Cranfield

College of Aeronautics Report No. 9005
April 1990

CRANFIELD INSTITUTE OF TECHNOLOGY
LIBRARY

Riemann Problems and the WAF Method for the Two-Dimensional Shallow Water Equations

E F Toro

The Department of Aerodynamics
College of Aeronautics
Cranfield Institute of Technology
Cranfield. Bedford MK43 OAL. England

3 8006 10128 6857

Cranfield

College of Aeronautics Report No. 9005
April 1990

CRANFIELD INSTITUTE OF TECHNOLOGY
LIBRARY

Cranfield Information and Library Service
Cranfield University
Bedford MK43 0AL
(01234) 754444

£8.00

"The views expressed herein are those of the authors alone and do not necessarily represent those of the Institute".

RIEMANN PROBLEMS AND THE WAF METHOD FOR THE TWO-DIMENSIONAL SHALLOW WATER EQUATIONS

E.F.TORO *

Aerodynamics Department, College of Aeronautics,
Cranfield Institute of Technology, England.

ABSTRACT

An exact Riemann solver for the shallow water equations along with several approximate Riemann solvers are presented. These solutions are used locally to compute numerically the global solution of the general initial boundary value problem for the shallow water equations. The numerical method used is the weighted average flux method (WAF) proposed by the author in Ref. 3. This is a conservative, shock capturing high resolution method. For shallow water flows where non-linear effects are important or where abrupt changes (hydraulic jumps) are to be expected the present algorithms can be useful in practice.

* Part of this work was carried out while the author was a visitor of the Mathematics Department, University of Trento, Italy.

CONTENTS

1. Introduction

2. Derivation of the shallow-water equations

3. The Riemann problem
   - 3.1 An exact Riemann solver
   - 3.2 Approximate Riemann solvers
     - the two-rarefaction approximation
     - the two-shock approximation
     - Roe-type approximations
     - HLL-type approximations

4. The WAF method
   - 4.1 Formulation of the WAF method
   - 4.2 Sonic flow
   - 4.3 A TVD version of WAF
   - 4.4 The CFL condition
   - 4.5 Boundary conditions
     - reflective boundaries
     - transmissive boundaries
   - 4.6 Extensions of the method
     - source terms
     - multidimensions
   - 4.7 An algorithm for the one-dimensional case.

5. Test problems

6. Conclusions

7. Appendix

Figures

1. INTRODUCTION

A wide variety of physical phenomena are governed by the shallow water equations. Some examples are tides in oceans, breaking of waves in shallow beaches, open-channel flow problems such as roll waves, flood waves in rivers and surges. Also, the equations can be reinterpreted and used to model flows in the atmosphere.

The shallow water equations are a set of non-linear hyperbolic equations and are an approximation to the full free-surface gravity flow problem with viscosity and surface tension effects neglected. The key assumption contained in the shallow-water equations is that the vertical component of the acceleration of the water particles has a negligible effect on the pressure, or equivalently that the pressure is given as in hydrostatics. The same governing equations result as the approximation of lowest order in a perturbation procedure. This involves a formal development of all quantities in powers of a parameter ϵ which is the ratio of the water depth to some other characteristic length associated with the horizontal direction.

The second approach to deriving the shallow water equations makes evident the role played by the undisturbed water depth in determining the accuracy of the approximation. More details can be found in Ref. 1.

The non-linear character of the equations means that the use of analytical techniques to solve them can only be successful in very special circumstances. Numerical methods must be employed to obtain solutions to realistic problems of scientific or engineering interest. But it is the hyperbolic character of the shallow water equations that makes the mathematical and numerical problems of finding solutions difficult.

Hyperbolic equations admit, in addition to smooth or classical solutions, discontinuous solutions. Even for the case in which the initial data is smooth everywhere, the non-linear character combined with the hyperbolic type of the equations can lead to

discontinuous solutions in a finite time. In the field of Gas Dynamics, the discontinuities are associated with shock waves and contact surfaces. In the context of the shallow water equations the discontinuities are associated with hydraulic jumps and bores in water or the propagation of sharp fronts in the atmosphere.

For a very large variety of flow situations one may assume that solutions remain smooth for all times. The development and appliction of numerical methods for the shallow water equations in these circumstances form an impressive body of research work. An up-to-date survey in this area is contained in the paper by Casulli (Ref. 2).

In this paper we are concerned with numerical methods for simulating flow situations in which discontinuities are present and are important to model. In particular, we shall describe the application of the weighted average flux (WAF for short) method to the two-dimensional shallow water equations. This is a shock-capturing, high-resolution conservative method that is based on solutions to local Riemann problems. The WAF approach was presented by this author in Ref. 3, as applied to systems of non-linear hyperbolic conservation laws. Applications of the method to gas-dynamics related problems with shock waves have been very successful (refs. 4 - 5). We can now use the resulting experience to exploit the specific features of the shallow water equations to formulate simple and robust numerical procedures based on the WAF principle.

This paper is organised as follows: Section 2 contains a derivation of the two-dimensional shallow water equations in conservative form (with "source" terms). In section 3 we formulate the Riemann problem, present an exact solver and a variety of approximate solution procedures. In section 4 we present the WAF numerical method and its practical implementation for the shallow water equations. In section 5 we apply the method to test problems in one and two dimensions. Conclusions are drawn in section 6.

2. DERIVATION OF THE SHALLOW WATER EQUATIONS

This section follows closely the presentation contained in the book by Stoker (Ref. 1) and that in the report by Glaister (Ref. 7). We consider a flow situation as depicted in Fig. 1.

The x - z plane is the horizontal plane that coincides with the position of the undisturbed depth of the water, which we denote by $h = h(x, z)$. The bottom of the channel, or sea, is given by $y = h(x,z)$ and the elevation of the free surface of the water by $y = \eta(x,z,t)$. The vertical direction is given by the y-axis..

We begin by considering the equations of motion in three dimensions in Lagrangian form

$$\left.\begin{aligned} -\frac{1}{\rho} p_x + X &= R_{(x)} \\ -\frac{1}{\rho} p_y + Y &= R_{(y)} \\ -\frac{1}{\rho} p_z + Z &= R_{(z)} \end{aligned}\right\} \qquad (1)$$

where ρ is the density (assumed constant), p is pressure, $T = (X,Y,Z)$ is the body force vector and $R = (R_{(x)}, R_{(y)}, R_{(z)})$ is the acceleration vector. For the case of interest here $T = (0, -g, 0)$ where g is the acceleration due to gravity. The subscript x denotes partial differentiation with respect to x and (x) denotes component in the x-direction. The meaning for the other subscripts is analogous. It is more useful to express equations (1) in Eulerian form via the Euler variables u, v, w, which are the components of the velocity field $V = (u,v,w)$. To this end it is convenient to introduce the <u>particle derivative</u> of a function $Q(x,y,z,t)$ associated with a particle as

$$\frac{DQ}{Dt} = u Q_x + v Q_y + w Q_z + Q_t \qquad (2)$$

The acceleration of a particle is the vector

$$R = (R_{(x)}, R_{(y)}, R_{(z)}) = \left(\frac{Du}{Dt}, \frac{Dv}{Dt}, \frac{Dw}{Dt} \right)$$

so that equations (1) become

$$\left.\begin{aligned} \frac{Du}{Dt} &= u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x \\ \frac{Dv}{Dt} &= v_t + uv_x + uv_y + wv_t = -\frac{1}{\rho} p_y - g \\ \frac{Dw}{Dt} &= w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z \end{aligned}\right\} \qquad (3)$$

Another equation is given by the principle of conservation of mass, or continuity equation. For incompressible fluids (ρ = constant) this becomes

$$u_x + v_y + w_z = 0 \qquad (4)$$

There are now four equations and four unknowns, namely u, v, w and p. In principle, given initial and boundary conditions one shuld be able to solve (3) - (4) for u, v, w and p as functions of space x, y, z and time t.

Boundary conditions are required at the bottom $y = -h(x,z)$ and on the free surface under gravity $y = \eta(x,z,t)$. Two boundary conditions are given for the free surface

$$\left.\begin{aligned} &\frac{D}{Dt}(\eta - y) = 0 && (5a) \\ &p = p_{atm} && (5b) \end{aligned}\right\} \quad y = \eta(x,z,t)$$

For the bottom boundary one takes

$$\frac{D}{Dt}(h + y) = 0 \quad , \quad y = -h(x,z) \tag{6}$$

In equation (5b) p_{atm} is the atmospheric pressure, which for convenience is assumed to be zero. Equation (6) states that the normal component of velocity vanishes, i.e. no flow through the bottom.

The shallow water equations result from the assumption that the vertical component of the acceleration $R_{(y)}$ is negligible, i.e. $\frac{Dv}{Dt} \equiv 0$. Thus the second equation in (3) gives

$$p = g\rho(\eta - y) \tag{7}$$

This is called the hydrostatic pressure relation.
Differentiation of p with respect to x and z gives

$$p_x = g\rho\eta_x \tag{8}$$

$$p_z = g\rho\eta_z \tag{9}$$

i.e. both p_x and p_z are independent of y and so the x and z components of the acceleration $\frac{Du}{Dt}$ and $\frac{Dw}{dt}$ in equations (3) are independent of y. Thus the x and z components of velocity are also independent of y for all t if they were at a given time, t = 0, say.

Hence the first and third equations in (3), after using (8) - (9), become

$$u_t + uu_x + wu_z = -g\eta_x \tag{10}$$

$$w_t + uw_x + ww_z = -g\eta_z \tag{11}$$

Next, we integrate the continuity equation (4) with respect to y to obtain

$$\int_{-h}^{\eta} (u_x)dy + \int_{-h}^{\eta} (w_z)\, dy + v\,\Big|_{-h}^{\eta} = 0 \qquad (12)$$

Expanding the boundary conditions (5a) and (6) according to (2) we obtain

$$(\eta_t + u\eta_x + w\eta_z - v)\big|_{y = \eta} = 0 \qquad (13)$$

$$(uh_x + wh_z + v)\big|_{y = -h} = 0 \qquad (14)$$

which, if used in (12), give

$$\int_{-h}^{\eta} (u_x)\, dy + \int_{-h}^{\eta} (w_z)\, dy + \eta_t + (u|_{y = \eta})\, \eta_x$$

$$+ \ (w|_{y = \eta})\, \eta_z + (u|_{y = -h})\, h_x + (w|_{y = -h})\, h_z = 0 \qquad (15)$$

Equation (15) can finally be expressed as

$$\eta_t + \frac{\partial}{\partial x}\int_{-h}^{\eta} udy + \frac{\partial}{\partial z}\int_{-h}^{\eta} wdy = 0 \qquad (16)$$

This result follows by using the relations

$$\frac{\partial}{\partial x}\int_{-h(x,z)}^{\eta(x,z,t)} udy = \int_{-h}^{\eta} (u_x)dy + (u|_{y = \eta})\eta_x + (u|_{y = -h})h_x$$

$$\frac{\partial}{\partial z}\int_{-h}^{\eta} wdy = \int_{-h}^{\eta} (w_z)dy + (w|_{y = \eta})\eta_z + (w|_{y = -h})h_z$$

Equation (16) can be simplified further. This follows from the observation that both u and w are independent of y and so (16) becomes

$$\eta_t + [u(\eta + h)]_x + [w(\eta + h)]_z = 0 \qquad (17)$$

The governing two-dimensional shallow water equations are (10), (11) and (17). For the purpose of this paper it is necessary to express these equations in conservation law form.

Since h(x,z) is independent of time t $h_t = 0$ and so equation (17) can be re-written as

$$(\eta + h)_t + [u(\eta + h)]_x + [w(\eta + h)]_z = 0 \qquad (18)$$

which if multiplied by u and added to (10), premultiplied by η + h, gives

$$[u(\eta + h)]_t + [u^2(\eta + h)]_x + [uw(\eta + h)]_z = -g(\eta + h)\eta_x \qquad (19)$$

In an analogous way equations (11) and (17) give

$$[w(\eta + h)]_t + [uw(\eta + h)]_x + [w^2(\eta + h)]_z = -g(\eta + h)\eta_z \qquad (20)$$

The right-hand side term of equation (19) can be rewritten as

$$-g(\eta + h)\eta_x = g(\eta + h)h_x - \frac{1}{2} g[(\eta + h)^2]_x$$

and so (19) becomes

$$(\phi u)_t + [\phi u^2 + \frac{\phi^2}{2}]_x + [\phi uw]_z = g\phi h_x \qquad (21)$$

Similarly, equation (20) becomes

$$(\phi w)_t + [\phi uw]_x + [\phi w^2 + \frac{\phi^2}{2}]_z = g\phi h_z \qquad (22)$$

where

$$\phi = g(\eta + h) \qquad (23)$$

The two-dimensional shallow water equations are now written in conservation form with source terms, i.e. equations (18), (21) and (22) can be expressed as

$$U_t + F_x + G_z = S(U) \tag{24}$$

with

$$U = \begin{bmatrix} \phi \\ \phi u \\ \phi w \end{bmatrix}, \quad F = \begin{bmatrix} \phi u \\ \phi u^2 + \frac{\phi^2}{2} \\ \phi uw \end{bmatrix}, \quad G = \begin{bmatrix} \phi w \\ \phi uw \\ \phi w^2 + \frac{\phi^2}{2} \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ g\phi h_x \\ g\phi h_z \end{bmatrix}$$

In equation (24) U is the vector of "conserved" variables, F(U) and G(U) are flux vectors and S(U) is a "source" term vector.

For many applications there will be additional terms in the vector S(U) to account, say, for Coriolis forces, bottom friction, etc. The numerical solution procedure described in sections 3 to 5 will deal essentially with the homogeneous part of equation (24). The treatment of the source terms is a relatively standard process via time operator splitting. Moreover, the two-dimensional homogeneous problem will be solved using space splitting, or method of fractional steps, whereby the two-dimensional problem is reduced to a sequence of one-dimensional problems. It is therefore sufficient to consider

$$U_t + F_x = 0 \tag{25}$$

This homogenous conservation law can be rewritten in <u>integral form</u> as

$$\oint (Udx - Fdt) = 0 \tag{26}$$

which is more general than (25), for it admits discontinuous solutions.

3. THE RIEMANN PROBLEM

The numerical technique WAF described in this paper uses solutions to local Riemann problems to evolve the global solution in time. For simplicity we shall consider first the Riemann problem for the purely one-dimensional case

$$\begin{bmatrix} \phi \\ \phi u \end{bmatrix}_t + \begin{bmatrix} \phi u \\ \phi u^2 + \frac{\phi^2}{2} \end{bmatrix}_x = 0 \ , \ t > 0 \ , \ -\infty < x < \infty \qquad (27)$$

with ϕ as given by (23)

The Riemann problem for (27) is the initial value problem (IVP) for (27) with piece-wise constant initial data.

$$U(0, x) = \begin{cases} U_L \ , & x < 0 \\ U_R \ , & x > 0 \end{cases} \qquad (28)$$

The subscripts L and R denote left and right states respectively.

It is easy to see that the system (27) is hyperbolic with real associated eigenvalues

$$\lambda_1 = u - a \ , \quad \lambda_2 = u + a \qquad (29)$$

where

$$a = \sqrt{\phi} = \sqrt{g(\eta + h)} \qquad (30)$$

denotes the "sound" speed, or celerity.

The IVP (27) - (28) can be solved exactly. This is possible because of the simplicity of the initial condition (28). The solution can be represented in the x - t plane as shown in Fig. 2. There are two waves, one travels to the left and the other to the right. They can either be shock or rarefaction waves. Fig. 3 shows all four possible wave patterns. The waves separate three constant states, namely U_L (data) to the left of the left wave, U_R (data) to the right of the right wave and a new constant state U^* valid in the wedge between the left and right waves. We shall denote this wedge by the star region. The solution through rarefaction waves, at any given time t^*, varies smoothly with x, while through shock waves it jumps discontinuously.

