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PhD THESIS

Academic Year 2001-2004

RIZAL ZAHARI

Progressive Damage Analysis of Composite Layered Plates and Shells using Finite Strip Methods

Supervisor: Dr. A. El-Zafrany

April 2005

This thesis is submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy

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Abstract

As far as design of composite structures is concerned, it is important that we are able to accurately determine the failure modes and damage progression of structures made of composite materials. This will in turn, aid structural designers to develop reliable and safe designs which will exploit the advantages offered by composite materials. With the development of numerical analysis strategies towards non-linear progressive failure algorithms and the increase in computing capacity, it has now become possible to replace the existing experimental data with numerical results. Verified numerical models offer a practical method for exploring parametric study. Hence, it is essential to devise a reliable progressive damage scheme in order to predict the damage propagation in composite laminated structures using efficient computational tools such as the finite strip method.

In this thesis, a new finite strip method, for non-linear stress analysis based on the tangential stiffness matrix has been developed using the new concept of polynomial and spline-type finite strip elements, with Mindlin and Reissner plate-bending theories for composite plates and shells. A progressive damage methodology and algorithm for composite laminates was successfully developed for the new finite strip methods using stress-based failure criteria.

A finite strip analysis programming package which is capable of performing non-linear progressive damage analysis for composite plates and facetted shells, has also been developed with *Mindlin* and *Reissner* plate bending elements.

Validation of the developed finite strip package has been successfully carried out by comparing the results with corresponding results obtained with the finite element analysis using ABAQUS and with some published experimental results. Good comparison with the finite element results and experimental results were observed through various test cases, confirming the accuracy and reliability of the new derivations and the programming package.

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List of Notations and Abbreviations

$\underline{B}$	Strain-displacement matrix
$\underline{D}$	Material stiffness matrix
E	Young's Modulus
h	Plate thickness
$\overline{U}$	Strain energy
l, m, n	Directional cosines
v	Poisson's ratio
$\underline{\sigma}$	Stress vector
$\underline{\varepsilon}$	Strain vector
$\underline{\tau}$	Shear stress vector
$\underline{\mu}$	Shear modulus
N	Shape functions
$\underline{\delta}$	Nodal displacement vector
γ	Shear strain
u, v, w	Displacement component in x, y and z directions, respectively
$\underline{R}$	Residual force vector
$\underline{K}$	Element stiffness matrix
θ	Slope angle
$\underline{\theta}$	Rotational vector
CPT	Classical plate theory
FSM	finite strip method
F.I	Failure index
SRF	Stiffness reduction factor
B	Width in-y direction
L	Length in x-direction
$\Re$	Element rotation matrix

Some other parameters are defined within the text.

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CHAPTER 1

1. Introduction

1.1 General Introduction

The finite strip method (FSM) has been well known for its efficient and accurate procedures in analyzing the structural response and behaviour of structures with regular shapes, such as prismatic plates and shells structures. A full three-dimensional finite element analysis of such structures is often unnecessary and is computationally expensive and less efficient as compared with the FSM method. In addition, the FSM only requires one-dimensional mesh which makes the data preparation much easier than the finite element method.

The finite strip method can be regarded generally as a specialization of the finite element method using the displacement approach in conjunction with the virtual work principles. The FSM uses simple polynomial functions in some directions and a continuously differentiable smooth series in other directions, which satisfies a priori the boundary conditions at the ends of the strips. In contrast, the finite element method uses polynomial displacement functions in all directions.

In recent years, the application of composite materials in plate and shell structures has been growing at a rapid pace. A substantial amount of research which has been carried out in the field of composite structures has shown that these materials are ideal for structural applications in a number of engineering disciplines such as aerospace, automotive, civil and mechanical engineering. This is due to the fact that composite materials possess high strength-to-weight and high stiffness-to-weight ratios. In addition, composite materials have better fatigue characteristics and corrosion resistance than the conventional metallic materials.

One of the advantages of using composite materials is that their strength and stiffness can be tailored to meet specific design requirement. This can be done simply by varying the fibre orientation or stacking sequence of the fibres and matrix materials.

Unlike conventional metallic structures, composite structures may fail under several failure modes such as matrix cracking, fibre-matrix debonds, fibre breakage, and delamination. The damage initiation in a composite laminate often occurs when one or more layer in the laminate fails in one of the above mentioned failure modes. It is important to note that initiation of damage does not necessary mean that the structure can no longer carry any further load. Instead, the final failure of the laminate often occurs due to the propagation of local damage as the load is increased. This is also due to the fact that some of the failure modes may be present locally in the structure without affecting the overall strength of the overall structure.

The ability to accurately determine the failure modes and their progression is important for predicting the performance of composite structures which will in turn, aid structural designers to develop reliable, and safe designs which will exploit the advantages offered by composite materials. Hence, it is essential to devise a reliable progressive damage scheme in order to predict the damage propagation in composite laminated structures using efficient computational tools such as the finite strip methods.

1.2 Research Objectives

The overall objective of the PhD research is to develop a progressive damage analysis methodology for stress analysis of composite layered plate and faceted shells using new derivations of finite strip methods. The newly derived finite strip methods were carried out using a new type of finite strip elements (Razzaq, 2003), where in this work, *Mindlin-type* and *Reissner-type* (both *Hermitian* and *Spline* interpolations) were considered. The progressive damage phenomenon was modelled using stress-based failure criteria and an appropriate material degradation method. In order to achieve the research objectives the following tasks have been performed:

i) Derivations of improved finite strip equations for non-linear stress analysis of composite laminated plates and faceted shells using Mindlin and Reissner plate bending theories.

ii) Development of the progressive damage analysis algorithm which involves the material degradation model implementation for composite layered plates and shells using the newly derived finite strip methods.

iii) Development of a computer program for non-linear stress analysis based on the new finite strip equations which include the progressive damage analysis.

iv) Validation of the developed work via several case studies which include the comparison of results with those obtained via the commercial finite element software ABAQUS and with existing experimental results.

1.3 Thesis Layout

Chapter 1 gives the general introduction about the finite strip methods, the importance of composite material and the progressive failure damage of composite plates and shells, which leads to the research objectives.

Chapter 2 presents the literature review of the recent development of finite strip analysis of composites and the non-linear progressive damage methodology of composite plates and shells using finite element and finite strip methods.

Chapter 3 gives the overview of composite materials which explains the importance of composite materials and the fundamental terms used in composite materials. The constitutive stress-strain equations for a layer of composite laminate with respect to local and material axes are also summarized. It also discusses the commonly used failure criteria, and the chapter ends with some discussions regarding the progressive failure methodology for composite laminates.

Chapter 4 reviews the derivations for *Mindlin-type* finite strip equations for composite plates and shells based on the Mindlin plate bending theory. The derivations are based on a new *Mindlin-type* finite strip element using the tangential stiffness matrix approach. It also contains a special case of large deflections which due only to the out-of-plane displacements.

Chapter 5 presents the finite strip analysis of composite plates and shells, which is based on a new derivation of the higher order shear element (*Reissner-type* element) using the tangential stiffness matrix approach. Both the *Hermitan* and *Spline* interpolations are employed for this type of element. It also contains the derivations for the special case of symmetric composites.

Chapter 6 discusses the progressive damage methodology and how it can be implemented into a computer program. It begins with the summary of the faceted shell finite strip elements which are employed in the analysis of shell structures such as box structures and stiffened plates, a non-linear static stress analysis algorithm, based on the new derived equations given in chapters 4 and 5 is also presented. Finally, a progressive failure algorithm and a damage model implemented in the finite strip analysis are also explained.

Chapter 7 explains the structure of the developed finite strip package in detail. The computer package was coded in FORTRAN 77 and consists of several modules. This chapter also illustrates the links between main subroutines for Mindlin-type, and Reissner-type (Hermitian and Spline) finite strip elements.

Chapter 8 discusses the results obtained from non-linear finite strip analysis using the developed computer program. Validation of the computer program is carried out by comparing the results of several test cases with the results obtained from the finite element analysis using ABAQUS software. Parametric studies were also performed on a number of selected case studies in order to investigate the effect of certain parameters in the composite plates and faceted shells.

Chapter 9 summarises the conclusions of the PhD research and several recommendations for future work.

Appendix A explains why certain integrated $\underline{D}$ matrices become zero in the analysis of symmetric composite laminates of the Reissner-type elements.

Appendix B shows several samples of input data file for each type of finite strip elements.

Appendix C presents the explanation of the user written subroutine USDFLD which is used in ABAQUS to model the progressive damage analysis.

CHAPTER 2

2. Literature Review

2.1 Introduction

This chapter presents the summary of the past work which has been carried out in the field of composite laminated structures. Past and recent publications on the development of finite strip methods to analyse composite laminates, and progressive failure or damage analysis of composite laminated structures using finite element and finite strip analysis have been reviewed.

2.2 Development of finite strip method

The finite strip method (FSM) can be regarded generally as a specialization of the finite element method (FEM) and both methods implement the multi-field forms of the Rayleigh-Ritz method as explained by Dawe (2002). The properties of a strip are based on the use of an assumed displacement field in conjunction with the potential energy or virtual work principles. The first publication on the FSM was by Cheung (1968), which concerned only the linear static analysis of single rectangular plates having a pair of opposite ends simply supported.

Wittrick (1968a, 1968b) and Smith (1968) were the pioneers in the development of the exact FSM for predicting the buckling stresses and natural frequencies of flat structures. The strip properties are based on the direct solution of the governing differential equations of classical plate theory (CPT).

There are many types of the FSM which exist so far as described by Dawe (2002). The two major types may be referred to as the semi-analytical finite strip method and the spline finite strip method. These are distinguished from one another by the nature of the variation of the displacement quantities along the width of the strip. In the semi-analytical FSM, a multi-term series of analytical functions such as beam eigenfuctions and trigonometric functions are used, whilst in the spline FSM, series of polynomial spline functions, such as the B-spline functions are employed.

There are advantages and disadvantages of each method. For instance, the semi-analytical FSM can be more accurate than spline FSM for orthotropic structures with diaphragm ends. Also, where the true mode shape of buckling or vibration is purely sinusoidal longitudinally, the semi-analytical FSM can be based on the use of only a single term representation of the displacements over one longitudinal half wave. Although it is efficient, the semi-analytical FSM suffers from drawbacks when dealing with plate problems involving anisotropic materials, abruptly changing properties, concentrated loading, continuous span, internal openings and interior support. In order to overcome these problems, the spline FSM was introduced.

Thus, in more general situations, the spline FSM is much more versatile method in incorporating anisotropic material properties, abrupt loading conditions and complex boundary conditions.

The semi-analytical FSM in the context of (CPT), has been used widely for the analysis of buckling and vibrations of flat plate structures which has been considered

by many researchers such as, Przemieniecki (1973), Turvey and Wittrick (1973), and Plank and Wittrick (1974) who employed single term type analyses corresponding strictly to diaphragm end conditions. On the other hand, multi- term type analyses has been considered by Cheung and Cheung (1971), Petyt (1977), and Graves-Smith and Sridharan (1978).

The spline FSM which uses cubic B-spline interpolation in the longitudinal direction of the strip was first introduced by Cheung and Fan (1983) and Fan and Cheung (1984) for analyzing static and vibrations using classical plate theory (CPT).Lau and Hancock (1986) and Hancock et al. (1990) in their work have used similar B-spline FSM to perform buckling analysis of flat plate structures under the applied direct and shear stresses. Wang and Dawe (1987) first used B-splines to study the vibration of anisotropic laminated plates using the shear deformation plate theory.

The geometrically non-linear analysis has been introduced by Graves-Smith (1978) who employed semi-analytical FSM to study the post-buckling of single and prismatic plate structures made of isotropic material. On the other hand, Dawe and Wang (1998) have developed the B-spline FSM for non-linear postbuckling analysis of a single composite laminated rectangular plates.

Sheikh and Mukhopadhyay (2000) investigated the geometric nonlinear analysis of stiffened plates by finite strip methods. They adopted von Karman's nonlinear plate theory and the formulation was made in total Lagrangian coordinate system.

2.3 Finite Strip Analysis of composite Laminates

Kermanidis and Labeas (1995) developed an anisotropic finite strip element based on the semi-analytical method, for static and stability analysis of thin composite laminates. The element was formulated based on the transfer matrix method. Several numerical results were presented and were in excellent agreement with results obtained from analytical and finite element results. The authors concluded that, the main advantage of the proposed element was the semi-analytical formulation which was reflected in the accuracy of the results and the reduction of the number of degrees of freedom required in the analysis, compared to the finite element method.

Kong and Cheung (1995) developed a finite strip method based on the modified B3-spline function, which is capable of modelling plates with cut-outs. The formulation is based upon a third order plate theory which is applicable to both isotropic and laminated composite plates. Numerical examples which include linear and nonlinear static analysis, vibrations and postbuckling analysis have been performed and good results have been observed when comparing with the existing results. However, only linear static analysis was carried out for laminated composite plates.

Wang et al. (1996) investigated the critical buckling load of composite laminated plates using a developed higher order shear deformable plate finite strip element. The warping of cross-section and transverse shear deformation can be predicted by the higher order plate theories. The authors have concluded that the results of the finite strip analysis are similar or better than the finite element results, and that the formulation is capable of performing stability analysis of both thin and thick laminated plates efficiently and precisely.

Akhras et al. (1998) have developed a new finite strip method for the large deflection analysis of composite laminates. The method incorporates additional degrees of freedom for each nodal line and uses the higher order shear deformation theory. The author used the modified Newton-Raphson method for the non-linear solution and divergence was prevented by the relaxation technique. The new method produced results in a good agreement with existing solutions.

Zhong et al. (1998) introduced a new development of finite strip method which may produce very precise results. The displacement function is computed by integration instead of using traditional series which are derived from beam vibration eigenfunction. The numerical results were very precise, almost up to computer precision. However, only isotropic plates and sandwich panels were considered.

Loughlan (1999, 2001) employed semi-analytical finite strip method to investigate the effect of bend-twist coupling in symmetric angle ply laminates. The shear buckling of laminated rectangular plates and stiffened panels were studied. He then (Loughlan, 2000) carried out the same study on a square anti-symmetric angle-ply laminated plate under compressive buckling. He mentioned that the effect of bend-twist couplings can reduce the shear and compressive buckling capability of a laminated plate especially for thinner plates. He concluded from the buckling point of view that, it is important to optimize the lay-up configuration in order to minimize the degrading effect of bend-twist coupling and to make appropriate choice of stiffener depth.

Lam and Zou (2001) developed a higher order shear deformable finite strip element for linear static analysis of composite laminated plates. Transverse shear effects and warping of the cross-section have been accurately modelled. Good comparison between the finite element results and analytical solutions was obtained through various test cases. The comparison also demonstrated the accuracy of the proposed nine variables higher order shear deformation finite strip analysis.

Soares et al. (2001) developed higher – and first order isoparametric B-spline finite strip models, applied to static and free vibration analysis of composite laminated plate and shell structures, which can have arbitrary lay-ups, loading and boundary conditions. The model also consider the possibility of existing embedded or surface bonded piezoelectric actuators and sensors, enabling the passive control of displacements and natural frequencies of this type of structures. Numerical results have shown that the models are very versatile when compared to the Fourier-based finite strip models.

Dawe (2002) has presented a comprehensive review of aspects of the use of the finite strip method (FSM) in predicting a number of types of linear and non-linear response of composite laminated plate and shell structures. He discussed the use of two different types of FSM, i.e. the semi-analytical and spline method. He concluded that, the semi-analytical approach is capable of achieving greater accuracy and efficiency in some relatively simple situations, including the single-term analysis of "long" structures, whilst the spline approach is the more versatile procedure in incorporating anisotropic material properties and different boundary conditions. He also suggested that further work needs to be conducted in the areas of non-linear dynamic response, thermomechanical behaviour, buckling and vibration of single sandwich plate and dynamic buckling behaviour of plates and shells.

Loja et al. (2002) extended the work of Soares et al. (2001,) by using the B-spline finite strip which was applied to plates and shells embedded and surface bonded piezoelectric sensors. The authors employed simulated annealing, as an optimization technique in order to improve the performance of composite adaptive structures.

Roufaeil and Tran-Cong (2002) developed a finite strip (Mindlin element) solution for composite laminated plates using two-node and three-node strip using layerwise model. The displacement field w is assumed to be constant through the thickness while the in-plane displacements u and v are assumed to vary linearly through lamina thickness. Some examples were considered and the results show excellent agreement with the corresponding exact solutions.

A new concept of finite strip elements has been introduced by Razzaq (2003) in his PhD thesis. Hermitian, Lagrangian and spline-type interpolations have been employed independently along the two axes of the plate mid-plane. Mindlin, Reissner and Kirchhoff theories have been applied in the derivations of the finite strips, for isotropic and composite materials. The new elements have also been extended to work as faceted shell elements. However, non-linear equations developed are sensitive to loads and diverge at smaller loads than expected. No damage assessment was included in his work.

Wang and Zhang (2005) developed a layerwise B-spline finite strip method for free vibration analysis of thick and thin composite laminates. The development was based on the first order shear deformable theory. In this method the composite laminated plates were divided into a number of numerical layers in the thickness direction. A linear variation of in-plane displacements were assumed within each numerical layer to represent the through thickness shear warping. Numerical results showed that the method was accurate for the prediction of natural frequencies of truly thick and thin composite laminates.

2.4 Progressive Damage Analysis

The failure analysis of composites which was performed by Lee (1982) can be regarded as one of the first finite element based failure analysis. He performed a three-dimensional finite element analysis and employed his own direct-mode determining failure criterion to predict the failures. He evaluated the stresses at the center of each element and the stresses at the centre of the interface of each element to identify the failure. The element stiffness matrix was modified depending on the modes of failure, and equilibrium was then re-established to give a new stress distribution and subsequent failure zones. This procedure was repeated until the maximum load carrying capability of the laminate was achieved. The failure methodology was applied to a plate with central hole subjected to uniaxial and biaxial loadings. However, no validation procedure was carried out.

Chang and Chang (1987) studied the progressive damage analysis of notched laminated composites subjected to tensile loading. Stiffness reduction method was employed in the progressive damage model in which after matrix failure the ply in the damage region was assumed to loose its transverse load carrying capability. For fibre failure, a micromechanics approach was used where the stiffness of damaged layer was assumed to depend on the extent of damage. Classical laminated theory was used

for stress analysis but geometric non-linearity was not considered.

Ochoa and Engblom (1987) presented a progressive failure methodology for composite laminates under uniaxial tension using a higher order plate theory with shear deformable elements. Hashin's failure criterion was used to predict fibre and matrix failures while Lee's failure criterion was used to predict delaminations.

Chang and Lesard (1991) studied the damage in laminated composite plates containing an open hole and subjected to compressive loading. They used the same stiffness reduction method as the one used by Chang and Chang (1987) but ignored the fibre breakage failure mode. The numerical results were compared with the results from experiments by Lesard and Chang (1991), and were found to be in good agreement with each other.

Tan (1991) has developed a progressive failure analysis using finite element method to analyse damage progression, stress-strain states and ultimate strength in composite laminates under tensile loading with centrally located holes. In his study, environmental effects in the form of thermal residual stresses due to moisture content were considered in the progressive failure analysis. Tsai-Wu failure criterion has been employed in the progressive failure analysis. Experimental comparisons have shown good correlation with ultimate strengths, damage progression and stress-strain states.

Reddy and Reddy (1992) devised a finite element method to calculate the first-ply failure load using both linear and non-linear static analyses on composite plates. The finite element model was based on the first order shear deformation theory. The Tsai-Hill, Tsai-Wu, Hoffman, maximum stress and maximum strain failure theories were used as the failure criteria to predict the failure of composite plates subject to in-plane tensile and transverse loadings. The authors concluded that the difference between the linear and non-linear loads was found much larger for the cases involving transverse loading than for the cases involving in-plane tensile loading.

Cheung and Akhras (1995) devised a finite strip method for the progressive failure analysis of composite laminates based on higher order shear deformation theory and Lee's failure criterion. The results of the finite strip method were in good agreement with existing analytical and numerical solutions. They also investigated the effects of fibre orientation and the number of plies on the load carrying capacity of composite laminates. However, only infinitesimal strain was considered in the formulation of the finite strip equations.

Reddy et al. (1995) developed a non-linear progressive failure algorithm where the Generalized Layerwise Plate Thoery (GLPT) of Reddy was used for the kinemetic description and the material was modelled a stable progressively fracturing solid. The geometric non-linearity was taken into consideration in the von Karman sense, and the extensibility of transverse normals was included in the finite element formulation. The progressive failure methodology was used to investigate the effect of the geometric non-linearity, span-to-depth ratio, stacking sequence and the boundary conditions at the supports on the first ply and the ultimate failure loads of composite laminates in bending. They also proposed a new stiffness reduction scheme, in which stiffness properties of the failed element were reduced gradually, resulting in partial unloading depending on the amount of damage it accumulates.

Eason and Ochoa, (1996) developed a shear deformable composite element with built-in progressive damage capability which can be incorporated into a commercial finite element program, ABAQUS, as a user element. The capability of the user element was demonstrated by comparing the finite element results with experimental results. The results were in good agreement with each other.

Kim et al. (1996) developed a progressive failure analysis for laminated composite beams using a beam finite element. In the damage assessment, Tsai-Wu and maximum stress failure criteria were employed to evaluate fibre breakage and matrix cracking. The element stiffness relations were reduced using degradation factors which were chosen using parametric studies and by analytic comparison with experimental results. Good correlations between the progressive failure analysis and the experimental results have been obtained. Generally, good results have been achieved in the prediction of progressive failure reponse and ultimate load, although the ultimate displacement varied by as much as 30 percent from experimental results.

Sleight et al. (1997) developed a progressive failure analysis procedure based on classical theory, using C1 plate and shell elements. A Newton-Raphson solution technique was used to solve the non-linear equations. Hashin, Christensen and maximum strain failure criteria were compared with converged stresses. If failures were detected, either an instantaneous or gradual reduction technique was used. Comparison between analytical solutions and experimental data has shown that good agreement was achieved. The overprediction of the final failure load for all failure criteria was also taken into account.

A progressive failure analysis approach was presented by Gummandi and Palazotto (1998) for cylindrical shells. A geometrically non-linear finite element formulation with large rotations capability based on the total Lagrangian approach has been used to predict the onset of various failure modes in laminated composite plates and shells. Hashin's and Lee's failure criteria have been employed to predict the failure modes including matrix cracking, fibre breakage and delamination. Load-carrying capabilities of the composite cylindrical shell structures were then discussed.

Padhi et al. (1998) presented a method to study the non-linear behaviour, first-ply failure and ultimate collapse of laminated composite plates with clamped edges, subjected to transverse pressure using the commercial finite element software, ABAQUS. Several failure criteria, including Hashin's and Tsai-Wu's were employed to predict the failure modes and were included in a user written subroutine called USDFLD. The prediction of the model agreed well with experimental results for different aspect ratios of the plate.

Naik and Murty (1999) suggested that a simple failure criterion can be used to identify the dominant failure mechanism and laminates can be designed using appropriate failure mechanism based criteria. The results from standard failure criteria indicate a significant difference in predicting the first ply failure. It was concluded that each failure criterion is tailored for a specific initiating failure mechanism.

Singh and Kumar (1999) studied the finite element analysis of post-buckling behaviour and progressive failure response of thin, symmetric laminates under

uniaxial compression and uniaxial compression combined with in-plane shear (positive and negative). They employed first order shear deformation theory. Tsai-Hill failure criterion was used to predict failure of a lamina and the maximum stress to predict the onset of delamination.

A progressive failure analysis for finite element method has been developed by Sleight (1999) for predicting the failure of composite laminated structures under geometrically non-linear deformations. Several failure criteria, including the maximum strain criterion, Hashin's criterion, Christensen's criterion, were used to degrade the material properties after failures. In general, the results obtained from the progressive failure analysis indicate good agreement with the existing test data.

Prusty et al. (2001) developed a first-ply failure analysis method for predicting the failure load of the laminated composite stiffened panels under various loading conditions. Eight-noded isoparametric elements for shells and three-noded curved beam elements for the stiffener are used for the analysis of stiffened panels. Several failure theories have been incorporated for the prediction of failure load.

Spottswood and Palazotto (2001) studied the material failure of a thin, curved composite panel subjected to transverse loading. The cause of the material failure was determined through a failure criterion based on non-linear finite element analysis technique known as the simplified large displacement/rotation (SLR) theory, which allows for large displacements but assumes small to moderate rotations. Results obtained using the SLR theory both with and without the addition of progressive failure criteria, were compared with existing experimental results. Results generated for the case incorporating a progressive failure criterion were found to be closer to the experimental data because of the element stiffness reduction which took place as a result of failure as deflection increased.

Zhang et al. (2003) developed a B-spline finite strip method to simulate large deflection and delamination behaviour of laminated composite plates subjected to transverse loading. In order to model progressive failure phenomena of composite laminates, both stress-based criteria and fracture mechanics-based criteria were employed, and the former for delamination whilst the later for delamination propagation. Good agreement between the finite strip method and the available experimental and analytical results has been achieved.

Ambur et al. (2004a) carried out progressive damage analysis of composite curved panel with or without circular cut-outs, using commercial finite element software, ABAQUS. The finite element implementation of this progressive failure analysis was developed for ABAQUS using a user subroutine called USDFLD, and Hashin's failure theory. The panels were subjected to axial compression loading well into their postbucking regime. Initial geometric imperfections were introduced in the model for certain test cases in order to obtain accurate results. Good agreement between experimental data and finite element results have been observed for most part of the loading range for the structural configuration considered.

Ambur et al. (2004b) extended their work by presenting experimental and finite element results for progressive failure of composite laminated stiffened panels with and without a notch and subjected to in-plane shear loading well into the postbuckling

region. Again, ABAQUS has been used together with the user subroutine USDFLD in order to model the material nonlinearity in the composite stiffened panels. The mode of failures (Hashin criterion) has been modelled by degrading the material properties. In general, good correlation between experimental results and finite element results has been noted.

Goyal et al. (2004) developed a progressive failure methodology to simulate the material degradation of a flat laminated panel with circular cut-outs due to intralaminar and interlaminar failures. Hashin failure criterion have been used to evaluate intralaminar failure, whereas decohesion elements coupled with fracture mechanics approach were used to predict failure due to delamination. The progressive failure methodology is implemented in the structural analysis code, ABAQUS using the user subroutine USDFLD. The methodology was validated using existing experimental data on the failure of quasi-isotropic panel with circular cut-outs loaded into the postbuckling region. Excellent agreement between the progressive failure analyses and the experimental results was achieved.

Kelly and Hallstrom (2004) investigated the behaviour of composite laminates subjected to transverse load experimentally and numerically. Two different types of square laminates with centrally located holes were tested. Finite element analysis was carried out to determine the stress distribution within the laminate using the ABAQUS finite element software. Hashin's three dimensional form of failure criterion has been used to predict the first ply failure of the laminates. The results from the finite element model were found to be in good agreement with the experimental observations.

2.5 Conclusion

A literature survey has been carried out in order to gather information and knowledge of the past and latest developments in the finite strip methods and the progressive damage analysis of composite laminated structures. To the author's conclusion, in the literature concerning the damage assessments of composites, very little work was reported which studied the non-linear damage progression in composite laminates using finite strip methods. Thus, it is one of the objectives of the author to develop the progressive damage analysis of composite laminates using finite strips equations. The non-linear equations will be derived using the tangential stiffness matrix approach, which is an improvement to the work of Razzaq (2003) with all integrations over the plate thickness carried out analytically.

CHAPTER 3

3. Composite Materials

3.1 Introduction

The advantage of the high strength-to-weight and high stiffness-to-weight ratios of composite materials has attracted designers of automotive, aerospace and marine industries. With the advent of powerful computing technology, there has been an increase in the numerical modelling of composite thin-walled structures. Unlike metallic structures, composite structures typically have a complex behaviour and response to loading. Therefore, it is essential to understand the behaviour of composite structures subject to various loading conditions before they can be safely designed.

This chapter begins with an overview of composite materials which explains some basic terminologies. The constitutive stress-strain equations for a lamina or layer of a composite plate with respect to its material and local axes are summarized. Next, classical theories of failures for composite materials are also discussed. Finally, the progressive failure methodology employing the stress based failure criteria applied to composite laminates is discussed.

3.2 Overview of Composite Material

A composite material is formed when two or more materials are combined so that the properties are different (and usually better) from those of individual constituents.

Most of the composites are made up of a continuous medium which is called the matrix and a discontinuous medium, which is usually stronger and harder one, is called the reinforcement fibre. The properties of a composite are dependent on the properties of the constituent materials, their distribution and interaction.

Common types of composite materials can be summarized as follows:

1) **Particulate composites**: particles in a matrix for example concrete, silver flakes in paint, carbon based cermets and mica flakes in glass matrix.

2) **Fibrous composites**: fibres in a matrix such as glass, carbon (graphite), Kevlar and Boron.

3) **Laminated composites**: layers of various materials bonded together for instance, laminated glass, clad metals and laminated fibrous composites.

