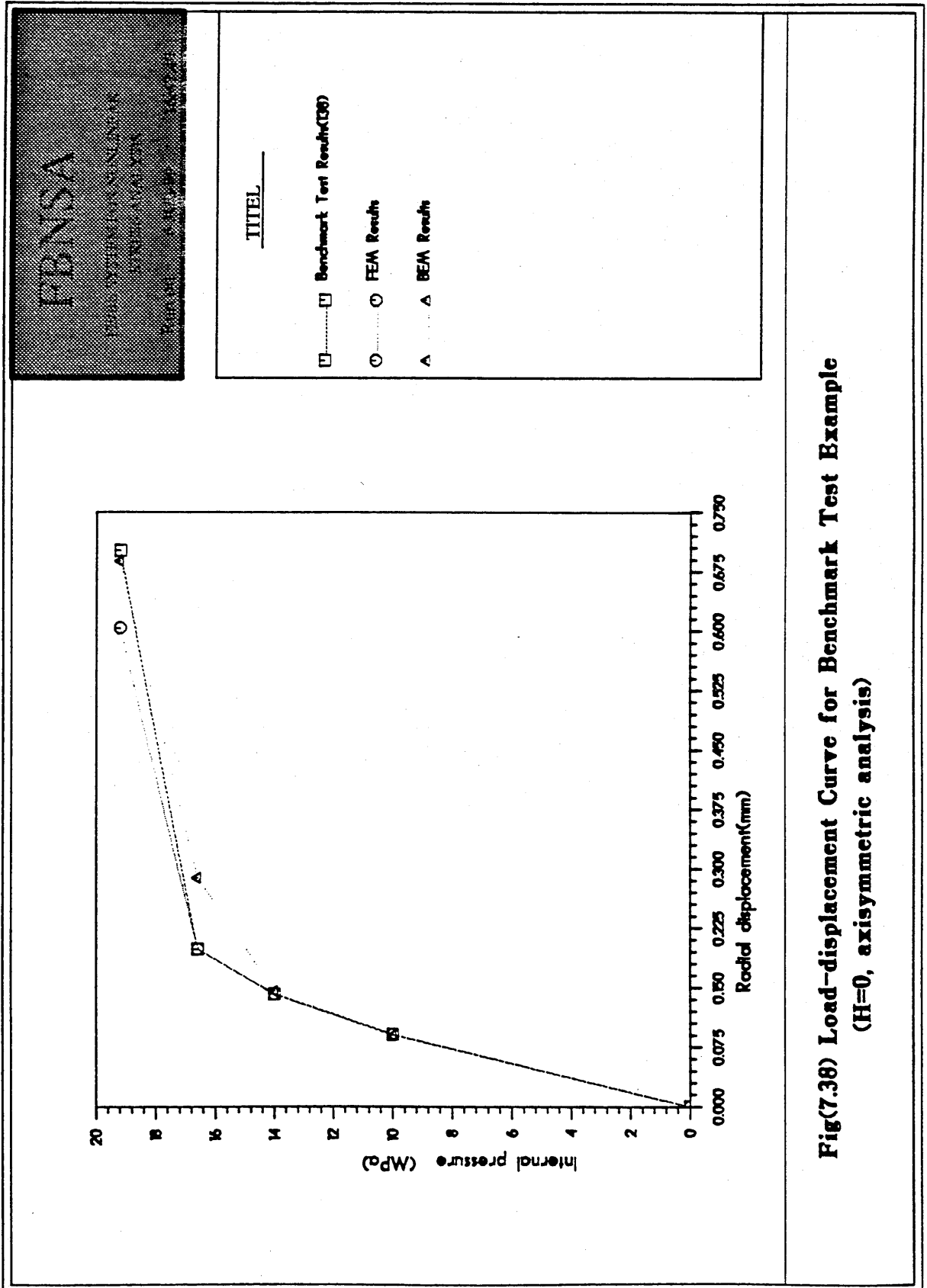
Fig(7.33) Variation of Displacement Ratio(12Ua/1) with applied Pressure Ratio(q/C) for Flexible Foundation Strip (FEM Results)