3.1 AN EXACT RIEMANN SOLVER

The main step in solving the Riemann problem is to find the constant state U^* in the star region. This depends on appropriate relations through the left and right waves connecting U^* to the data states U_L and U_R respectively. Such relations depend on the type of waves present. But this is not known a priori. It could be any of the four patterns shown in Fig. 3. The determination of the wave type is part of the solution procedure, which is iterative.

The exact Riemann solver to be described next follows the ideas behind the Riemann solver for covolume gases presented in Ref. 7. Essentially, we derive a single (non-linear) algebraic equation for ϕ^*, the value of ϕ in the star region, namely

$$f(\phi^*) = f_L(\phi^*, \phi_L) = f_R(\phi^*, \phi_R) + u_R - u_L = 0 \qquad (31)$$

Here u_L amd u_R are velocity on left and right states respectively (data); f_L and f_R are functions connecting the left and right states respectively, to the star region. In what follows we shall derive expressions for the function f_L and f_R. As already

mentioned, they depend on the type of waves present. We shall consider each case separately.

A LEFT SHOCK WAVE

Here we suppose that the left wave is a shock wave travelling with velocity S_L. It is convenient to transform coordinates to a stationary frame as shown in Fig. 4, where

$$v_L = u_L - S_L \ , \quad v^* = u^* - s_L \tag{32}$$

In this frame the conservation equations (27) give

$$\phi_L v_L = \phi^* v^* = M_L \tag{33}$$

$$\phi_L v_L^2 + \frac{\phi_L^2}{2} = \phi^* v^{*2} + \frac{\phi^{*2}}{2} \tag{34}$$

Equation (34) can be rewritten as

$$(\phi_L v_L)\, v_L - (\phi^* v^*)\, v^* = \frac{\phi^{*2} - \phi_L^2}{2}$$

which after inserting M_L from (33) gives

$$M_L = -\frac{(\phi^{*2} - \phi_L^2)}{2(v^* - v_L)} = -\frac{(\phi^{*2} - \phi_L^2)}{2(u^* - u_L)} \tag{35}$$

From equation (33) again we have

$$v^* = M_L/\phi^* \quad \text{and} \quad v_L = M_L/\phi_L$$

and so (35), after some manipulations, becomes

$$M_L = \left[\frac{1}{2} \phi_L \phi^* (\phi_L + \phi^*) \right]^{1/2} \qquad (36)$$

From equation (35) we obtain

$$u^* = u_L - \frac{1}{2} \frac{(\phi^{*2} - \phi_L^2)}{M_L}$$

or

$$u^* = u_L - f_L (\phi^*, \phi_L) \qquad (37)$$

where

$$f_L(\phi^*, \phi_L) = (\phi^* - \phi_L) \left[\frac{\phi^* + \phi_L}{2\phi^* \phi_L} \right]^{1/2} \qquad (38)$$

We have obtained the function f_L for the case in which the left wave is a shock wave.

A RIGHT SHOCK WAVE

We proceed in a analogous way as in the case of a left shock. Suppose the right shock travels with speed S_R. In the stationary frame of reference we define

$$v^* = u^* - S_R \,, \quad v_R = u_R - S_R \qquad (39)$$

so that the conservation equations (27) become

$$\phi^* v^* = \phi_R v_R = - M_R \tag{40}$$

$$\phi^* v^{*2} + \frac{\phi^{*2}}{2} = \phi_R v_R^2 + \frac{\phi_R^2}{2} \tag{41}$$

Equations (39) - (41) give

$$M_R = \frac{(\phi^{*2} - \phi_R^2)}{2(u^* - u_R)} = \frac{(\phi^{*2} - \phi_R^2)}{2(v^* - v_R)} \tag{42}$$

or

$$M_R = \left[\frac{1}{2} \phi^* \phi_R (\phi^* + \phi_R) \right]^{1/2} \tag{43}$$

From equation (42) we obtain

$$u^* = u_R + \frac{1}{2} \frac{(\phi^{*2} - \phi_R^2)}{M_R}$$

or

$$u^* = u_R + f_R (\phi^*, \phi_R) \tag{44}$$

where

$$f_R(\phi^*, \phi_R) = (\phi^* - \phi_R)\left[\frac{\phi^* + \phi_R}{2\phi^*\phi_R}\right]^{1/2} \qquad (45)$$

This is the function f_R for the case of a right shock wave. Note that by subtracting (37) from (44) u^* is eliminated leaving a single algebraic equation of the form (31) for the unknown ϕ^*.

RAREFACTION WAVES

The case of rarefaction waves is most easily dealt with by rewriting equations (27) in non-conservative form as

$$2a_t + 2uu_x + au_x = 0$$

$$u_t + uu_x + 2aa_x = 0$$

and noting that the Riemann invariants (Ref. 8) are

$$u - 2a = I_L \text{ along curves } \frac{dx}{dt} = u - a \qquad (46)$$

$$u + 2a = I_R \text{ along curves } \frac{dx}{dt} = u + a \qquad (47)$$

where I_L and I_R are constant.

If the left wave is a rarefaction wave we can connect the left state U_L to the unknown state U^* across the left wave by transversing it with a wave of the family $\frac{dx}{dt} = u + a$, along which the right Riemann invariant I_R in (47) is constant, i.e.

$$u_L + 2a_L = u^* + 2a^*$$

Hence

$$u^* = u_L - f_L(\phi^*, \phi_L) \tag{48}$$

$$f_L = 2\left(\sqrt{\phi^*} - \sqrt{\phi_L}\right) \tag{49}$$

Similarly for a right rarefaction wave we have

$$u^* = u_R + f_R(\phi^*, \phi_R) \tag{50}$$

with

$$f_R = 2\left(\sqrt{\phi^*} - \sqrt{\phi_R}\right) \tag{51}$$

The equation (31) for the unknown ϕ^* is now completely determined. For convenience we summarise the expressions for f_L and f_R, namely

$$f_L = \begin{cases} 2\left(\sqrt{\phi^*} - \sqrt{\phi_L}\right), & \text{left rarefaction, } \phi^* \le \phi_L \\ \left(\phi^* - \phi_L\right)\sqrt{\dfrac{\phi^* + \phi_L}{2\phi^*\,\phi_L}}, & \text{left shock, } \phi^* > \phi_L \end{cases} \tag{52}$$

$$f_R = \begin{cases} 2\left(\sqrt{\phi^*} - \sqrt{\phi_R}\right) & \text{, right rarefaction, } \phi^* \le \phi_R \\ \left(\phi^* - \phi_R\right)\sqrt{\dfrac{\phi^* + \phi_R}{2\phi^*\,\phi_R}} & \text{, right shock , } \phi^* > \phi_R \end{cases} \tag{53}$$

The non-linear equation (31) can easily be solved using a Newton Raphson iteration procedure

$$\phi^*_{(k)} = \phi^*_{(k-1)} + \sigma_{(k-1)} \tag{54}$$

where

$$\sigma_{(k)} = - f_{(k)} / f'_{(k)} \tag{55}$$

with k denoting the iteration number, and $f' = \dfrac{df}{d\phi^*} = f'_L + f'_R$ denoting the first derivative of f with respect to ϕ^*. From (52) - (53) f_L' and f_R' are found to be

$$f'_L = \begin{cases} \dfrac{1}{\sqrt{\phi^*}} & \text{, left rarefaction} \\ D_L - \dfrac{(\phi^* - \phi_L)}{4D_L\,\phi^{*2}} & \text{, left shock} \end{cases} \tag{56}$$

$$f'_R = \begin{cases} \dfrac{1}{\sqrt{\phi^*}} & , \text{ right rarefaction} \\ D_R - \dfrac{(\phi^* - \phi_R)}{4D_R\,\phi^{*2}} & , \text{ right shock} \end{cases} \tag{57}$$

where

$$D_L = \sqrt{\frac{\phi^* + \phi_L}{2\phi^*\,\phi_L}} \quad , \qquad D_R = \sqrt{\frac{\phi^* + \phi_R}{2\phi^*\,\phi_R}} \tag{58}$$

The iteration (54) requires an initial guess $\phi^*_{(0)}$ for ϕ^*. One could take $\phi_{(0)} = \frac{1}{2}(\phi_L + \phi_R)$ for instance, which although simple is not very accurate. The initial guess

$$\phi^*_{(0)} = \left[\frac{1}{2}\left(\sqrt{\phi_L} + \sqrt{\phi_R}\right) + \frac{1}{4}\left(u_L - u_R\right)\right]^2 \tag{59}$$

is found to be very accurate for most pairs of states (U_L, U_R) in a typical flow field. However, it is costly to compute.

The iteration (54) is stopped whenever

$$\sigma = \frac{|\phi^*_{(k)} - \phi^*_{(k-1)}|}{\phi^*_{(k)}} \leq \text{TOL}$$

where TOL is a chosen tolerance. For most purposes TOL = 10^{-4} is good enough. Once ϕ^* is known, u^* can be computed from (48) or (50) or a combination of the two as

$$u^* = \frac{1}{2}(u_R + u_L) + \frac{1}{2}(f_R - f_L)$$

SHOCK WAVE SPEEDS

The speed S_L for a left shock can be calculated once ϕ^* is known. From equation (33) we have $\phi_L v_L = M_L$, where M_L, given by equation (36), is now known. From equation (32) $v_L = u_L - S_L$ and so $\phi_L (u_L - S_L) = M_L$, from which the speed S_L follows as

$$S_L = u_L - M_L/\phi_L \qquad (60)$$

Similar arguments give the speed S_R for a right shock as

$$S_R = u_R + M_R/\phi_R \qquad (61)$$

SOLUTION INSIDE RAREFACTION FANS

Suppose the left wave is a rarefaction wave. See Fig. 5. The head of the fan is given by the ray $\frac{dx}{dt} = u_L - a_L$ and the tail is given by the ray $\frac{dx}{dt} = u^* - a^*$. Suppose we wish to find the solution inside the fan at the point $(\hat{x}, \hat{t})$ say. Denote by u and a the unknown values for the particle speed and the celerity, or "sound speed", in analogy with gas dynamics. Consider the ray passing through (0,0) and $(\hat{x}, \hat{t})$ for which $\frac{dx}{dt} = u - a$. Then

$$\frac{\hat{x}}{\hat{t}} = u - a \qquad (62)$$

Also, using the right Riemann invariant (47) across the left rarefaction we obtain

$$u + 2a = u_L + 2a_L \tag{63}$$

the solution of (62) - (63) for u and a is

$$u = \frac{1}{3} \left(u_L + 2a_L + 2\hat{x}/\hat{t} \right) \tag{64}$$

$$a = \frac{1}{3} \left(u_L + 2a_L - \hat{x}/\hat{t} \right) \tag{65}$$

For the case of a right rarefaction the head is given by $\frac{dx}{dt} = u_R + a_R$ and the tail by $\frac{dx}{dt} = u^* + a^*$. The solution at a point $(\hat{x}, \hat{t})$ inside the fan is found to be

$$u = \frac{1}{3} (u_R - 2a_R + 2\hat{x}/\hat{t}) \tag{66}$$

$$a = \frac{1}{3} (-u_R + 2a_R + \hat{x}/\hat{t}) \tag{67}$$

The exact solution of the Riemann problem for the one-dimensional shallow water equations (27) is now complete.

THE RIEMANN PROBLEM FOR THE SPLIT TWO-DIMENSIONAL CASE

When extending the WAF method to two space dimensions one considers the associated Riemann problem for the equations in one direction only, say x. Then the equations to consider are

$$\begin{bmatrix} \phi \\ \phi u \\ \phi w \end{bmatrix}_t + \begin{bmatrix} \phi u \\ \phi u^2 + \frac{\phi^2}{2} \\ \phi uw \end{bmatrix}_x = 0 \tag{68}$$

This system is hyperbolic and has three distinct eigenvalues

$$\lambda_1 = u - a \ , \quad \lambda_2 = u \ , \quad \lambda_3 = u + a \tag{69}$$

It is not difficult to see that the third equation involving the z - component of velocity w is decoupled from the previous two equations; w is passively advected by the speed u. The variables u and ϕ are constant (as in one-dimensional case) in the star region between the acoustic waves $\lambda_1 = u-a$ and $\lambda_3 = u+a$, while w only changes (discontinuously) across $\frac{dx}{dt} = \lambda_2 = u$. So the exact solution of the Riemann problem for (68) with data (ϕ_L, u_L, w_L) and (ϕ_R, u_R, w_R) is exactly the same as in the one-dimensional case, for the unknowns ϕ^* and u^*, with the addition

$$w(x,t) = \begin{cases} w_L, & \text{if } x/t < u^* \\ w_R, & \text{otherwise} \end{cases} \tag{70}$$

This particularly simple structure of the solution of the Riemann problem associated with the two-dimensional shallow water equations will be exploited when searching for approximations.

3.2 APPROXIMATE RIEMANN SOLVERS

Although the exact Riemann solver presented in the previous section is very efficient, the search for approximate Riemann solvers is still justified. One would like to have to perform less and simpler operations. Perhaps more importantly, it is desirable to eliminate altogether the iteration procedure contained in the exact solver. This can result in significant additional computing savings in vector machines. The great advantage of using exact solvers is the added robustness of the codes under severe conditions. In the field of Gas Dynamics all the approximate Riemann solvers known to the author have limitations under certain circumstances. For the shallow water

equations however, it appears as if certain approximations can be very robust and simple to compute.

We shall present several approximate Riemann solvers.

A TWO RAREFACTION APPROXIMATE SOLVER (TR)

This approximation results from assuming a priori that the two 'acoustic' waves present in the solution of the Riemann problem are both rarefaction waves, (see Figs. 2 and 3). An immediate consequence is that the functions f_L and f_R in equation (31) are those corresponding to rarefaction waves. Thus $f(\phi^*)$ in equation (31) becomes

$$2\left(\sqrt{\phi^*} - \sqrt{\phi_L}\right) + 2\left(\sqrt{\phi^*} - \sqrt{\phi_R}\right) + u_R - u_L = 0$$

from which the solution for ϕ^* follows directly as

$$\phi^* = \left[\frac{1}{2}\left(\sqrt{\phi_L} + \sqrt{\phi_R}\right) + \frac{1}{4}\left(u_L - u_R\right)\right]^2 \qquad (71)$$

The solution for u^* follows as

$$u^* = \frac{1}{2}(u_L + u_R) + \sqrt{\phi_L} - \sqrt{\phi_R} \qquad (72)$$

For the case of the split two-dimensional Riemann problem given by equation (68) one only needs to add the solution for w as given by equation (70).

The TR approximation is exact for the case of two rarefaction waves in the Riemann problem, as well as for combination of

rarefactions and "Mach" waves (waves of zero strength). It is also found to be quite accurate even for cases involving shock waves, as it will be seen later. More importantly, for a typical, one-dimensional calculation say, we may have to solve 100 Riemann problems at each time step. The most likely event is that only a couple of those Riemann problems may contain shock waves. The rest of the flow field can be accurately modelled by the two-rarefaction approximation.

CHOICE OF WAVE SPEEDS

The WAF method, to be described in section 4, requires the wave speeds and the states separated by the waves. In the TR approximation the acoustic waves are assumed to be rarefactions and thus there is ambiguity when having to choose the representative wave speeds. We find that

$$\begin{aligned} \lambda_1 &= \min \{ u_L - a_L , \; u^* - a^* \} \\ \lambda_2 &= u^* \\ \lambda_3 &= \max \{ u_R + a_R , \; u^* + a^* \} \end{aligned} \tag{73}$$

is the best choice one can make from the information available in the TR approximation.