A *lamina* or ply is a typical sheet of composite material which represents a fundamental building block. A fibre reinforced lamina consists of many fibres embedded in a matrix material, which can be a metal like aluminium, or a non-metal like thermoset or thermoplastic polymer. On the other hand, fibres can be continuous or discontinuous, woven, unidirectional, bidirectional, or randomly distributed such as that shown in figure 3.1.

A *laminate* is a collection of *laminae* stacked to obtain the desired stiffness and thickness. For instance, unidirectional fibre-reinforced *laminae* can be stacked together in such a way that the fibres in each lamina are oriented in the same or different directions, such as shown in figure 3.2. The sequence of various orientations of a fibre-reinforced composite layer in a laminate is called the *stacking sequence* or *lamination scheme.*

The stacking sequence and the material properties of the individual lamina provide an added advantage and flexibility to structural designers to tailor the stiffness and strength of the laminate to achieve the desired structural stiffness and strength.

3.3 Review of Stress-Strain Equations

3.3.1 Constitutive Equations for Laminates

In formulating the constitutive equations of a lamina the following assumptions are considered:

1) A lamina is a continuum; i.e. no gaps or empty spaces exist.
2) A lamina behaves as a linear elastic material.

The first assumption implies that only macromechanical behaviour is considered in the formulation of the constitutive equations. The second assumption implies that the generalized Hooke's law is valid.

With reference to figure 3.3, a lamina is normally lies on or parallel to the x-y plane, with x, y, z being its geometrical or local axes. The x', y', z' are the material axes where x' is usually in the fibre direction. It differs from layer to layer depending upon the orientation of the fibres, which is defined by the angle of orientation, θ. The properties of each lamina can be defined in terms of:

i) E_{11}, Young's Modulus of elasticity in fibre direction

ii) E_{22}, Young's Modulus of elasticity in y' direction
(If E_{33} is required, the Young's modulus the z-direction, we take $E_{33}=E_{22}$)

iii) ν_{12}, which is the major Poisson's ratio (minor Poisson's ratio, ν_{21} can be obtained from $\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}}$

iv) μ_{12}, the shear modulus of $x' - y'$ plane

v) $\mu_{23} = \mu_{31}$, the transverse shear moduli (which are not always given)

For composite plates $\sigma_z = \sigma_{z'}$ and is negligible and shall not be considered in the constitutive equations. The lamina is orthotropic with respect to $x' - y' - z'$ axes, and the following Hooke's law equations can be employed

$$\varepsilon_{11} = \frac{1}{E_{11}}\sigma_{11} - \frac{\nu_{21}}{E_{22}}\sigma_{22} \tag{3.1}$$

$$\varepsilon_{22} = \frac{\nu_{12}}{E_{11}}\sigma_{11} - \frac{1}{E_{22}}\sigma_{22}, \sigma_{33} \approx 0 \tag{3.2}$$

$$\gamma_{12} = \frac{1}{\mu_{12}}\tau_{12}$$

$$\gamma_{23} = \frac{1}{\mu_{23}}\tau_{23} \tag{3.3}$$

$$\gamma_{31} = \frac{1}{\mu_{31}}\tau_{31}$$

where subscripts 1-2-3 are used to represent $x' - y' - z'$.

Hence

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \end{bmatrix} \tag{3.4}$$

and by matrix manipulation it can be shown that:

$$\sigma_{11} = \frac{\begin{vmatrix} \varepsilon_{11} & -\frac{\nu_{21}}{E_{22}} \\ \varepsilon_{22} & \frac{1}{E_{22}} \end{vmatrix}}{\begin{vmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} \end{vmatrix}} = \frac{\frac{1}{E_{22}}(\varepsilon_{11} + \nu_{21}\varepsilon_{22})}{\frac{1}{E_{11}E_{22}}(1 - \nu_{12}\nu_{21})}$$

Hence,

$$\sigma_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}(\varepsilon_{11} + \nu_{21}\varepsilon_{22}) \tag{3.5}$$

and,

$$\sigma_{22} = \frac{E_{22}}{\left(1 - \nu_{12}\nu_{21}\right)}\left(\varepsilon_{22} + \nu_{12}\varepsilon_{11}\right) \tag{3.6}$$

By using planar rotations

$$\sigma_{11} = \ell^2\sigma_x + m^2\sigma_y + 2\ell m\tau_{xy} \tag{3.6}$$

$$\sigma_{22} = m^2\sigma_x + \ell^2\sigma_y - 2\ell m\tau_{xy} \tag{3.7}$$

$$\tau_{12} = -\ell m\sigma_x + \ell m\sigma_y + \left(\ell^2 - m^2\right)\tau_{xy} \tag{3.8}$$

$$\tau_{31} = \ell\tau_{zx} + m\tau_{zy} \tag{3.9}$$

$$\tau_{32} = -m\tau_{zx} + \ell\tau_{zy} \tag{3.10}$$

$$\varepsilon_{11} = \ell^2\varepsilon_x + m^2\varepsilon_y + \ell m\gamma_{xy} \tag{3.11}$$

$$\varepsilon_{22} = m^2\varepsilon_x + n^2\varepsilon_y - \ell m\gamma_{xy} \tag{3.12}$$

$$\gamma_{12} = -2\ell m\varepsilon_x + 2\ell m\varepsilon_y + \left(\ell^2 - m^2\right)\gamma_{xy} \tag{3.13}$$

$$\gamma_{31} = \ell\gamma_{zx} + m\gamma_{zy} \tag{3.14}$$

$$\gamma_{32} = -m\gamma_{zx} + \ell\gamma_{zy} \tag{3.15}$$

3.3.1.1 The Transverse Shear Component

From equations (3.9) and (3.10)

$$\begin{bmatrix}\tau_{31}\\ \tau_{32}\end{bmatrix} = \begin{bmatrix}\ell & m\\ -m & \ell\end{bmatrix}\begin{bmatrix}\tau_{zx}\\ \tau_{zy}\end{bmatrix} = \begin{bmatrix}\mu_{31} & 0\\ 0 & \mu_{32}\end{bmatrix}\begin{bmatrix}\gamma_{31}\\ \gamma_{32}\end{bmatrix}$$

Hence, it can be deduced that:

$$\begin{bmatrix}\tau_{zx}\\ \tau_{zy}\end{bmatrix} = \begin{bmatrix}\ell & -m\\ m & \ell\end{bmatrix}\begin{bmatrix}\mu_{31} & 0\\ 0 & \mu_{32}\end{bmatrix}\begin{bmatrix}\ell & m\\ -m & \ell\end{bmatrix}\begin{bmatrix}\gamma_{zx}\\ \gamma_{zy}\end{bmatrix}$$

or

$$\begin{bmatrix}\tau_{zx}\\ \tau_{zy}\end{bmatrix} = \underline{\mu}\begin{bmatrix}\gamma_{zx}\\ \gamma_{zy}\end{bmatrix} \tag{3.16}$$

$$\underline{\mu} = \begin{bmatrix}\ell & -m\\ m & \ell\end{bmatrix}\begin{bmatrix}\mu_{31} & 0\\ 0 & \mu_{32}\end{bmatrix}\begin{bmatrix}\ell & m\\ -m & \ell\end{bmatrix}$$

i.e.

$$\underline{\mu} = \underline{\mu}^{(\ell)} = \begin{bmatrix}\ell^2\mu_{31} + m^2\mu_{32} & \ell m\left(\mu_{31} - \mu_{32}\right)\\ \ell m\left(\mu_{31} - \mu_{32}\right) & m^2\mu_{31} + \ell^2\mu_{32}\end{bmatrix} \tag{3.17}$$

for special case of $\mu_{31} = \mu_{32} = \mu$, it can be proved that:

$$\underline{\mu}^{(l)} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.18}$$

3.3.1.2 The x-y Components

From equations (3.6)-(3.8), it can be shown that:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \ell^2 & m^2 & 2\ell m \\ m^2 & \ell^2 & -2\ell m \\ -\ell m & \ell m & \ell^2 - m^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{3.19}$$

Similarly, it can be deduced from equations (3.11)-(3.13) that:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \ell^2 & m^2 & \ell m \\ m^2 & \ell^2 & -\ell m \\ -2\ell m & 2\ell m & \ell^2 - m^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{3.20}$$

Let

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} d'_{11} & d'_{12} & 0 \\ d'_{21} & d'_{22} & 0 \\ 0 & 0 & d'_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \tag{3.21}$$

where as shown in equations (3.5) and (3.6):

$$d'_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}, d'_{12} = \frac{E_{11}\nu_{21}}{(1 - \nu_{12}\nu_{21})}, d'_{21} = \frac{E_{22}\nu_{12}}{(1 - \nu_{12}\nu_{21})}, d'_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}, d'_{33} = \mu_{12}$$

From planar rotation, it can also be proven that:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \ell^2 & m^2 & -2\ell m \\ m^2 & \ell^2 & 2\ell m \\ \ell m & -\ell m & \ell^2 - m^2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} \tag{3.22a}$$

Substituting from equations (3.20) and (3.21) into (3.22a) then, it can be deduced that:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \underline{D}^{(\ell)}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{3.22b}$$

where

$$\underline{D} = \begin{bmatrix}\ell^2 & m^2 & -2\ell m\\ m^2 & \ell^2 & 2\ell m\\ \ell m & -\ell m & \ell^2 - m^2\end{bmatrix}\begin{bmatrix}d_{11}' & d_{12}' & 0\\ d_{21}' & d_{22}' & 0\\ 0 & 0 & d_{33}'\end{bmatrix}\begin{bmatrix}\ell^2 & m^2 & \ell m\\ m^2 & \ell^2 & -\ell m\\ -2\ell m & 2\ell m & \ell^2 - m^2\end{bmatrix} \tag{3.23}$$

with

$$\underline{D} = \underline{D}^{(l)} = \begin{bmatrix}d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{32} & d_{33}\end{bmatrix} \tag{3.24}$$

then it can be deduced from equation (3.23) that:

$$\begin{aligned}
d_{11} &= \ell^4 d_{11}' + m^4 d_{22}' + 2\ell^2 m^2\left(d_{12}' + 2d_{33}'\right)\\
d_{22} &= m^4 d_{11}' + \ell^4 d_{22}' + 2\ell^2 m^2\left(d_{12}' + 2d_{33}'\right)\\
d_{12} &= d_{21} = \left(\ell^4 + m^4\right)d_{12}' + \ell^2 m^2\left(d_{11}' + d_{22}' - 4d_{33}'\right)\\
d_{33} &= \left(\ell^2 - m^2\right)d_{33}' + \ell^2 m^2\left(d_{11}' + d_{22}' - 2d_{33}'\right)\\
d_{13} &= d_{31} = \ell m\left[\ell^2 d_{11}' - m^2 d_{22}' - \left(\ell^2 - m^2\right)\left(d_{22}' + 2d_{33}'\right)\right]\\
d_{23} &= d_{32} = \ell m\left[m^2 d_{11}' - \ell^2 d_{22}' + \left(\ell^2 - m^2\right)\left(d_{12}' + 2d_{33}'\right)\right]
\end{aligned} \tag{3.25}$$

Notice that for l^{th} layer with $\theta = \theta_l$:

$$l = \cos\theta_l \ , m = \sin\theta_l \tag{3.26}$$

and the values of $d_{11}' \ , d_{12}' \ , d_{21}' \ , d_{22}' \ , d_{33}' \ , \mu_{31} \ , \mu_{32}$ are the same for all layers.

Define the stress components in terms of two vectors:

(a) Transverse shear stress:

$$\underline{\tau} = \begin{bmatrix}\tau_{zx}\\ \tau_{zy}\end{bmatrix} \tag{3.27}$$

(b) X-Y Stress Component

$$\underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{3.28}$$

Hence, the stress and strain equations for a point inside the layers can be partitioned as follows:

$$\underline{\sigma} = \underline{D}^{(\ell)} \underline{\varepsilon} \tag{3.29}$$

$$\underline{\tau} = \underline{\mu}^{(\ell)} \underline{\gamma} \tag{3.30}$$

where $\underline{D}^{(l)}$ and $\underline{\mu}^{(l)}$ are as given by equations (3.24) and (3.18) respectively.

3.4 Strength Analysis of Fibrous Composite Ply

For design purposes it is important to be able to reliably and quickly predict the composite strength for various combinations of loading in terms of a number of basic strength parameters with reference to the material directions. The basic strength parameters are determined by experiment. For these reasons the problem of macromechanical approach to composite strength analysis is preferable than the micromechanical approach which is more complicated.

The strength, such as the stiffness of a composite laminate depends on the individual ply properties, ply orientations and ply stacking sequences. Thus, the macromechanics stiffness and strength analyses of a ply coupled with the macromechanics stiffness analysis of a composite laminate are used to perform the strength analysis of a composite laminate. An assumption is made, in addition to those made in developing the ply stiffness and strength analyses and the laminate constitutive equation, that the strength behaviour of an individual ply is the same as the strength behaviour when the ply is part of the laminate. On macroscopic scale and within the limitations of the thin laminate theory, failure of an individual ply occurs in one, or in a combination, of the following failure modes:

1) longitudinal tension
2) longitudinal compression
3) transverse tension
4) transverse compression
5) in-plane shear
6) delamination

The strength analysis of a ply at a macroscopic level has the following assumptions

1) Theory of linear elasticity based on *Hooke's* Law
2) Plane stress
3) Macroscopic homogeneity.

The strength analysis of a ply based on a failure criterion involves the comparison of the actual stress at a point with the allowable values. The allowable values are some fraction of the material failure stresses values (failure strengths).

For an isotropic ply there are three basic failure strengths namely tensile, compressive and shear are required to completely define the ply failure strength under a multi-axial stress system. On the other hand, an orthotropic ply usually has five basic failure strengths required (measured in the material axes), which are obtained from tests:

1) X_t – longitudinal (fibre direction) tensile failure strength.
2) X_c - longitudinal (fibre direction) compressive failure strength.
3) Y_t – transverse (matrix direction) tensile failure strength.
4) Y_c - transverse (matrix direction) compressive failure strength.
5) S – in-plane shear failure strength.

These can be shown schematically in figure 3.4.

There are five most commonly used failure criteria and they are divided into non-interactive and interactive criteria. The non-interactive criteria assess the ply strength by comparing the actual stress (strain) in a direction with its corresponding basic failure strength (strain). They do not take into account the interaction between the other stresses (strains) in a multi-axial stress (strain) system. On the other hand, the interactive failure criteria take into account all the components of stress (strains) in a multi-axial stress system when assessing the ply strength.

The five common failure criteria are:

1) Maximum stress theories (non-interactive theory)

Ply failure occurs if any stress in the material axes directions exceeds the respective basic failure strength.

For tensile stresses: $\sigma_{11} < X_t$

$\sigma_{22} < Y_t$

For compressive stresses: $|\sigma_{11}| < |X_c|$

$|\sigma_{22}| < |Y_c|$

For shear stresses: $|\tau_{12}| < S$

If any of the above inequalities is not satisfied, then it is assumed that the ply has failed by a mechanism associated with X_t, X_c, Y_t, Y_c or S respectively.

2) Maximum strain theory (non-interactive)

It is similar to the maximum stress theory except that the strains are now the limiting values:

For tensile strains: $e_{11} < e_{x_t}$

$e_{22} < e_{y_t}$

For compressive strains: $|e_{11}| < |e_{x_c}|$

$|e_{22}| < |e_{y_c}|$

For shear strains: $|e_{12}| < e_s$

Where

e_{x_t} - longitudinal (fibre direction) tensile failure strain.

e_{x_c} - longitudinal (fibre direction) compressive failure strain.

e_{y_t} - transverse (matrix direction) tensile failure strain.

e_{y_c} - transverse (matrix direction) compressive failure strain.

e_s - in-plane shear failure strain.

3) Tsai-Hill theory (interactive theory)

Failure occurs when the following inequality is not satisfied:

$$\left[\frac{\sigma_{11}}{X}\right]^2 + \left[\frac{\sigma_{22}}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 - \left[\frac{\sigma_{11}}{X}\right]\left[\frac{\sigma_{22}}{X}\right] < 1$$

where $X = X_t$ or X_c and $Y = Y_t$ or Y_c depending on the signs of σ_{11} and σ_{22} and X_c and Y_c are absolute values. The left hand side expression in the above inequalitiy is denoted as the *Failure Index (F.I.)* and failure occurs when

$$F.I. \geq 1$$

4) Hoffman Theory

Failure occurs when the following inequality is not satisfied:

$$F_1 \sigma_{11} + F_2 \sigma_{22} + F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{33} \tau_{12}^2 + 2F_{12} \sigma_{11} \sigma_{22} < 1$$

where

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}, \qquad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}, \qquad F_{11} = \frac{1}{X_t X_c},$$

$$F_{22} = \frac{1}{Y_t Y_c} \qquad F_{33} = \frac{1}{S^2} \qquad F_{12} = \frac{-1}{2X_t X_c}$$

and X_c and Y_c are absolute values.

The left hand side expression in the above inequalitiy is denoted as the *Failure Index (F.I.)* and failure occurs when

$$F.I. \geq 1$$

5) Tsai-Wu Stress Theory

Failure occurs when the following inequality is not satisfied:

$$F_1 \sigma_{11} + F_2 \sigma_{22} + F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{33} \tau_{12}^2 + 2F_{12} \sigma_{11} \sigma_{22} < 1$$

where

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c} \qquad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \qquad F_{11} = \frac{1}{X_t X_c}$$

$$F_{22} = \frac{1}{Y_t Y_c} \qquad F_{33} = \frac{1}{S^2}$$

$$F_{12} = F_{12}^* \sqrt{F_{11} F_{22}} = \frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}$$

where $F_{12}^* = -1$

and X_c and Y_c are absolute values.

The left hand side expression in the above inequalitiy is denoted as the *Failure Index (F.I.)* and failure occurs when

$$F.I. \geq 1$$

3.5 Progressive Failure Analysis

It has been proved in a composite structure that there is a significant load carrying capability between the first material failure mode and the final failure mode. This is due to the fact that some of the failure modes may be present locally in the structure without affecting the overall strength of the overall structure.

Laminated composite structures may fail by matrix cracking, fibre breakage or by delamination of layers. The mode of failure depends upon loading, stacking sequence, and specimen geometry. Matrix cracking occurs when the applied loading exceeds the strength (Y_c or Y_t) of the matrix material. Similarly, fibre breakage occurs when the applied loading exceeds the longitudinal strength (X_c or X_t) of the fibre material. Delamination is a very complex phenomenon and can occur as peeling or opening, as a result of loading normal to the lamina. Delamination can also happen as ply shearing or tearing. Depending on the layup of the composite, lamina failure can occur individually or as a combination of all the three failure modes.

Failure of a composite laminate may be considered in two stages: first ply failure and post first ply failure.

The first ply failure method stipulates that a laminate is said to have failed when any individual ply in the laminate has failed due to the applied loads. The method to determine the load which causes the first ply failure in the laminate is relatively straightforward. For given low loads, strains or stresses in the plies are determined and a ply failure criterion is used to predict if any ply has failed. If no failure is predicted, the loads are increased and the process repeated until the first ply failure in the laminate is predicted. It is analogous to the yield limit in an isotropic material.

However, unlike an isotropic material, the first ply failure does not necessarily mean that the composite laminate cannot carry any additional load. It has been observed that even after the first ply failure, the laminate can still carry a significant amount of load, depending on the nature of the multidirectional laminate lay-ups. The last ply failure or ultimate failure is the load at which the laminate can no longer carry any further load. Between the first ply failure and ultimate failure, the laminate can carry a significant amount of load. This means that after first ply failure occurs, failure progression takes place until the whole laminate fails.

The post first ply failure method stipulates that after the first ply failure in a laminate, the load is redistributed and carried by the remaining intact plies in the now reduced stiffness laminate. The process of determining a load that will cause second or successive ply failures in the laminate is a more involved one than that used in determining the first ply failure load. This is due to the complexities as to exactly how the load is redistributed within the plies in a laminate in a post first ply failure situation.

There are two classical approaches which are used in order to analyse the post first ply failure of a laminate namely, the partial ply failure and complete ply failure assumptions (Loughlan, 2000).

The partial ply failure approach considers each individual mode of ply failure separately. For example, if a ply fails in a transverse (matrix) mode then a decision has to be made if the ply is still capable of carrying any loads in shear and in the longitudinal directions. Alternatively, if ply failure occurs in the shear mode, a decision has to be made again whether the ply is still capable of carrying loads in transverse or longitudinal directions.

It is generally assumed that with a partial ply failure:

i) A transverse mode of failure (matrix failure) also implies a shear failure, but the load can still be carried in the longitudinal (fibre) direction.

ii) A shear failure also implies a transverse mode of failure, but the load can still be carried in the longitudinal direction.

iii) A longitudinal mode of failure (fibre failures) implies a total ply failure, i.e. the load cannot be carried in the longitudinal, transverse or shear.

The complete ply failure approaches assume that any mode of failure in a ply implies a complete ply failure. In other words, a shear, transverse or longitudinal mode of

failure renders the whole ply incapable of carrying any further loads. The complete ply failure assumption gives a conservative design.

Assuming a partial or a complete ply failure approach, the process of determining the load that will cause second, or subsequent, ply failures may be determined. Upon the first ply failure of a laminate, the new (reduced) stiffness terms are recalculated by downgrading the failed ply elastic properties. The values of these reduced stiffness terms will depend on the choice of the ply failure assumptions.

If a partial ply failure approach is used then the elastic constant values corresponding to the ply mode of failure are set to zero

- For transverse or shear failure: $E_{22} = G_{12} = 0$.
- For longitudinal failure: $E_{11} = E_{22} = G_{12} = 0$.

If a complete ply failure approach is used then all the three elastic properties are set to zero, $E_{11} = E_{22} = G_{12} = 0$.

The failed ply still remains at the same position in the laminate, but setting an appropriate elastic constant value to zero accomplishes the fact that the ply cannot carry loads in the corresponding directions.

The load which causes the preceding ply failure must now be applied to this new reduced stiffness laminate to check that this new laminate can still sustain the load that causes the preceding ply failure. If no ply failure is predicted, then the load is increased until another ply failure occurs. The complete cycle is repeated for the determination of loads that will cause subsequent ply failures. This step by step process of successive ply failures is also known as the progressive ply failures. This process is continued until all the plies have failed, i.e., the total laminate failure occurs. This process can be shown schematically in the flowchart diagram in figure 3.5.

CHAPTER 4

4. Non-linear Stress Analysis Using Mindlin-Type Finite Strip Elements

4.1 Introduction

In many structural engineering applications, the solution of stress and strain is always required. Special cases of such applications may range from two-dimensional plane stress or plane strain distributions, axisymmetric solids, plate bending and shells and fully three-dimensional solids.

Mindlin plate bending theory (Mindlin, 1951) is one of the classical plate bending theories which is used widely for the analysis of plate structures. It takes into account the transverse shear effect where the transverse shear stresses and strains are averaged over the thickness. Transverse shear effects become very important in the analysis of composite materials with low ratios of transverse shear to in-plane modulus. In *Mindlin* theory, the reduced integration scheme is recommended with thin plates to avoid the so called 'shear locking' phenomena, which would reduce the accuracy of the results.

In this chapter, the finite strip analysis of composite plates and shells, which is based on a new derivation of the first order shear element (*Mindlin type* element) is presented taking into account the geometric non-linearity. This derivation follows concepts similar to those employed in the finite element analysis. The derivations are based on a new *Mindlin-type* finite strip element (Razzaq, 2003) in which one-dimensional *Lagrangian* interpolation will be employed along both the x and y directions for the parameters u, v, w, θx θy together with appropriate reduced integration schemes.

4.2 Mindlin Plate Bending Theory

Consider a plate defined in terms of a midplane in the Cartesian x-y plane, and a thickness distribution $h(x,y)$ measured in the z-direction, such as that shown in figure 4.1. The upper surface of the plate ($z = h/2$) is considered to be under a shear force per unit area of $qu\ (x, y)$ and the lower surface is subjected to a shear force per unit area $ql\ (x, y)$. Other types of loading can also be considered (Razzaq, 2003).

The derivations of the Mindlin-type element are based upon the following basic assumptions and approximations of classical plate bending theories:

a) The composite material of the plate is homogeneous, and the material behaves linearly.

b) Displacement gradients are finite, such that Cauchy's strain-displacements relationships are no longer applicable and large deflection analysis will be considered. However, the transverse shear strains are assumed infinitesimal (linear) and continuous along the plate width.

c) The mid-plane remains unstrained after bending, i.e. it is the neutral plane in bending.

d) The lateral deflection w, which is the displacement component in the z-direction, is independent of z, i.e. $w \approx w(x, y)$.

The first element to employ *Mindlin's* theory was introduced by Hinton *et al.* (1975). *Mindlin* plate bending elements are

a) More accurate for thick plates, since they take into account the transverse shear effects.

b) Derived using *Lagrangian* shape functions which are simpler than the *Hermitian* ones required for *Reissner* elements.

4.3 Displacement Components of Mindlin-Type Element

The composite plate is represented by N_l layers (figure 4.2) which are parallel to the x-y plane such that for the l^{th} layer

$$Z = Z_L^{(l)} \tag{4.1}$$

and

$$Z = Z_U^{(l)} \tag{4.2}$$

represent the equations of the lower and the upper surfaces respectively. Hence, the thickness of the l^{th} layer is defined as

$$h^{(l)} = Z_U^{(l)} - Z_L^{(l)} \tag{4.3}$$

and the total thickness of the plate is

$$h = \sum_{l=1}^{Nl} h^{(l)} \tag{4.4}$$

Using the three-point *Lagrangian* interpolation (Hegaze, 2002) then parabolic distributions of γ_{xz} and γ_{yz} can be obtained in terms of their values at three points as follows

(i) at the lower surface of the plate:

$$Z_1 = -\frac{h}{2}, \qquad \gamma_{xz} = 0, \qquad \gamma_{yz} = 0$$

(ii) at the mid-surface of the plate:

$$Z_2 = 0, \qquad \gamma_{xz} = \gamma^{\circ}_{xz}, \qquad \gamma_{yz} = \gamma^{\circ}_{yz}$$

(iii) at the upper surface of the plate

$$Z_3 = +\frac{h}{2}, \qquad \gamma_{xz} = 0, \qquad \gamma_{yz} = 0$$

hence, it can be deduced that:

$$\gamma_{xz} = \gamma^{\circ}_{xz}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{4.5}$$

$$\gamma_{yz} = \gamma^{\circ}_{yz}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{4.6}$$

The average values of γ_{xz} and γ_{yz} over the thickness can be obtained as follows:

$$\bar{\gamma}_{xz}(x,y) = \frac{1}{h}\int_{-h/2}^{h/2} \gamma_{xz}\, dz \equiv \frac{2}{3}\gamma^{\circ}_{xz} \tag{4.7}$$

$$\bar{\gamma}_{yz}(x,y) = \frac{1}{h}\int_{-h/2}^{h/2} \gamma_{xz}\, dz \equiv \frac{2}{3}\gamma^{\circ}_{xz} \tag{4.8}$$

and equations (4.5) and (4.6) can be rewritten as follows:

$$\gamma_{xz}(x,y,z) = \frac{3}{2}\bar{\gamma}_{xz}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{4.9}$$

$$\gamma_{yz}(x,y,z) = \frac{3}{2}\bar{\gamma}_{yz}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{4.10}$$

Approximate (z-linearized) distributions of displacement components in x, y directions, u and v can be assumed such that:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \approx \bar{\gamma}_{xz} \tag{4.11}$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\approx \bar{\gamma}_{xz} \tag{4.12}$$

leading to:

$$\frac{\partial u}{\partial z}=\bar{\gamma}_{xz}-\frac{\partial w}{\partial x}\equiv \theta_y(x,y) \tag{4.13}$$

$$\frac{\partial v}{\partial z}=\bar{\gamma}_{yz}-\frac{\partial w}{\partial y}\equiv -\theta_x(x,y) \tag{4.14}$$

where θ_x, θ_y represent average slope angles defined as follows:

$$\theta_x=+\left[\frac{\partial w}{\partial y}-\bar{\gamma}_{yz}\right] \tag{4.15}$$

$$\theta_y=-\left[\frac{\partial w}{\partial x}-\bar{\gamma}_{xz}\right] \tag{4.16}$$

Integrating equations (4.13) and (4.14) with respect to z, then:

$$u(x,y,z)=u^{\circ}(x,y)+z\,\theta_y(x,y) \tag{4.17}$$

$$v(x,y,z)=v^{\circ}(x,y)-z\,\theta_x(x,y) \tag{4.18}$$

where u°, v° are the values of *u, v* at the mid-plane of the plate ($z = 0$). Notice also from the basic assumptions of the plate theory:

$$w\,(x, y, z)\approx w(x, y) \tag{4.19}$$

Equation (4.17), (4.18), (4.19) represent the displacement components at any point (*x, y, z*) inside the plate.