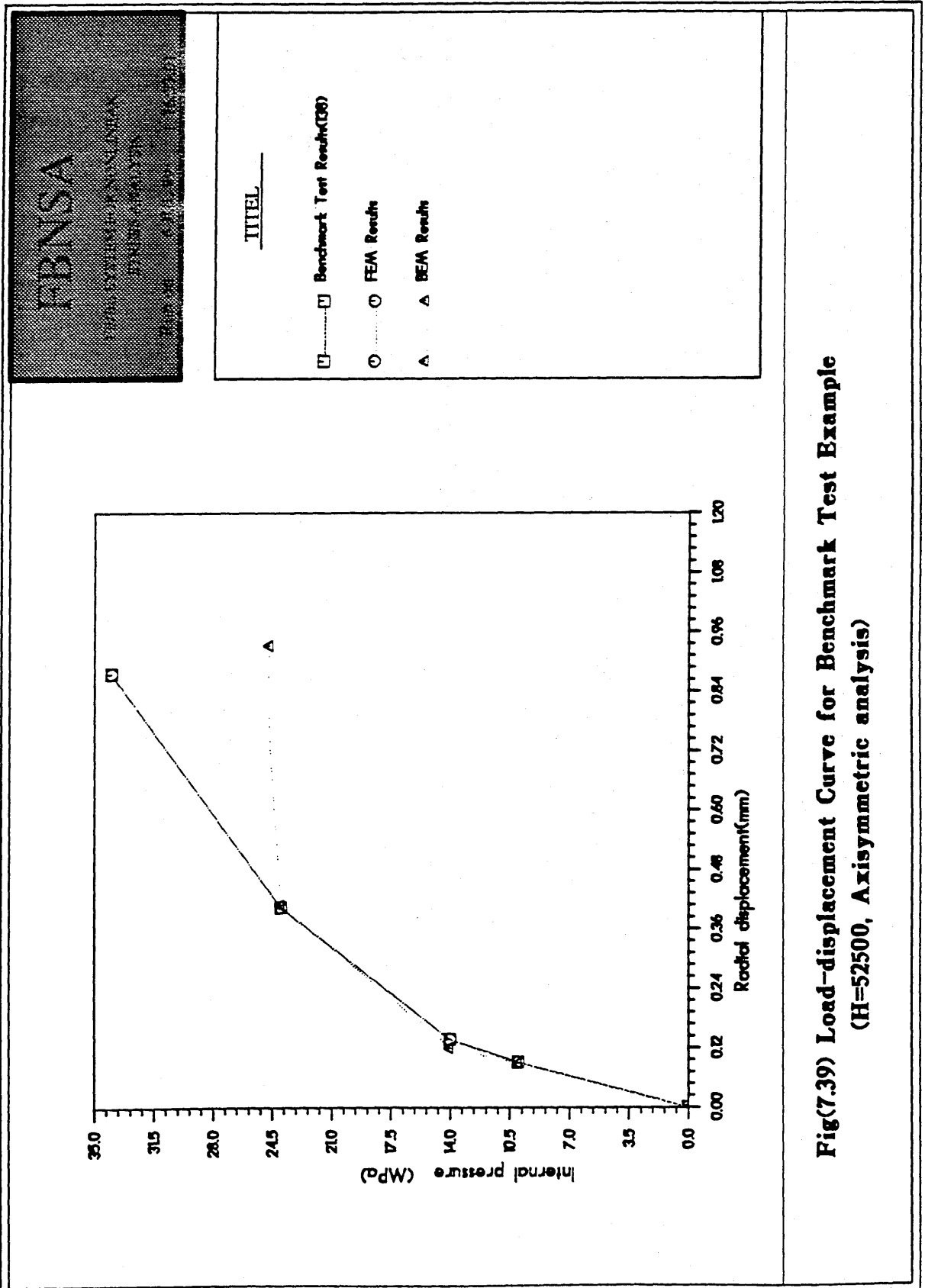




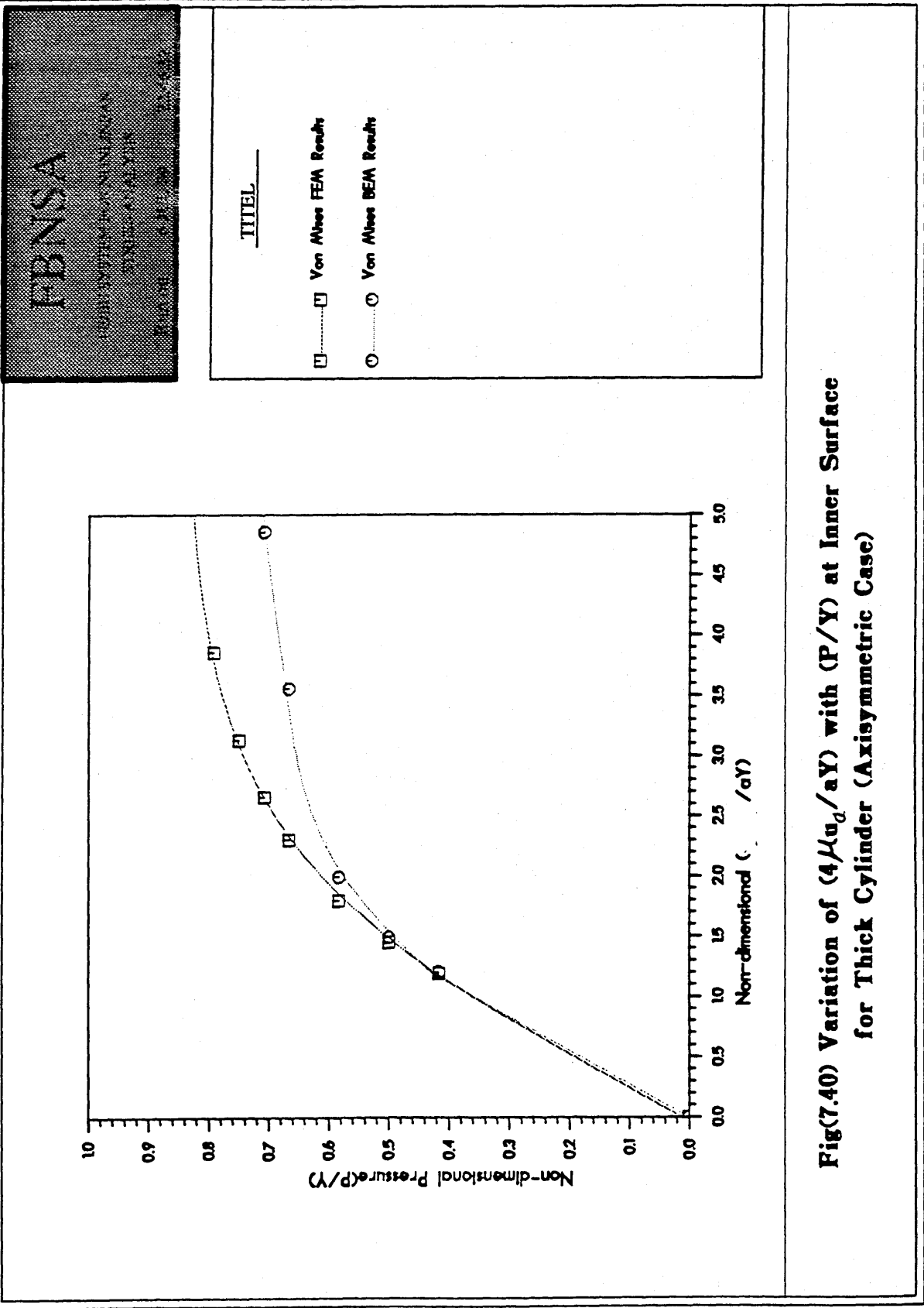








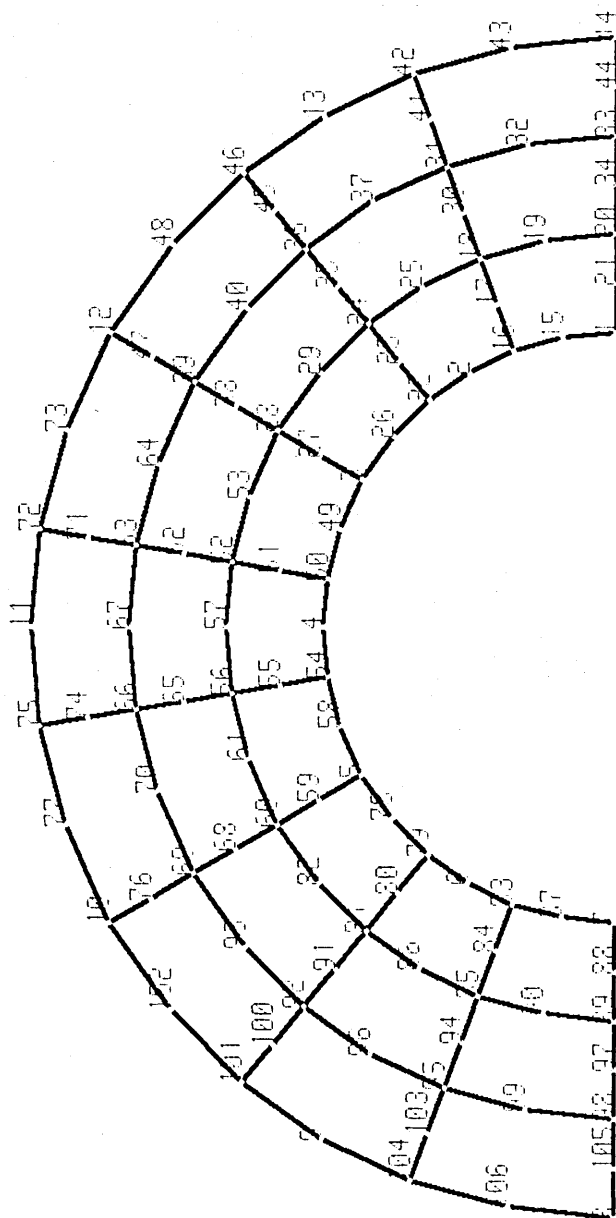
**Fig(7.39) Load-displacement Curve for Benchmark Test Example
(H=52500, Axisymmetric analysis)**