THE TWO-SHOCK APPROXIMATION (TS)

This approximation results from assuming that the acoustic waves in the Riemann problem are both shock waves. The functions f_L and f_R in equation (31) are those corresponding to shocks in equations (52) - (53) and thus we obtain

$$(\phi^* - \phi_L)\sqrt{\frac{\phi^* + \phi_L}{2\phi^*\,\phi_L}} + (\phi^* - \phi_R)\sqrt{\frac{\phi^* + \phi_R}{2\phi^*\,\phi_R}} + u_R - u_L = 0 \quad (74)$$

Unfortunately, this is still a non-linear equation for ϕ^* and I have not been able to find a close-form solution for it. The advantage of using the TS approximation instead of the exact solver is that the logic associated with the type of waves is not present.

A possible further approximation to the TS approximation can be obtained by expanding (74) in a Taylor series about a point $\phi^*_{(0)}$ and retaining first order terms. The point $\phi^*_{(0)}$ can be taken from the TS approximation (71). We find that we still need to solve a quadratic equation with coefficients involving many operations. We do not regard this approach as useful, although the solution for ϕ^* is then direct.

Once ϕ^* is obtained from solving (74) u^* follows from equation (37), or (44) or a mean value

$$u^* = \frac{1}{2}(u_R + u_L) + \frac{1}{2}(f_R - f_L) \quad (75)$$

where, for consistency with the TS approximation, one takes f_L and f_R from the shock branches in (52) - (53).

The wave speeds λ_1, λ_2, λ_3 can be taken as those given by (73). Alternatively, one can test the ratios ϕ^*/ϕ_L , ϕ^*/ϕ_R. If either of these is greater than unity one could choose the exact shock-wave speeds S_L or S_R , given by (60) - (61), for λ_1 or λ_3.

For the two-dimensional split case, again we choose $\lambda_2 = u^*$ and $w(x,t)$ as given by (70).

ROE-TYPE APPROXIMATIONS

Glaister (ref. 7) derived a Roe-type approximation for the shallow water equations. There are two possible interpretations for this approximation. The first, is the original concept, due to Roe (Ref. 9). We shall denote this interpretation by RS1. This is the interpretation to be used in Roe's method. The second interpretation is new and can be used in the context of any Godunov-type methods, including WAF. We shall denote this approximation by RS2.

FIRST INTERPRETATION OF ROE'S SOLVER (RS1)

For the purpose of using Roe's method one must obtain the Roe approximation to the solution of the Riemann problem. This makes use of the fact that for a linear hyperbolic system of the form (25) the flux difference $\Delta F = F_R - F_L$ between a left (L) and right (R) states can be written as

$$\Delta F = \sum_{k=1}^{N} \alpha_k \lambda_k r_k \qquad (76)$$

where N is the number of waves in the Riemann problem with data U_L and U_R, α_K are the wave strengths, λ_k are the wave speeds or eigenvalues of the Jacobian matrix $J = \partial F/\partial U$ and r_k are the right eigenvectors corresponding to the eigenvalues λ_k . Note that (76) is in general a vector equation.

For linear systems with constant coefficients the Jacobian matrix J is constant. Roe's approximation is based on the assumption that the matrix J is still constant for non-linear systems. It is then a question of constructing appropriate averaged values $\tilde{U}$ in terms of the data states U_L and U_R such that $\tilde{J} = \tilde{J}(\tilde{U}) = \tilde{J}(U_L, U_R)$.

Glaister (Ref. 7) provided the Roe averaged values for the two-dimensional shallow water equations. We simply state the result here.

$$\tilde{u} = \frac{\sqrt{\phi_L}\ u_L + \sqrt{\phi_R}\ u_R}{\sqrt{\phi_L} + \sqrt{\phi_R}} \tag{77a}$$

$$\tilde{w} = \frac{\sqrt{\phi_L}\ w_L + \sqrt{\phi_R}\ w_R}{\sqrt{\phi_L} + \sqrt{\phi_R}} \tag{77b}$$

$$\tilde{a} = \sqrt{\frac{1}{2}\left(\phi_L + \phi_R\right)} \tag{77c}$$

$$\tilde{\phi} = \sqrt{\phi_L\ \phi_R} \tag{77d}$$

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{a}\ , \quad \tilde{\lambda}_2 = \tilde{u}\ , \quad \tilde{\lambda}_3 = \tilde{u} + \tilde{a} \tag{77e}$$

$$\left.\begin{aligned} \tilde{\alpha}_1 &= \frac{1}{2}\Delta\phi - \frac{1}{2}\ \frac{\tilde{\phi}\ \Delta\ u}{\tilde{a}} \\ \tilde{\alpha}_2 &= \tilde{\phi}\ \Delta\ w \\ \tilde{\alpha}_3 &= \frac{1}{2}\Delta\phi + \frac{1}{2}\ \frac{\tilde{\phi}\ \Delta\ u}{\tilde{a}} \end{aligned}\right\} \tag{77f}$$

where $\Delta\phi = \phi_R - \phi_L$, $\Delta u = u_R - u_L$, $\Delta w = w_R - w_L$.

The right eigenvectors are

$$\tilde{e}_1 = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{a} \\ \tilde{w} \end{bmatrix} , \quad \tilde{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \quad \tilde{e}_3 = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{a} \\ \tilde{w} \end{bmatrix} \qquad (77g)$$

The Roe-type solution (77) is valid for the x-split of the two-dimensional shallow water equations (68). For the z-sweep one interchanges the roles of the velocity components u and w.

It is worth mentioning that the averages (77), in addition to (76), also satisfy

$$\Delta U = U_R - U_R = \sum_{k=1}^{N} \tilde{\alpha}_k \tilde{r}_k \qquad (78)$$

which means that the jump in states across the wave k is given by $\tilde{\alpha}_k \tilde{r}_k$, the product of the wave strength $\tilde{\alpha}_k$ and the appropriate component of the right eigenvector. The same interpretation applies to the flux difference ΔF in (76). This is split into flux differences across all the waves present in the Riemann problem. The jump in flux across wave k is $\tilde{\alpha}_k \tilde{\lambda}_k \tilde{r}_k$.

Roe's numerical method makes use of (76) - (78). See Ref. 7 for details.

SECOND INTERPRETATION OF ROE'S SOLVER (RS2)

The information contained in (77) can be used directly to provide an approximate solution to the Riemann problem in the sense of section 3.1, namely, an approximation to the star state U^* and the wave speeds.

Equation (78) applied to the two-dimensional equations (68) gives

$$\phi^* - \phi_L = \tilde{\alpha}_1 , \qquad \phi_R - \phi^* = \tilde{\alpha}_3$$

$$\phi^* u^* - \phi_L u_L = \tilde{\alpha}_1(\tilde{u} - \tilde{a}) , \quad \phi_R u_R - \phi^* u^* = \tilde{\alpha}_3(\tilde{u} + \tilde{a})$$

It follows that

$$\phi^* = \frac{1}{2}(\phi_L + \phi_R + \tilde{\alpha}_1 - \tilde{\alpha}_3) \tag{79}$$

$$u^* = \frac{\phi_L u_L + \phi_R u_R + \tilde{\alpha}_1(\tilde{u} - \tilde{a}) - \tilde{\alpha}_3(\tilde{u} + \tilde{a})}{\phi_L + \phi_R + \tilde{\alpha}_1 - \tilde{\alpha}_3} \tag{80}$$

One could also obtain values for w_L^* and w_R^* using the Roe averages but this solution would be incorrect. By inspecting (77f) and (77g) one can see that w changes across the acoustic waves by $\tilde{\alpha}_1\tilde{w}$ and $\tilde{\alpha}_3\tilde{w}$ respectively. In the exact solution w only changes across the contact wave. This defect in the Roe approximation can lead to some numerical problems. The same remark applies to the Euler equations in two and three dimensions. The problem becomes apparent in shear waves. We are currently investigating this issue in detail.

For a method that is independent of the Riemann approximation, such as WAF, one can easily modify the Roe solution for the passive velocity w. We simply set $w = w_L$ to the left of the wave $\frac{dx}{dt} = u^*$ and $w = w_R$ otherwise as in the exact solution (70).

For the wave speeds in this interpretation of the Roe solver we take λ_1, λ_2 and λ_3 as given by (73).

HLL - TYPE APPROXIMATIONS

Harten Lax and van Leer (Ref. 10) suggested some very simple types of approximations to the solution of Riemann problems. We denote this approach by HLL. They begin by considering estimates for bounds on the smallest and largest signal velocities in the solution of the Riemann problem.

Suppose these estimates, E_L and E_R say, are available and that $E_L < 0$ and $E_R > 0$ (see Fig. 6). Consider first the one-dimensional case of equation (27) in the integral form (26). Evaluation of (26) round the rectangle ABCD in Fig. 6 gives

$$U^* = \frac{E_R U_R - E_L U_L + F_L - F_R}{E_R - E_L} \tag{81}$$

One could also compute the star fluxes directly as

$$F^* = \frac{E_R F_L - E_L F_R + E_L E_R (U_R - U_L)}{E_R - E_L} \tag{82}$$

There is freedom in choosing the estimates E_L and E_R. Davis (Ref. 11) proposed various alternatives. Here we suggest

$$E_L = \lambda_1 = \min \left\{ u_L - a_L \, , \; u^*_{TR} - a^*_{TR} \right\}$$

$$E_R = \lambda_2 = \max \left\{ u_R - a_R \, , \; u^*_{TR} + a^*_{TR} \right\} \tag{83}$$

Where u^*_{TR} and a^*_{TR} are the values given by the two-rarefaction approximation (71) - (72).

For the two-dimensional split equations (68) the HLL solver is extended by setting $\lambda_3 = E_R$ in (83) and adding

$$\left.\begin{array}{l} \lambda_2 = u^* \\ w(x,\ t) \begin{cases} w_L & \text{if} \quad x/t < u^* \\ w_R & \text{if} \quad x/t > u^* \end{cases} \end{array}\right\} \tag{84}$$

The HLL-type approximation (81) and (83) is very simple and robust. For the shallow water equations this type of approximation is bound to be more successful than for the Euler equations, for which the neglecting of the contact wave leads to excessive smearing of contact surfaces. For the shallow water equations the acknowledgement of only two waves is correct, provided the solution for the passive velocity w is then taken as in (84), for the two-dimensional case.

For fully subcritical (E_L, $E_R < 0$) and fully supercritical (E_L, $E_R < 0$) one can still carry out the integration (26). The result is, in either case, a pair of non-linear algebraic equations for u^* and ϕ^*.

Instead we propose the following procedure: acknowledge a single wave of speed λ and take the solution

$$U(x,t) = \begin{cases} U_L\ , & \text{if} \quad x/t \le \lambda \\ U_R\ , & \text{if} \quad x/t > \lambda \end{cases} \tag{85}$$

with

$$\lambda = \begin{cases} \min \left\{ E_L,\ E_R \right\}\ , & \text{if} \quad E_L,\ E_R \le 0 \\ \max \left\{ E_L,\ E_R \right\}\ , & \text{if} \quad E_L,\ E_R \ge 0 \end{cases} \tag{86}$$

Note that (85) - (86) could be applied to all conserved variables, including the passive velocity.

4. THE WEIGHTED AVERAGE FLUX METHOD (WAF)

Here we escribe a conservative shock capturing method that is applicable to the two-dimensional shallow water equations (24) and uses, locally, the solution of the relevant Riemann problem. We call the method WAF, which stands for weighted average flux. Further details can be found in Refs. 3 - 5.

The method is directly applicable to hyperbolic conservation laws of the form (25), or (26). Consider a computational grid in the x - t plane as shown in Fig. 7. Application of the integral form of the conservation law (26) to the rectangle ABCD in Fig. 7 gives the explicit conservative method

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[F_{i-1/2} - F_{i+1/2} \right] \qquad (87)$$

4.1 FORMULATIONS OF THE WAF METHOD

The WAF method uses an intercell flux $F_{i+1/2}$ that is obtained from an integral average of the flux function F(U) across the complete wave structure of the local Riemann problem with piece-wise constant data (U_i^n , U_{i+1}^n). That is to say

$$F_{i+1/2} = \frac{1}{\frac{1}{2}\Delta x_i}\int_{-\frac{1}{2}\Delta x_i}^{0} F(U^*)dx + \frac{1}{\frac{1}{2}\Delta x_{i+1}}\int_{0}^{\frac{1}{2}\Delta x_{i+1}} F(U^*)dx \qquad (88)$$

where $U^* = U^*(x/\frac{\Delta t}{2}\ U_i^n,\ U_{i+1}^n)$ is the solution of the Riemann problem with data U_i^n , U_{i+1}^n at the half-time level n + 1/2. The Riemann problem is centred at x = 0 and its solution depends on

the similarity variable x/t. In (88) we are allowing for irregular grids; Δx_i is the spacing for the computing cell i. Fig. 8 illustrates the WAF-flux evaluation for a 2 x 2 hyperbolic system, such as the one-dimensional shallow water equations (27).

The integration in (88) could be done exactly, but that would be costly and unnecessary. If we assume that all waves are represented by single rays through the origin then (88) becomes a summation, namely

$$F^{WAF}_{i+1/2} = \sum_{k=1}^{N+1} W_k \, F^{(k)}_{i+1/2} \qquad (89)$$

where, for generality, N represents the number of waves in the Riemann problem. For the one-dimensional shallow water equations N = 2. The coefficient W_k in (89) represents the geometric extent of the region k in which the flux function $F(U^*)$ takes on the constant value $F^{(k)}_{i+1/2}$. Effectively, W_k is a weight and can be easily seen to be given by

$$W_k = \frac{1}{2} (\nu_k - \nu_{k-1}) \ , \ \nu_0 = -1 \ , \ \nu_{N+1} = +1 \qquad (90)$$

Note that $W_k \geq 0$ for all k and $\sum_{k=1}^{N+1} W_k = 1$.

The WAF flux is an extension of the Godunov flux, which is defined by

$$F^{GOD}_{i+1/2} = F\left[U^*(0, U^n_i, U^n_{i+1}) \right] \qquad (91)$$

i.e. U^* is evaluated along the ray x/t = 0. Both fluxes use the same Riemann problem, but $F^{GOD}_{i+1/2}$ gives a first-order accurate method, while $F^{WAF}_{i+1/2}$ gives a second-order accurate method.

The formulae (89) - (90) give a more revealing expression for the WAF flux, namely

$$F^{WAF}_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} \nu_k \Delta F^{k)}_{i+1/2} \qquad (92)$$

with

$$\nu_k = \Delta t \lambda_k / \Delta x \ , \ \Delta F^{(k)}_{i+1/2} = F^{(k+1)}_{i+1/2} - F^{(k)}_{i+1/2} \qquad (93)$$

representing the Courant number associated with the wave k, of speed λ_k and the flux jump across the same wave k.

Instead of obtaining an intercell averaged flux one could obtain an averaged state $\bar{V}_{i+1/2}$ as

$$\bar{V}_{i+1/2} = \frac{1}{2}(V^n_i + V^n_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} \nu_k \ \Delta V^{(k)}_{i+1/2} \qquad (94)$$

and then set

$$F_{i+1/2} = F(\bar{V}_{i+1/2}) \qquad (95)$$

For linear problems (92) and (95) are identical. For non-linear problems they formally differ, but numerical results are virtually identical. Formula (95) may, in some cases, be computationally cheaper, but this will depend on the way the Riemann problem solution is made available by the solver.