Equations (4.15) and (4.16) have led finite element researchers to use the C^o-continuous *Lagrangian* interpolation for the paramenters *w*, θ_x θ_y, thus maintaining the C^o continuity of u and v without using the more sophisticated *Hermitian* interpolation. However, when the thickness of the plate is small, then the parameters w, θ_x θ_y can no longer be interpolated independently leading to wrong answers. This phenomenon is called shear locking, and has been overcome in *Mindlin* finite elements by using reduced integration (Zeinkiewicz & Taylor, 2000).

4.4 Strain Components

4.4.1 Transverse Shear Strains

Transverse shear strains are always assumed infinitesimal and are based on equations (4.9) and (4.10), which can be written in the following matrix form

$$\underline{\gamma}(x,y,z)=\begin{bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{bmatrix}=\frac{3}{2}\left(1-\frac{4z^2}{h^2}\right)\begin{bmatrix}\bar{\gamma}_{xz}\\ \bar{\gamma}_{yz}\end{bmatrix} \tag{4.20}$$

It can be deduced from equations (4.13) and (4.14) that:

$$\bar{\gamma}_{xz}=\frac{\partial w}{\partial x}+\theta_y \tag{4.21a}$$

$$\bar{\gamma}_{yz}=\frac{\partial w}{\partial y}-\theta_x \tag{4.21b}$$

Substituting from (4.21a) and (4.21b) into (4.20) then

$$\underline{\gamma}(x,y,z)=f_\gamma(z)\begin{bmatrix}\frac{\partial w}{\partial x}+\theta_y\\ \frac{\partial w}{\partial y}-\theta_x\end{bmatrix} \tag{4.22a}$$

we can also define

$$\underline{\gamma}(x,y,z)=f_\gamma(z)\hat{\underline{\gamma}}(x,y) \tag{4.22b}$$

where

$$f_\gamma(z)=\frac{3}{2}\left(1-\frac{4z^2}{h^2}\right) \tag{4.23}$$

$$\hat{\underline{\gamma}}(x,y,z)=\begin{bmatrix}\frac{\partial w}{\partial x}+\theta_y\\ \frac{\partial w}{\partial y}-\theta_x\end{bmatrix} \tag{4.24}$$

In this work the transverse shear strains are always assumed infinitesimal. They are very small for thin plates, and completely negligible in *Kirchhoff* plate bending theory.

4.4.2 *X-Y* Strain Components

These are the ε_x, ε_y and γ_{xy}, which will be represented by vector $\underline{\varepsilon}$, where

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{4.25}$$

Using Green's strain displacement equations, then:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{4.26}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{4.27}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{4.28}$$

The strain vector can be partitioned into two parts:

(i) infinitesimal strain vector (small strain)

$$\underline{\varepsilon}_s = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} \tag{4.29}$$

(ii) the additional (non-linear) large strain terms:

$$\underline{\varepsilon}_L = \frac{1}{2}\begin{bmatrix} \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2 \\ \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 \\ 2\left(\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\right) \end{bmatrix} \tag{4.30}$$

Hence

$$\underline{\varepsilon} = \underline{\varepsilon}_S + \underline{\varepsilon}_L \tag{4.31}$$

4.4.2.1 Small Strain Components

Substituting from equations (4.17)-(4.19) into (4.29), then

$$\underline{\varepsilon}_s = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u^\circ}{\partial x} + z\dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial v^\circ}{\partial y} - z\dfrac{\partial \theta_x}{\partial y} \\ \left(\dfrac{\partial u^\circ}{\partial y} + \dfrac{\partial v^\circ}{\partial x}\right) - z\left(\dfrac{\partial \theta_x}{\partial x} - \dfrac{\partial \theta_y}{\partial y}\right) \end{bmatrix} \tag{4.32}$$

which can be rewritten as follows:

$$\underline{\varepsilon}_S = \underline{\varepsilon}_o - z\,\underline{\hat{\varepsilon}}_b \tag{4.33}$$

where

$$\underline{\varepsilon}_o = \begin{bmatrix} \dfrac{\partial u^\circ}{\partial x} \\ \dfrac{\partial v^\circ}{\partial y} \\ \dfrac{\partial u^\circ}{\partial y} + \dfrac{\partial v^\circ}{\partial x} \end{bmatrix} \tag{4.34}$$

$$\underline{\varepsilon}_b = \begin{bmatrix} -\dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} \\ \dfrac{\partial \theta_x}{\partial x} - \dfrac{\partial \theta_y}{\partial y} \end{bmatrix} \tag{4.35}$$

4.4.2.2 Additional Large Strain Components

$$\text{Let} \quad \underline{\varepsilon}_L \equiv \left\{ \varepsilon_x^L \quad \varepsilon_y^L \quad \gamma_{xy}^L \right\} \tag{4.36}$$

then, comparing (4.36) with (4.30), it can be deduced that:

$$\varepsilon_x^L = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{4.37}$$

$$\varepsilon_y^L = \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{4.38}$$

$$\gamma_{xy}^L = \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{4.39}$$

Substituting from equations (4.17)-(4.19) into (4.37), then:

$$\begin{aligned}\varepsilon_x^L &= \frac{1}{2}\left\{\left(\frac{\partial u^\circ}{\partial x} + z\frac{\partial \theta_y}{\partial x}\right)^2 + \left(\frac{\partial v^\circ}{\partial x} - z\frac{\partial \theta_x}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\ &= \frac{1}{2}\left\{\left[\left(\frac{\partial u^\circ}{\partial x}\right)^2 + \left(\frac{\partial v^\circ}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] - 2z\left[\frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \theta_y}{\partial x}\right) + \frac{\partial v^\circ}{\partial x}\left(\frac{\partial \theta_x}{\partial x}\right)\right]\right. \\ &\left. + z^2\left[\left(\frac{\partial \theta_x}{\partial x}\right)^2 + \left(\frac{\partial \theta_y}{\partial x}\right)^2\right]\right\}\end{aligned} \tag{4.40}$$

Similarly, it can be deduced that

$$\begin{aligned}\varepsilon_y^L &= \frac{1}{2}\left\{\left[\left(\frac{\partial u^\circ}{\partial y}\right)^2 + \left(\frac{\partial v^\circ}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - 2z\left[\frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \theta_y}{\partial y}\right) + \frac{\partial v^\circ}{\partial y}\left(\frac{\partial \theta_x}{\partial y}\right)\right]\right. \\ &\left. + z^2\left[\left(\frac{\partial \theta_x}{\partial y}\right)^2 + \left(\frac{\partial \theta_y}{\partial y}\right)^2\right]\right\}\end{aligned} \tag{4.41}$$

Substituting from equations (4.17)-(4.19) into (4.39), then:

$$\begin{aligned}\gamma_{xy}^L &= \left[\frac{\partial u^\circ}{\partial x} + z\frac{\partial \theta_y}{\partial x}\right]\left[\left[\frac{\partial u^\circ}{\partial y} + z\frac{\partial \theta_y}{\partial y}\right]\right] + \left[\frac{\partial v^\circ}{\partial x} - z\frac{\partial \theta_x}{\partial x}\right]\left[\frac{\partial v^\circ}{\partial y} - z\frac{\partial \theta_x}{\partial y}\right] + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ &= \left[\frac{\partial u^\circ}{\partial x}\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}\frac{\partial v^\circ}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - z\left[\frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \theta_y}{\partial y}\right) + \frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \theta_y}{\partial x}\right)\right. \\ &\left. + \frac{\partial v^\circ}{\partial x}\frac{\partial \theta_x}{\partial y} + \frac{\partial v^\circ}{\partial y}\frac{\partial \theta_x}{\partial x}\right] + z^2\left[\frac{\partial \theta_x}{\partial x}\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\frac{\partial \theta_y}{\partial y}\right]\end{aligned} \tag{4.42}$$

Equations (4.40) ,(4.41),(4.42) can be expressed in the following matrix form:

$$\underline{\varepsilon}_L = \underline{\varepsilon}_m + \underline{\varepsilon}_w - z\,\underline{\hat{\varepsilon}}_{m\theta} + z^2\,\underline{\hat{\varepsilon}}_\theta \tag{4.43}$$

where

$$\underline{\varepsilon}_m = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u^\circ}{\partial x}\right)^2 + \left(\frac{\partial v^\circ}{\partial x}\right)^2 \\ \left(\frac{\partial u^\circ}{\partial y}\right)^2 + \left(\frac{\partial v^\circ}{\partial y}\right)^2 \\ 2\left(\frac{\partial u^\circ}{\partial x}\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}\frac{\partial v^\circ}{\partial y}\right) \end{bmatrix} \tag{4.44}$$

$$\underline{\varepsilon}_w = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial w}{\partial y}\right)^2 \\ 2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \end{bmatrix} \tag{4.45}$$

$$\hat{\underline{\varepsilon}}_{m\theta} = \begin{bmatrix} \frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \theta_y}{\partial x}\right) + \frac{\partial v^\circ}{\partial x}\frac{\partial \theta_x}{\partial x} \\ \frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \theta_y}{\partial y}\right) + \frac{\partial v^\circ}{\partial y}\frac{\partial \theta_x}{\partial y} \\ \frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \theta_y}{\partial y}\right) + \frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \theta_y}{\partial x}\right) + \frac{\partial v^\circ}{\partial x}\left(-\frac{\partial \theta_x}{\partial y}\right) + \frac{\partial v^\circ}{\partial y}\left(-\frac{\partial \theta_x}{\partial x}\right) \end{bmatrix} \tag{4.46}$$

$$\hat{\underline{\varepsilon}}_\theta = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial \theta_x}{\partial x}\right)^2 + \left(\frac{\partial \theta_y}{\partial x}\right)^2 \\ \left(\frac{\partial \theta_x}{\partial y}\right)^2 + \left(\frac{\partial \theta_y}{\partial y}\right)^2 \\ 2\left[\frac{\partial \theta_x}{\partial x}\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\frac{\partial \theta_y}{\partial y}\right] \end{bmatrix} \tag{4.47}$$

The total strain vector can be obtained from equations (4.31), (4.33) and (4.43) as:

$$\underline{\varepsilon} = \underline{\varepsilon}_S + \underline{\varepsilon}_L = \underline{\varepsilon}_o - z\,\hat{\underline{\varepsilon}}_b + \underline{\varepsilon}_m + \underline{\varepsilon}_w - z\,\hat{\underline{\varepsilon}}_{m\theta} + z^2\,\hat{\underline{\varepsilon}}_\theta$$

i.e.

$$\underline{\varepsilon} = \left(\underline{\varepsilon}_o + \underline{\varepsilon}_m + \underline{\varepsilon}_w\right) - z\left(\hat{\underline{\varepsilon}}_b + \hat{\underline{\varepsilon}}_{m\theta}\right) + z^2\,\hat{\underline{\varepsilon}}_\theta \tag{4.48}$$

Notice also that $\underline{\varepsilon}_o, \underline{\varepsilon}_m, \underline{\varepsilon}_w, \hat{\underline{\varepsilon}}_b, \hat{\underline{\varepsilon}}_{m\theta}$, and $\hat{\underline{\varepsilon}}_\theta$ are functions of *(x,y)* only, i.e. they are mid-plane parameters.

4.4.3 Matrix Representation of $\underline{\varepsilon}_L$ Parts

This can be achieved in terms of three "Rotational Vectors" defined as follows:

$$\underline{\theta}_m^t = \begin{bmatrix} \frac{\partial u^\circ}{\partial x} & \frac{\partial v^\circ}{\partial x} & \frac{\partial u^\circ}{\partial y} & \frac{\partial v^\circ}{\partial y} \end{bmatrix} \tag{4.49}$$

$$\underline{\theta}_w^t = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \tag{4.50}$$

$$\underline{\theta}_\theta^t = \begin{bmatrix} \left(-\frac{\partial \theta_y}{\partial x}\right) & \left(\frac{\partial \theta_x}{\partial x}\right) & \left(-\frac{\partial \theta_y}{\partial y}\right) & \left(\frac{\partial \theta_x}{\partial y}\right) \end{bmatrix} \tag{4.51}$$

Using matrix manipulation, it can be proved that

$$\underline{\varepsilon}_m = \frac{1}{2} \underline{A}_m \underline{\theta}_m \tag{4.52}$$

$$\underline{\varepsilon}_w = \frac{1}{2} \underline{A}_w \underline{\theta}_w \tag{4.53}$$

$$\underline{\hat{\varepsilon}}_\theta = \frac{1}{2} \underline{A}_\theta \underline{\theta}_\theta \tag{4.54}$$

$$\underline{\hat{\varepsilon}}_{m\theta} = \frac{1}{2} \left(\underline{A}_m \underline{\theta}_\theta + \underline{A}_\theta \underline{\theta}_m \right) \tag{4.55}$$

where

$$\underline{A}_m = \begin{bmatrix} \frac{\partial u^\circ}{\partial x} & \frac{\partial v^\circ}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial u^\circ}{\partial y} & \frac{\partial v^\circ}{\partial y} \\ \frac{\partial u^\circ}{\partial y} & \frac{\partial v^\circ}{\partial y} & \frac{\partial u^\circ}{\partial x} & \frac{\partial v^\circ}{\partial x} \end{bmatrix} \tag{4.56}$$

$$\underline{A}_w = \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix} \tag{4.57}$$

$$\underline{A}_\theta = \begin{bmatrix} \left(-\frac{\partial \theta_y}{\partial x}\right) & \frac{\partial \theta_x}{\partial x} & 0 & 0 \\ 0 & 0 & \left(-\frac{\partial \theta_y}{\partial y}\right) & \frac{\partial \theta_x}{\partial y} \\ \left(-\frac{\partial \theta_y}{\partial y}\right) & \frac{\partial \theta_x}{\partial y} & \left(-\frac{\partial \theta_y}{\partial x}\right) & \frac{\partial \theta_x}{\partial x} \end{bmatrix} \tag{4.58}$$

Finally the total strain vector can be expressed as follows:

$$\underline{\varepsilon} = \left(\underline{\varepsilon}_o + \frac{1}{2}\underline{A}_m \underline{\theta}_m + \frac{1}{2}\underline{A}_w \underline{\theta}_w\right) - z\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}\left(\underline{A}_m \underline{\theta}_\theta + \underline{A}_\theta \underline{\theta}_m\right)\right] + \frac{1}{2} z^2 \underline{A}_\theta \underline{\theta}_\theta \tag{4.59}$$

4.4.4 Strain Variation

Notice that:

$$d\underline{\varepsilon}_s = d\underline{\varepsilon}_o - d\hat{\underline{\varepsilon}}_b \tag{4.60}$$

From equations (4.44), together with the corresponding matrix representations, it can be shown that:

$$d\underline{\varepsilon}_m = \begin{bmatrix} \left(\frac{\partial u^\circ}{\partial x}\right) d\left(\frac{\partial u^\circ}{\partial x}\right) + \left(\frac{\partial v^\circ}{\partial x}\right) d\left(\frac{\partial v^\circ}{\partial x}\right) \\ \left(\frac{\partial u^\circ}{\partial y}\right) d\left(\frac{\partial u^\circ}{\partial y}\right) + \left(\frac{\partial v^\circ}{\partial y}\right) d\left(\frac{\partial v^\circ}{\partial y}\right) \\ \left(\frac{\partial u^\circ}{\partial x}\right) d\left(\frac{\partial u^\circ}{\partial y}\right) + \left(\frac{\partial u^\circ}{\partial y}\right) d\left(\frac{\partial u^\circ}{\partial x}\right) + \ldots\ldots\ldots \end{bmatrix} \equiv \underline{A}_m d\underline{\theta}_m \tag{4.61}$$

Similarly, it can be deduced that:

$$d\underline{\varepsilon}_w = \underline{A}_w d\underline{\theta}_w \tag{4.62}$$

$$d\hat{\underline{\varepsilon}}_\theta = \underline{A}_\theta d\underline{\theta}_\theta \tag{4.63}$$

$$d\hat{\underline{\varepsilon}}_{m\theta} = \underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m \tag{4.64}$$

i.e.

$$d\underline{\varepsilon}_L = \left(\underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w\right) - z\left(\underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right) + z^2 \underline{A}_\theta d\underline{\theta}_\theta \tag{4.65}$$

Finally

$$\begin{aligned} d\underline{\varepsilon} &= d\underline{\varepsilon}_S + d\underline{\varepsilon}_L \\ &= \left(d\underline{\varepsilon}_o + \underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w\right) - z\left(d\hat{\underline{\varepsilon}}_b + \underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right) + z^2 \underline{A}_\theta d\underline{\theta}_\theta \end{aligned} \tag{4.66}$$

4.5 Stress Components

The stress components at a point in the l^{th} layer, can also be partitioned and represented by the following vectors:

$$\underline{\sigma}^{(l)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{4.67}$$

$$\underline{\tau}^{(l)} = \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{4.68}$$

Using stress-strain relations for the l^{th} layer (refer to section in chapter 3), then:

$$\underline{\sigma}^{(l)} = \underline{D}^{(l)} \underline{\varepsilon} \tag{4.69}$$

$$\underline{\tau}^{(l)} = \underline{\mu}^{(l)} \underline{\gamma} \tag{4.70}$$

where $\underline{D}^{(l)}, \underline{\mu}^{(l)}$ are matrices with respect to plate axes, for the l^{th} layer, as defined by equations (3.24), (3.17) repectively.

4.6 Strain Energy Variation

The variation of the strain energy density at any point inside the l^{th} layer, can be written as follows:

$$d\overline{U}^{(l)} = d\underline{\varepsilon}^t \underline{\sigma}^{(l)} + d\underline{\gamma}^t \underline{\tau}^{(l)} \equiv d\overline{U}_{xy}^{(l)} + d\overline{U}_S^{(l)} \tag{4.71}$$

where $d\overline{U}_{xy}^{(l)}$ represents the variation due to x-y components, and $d\overline{U}_S^{(l)}$ represents the variation due to transverse shear components.

From equation (4.22), it can be deduced that:

$$d\overline{U}_S^{(l)} = d\underline{\gamma}^t \underline{\tau}^{(l)} = f_\gamma(z) d\hat{\underline{\gamma}}^t \underline{\tau}^{(l)} \tag{4.72}$$

and from equation (4.66), it can be shown that:

$$d\overline{U}_{xy}^{(l)} = \left[\left(d\underline{\varepsilon}_o^t + d\underline{\theta}_m^t \underline{A}_m^t + d\underline{\theta}_w^t \underline{A}_w^t\right) - z\left(d\hat{\underline{\varepsilon}}_b^t + d\underline{\theta}_\theta^t \underline{A}_m^t + d\underline{\theta}_m^t \underline{A}_\theta^t\right)\right.$$

$$+z^2 d\underline{\theta}_\theta^t \underline{A}_\theta^t \Big] \underline{\sigma}^{(l)} \tag{4.73}$$

Hence, the variation of strain energy is:

$$dU = \iint_{x-y} \left(\int_{-h/2}^{h/2} d\bar{U}^{(l)} dz \right) dxdy \tag{4.74}$$

It may be useful to define the strain energy variation per unit area as:

$$dU' = \left(\int_{-h/2}^{h/2} d\bar{U}^{(l)} dz \right) \equiv dU_S' + dU_{xy}' \tag{4.75}$$

Hence,

$$dU_S' = \int_{-h/2}^{h/2} f_\gamma(z) d\,\hat{\underline{\gamma}}^t \underline{\tau}^{(l)} dz = \int_{-h/2}^{h/2} f_\gamma^{\,2}(z) d\,\hat{\underline{\gamma}}^t \underline{\mu}^{(l)} \hat{\underline{\gamma}}\; dz \tag{4.76}$$

Define

$$\underline{\mu}_{\gamma\gamma} = \int_{-h/2}^{h/2} f_\gamma^{\,2}(z) \underline{\mu}^{(l)} dz \tag{4.77a}$$

$$= \int_{-h/2}^{h/2} \frac{9}{4} \left(1 - \frac{4z^2}{h^2} \right)^2 \underline{\mu}^{(l)} dz \tag{4.77b}$$

and for transversely isotropic composites: $(\mu_{xz} = \mu_{yz} = \mu)$

$$\underline{\mu}^{(l)} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.78}$$

Hence,

$$\underline{\mu}_{\gamma\gamma} = \frac{6}{5} h\mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.79}$$

and

$$dU_S' = d\hat{\underline{\gamma}}^t \underline{\mu}_{\gamma\gamma} \hat{\underline{\gamma}} \tag{4.80}$$

We can also define

$$\tilde{\underline{\tau}} = \underline{\mu}_{\gamma\gamma} \hat{\underline{\gamma}} \equiv \int_{-h/2}^{h/2} f_\gamma(z) \underline{\tau}^{(l)} dz \tag{4.81}$$

then

$$dU_S' = d\hat{\gamma}^t \tilde{\tau} \tag{4.82}$$

Integrating equation (4.73) with respect to z, then:

$$dU_{xy}' = \left(d\,\underline{\varepsilon}_o^t + d\,\underline{\theta}_m^t \underline{A}_m^t + d\,\underline{\theta}_w^t \underline{A}_w^t\right) \int_{-h/2}^{h/2} \underline{\sigma}^{(l)} dz$$

$$-\left(d\,\hat{\underline{\varepsilon}}_b^t + d\,\underline{\theta}_\theta^t \underline{A}_m^t + d\,\underline{\theta}_m^t \underline{A}_\theta^t\right) \int_{-h/2}^{h/2} z\,\underline{\sigma}^{(l)} dz + d\,\underline{\theta}_\theta^t \underline{A}_\theta^t \int_{-h/2}^{h/2} z^2\,\underline{\sigma}^{(l)} dz \tag{4.83}$$

Define the following integrated stress:

$$\underline{\sigma}_{(n)} = (-1)^n \int_{-h/2}^{h/2} z^n \underline{\sigma}^{(l)} dz \tag{4.84}$$

then

$$\underline{\sigma}_{(n)} = (-1)^n \left\{ \underline{D}_{(n)} \left[\underline{\varepsilon}_o + \frac{1}{2}\underline{A}_m \underline{\theta}_m + \frac{1}{2}\underline{A}_w \underline{\theta}_w \right] - \underline{D}_{(n+1)} \left[\underline{\varepsilon}_b + \frac{1}{2}\left(\underline{A}_m \underline{\theta}_\theta + \underline{A}_\theta \underline{\theta}_m\right) \right] \right.$$

$$\left. + \frac{1}{2}\underline{D}_{(n+2)} \underline{A}_\theta \underline{\theta}_\theta \right\} \tag{4.85}$$

where

$$\underline{D}_{(n)} = \int_{-h/2}^{h/2} z^n \underline{D}^{(l)} dz \tag{4.86}$$

Hence equation (4.83) can be rewritten as follows:

$$dU_{xy}' = \left(d\,\underline{\varepsilon}_o^t + d\,\underline{\theta}_m^t \underline{A}_m^t + d\,\underline{\theta}_w^t \underline{A}_w^t\right) \underline{\sigma}_{(0)}$$

$$+ \left(d\,\hat{\underline{\varepsilon}}_b^t + d\,\underline{\theta}_\theta^t \underline{A}_m^t + d\,\underline{\theta}_m^t \underline{A}_\theta^t\right) \underline{\sigma}_{(1)} + d\,\underline{\theta}_\theta^t \underline{A}_\theta^t \underline{\sigma}_{(2)} \tag{4.87}$$

4.7 Interpolated Displacement Equations

4.7.1 Interpolation in x-Direction

For an n-node finite strip element, as shown in Figure 4.3 we define intrinsic coordinates ξ and η such that:

$$\xi = \frac{x - x_1}{a}, \quad \eta = \frac{y - y_1(x)}{b(x)} \tag{4.88a}$$

Hence

$$\frac{dx}{d\xi} = a \equiv J_x, \quad \frac{dy}{d\eta} = b \equiv J_y \tag{4.88b}$$

where a,b are element length and width in x,y directions respectively, as shown in Figure (4.3).

For trapezoidal elements:

$$b(\xi) = b_1 + \xi(b_2 - b_1) \tag{4.89}$$

where b_1, b_2 are the widths at $\xi = 0, \xi = 1$ respectively, and

$$d\eta = \frac{dy}{b(\xi)} \tag{4.90}$$

For interpolation along –x, we use nodal values, i.e.

$$P(x,y) = \sum_{i=1}^{n} N_i(\xi) P(x_i, y) \tag{4.91}$$

where

$$N_i(\xi) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(n-1)\xi - (j-1)}{i-j} \tag{4.92}$$

which represents one dimensional *Lagrangian* shape functions.

Mindlin-type elements only require C^0 continuity and can be interpolated in x-direction via *Lagrangian* interpolation i.e. for an n-node strip:

$$u^\circ(x,y) = \sum_{i=1}^{n} N_i(\xi) u_i^\circ(y) \tag{4.93}$$

$$v^\circ(x,y) = \sum_{i=1}^{n} N_i(\xi) v_i^\circ(y) \tag{4.94}$$

$$w(x,y) = \sum_{i=1}^{n} N_i(\xi) w_i(y) \tag{4.95}$$

$$\theta_x(x,y) = \sum_{i=1}^{n} N_i(\xi) \left(\theta_x(y)\right)_i \tag{4.96}$$

$$\theta_y(x,y) = \sum_{i=1}^{n} N_i(\xi) \left(\theta_y(y)\right)_i \tag{4.97}$$

where $u_i^\circ(y), v_i^\circ(y), w_i(y), (\theta_x(y))_i$, and $\left(\theta_y(y)\right)_i$ represent values at $x = x_i$ at any given y value.