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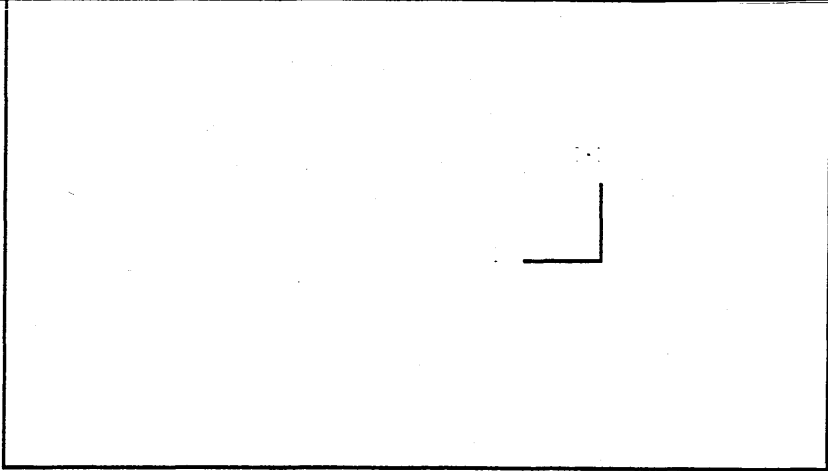
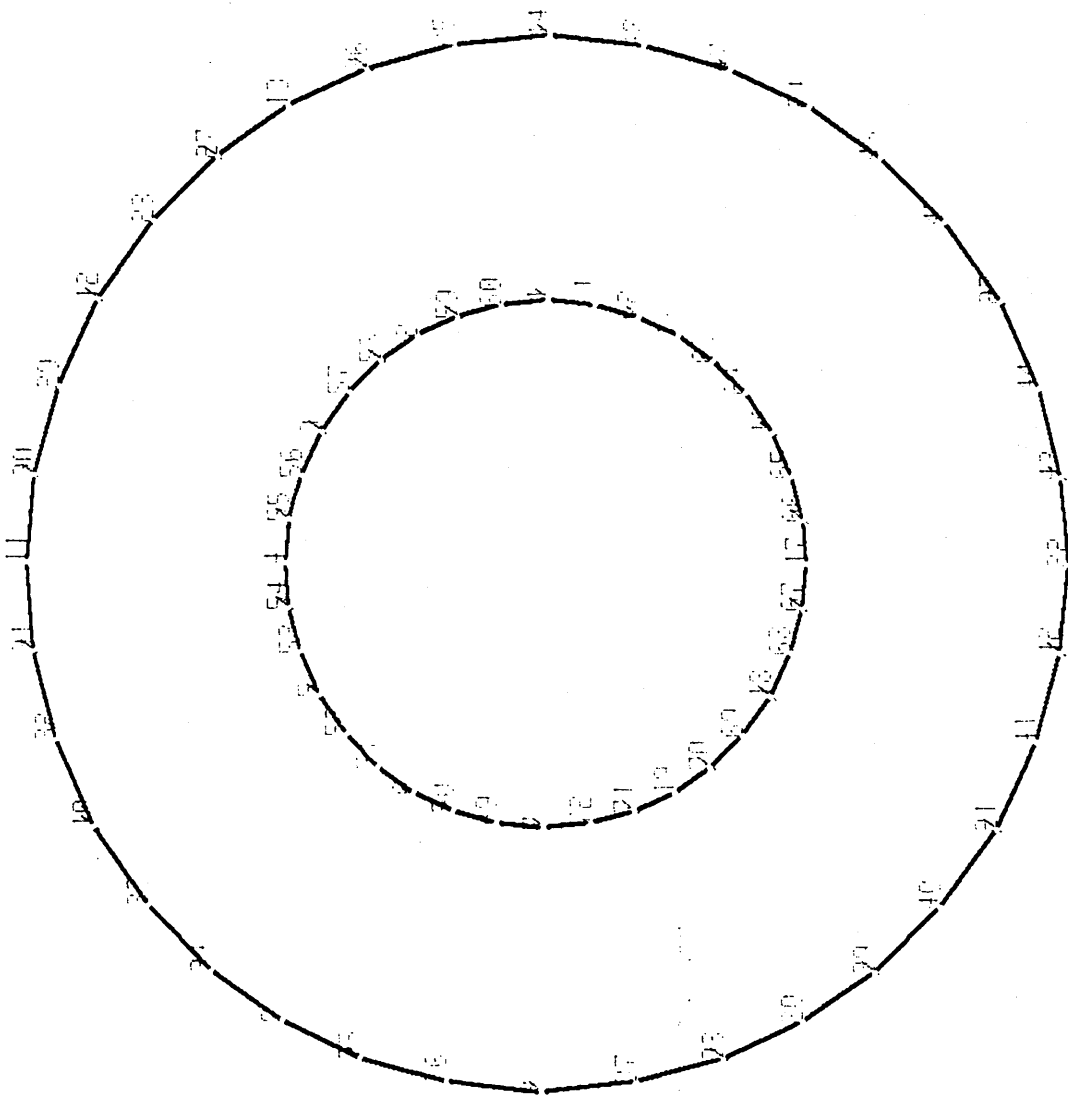
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Fig(7.41) Finite Element Mesh for Toroidal Example using
8-noded Serendipity Elements