4.2 SONIC FLOW

In deriving the approximate expression (92), or (94) - (95), for the WAF intercell flux it was assumed that the waves in the

$$A_k = \begin{cases} \dfrac{1 - 2(1 - |\nu_k|)}{|\nu_k|}, & r_i \geq 2 \\ \dfrac{1 - r_i(1 - |\nu_k|)}{|\nu_k|}, & 1 \leq r_i \leq 2 \\ 1, & \frac{1}{2} \leq r_i \leq 1 \\ \dfrac{1 - 2r_i(1 - |\nu_k|)}{|\nu_k|}, & 0 \leq r_i \leq \frac{1}{2} \\ \dfrac{1}{|\nu_k|}, & r_i \leq 0 \end{cases} \tag{98}$$

The definitions (97) and (98) for A_k are for a single wave k with Courant number ν_k. The flow parameter r_i is

$$r_i = \begin{cases} \dfrac{\Delta Q^{(k)}_{i-1/2}}{\Delta Q^{(k)}_{i+1/2}} & \text{if } \nu_k \geq 0 \\ \dfrac{\Delta Q^{(k)}_{i+3/2}}{\Delta Q^{(k)}_{i+1/2}} & \text{if } \nu_k < 0 \end{cases} \tag{99}$$

where Q is a suitable variable. For the shallow water equations we use $Q = \phi$ for the acoustic waves and $Q = w$ for the contact wave. The notation $\Delta Q^{(k)}_{i+1/2}$ means the jump in Q across the wave k in the Riemann problem with data (U_i, U_{i+1}). The denominator in (99) is always the jump through wave k in the local Riemann problem (i, i+1), whereas the numerator looks at the upstream or upwind direction to find the appropriate jump. Since the waves in the Riemann problem generally travel in different directions one must construct a function A_k for each wave k.

If the version (94) - (95) of WAF is to be used then the oscillation-free average state $\bar{V}_{i+1/2}$ is given by

$$\bar{V}_{i+1/2} = \frac{1}{2}\,(V_i^n + V_{i+1}^n) - \frac{1}{2}\sum_{k=1}^{N} A_k\,\nu_k\,\Delta V_{i+1/2}^{(k)} \qquad (100)$$

4.4 THE CFL CONDITION

The WAF method (87) and (96), or (100), is an explicit time-marching scheme. The stable rate of advance in time is given by the Courant condition. Here we suggest a non-linear CFL condition as proposed in Ref. 11.

For a given cell i there is time step Δt_i that is the time it takes for the right acoustic wave of speed $S^R_{i-1/2}$ from the left Riemann problem (i-1, i) to intersect the left acoustic wave of speed $S^L_{i+1/2}$ of the right Riemann problem (i, i+1). If the size of the cell i is given by Δx_i then

$$\Delta t_i = \frac{\Delta x_i}{S^L_{i+1/2} - S^R_{i-1/2}} \qquad (101)$$

Then we take Δt, the time-steps size for the scheme at the given time level, as $\Delta t = \min_i \{\Delta t_i\}$. Due to the denominator in (101), for programming purposes it is best to take

$$\left.\begin{aligned} \Delta t &= \Delta x / S_{max} \\ S_{max} &= \max_i \{ S^L_{i+1/2} - S^R_{i-1/2} \} \end{aligned}\right\} \qquad (102)$$

A simpler, but less reliable CFL condition is

solution of the Riemann problem were single rays. For contacts and shocks this assumption is correct, but it is not for rarefaction waves, which have a fan-like configuration. This simplification only appears to cause difficulties when the rarefaction fan is centered around the t-axis, i.e. when for one of the characteristic rays of the rarefaction wave one has $\frac{dx}{dt} = u-a = 0$ (for a left rarefaction) or $\frac{dx}{at} = u+a = 0$ (for a right rarefaction). In such circumstances one speaks of locally sonic flow.

The two possible cases for the one-dimensional shallow water equations are illustrated in Fig. 9.

Locally sonic flow is related to the property of entropy satisfaction for conservative schemes. See Ref. 10 for details. Entropy violating schemes can compute "rarefaction shocks", which are unphysical. The WAF method with the exact Riemann solver is entropy satisfying provided one replaces the U^* constant state between the acoustic waves by the point value $U^*(0, U_i, U_{i+1})$, which is the value at which the Godunov flux is evaluated. For the shallow water equations the exact solution at $x/t = 0$ is given by equations (64) - (67).

In general, it is the Riemann-problem solution that plays the crucial role. Certain approximations, such as the Roe approximation (77), are known to be entropy violating. Special entropy fixes are then required.

The inherent second-order accuracy of the WAF method leads to spurious oscillations near shock waves or other discontinuities. The local wave structure of the Riemann problem can be used to construct an oscillation free version of the method.

4.3 A TVD VERSION OF WAF

Given the discrete solution $\{U_i^n\}$ at time level n the total variation of $\{U_i^n\}$ is defined as

$$TV(U^n) = \sum_i | U^n_{i+1} - U^n_i |$$

This is essentially a measure of the oscillatory character of the solution. Many useful difference schemes are total variation diminishing, or TVD for short. That is the total variation diminishes with time, or

$$TV(U^{n+1}) \leq TV(U^n)$$

For the model linear advection equation

$$U_t + a\, U_x = 0$$

We can construct rigorously a TVD version of WAF. See Ref. 5 for details. For non-linear systems the TVD procedure is empirical, but in practice it works very well as it will be demonstrated in section 5.

The principle is to amplify the wave speeds in the solution of the local Riemann problem. This is achieved by multiplying the Courant numbers ν_k by an amplifying function A_k. Thus the oscillation-free WAF flux (92) becomes

$$F_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} A_k \nu_k \Delta F^{(k)}_{i+1/2} \qquad (96)$$

The amplifying function $A_k = A_k(U)$ is a function of the flow features. Here we give two examples, namely

$$A_k = \begin{cases} \dfrac{1}{|\nu_k|} & \text{if } r_i \leq 0 \\ \dfrac{1 - (1 - |\nu_k|)\, r_i}{|\nu_k|}, & 0 \leq r_i \leq 1 \\ 1, & \text{if } r_i \geq 1 \end{cases} \qquad (97)$$

$$\Delta t = \min_i \left\{ \frac{\Delta x_i}{2a_i^n} \right\} \qquad (103)$$

where a_i^n is the sound speed at i at the data time level. This is a special case of (101) - (102) with $S^L_{i+1/2} = u_i^n - a_i^n$, $S^R_{i-1/2} = u_i^n + a_i^n$. Both (102) and (103) are better than the commonly used empirical formula $\Delta t = C\,\Delta x/S$ where S is the maximum wave speed present in the data, usually taken as $S = \max_i\{|u_i^n| + a_i^n\}$. C is an empirical coefficient with $0 < C \leq 1$. For shock-tube like initial data ($u_i^n \equiv 0$) this condition fails and can lead to serious stability problems at the beginning of the calculations.

4.5 BOUNDARY CONDITIONS

We describe two types of boundary conditions that are applicable to one-dimensional problems.

REFLECTIVE BOUNDARIES

In the presence of solid walls, fixed or moving, reflective boundary conditions must be imposed. Consider the boundary on the right hand side of the computational domain, as shown in Fig. 10. The last two cells inside the computational domain are M - 1 and M. Two fictitious computational cells are added to the right-hand side of the solid boundary; these are denoted by M + 1 and M + 2. For simplicity consider the Riemann problem RP(M,M+1). The correct Riemann problem is that whose solution has $u^* = v_B$ = velocity of the solid boundary . It can be easily verified that for this to happen the fictitious state M + 1 must be given by

$$\phi_{M+1} = \phi_M \,, \quad u_{M+1} = -u_M + 2v_B \qquad (104a)$$

To preserve second-order accuracy we also need

$$\phi_{M+2} = \phi_{M-1} \ , \ u_{M+2} = \ - u_{M-1} + 2v_B \qquad (104b)$$

TRANSMISSIVE BOUNDARIES

There are cases in which one is only interested in the local behaviour of the solution, or one may wish to simulate a boundary at infinity or a transmissive boundary. This can be achieved by placing "transparent" boundaries that allow waves to pass through.

With reference to Fig. 10 one imposes

$$\begin{aligned} \phi_{M+1} &= \phi_M \ , \quad u_{M+1} = u_M \\ \\ \phi_{M+2} &= \phi_{M-1} \ , \ u_{M+2} = u_{M-1} \end{aligned} \qquad (105)$$

Note that the solution of the Riemann problem RP(M, M+1) is trivial. It is important to realise that the boundary condition (105) does not impose zero gradient of the solution at the boundary, which would be incorrect.

4.6 EXTENSIONS OF THE METHOD

We describe two extensions of the WAF method so that the full two-dimensional problem with source terms

$$U_t + F_x + G_z = S(U) \qquad (106)$$

can be solved.

TREATMENT OF SOURCE TERMS

Consider the one-dimensional inhomogeneous problem

$$U_t + F_x = S(U) \qquad (107)$$

The source term S(U) can include $[0, g\phi h_x]^T$ as well as some other effects such as Coriolis forces and bottom roughness.

We use time-operator splitting so that the problem (107) with initial data U^n is split into two sub-problems. One proceeds as follows:

(a) Solve the homogeneous problem

$$U_t + F_x = 0$$

with data U^n to obtain a provisional solution $\bar{U}^{n+1}$ for the next time level.

(b) Solve the system of ordinary differential equations (ODE's)

$$U_t = S(U)$$

using as initial condition the provisional solution $\bar{U}^{n+1}$ of the previous step. This gives the final solution U^{n+1} for the new time level n+1.

The choice of the numerical method for solving the ODE's depends very much on the character of the sources. For most problems a Runge-Kutta method is good enough, although the simpler first-order Euler step

$$U_i^{n+1} = \bar{U}_i^{n+1} + \Delta t\ S(\bar{U}_i^{n+1})$$

is used very frequently. Here Δt is the time-step size given by the CFL condition in the homogeneous step (1). If the ODE's are a stiff system an implicit method must be used.

The terms involving derivatives, such as h_x can be approximated using central differences, say. If $h(x)$ is known at the intercell boundaries then a second-order approximation at the cell centre would be $h_x \approx (h_{i+1/2} - h_{i-1/2})/\Delta x$.

TREATMENT OF A SECOND DIMENSION

Consider the two-dimensional problem

$$U_t + F_x + G_y = 0$$

with data U^n. We use space-operator splitting whereby the problem is solved in two sweeps.

x-sweep: solve

$$U_t + F_x = 0$$

with data U^n to obtain a provisional solution $U^{n+1/2}$.

Note that this problem is an extended one-dimensional problem. See equation (24). There is a third equation involving the "passive" component of velocity w. The extended Riemann-problem solution must be employed.

z-sweep: solve

$$U_t + G_z = 0$$

with initial data $U^{n+1/2}$. The solution of this step is the final solution U^{n+1} for the time level. The time step size Δt is chosen at the beginning of the splitting procedure and it is common to both the x and z sweeps. If sources were present one would require another step, as described previously.

There are more sophisticated combinations of the x and z sweeps that would ensure formal second-order accuracy. See Ref. 12 for details.

4.7 AN ALGORITHM FOR THE ONE-DIMENSIONAL CASE

We summarise the main steps involved in the implementation of WAF as applied to the homogeneous one-dimensional shallow water equations (27).

Having specified the domain length, the number of computing cells M and the grid size Δx the following operations are performed at every time step n:

(a) Solve the Riemann problem RP(i,i+1) and store:
- the wave speeds into WS(1,i), WS(2,i)
- the ϕ-jumps across each wave into WJ(1,i), WJ(2,i)
- the flux values in the star region into FS(1, i), FS(1,i).

These may have to be computed from the star values ϕ^*, u^* or they may be directly available from the solution of the Riemann problem, as when using the HLL approximation given by equation (82), for instance.

Here the loop runs from i = -1 to M+1.

(b) Apply the CFL condition (102) to find Δt.

(c) For each i, i=0 to M

- compute the local Courant numbers

  ν_k = WS(k,i) $\Delta t/\Delta x$, k = 1, 2

- compute the amplifiers A_k (using equation 97, say), k = 1, 2
- modify Courant numbers, $\nu_k = A_k \nu_k$
- compute the intercell fluxes according to (96), say. Store values into FI(1, i), FI(2, i).

(d) Advance to the next time level n+1 using the conservative formula (87).

A sample program in FORTRAN 77 for the homogeneous one-dimensional equations (27) is included in the appendix. Four choices of Riemann solvers are available, including the exact solver.

5. TEST PROBLEMS

Three test problems are used to assess the performance of the WAF method using the various Riemann solvers described in section 3. Test problems 1 and 2 are one-dimensional cases, for which the exact solution is known. Comparison between the numerical and the exact solution is essential in assessing the performance of the approximate numerical techniques. Test problem 3 is a two-dimensional time-dependent problem, whose exact solution is unknown to us. There are however, certain features of the solution which can be used to judge the numerical results.

TEST 1:

A one-dimensional domain of length 1.0 is chosen. We take M = 100 so that $\Delta x = 0.01$. The initial data is that for a dam-break problem

$$\phi(x, 0) = \begin{cases} \phi_L = 1.0 \quad , & 0 \le x \le 1/2 \\ \phi_R = 0.1 \quad , & \frac{1}{2} < x \le 1.0 \end{cases}$$

and $u(x, 0) = 0$ for all x in [0, 1]. The CFL condition is that given by equation (102). This non-linear CFL condition gives the maximum time-step size without wave interaction within a cell.

The computed solution is evolved to time t = 0.35 units. Fig. 11 shows the exact solution in full line for the flow variable $\phi(x, t)$ and the speed $u(x, t)$ together with the numerical solution, shown in symbols plus a full line through them. The numerical results of Fig. 11 were obtained by using the first order Godunov's method together with the exact Riemann solver.

The shock wave is reasonably well resolved by 4 to 5 grid points. There are no oscillations behind the shock. The rarefaction wave however is very poorly represented. The excessive numerical diffusion inherent in this first-order accurate method is clearly manifested near "corners", where the solution has a discontinuity in derivative. Also the entropy problem of Godunov's method can be seen around x = 0 in Fig. 11. This method is theoretically entropy satisfying and thus the entropy glitch should vanish as the mesh size vanishes. For finite grids however the glitch is visible although it is not a serious problem. Fig. 12 shows the results obtained by the fully second-order method of Richtmyer and Morton. The accuracy in smooth parts of the flow is now better. Also, the discontinuities are more sharply resolved, including the head and the tail of the rarefaction. The entropy glitch is worse than that of Godunov's method, as expected. The most serious problem however is that of the spurious oscillations. The overall solution is unacceptable, by any standards. Fig. 13 shows the results of the WAF method using the exact Riemann solver and the amplifying function A of equation (98). This result is very good all round. The quality of the solution is preserved even if approximate Riemann solvers are used. For this test problem the results obtained by using the approximate Riemann solvers are, to plotting accuracy, indistinguishable from those of the exact Riemann solver. The only exception was the HLL - approximate Riemann solver. The flow is supercritical here. In such circumstances the HLL solver replaces the star values by the upwind data values. This explains the smearing of the tail of the rarefaction.

TEST 2:

For this test all conditions are as in Test 1 but the variable ϕ is

$$\phi(x, 0) = \begin{cases} \phi_L = 10.0 , & 0 \le x \le 1/2 \\ \phi_R = 0.1 , & \text{otherwise} \end{cases}$$

This is a very severe test problem, with the ratio $\phi_L/\phi_R = 100$. The resulting shock wave has a ratio ϕ^*/ϕ_R close to 20. Fig. 14 shows a comparison between the exact solution (full line) and the numerical solution (symbol and full line) obtained by the WAF method using the HLL approximate Riemann solver. This result is satisfactory but improved results can be obtained using other Riemann solvers.