4.7.2 Interpolation in y-Direction

If we also use m values along y (at any ξ), then using *Lagrangian* interpolation:

$$P(x_i, y) = \sum_{r=1}^{m} \mu_r(\eta) P(x_i, y_r) \tag{4.98}$$

where

$$\mu_r(\eta) = \prod_{\substack{s=1 \\ s \neq r}}^{m} \frac{(m-1)\eta - (s-1)}{r-s} \tag{4.99}$$

The interpolations in y-direction are defined as follows:

$$u_r^{\circ}(y) = \sum_{r=1}^{m} \mu_r(\eta) u_i^r \tag{4.100}$$

$$u_i^{\circ}(y) = \sum_{r=1}^{m} \mu_r(\eta) v_i^r \tag{4.101}$$

$$w_i(y) = \sum_{r=1}^{m} \mu_r(\eta) w_i^r \tag{4.102}$$

$$(\theta_x(y))_i = \sum_{r=1}^{m} \mu_r(\eta) (\theta_x^r)_i \tag{4.103}$$

$$(\theta_y(y))_i = \sum_{r=1}^{m} \mu_r(\eta) (\theta_y^r)_i \tag{4.104}$$

where

$$\begin{aligned} u_i^r &\equiv u^{\circ}(x_i, y_r) \\ v_i^r &\equiv v^{\circ}(x_i, y_r) \\ w_i^r &\equiv w(x_i, y_r) \\ (\theta_x^r)_i &\equiv \theta_x(x_i, y_r) \\ (\theta_y^r)_i &\equiv \theta_y(x_i, y_r) \end{aligned} \tag{4.105}$$

4.7.3 Interpolation in x-y-Plane

For the interpolation in the x-y-plane, defining:

$$P(x_i, y_r) \equiv P_i^r \tag{4.106}$$

and using equations (4.91) and (4.98), it can be deduced that:

$$P(x, y) = \sum_{i=1}^{n} \sum_{r=1}^{m} N_i(\xi) \mu_r(\eta) P_i^r \tag{4.107}$$

Hence, based on this analysis, the full x-y interpolated parameters can be expressed for an n-node strip as follows:

$$u^{\circ}(x,y)=\sum_{i=1}^{n}\sum_{r=1}^{m}N_i(\xi)\mu_r(\eta)u_i^r \tag{4.108}$$

$$v^{\circ}(x,y)=\sum_{i=1}^{n}\sum_{r=1}^{m}N_i(\xi)\mu_r(\eta)v_i^r \tag{4.109}$$

$$w(x,y)=\sum_{i=1}^{n}\sum_{r=1}^{m}N_i(\xi)\mu_r(\eta)w_i^r \tag{4.110}$$

$$\theta_x(x,y)=\sum_{i=1}^{n}\sum_{r=1}^{m}N_i(\xi)\mu_r(\eta)(\theta_x^r)_i \tag{4.111}$$

$$\theta_y(x,y)=\sum_{i=1}^{n}\sum_{r=1}^{m}N_i(\xi)\mu_r(\eta)(\theta_y^r)_i \tag{4.112}$$

4.8 Interpolation of Strain Components

4.8.1 Transverse Shear Strains

Substituting from equations (4.110)-(4.112) into (4.24) then:

$$\hat{\underline{\gamma}}(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}\begin{bmatrix}\mu_r\dfrac{\partial N_i}{\partial x}w_i^r+N_i\mu_r(\theta_y^r)_i\\ \dfrac{\partial\mu_r}{\partial y}N_iw_i^r-N_i\mu_r(\theta_x^r)_i\end{bmatrix} \tag{4.113}$$

Notice that:

$$\frac{\partial N_i}{\partial x}=\frac{dN_i}{d\xi}\frac{d\xi}{dx}=\frac{1}{J_x}N_r'(\xi) \tag{4.114}$$

$$\frac{\partial\mu_r}{\partial y}=\frac{dN_i}{d\eta}\frac{d\eta}{dy}=\frac{1}{J_y}\mu_r'(\eta) \tag{4.115}$$

where

$$N_i'(\xi)=\frac{dN_i}{d\xi},\quad \mu_r'(\eta)=\frac{d\mu_r}{d\eta} \tag{4.116}$$

Defining

$$\left(\underline{\delta}_b^r\right)^t=\left[w_1^r\ \ (\theta_x^r)_1\ \ (\theta_y^r)_1\cdots\cdots w_n^r\ \ (\theta_x^r)_n\ \ (\theta_y^r)_n\right] \tag{4.117}$$

then equation (4.113) can be rewritten in the following matrix form:

$$\underline{\hat{\gamma}}(x,y)=\sum_{r=1}^{m}\underline{B}_{\gamma}^{r}(x,y)\underline{\delta}_{b}^{r} \tag{4.118}$$

where

$$\underline{B}_{\gamma}^{r}=\begin{bmatrix} \cdots & \dfrac{N_i'\mu_r}{J_x} & 0 & N_i\mu_r & \cdots \\ \cdots & \dfrac{N_i\mu_r'}{J_y} & -N_i\mu_r & 0 & \cdots \end{bmatrix} \tag{4.119}$$

Notice also from equations (4.22a) and (4.118):

$$\underline{\gamma}(x,y,z)=f_{\gamma}(z)\sum_{r=1}^{m}\underline{B}_{\gamma}^{r}(x,y)\underline{\delta}_{b}^{r} \tag{4.120}$$

4.8.2 Infinitesimal x-y Strain Components

Substituting from equations (4.108)-(4.112) into (4.34), it can be proved that:

$$\underline{\varepsilon}_{o}(x,y)=\begin{bmatrix} \dfrac{\partial u^{\circ}}{\partial x} \\ \dfrac{\partial v^{\circ}}{\partial y} \\ \dfrac{\partial u^{\circ}}{\partial y}+\dfrac{\partial v^{\circ}}{\partial x} \end{bmatrix}\equiv\sum_{r=1}^{m}\underline{B}_{o}^{r}(x,y)\underline{\delta}_{o}^{r} \tag{4.121}$$

where

$$\left(\underline{\delta}_{o}^{r}\right)^{t}=\begin{bmatrix} u_1^r & v_1^r & u_2^r & v_2^r & \cdots\cdots & u_n^r & v_n^r \end{bmatrix} \tag{4.122}$$

and

$$\underline{B}_{o}^{r}(x,y)=\begin{bmatrix} \cdots & \dfrac{N_i'\mu_r}{J_x} & 0 & \cdots \\ \cdots & 0 & \dfrac{N_i\mu_r'}{J_y} & \cdots \\ \cdots & \dfrac{N_i\mu_r'}{J_y} & \dfrac{N_i'\mu_r}{J_x} & \cdots \end{bmatrix} \tag{4.123}$$

Similarly it can be deduced that:

$$\underline{\hat{\varepsilon}}_b(x,y) = \sum_{r=1}^{m} \underline{B}_b^r(x,y)\underline{\delta}_b^r \tag{4.124}$$

where

$$\underline{B}_b^r(x,y) = \begin{bmatrix} \cdots & 0 & 0 & -\dfrac{N_i'\mu_r}{J_x} & \cdots \\ \cdots & 0 & \dfrac{N_i\mu_r'}{J_y} & 0 & \cdots \\ \cdots & 0 & \dfrac{N_i'\mu_r}{J_x} & -\dfrac{N_i\mu_r'}{J_y} & \cdots \end{bmatrix} \tag{4.125}$$

4.8.3 Additional Non-Linear Strain Components

(a) Rotation vectors

Substituting from equations (4.108), (4.109) into (4.49) gives:

$$\underline{\theta}_m(x,y) = \begin{bmatrix} \dfrac{\partial u^\circ}{\partial x} \\ \dfrac{\partial v^\circ}{\partial x} \\ \dfrac{\partial u^\circ}{\partial y} \\ \dfrac{\partial v^\circ}{\partial y} \end{bmatrix} = \sum_{r=1}^{m}\sum_{i=1}^{n} \begin{bmatrix} \dfrac{N_i'}{J_x} & \mu_r & u_i^r \\ \dfrac{N_i'}{J_x} & \mu_r & v_i^r \\ N_i & \dfrac{\mu_r'}{J_y} & u_i^r \\ N_i & \dfrac{\mu_r'}{J_y} & v_i^r \end{bmatrix} \tag{4.126}$$

which can be rewritten as follows:

$$\underline{\theta}_m(x,y) = \sum_{r=1}^{m} \underline{G}_m^r(x,y)\underline{\delta}_o^r \tag{4.127}$$

where

$$\underline{G}_m^r(x,y)=\begin{bmatrix} \cdots & \frac{N_i\mu_r'}{J_x} & 0 & \cdots \\ \cdots & 0 & \frac{N_i\mu_r'}{J_x} & \cdots \\ \cdots & \frac{N_i\mu_r'}{J_y} & 0 & \cdots \\ \cdots & 0 & \frac{N_i\mu_r'}{J_y} & \cdots \end{bmatrix} \tag{4.128}$$

Similarly, it can be proved by using the same technique that:

$$\underline{\theta}_w(x,y)=\sum_{r=1}^{m}\underline{G}_w^r(x,y)\underline{\delta}_b^r \tag{4.129}$$

$$\underline{\theta}_\theta(x,y)=\sum_{r=1}^{m}\underline{G}_\theta^r(x,y)\underline{\delta}_b^r \tag{4.130}$$

where

$$\underline{G}_w^r(x,y)=\begin{bmatrix} \cdots & \frac{N_i\mu_r'}{J_x} & 0 & 0 & \cdots \\ \cdots & \frac{N_i\mu_r'}{J_y} & 0 & 0 & \cdots \end{bmatrix} \tag{4.131}$$

and

$$\underline{G}_\theta^r(x,y)=\begin{bmatrix} \cdots & 0 & 0 & -\frac{N_i\mu_r'}{J_x} & \cdots \\ \cdots & 0 & \frac{N_i\mu_r'}{J_x} & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{N_i\mu_r'}{J_y} & \cdots \\ \cdots & 0 & \frac{N_i\mu_r'}{J_y} & 0 & \cdots \end{bmatrix} \tag{4.132}$$

(b) Non-linear Strain Components

Substituting from equations (4.127), (4.129) and (4.130) into equations (4.52)-(4.55), then it can be deduced that:

$$\underline{\varepsilon}_m = \frac{1}{2}\sum_{r=1}^{m} \underline{A}_m \, \underline{G}_m^r \, \underline{\delta}_o^r \tag{4.133}$$

$$\underline{\varepsilon}_w = \frac{1}{2}\sum_{r=1}^{m} \underline{A}_w \, \underline{G}_w^r \, \underline{\delta}_b^r \tag{4.134}$$

$$\underline{\hat{\varepsilon}}_\theta = \frac{1}{2}\sum_{r=1}^{m} \underline{A}_\theta \, \underline{G}_\theta^r \, \underline{\delta}_b^r \tag{4.135}$$

$$\underline{\hat{\varepsilon}}_{m\theta} = \frac{1}{2}\sum_{r=1}^{m} \left[\underline{A}_m \, \underline{G}_\theta^r \, \underline{\delta}_b^r + \underline{A}_\theta \, \underline{G}_m^r \, \underline{\delta}_o^r\right] \tag{4.136}$$

4.8.4 Total x-y Strain and its Variation

Substituting from equations (4.121), (4.124), (4.133)-(4.136) into equation (4.59) then, it can be deduced that:

$$\underline{\varepsilon}(x,y,z) = \sum_{r=1}^{m} \left\{ \left(\underline{B}_o^r \underline{\delta}_o^r + \frac{1}{2}\underline{A}_m \, \underline{G}_m^r \, \underline{\delta}_o^r + \frac{1}{2}\underline{A}_w \, \underline{G}_w^r \, \underline{\delta}_b^r \right) \right.$$
$$\left. - z\left(\underline{B}_b^r \underline{\delta}_b^r + \frac{1}{2}\underline{A}_\theta \, \underline{G}_m^r \, \underline{\delta}_o^r + \frac{1}{2}\underline{A}_m \, \underline{G}_\theta^r \, \underline{\delta}_b^r \right) + \frac{1}{2}z^2 \underline{A}_\theta \, \underline{G}_\theta^r \, \underline{\delta}_b^r \right\}$$

(4.137)

Non-linear $\underline{B}$ matrices can be defined as follows where, double-subscript is used for all the non-linear $\underline{B}$ matrices:

$$\underline{B}_{mm}^r(x,y) = \underline{A}_m \underline{G}_m^r \tag{4.138}$$

$$\underline{B}_{ww}^r(x,y) = \underline{A}_w \underline{G}_w^r \tag{4.139}$$

$$\underline{B}_{\theta m}^r(x,y) = \underline{A}_\theta \underline{G}_m^r \tag{4.140}$$

$$\underline{B}_{m\theta}^r(x,y) = \underline{A}_m \underline{G}_\theta^r \tag{4.141}$$

$$\underline{B}_{\theta\theta}^r(x,y) = \underline{A}_\theta \underline{G}_\theta^r \tag{4.142}$$

Hence equation (4.137) can be rewritten as follows:

$$\underline{\varepsilon}(x,y,z) = \sum_{r=1}^{m} \left\{ \left[\left(\underline{B}_o^r + \frac{1}{2}\underline{B}_{mm}^r \right) \underline{\delta}_o^r + \frac{1}{2}\underline{B}_{ww}^r \underline{\delta}_b^r \right] \right.$$
$$\left. - z\left[\frac{1}{2}\underline{B}_{\theta m}^r \underline{\delta}_o^r + \left(\underline{B}_b^r + \frac{1}{2}\underline{B}_{m\theta}^r \right) \underline{\delta}_b^r \right] + \frac{1}{2}z^2 \underline{B}_{\theta\theta}^r \underline{\delta}_b^r \right\} \tag{4.143}$$

Notice also that from equation (4.66) and previous definitions, it can be deduced that:

$$d\underline{\varepsilon}(x,y,z)=\sum_{r=1}^{m}\left\{\left[\left(\underline{B}_o^r+\underline{A}_m\,\underline{G}_m^r\right)d\underline{\delta}_o^r+\underline{A}_w\,\underline{G}_w^r d\underline{\delta}_b^r\right]\right.$$
$$\left.-z\left[\underline{A}_\theta\,\underline{G}_m^r d\underline{\delta}_o^r+\left(\underline{B}_b^r+\underline{A}_m\,\underline{G}_\theta^r\right)d\underline{\delta}_b^r\right]+z^2\underline{A}_\theta\,\underline{G}_\theta^r d\underline{\delta}_b^r\right\} \tag{4.144}$$

which also can be rearranged as follows:

$$d\underline{\varepsilon}(x,y,z)=\sum_{r=1}^{m}\left\{\left[\left(\underline{B}_o^r+\underline{B}_{mm}^r\right)d\underline{\delta}_o^r+\underline{B}_{ww}^r d\underline{\delta}_b^r\right]\right.$$
$$\left.-z\left[\underline{B}_{\theta m}^r d\underline{\delta}_o^r+\left(\underline{B}_b^r+\underline{B}_{m\theta}^r\right)d\underline{\delta}_b^r\right]+z^2\underline{B}_{\theta\theta}^r d\underline{\delta}_b^r\right\} \tag{4.145}$$

4.9 Interpolated Strain Energy Variation

Stress vectors can also be expressed as follows:

$$\underline{\tau}^{(l)}=\underline{\mu}^{(l)}\underline{\gamma}=\underline{f}_\gamma(z)\underline{\mu}^{(l)}\underline{\gamma}^{(l)}=\underline{f}_\gamma(z)\underline{\mu}^{(l)}\sum_{r=1}^{m}\underline{B}_\gamma^r(x,y)\underline{\delta}_b^r \tag{4.146}$$

and

$$\underline{\sigma}^{(l)}=\underline{D}^{(l)}\underline{\varepsilon}=\underline{D}^{(l)}\sum_{r=1}^{m}\left\{\left[\left(\underline{B}_o^r+\frac{1}{2}\underline{B}_{mm}^r\right)\underline{\delta}_o^r+\frac{1}{2}\underline{B}_{ww}^r\,\underline{\delta}_b^r\right]\right.$$
$$\left.-z\left[\frac{1}{2}\underline{B}_{\theta m}^r\underline{\delta}_o^r+\left(\underline{B}_b^r+\frac{1}{2}\underline{B}_{m\theta}^r\right)\underline{\delta}_b^r+\frac{1}{2}z^2\underline{B}_{\theta\theta}^r\underline{\delta}_b^r\right]\right\} \tag{4.147}$$

Notice also that:

$$d\underline{\gamma}=\sum_{s=1}^{m}f_\gamma(z)\underline{B}_\gamma^s(x,y)d\underline{\delta}_b^s \tag{4.148}$$

$$d\underline{\varepsilon}=\sum_{s=1}^{m}\left\{\left[\left(\underline{B}_o^s+\underline{A}_m\underline{G}_m^s\right)d\underline{\delta}_o^s+\underline{A}_w\underline{G}_w^s d\underline{\delta}_b^s\right]\right.$$
$$\left.-z\left[\underline{A}_\theta\underline{G}_m^s d\underline{\delta}_o^s+\left(\underline{B}_b^s+\underline{A}_m\underline{G}_\theta^s\right)d\underline{\delta}_b^s\right]+z^2\underline{A}_\theta\underline{G}_\theta^s d\underline{\delta}_b^s\right\} \tag{4.149}$$

Hence, it can be deduced that:

$$d\underline{\tau}^{(l)}=\underline{f}_\gamma(z)\underline{\mu}^{(l)}\sum_{r=1}^{m}\underline{B}_\gamma^r(x,y)\underline{\delta}_b^r \tag{4.150}$$

$$d\underline{\sigma}^{(l)}=\underline{D}^{(l)}\sum_{r=1}^{m}\left\{\left[\left(\underline{B}_o^r+\underline{B}_{mm}^r\right)d\underline{\delta}_o^r+\underline{B}_{ww}^r d\underline{\delta}_b^r\right]\right.$$
$$\left.-z\left[\left(\underline{B}_b^r+\underline{B}_{m\theta}^r\right)d\underline{\delta}_b^r+\underline{B}_{\theta m}^r d\underline{\delta}_o^r\right]+z^2\underline{B}_{\theta\theta}^r d\underline{\delta}_b^r\right\} \tag{4.151}$$

The variation of strain energy density, as defined by equation (4.71) can now be rewritten as follows:

$$d\bar{U}^{(l)} = \sum_{s=1}^{m} \left\{ \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_\gamma^s\right)' f_\gamma(z)\, \underline{\tau}^{(l)} + \left[\left(d\underline{\delta}_o^s\right)' \left(\left(\underline{B}_o^s\right)' + \left(\underline{B}_{mm}^s\right)' \right) + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_{ww}^s\right)' \right] \underline{\sigma}^{(l)} \right.$$

$$\left. + \left[\left(d\underline{\delta}_o^s\right)' \left(\underline{B}_{\theta m}^s\right)' + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_b^s\right)' + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_{m\theta}^s\right)' \right] \left(-z\underline{\sigma}^{(l)}\right) + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_{\theta\theta}^s\right)' \left(z^2 \underline{\sigma}^{(l)}\right) \right\} \tag{4.152}$$

Similarly, it can be shown that the variation of strain energy per unit area can be expressed as follows:

$$dU' = \int_{-h/2}^{h/2} d\bar{U}^{(l)}\, dz$$

$$= \sum_{s=1}^{m} \left\{ \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_\gamma^s\right)' \underline{\tilde{\tau}} + \left[\left(d\underline{\delta}_o^s\right)' \left(\underline{B}_o^s + \underline{B}_{mm}^s\right)' + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_{ww}^s\right)' \right] \underline{\sigma}_{(0)} \right.$$

$$\left. + \left[\left(d\underline{\delta}_o^s\right)' \left(\underline{B}_{\theta m}^s\right)' + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)' \right] \underline{\sigma}_{(1)} + \left(d\underline{\delta}_b^s\right)' \left(\underline{B}_{\theta\theta}^s\right)' \underline{\sigma}_{(2)} \right\} \tag{4.153}$$

4.10 Generalized Equations of Equilibrium

Notice that equivalent nodal loading can be defined such that the work done by actual loads is equal to the work done by equivalent nodal loading, i.e.:

$$dW = \sum_{s=1}^{m} \left[\left(d\underline{\delta}_o^s\right)' \underline{F}_o^s + \left(d\underline{\delta}_b^s\right)' \underline{F}_b^s \right] \tag{4.154}$$

where

$$\left(\underline{F}_o^s\right)' = \left[\left(F_x^s\right)_1 \quad \left(F_y^s\right)_1 \cdots\cdots \left(F_x^s\right)_n \quad \left(F_y^s\right)_n \right] \tag{4.155}$$

$$\left(\underline{F}_b^s\right)' = \left[\left(F_z^s\right)_1 \quad \left(M_x^s\right)_1 \quad \left(M_y^s\right)_1 \cdots\cdots \left(F_z^s\right)_n \quad \left(M_x^s\right)_n \quad \left(M_y^s\right)_n \right] \tag{4.156}$$

which represent components of equivalent nodal loadings.

Notice also that :

$$dU = \iiint d\bar{U}\, dx\, dy\, dz = \sum_{strips} \iint_{\substack{x-y \\ strip}} dU'\, dx\, dy \tag{4.157}$$

From the principle of virtual work:

$$d\chi = dU - dW = 0 \tag{4.158}$$

Hence

$$\left[\sum_{strips}\iint_{strip} dU' \, dx\,dy\right] - dW = 0 \tag{4.159}$$

Substituting from equations (4.153) and (4.154) into (4.159), then it can be deduced that:

$$\sum_{s=1}^{m}\left\{\sum_{strips}\iint_{strip}\left[\left(d\underline{\delta}_b^s\right)'\left(\underline{B}_\gamma^s\right)'\underline{\tau} + \left[\left(d\underline{\delta}_o^s\right)'\left(\underline{B}_o^s + \underline{B}_{mm}^s\right)' + \left(d\underline{\delta}_b^s\right)'\left(\underline{B}_{ww}^s\right)'\right]\underline{\sigma}_{(o)}\right.\right.$$

$$+\left[\left(d\underline{\delta}_o^s\right)'\left(\underline{B}_{\theta m}^s\right)' + \left(d\underline{\delta}_b^s\right)'\left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)'\right]\underline{\sigma}_{(1)} + \left(d\underline{\delta}_b^s\right)'\left(\underline{B}_{\theta\theta}^s\right)'\underline{\sigma}_{(2)}\left.\right]dx\,dy$$

$$\left.\left. - \left(d\underline{\delta}_o^s\right)'\underline{F}_o^s - \left(d\underline{\delta}_b^s\right)'\underline{F}_b^s\right\} = 0 \tag{4.160}$$

Since $d\underline{\delta}_o^s$, $d\underline{\delta}_b^s$ $(s = 1,2,3,...,m)$ represent arbitrary changes, then their coefficients vanish,, i.e.:

$$\sum_{strips}\iint_{strip}\left[\left(\underline{B}_o^s + \underline{B}_{mm}^s\right)'\underline{\sigma}_{(0)} + \left(\underline{B}_{\theta m}^s\right)'\underline{\sigma}_{(1)}\right]dx\,dy - \underline{F}_o^s = \underline{0} \tag{4.161a}$$

$$\sum_{strips}\iint_{strip}\left\{\left(\underline{B}_\gamma^s\right)'\underline{\tau} + \left(\underline{B}_{ww}^s\right)'\underline{\sigma}_{(0)} + \left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)'\underline{\sigma}_{(1)} + \left(\underline{B}_{\theta\theta}^s\right)'\underline{\sigma}_{(2)}\right\}dx\,dy - \underline{F}_b^s = \underline{0} \tag{4.161b}$$

where s= 1, 2,3, ..., m)

which represents the generalized equations of equilibrium, with $\underline{\tau}$ and $\underline{\sigma}_{(n)}$ as defined by equations (4.81) and (4.84), respectively.

4.11 Linearization of Equations of Equilibrium

For an approximate solution, residual force vectors are obtained from (4.161) as follows

$$\underline{R}_o^s = \underline{F}_o^s - \sum_{strips}\iint_{strips}\left[\left(\underline{B}_o^s + \underline{B}_{mm}^s\right)'\underline{\sigma}_{(0)} + \left(\underline{B}_{\theta m}^s\right)'\underline{\sigma}_{(1)}\right]dx\,dy \tag{4.162a}$$

$$\underline{R}_b^s = \underline{F}_b^s - \sum_{strips}\iint_{strip}\left\{\left(\underline{B}_\gamma^s\right)'\underline{\tau} + \left(\underline{B}_{ww}^s\right)'\underline{\sigma}_{(0)} + \left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)'\underline{\sigma}_{(1)} + \left(\underline{B}_{\theta\theta}^s\right)'\underline{\sigma}_{(2)}\right\}dx\,dy \tag{4.162b}$$

In order to recognize non-linear terms, equations (4.162a) and (4.62b) can also be rewritten as follows:

$$\underline{R}_o^s = \underline{F}_o^s - \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left[\left(\underline{B}_o^s\right)^t + \left(\underline{G}_m^s\right)^t \underline{A}_m^t \right] \underline{\sigma}^{(l)} + \left(\underline{G}_m^s\right)^t \underline{A}_\theta^t \left(-z\underline{\sigma}^{(l)}\right) \right\} dx\,dy\,dz \qquad (4.163a)$$

$$\underline{R}_o^s = \underline{F}_o^s - \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\underline{\tau}^{(l)} + \left(\underline{G}_w^s\right)^t \underline{A}_w^t \underline{\sigma}^{(l)} + \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t \right] \left(-z\underline{\sigma}^{(l)}\right) + \left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t \left(z^2 \underline{\sigma}^{(l)}\right) \right\} dz\,dx\,dy \qquad (4.163b)$$

To restore equilibrium:

$$\underline{\tau}^{(l)} \to \underline{\tau}^{(l)} + d\underline{\tau}^{(l)}$$
$$\underline{\sigma}^{(l)} \to \underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}$$
$$\underline{A}_p \to \underline{A}_p + d\underline{A}_p \quad (p \equiv m, w, \theta)$$

Hence,

$$0 = \underline{F}_o^s - \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left[\left(\underline{B}_o^s\right)^t + \left(\underline{G}_m^s\right)^t \left(\underline{A}_m^t + d\underline{A}_m^t\right) \right] \left(\underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}\right) + \left(\underline{G}_m^s\right)^t \left(\underline{A}_\theta^t + d\underline{A}_\theta^t\right)(-z)\left(\underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}\right) \right\} dz\,dx\,dy \qquad (4.164a)$$

$$0 = \underline{F}_b^s - \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\left(\underline{\tau}^{(l)} + d\underline{\tau}^{(l)}\right) + \left(\underline{G}_w^s\right)^t \left(\underline{A}_w^t + d\underline{A}_w^t\right)\left(\underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}\right) + \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \left(\underline{A}_m^t + d\underline{A}_m^t\right)(-z)\left(\underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}\right) \right] + \left(\underline{G}_\theta^s\right)^t \left(\underline{A}_\theta^t + d\underline{A}_\theta^t\right)\left(z^2\right)\left(\underline{\sigma}^{(l)} + d\underline{\sigma}^{(l)}\right) \right\} dz\,dx\,dy \qquad (4.164b)$$

Ignoring terms which contain $d\underline{\sigma} \bullet d\underline{A}$, and using $\underline{R}_o^s, \underline{R}_b^s$ as defined earlier, then it can be deduced from equations (4.164a) and (4.164b) that

$$\underline{R}_o^s = \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left[\left(\underline{B}_o^s\right)^t + \left(\underline{G}_m^s\right)^t \underline{A}_m^t \right] d\underline{\sigma}^{(l)} + \left(\underline{G}_m^s\right)^t d\underline{A}_m^t \underline{\sigma}^{(l)} + \left(\underline{G}_m^s\right)^t \underline{A}_\theta^t (-z) d\underline{\sigma}^{(l)} + \left(\underline{G}_m^s\right)^t d\underline{A}_\theta^t (-z)\underline{\sigma}^{(l)} \right\} dz\,dx\,dy \qquad (4.165)$$

$$\underline{R}_b^s = \sum_{strips} \iint_{strip} \int_{-h/2}^{h/2} \left\{ \left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\, d\underline{\tau}^{(l)} + \left(\underline{G}_w^s\right)^t \underline{A}_w^t d\underline{\sigma}^{(l)} + \left(\underline{G}_w^s\right)^t d\underline{A}_w^t\, \underline{\sigma}^{(l)} \right.$$

$$+\left[\left(\underline{B}_b^s\right)' + \left(\underline{G}_\theta^s\right)' \underline{A}_m^t\right](-z)d\underline{\sigma}^{(l)} + \left(\underline{G}_\theta^s\right)' d\underline{A}_m^t(-z)\underline{\sigma}^{(l)} + \left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t(z^2)d\underline{\sigma}^{(l)}$$

$$+\left(\underline{G}_\theta^s\right)' d\underline{A}_\theta^t(z^2)\underline{\sigma}^{(l)}\Big\}dz\,dx\,dy \qquad (4.166)$$

Notice from equation (4.150) that:

$$\int_{-h/2}^{h/2} f_\gamma(z)\, d\underline{\tau}^{(l)}\, dz = \underline{\mu}_{\gamma\gamma} \sum_{r=1}^{m} \underline{B}_\gamma^r\, d\underline{\delta}_b^r \qquad (4.167)$$

where from equation (4.77)

$$\underline{\mu}_{\gamma\gamma} = \int_{-h/2}^{h/2} f_\gamma^2(z)\, \underline{\mu}^{(l)}\, dz$$

Similarly, it can be shown that:

$$(-1)^n \int_{-h/2}^{h/2} z^n d\underline{\sigma}^{(l)} dz = (-1)^n \sum_{r=1}^{m} \Big\{ D_{(n)} \left[\left(\underline{B}_o^r + \underline{B}_{mm}^r\right) d\underline{\delta}_o^r + \underline{B}_{ww}^r d\underline{\delta}_b^r\right]$$

$$- D_{(n+1)} \left[\left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) d\underline{\delta}_b^r + \underline{B}_{\theta m}^r d\underline{\delta}_o^r\right] + D_{(n+2)} \underline{B}_{\theta\theta}^r\, d\underline{\delta}_b^r \Big\} \qquad (4.168)$$

and from (4.86)

$$\underline{D}_{(n)} = \int_{-h/2}^{h/2} z^n \underline{D}^{(l)}\, dz$$

Substituting from equations (4.167) and (4.168) into (4.165) and (4.166) then, it can be deduced that:

$$\underline{R}_o^s = \underline{R}_{ok}^s + \underline{R}_{o\sigma}^s \qquad (4.169a)$$

$$\underline{R}_b^s = \underline{R}_{bk}^s + \underline{R}_{b\sigma}^s \qquad (4.169b)$$

where

$$\underline{R}_{ok}^s = \sum_{strips} \sum_{r=1}^{m} \iint_{strip} \Big\{ \left(\underline{B}_o^s + \underline{B}_{mm}^s\right)' \left\{ \underline{D}_{(o)} \left[\left(\underline{B}_o^r + \underline{B}_{mm}^r\right) d\underline{\delta}_o^r + \underline{B}_{ww}^r\, d\underline{\delta}_b^r\right]$$

$$- \underline{D}_{(1)} \left[\underline{B}_{\theta m}^r\, d\underline{\delta}_o^r + \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) d\underline{\delta}_b^r\right] + \underline{D}_{(2)} \underline{B}_{\theta\theta}^r\, d\underline{\delta}_b^r \Big\} + \left(\underline{B}_{\theta m}^s\right)' \Big\{ - \underline{D}_{(1)} \left[\left(\underline{B}_o^r + \underline{B}_{mm}^r\right) d\underline{\delta}_o^r + \underline{B}_{ww}^r\, d\underline{\delta}_b^r\right]$$

$$+ \underline{D}_{(2)} \left[\underline{B}_{\theta m}^r\, d\underline{\delta}_o^r + \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) d\underline{\delta}_b^r\right] - \underline{D}_{(3)} \underline{B}_{\theta\theta}^r\, d\underline{\delta}_b^r \Big\} \Big\} dx\, dy \qquad (4.170)$$

$$\underline{R}^s_{o\sigma} = \sum_{strips} \iint_{strip} \left[\left(\underline{G}^s_m\right)' d\,\underline{A}^t_m \underline{\sigma}_{(o)} + \left(\underline{G}^s_m\right)' d\,\underline{A}^t_\theta \underline{\sigma}_{(1)} \right] dx\,dy \tag{4.171}$$

$$\begin{aligned}
\underline{R}^s_{bk} = \sum_{strips} \sum_{r=1}^{m} \iint_{strip} \Big\{ & \left(\underline{B}^s_\gamma\right)' \underline{\mu}_{\gamma\gamma} \underline{B}^r_\gamma d\,\underline{\delta}^r_b + \left(\underline{B}^s_{ww}\right)' \Big\{ \underline{D}_{(o)} \left[\left(\underline{B}^r_o + \underline{B}^r_{mm}\right) d\,\underline{\delta}^r_o + \underline{B}^r_{ww}\, d\,\underline{\delta}^r_b \right] \\
& - \underline{D}_{(1)} \left[\underline{B}^r_{\theta m}\, d\,\underline{\delta}^r_o + \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) d\,\underline{\delta}^r_b \right] + \underline{D}_{(2)} \underline{B}^r_{\theta\theta}\, d\,\underline{\delta}^r_o \Big\} \\
& + \left(\underline{B}^s_b + \underline{B}^s_{m\theta}\right)' \Big\{ -\underline{D}_{(1)} \left[\left(\underline{B}^r_o + \underline{B}^r_{mm}\right) d\,\underline{\delta}^r_o + \underline{B}^r_{ww}\, d\,\underline{\delta}^r_b \right] \\
& + \underline{D}_{(2)} \left[\underline{B}^r_{\theta m}\, d\,\underline{\delta}^r_o + \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) d\,\underline{\delta}^r_b \right] - \underline{D}_{(3)} \underline{B}^r_{\theta\theta}\, d\,\underline{\delta}^r_b \Big\} \\
& + \left(\underline{B}^s_{\theta\theta}\right)' \Big\{ \underline{D}_{(2)} \left[\left(\underline{B}^r_o + \underline{B}^r_{mm}\right) d\,\underline{\delta}^r_o + \underline{B}^r_{ww}\, d\,\underline{\delta}^r_b \right] - \underline{D}_{(3)} \Big[\underline{B}^r_{\theta m}\, d\,\underline{\delta}^r_o + \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) d\,\underline{\delta}^r_b \\
& \qquad + \underline{D}_{(4)} \underline{B}^r_{\theta\theta}\, d\,\underline{\delta}^r_b \Big] \Big\} \Big\} dx\,dy
\end{aligned} \tag{4.172}$$

$$\underline{R}^s_{b\sigma} = \sum_{strips} \iint_{strip} \left[\left(\underline{G}^s_w\right)' d\,\underline{A}^t_w \underline{\sigma}_{(o)} + \left(\underline{G}^s_\theta\right)' d\,\underline{A}^t_m \underline{\sigma}_{(1)} + \left(\underline{G}^s_\theta\right)' d\,\underline{A}^t_\theta \underline{\sigma}_{(2)} \right] dx\,dy \tag{4.173}$$

Each of the four terms can be simplified and can be represented in matrix form.