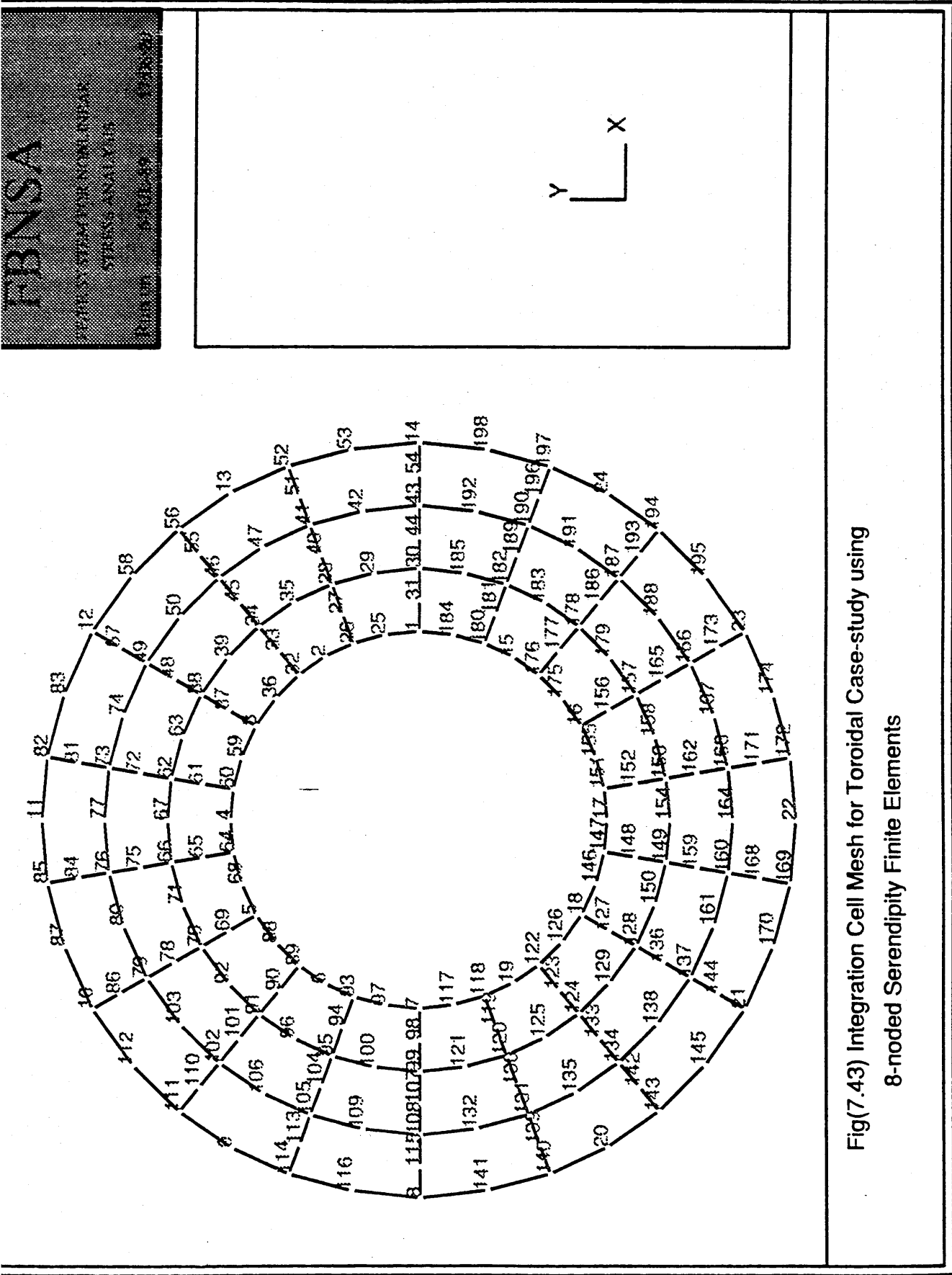
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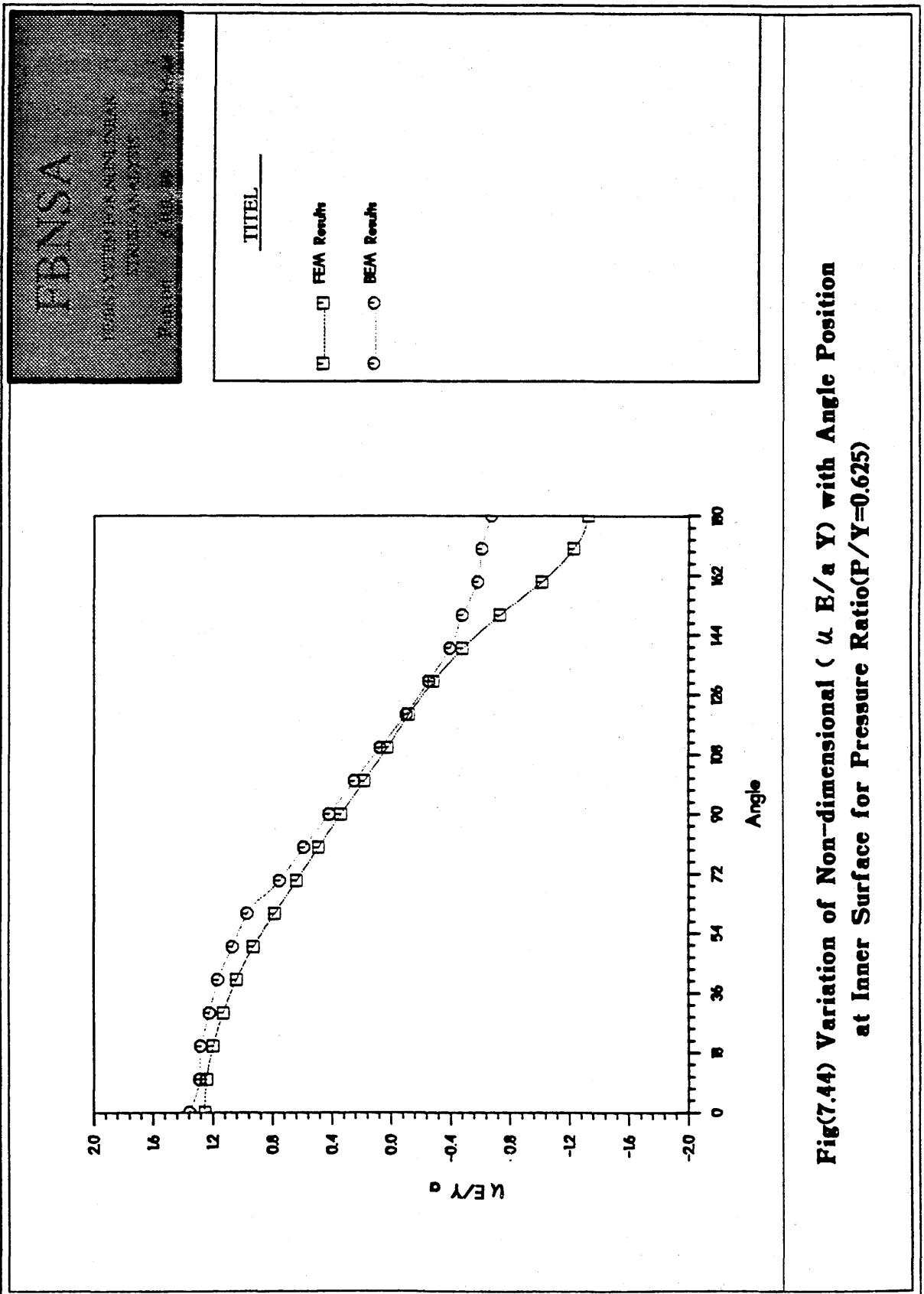