Fig. 15 shows the WAF solution with the two-rarefaction approximation to the Riemann problem. The same result, to plotting accuracy, is obtained when using the two-shock approximation TS, the RS2 solver derived from the Roe approximation and the exact Riemann solver. This is a pleasing observation because it shows that the simplest, and therefore computationally cheapest, approximation can be used reliably in severe flow conditions. Fig. 16 shows the results obtained when the exact Riemann solver is used in the WAF method. This confirms our previous comments. Figs. 17 and 18 show ϕ and u at time t = 0.8 after reflection from the left and right solid walls.

TEST 3:

This is a two-dimensional problem that simulates the collapse of a circular dam. The computational domain is [0, 2] x [0, 2] in the x - z horizontal plane. The dam is centered at (1, 1) and its radius is 0.35. The initial conditions are u(x, z, 0) = w(x, z, 0) = 0 and

$$\phi(x, z, 0) = \begin{cases} 1.0, & \text{if } (x-1)^2 + (z-1)^2 = (0.35)^2 \\ 0.1 & \text{otherwise} \end{cases}$$

The computed solution for $\phi(x, z, t)$ at time t = 0.5 is shown in Fig. 17. The grid used is 100 x 100.

The leading circular shock wave is attenuated as time evolves as one would expect. The mechanism that makes that possible is the

rarefaction wave overtaking the shock, i.e. ϕ decreases from the shock towards the centre.

6. CONCLUSIONS

Several Riemann solvers for the shallow-water equations have been presented. These have been used locally in the weighted average flux method (WAF) to compute the global solution to the general initial-boundary value problem for the unsteady, two-dimensional shallow water equations. The algorithms are conservative and have the ability to capture discontinuities with high resolution without the spurious oscillations of traditional high-order finite difference methods. This is illustrated by the preliminary numerical results presented. A systematic assessment of WAF with the various Riemann solvers available is still a pending task. The results so far however, suggest that for applications in which the solution exhibits high gradients and non-linear effects are important the WAF approach presented here can provide very accurate numerical solutions.

REFERENCES

1. Stoker, J.J. 1957
Water Waves. Intescience Publishers, New York.

2. Casulli, V. 1990
Nuerical simulation of shallow water flow. Invited Lecture to appear in the Proc. VIII Conference on Computational Methods in Water Resources, Venice, Italy, June 1990.

3. Toro, E.F. 1989.a
A weighted average flux method for hyperbolic conservation laws. Proc. Roy. Soc. London, A423, 401-418.

4. Toro, E.F. 1989.b
Riemann-problem methods for computing reactive two-phase flows. Lecture Notes in Physics, 351 pp. 472-481: Numerical Combustion, Proceedings, Juan Les Dius, Antibes, France, 1989. Dervieux and Larrouturou (Eds.)

5. Toro, E.F. 1990.
Numerical solution of the Euler equations by the weighted average flux method. CoA report 9001, College of Aeronautics, Cranfield Institute of Technology, U.K., March 1990.

6. Toro, E.F. 1989.c
A fast Riemann solver with constant covolume applied to the Random Choice Method. Int. J. of Numer. Meth. in Fluids. Vol.9, pp. 1145-1164.

7. Glaister, P. 1987.
Difference schemes for the shallow water equations. Numerical analysis report 9/87. Department of Mathematics, University of Reading, U.K.

8. Jeffrey, A. 1976.
Quasilinear hyperbolic systems and waves. Research notes in Mathematics. Pitman Publishing.

9. Roe, P.L. 1981.
Approximate Riemann solvers, parameter vectors, and difference schemes. J. Comput. Physics. 43, pp. 357-372.

10. Harten, A., Lax, P.D. and van Leer, B. 1983.
On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. SIAM Review, 25, pp. 35-61.

11. Toro, E.F. 1989.d
A CFL condition for characteristic based methods.
Applied mathematics letters, Vol. 2, No. 1, pp. 57-60.

12. Strang, G. 1968.
On the construction and comparison of finite difference schemes.
SIAM, J. Numer. Anal. Vol. 5, pp. 506-517.

ACKNOWLEDGEMENTS

A significant part of this work was carried out at the Mathematics Department, University of Trento, Italy, while the author was visiting Trento at the kind invitation of Prof. Vincenzo Casulli. The support provided is greatfully acknowledged.

LEGEND TO FIGURES

Fig.1 Free-surface flow configuration. (x, z) is horizontal plane, y is vertical direction, h(x, z) is depth of undisturbed water, η(x, z, t) is free surface disturbance.

Fig.2 Wave pattern of solution of the Riemann problem for the one-dimensional shallow-water equations.

Fig.3 All possible wave configurations in the solution of the Riemann problem for the one-dimensional shallow-water equations.

Fig.4 Transformation to a steady frame of reference moving with the shock speed.

Fig.5 Determination of the solution inside a rarefaction fan.

Fig.6 HLL solution of the Riemann problem. E_L amd E_R are estimates for the smallest and largest signal velocities.

Fig.7 Computational grid in the x-t plane. The computing cell i has dimensions Δx by Δt.

Fig.8 Integration path for the WAF-flux evaluation. The illustration is for a 2 x 2 system such as the one dimensional shallow water equations.

Fig.9 Locally sonic flow. Case (a) shows the solution of a Riemann problem where the left wave is a rarefaction centered around the t-axis, i.e. u = a along the t axis. Case (b) illustrates the sonic-flow situation for a right rarefaction.

Fig.10 Right-hand boundary. Two fictitious states M+1 and M+2 are required.

Fig.11 Test 1 solved by the Godunov's method with the exact Riemann solver (symbol). The exact solution (shown by the full line) is also given for comparison.

Fig.12 Test 1 solved by the Richtmyer-Morton scheme (symbol). The exact solution (line) is shown for comparison.

Fig.13 Test 1 solved by the WAF method (symbol). The exact solution is given by the single full line.

Fig.14 Test 2 solved by the WAF method using the HLL Riemann solver (symbol). The exact solution is given by the full line alone.

Fig.15 Test 2 solved by WAF using the TR approximate Riemann solver (symbol). The exact solution is given by the single full line.

Fig.16 Test 2 solved by WAF with the exact Riemann solver (symbol). The single line shows the exact solution.

Fig.17 Test 2 solved by WAF with the TR approximation at time t = 0.8 after reflection from solid walls. Quantity shown is depth variable ϕ.

Fig.18 Test 2 solved by the WAF method with the TR approximation at time t = 0.8 after reflection from solid walls. Quantity shown is velocity.