Equation (4.170) can be rearranged and written as follows:

$$\begin{aligned}
\underline{R}^s_{ok} = \sum_{strip} \sum_{r=1}^{m} \Big[& \iint_{strip} \Big\{ \left(\underline{B}^s_o + \underline{B}^s_{mm}\right)' \left[\underline{D}_{(o)} \left(\underline{B}^r_o + \underline{B}^r_{mm}\right) - \underline{D}_{(1)} \underline{B}^r_{\theta m} \right] \\
& + \left(\underline{B}^s_{\theta m}\right)' \left[-\underline{D}_{(1)} \left(\underline{B}^r_o + \underline{B}^r_{mm}\right) + \underline{D}_{(2)} \underline{B}^r_{\theta m} \right] \Big\} dx\,dy\,d\,\underline{\delta}^r_o \\
& + \iint_{strip} \Big\{ \left(\underline{B}^s_o + \underline{B}^s_{mm}\right)' \left[\underline{D}_{(0)} \underline{B}^r_{ww} - \underline{D}_{(1)} \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) + \underline{D}_{(2)} \underline{B}^r_{\theta\theta} \right] \\
& \quad + \left(\underline{B}^s_{\theta m}\right)' \left[-\underline{D}_{(1)} \underline{B}^r_{ww} + \underline{D}_{(2)} \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) - \underline{D}_{(3)} \underline{B}^r_{\theta\theta} \right] \Big\} dx\,dy\,d\,\underline{\delta}^r_b \Big]
\end{aligned} \tag{4.174}$$

Similarly equation (4.172) can be rearranged as follows:

$$\begin{aligned}
\underline{R}^s_{bk} = \sum_{strip} \sum_{r=1}^{m} \Big[& \iint_{strip} \Big\{ \left(\underline{B}^s_{ww}\right)' \left[\underline{D}_{(o)} \left(\underline{B}^r_o + \underline{B}^r_{mm}\right) - \underline{D}_{(1)} \underline{B}^r_{\theta m} \right] + \left(\underline{B}^s_b + \underline{B}^s_{m\theta}\right)' \left[-\underline{D}_{(1)} \left(\underline{B}^r_o + \underline{B}^r_{mm}\right) \right. \\
& \left. + \underline{D}_{(2)} \underline{B}^r_{\theta m} \right] + \left(\underline{B}^s_{\theta\theta}\right)' \left[\underline{D}_{(2)} \left(\underline{B}^r_o + \underline{B}^r_{mm}\right) - \underline{D}_{(3)} \underline{B}^r_{\theta m} \right] \Big\} dx\,dy\,d\,\underline{\delta}^r_o \\
& + \iint_{strip} \Big\{ \left(\underline{B}^s_\gamma\right)' \underline{\mu}_{\gamma\gamma} \underline{B}^r_\gamma + \left(\underline{B}^s_{ww}\right)' \left[\underline{D}_{(0)} \underline{B}^r_{ww} - \underline{D}_{(1)} \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) + \underline{D}_{(2)} \underline{B}^r_{\theta\theta} \right] \\
& + \left(\underline{B}^s_b + \underline{B}^s_{m\theta}\right)' \left[-\underline{D}_{(1)} \underline{B}^r_{ww} + \underline{D}_{(2)} \left(\underline{B}^r_b + \underline{B}^r_{m\theta}\right) - \underline{D}_{(3)} \underline{B}^r_{\theta\theta} \right]
\end{aligned}$$

$$+\left(\underline{B}^{s}_{\theta\theta}\right)^{t}\left[\underline{D}_{(2)}\underline{B}^{r}_{ww}-\underline{D}_{(3)}\left(\underline{B}^{r}_{b}+\underline{B}^{r}_{m\theta}\right)+\underline{D}_{(4)}\underline{B}^{r}_{\theta\theta}\right]\Big\}dx\,dy\,d\underline{\delta}^{r}_{b}\Big] \tag{4.175}$$

To simplify equations (4.171) and (4.173) we use the following theorems:
If

$$\underline{\sigma}_{(0)}\equiv\left\{\sigma_x^{(0)}\quad\sigma_y^{(0)}\quad\tau_{xy}^{(0)}\right\} \tag{4.176}$$

$$\underline{\sigma}_{(1)}\equiv\left\{\sigma_x^{(1)}\quad\sigma_y^{(1)}\quad\tau_{xy}^{(1)}\right\} \tag{4.177}$$

$$\underline{\sigma}_{(2)}\equiv\left\{\sigma_x^{(2)}\quad\sigma_y^{(2)}\quad\tau_{xy}^{(2)}\right\} \tag{4.178}$$

then, by matrix manipulation, it can be proved that:

$$d\underline{A}^{t}_{m}\underline{\sigma}_{(0)}\equiv\underline{S}_{(0)}d\underline{\theta}_{m}=\sum_{r=1}^{m}\underline{S}_{(0)}\underline{G}^{r}_{m}d\underline{\delta}_{o} \tag{4.179}$$

$$d\underline{A}^{t}_{\theta}\underline{\sigma}_{(1)}\equiv\underline{S}_{(1)}d\underline{\theta}_{\theta}=\sum_{r=1}^{m}\underline{S}_{(1)}\underline{G}^{r}_{\theta}d\underline{\delta}_{b} \tag{4.180}$$

$$d\underline{A}^{t}_{w}\underline{\sigma}_{(0)}\equiv\underline{S}_{w}d\underline{\theta}_{w}=\sum_{r=1}^{m}\underline{S}_{w}\underline{G}^{r}_{w}d\underline{\delta}_{b} \tag{4.181}$$

$$d\underline{A}^{t}_{m}\underline{\sigma}_{(1)}\equiv\underline{S}_{(1)}d\underline{\theta}_{m}=\sum_{r=1}^{m}\underline{S}_{(1)}\underline{G}^{r}_{m}d\underline{\delta}_{o} \tag{4.182}$$

$$d\underline{A}^{t}_{\theta}\underline{\sigma}_{(2)}\equiv\underline{S}_{(2)}d\underline{\theta}_{\theta}=\sum_{r=1}^{m}\underline{S}_{(2)}\underline{G}^{r}_{\theta}d\underline{\delta}_{b} \tag{4.183}$$

where

$$\underline{S}_{(n)}=\begin{bmatrix}\sigma_x^{(n)} & 0 & \tau_{xy}^{(n)} & 0\\ 0 & \sigma_x^{(n)} & 0 & \tau_{xy}^{(n)}\\ \tau_{xy}^{(n)} & 0 & \sigma_y^{(n)} & 0\\ 0 & \tau_{xy}^{(n)} & 0 & \sigma_y^{(n)}\end{bmatrix} \tag{4.184}$$

and

$$\underline{S}_w = \begin{bmatrix} \sigma_x^{(0)} & \tau_{xy}^{(0)} \\ \\ \tau_{xy}^{(0)} & \sigma_y^{(0)} \end{bmatrix} \tag{4.185}$$

Substituting from equations (4.179), (4.180) into (4.171), then

$$\underline{R}_{o\sigma}^{s} = \sum_{strips} \sum_{r=1}^{m} \iint_{strip} \left[\left(\underline{G}_m^s\right)^t \underline{S}_{(o)} \underline{G}_m^r \, d\underline{\delta}_o^r + \left(\underline{G}_m^s\right)^t \underline{S}_{(1)} \underline{G}_\theta^r \, d\underline{\delta}_b^r \right] dx\, dy \tag{4.186}$$

Similarly substituting from equations (4.181)-(4.183) into (4.173) gives

$$\underline{R}_{b\sigma}^{s} = \sum_{strips} \sum_{r=1}^{m} \iint_{strip} \left\{ \left[\left(\underline{G}_\theta^s\right)^t \underline{S}_{(1)} \underline{G}_m^r \right] d\underline{\delta}_o^r + \left[\left(\underline{G}_w^s\right)^t \underline{S}_w \underline{G}_w^r + \left(\underline{G}_\theta^s\right)^t \underline{S}_{(2)} \underline{G}_\theta^r \right] d\underline{\delta}_b^r \right\} dx\, dy \tag{4.187}$$

4.11.1 Defining of $\underline{K}^{sr}$ Terms

From equations (4.174) and (4.175) the following matrices are defined for one strip:

$$\underline{K}_{oo}^{sr} = \iint_{strip} \left\{ \left(\underline{B}_o^s + \underline{B}_{mm}^s\right)^t \left[\underline{D}_{(0)} \left(\underline{B}_o^r + \underline{B}_{mm}^r\right) - \underline{D}_{(1)} \underline{B}_{\theta m}^r \right] + \left(\underline{B}_{\theta m}^s\right)^t \left[\underline{D}_{(2)} \underline{B}_{\theta m}^r - \underline{D}_{(1)} \left(\underline{B}_o^r + \underline{B}_{mm}^r\right) \right] \right\} dx\, dy \tag{4.188}$$

$$\underline{K}_{ob}^{sr} = \iint_{strip} \left\{ \left(\underline{B}_o^s + \underline{B}_{mm}^s\right)^t \left[\underline{D}_{(0)} \underline{B}_{ww}^r - \underline{D}_{(1)} \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) + \underline{D}_{(2)} \underline{B}_{\theta\theta}^r \right] + \left(\underline{B}_{\theta m}^s\right)^t \left[-\underline{D}_{(1)} \underline{B}_{ww}^r + \underline{D}_{(2)} \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) - \underline{D}_{(3)} \underline{B}_{\theta\theta}^r \right] \right\} dx\, dy \tag{4.189}$$

$$\underline{K}_{bo}^{sr} = \iint_{strip} \left\{ \left(\underline{B}_{ww}^s\right)^t \left[\underline{D}_{(0)} \left(\underline{B}_o^r + \underline{B}_{mm}^r\right) - \underline{D}_{(1)} \underline{B}_{\theta m}^r \right] + \left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)^t \left[-\underline{D}_{(1)} \left(\underline{B}_o^r + \underline{B}_{mm}^r\right) + \underline{D}_{(2)} \underline{B}_{\theta m}^r + \left(\underline{B}_{\theta\theta}^s\right)^t \left[\underline{D}_{(2)} \left(\underline{B}_o^r + \underline{B}_{mm}^r\right) - \underline{D}_{(3)} \underline{B}_{\theta m}^r \right] \right\} dx\, dy \tag{4.190}$$

$$\underline{K}_{\gamma\gamma}^{sr} = \iint_{strip} \left(\underline{B}_\gamma^s\right)^t \underline{\mu}_{\gamma\gamma} \underline{B}_\gamma^r \, dx\, dy \tag{4.191}$$

$$\underline{K}_{bb}^{sr} = \iint_{strip} \left\{ \left(\underline{B}_{ww}^s\right)^t \left[\underline{D}_{(0)} \underline{B}_{ww}^r - \underline{D}_{(1)} \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) + \underline{D}_{(2)} \underline{B}_{\theta\theta}^r \right] + \left(\underline{B}_b^s + \underline{B}_{m\theta}^s\right)^t \left[-\underline{D}_{(1)} \underline{B}_{ww}^r + \underline{D}_{(2)} \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) - \underline{D}_{(3)} \underline{B}_{\theta\theta}^r \right] + \left(\underline{B}_{\theta\theta}^s\right)^t \left[\underline{D}_{(2)} \underline{B}_{ww}^r - \underline{D}_{(3)} \left(\underline{B}_b^r + \underline{B}_{m\theta}^r\right) + \underline{D}_{(4)} \underline{B}_{\theta\theta}^r \right] \right\} dx\, dy \tag{4.192}$$

4.11.2 Definition of $\underline{K}_{\sigma}^{sr}$ Terms for a Strip

The following matrices can be defined from (4.186) and (4.187):

$$\underline{K}_{mm}^{sr} = \iint_{strip} \left(\underline{G}_m^s\right)^t \underline{S}_{(0)}\, \underline{G}_m^r \, dx\, dy \tag{4.193}$$

$$\underline{K}_{m\theta}^{sr} = \iint_{strip} \left(\underline{G}_m^s\right)^t \underline{S}_{(1)}\, \underline{G}_\theta^r \, dx\, dy \tag{4.194}$$

$$\underline{K}_{\theta m}^{sr} = \iint_{strip} \left(\underline{G}_\theta^s\right)^t \underline{S}_{(1)}\, \underline{G}_m^r \, dx\, dy \tag{4.195}$$

$$\underline{K}_{ww}^{sr} = \iint_{strip} \left(\underline{G}_w^s\right)^t \underline{S}_w\, \underline{G}_w^r \, dx\, dy \tag{4.196}$$

$$\underline{K}_{\theta\theta}^{sr} = \iint_{strip} \left(\underline{G}_\theta^s\right)^t \underline{S}_{(2)}\, \underline{G}_\theta^r \, dx\, dy \tag{4.197}$$

4.11.3 Matrix Form of Linearized Equations

Substituting from equations (4.188)-(4.192) into (4.174), (4.175) then:

$$\underline{R}_{ok}^s = \sum_{strip} \sum_{r=1}^{m} \left[\underline{K}_{oo}^{sr} d\underline{\delta}_o^r + \underline{K}_{ob}^{sr} d\underline{\delta}_b^r \right] \tag{4.198}$$

$$\underline{R}_{bk}^s = \sum_{strip} \sum_{r=1}^{m} \left[\underline{K}_{bo}^{sr} d\underline{\delta}_o^r + \left(\underline{K}_{\gamma\gamma}^{sr} + \underline{K}_{bb}^{sr}\right) d\underline{\delta}_b^r \right] \tag{4.199}$$

Similarly, substituting from equations (4.193)-(4.197) into (4.186) and (4.187) :

$$\underline{R}_{o\sigma}^s = \sum_{strip} \sum_{r=1}^{m} \left(\underline{K}_{mm}^{sr}\, d\underline{\delta}_o^r + \underline{K}_{m\theta}^{sr}\, d\underline{\delta}_b^r \right) \tag{4.200}$$

$$\underline{R}_{b\sigma}^s = \sum_{strip} \sum_{r=1}^{m} \left[\underline{K}_{\theta m}^{sr}\, d\underline{\delta}_o^r + \left(\underline{K}_{ww}^{sr} + \underline{K}_{\theta\theta}^{sr}\right) d\underline{\delta}_b^r \right] \tag{4.201}$$

Substituting from equations (4.198)-(4.201) into (4.169a) and (4.169b), it can be deduced that:

$$\sum_{strip} \sum_{r=1}^{m} \left\{ \left[\underline{K}_{oo}^{sr} d\underline{\delta}_o^r + \underline{K}_{ob}^{sr} d\underline{\delta}_b^r \right] + \left[\underline{K}_{mm}^{sr} d\underline{\delta}_o^r + \underline{K}_{m\theta}^{sr} d\underline{\delta}_b^r \right] \right\} = \underline{R}_o^s \tag{4.202}$$

$$\sum_{strip}\sum_{r=1}^{m}\left\{\left[\underline{K}_{bo}^{sr}\,d\underline{\delta}_o^r+\left(\underline{K}_{\gamma\gamma}^{sr}+\underline{K}_{bb}^{sr}\right)d\underline{\delta}_b^r\right]+\left[\underline{K}_{\theta m}^{sr}\,d\underline{\delta}_o^r+\left(\underline{K}_{ww}^{sr}+\underline{K}_{\theta\theta}^{sr}\right)d\underline{\delta}_b^r\right]\right\}=\underline{R}_b^s \quad (4.203)$$

for s = 1, 2,3, ….. , m.

Equations (4.202), (4.203) can also be written in the following matrix form:

$$\sum_{strip}\left(\underline{K}+\underline{K}_\sigma\right)d\underline{\delta}=\underline{R} \quad (4.204)$$

$$\text{where } d\underline{\delta}^t=\left[d\underline{\delta}^1 \quad d\underline{\delta}^2\cdots d\underline{\delta}^r\cdots d\underline{\delta}^m\right] \quad (4.205)$$

$$\underline{R}^t=\left[\underline{R}^1 \quad \underline{R}^2\cdots\underline{R}^s\cdots\underline{R}^m\right] \quad (4.206)$$

$$\underline{K}=\begin{bmatrix}\underline{K}^{11} & \cdots & \underline{K}^{1r} & \cdots & \underline{K}^{1m}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \underline{K}^{s1} & \cdots & \underline{K}^{sr} & \cdots & \underline{K}^{sm}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \underline{K}^{m1} & \cdots & \underline{K}^{mr} & \cdots & \underline{K}^{mm}\end{bmatrix} \quad (4.207)$$

If $d\underline{\delta}^r$, $\underline{R}^s$ are defined such that:

$$\left(d\underline{\delta}^r\right)=\begin{bmatrix}d\underline{\delta}_o^r\\ \\ d\underline{\delta}_b^r\end{bmatrix} \quad (4.208)$$

$$\underline{R}^s=\begin{bmatrix}\underline{R}_o^s\\ \\ \underline{R}_b^s\end{bmatrix} \quad (4.209)$$

therefore, the element matrices can be defined as follows:

$$\underline{K}^{sr}=\begin{bmatrix}\underline{K}_{oo}^{sr} & \underline{K}_{ob}^{sr}\\ \\ \underline{K}_{bo}^{sr} & \left(\underline{K}_{\gamma\gamma}^{sr}+\underline{K}_{bb}^{sr}\right)\end{bmatrix} \quad (4.210)$$

$$\underline{K}_{\sigma}^{sr} = \begin{bmatrix} \underline{K}_{mm}^{sr} & \underline{K}_{m\theta}^{sr} \\ \underline{K}_{\theta m}^{sr} & \left(\underline{K}_{ww}^{sr} + \underline{K}_{\theta\theta}^{sr}\right) \end{bmatrix} \tag{4.211}$$

4.12. Special Case of Large Deflection due to Out-of-Plane Bending (w) Only

In this case, only the out-of-plane displacement, w will be taken into account in the large deflection analysis.

This means that equations (4.37)-(4.39) will be simplified as follows:

$$\varepsilon_x^L = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{4.212}$$

$$\varepsilon_y^L = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{4.213}$$

$$\gamma_{xy}^L = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{4.214}$$

then, it can be shown that:

$$\underline{\varepsilon}_L = \underline{\varepsilon}_w = \frac{1}{2}\,\underline{A}_w\,\underline{\theta}_w \tag{4.215}$$

and all other terms are negligible.

Hence, different stiffness matrix terms are simplified as follows:

$$\underline{K}_{oo}^{sr} = \iint_{strip}\left(\underline{B}_o^s\right)^t \underline{D}_{(0)}\,\underline{B}_o^r\,dx\,dy \tag{4.216}$$

$$\underline{K}_{ob}^{sr} = \iint_{strip}\left(\underline{B}_o^s\right)^t \left[\underline{D}_{(0)}\,\underline{B}_{ww}^r - \underline{D}_{(1)}\,\underline{B}_b^r\right] dx\,dy \tag{4.217}$$

$$\underline{K}_{bo}^{sr} = \iint_{strip}\left[\left(\underline{B}_{ww}^s\right)^t \underline{D}_{(0)}\,\underline{B}_o^r - \left(\underline{B}_b^s\right)^t \underline{D}_{(1)}\,\underline{B}_o^r\right] dx\,dy \tag{4.218}$$

$$\underline{K}_{\gamma\gamma}^{sr} = \iint_{strip}\left(\underline{B}_\gamma^s\right)^t \underline{\mu}_{\gamma\gamma}\,\underline{B}_\gamma^r\,dx\,dy \tag{4.219}$$

$$\underline{K}_{bb}^{sr} = \iint_{strip}\left\{\left(\underline{B}_{ww}^s\right)^t \left[\underline{D}_{(0)}\,\underline{B}_{ww}^r - \underline{D}_{(1)}\,\underline{B}_b^r\right] + \left(\underline{B}_b^s\right)^t \left[-\underline{D}_{(1)}\,\underline{B}_{ww}^r + \underline{D}_{(2)}\,\underline{B}_b^r\right]\right\} dx\,dy \tag{4.220}$$

$$\underline{\underline{K}}_{mm}^{sr} = \underline{\underline{0}} \tag{4.221}$$

$$\underline{\underline{K}}_{m\theta}^{sr} = \underline{\underline{0}} \tag{4.222}$$

$$\underline{\underline{K}}_{\theta m}^{sr} = \underline{\underline{0}} \tag{4.223}$$

$$\underline{\underline{K}}_{\theta\theta}^{sr} = \underline{\underline{0}} \tag{4.224}$$

$$\underline{\underline{K}}_{ww}^{sr} = \iint_{strip} \left(\underline{G}_w^s\right)^t \underline{S}_w \underline{G}_w^r \, dx\, dy \tag{4.225}$$

CHAPTER 5

5. Reissner-Type finite Strip Elements

5.1 Introduction

Reissner-type elements are higher order shear elements, in which the transverse shear stresses are modelled accurately according to the equations derived by Reissner (1945). The transverse shear strains distributions are assumed to be parabolic. The interpolation theorems employed along the length of the plate are one-dimensional *Lagrangian* interpolation for in-plane components and *Hermitian* interpolation for the lateral deflection components. One dimensional polynomial interpolation is applied along the width of the plate. Practically, *Reissner-type* elements are used for the analysis of isotropic or composite structures made of thick plates, in which the transverse shear effects are important.

This chapter presents the finite strip analysis of composite plates and shells, which is based on a new derivation of the higher order shear element (*Reissner-type* element) taking into account the geometric non-linearity. This derivation follows concepts similar to those employed in the finite element analysis, and are based on a new *Reissner-type* finite strip element (Razzaq, 2003).

5.2 Reissner Plate Bending Theory

Consider a plate defined in terms of a midplane in the Cartesian x-y plane, and a thickness distribution $h(x,y)$ measured in the z-direction, such as that shown previously in figure (4.1). The upper surface of the plate ($z = h/2$) is considered to be under a shear force per unit area of $qu\ (x, y)$ and the lower surface is subjected to a shear force per unit area $ql\ (x, y)$. Other types of loading can also be considered (Razzaq, 2003).

The derivations of the *Reissner-type* element are based upon the following basic assumptions and approximations of plate bending theories:

a) The composite material of the plate is homogeneous, and the material behaves linearly.

b) Displacement gradients are finite, such that Cauchy's strain-displacements relationships are no longer applicable and large deflection analysis will be considered. However, the transverse shear strains are assumed infinitesimal (linear) and continuous along the plate width.

c) The mid-plane remains unstrained after bending, i.e. it is the neutral plane in bending.

d) The lateral deflection w, which is the displacement component in the z-direction, is independent of z, i.e. $w \approx w(x, y)$.

5.3 Displacement Components of Reissner-Type Element

The composite plate consist of N_l layers each of thickness $h^{(l)}$, and they are parallel to the x-y plane (z = 0) which is assumed as the mid-plane of the plate, such as shown in figure (4.2). Each layer l is defined by its lower and upper surface equations:

$$Z = Z_L^{(l)} \tag{5.1}$$

and

$$Z = Z_U^{(l)} \tag{5.2}$$

with the layer thickness:

$$h^{(l)} = Z_U^{(l)} - Z_L^{(l)} \tag{5.3}$$

and the total plate thickness will be:

$$h = \sum_{l=1}^{N_l} h^{(l)} \tag{5.4}$$

By considering composite plates to be transversely isotropic, i.e.

$$\mu_{13} = \mu_{23} \tag{5.5a}$$

then

$$\mu_{xz} = \mu_{yz} \tag{5.5b}$$

which will lead to continuous transverse shear stress and strain distributions over the plate thickness.

Using a three-point *Lagrangian* interpolation, as described in chapter 4, the transverse shear strain distributions over the thickness of the plate can be described as follows:

$$\gamma_{xz}(x,y,z) = \gamma_{xz}^{\circ}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{5.6}$$

$$\gamma_{yz}(x,y,z) = \gamma_{yz}^{\circ}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \tag{5.7}$$

where $\gamma_{xz}^{\circ}, \gamma_{yz}^{\circ}$ are the values of γ_{xz}, γ_{yz} at the mid-plane (z = 0) of the plate. The average values of γ_{xz}, γ_{yz} over the plate thickness can be defined as follows:

$$\bar{\gamma}_{xz}(x,y) = \frac{1}{h}\int_{-h/2}^{h/2} \gamma_{xz}\, dz = \tfrac{2}{3}\gamma_{xz}^{\circ} \tag{5.8}$$

$$\bar{\gamma}_{yz}(x,y) = \frac{1}{h}\int_{-h/2}^{h/2} \gamma_{yz}\, dz = \tfrac{2}{3}\gamma^{\circ}_{yz} \tag{5.9}$$

and equations (5.6) and (5.7) can be rewritten as follows:

$$\gamma_{xz}(x,y,z) = \frac{3}{2}\bar{\gamma}_{xz}(x,y)\left[1-\frac{4z^2}{h^2}\right] \tag{5.10a}$$

$$\gamma_{yz}(x,y,z) = \frac{3}{2}\bar{\gamma}_{yz}(x,y)\left[1-\frac{4z^2}{h^2}\right] \tag{5.10b}$$

From *Mindlin's* plate bending theory, the average slope angles are defined as follows:

$$\theta_x = \frac{\partial w}{\partial y} - \bar{\gamma}_{yz} \equiv \frac{\partial w}{\partial y} - \psi_x \tag{5.11a}$$

$$\theta_y = -\frac{\partial w}{\partial x} + \bar{\gamma}_{xz} \equiv -\frac{\partial w}{\partial x} - \psi_y \tag{5.11b}$$

$$\text{where } \psi_x = \bar{\gamma}_{yz},\ \psi_y = -\bar{\gamma}_{xz} \tag{5.12}$$

Hence, equations (5.10a) and (5.10b) can be rewritten as follows:

$$\gamma_{xz}(x,y,z) = -\frac{3}{2}\psi_y(x,y)\left[1-\frac{4z^2}{h^2}\right] \tag{5.13a}$$

$$\gamma_{yz}(x,y,z) = \frac{3}{2}\psi_x(x,y)\left[1-\frac{4z^2}{h^2}\right] \tag{5.13b}$$

Equations (5.13a), (5.13b) can be written in the following matrix form:

$$\underline{\gamma} = \begin{bmatrix} \gamma_{xz} \\ \\ \gamma_{yz} \end{bmatrix} = f_{\gamma}(z)\hat{\underline{\gamma}}(x,y) \tag{5.14}$$

$$\text{where } f_{\gamma}(z) = \frac{3}{2}\left[1-\frac{4z^2}{h^2}\right] \tag{5.15}$$

and $$\hat{\gamma}(x,y)=\begin{bmatrix}-\psi_y\\ \\ \psi_x\end{bmatrix} \tag{5.16}$$

The following two approximations are considered in this theory:

(i) The effect of z on the lateral deflection w is negligible i.e.:

$$w(x,y,z)\approx w(x,y) \tag{5.17}$$

(ii) The transverse shear strains are infinitesimal which will lead to:

$$\gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \tag{5.18}$$

and

$$\gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \tag{5.19}$$

Substituting from equations (5.13a) and (5.13b) into (5.18) and (5.19), then it can be deduced that:

$$\frac{\partial u}{\partial z}=-\frac{\partial w}{\partial x}-\frac{3}{2}\psi_y\left[1-\frac{4z^2}{h^2}\right] \tag{5.20}$$

$$\frac{\partial v}{\partial z}=-\frac{\partial w}{\partial y}+\frac{3}{2}\psi_x\left[1-\frac{4z^2}{h^2}\right] \tag{5.21}$$

Integrating equations (5.20) and (5.21) with respect to z, then:

$$u(x,y,z)=u^{\circ}(x,y)-z\frac{\partial w}{\partial x}-\psi_y\left(\frac{3}{2}z-\frac{2z^3}{h^2}\right) \tag{5.22}$$

$$v(x,y,z)=v^{\circ}(x,y)-z\frac{\partial w}{\partial y}+\psi_x\left(\frac{3}{2}z-\frac{2z^3}{h^2}\right) \tag{5.23}$$

where u°, v° represent the values of u, v at the mid-plane $z = 0$.