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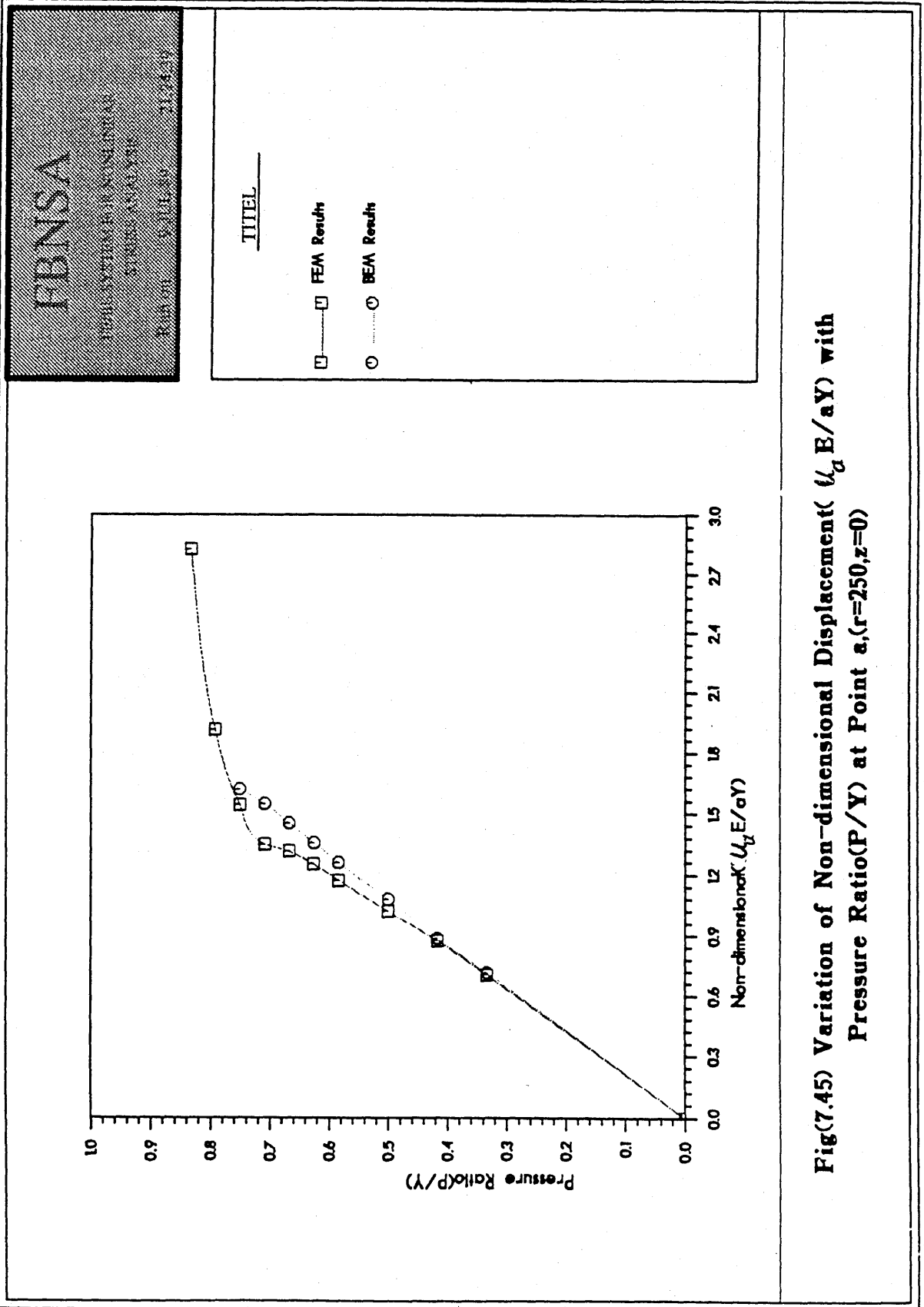