Fig.19 Numerical solution of Test 3 $\phi(x, z)$ at time t = 0.25

```
C-----------------------------------------------------------------------C
C                                                                       C
C                                                                       C
C                                                                       C
C              APPENDIX :     FORTRAN 77 PROGRAM                        C
C                                                                       C
C                                                                       C
C                                                                       C
C          THE WAF METHOD FOR THE 1D SHALLOW WATER EQUATIONS            C
C            WITH A CHOICE OF FOUR RIEMANN SOLVERS AND TWO              C
C           AMPLIFIERS. PROGRAM IS SETUP TO SOLVE DAM-BREAK              C
C                         PROBLEMS.                                     C
C                                                                       C
C                                                                       C
C                DATA REQUIRED:                                         C
C                                                                       C
C                                                                       C
C          TUBLEN : LENGHT OF DOMAIN                                    C
C          M      : NUMBER OF COMPUTING CELLS                           C
C          NOTIST : NUMBER OF TIME STEPS                                C
C          TIMEOUT: SELECTED TIME FOR OUTPUT                            C
C          IRP    : SELECTS RIEMANN SOLVER                              C
C               =1: TWO-RAREFACTION APPROXIMATION                       C
C               =2: RE-INTERPRETATION OF ROE'S SOLVER                   C
C               =3: TWO-SHOCK APPROXIMATION                             C
C               =4: EXACT RIEMANN SOLVER                                C
C          IAM    : SELECTS AMPLIFYING FUNCTION                         C
C               =1: SELECTS MINAAA                                      C
C               =2: SELECTS SUPERA                                      C
C          DL     : DEPTH VARIABLE ON LEFT SIDE OF TUBLEN/2             C
C          UL     : VELOCITY ON LEFT SIDE OF TUBLEN/2                   C
C          DR     : DEPTH VARIABLE ON RIGHT SIDE OF TUBLEN/2            C
C          UR     : VELOCITY ON RIGHT SIDE OF TUBLEN/2                  C
C                                                                       C
C-----------------------------------------------------------------------C
      PARAMETER (MD=1000)
      DIMENSION D(-1:MD+2),U(-1:MD+2),C(-1:MD+2),CS(2,-1:MD+2)
      DIMENSION FD(2,-1:MD+2),WS(2,-1:MD+2)
      COMMON /SOLUTI/D,U,C,CS
      COMMON /RIESO1/FD,WS
      COMMON /GRIDMS/M,MM1,MP1,MP2,DT,DX,DTODX

      READ(91,*)TUBLEN,M,NOTIST,TIMEOUT,IRP,IAM
      READ(91,*)DL,UL,DR,UR

      TIME=0.0
      MM1=M-1
      MP1=M+1
      MP2=M+2

C     SET INITIAL DATA

      DX=TUBLEN/REAL(M)
      MH=M/2
      DO 0001 I=-1,MP2
         IF(I.LE.MH)THEN
            D(I)=DL
            U(I)=UL
         ELSE
            D(I)=DR
            U(I)=UR
         ENDIF
         CS(1,I)=D(I)
         CS(2,I)=D(I)*U(I)
```

```
 0001    CONTINUE

         DO 0002 N=1,NOTIST

C           COMPUTE CELERITY C(I)

            DO 0003 I=1,M
               C(I)=SQRT(D(I))
 0003       CONTINUE

C           APPLY REFLECTIVE BOUNDARY CONDITIONS

            D(0)   = D(1)
            U(0)   =-U(1)
            C(0)   = C(1)

            D(-1) = D(2)
            U(-1) =-U(2)
            C(-1) = C(2)

            D(MP1) = D(M)
            U(MP1) =-U(M)
            C(MP1) = C(M)

            D(MP2) = D(MM1)
            U(MP2) =-U(MM1)
            C(MP2) = C(MM1)

C           COMPUTE FLUXES ON DATA

            DO 0004 I=-1,MP2
               CONS2  =D(I)*U(I)
               FD(1,I)=CONS2
               FD(2,I)=CONS2*U(I)+0.5*D(I)*D(I)
 0004       CONTINUE

            CALL RIEMANN(IRP)

C           COMPUTE TIME-STEP SIZE DT

            WSMAX=-1.0E+06
            DO 0005 I=1,M
               WSDIF=WS(2,I)-WS(1,I+1)
               IF(WSDIF.GT.WSMAX)WSMAX=WSDIF
 0005       CONTINUE
            DT=DX/WSMAX

             IF((TIME+DT).GT.TIMEOUT)THEN
                 WRITE(6,*)N,TIME
                 DT=TIMEOUT-TIME
             ENDIF
             TIME=TIME+DT
             DTODX=DT/DX

             CALL SOLVER(IAM)

             TDIFF=ABS(TIME-TIMEOUT)
             IF(TDIFF.LE.1.0E-06)THEN
                WRITE(6,*)TIME,M
                WRITE(1,*)TIME,M
                DO 0006 I=1,M
                   WRITE(6,*)I,D(I),U(I)
                   WRITE(1,*)I,D(I),U(I)
 0006           CONTINUE
                STOP
             ENDIF
```

```
 0002    CONTINUE

         END

C--------------------------------------------------------------------C

         SUBROUTINE SOLVER(IAM)
         PARAMETER (MD=1000)
         DIMENSION D(-1:MD+2),U(-1:MD+2),C(-1:MD+2),CS(2,-1:MD+2)
         DIMENSION FS(2,-1:MD+2),WS(2,-1:MD+2),WJ(2,-1:MD+2)
         DIMENSION FD(2,-1:MD+2),FI(2,-1:MD+2),CN(2)
         COMMON /SOLUTI/D,U,C,CS
         COMMON /RIESO1/FD,WS
         COMMON /RIESO2/FS,WJ
         COMMON /GRIDMS/M,MM1,MP1,MP2,DT,DX,DTODX
         DATA TOL/1.0E-04/

C        COMPUTE INTERFACE FLUXES FI

         DO 0001 I=0,M

C           MODIFY COURANT NUMBERS

            DO 0002 K=1,2
               CN(K)=WS(K,I)*DTODX
               IUPW=-1
               IF(CN(K).LT.0.0)IUPW=1
               DLOC=WJ(K,I)
               DUPW=WJ(K,I+IUPW)
               IF(ABS(DUPW).LT.TOL)DUPW=TOL*SIGN(1.0,DUPW)
               IF(ABS(DLOC).LT.TOL)DLOC=TOL*SIGN(1.0,DLOC)
               RATIO=DUPW/DLOC
               CNABS=ABS(CN(K))
               IF(CNABS.LE.TOL)CNABS=TOL
               IF(IAM.EQ.1)CALL MINAAA(RATIO,CNABS,AA)
               IF(IAM.EQ.2)CALL SUPERA(RATIO,CNABS,AA)
               CN(K)=AA*CN(K)
 0002       CONTINUE

C           COMPUTE WEIGHTS WL,WM,WR

            WL =0.5*(1.0+CN(1))
            WM =0.5*(CN(2)-CN(1))
            WR =0.5*(1.0-CN(2))

C           COMPUTE WEIGHTED AVERAGE FLUX FI

            DO 0003 K=1,2
               FI(K,I)=WL*FD(K,I) + WM*FS(K,I) + WR*FD(K,I+1)
 0003       CONTINUE

 0001    CONTINUE

C        UPDATE CONSERVED VARIABLES TO NEW TIME LEVEL

         DO 0004 I=1,M
            DO 0005 K=1,2
               CS(K,I)=CS(K,I)-DTODX*(FI(K,I)-FI(K,I-1))
 0005       CONTINUE

C           COMPUTE PHYSICAL VARIABLES

            D(I)=CS(1,I)
            U(I)=CS(2,I)/D(I)
 0004    CONTINUE
```

```
      RETURN
      END

C-----------------------------------------------------------------------C

      SUBROUTINE MINAAA(R,C,A)
      B=1.0
      IF(R.LE.1.0)B=R
      IF(R.LE.0.0)B=0.0
      A=(1.0-(1.0-C)*B)/C
      RETURN
      END

C-----------------------------------------------------------------------C

      SUBROUTINE SUPERA(R,C,A)
      B=2.0
      IF(R.LE.2.0)B=R
      IF(R.LE.1.0)B=1.0
      IF(R.LE.0.5)B=2.0*R
      IF(R.LE.0.0)B=0.0
      A=(1.0-(1.0-C)*B)/C
      RETURN
      END

C-----------------------------------------------------------------------C

      SUBROUTINE RIEMANN(IRP)
      PARAMETER (MD=1000)
      DIMENSION D(-1:MD+2),U(-1:MD+2),C(-1:MD+2),CS(2,-1:MD+2)
      DIMENSION FS(2,-1:MD+2),WS(2,-1:MD+2),WJ(2,-1:MD+2)
      DIMENSION FD(2,-1:MD+2)
      COMMON /GRIDMS/M,MM1,MP1,MP2,DT,DX,DTODX
      COMMON /STATES/DL,UL,CL,DR,UR,CR,DS,US,SL,SR
      COMMON /SOLUTI/D,U,C,CS
      COMMON /RIESO1/FD,WS
      COMMON /RIESO2/FS,WJ

      DO 0001 I=-1,MP1
          DL=D(I)
          UL=U(I)
          CL=C(I)
          DR=D(I+1)
          UR=U(I+1)
          CR=C(I+1)

          IF(IRP.EQ.1)CALL RIETRA

          IF(IRP.EQ.2)CALL RIERS2

          IF(IRP.EQ.3)CALL RIETSH

          IF(IRP.EQ.4)CALL RIEEXA

C         COMPUTE WAVE JUMPS IN D

          WJ(1,I)=DS-DL
          WJ(2,I)=DR-DS

          WS(1,I)=SL
          WS(2,I)=SR

C         COMPUTE STAR FLUXES FS

          RMOM=DS*US
          FS(1,I)=RMOM
```

```
            FS(2,I)=RMOM*US+0.5*DS*DS
 0001       CONTINUE
            RETURN
            END

C-----------------------------------------------------------------------C

            SUBROUTINE RIETRA
            COMMON /STATES/DL,UL,CL,DR,UR,CR,DS,US,SL,SR

C           TWO-RAREFACTION APPROXIMATION

            CS=0.5*(CL+CR)+0.25*(UL-UR)
            US=0.5*(UL+UR)+CL-CR

C           COMPUTE WAVE SPEEDS

            SL=AMIN1(UL-CL,US-CS)
            SR=AMAX1(UR+CR,US+CS)

C           CHECK FOR SONIC FLOW

            IF((UL-CL).LE.0.0.AND.(US-CS).GE.0.0)THEN
               US=(UL+2.0*CL)/3.0
               CS=US
            ENDIF

            IF((US+CS).LE.0.0.AND.(UR+CR).GE.0.0)THEN
               US=(UR-2.0*CR)/3.0
               CS=-US
            ENDIF

            DS=CS*CS

            RETURN
            END

C-----------------------------------------------------------------------C

            SUBROUTINE RIERS2
            COMMON /STATES/DL,UL,CL,DR,UR,CR,DS,US,SL,SR

C           ROE RS2 APPROXIMATION

            CLPCR=CL+CR
            UB    =(CL*UL+CR*UR)/CLPCR
            AB    =SQRT(0.5*(DL+DR))
            DB    =SQRT(DL*DR)
            DELD  =DR-DL
            DELU  =UR-UL
            ALF1  =0.5*(DELD-DB*DELU/AB)
            ALF2  =0.5*(DELD+DB*DELU/AB)

            DS    =0.5*(DL+DR+ALF1-ALF2)
            RNUME=DL*UL+DR*UR+ALF1*(UB-AB)-ALF2*(UB+AB)
            RDENO=DL+DR+ALF1-ALF2
            US    =RNUME/RDENO
            CS    =SQRT(DS)

C           COMPUTE WAVE SPEEDS

            SL=AMIN1(UL-CL,US-CS)
            SR=AMAX1(UR+CR,US+CS)

C           CHECK FOR SONIC FLOW
```

```
      IF((UL-CL).LE.0.0.AND.(US-CS).GE.0.0)THEN
         US=(UL+2.0*CL)/3.0
         DS=US*US
      ENDIF

      IF((US+CS).LE.0.0.AND.(UR+CR).GE.0.0)THEN
         US=(UR-2.0*CR)/3.0
         DS=US*US
      ENDIF

      RETURN
      END

C-----------------------------------------------------------------------C

      SUBROUTINE RIETSH
      COMMON /STATES/DL,UL,CL,DR,UR,CR,DS,US,SL,SR
      DATA TOL/1.0E-04/

C     TWO-SHOCK APPROXIMATION

      DELU=UR-UL
C     GUESSED VALUE FROM TR APROXIMATION
      SQDS=0.5*(CL+CR)-0.25*DELU
      DS  =SQDS*SQDS
      DS0  =DS

C     START ITERATION

      DO 0001 IT=1,50

C        LEFT WAVE

         AL      =SQRT(0.5*(DS+DL)/(DS*DL))
         DELDSDL=DS-DL
         FLEFVAL=DELDSDL*AL
         FLEFDER=AL-0.25*(DELDSDL)/(AL*DS*DS)

C        RIGHT WAVE

         AR      =SQRT(0.5*(DS+DR)/(DS*DR))
         DELDSDR=DS-DR
         FRIGVAL=DELDSDR*AR
         FRIGDER=AR-0.25*(DELDSDR)/(AR*DS*DS)

         FUNVAL=FLEFVAL+FRIGVAL+DELU
         FUNDER=FLEFDER+FRIGDER
         DS     =DS-FUNVAL/FUNDER
         TESTDS  =ABS((DS-DS0)/DS)
         IF(TESTDS.LE.TOL)GOTO 0003
         IF(DS.LT.TOL)DS=TOL
         DS0=DS
 0001    CONTINUE

      WRITE(6,0004)IT
      STOP
 0004 FORMAT(' DIVERGENCE, ITERATION  NO. =',I4)

 0003 US=0.5*(FRIGVAL-FLEFVAL+UL+UR)

C     COMPUTE WAVE SPEEDS SL AND SR

      IF(DS/DL.LE.1.0)THEN
         SL=UL-CL
      ELSE
         SL=UL-SQRT(0.5*DL*DS*(DL+DS))/DL
```

```
      ENDIF

      IF(DS/DR.LE.1.0)THEN
         SR=UR+CR
      ELSE
         SR=UR+SQRT(0.5*DR*DS*(DR+DS))/DR
      ENDIF

C     CHECK FOR SONIC FLOW

      CS=SQRT(DS)

      IF((UL-CL).LE.0.0.AND.(US-CS).GE.0.0)THEN
         US=(UL+2.0*CL)/3.0
         DS=US*US
      ENDIF

      IF((US+CS).LE.0.0.AND.(UR+CR).GE.0.0)THEN
         US=(UR-2.0*CR)/3.0
         DS=US*US
      ENDIF

      RETURN
      END
C-----------------------------------------------------------------------C

      SUBROUTINE RIEEXA
      COMMON /STATES/DL,UL,CL,DR,UR,CR,DS,US,SL,SR
      DATA TOL/1.0E-04/

C     EXACT RIEMANN SOLVER

      DELU=UR-UL
C     GUESSED VALUE FROM TR APROXIMATION
      SQDS=0.5*(CL+CR)-0.25*DELU
      DS  =SQDS*SQDS
      DS0  =DS

C     START ITERATION

      DO 0001 IT=1,50

C     LEFT WAVE

         IF(DL.LT.DS)THEN
C           LEFT SHOCK
            AL      =SQRT(0.5*(DS+DL)/(DS*DL))
            DELDSDL=DS-DL
            FLEFVAL=DELDSDL*AL
            FLEFDER=AL-0.25*(DELDSDL)/(AL*DS*DS)
         ELSE
C           LEFT RAREFACTION
            SQDS    =SQRT(DS)
            FLEFVAL=2.0*(SQDS-SQRT(DL))
            FLEFDER=1.0/SQDS
         ENDIF

C     RIGHT WAVE

         IF(DR.LT.DS)THEN
C           RIGHT SHOCK
            AR      =SQRT(0.5*(DS+DR)/(DS*DR))
            DELDSDR=DS-DR
            FRIGVAL=DELDSDR*AR
            FRIGDER=AR-0.25*(DELDSDR)/(AR*DS*DS)
         ELSE
```

```
C              RIGHT RAREFACTION
               SQDS   =SQRT(DS)
               FRIGVAL=2.0*(SQDS-SQRT(DR))
               FRIGDER=1.0/SQDS
            ENDIF

            FUNVAL=FLEFVAL+FRIGVAL+DELU
            FUNDER=FLEFDER+FRIGDER
            DS     =DS-FUNVAL/FUNDER
            TESTDS  =ABS((DS-DS0)/DS)
            IF(TESTDS.LE.TOL)GOTO 0003
            IF(DS.LT.TOL)DS=TOL
            DS0=DS
 0001    CONTINUE
         WRITE(6,0004)IT
         STOP
 0004    FORMAT(' DIVERGENCE, ITERATION  NO. =',I4)

 0003    US=0.5*(FRIGVAL-FLEFVAL+UL+UR)

C        COMPUTE WAVE SPEEDS SL AND SR

         IF(DS/DL.LE.1.0)THEN
            SL=UL-CL
         ELSE
            SL=UL-SQRT(0.5*DL*DS*(DL+DS))/DL
         ENDIF

         IF(DS/DR.LE.1.0)THEN
            SR=UR+CR
         ELSE
            SR=UR+SQRT(0.5*DR*DS*(DR+DS))/DR
         ENDIF

C        CHECK FOR SONIC FLOW

         CS=SQRT(DS)

         IF((UL-CL).LE.0.0.AND.(US-CS).GE.0.0)THEN
            US=(UL+2.0*CL)/3.0
            DS=US*US
         ENDIF

         IF((US+CS).LE.0.0.AND.(UR+CR).GE.0.0)THEN
            US=(UR-2.0*CR)/3.0
            DS=US*US
         ENDIF

         RETURN
         END

C-------------------------------------------------------------------------C
```

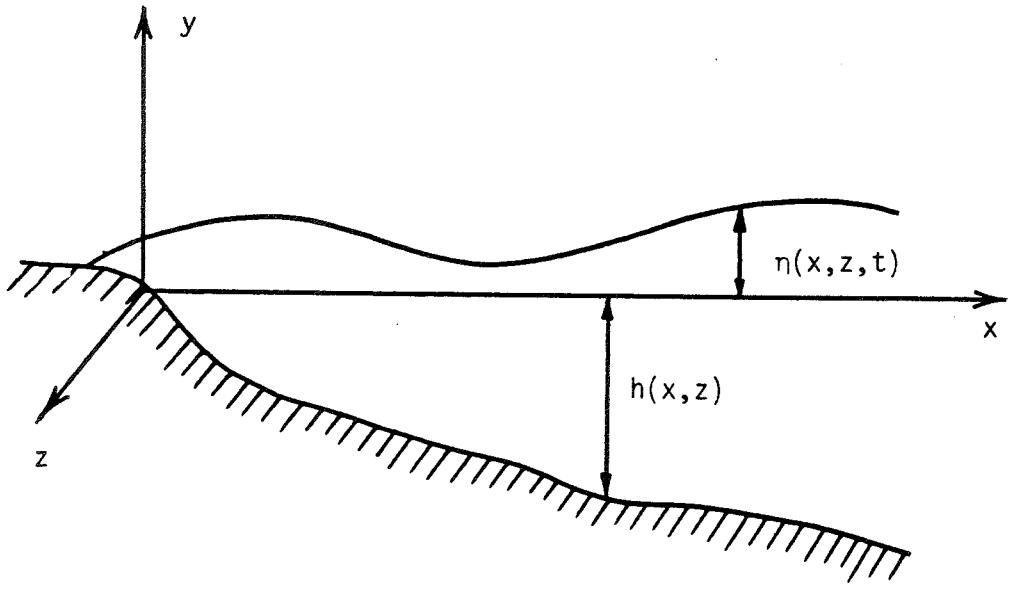

Fig.1 Free-surface flow configuration. (x,z) is horizontal plane, y is vertical direction, h(x,z) is depth of indisturbed water, h(x,z,t) is free-surface disturbance.

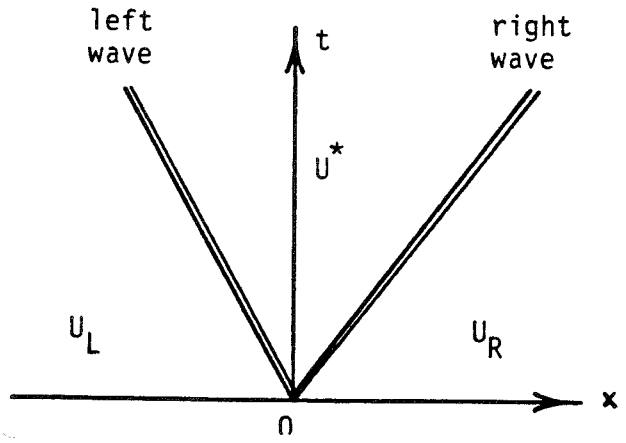

Fig.2 Wave pattern of solution of the Riemann problem for the one-dimensional shallow-water equations.

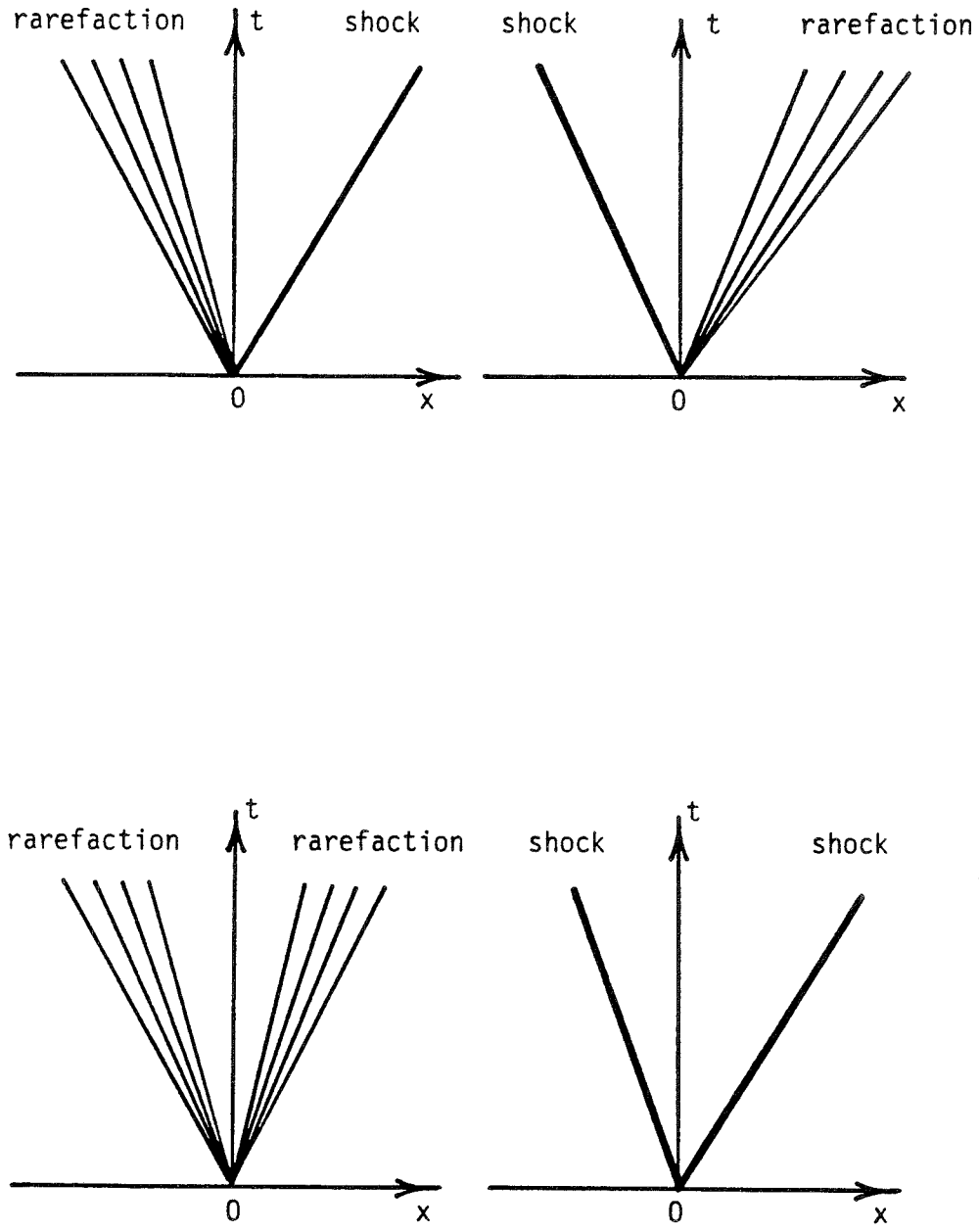

Fig.3: All possible wave configurations in the solution of the Riemann problem for the one-dimensional shallow-water equations.