5.4 Strain Components

5.4.1 X-Y strain Components

These are $\varepsilon_x, \varepsilon_y$ and γ_{xy}, which will be represented by the vector $\underline{\varepsilon}$, where

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{5.24}$$

Using Green's strain-displacement equations, then:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right] \tag{5.25}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] \tag{5.26}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right) \tag{5.27}$$

The x-y strain vector can be partitioned into two parts:

$$\underline{\varepsilon} = \underline{\varepsilon}_S + \underline{\varepsilon}_L \tag{5.28}$$

where

$$\underline{\varepsilon}_S = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} \tag{5.29}$$

which represents the infinitesimal components, and

$$\underline{\varepsilon}_L = \frac{1}{2}\begin{bmatrix} \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2 \\ \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 \\ 2\left(\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \right) \end{bmatrix} \tag{5.30}$$

which represents the additional (non-linear) large strain terms.

5.4.1.1 Infinitesimal Strain Components

Substituting from equations (5.22), (5.23) into (5.29) then

$$\underline{\varepsilon}_S = \begin{bmatrix} \dfrac{\partial u}{\partial x}^{\circ} - z\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial \psi_y}{\partial x}\left(\dfrac{3}{2}z - \dfrac{2z^3}{h^2}\right) \\ \dfrac{\partial v}{\partial y}^{\circ} - z\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial \psi_x}{\partial y}\left(\dfrac{3}{2}z - \dfrac{2z^3}{h^2}\right) \\ \left(\dfrac{\partial u}{\partial y}^{\circ} + \dfrac{\partial v}{\partial x}^{\circ}\right) - 2z\dfrac{\partial^2 w}{\partial x \partial y} + \left(\dfrac{\partial \psi_x}{\partial x} - \dfrac{\partial \psi_y}{\partial y}\right)\left(\dfrac{3}{2}z - \dfrac{2z^3}{h^2}\right) \end{bmatrix} \tag{5.31}$$

which can be rewritten as follows:

$$\underline{\varepsilon}_S(x,y,z) = \underline{\varepsilon}_o(x,y) - z\hat{\varepsilon}_b(x,y) + f_\psi(z)\hat{\underline{\varepsilon}}_\psi(x,y) \tag{5.32}$$

$$\text{where} \quad f_\psi(z) = \frac{3}{2}z - \frac{2z^3}{h^2} \tag{5.33}$$

$$\underline{\varepsilon}_o(x,y) = \begin{bmatrix} \dfrac{\partial u}{\partial x}^{\circ} \\ \dfrac{\partial v}{\partial y}^{\circ} \\ \dfrac{\partial u}{\partial y}^{\circ} + \dfrac{\partial v}{\partial x}^{\circ} \end{bmatrix} \tag{5.34}$$

$$\underline{\varepsilon}_b(x,y) = \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{5.35}$$

$$\hat{\underline{\varepsilon}}_{\psi}(x,y)=\begin{bmatrix} -\dfrac{\partial \psi_y}{\partial x} \\ \dfrac{\partial \psi_x}{\partial y} \\ \dfrac{\partial \psi_x}{\partial x}-\dfrac{\partial \psi_y}{\partial y} \end{bmatrix} \tag{5.36}$$

5.4.1.2 Additional Large Strain Components

$$\text{Let} \qquad \underline{\varepsilon}_L=\left\{ \varepsilon_x^L \quad \varepsilon_y^L \quad \gamma_{xy}^L \right\} \tag{5.37}$$

then, comparing (5.37) with (5.30), it can be shown that:

$$\varepsilon_x^L=\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{5.38}$$

$$\varepsilon_y^L=\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \tag{5.39}$$

$$\gamma_{xy}^L=\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{5.40}$$

Substituting from equations (5.22), (5.23) into (5.38) then:

$$\varepsilon_x^L=\frac{1}{2}\left[\frac{\partial u^\circ}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+f_\psi\left(-\frac{\partial \psi_y}{\partial x}\right)\right]^2+\frac{1}{2}\left[\frac{\partial v^\circ}{\partial x}-z\frac{\partial^2 w}{\partial x\partial y}+f_\psi\frac{\partial \psi_x}{\partial x}\right]^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$=\frac{1}{2}\left[\left(\frac{\partial u^\circ}{\partial x}\right)^2+\left(\frac{\partial v^\circ}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{1}{2}z^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]$$

$$+\frac{1}{2}f_\psi^2\left[\left(\frac{\partial \psi_x}{\partial x}\right)^2+\left(\frac{\partial \psi_y}{\partial x}\right)^2\right]-z\left(\frac{\partial u^\circ}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial v^\circ}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right)$$

$$+f_\psi\left[\frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \psi_y}{\partial x}\right)+\frac{\partial v^\circ}{\partial x}\left(\frac{\partial \psi_x}{\partial x}\right)\right]-zf_\psi\left[\frac{\partial^2 w}{\partial x^2}\left(-\frac{\partial \psi_y}{\partial x}\right)+\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial \psi_x}{\partial x}\right)\right] \tag{5.41}$$

Similarly:

$$\varepsilon_y^L = \frac{1}{2}\left[\frac{\partial u^\circ}{\partial y} - z\frac{\partial^2 w}{\partial x \partial y} - f_\psi \frac{\partial \psi_y}{\partial y}\right]^2 + \frac{1}{2}\left[\frac{\partial v^\circ}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + f_\psi \frac{\partial \psi_x}{\partial x}\right]^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$= \frac{1}{2}\left[\left(\frac{\partial u^\circ}{\partial y}\right)^2 + \left(\frac{\partial v^\circ}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{2}z^2\left[\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]$$

$$+ \frac{1}{2}f_\psi^2\left[\left(\frac{\partial \psi_x}{\partial y}\right)^2 + \left(\frac{\partial \psi_y}{\partial y}\right)^2\right] - z\left(\frac{\partial u^\circ}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial v^\circ}{\partial y}\frac{\partial^2 w}{\partial y^2}\right)$$

$$+ f_\psi\left[\frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \psi_y}{\partial y}\right) + \frac{\partial v^\circ}{\partial y}\left(\frac{\partial \psi_x}{\partial y}\right)\right] - zf_\psi\left[\frac{\partial^2 w}{\partial x \partial y}\left(-\frac{\partial \psi_y}{\partial y}\right) + \frac{\partial^2 w}{\partial y^2}\frac{\partial \psi_x}{\partial y}\right] \tag{5.42}$$

and

$$\gamma_{xy}^L = \left(\frac{\partial u^\circ}{\partial y} - z\frac{\partial^2 w}{\partial x \partial y} - f_\psi \frac{\partial \psi_y}{\partial y}\right)\left(\frac{\partial u^\circ}{\partial x} - z\frac{\partial^2 w}{\partial x^2} - f_\psi \frac{\partial \psi_y}{\partial x}\right)$$

$$+ \left(\frac{\partial v^\circ}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y} + f_\psi \frac{\partial \psi_x}{\partial x}\right)\left(\frac{\partial v^\circ}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + f_\psi \frac{\partial \psi_x}{\partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$= \left(\frac{\partial u^\circ}{\partial x}\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}\frac{\partial v^\circ}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + z^2\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 w}{\partial x \partial y}\right)\right]$$

$$+ f_\psi^2\left[\frac{\partial \psi_x}{\partial x}\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\frac{\partial \psi_y}{\partial y}\right] - z\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial u^\circ}{\partial y} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial u^\circ}{\partial x} + \frac{\partial^2 w}{\partial y^2}\frac{\partial v^\circ}{\partial x} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial v^\circ}{\partial y}\right]$$

$$+ f_\psi\left[\frac{\partial u^\circ}{\partial x}\left(-\frac{\partial \psi_y}{\partial y}\right) + \frac{\partial u^\circ}{\partial y}\left(-\frac{\partial \psi_y}{\partial x}\right) + \frac{\partial v^\circ}{\partial x}\frac{\partial \psi_x}{\partial y} + \frac{\partial v^\circ}{\partial y}\frac{\partial \psi_x}{\partial x}\right]$$

$$- zf_\psi\left[\left(-\frac{\partial \psi_y}{\partial x}\right)\frac{\partial^2 w}{\partial x \partial y} + \left(-\frac{\partial \psi_y}{\partial y}\right)\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial \psi_x}{\partial x}\right)\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial \psi_x}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y}\right] \tag{5.43}$$

Equations (5.41), (5.42), (5.43) can be written in the following matrix form:

$$\underline{\varepsilon}_L(x,y,z) = \underline{\varepsilon}_{mm}(x,y) + \underline{\varepsilon}_{ww}(x,y) + z^2\hat{\underline{\varepsilon}}_{\theta\theta} + f_\psi{}^2\hat{\underline{\varepsilon}}_{\psi\psi} - z\hat{\underline{\varepsilon}}_{m\theta} + f_\psi\hat{\underline{\varepsilon}}_{m\psi} - zf_\psi\hat{\underline{\varepsilon}}_{\theta\psi} \tag{5.44}$$

where

$$\underline{\varepsilon}_{mm}(x,y)=\frac{1}{2}\begin{bmatrix}\left(\frac{\partial u^{\circ}}{\partial x}\right)^2+\left(\frac{\partial v^{\circ}}{\partial x}\right)^2\\ \left(\frac{\partial u^{\circ}}{\partial y}\right)^2+\left(\frac{\partial v^{\circ}}{\partial y}\right)^2\\ 2\left(\frac{\partial u^{\circ}}{\partial x}\frac{\partial u^{\circ}}{\partial y}+\frac{\partial v^{\circ}}{\partial x}\frac{\partial v^{\circ}}{\partial y}\right)\end{bmatrix} \tag{5.45}$$

$$\underline{\varepsilon}_{ww}(x,y)=\frac{1}{2}\begin{bmatrix}\left(\frac{\partial w}{\partial x}\right)^2\\ \left(\frac{\partial w}{\partial y}\right)^2\\ 2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\end{bmatrix} \tag{5.46}$$

$$\hat{\underline{\varepsilon}}_{\theta\theta}(x,y)=\frac{1}{2}\begin{bmatrix}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\\ \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\\ 2\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w}{\partial y^2}\right)\end{bmatrix} \tag{5.47}$$

$$\hat{\underline{\varepsilon}}_{\psi\psi}(x,y)=\frac{1}{2}\begin{bmatrix}\left(\frac{\partial \psi_x}{\partial x}\right)^2+\left(\frac{\partial \psi_y}{\partial x}\right)^2\\ \left(\frac{\partial \psi_x}{\partial y}\right)^2+\left(\frac{\partial \psi_y}{\partial y}\right)^2\\ 2\left(\frac{\partial \psi_x}{\partial x}\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\frac{\partial \psi_y}{\partial y}\right)\end{bmatrix} \tag{5.48}$$

$$\hat{\underline{\varepsilon}}_{m\theta}(x,y)=\begin{bmatrix}\frac{\partial u^{\circ}}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial v^{\circ}}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\\ \frac{\partial u^{\circ}}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial v^{\circ}}{\partial y}\frac{\partial^2 w}{\partial y^2}\\ \frac{\partial u^{\circ}}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial u^{\circ}}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{\partial v^{\circ}}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\partial v^{\circ}}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\end{bmatrix} \tag{5.49}$$

$$\hat{\underline{\varepsilon}}_{m\psi}(x,y)=\begin{bmatrix} \dfrac{\partial u^{\circ}}{\partial x}\left(-\dfrac{\partial \psi_y}{\partial x}\right)+\dfrac{\partial v^{\circ}}{\partial x}\left(\dfrac{\partial \psi_x}{\partial x}\right) \\ \dfrac{\partial u^{\circ}}{\partial y}\left(-\dfrac{\partial \psi_y}{\partial y}\right)+\dfrac{\partial v^{\circ}}{\partial y}\left(\dfrac{\partial \psi_x}{\partial y}\right) \\ \dfrac{\partial u^{\circ}}{\partial x}\left(-\dfrac{\partial \psi_y}{\partial y}\right)+\dfrac{\partial u^{\circ}}{\partial y}\left(-\dfrac{\partial \psi_y}{\partial x}\right)+\dfrac{\partial v^{\circ}}{\partial x}\dfrac{\partial \psi_x}{\partial y}+\dfrac{\partial v^{\circ}}{\partial y}\dfrac{\partial \psi_x}{\partial x} \end{bmatrix} \tag{5.50}$$

$$\hat{\underline{\varepsilon}}_{m\psi}(x,y)=\begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2}\left(-\dfrac{\partial \psi_y}{\partial x}\right)+\dfrac{\partial^2 w}{\partial x\partial y}\left(\dfrac{\partial \psi_x}{\partial x}\right) \\ \dfrac{\partial^2 w}{\partial x\partial y}\left(-\dfrac{\partial \psi_y}{\partial y}\right)+\dfrac{\partial^2 w}{\partial y^2}\left(\dfrac{\partial \psi_x}{\partial y}\right) \\ \dfrac{\partial^2 w}{\partial x\partial y}\left(-\dfrac{\partial \psi_y}{\partial x}\right)+\dfrac{\partial^2 w}{\partial x^2}\left(-\dfrac{\partial \psi_y}{\partial y}\right)+\dfrac{\partial^2 w}{\partial y^2}\dfrac{\partial \psi_x}{\partial x}+\dfrac{\partial^2 w}{\partial x\partial y}\dfrac{\partial \psi_x}{\partial y} \end{bmatrix} \tag{5.51}$$

Substituting from equations (5.32) and (5.44) into (5.28), then

$$\underline{\varepsilon}=\underline{\varepsilon}_o+\underline{\varepsilon}_{mm}+\underline{\varepsilon}_{ww}-z\left(\hat{\underline{\varepsilon}}_b+\hat{\underline{\varepsilon}}_{m\theta}\right)+f_{\psi}\left(\hat{\underline{\varepsilon}}_{\psi}+\hat{\underline{\varepsilon}}_{m\psi}\right)-zf_{\psi}\hat{\underline{\varepsilon}}_{\theta\psi}+z^2\hat{\varepsilon}_{\theta\theta}+f_{\psi}{}^2\hat{\underline{\varepsilon}}_{\psi\psi} \tag{5.52}$$

5.4.2 Matrix Representation of $\underline{\varepsilon}_L$ Parts

This can be achieved in terms of the following rotation vectors:

$$\underline{\theta}_m=\left\{\frac{\partial u^{\circ}}{\partial x}\quad\frac{\partial v^{\circ}}{\partial x}\quad\frac{\partial u^{\circ}}{\partial y}\quad\frac{\partial v^{\circ}}{\partial y}\right\} \tag{5.53}$$

$$\underline{\theta}_w=\left\{\frac{\partial w}{\partial x}\quad\frac{\partial w}{\partial y}\right\} \tag{5.54}$$

$$\underline{\theta}_\theta=\left\{\frac{\partial^2 w}{\partial x^2}\quad\frac{\partial^2 w}{\partial x\partial y}\quad\frac{\partial^2 w}{\partial y\partial x}\quad\frac{\partial^2 w}{\partial y^2}\right\} \tag{5.55}$$

$$\underline{\theta}_\psi=\left\{\left(-\frac{\partial \psi_y}{\partial x}\right)\quad\frac{\partial \psi_x}{\partial x}\quad\left(-\frac{\partial \psi_y}{\partial y}\right)\quad\frac{\partial \psi_x}{\partial y}\right\} \tag{5.56}$$

Hence, by matrix manipulation, it can be proved that:

i) $$\underline{\varepsilon}_m=\frac{1}{2}\underline{A}_m\underline{\theta}_m,\quad d\underline{\varepsilon}_m=\underline{A}_m d\underline{\theta}_m \tag{5.57}$$

where

$$A_m = \begin{bmatrix} \frac{\partial u^\circ}{\partial x} & \frac{\partial v^\circ}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial u^\circ}{\partial y} & \frac{\partial v^\circ}{\partial y} \\ \frac{\partial u^\circ}{\partial y} & \frac{\partial v^\circ}{\partial y} & \frac{\partial u^\circ}{\partial x} & \frac{\partial v^\circ}{\partial x} \end{bmatrix} \tag{5.58}$$

ii) $$\underline{\varepsilon}_w = \frac{1}{2} A_w \underline{\theta}_w, \quad d\underline{\varepsilon}_w = A_w d\underline{\theta}_w \tag{5.59a}$$

where

$$\underline{A}_w = \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix} \tag{5.59b}$$

iii) $$\hat{\underline{\varepsilon}}_{\theta\theta} = \frac{1}{2} A_\theta \underline{\theta}_\theta, \quad d\hat{\underline{\varepsilon}}_{\theta\theta} = A_\theta d\underline{\theta}_\theta \tag{5.60}$$

where

$$A_\theta = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 w}{\partial y \partial x} & \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial y \partial x} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{5.61}$$

iv) $$\hat{\underline{\varepsilon}}_{\psi\psi} = \frac{1}{2} A_\psi \underline{\theta}_\psi, \quad d\hat{\underline{\varepsilon}}_{\psi\psi} = \underline{A}_\psi d\underline{\theta}_\psi \tag{5.62}$$

where

$$A_\psi = \begin{bmatrix} \left(-\frac{\partial \psi_y}{\partial x}\right) & \frac{\partial \psi_x}{\partial x} & 0 & 0 \\ 0 & 0 & \left(-\frac{\partial \psi_y}{\partial y}\right) & \frac{\partial \psi_x}{\partial y} \\ \left(-\frac{\partial \psi_y}{\partial y}\right) & \frac{\partial \psi_x}{\partial y} & \left(-\frac{\partial \psi_y}{\partial x}\right) & \frac{\partial \psi_x}{\partial x} \end{bmatrix} \tag{5.63}$$

v) $$\hat{\underline{\varepsilon}}_{m\theta} = \underline{A}_m \underline{\theta}_\theta = \underline{A}_\theta \underline{\theta}_m, \qquad d\hat{\underline{\varepsilon}}_{m\theta} = \underline{A}_\theta d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\theta \tag{5.64}$$

vi) $$\hat{\underline{\varepsilon}}_{m\psi} = \underline{A}_m \underline{\theta}_\psi = \underline{A}_\psi \underline{\theta}_m, \qquad d\hat{\underline{\varepsilon}}_{m\psi} = \underline{A}_\psi d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\psi \tag{5.65}$$

vii) $$\hat{\underline{\varepsilon}}_{\theta\psi} = \underline{A}_\theta \underline{\theta}_\psi = \underline{A}_\psi \underline{\theta}_\theta, \qquad d\hat{\underline{\varepsilon}}_{\theta\psi} = \underline{A}_\psi d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_\psi \tag{5.66}$$

Hence,

$$d\underline{\varepsilon} = d\underline{\varepsilon}_o + \underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w - z\left[d\hat{\underline{\varepsilon}}_b + \underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right] + f_\psi\left[d\hat{\varepsilon}_\psi + \underline{A}_\psi d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\psi\right] - zf_\psi\left[\underline{A}_\psi d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_\psi\right] + z^2 \underline{A}_\theta d\underline{\theta}_\theta + f_\psi{}^2 \underline{A}_\psi d\underline{\theta}_\psi \tag{5.67}$$

For strain calculation:

$$\underline{\varepsilon}(x,y,z) = \underline{\varepsilon}_o + \frac{1}{2}\left(\underline{A}_m \underline{\theta}_m + \underline{A}_w \underline{\theta}_w\right) - z\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}\left(\underline{A}_m \underline{\theta}_\theta + \underline{A}_\theta \underline{\theta}_m\right)\right] + f_\psi\left[\hat{\varepsilon}_\psi + \frac{1}{2}\left(\underline{A}_\psi \underline{\theta}_m + \underline{A}_m \underline{\theta}_\psi\right)\right] - \frac{zf_\psi}{2}\left[\underline{A}_\psi \underline{\theta}_\theta + \underline{A}_\theta \underline{\theta}_\psi\right] + \frac{1}{2} z^2 \underline{A}_\theta \underline{\theta}_\theta + \frac{1}{2} f_\psi^2 \underline{A}_\psi \underline{\theta}_\psi \tag{5.68}$$

From equation (5.28):

$$d\underline{\varepsilon} = d\underline{\varepsilon}_S + d\underline{\varepsilon}_L \tag{5.69}$$

where

$$d\underline{\varepsilon}_S = d\underline{\varepsilon}_o - z d\hat{\underline{\varepsilon}}_b + f_\psi(z) d\hat{\varepsilon}_\psi \tag{5.70a}$$

$$d\underline{\varepsilon}_L = \underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w - z\left(\underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right) + f_\psi(z)\left(\underline{A}_\psi d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\psi\right) - zf_\psi(z)\left(\underline{A}_\psi d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_\psi\right) + z^2 \underline{A}_\theta d\underline{\theta}_\theta + f_\psi^2(z) \underline{A}_\psi d\underline{\theta}_\psi \tag{5.70b}$$

Also, from equation (5.14):

$$d\underline{\gamma} = f_\gamma(z) d\hat{\underline{\gamma}} \tag{5.70c}$$

5.5 Stress Components

The stress components at any point (x, y, z) in the l^{th} layer of the composite plate can be represented by two vectors for x-y components and transverse shear stress components, defined as follows:

$$\underline{\sigma}^{(l)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{5.71a}$$

and

$$\underline{\tau}^{(l)} = \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{5.71b}$$

Using stress-strain relations for the l^{th} layer, then:

$$\underline{\sigma}^{(l)} = \underline{D}^{(l)} \underline{\varepsilon} \tag{5.72a}$$

$$\underline{\tau}^{(l)} = \underline{\mu}^{(l)} \underline{\gamma} \tag{5.72b}$$

where $\underline{D}^{(l)}$, $\underline{\mu}^{(l)}$ are stress-strain matrices defined with respect to plate axes for the l^{th} layer (Razzaq, 2003). Notice also that by substituting from equation (5.14) into (5.72b):

$$\underline{\tau}^{(l)} = f_\gamma(z) \underline{\mu}^{(l)} \hat{\underline{\gamma}} \tag{5.73}$$

where $f_\gamma(z)$, $\hat{\underline{\gamma}}$ are as defined by equation (5.15) and (5.16), respectively.

5.6 Strain Energy Variation

The variation of strain energy density at any point inside a composite plate is defined as follows:

$$\delta \bar{U} = \delta\varepsilon_x \sigma_x + \delta\varepsilon_y \sigma_y + \delta\varepsilon_z \sigma_z + \delta\gamma_{xy} \tau_{xy} + \delta\gamma_{yz} \tau_{yz} + \delta\gamma_{zx} \tau_{zx} \equiv \delta \underline{\varepsilon}^t \, \underline{\sigma}^{(l)} + \delta \underline{\gamma}^t \, \underline{\tau}^{(l)} \tag{5.74}$$

To facilitate the lengthy derivations expected we shall divide the variation of strain energy density for the composite plate (at any point inside the l^{th} layer) into 3 parts, and we can add them together in the final equations, i.e.

$$\delta\overline{U}^{(l)} = \delta\overline{U}_S^{(l)} + \delta\overline{U}_L^{(l)} + \delta\overline{U}_\gamma^{(l)} \tag{5.75}$$

where

$$\delta\overline{U}_S^{(l)} = \delta\underline{\varepsilon}_S^t\ \underline{\sigma}^{(l)} \tag{5.76}$$

$$\delta\overline{U}_L^{(l)} = \delta\underline{\varepsilon}_L^t\ \underline{\sigma}^{(l)} \tag{5.77}$$

$$\delta\overline{U}_\gamma^{(l)} = \delta\underline{\gamma}^t\ \underline{\tau}^{(l)} \tag{5.78}$$

and

$\delta\overline{U}_S^{(l)}$ represents the variation of strain energy density due to a variation in the infinitesimal *x-y* strains,

$\delta\overline{U}_L^{(l)}$ represents the variation of strain energy density due to a variation in the additional non-linear strain,

$\delta\overline{U}_\gamma^{(l)}$ represents the variation of strain energy density due to a variation in the transverse shear strains.

We shall examine every term, and integrate it with respect to z over the composite plate layers.