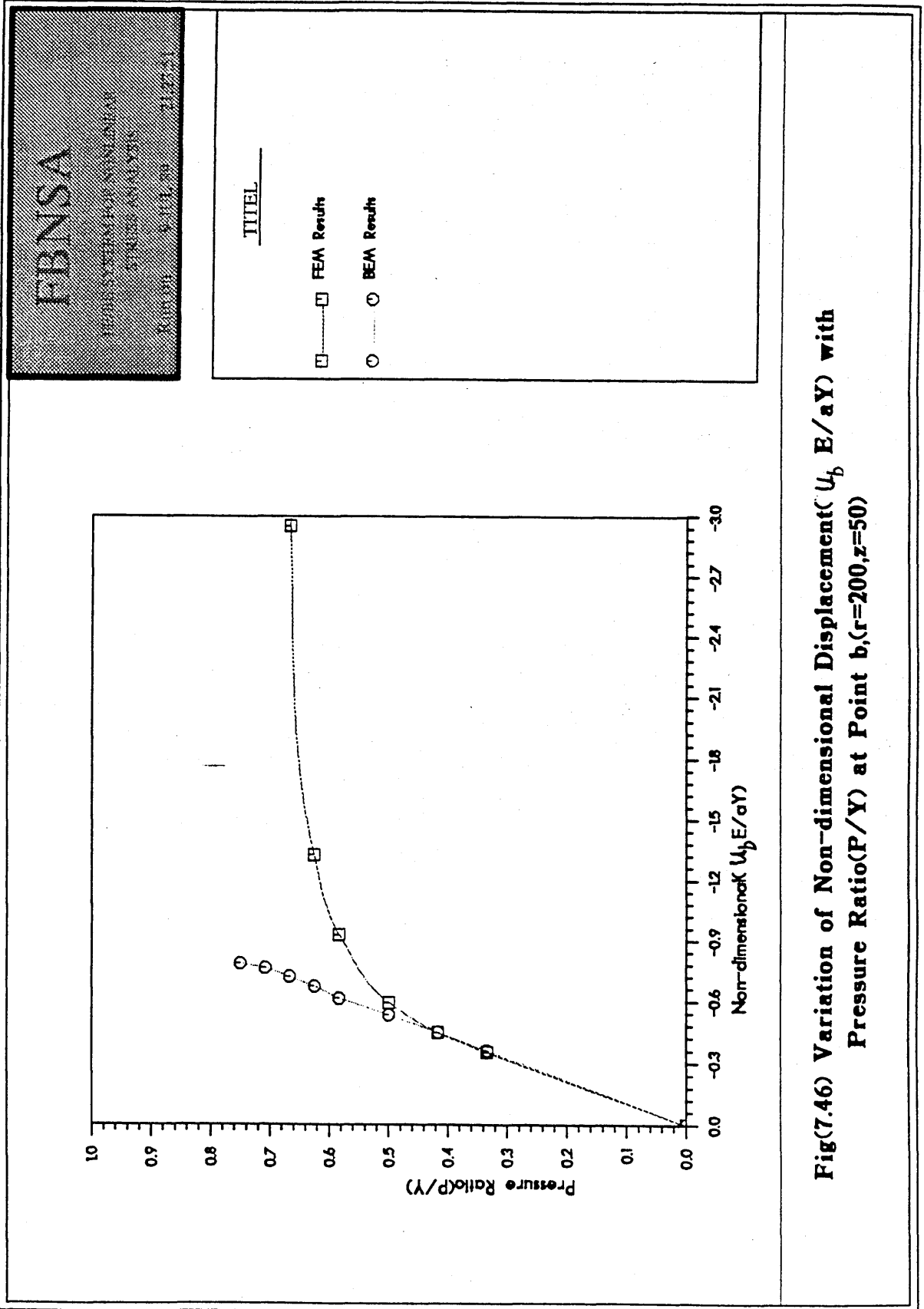
Fig(7.43) BEM Mesh for Toroidal Case-study using
3-noded Isoparametric Elements