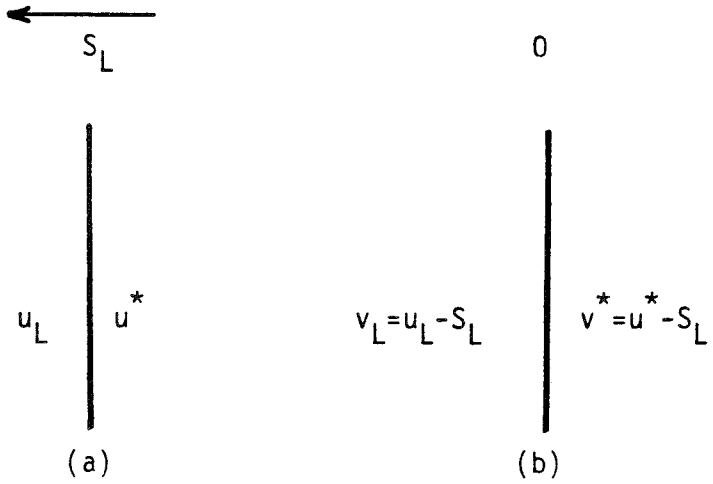

Fig.4: Transformation to a steady frame of reference moving with the shock speed.

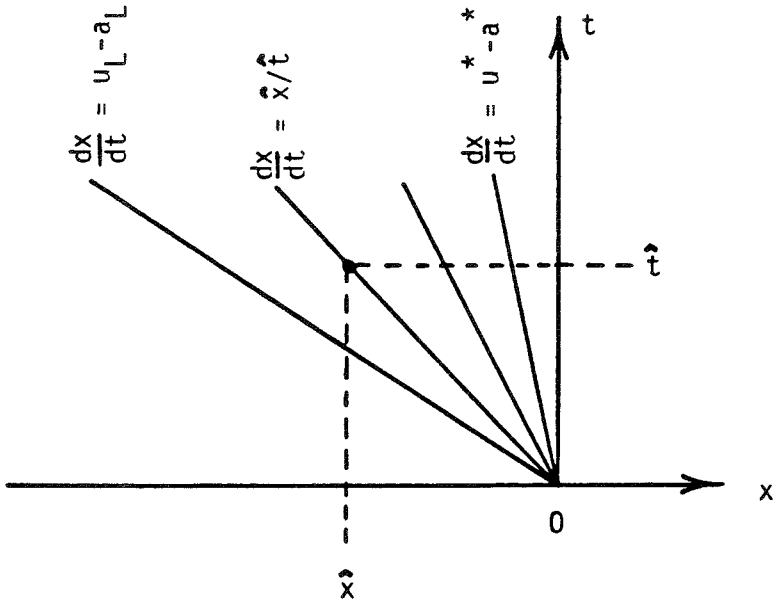

Fig.5: Determination of the solution inside a rarefaction fan.

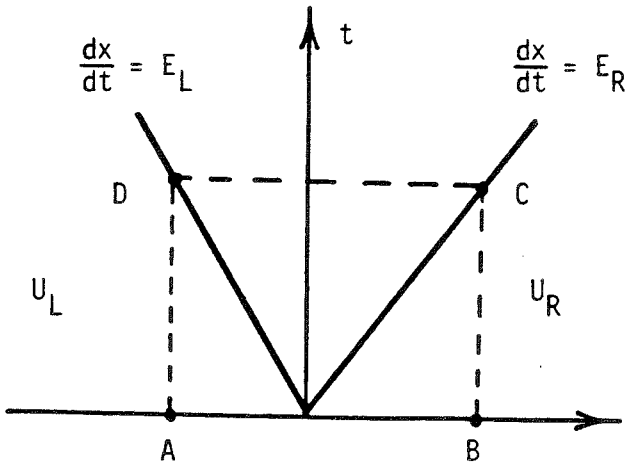

Fig.6: HLL Solution of the Riemann problem. E_L and E_R are estimates for the smallest and largest signal velocities.

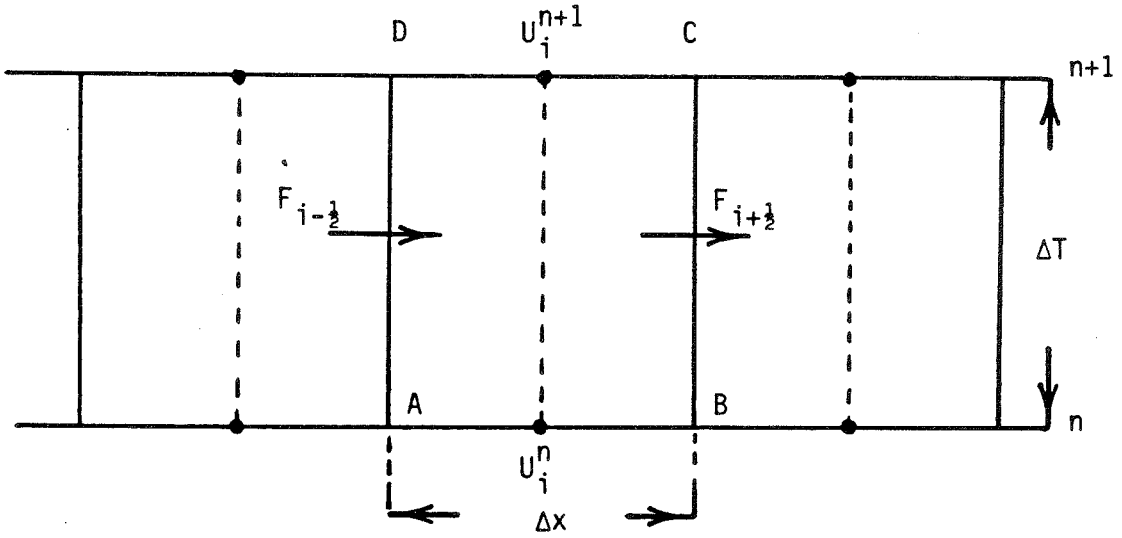

Fig.7: Computational grid in the x-t plane. The computing cell i has dimensions ΔX by ΔT.

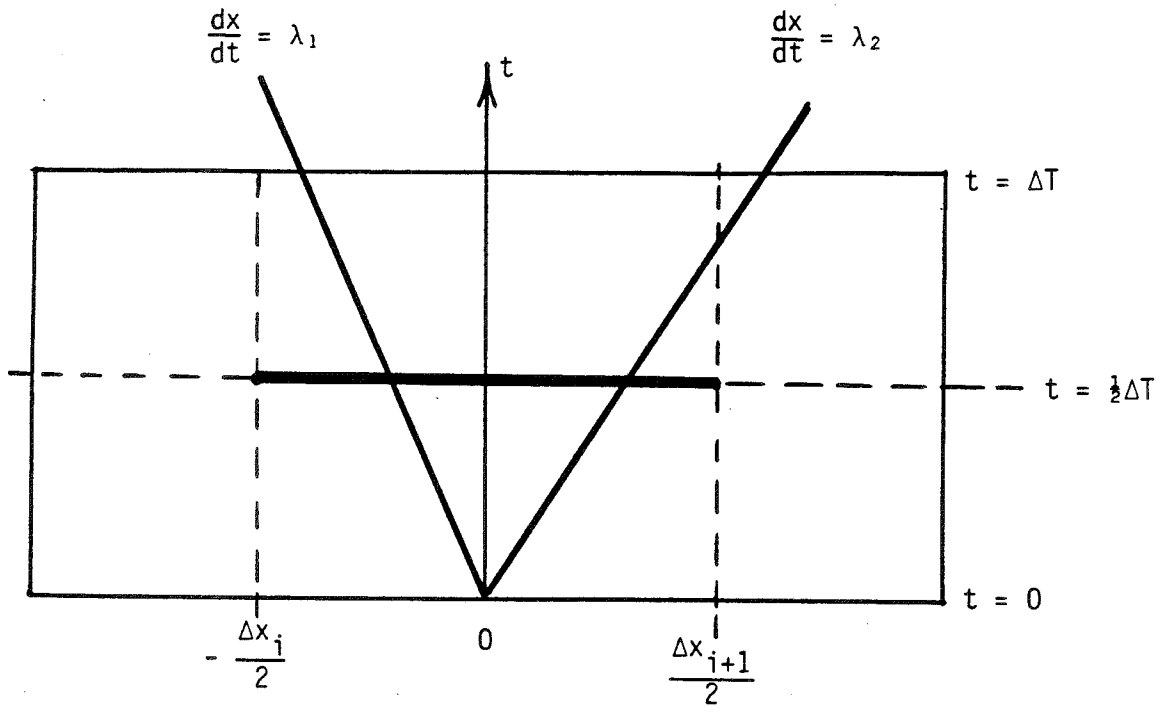

Fig.8: Integration path for the WAF-flux evaluation. The illustration is for a 2 x 2 system such as the one dimensional shallow water equations.

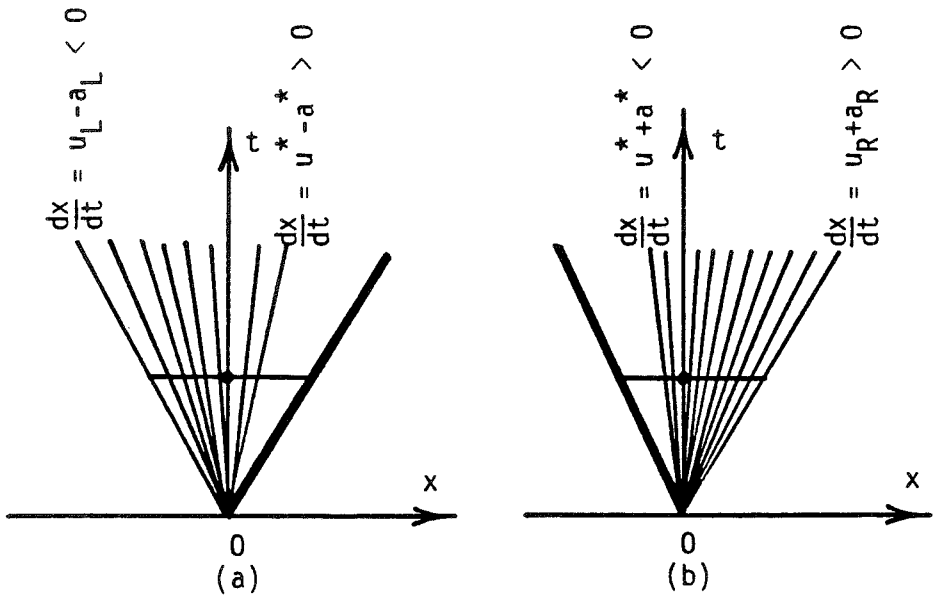

Fig.9: Locally sonic flow. Case (a) shows the solution of a Riemann problem where the left wave is a rarefaction centered around the t-axis, i.e. u = a along the t axis. Case (b) illustrates the sonic-flow situation for a right rarefaction.

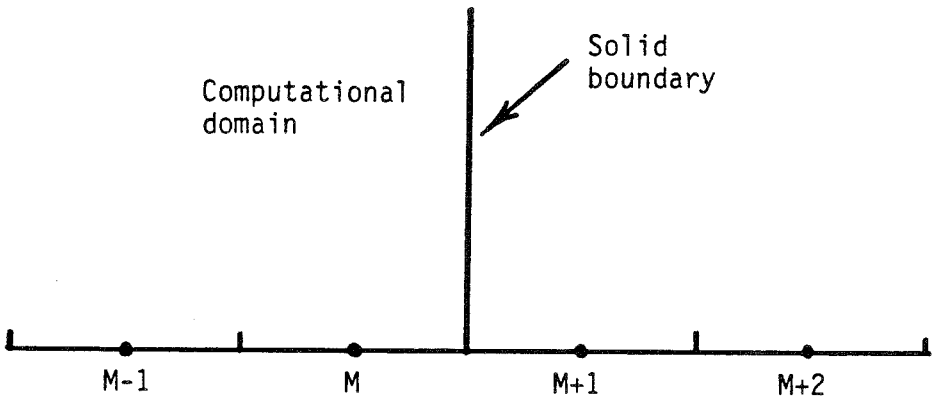

Fig.10 Right-hand boundary. Two fictitious states M+1 and M+2 are required.

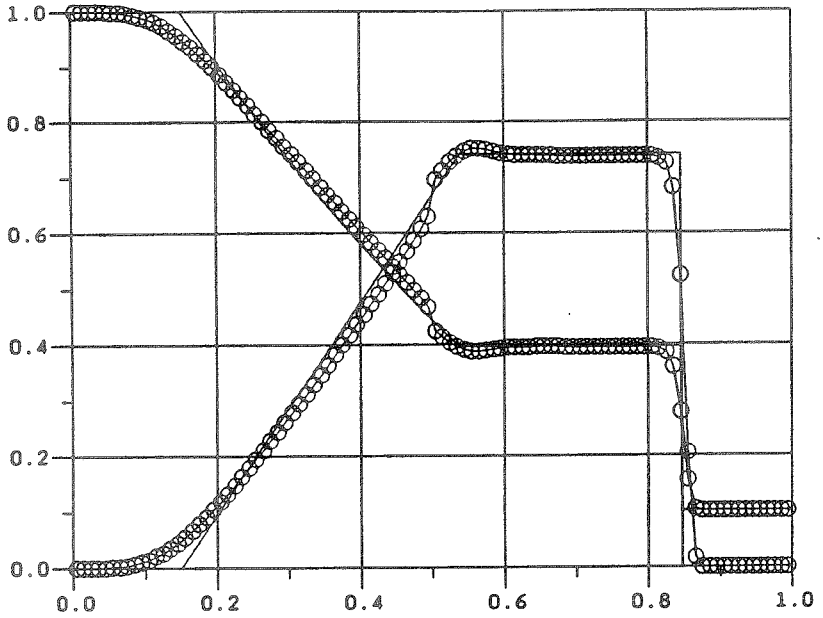

Fig.11: Test 1 solved by the Godunov's Method with the exact Riemann solver (symbol). The exact solution (shown by the full line) is also given for comparison.

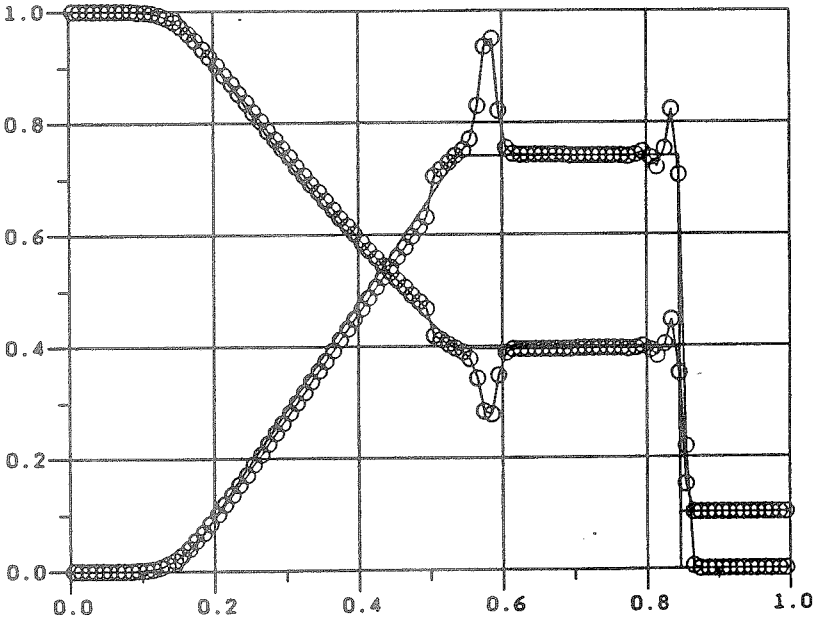

Fig.12: Test 1 solved by the Richtmyer-Morton scheme (symbol). The exact solution (line) is shown for comparison.