5.6.1 The Term $\delta\overline{U}_S^{(l)}$

Substituting from equation (5.70a) into (5.76) then:

$$\delta\overline{U}_S^{(l)} = \left[\delta\underline{\varepsilon}_o^t - z\delta\underline{\hat{\varepsilon}}_b^t + f_\psi(z)\delta\hat{\varepsilon}_\psi^t\right]\underline{\sigma}^{(l)} \tag{5.79}$$

Because of different terms involved, we shall define the following integrated stresses:

$$\underline{\sigma}_{(n)} = (-1)^n \int_{-h/2}^{h/2} z^n\ \underline{\sigma}^{(l)}\, dz \tag{5.80}$$

$$\underline{\sigma}_{\psi n} = (-1)^n \int_{-h/2}^{h/2} z^n f_\psi(z)\underline{\sigma}^{(l)}\, dz \tag{5.81}$$

$$\underline{\sigma}_{\psi\psi n} = (-1)^n \int_{-h/2}^{h/2} z^n f_\psi^2(z)\underline{\sigma}^{(l)}\, dz \tag{5.82}$$

Integrating equation (5.79) with respect to z, then it can be deduced that:

$$\delta U_S' = \int_{-h/2}^{h/2} \delta \overline{U}_S^{(l)}\, dz \equiv \delta \underline{\varepsilon}_o^t\, \underline{\sigma}_{(o)} + \delta \hat{\underline{\varepsilon}}_b^t\, \underline{\sigma}_{(1)} + \delta \hat{\underline{\varepsilon}}_\psi^t\, \underline{\sigma}_{\psi o} \tag{5.83}$$

and the integrated stresses can be defined from strains as follows:

$$\underline{\sigma}_{(o)} = \int_{-h/2}^{h/2} \underline{\sigma}^{(l)}\, dz = \int_{-h/2}^{h/2} \underline{D}^{(l)}\, \underline{\varepsilon}\, dz \tag{5.84}$$

Substituting from equation (5.68) into (5.84) and integrating with respect to z, then:

$$\begin{aligned}
\underline{\sigma}_o(x,y) = &\left[\int_{-h/2}^{h/2} \underline{D}^{(l)}\, dz\right]\left[\underline{\varepsilon}_o + \frac{1}{2}(\underline{A}_m\, \underline{\theta}_m + \underline{A}_w\, \underline{\theta}_w)\right] \\
&+\left[-\int_{-h/2}^{h/2} z\, \underline{D}^{(l)}\, dz\right]\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}(\underline{A}_m\, \underline{\theta}_\theta + \underline{A}_\theta\, \underline{\theta}_m)\right] \\
&+\left[\int_{-h/2}^{h/2} f_\psi(z)\, \underline{D}^{(l)}\, dz\right]\left[\hat{\varepsilon}_\psi + \frac{1}{2}(\underline{A}_\psi\, \underline{\theta}_m + \underline{A}_m\, \underline{\theta}_\psi)\right] \\
&+\left[-\int_{-h/2}^{h/2} z f_\psi(z)\, \underline{D}^{(l)}\, dz\right]\left[\frac{1}{2}(\underline{A}_\psi\, \underline{\theta}_\theta + \underline{A}_\theta\, \underline{\theta}_\psi)\right] \\
&+\left[\int_{-h/2}^{h/2} z^2\, \underline{D}^{(l)}\, dz\right]\left[\frac{1}{2}\underline{A}_\theta\, \underline{\theta}_\theta\right] + \left[\int_{-h/2}^{h/2} f_\psi^2\, \underline{D}^{(l)}\, dz\right]\left[\frac{1}{2}\underline{A}_\psi\, \underline{\theta}_\psi\right]
\end{aligned} \tag{5.85}$$

The following notation will be used for the integrated $\underline{D}$ matrices:

$$\underline{D}_{(n)} = \int_{-h/2}^{h/2} z^n\, \underline{D}^{(l)}\, dz \tag{5.86}$$

$$\underline{D}_{\psi m} = \int_{-h/2}^{h/2} z^n f_\psi(z)\, \underline{D}^{(l)}\, dz \tag{5.87}$$

$$\underline{D}_{\psi\psi} = \int_{-h/2}^{h/2} f_\psi^2(z)\, \underline{D}^{(l)}\, dz \equiv \underline{D}_{\psi\psi o} \tag{5.88}$$

Hence,

$$\begin{aligned}
\underline{\sigma}_o(x,y) = &\underline{D}_{(o)}\left[\underline{\varepsilon}_o + \frac{1}{2}(\underline{A}_m\, \underline{\theta}_m + \underline{A}_w\, \underline{\theta}_w)\right] \\
&-\underline{D}_{(1)}\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}(\underline{A}_m\, \underline{\theta}_\theta + \underline{A}_\theta\, \underline{\theta}_m)\right] + \underline{D}_{\psi o}\left[\hat{\varepsilon}_\psi + \frac{1}{2}(\underline{A}_\psi\, \underline{\theta}_m + \underline{A}_m\, \underline{\theta}_\psi)\right] \\
&-\frac{1}{2}\underline{D}_{\psi 1}(\underline{A}_\psi\, \underline{\theta}_\theta + \underline{A}_\theta\, \underline{\theta}_\psi) + \frac{1}{2}\underline{D}_{(2)}\, \underline{A}_\theta\, \underline{\theta}_\theta + \frac{1}{2}\underline{D}_{\psi\psi}\, \underline{A}_\psi\, \underline{\theta}_\psi
\end{aligned} \tag{5.89}$$

Similarly,

$$\underline{\sigma}_{(1)}(x,y) = -\int_{-h/2}^{h/2} z\,\underline{\sigma}^{(l)}\,dz = -\int_{-h/2}^{h/2} z\,\underline{D}^{(l)}\,\underline{\varepsilon}\,dz \tag{5.90}$$

Substituting from equation (5.68) into (5.90), and with the following additional definitions:

$$\underline{D}_{\psi\psi n} = \int_{-h/2}^{h/2} z^n f_\psi^2(z)\,\underline{D}^{(l)}\,dz \tag{5.91}$$

then, it can be deduced that:

$$\underline{\sigma}_{(1)} = -\underline{D}_{(1)}\left[\underline{\varepsilon}_o + \frac{1}{2}(\underline{A}_m\,\underline{\theta}_m + \underline{A}_w\,\underline{\theta}_w)\right] + \underline{D}_{(2)}\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}(\underline{A}_m\,\underline{\theta}_\theta + \underline{A}_\theta\,\underline{\theta}_m)\right]$$

$$-\underline{D}_{\psi 1}\left[\hat{\underline{\varepsilon}}_\psi + \frac{1}{2}(\underline{A}_\psi\,\underline{\theta}_m + \underline{A}_m\,\underline{\theta}_\psi)\right] + \frac{1}{2}\underline{D}_{\psi 2}(\underline{A}_\psi\,\underline{\theta}_\theta + \underline{A}_\theta\,\underline{\theta}_\psi)$$

$$-\frac{1}{2}\underline{D}_{(3)}\,\underline{A}_\theta\,\underline{\theta}_\theta - \frac{1}{2}\underline{D}_{\psi\psi 1}\,\underline{A}_\psi\,\underline{\theta}_\psi \tag{5.92}$$

Similarly,

$$\underline{\sigma}_{\psi o} = \underline{D}_{\psi o}\left[\underline{\varepsilon}_o + \frac{1}{2}(\underline{A}_m\,\underline{\theta}_m + \underline{A}_w\,\underline{\theta}_w)\right] - \underline{D}_{\psi 1}\left[\hat{\underline{\varepsilon}}_b + \frac{1}{2}(\underline{A}_m\,\underline{\theta}_\theta + \underline{A}_\theta\,\underline{\theta}_m)\right]$$

$$+\underline{D}_{\psi\psi o}\left[\hat{\underline{\varepsilon}}_\psi + \frac{1}{2}(\underline{A}_\psi\,\underline{\theta}_m + \underline{A}_m\,\underline{\theta}_\psi)\right] - \frac{1}{2}\underline{D}_{\psi\psi 1}(\underline{A}_\psi\,\underline{\theta}_\theta + \underline{A}_\theta\,\underline{\theta}_\psi)$$

$$+\frac{1}{2}\underline{D}_{\psi 2}\underline{A}_\theta\underline{\theta}_\theta + \frac{1}{2}\underline{D}_{\psi\psi\psi}\underline{A}_\psi\underline{\theta}_\psi \tag{5.93}$$

where

$$\underline{D}_{\psi\psi\psi} = \int_{-h/2}^{h/2} f_\psi^3(z)\,\underline{D}^{(l)}\,dz \tag{5.94}$$

5.6.2 The Term $\delta \overline{U}_\gamma^{(l)}$

Substituting from equation (5.70c) into (5.78), then:

$$\delta \overline{U}_\gamma^{(l)} = f_\gamma(z)\,\delta\hat{\underline{\gamma}}^t\,\underline{\tau}^{(l)} \tag{5.95}$$

Hence,

$$\delta U'_\gamma = \int_{-h/2}^{h/2} \delta \overline{U}_\gamma^{(l)}\,dz = \delta\hat{\underline{\gamma}}^t \int_{-h/2}^{h/2} f_\gamma(z)\,\underline{\tau}^{(l)}\,dz \tag{5.96}$$

Defining

$$\underline{\tilde{\tau}} = \int_{-h/2}^{h/2} f_\gamma(z)\,\underline{\tau}^{(l)}\,dz \tag{5.97}$$

then

$$\delta U_\gamma' = \delta \underline{\hat{\gamma}}^t\ \underline{\tilde{\tau}}(x,y) \tag{5.98}$$

Substituting from equation (5.73) into (5.97), then

$$\underline{\tilde{\tau}} = \int_{-h/2}^{h/2} f_\gamma^2(z)\,\underline{\mu}^{(l)}\ \underline{\hat{\gamma}}\,dz \tag{5.99}$$

Defining

$$\underline{\mu}_{\gamma\gamma} = \int_{-h/2}^{h/2} f_\gamma^2(z)\,\underline{\mu}^{(l)}\,dz \tag{5.100}$$

Then $\underline{\tilde{\tau}} = \underline{\mu}_{\gamma\gamma}\ \underline{\hat{\gamma}}$ (5.101)

Notice also that for transversely isotropic composites, $(\mu_{xz} = \mu_{yz} = \mu)$, then:

$$\underline{\mu}^{(l)} = \mu\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.102}$$

which is the same for every layer, and

$$\underline{\mu}_{\gamma\gamma} = \left(\int_{-h/2}^{h/2} f_\gamma^2(z)\,dz\right)\underline{\mu}^{(l)} = \frac{6}{5}h\mu\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.103}$$

5.6.3 The Term $\delta\overline{U}_L^{(l)}$

Substituting from equation (5.70b) into (5.77), then:

$$\delta\overline{U}_L = \left[\left(\underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w\right) - z\left(\underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right) + f_\psi(z)\left(\underline{A}_\psi d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\psi\right)\right.$$
$$\left. - z f_\psi(z)\left(\underline{A}_\psi d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_\psi\right) + z^2\underline{A}_\theta d\underline{\theta}_\theta + f_\psi^2(z)\underline{A}_\psi d\underline{\theta}_\psi\right]\underline{\sigma}^{(l)} \tag{5.104}$$

Using the definitions of integrated $\underline{\sigma}$ given by equations (5.80)-(5.82), then equation (5.104) can be integrated with respect to z as follows:

$$\delta U_L' = \left(\underline{A}_m d\underline{\theta}_m + \underline{A}_w d\underline{\theta}_w\right)^t \underline{\sigma}_{(o)} + \left(\underline{A}_m d\underline{\theta}_\theta + \underline{A}_\theta d\underline{\theta}_m\right)^t \underline{\sigma}_{(1)} + \left(\underline{A}_\psi d\underline{\theta}_m + \underline{A}_m d\underline{\theta}_\psi\right)^t \underline{\sigma}_{\psi o}$$

$$+\left(\underline{A}_{\psi}\, d\,\underline{\theta}_{\theta}+\underline{A}_{\theta}\, d\,\underline{\theta}_{\psi}\right)^{t}\underline{\sigma}_{\psi 1}+\left(\underline{A}_{\theta}\, d\,\underline{\theta}_{\theta}\right)^{t}\underline{\sigma}_{(2)}+\left(\underline{A}_{\psi}\, d\,\underline{\theta}_{\psi}\right)^{t}\underline{\sigma}_{\psi\psi} \tag{5.105}$$

$\underline{\sigma}_{(o)}$, $\underline{\sigma}_{(1)}$ and $\underline{\sigma}_{\psi o}$ are as defined by equations (5.89), (5.92), and (5.93) respectively. The other terms are defined as follows:

$$\underline{\sigma}_{(2)}=\int_{-h/2}^{h/2} z^{2}\underline{D}^{(l)}\underline{\varepsilon}\, dz \tag{5.106}$$

Substituting from equation (5.68) and integrating with respect to z, then it can be deduced that:

$$\underline{\sigma}_{(2)}=\underline{D}_{(2)}\left[\underline{\varepsilon}_{o}+\frac{1}{2}\left(\underline{A}_{m}\underline{\theta}_{m}+\underline{A}_{w}\underline{\theta}_{w}\right)\right]-\underline{D}_{(3)}\left[\hat{\underline{\varepsilon}}_{b}+\frac{1}{2}\left(\underline{A}_{m}\underline{\theta}_{\theta}+\underline{A}_{\theta}\underline{\theta}_{m}\right)\right]$$

$$+\underline{D}_{\psi 2}\left[\hat{\underline{\varepsilon}}_{\psi}+\frac{1}{2}\left(\underline{A}_{\psi}\underline{\theta}_{m}+\underline{A}_{m}\underline{\theta}_{\psi}\right)\right]-\frac{1}{2}\underline{D}_{\psi 3}\left(\underline{A}_{\psi}\underline{\theta}_{\theta}+\underline{A}_{\theta}\underline{\theta}_{\psi}\right)+\frac{1}{2}\underline{D}_{(4)}\underline{A}_{\theta}\underline{\theta}_{\theta}+\frac{1}{2}\underline{D}_{\psi\psi 2}\underline{A}_{\psi}\underline{\theta}_{\psi} \tag{5.107}$$

Similarly,

$$\underline{\sigma}_{\psi 1}=-\int_{-h/2}^{h/2} z f_{\psi}(z)\underline{\sigma}^{(l)}\, dz=-\int_{-h/2}^{h/2} z f_{\psi}(z)\underline{D}^{(l)}\underline{\varepsilon}\, dz \tag{5.108}$$

Substituting from equation (5.68) and integrating with respect to z, then:

$$\underline{\sigma}_{\psi 1}=-\underline{D}_{\psi 1}\left[\underline{\varepsilon}_{o}+\frac{1}{2}\left(\underline{A}_{m}\underline{\theta}_{m}+\underline{A}_{w}\underline{\theta}_{w}\right)\right]+\underline{D}_{\psi 2}\left[\hat{\underline{\varepsilon}}_{b}+\frac{1}{2}\left(\underline{A}_{m}\underline{\theta}_{\theta}+\underline{A}_{\theta}\underline{\theta}_{m}\right)\right]$$

$$-\underline{D}_{\psi\psi 1}\left[\hat{\underline{\varepsilon}}_{\psi}+\frac{1}{2}\left(\underline{A}_{\psi}\underline{\theta}_{m}+\underline{A}_{m}\underline{\theta}_{\psi}\right)\right]+\frac{1}{2}\underline{D}_{\psi\psi 2}\left(\underline{A}_{\psi}\underline{\theta}_{\theta}+\underline{A}_{\theta}\underline{\theta}_{\psi}\right)-\frac{1}{2}\underline{D}_{\psi 3}\underline{A}_{\theta}\underline{\theta}_{\theta}+\frac{1}{2}\underline{D}_{\psi\psi\psi 1}\underline{A}_{\psi}\underline{\theta}_{\psi} \tag{5.109}$$

where

$$\underline{D}_{\psi\psi n}=\int_{-h/2}^{h/2} z^{n} f_{\psi}^{2}(z)\underline{D}^{(l)}\, dz \tag{5.110a}$$

with $\underline{D}_{\psi\psi}\equiv\underline{D}_{\psi\psi o}$ (5.110b)

and $$\underline{D}_{\psi\psi\psi n}=\int_{-h/2}^{h/2} z^{n} f_{\psi}^{3}(z)\underline{D}^{(l)}\, dz \tag{5.110c}$$

with $\underline{D}_{\psi\psi\psi} \equiv \underline{D}_{\psi\psi\psi o}$ (5.110d)

Similarly,

$$\underline{\sigma}_{\psi\psi} = \int_{-h/2}^{h/2} f_{\psi}^{2}(z)\underline{\sigma}^{(l)}\,dz = \int_{-h/2}^{h/2} f_{\psi}^{2}(z)\underline{D}^{(l)}\,\underline{\varepsilon}\,dz \tag{5.111}$$

Substituting from equation (5.68) into (5.111) and integrating with respect to z, then it can be proved that:

$$\underline{\sigma}_{\psi\psi} = -\underline{D}_{\psi\psi o}\left[\underline{\varepsilon}_{o} + \frac{1}{2}(\underline{A}_{m}\,\underline{\theta}_{m} + \underline{A}_{w}\,\underline{\theta}_{w})\right] - \underline{D}_{\psi\psi 1}\left[\hat{\underline{\varepsilon}}_{b} + \frac{1}{2}(\underline{A}_{m}\,\underline{\theta}_{\theta} + \underline{A}_{\theta}\,\underline{\theta}_{m})\right]$$
$$+\underline{D}_{\psi\psi\psi o}\left[\hat{\underline{\varepsilon}}_{\psi} + \frac{1}{2}(\underline{A}_{\psi}\,\underline{\theta}_{m} + \underline{A}_{m}\,\underline{\theta}_{\psi})\right] - \frac{1}{2}\underline{D}_{\psi\psi\psi 1}(\underline{A}_{\psi}\,\underline{\theta}_{\theta} + \underline{A}_{\theta}\,\underline{\theta}_{\psi})$$
$$+\frac{1}{2}\underline{D}_{\psi\psi 2}\,\underline{A}_{\theta}\,\underline{\theta}_{\theta} + \frac{1}{2}\underline{D}_{\psi\psi\psi\psi}\,\underline{A}_{\psi}\,\underline{\theta}_{\psi} \tag{5.112}$$

where

$$\underline{D}_{\psi\psi\psi\psi} = \int_{-h/2}^{h/2} f_{\psi}^{4}(z)\underline{D}^{(l)}\,dz \tag{5.113}$$

5.7 Interpolated Equations of displacements

Three different interpolation theorems (one dimensional) will be employed in the *Reissner*-type element, as will be discussed in the next sections.

5.7.1 Lagrangian Interpolation Theorem

Given $f_1, f_2, \ldots f_n$ at $\xi_1, \xi_2, \ldots \xi_n$ then according to Lagrange's interpolation theorem,

$$f(\xi) = \sum_{i=1}^{n} L_i^n(\xi) f_i \tag{5.114}$$

where

$$L_i^n(\xi) = \prod_{\substack{r=1 \\ r\neq i}}^{n} \frac{\xi - \xi_r}{\xi_i - \xi_r} \tag{5.115}$$

If $\xi_2 - \xi_1 = \xi_3 - \xi_2 = \ldots = \xi_n - \xi_{n-1}$, with $\xi_1 = 0$, $\xi_n = 1$, then it can be shown that:

$$\xi_r = \frac{r-1}{n-1} \tag{5.116}$$

and

$$L_i^n(\xi) = \prod_{\substack{r=1 \\ r\neq i}}^{n} \frac{(n-1)\xi-(r-1)}{i-r} \tag{5.117}$$

5.7.2 Hermitian Interpolation Theorem

$$\text{If} \quad f_{i,\xi} \equiv \left.\frac{\partial f}{\partial \xi}\right|_{at\xi=\xi_i} \tag{5.118}$$

and $f_1, f_{1,\xi}, f_2, f_{2,\xi}, \ldots f_n, f_{n,\xi}$ are given at $\xi_1, \xi_2, \ldots \xi_n$, then according to Hermite's interpolation theorem:

$$f(\xi) = \sum_{i=1}^{n} \left[g_i(\xi) f_i + h_i(\xi) f_{i,\xi} \right] \tag{5.119}$$

where

$$g_i(\xi) = [1-2\alpha_i(\xi-\xi_i)]\left[L_i^n(\xi)\right]^2 \tag{5.120}$$

$$h_i(\xi) = (\xi-\xi_i)\left[L_i^n(\xi)\right]^2 \tag{5.121}$$

$$\alpha_i = \sum_{\substack{r=1 \\ r\neq i}}^{n} \frac{1}{\xi_i - \xi_r} \tag{5.122}$$

5.7.3 Spline-Type Interpolation Theorem

This is based on given

$f_1, f_{1,\xi}$, at $\xi = \xi_1$

$f_n, f_{n,\xi}$, at $\xi = \xi_n$

and f_2, f_3, f_{n-1} at $\xi_2, \xi_3, \ldots \xi_{n-1}$

It can be written in a formula similar to that of the *Hermitian* interpolation as follows:

$$f(\xi) = \sum_{i=1}^{n} \left[g_i^s(\xi) f_i + h_i^s(\xi) f_{i,\xi} \right] \tag{5.123}$$

where

$$g_1^s(\xi)=[1+(1-\alpha_1)\xi](1-\xi)L_1^n(\xi) \tag{5.124a}$$

$$h_1^s(\xi)=\xi(1-\xi)L_1^n(\xi) \tag{5.124b}$$

$$g_n^s(\xi)=[1+(1+\alpha_n)(1-\xi)]\xi L_n^n(\xi) \tag{5.124c}$$

$$h_n^s(\xi)=-\xi(1-\xi)L_n^n(\xi) \tag{5.124d}$$

and for $1<i<n$

$$g_i^s(\xi)=\frac{\xi(1-\xi)}{\xi_i(1-\xi_i)}L_i^n(\xi) \tag{5.124e}$$

$$h_i^s(\xi)=0 \tag{5.124f}$$

α_1 ,α_n as obtained from equation (5.122).

5.7.4 Interpolation in x-Direction

At any given y, the values of displacement parameters (in the mid-plane) at element nodes $(x=x_1, x=x_2, \ldots, x=x_n)$ for an *n*-node element will be used for interpolation.

5.7.4.1 Interpolation of Lateral deflection w

In order to maintain C^1 continuity over the whole plate, only *Hermitian or Spline*-type can be used. The interpolated equation of w, at any y, can generally be written as follows:

$$w(x,y)=\sum_{i=1}^{n}[G_i(\xi)w_i(y)+H_i(\xi)w_{i,x}(y)] \tag{5.125}$$

Notice that

$$w_{i,x}=\left.\frac{\partial w}{\partial x}\right|_{at\ x=x_i}\equiv J_x w_{i,\xi} \tag{5.126}$$

where *Jx* is as given by equation (4.88b).

Hence for *Hermitian* interpolation:

$$G_i(\xi)=g_i(\xi),\ H_i(\xi)=J_x h_i(\xi) \tag{5.127}$$

and for *Spline*-Type interpolation:

$$G_i(\xi) = g_i^s(\xi),\ H_i(\xi) = J_x h_i^s(\xi) \tag{5.128}$$

5.7.4.2 Interpolation of Other Components

All other displacement components are interpolated in the x-direction by means of *Lagrangian* interpolation, as there is no need for more than C° continuity required. Hence, at any y:

$$u^\circ(x,y) = \sum_{i=1}^{n} N_i(\xi) u_i(y) \tag{5.129}$$

$$v^\circ(x,y) = \sum_{i=1}^{n} N_i(\xi) v_i(y) \tag{5.130}$$

$$\psi_x(x,y) = \sum_{i=1}^{n} N_i(\xi) \left(\psi_x(y)\right)_i \tag{5.131}$$

$$\psi_y(x,y) = \sum_{i=1}^{n} N_i(\xi) \left(\psi_y(y)\right)_i \tag{5.132}$$

where

$$\begin{aligned} u_i(y) &= u^\circ(x_i, y) \\ v_i(y) &= v^\circ(x_i, y) \\ \left(\psi_x(y)\right)_i &= \psi_x(x_i, y) \\ \left(\psi_y(y)\right)_i &= \psi_y(x_i, y) \end{aligned} \tag{5.133}$$

and $$N_i(\xi) = L_i^n(\xi) \tag{5.134}$$

Notice also, the following notations will be used for *Hermitian* and *Spline* interpolation functions

$$\begin{aligned} G_i(n,\xi) &\equiv G_i(\xi) \\ H_i(n,\xi) &\equiv H_i(\xi) \end{aligned} \tag{5.135}$$

only wherever there is any doubt about the number of points used in the interpolations.

5.7.5 Interpolation in y-Direction

5.7.5.1 Lateral Deflection w

Based on several tests, it was found that the best accuracy is obtained when the interpolation of w along the y direction is based upon the same interpolation theorem used in x-direction.

Hence, for m' y-points, we can write these generalized interpolation equations along y:

$$w_i(y) = \sum_{s=1}^{m'} \left[G_s(m',\eta) w_i(y_s) + H_s(m',\eta) \frac{\partial w_i}{\partial y}\bigg|_{\text{at } y_s} \right] \tag{5.136}$$

and

$$w_{i,x}(y) = \sum_{s=1}^{m'} \left[G_s(m',\eta) w_{i,x}(y_s) + H_s(m',\eta) \frac{\partial w_{i,x}}{\partial y}\bigg|_{\text{at } y_s} \right] \tag{5.137}$$

For practical programming purposes, we shall represent G_s, H_s by one generalized function as follows:

$$w_i(y) = \sum_{r=1}^{m} S_r(\eta)\, w_i^r \tag{5.138}$$

$$w_{i,x}(y) = \sum_{r=1}^{m} S_r(\eta)\, w_{i,x}^r \tag{5.139}$$

with the following definitions and notations:

(a) *Hermitian* Interpolation

$$m = 2m' \tag{5.140}$$

$$s = \text{Integer of} \left(\frac{r+1}{2} \right) \tag{5.141}$$

$$S_r(\eta) \equiv G_s(m',\eta) \quad \text{for } r = 1,3,5 \ldots\ldots \tag{5.142a}$$

$$S_r(\eta) \equiv H_s(m',\eta) \quad \text{for } r = 2,4,6 \ldots\ldots \tag{5.142b}$$

(b) *Spline*-Type interpolation

$$m = m'+2 \tag{5.143}$$

$$s = r-1 \tag{5.144}$$

and definitions similar to equations (5.142a) and (5.142b) are used, but specifically:

$$S_1(\eta) \equiv g_1^s(\eta) = \left[1 + \left(1-\alpha_1'\right)\eta\right](1-\eta) L_1^{m'}(\eta) \tag{5.145a}$$

$$S_2(\eta) \equiv J_y h_1^s(\eta) = J_y \eta (1-\eta) L_1^{m'}(\eta) \tag{5.145b}$$

$$S_{m-1}(\eta) \equiv g^s_{m'}(\eta) = \left[1 + \left(1 + \alpha'_{m'}\right)\right](1-\eta)\eta\, L^{m'}_{m'}(\eta) \tag{5.145c}$$

$$S_m(\eta) \equiv J_y h^s_{m'}(\eta) = -J_y \eta(1-\eta) L^{m'}_{m'}(\eta) \tag{5.145d}$$

and

$$S_r(\eta) \equiv g^s_{r'}(\eta) = \frac{\eta(1-\eta)}{\eta_{r'}(1-\eta_{r'})} L^{m'}_{r'}(\eta) \tag{5.145e}$$

for $r = 3,4, \ldots\ldots, m-2$

where

$$r' = r - 1$$

$$\alpha'_j = \sum_{\substack{k=1, \\ k \neq j}}^{m'} \frac{m'-1}{j-k}$$

(c) Boundary Conditions

Comparing equations (5.138) with (5.136) and (5.139) with (5.137), then the followings represent values at edges $\eta = 0, \eta = 1$:

w^1_i = The value of w, at $x = x_i$, and $\eta = 0$,

$w^1_{i,x}$ = The value of $\frac{dw}{dx}$, at $x = x_i$, and $\eta = 0$,

w^2_i = The value of $\frac{dw}{dy}$, at $x = x_i$, and $\eta = 0$,

$w^2_{i,x}$ = The value of $\frac{d^2w}{dxdy}$, at $x = x_i$, and $\eta = 0$,

w^{m-1}_i = The value of w, at $x = x_i$, and $\eta = 1$,

$w^{m-1}_{i,x}$ = The value of $\frac{dw}{dx}$, at $x = x_i$, and $\eta = 1$,

w^m_i = The value of $\frac{dw}{dy}$, at $x = x_i$, and $\eta = 1$,

$w^m_{i,x}$ = The value of $\frac{d^2w}{dxdy}$, at $x = x_i$, and $\eta = 1$,

Notice that for *Hermitian* interpolation:

w^{2r-1}_i = The value of w, at $x = x_i$, and $\eta = \eta_r$,

$w_{i,x}^{2r-1}$ = The value of $\frac{dw}{dx}$, at $x = x_i$, and $\eta = \eta_r$,

w_i^{2r} = The value of $\frac{dw}{dy}$, at $x = x_i$, and $\eta = \eta_r$,

$w_{i,x}^{2r}$ = The value of $\frac{d^2w}{dxdy}$, at $x = x_i$, and $\eta = \eta_r$,

For *Spline-type* interpolation, $1 < r < m-1$

w_i^{r+1} = The value of w, at $x = x_i$, and $\eta = \eta_r$,

$w_{i,x}^{r+1}$ = The value of $\frac{dw}{dx}$, at $x = x_i$, and $\eta = \eta_r$ (not required).

5.7.5.2 Interpolation of other components

All other components have the same interpolation, and for simplicity the interpolated equations along y are written as follows:

$$u_i(y) = \sum_{r=1}^{m} \mu_r(\eta) u_i^r \tag{5.146}$$

$$v_i(y) = \sum_{r=1}^{m} \mu_r(\eta) v_i^r \tag{5.147}$$

$$\psi_x(y) = \sum_{r=1}^{m} \mu_r(\eta) \left(\psi_x^r\right)_i \tag{5.148}$$

$$\psi_y(y) = \sum_{r=1}^{m} \mu_r(\eta) \left(\psi_y^r\right)_i \tag{5.149}$$

where, for *Lagrangian* interpolation:

$$u_i^r \equiv u^\circ(x_i, y_r)$$
$$v_i^r \equiv v^\circ(x_i, y_r), \ldots\ldots \text{ etc.} \tag{5.150}$$

(a) For *Lagrangian* interpolation along y direction

$$\mu_r(\eta) = L_r^m(\eta) \tag{5.151}$$

with $\eta_1 = 0, \eta_m = 1$, the values of η's

$$\eta_r = \frac{r-1}{m-1} \tag{5.152}$$

i.e.

$$\mu_r(\eta) = \prod_{\substack{j=1,\\ j\neq r}}^{m} \frac{(m-1)\eta - (j-1)}{r-j} \tag{5.153}$$

and from equation (4.88a)

$$y_r = y_L(\xi) + b(\xi)\eta \tag{5.154}$$

(b) For *Hermitian* interpolation

$$\text{We put:} \quad \mu_r(\eta) \equiv S_r(\eta) \tag{5.155}$$

as defined by equation (5.142).

Notice that with $m = 2m'$, $r' = \text{Integer}\left(\frac{r+1}{2}\right)$,

$$y_r = y_{r+1} \qquad \text{for } r = 1, 3, 5, \ldots\ldots$$

$$\eta_r = \eta_{r+1} = \frac{r'-1}{m'-1} \tag{5.156}$$

(c) *Spline-Type* Interpolation

$$\mu_r(\eta) = S_r(\eta) \tag{5.157}$$

as defined by equation (5.145), $m' = m-2$

$$\eta_1 = \eta_2 = 0, \quad \eta_{m-1} = \eta_m = 1 \tag{5.158a}$$

$$\eta_r = \frac{r-2}{m'-1}, \quad r = 2,3, \ldots\ldots, m-1 \tag{5.158b}$$

This means that:

$$y_1 = y_2 = y_L \tag{5.159a}$$

$$y_m = y_{m-1} = y_L + b \tag{5.159b}$$

$$y_r = y_L + \left(\frac{r-2}{m-3}\right)\eta b, \quad \text{for } r = 2, 3, \ldots\ldots, m-1 \tag{5.159c}$$

The nodal values for *Hermitian* and *Spline-type* will be as defined for lateral deflection, *w*.