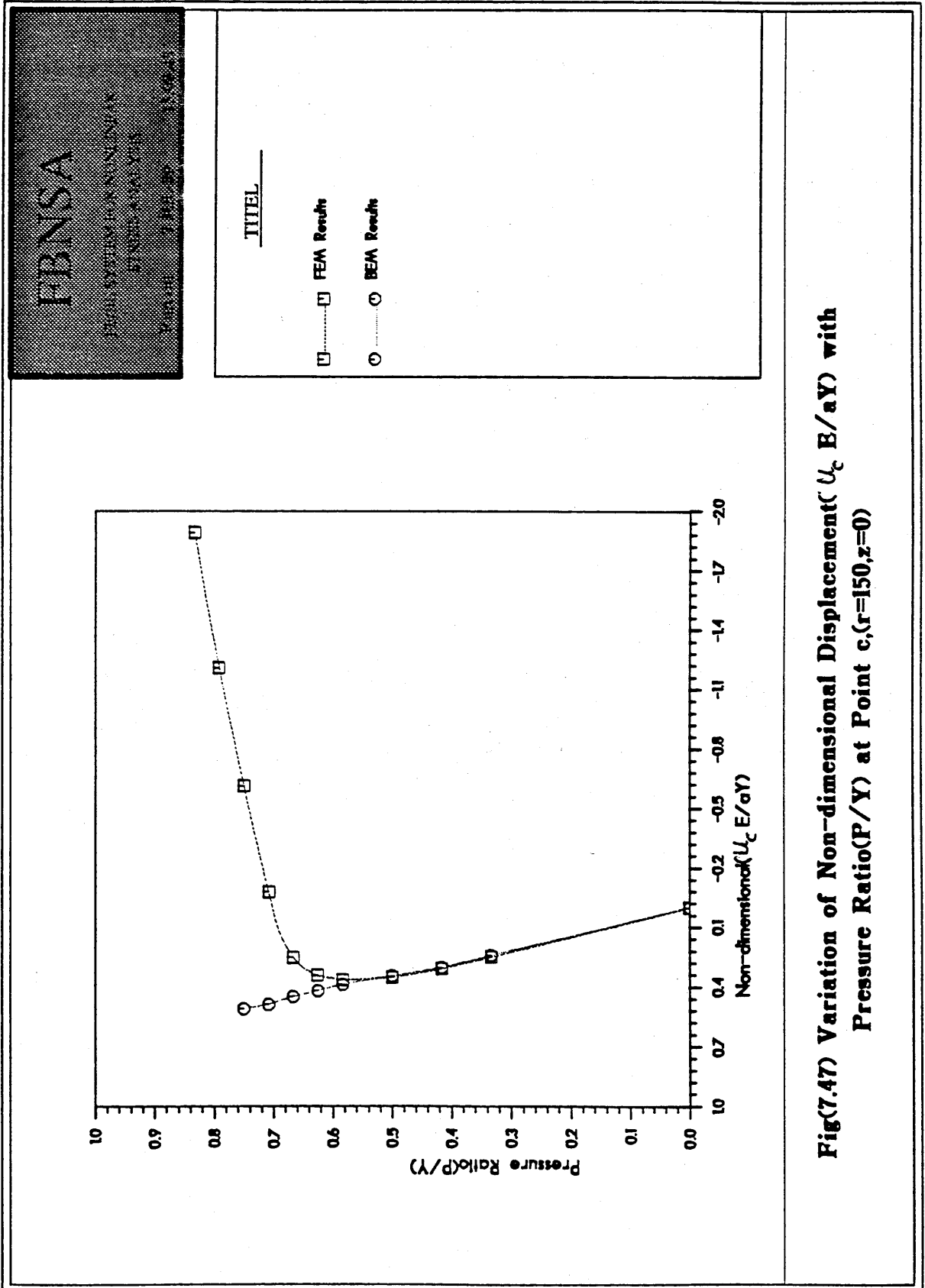


Fig(7.43) Integration Cell Mesh for Toroidal Case-study using
8-noded Serendipity Finite Elements









Load No.	P/Y	CPU time (sec)					
		FEM Solution			BEM Solution		
		Tresca Polynomial	Tresca Trigonometric	Tresca Trigonometric	Tresca Polynomial	Tresca Trigonometric	Tresca Trigonometric
1	.3333	22.89	23.5	4.3	4.35		
2	.41666	37.4	38.77	14.34	14.01		
3	.5	44.67	45.98	41.66	39.3		
4	.54166	59.33	59.97	91.04	85.19		
5	.58333	84.18	86.95	128.06	120.75		
6	.625	90.85	91.82	277.7	263.06		
7	.70833	138.64	139.15	683.58	650.55		
8	.75	198.19	200.73	711.7	670		

Table(7.1) CPU Time Required for each Load Increment When using the Tresca
Polynomial and Tresca Trigonometric Yield Criterion
(Error in BEM= 1.0e-8, FE= 1.0 e-8)

Load No.	σ_r	σ_θ	τ_{rz}
1	-7.608357	14.27502	2.000000
2	-11.03446	16.66503	2.070447
3	-13.79300	13.91508	0.5046385
4	-16.39584	11.31697	-2.520069

a)Results from NAFEMS[138]

Load No.	σ_r	σ_θ	τ_{rz}
1	-7.608357	14.27502	2.000000
2	-11.05071	16.64784	2.028756
3	-13.79133	13.91508	0.5777365
4	-16.37780	11.33472	-2.410570

b)Results from FEM axisymmetric program

Table(7.2) Stress components at inner Gauss points for material
with H=0 (Axisymmetric Solution, Benchmark Test)

Load No.	σ_r	σ_θ	τ_{rz}
1	-0.7608357E+01	0.1427502E+02	0.2000000E+01
2	-0.1089933E+02	0.178872E+02	0.2350480E+01
3	-0.1961019E+02	0.2205203E+02	0.8896005E+00
4	-0.2723264E+02	0.3624982E+02	0.3751026E+01

a)Results from NAFEMS[138]

Load No.	σ_r	σ_θ	τ_{rz}
1	-0.7608357E+01	0.1427502E+02	0.2000000E+01
2	-0.1090363E+02	0.1787748E+02	0.2335361E+01
3	-0.1961679E+02	0.2203865E+02	0.1695233E+01
4	-0.2722934E+02	0.3623482E+02	0.4303939E+01

b)Results from FEM axisymmetric program

Table(7.3) Stress components at inner Gauss points for material with
H=525 MPa (axisymmetric solution, Benchmark Test)

CHAPTER EIGHT

CONCLUSIONS

Chapter 8

CONCLUSIONS

It is clear from the previous chapters that the different techniques developed in this work have led to an accurate elasto-plastic analysis. The different accuracy measures employed with the BEM have improved the accuracy of the results for the cases of elasticity and low plasticity. Developed implementations of yield criteria and hardening rules have worked satisfactorily with both the FEM and BEM analyses.

The package developed in this work, with its own pre- and post-processors, has proved to be reliable. The wide range of isoparametric finite and boundary elements has proved useful for the selection of optimum finite-element, boundary-element, and integration cell meshes.

Comparing the FEM with BEM for elasto-plastic analysis, the following conclusions can be drawn:-

- 1-The FEM is more stable and reliable than the BEM, for practical engineering problems.
- 2-The BEM can be more economical than the FEM for elasto-plastic analysis of problems with small local plastic zones.
- 3-For large plastic deformation, the BEM program given in this work is unreliable
- 4-The BEM is still not a fool-proof method and requires an experienced user so as to obtain reliable answers.

Chapter 8

Recomendations for Future Work

- 1-More BEM research is required so as to make the method more reliable and fool-proof. The accuracy of the BEM elasto-plastic solution may be improved if the stresses can be evaluated accurately at points very near to the boundary.
- 2-With little effort, the programming package can be modified so as to deal with unloading and reloading problems.
- 3-The programming module for elasto-plastic analysis has already been designed for fully 3-dimensional problems. The FEM and BEM programs can, therefore, be developed so as to include 3-dimensional elements without need to change the present plastic module.
- 4-If it is thought that the BEM will always be inferior to the FEM for elasto-plastic analysis, then a hybrid procedure may be optimum. The FEM formulation could be employed within expected local plastic zones, whilst a B.I.E formulation could be used for the remaining elastic part of the structure.
- 5-The programming package can be interfaced with powerful pre- and post-processors, such as I-DEAS, so as to obtain high-quality graphical output or to facilitate data preparation.

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