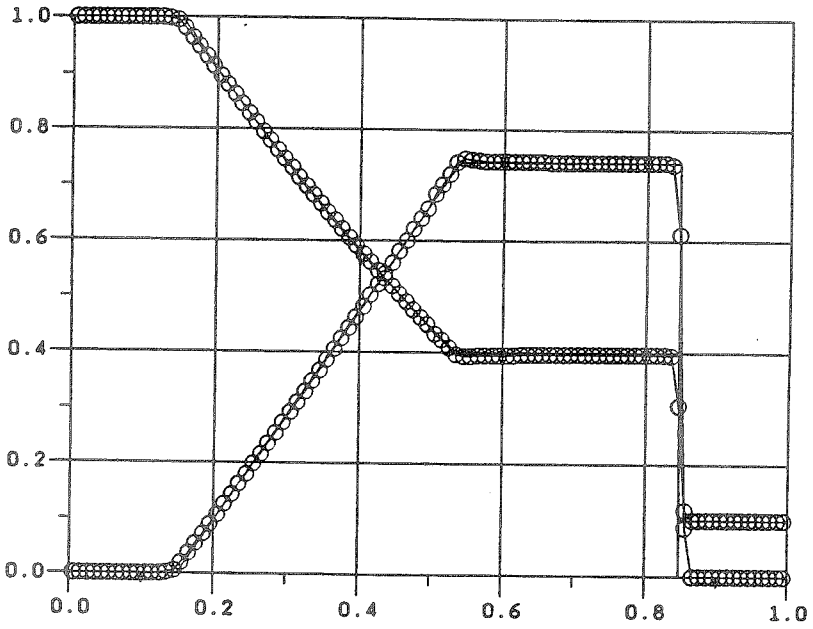

Fig.13: Test 1 solved by the WAF method (symbol). The exact solution isgiven by the single full line.

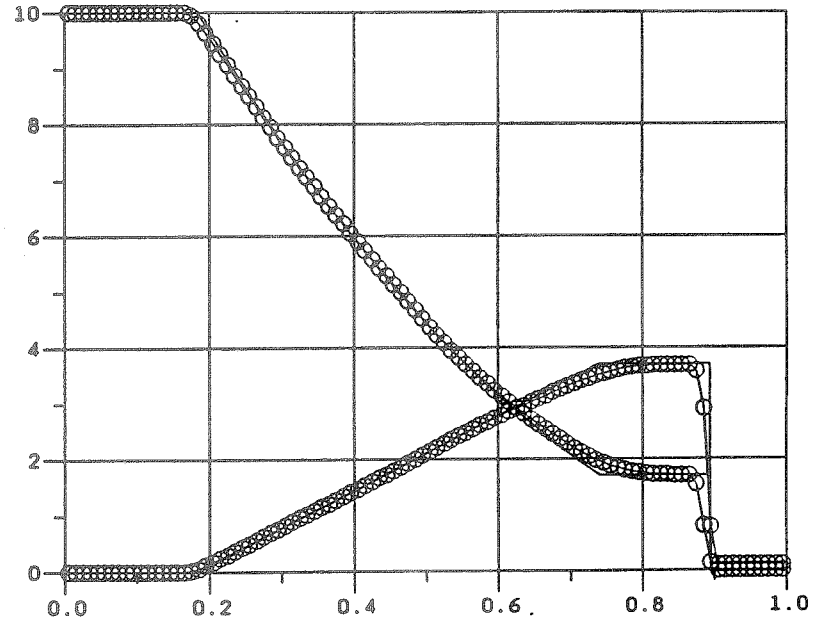

Fig.14: Test 2 solved by the WAF Method using the HLL Riemann Solver (symbol). The exact solution is given by the full line alone.

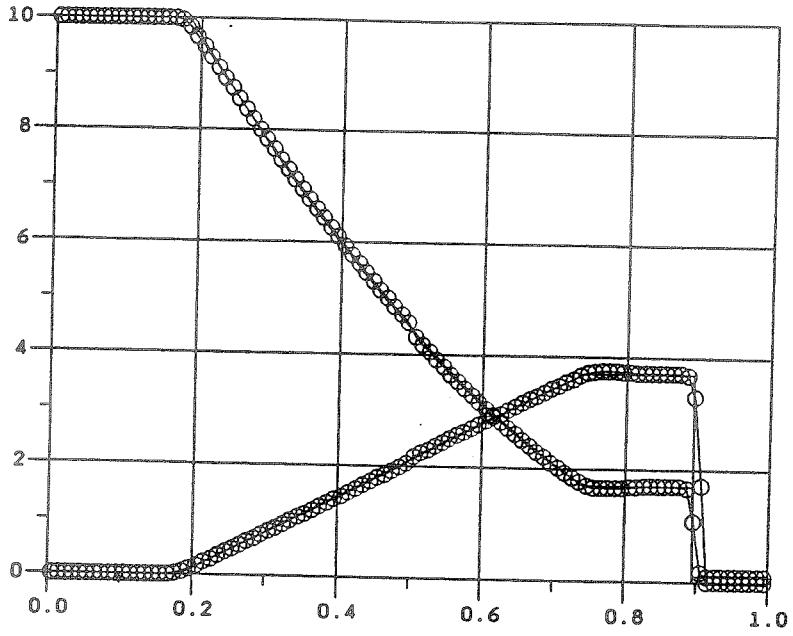

Fig.15: Test 2 solved by WAF using the TR approximate Riemann solver (symbol). The exact solution is given by the single full line.

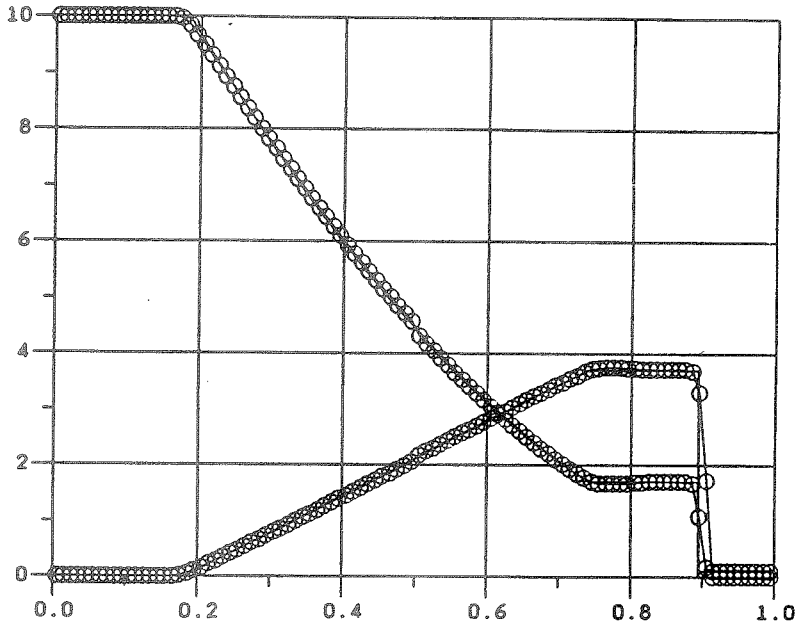

Fig.16: Test 2 solved by WAF with the exact Riemann solver (symbol). The single line shows the exact solution.

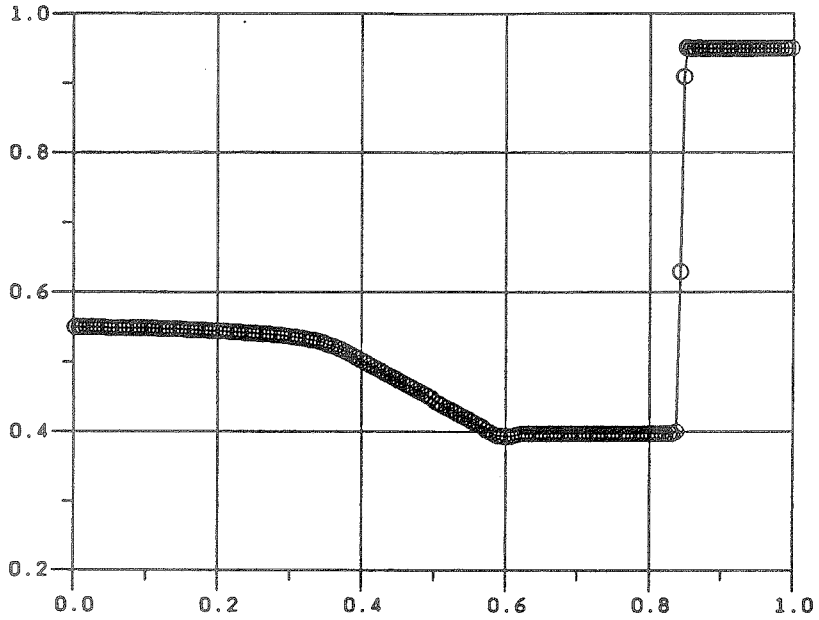

Fig.17: Test 2 solved by WAF with the TR approximation at time t = 0.8 after reflection from solid walls. Quantity shown is depth variable Φ.

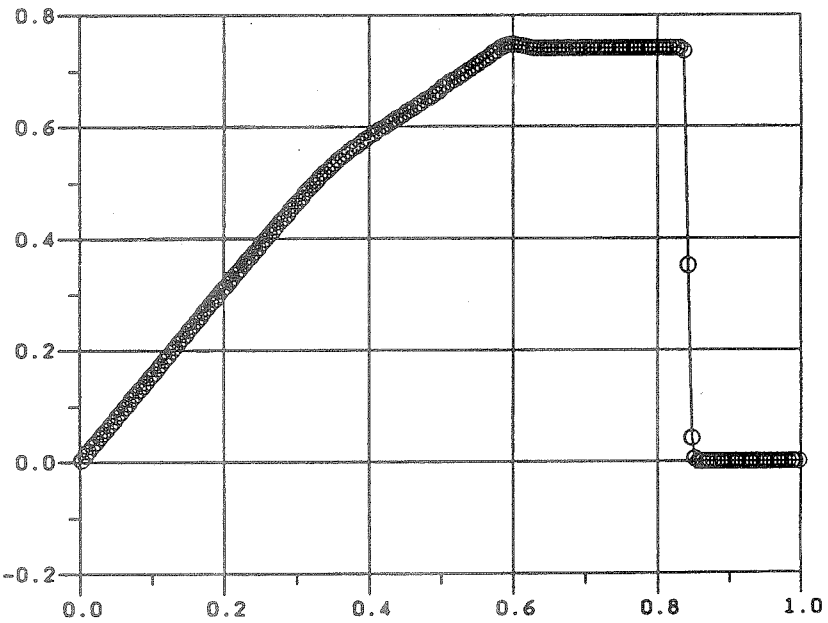

Fig.18: Test 2 solved by the WAF Method with the TR approximation at time t = 0.8 after refection from solid walls. Quantity shown is velocity.

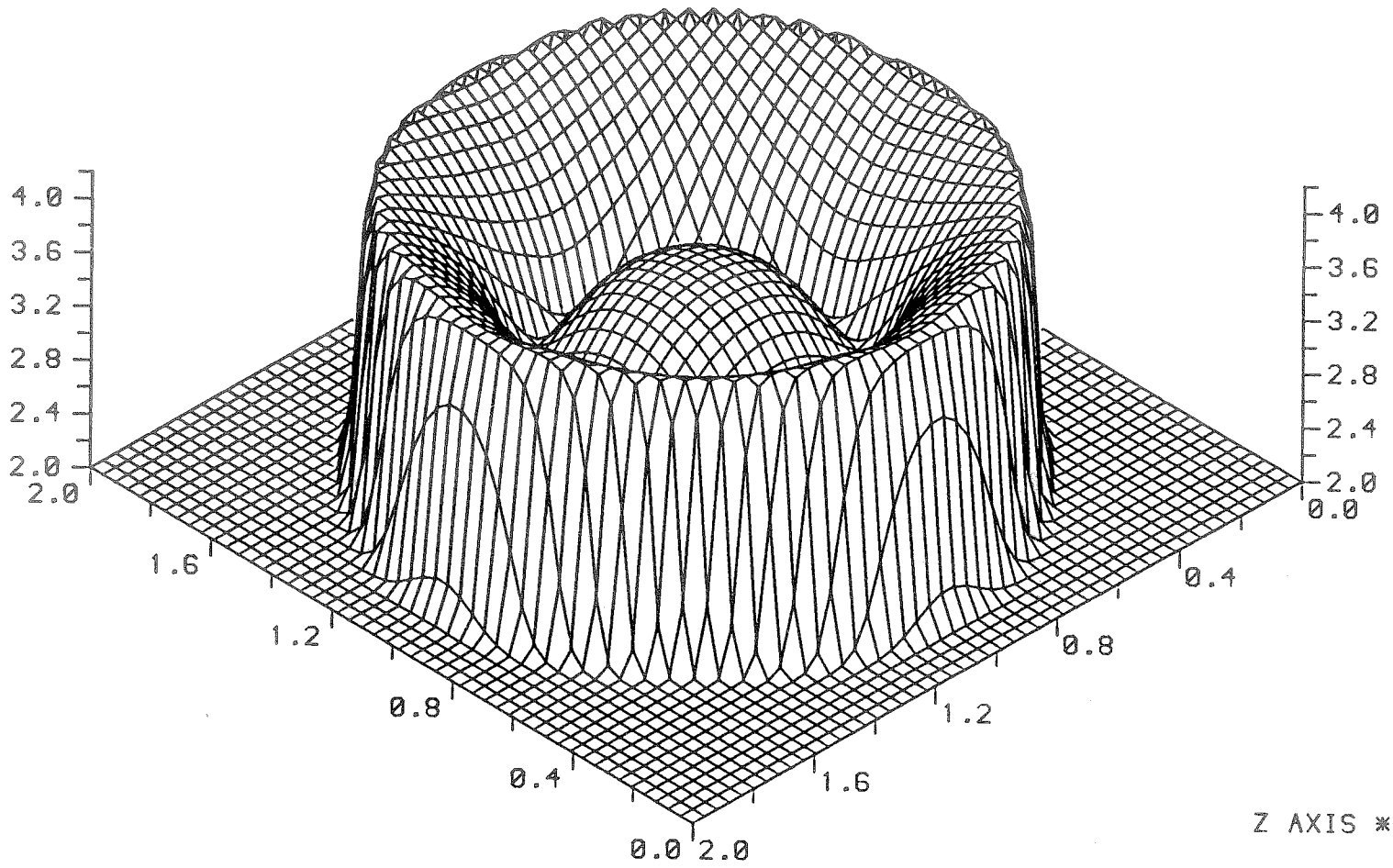

FIGURE : COLLAPSE OF A CIRCULAR DAM. NUMERICAL SOLUTION FOR DEPTH AT TIME = 0.5000 UNITS.