5.7.6 Interpolation of Displacement Components in x ,y directions

Substituting from equations (5.138), (5.139), (5.146)- (5.149) into (5.125) and (5.129)- (5.132), the interpolated displacement equations with respect to (x, y) can be expressed as follows:

$$u^{\circ}(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)u_i^r \tag{5.160}$$

$$v^{\circ}(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)v_i^r \tag{5.161}$$

$$\psi_x(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)(\psi_x^r)_i \tag{5.162}$$

$$\psi_y(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)(\psi_y^r)_i \tag{5.163}$$

$$w(x,y)=\sum_{r=1}^{m}\sum_{i=1}^{n}S_r(\eta)\left[G_i(\xi)w_i^r+H_i(\xi)w_{i,x}^r\right] \tag{5.164}$$

5.8 Interpolation of Strain Components

5.8.1 Transverse Shear Strains

Substituting from equations (5.162), (5.163) into (5.16), then:

$$\underline{\hat{\gamma}}(x,y)=\begin{bmatrix}-\psi_y\\ \\ \psi_x\end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix}-\sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)\left(\psi_y^r\right)_i\\ \\ \sum_{i=1}^{n}N_i(\xi)\mu_r(\eta)\left(\psi_x^r\right)_i\end{bmatrix} \tag{5.165}$$

Defining:

$$\left(\underline{\delta}_{\psi}^{r}\right)^{t}=\left[\left(\psi_x^r\right)_1\quad\left(\psi_y^r\right)_1\ .....\ \left(\psi_x^r\right)_n\quad\left(\psi_y^r\right)_n\right] \tag{5.166}$$

then equation (5.165) can be rewritten in the following matrix form:

$$\underline{\hat{\gamma}}(x,y)==\sum_{r=1}^{m}\underline{B}_{\gamma}^{r}(\xi,\eta)\underline{\delta}_{\psi}^{r} \tag{5.167a}$$

where

$$\underline{B}_{\gamma}^{r}(\xi,\eta)=\begin{bmatrix}\cdots & 0 & -N_i(\xi)\mu_r(\eta) & \cdots\\ \\ \cdots & N_i(\xi)\mu_r(\eta) & 0 & \cdots\end{bmatrix} \tag{5.167b}$$

Hence

$$\underline{\gamma}(x,y,z)=\begin{bmatrix}\gamma_{xz}\\ \\ \gamma_{yz}\end{bmatrix}=f_{\gamma}(z)\sum_{r=1}^{m}\underline{B}_{\gamma}^{r}(\xi,\eta)\underline{\delta}_{\psi}^{r} \tag{5.168a}$$

with

$$d\underline{\gamma}(x,y,z)=\sum_{s=1}^{m}f_{\gamma}(z)\underline{B}_{\gamma}^{s}(\xi,\eta)d\underline{\delta}_{\psi}^{s} \tag{5.168b}$$

5.8.2 Small X-Y Strain Components

(i) $\underline{\varepsilon}_o(x,y)$

Substituting from equations (5.160), (5.161) into (5.34), then

$$\underline{\varepsilon}_o(x,y)=\begin{bmatrix}\dfrac{\partial u}{\partial x}^{\circ}\\ \\ \dfrac{\partial v}{\partial y}^{\circ}\\ \\ \dfrac{\partial u}{\partial y}^{\circ}+\dfrac{\partial v}{\partial x}^{\circ}\end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix}\sum_{i=1}^{n}\dfrac{\partial N_i}{\partial x}\mu_r(\eta)u_i^r\\ \\ \sum_{i=1}^{n}N_i(\xi)\dfrac{\partial \mu_r}{\partial y}v_i^r\\ \\ \sum_{i=1}^{n}\left[N_i(\xi)\dfrac{\partial \mu_r}{\partial y}u_i^r+\dfrac{\partial N_i}{\partial x}\mu_r(\eta)v_i^r\right]\end{bmatrix} \tag{5.169}$$

Notice from the definition of ξ,η:

$$\frac{\partial N_i(\xi)}{\partial x}=\frac{1}{J_x}\frac{\partial N_i(\xi)}{\partial \xi}\equiv\frac{N_i'(\xi)}{J_x} \tag{5.170}$$

$$\frac{\partial \mu_r(\eta)}{\partial y}=\frac{1}{J_y}\frac{\partial \mu_r(\eta)}{\partial \eta}\equiv\frac{\mu'_r(\eta)}{J_y} \tag{5.171}$$

$$\text{Defining}:\ \left(\underline{\delta}_o^r\right)^t=\left[u_1^r v_1^r \quad u_2^r v_2^r \ \ldots\ldots\ u_n^r v_n^r\right] \tag{5.172}$$

Then equations (5.169) can be rewritten in the following matrix form:

$$\underline{\varepsilon}_o(x,y)=\sum_{r=1}^{m}\underline{B}_o^r(\xi,\eta)\underline{\delta}_o^r \tag{5.173}$$

where

$$\underline{B}_o^r(\xi,\eta)=\begin{bmatrix} \cdots & \frac{N_i^{'}(\xi)}{J_x}\mu_r(\eta) & 0 & \cdots \\ \cdots & 0 & N_i(\xi)\frac{\mu_r^{'}(\eta)}{J_y} & \cdots \\ \cdots & N_i(\xi)\frac{\mu_r^{'}(\eta)}{J_y} & \frac{N_i^{'}(\xi)}{J_x}\mu_r(\eta) & \cdots \end{bmatrix} \tag{5.174}$$

(ii) $\underline{\hat{\varepsilon}_b(x,y)}$

Substituting from equation (5.164) into (5.35), then:

$$\underline{\hat{\varepsilon}}_b=\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix} \sum_{i=1}^{n} S_r(\eta)\left(\frac{d^2 G_i}{dx^2}w_i^r+\frac{d^2 H_i}{dx^2}w_{i,x}^r\right) \\ \sum_{i=1}^{n}\frac{d^2 S_r}{dy^2}\left(G_i w_i^r+H_i w_{i,x}^r\right) \\ 2\sum_{i=1}^{n}\frac{dS_r}{dy}\left(\frac{dG_i}{dx}w_i^r+\frac{dH_i}{dx}w_{i,x}^r\right) \end{bmatrix} \tag{5.175}$$

Notice that:

$$\frac{d^2 G_i}{dx^2}=\frac{1}{J_x^2}\frac{d^2 G_i(\xi)}{d\xi^2}\equiv\frac{G_i^{''}(\xi)}{J_x^2} \tag{5.176a}$$

$$\frac{d^2 H_i}{dx^2}=\frac{1}{J_x^2}\frac{d^2 H_i(\xi)}{d\xi^2}\equiv\frac{H_i^{''}(\xi)}{J_x^2} \tag{5.176b}$$

Similarly

$$\frac{dG_i}{dx}=\frac{G_i^{'}(\xi)}{J_x} \tag{5.176c}$$

$$\frac{dH_i}{dx}=\frac{H_i^{'}(\xi)}{J_x} \tag{5.176d}$$

and

$$\frac{dS_r}{dy}=\frac{1}{J_y}\frac{dS_r}{d\eta}\equiv\frac{S_r^{'}(\eta)}{J_y} \tag{5.176e}$$

$$\frac{d^2 S_r}{dy^2}=\frac{1}{J_y^2}\frac{d^2 S_r}{d\eta^2}\equiv\frac{S_r^{''}(\eta)}{J_y^2} \tag{5.176f}$$

Defining: $\left(\underline{\delta}_b^r\right)^t = \left[w_1^r w_{1,x}^r \quad w_2^r w_{2,x}^r \ldots\ldots\ldots \quad w_n^r w_{n,x}^r\right]$ (5.177)

Then equation (5.175) can be rewritten in the following matrix form:

$$\underline{\hat{\varepsilon}}_b(x,y) = \sum_{r=1}^{m} \underline{B}_b^r(\xi,\eta)\underline{\delta}_b^r \tag{5.178}$$

where

$$\underline{B}_b^r(\xi,\eta) = \begin{bmatrix} \cdots & \dfrac{G_i''(\xi)}{J_x^{\,2}} S_r(\eta) & \dfrac{H_i''(\xi)}{J_x^{\,2}} S_r(\eta) & \cdots \\ \cdots & G_i(\xi)\dfrac{S_r''(\eta)}{J_y^{\,2}} & H_i(\xi)\dfrac{S_r''(\eta)}{J_y^{\,2}} & \cdots \\ \cdots & 2\dfrac{G_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & 2\dfrac{H_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & \cdots \end{bmatrix} \tag{5.179}$$

(ii) $\underline{\hat{\varepsilon}}_\psi(x,y)$

Similarly, by substituting from equations (5.162), (5.163) into (5.36), it can be proved that:

$$\underline{\hat{\varepsilon}}_\psi(x,y) = \sum_{r=1}^{m} \underline{B}_\psi^r(\xi,\eta)\underline{\delta}_\psi^r \tag{5.180}$$

where

$$\underline{B}_\psi^r(\xi,\eta) = \begin{bmatrix} \cdots & 0 & -\dfrac{N_i'(\xi)}{J_x}\mu_r(\eta) & \cdots \\ \cdots & N_i(\xi)\dfrac{\mu_r'(\eta)}{J_y} & 0 & \cdots \\ \cdots & \dfrac{N_i'(\xi)}{J_x}\mu_r(\eta) & -N_i(\xi)\dfrac{\mu_r'(\eta)}{J_y} & \cdots \end{bmatrix} \tag{5.181}$$

5.8.3 Additional Non-Linear Strain Components

5.8.3.1 Rotation Vectors

(i) $\underline{\theta}_m$

Substituting from equations (5.160), (5.161) into (5.53), then it can be deduced that:

$$\underline{\theta}_m(x,y)=\begin{bmatrix}\frac{\partial u}{\partial x}^{\circ}\\ \frac{\partial v}{\partial x}^{\circ}\\ \frac{\partial u}{\partial y}^{\circ}\\ \frac{\partial v}{\partial y}^{\circ}\end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix}\sum_{i=1}^{n}\frac{\partial N_i}{\partial x}\mu_r u_i^r\\ \sum_{i=1}^{n}\frac{\partial N_i}{\partial x}\mu_r v_i^r\\ \sum_{i=1}^{n}N_i\frac{\partial \mu_r}{\partial y}u_i^r\\ \sum_{i=1}^{n}N_i\frac{\partial \mu_r}{\partial y}v_i^r\end{bmatrix}\equiv\sum_{r=1}^{m}\underline{G}_m^r(\xi,\eta)\,\underline{\delta}_o^r \tag{5.182}$$

where

$$\underline{G}_m^r(\xi,\eta)=\begin{bmatrix}\cdots & \frac{N_i'(\xi)}{J_x}\mu_r(\eta) & 0 & \cdots\\ \cdots & 0 & \frac{N_i'(\xi)}{J_x}\mu_r(\eta) & \cdots\\ \cdots & N_i(\xi)\frac{\mu_r'(\eta)}{J_y} & 0 & \cdots\\ \cdots & 0 & N_i(\xi)\frac{\mu_r'(\eta)}{J_y} & \cdots\end{bmatrix} \tag{5.183}$$

(ii) $\underline{\theta}_w$

Substituting from equation (5.164) into (5.54), then

$$\underline{\theta}_w(x,y)=\begin{bmatrix}\frac{\partial w}{\partial x}\\ \frac{\partial w}{\partial y}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{n}S_r(\eta)\left(\frac{\partial G_i}{\partial x}w_i^r+\frac{\partial H_i}{\partial x}w_{i,x}^r\right)\\ \sum_{i=1}^{n}\frac{dS_r}{dy}\left[G_i w_i^r+H_i w_{i,x}^r\right]\end{bmatrix}$$

which can be written in the following matrix form:

$$\underline{\theta}_w(x,y)=\sum_{r=1}^{m}\underline{G}_w^r(\xi,\eta)\,\underline{\delta}_b^r \tag{5.184}$$

where

$$\underline{G}_w^r(\xi,\eta)=\begin{bmatrix} \cdots & \dfrac{G_i'(\xi)}{J_x}S_r(\eta) & \dfrac{H_i'(\xi)}{J_x}S_r(\eta) & \cdots \\ \cdots & G_i(\xi)\dfrac{S_r'(\eta)}{J_y} & H_i(\xi)\dfrac{S_r'(\eta)}{J_y} & \cdots \end{bmatrix} \tag{5.185}$$

(iii) $\underline{\theta}_\theta(x,y)$

Substituting from equation (5.164) into (5.55), then it cam be shown that:

$$\underline{\theta}_\theta(x,y)=\begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial x\partial y} \\ \dfrac{\partial^2 w}{\partial y\partial x} \\ \dfrac{\partial^2 w}{\partial y^2} \end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix} S_r(\eta)\displaystyle\sum_{i=1}^{n}\left[\frac{d^2G_i}{dx^2}w_i^r+\frac{d^2H_i}{dx^2}w_{i,x}^r\right] \\ \dfrac{dS_r}{dy}\displaystyle\sum_{i=1}^{n}\left[\frac{dG_i}{dx}w_i^r+\frac{dH_i}{dx}w_{i,x}^r\right] \\ \dfrac{dS_r}{dy}\displaystyle\sum_{i=1}^{n}\left[\frac{dG_i}{dx}w_i^r+\frac{dH_i}{dx}w_{i,x}^r\right] \\ \dfrac{d^2S_r}{dy^2}\displaystyle\sum_{i=1}^{n}\left(G_iw_i^r+H_iw_{i,x}^r\right) \end{bmatrix}\equiv\sum_{r=1}^{m}\underline{G}_\theta^r(\xi,\eta)\underline{\delta}_b^r \tag{5.186}$$

where

$$\underline{G}_\theta^r(\xi,\eta)=\begin{bmatrix} \cdots & \dfrac{G_i''(\xi)}{J_x^{\,2}}S_r(\eta) & \dfrac{H_i''(\xi)}{J_x^{\,2}}S_r(\eta) & \cdots \\ \cdots & \dfrac{G_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & \dfrac{H_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & \cdots \\ \cdots & \dfrac{G_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & \dfrac{H_i'(\xi)}{J_x}\dfrac{S_r'(\eta)}{J_y} & \cdots \\ \cdots & G_i(\xi)\dfrac{S_r''(\eta)}{J_y^{\,2}} & H_i(\xi)\dfrac{S_r''(\eta)}{J_y^{\,2}} & \cdots \end{bmatrix} \tag{5.187}$$

(iv) $\underline{\theta}_\psi(x,y)$

Substituting from equations (5.162) and (5.163) into (5.56), then

$$\underline{\theta}_\psi(x,y)=\begin{bmatrix} -\dfrac{\partial\psi_y}{\partial x} \\ \dfrac{\partial\psi_x}{\partial x} \\ -\dfrac{\partial\psi_y}{\partial y} \\ \dfrac{\partial\psi_x}{\partial y} \end{bmatrix}=\sum_{r=1}^{m}\begin{bmatrix} -\displaystyle\sum_{i=1}^{n}\frac{dN_i}{dx}\mu_r\left(\psi_y^r\right)_i \\ \displaystyle\sum_{i=1}^{n}\frac{dN_i}{dx}\mu_r\left(\psi_x^r\right)_i \\ -\displaystyle\sum_{i=1}^{n}N_i\frac{d\mu_r}{dy}\left(\psi_y^r\right)_i \\ \displaystyle\sum_{i=1}^{n}N_i\frac{d\mu_r}{dy}\left(\psi_x^r\right)_i \end{bmatrix}$$

which can be written in the following matrix form:

$$\underline{\theta}_{\psi}(x,y)=\sum_{r=1}^{m}\underline{G}_{\psi}^{r}(\xi,\eta)\underline{\delta}_{\psi}^{r} \tag{5.188}$$

where

$$\underline{G}_{\psi}^{r}(\xi,\eta)=\begin{bmatrix} \cdots & 0 & -\dfrac{N_i'(\xi)}{J_x}\mu_r(\eta) & \cdots \\ \cdots & \dfrac{N_i'(\xi)}{J_x}\mu_r(\eta) & 0 & \cdots \\ \cdots & 0 & -N_i(\xi)\dfrac{\mu_r'(\eta)}{J_y} & \cdots \\ \cdots & N_i(\xi)\dfrac{\mu_r'(\eta)}{J_y} & 0 & \cdots \end{bmatrix} \tag{5.189}$$

5.8.4 Total x-y Strain and its Variation

Substituting from (5.173), (5.178), (5.180), (5.182), (5.184), (5.186), (5.188), into (5.68), then the x-y strain vector can be expressed as follows:

$$\begin{aligned}\underline{\varepsilon}(x,y,z)=\sum_{r=1}^{m}\Bigg\{&\left[\underline{B}_o^r(\xi,\eta)\underline{\delta}_o^r+\frac{1}{2}A_m\underline{G}_m^r(\xi,\eta)\underline{\delta}_o^r+\frac{1}{2}A_w\underline{G}_w^r(\xi,\eta)\underline{\delta}_b^r\right]\\ &-z\left[\underline{B}_b^r(\xi,\eta)\underline{\delta}_b^r+\frac{1}{2}A_m\underline{G}_\theta^r(\xi,\eta)\underline{\delta}_b^r+\frac{1}{2}A_\theta\underline{G}_m^r(\xi,\eta)\underline{\delta}_o^r\right]\\ &+f_\psi(z)\left[\underline{B}_\psi^r(\xi,\eta)\underline{\delta}_\psi^r+\frac{1}{2}A_\psi\underline{G}_m^r(\xi,\eta)\underline{\delta}_o^r+\frac{1}{2}A_m\underline{G}_\psi^r(\xi,\eta)\underline{\delta}_\psi^r\right]\\ &-\frac{1}{2}zf_\psi(z)\left[A_\psi\underline{G}_\theta^r(\xi,\eta)\underline{\delta}_b^r+A_\theta\underline{G}_\psi^r(\xi,\eta)\underline{\delta}_\psi^r\right]+\frac{1}{2}z^2A_\theta\underline{G}_\theta^r(\xi,\eta)\underline{\delta}_b^r+\frac{1}{2}f_\psi^2(z)A_\psi\underline{G}_\psi^r(\xi,\eta)\underline{\delta}_\psi^r\Bigg\}\end{aligned} \tag{5.190}$$

we shall write its variation using a different summation index:

$$\begin{aligned}d\underline{\varepsilon}(x,y,z)=\sum_{s=1}^{m}\Bigg(&\underline{B}_o^s d\underline{\delta}_o^s+A_m\underline{G}_m^s d\underline{\delta}_o^s+A_w\underline{G}_w^s d\underline{\delta}_b^s\Big)-z\Big(\underline{B}_b^s d\underline{\delta}_b^s+A_m\underline{G}_\theta^s d\underline{\delta}_b^s+A_\theta\underline{G}_m^s d\underline{\delta}_o^s\Big)\\ &+f_\psi(z)\Big(\underline{B}_\psi^s d\underline{\delta}_\psi^s+A_\psi\underline{G}_m^s d\underline{\delta}_o^s+A_m\underline{G}_\psi^s d\underline{\delta}_\psi^s\Big)-zf_\psi(z)\Big(A_\psi\underline{G}_\theta^s d\underline{\delta}_b^s+A_\theta\underline{G}_\psi^s d\underline{\delta}_\psi^s\Big)\\ &+z^2A_\theta\underline{G}_\theta^s d\underline{\delta}_b^s+f_\psi^2(z)A_\psi\underline{G}_\psi^s d\underline{\delta}_\psi^s\Bigg\}\end{aligned} \tag{5.191}$$

5.9 Interpolated Strain Energy Variation

For simplicity, the strain energy variation will be divided into three parts first, before adding all final equations together. Notice also that equation (5.191) can be partitioned as follows:

$$d\underline{\varepsilon}(x,y,z) = d\underline{\varepsilon}_S(x,y,z) + d\underline{\varepsilon}_L(x,y,z) \tag{5.192}$$

where

$$d\underline{\varepsilon}_S(x,y,z) = \sum_{s=1}^{m}\left\{\underline{B}_o^s d\underline{\delta}_o^s - z\underline{B}_b^s d\underline{\delta}_b^s + f_\psi(z)\underline{B}_\psi^s d\underline{\delta}_\psi^s\right\} \tag{5.193}$$

and

$$d\underline{\varepsilon}_L(x,y,z) = \sum_{s=1}^{m}\Big\{\left(\underline{A}_m\,\underline{G}_m^s d\underline{\delta}_o^s + \underline{A}_w\,\underline{G}_w^s d\underline{\delta}_b^s\right) - z\left(\underline{A}_m\,\underline{G}_\theta^s d\underline{\delta}_b^s + \underline{A}_\theta\,\underline{G}_m^s d\underline{\delta}_o^s\right)$$

$$+ f_\psi(z)\left(\underline{A}_\psi \underline{G}_m^s d\underline{\delta}_o^s + \underline{A}_m \underline{G}_\psi^s d\underline{\delta}_\psi^s\right) - z f_\psi(z)\left(\underline{A}_\psi \underline{G}_\theta^s d\underline{\delta}_b^s + \underline{A}_\theta \underline{G}_\psi^s d\underline{\delta}_\psi^s\right)$$

$$+ z^2 \underline{A}_\theta \underline{G}_\theta^s d\underline{\delta}_b^s + f_\psi^2(z)\underline{A}_\psi \underline{G}_\psi^s d\underline{\delta}_\psi^s\Big\} \tag{5.194}$$

$$\text{with } f_\gamma(z)d\underline{\hat{\gamma}}(x,y,z) = \sum_{s=1}^{m} f_\gamma(z)\underline{B}_\gamma^s d\underline{\delta}_\psi^s \tag{5.195}$$

(i) dU_γ

Substituting from equation (5.195) into (5.78), then

$$d\overline{U}_\gamma^{(l)} = d\gamma^t\,\underline{\tau}^{(l)} = \sum_{s=1}^{m}\left(d\underline{\delta}_\psi^s\right)^t\left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\underline{\tau}^{(l)}$$

(5.196)

Similarly, substituting from equation (5.195) into (5.98) for $d\underline{\hat{\gamma}}$, then

$$dU_\gamma' = \int_{-h/2}^{h/2}\left(d\overline{U}_\gamma^{(l)}\right)dz = \sum_{s=1}^{m}\left(d\underline{\delta}_\psi^s\right)'\left(\underline{B}_\gamma^s\right)'\underline{\tilde{\tau}} \tag{5.197}$$

where $\underline{\tilde{\tau}}$ is defined by equation (5.97). Hence, for one finite strip:

$$dU_\gamma = \int_{-h/2}^{h/2}\iint_{strip}\left[\sum_{s=1}^{m}\left(d\underline{\delta}_\psi^s\right)'\left(\underline{B}_\gamma^s\right)' f_\gamma(z)\underline{\tau}^{(l)}\right]dxdydz \equiv \iint_{strip}\left[\sum_{s=1}^{m}\left(d\underline{\delta}_\psi^s\right)'\left(\underline{B}_\gamma^s\right)'\underline{\tilde{\tau}}\right]dxdy \tag{5.198}$$

(ii) dU_S

Substituting from equation (5.193) into (5.76), then it can be shown that:

$$d\bar{U}_s^{(l)} = \sum_{s=1}^{m}\left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{B}_o^s\right)^t + \left(d\underline{\delta}_b^s\right)^t\left(\underline{B}_b^s\right)^t(-z) + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{B}_\psi^s\right)^t f_\psi(z)\right]\underline{\sigma}^{(l)} \tag{5.199}$$

integrating with respect to z, then it can be deduced that:

$$dU_s^{'} = \int_{-h/2}^{h/2}\left(d\bar{U}_s^{(l)}\right)dz = \sum_{s=1}^{m}\left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{B}_o^s\right)^t\underline{\sigma}_{(o)} + \left(d\underline{\delta}_b^s\right)^t\left(\underline{B}_b^s\right)^t\underline{\sigma}_{(1)} + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{B}_\psi^s\right)^t\underline{\sigma}_{\psi o}\right] \tag{5.200}$$

where $\underline{\sigma}_{(o)}, \underline{\sigma}_{(1)}$, and $\underline{\sigma}_{\psi o}$ are as defined by equations (5.89), (5.92), and (5.93), respectively. dU_S can be obtained by integrating $d\bar{U}_s^{(l)}$ or $dU_s^{'}$ as in equation (5.198).

(iii) dU_L

Substituting from equation (5.194) into (5.77), then it can be shown that

$$\begin{aligned}
d\bar{U}_L^{(l)} = \sum_{s=1}^{m}\Bigg\{&\left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_m^t + \left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_w^s\right)^t\underline{A}_w^t\right]\underline{\sigma}^{(l)} \\
&+\left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_m^t + \left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_\theta^t\right]\left(-z\underline{\sigma}^{(l)}\right) + \left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_m^t\right]f_\psi(z)\underline{\sigma}^{(l)} \\
&+\left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_\theta^t\right]\left(-zf_\psi(z)\underline{\sigma}^{(l)}\right) + \left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_\theta^t\right]\left(z^2\underline{\sigma}^{(l)}\right) \\
&+\left[\left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_\psi^t\right]f_\psi^2\underline{\sigma}^{(l)}\Bigg\}
\end{aligned} \tag{5.201}$$

Integrating the previous equation with respect to z, and using the definition of integrated stresses, as given in section 5.6.1, then it can be proved that:

$$\begin{aligned}
dU_L^{'} &= \int_{-h/2}^{h/2}\left(d\bar{U}_L^{(l)}\right)dz \\
&= \sum_{s=1}^{m}\Bigg\{\left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_m^t + \left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_w^s\right)^t\underline{A}_w^t\right]\underline{\sigma}_{(o)} \\
&+\left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_m^t + \left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_\theta^t\right]\underline{\sigma}_{(1)} + \left[\left(d\underline{\delta}_o^s\right)^t\left(\underline{G}_m^s\right)^t\underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_m^t\right]\underline{\sigma}_{\psi o} \\
&+\left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_\theta^t\right]\underline{\sigma}_{\psi 1} + \left[\left(d\underline{\delta}_b^s\right)^t\left(\underline{G}_\theta^s\right)^t\underline{A}_\theta^t\right]\underline{\sigma}_{(2)} \\
&+\left[\left(d\underline{\delta}_\psi^s\right)^t\left(\underline{G}_\psi^s\right)^t\underline{A}_\psi^t\right]\underline{\sigma}_{\psi\psi}\Bigg\}
\end{aligned} \tag{5.202}$$

dU_L can be obtained by integration, as shown before.

The variation of the total strain energy density will be

$$d\overline{U}^{(l)} = d\overline{U}_{\gamma}^{(l)} + d\overline{U}_{S}^{(l)} + d\overline{U}_{L}^{(l)} \tag{5.203}$$

Substituting from equation (5.196), (5.199), (5.201) into (5.203) and rearranging terms, it can be deduced that:

$$\begin{aligned} d\overline{U}^{(l)} = \sum_{s=1}^{m} \Big\{ & \Big[\left(d\underline{\delta}_o^s\right)' \left(\underline{B}_o^s\right)' + \left(d\underline{\delta}_o^s\right)' \left(\underline{G}_m^s\right)' \underline{A}_m^t + \left(d\underline{\delta}_b^s\right)' \left(\underline{G}_w^s\right)' \underline{A}_w^t \Big] \underline{\sigma}^{(l)} \\ & + \Big[\left(d\underline{\delta}_b^s\right)' \left(\underline{B}_b^s\right)' + \left(d\underline{\delta}_o^s\right)' \left(\underline{G}_m^s\right)' \underline{A}_\theta^t + \left(d\underline{\delta}_b^s\right)' \left(\underline{G}_\theta^s\right)' \underline{A}_m^t \Big] \left(-z\underline{\sigma}^{(l)}\right) \\ & + \Big[\left(d\underline{\delta}_\psi^s\right)' \left(\underline{B}_\psi^s\right)' + \left(d\underline{\delta}_o^s\right)' \left(\underline{G}_m^s\right)' \underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)' \left(\underline{G}_\psi^s\right)' \underline{A}_m^t \Big] f_\psi \underline{\sigma}^{(l)} \\ & + \Big[\left(d\underline{\delta}_b^s\right)' \left(\underline{G}_\theta^s\right)' \underline{A}_\psi^t + \left(d\underline{\delta}_\psi^s\right)' \left(\underline{G}_\psi^s\right)' \underline{A}_\theta^t \Big] \left(-zf_\psi \underline{\sigma}^{(l)}\right) + \Big[\left(d\underline{\delta}_b^s\right)' \left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t \Big] \left(z^2 \underline{\sigma}^{(l)}\right) \\ & + \Big[\left(d\underline{\delta}_\psi^s\right)' \left(\underline{G}_\psi^s\right)' \underline{A}_\psi^t \Big] \left(f_\psi^2 \underline{\sigma}^{(l)}\right) + \Big[\left(d\underline{\delta}_\psi^s\right)' \left(\underline{B}_\gamma^s\right)' \Big] f_\gamma(z)\, \underline{\tau}^{(l)} \Big\} \end{aligned} \tag{5.204}$$

The variation of the strain energy of a finite strip will be:

$$dU = \int_{-b/2}^{b/2} \left[\iint_{strip} \left(d\overline{U}^{(l)}\right) dxdy \right] dz \tag{5.205}$$

$$\equiv \iint_{strip} dU' dxdy \tag{5.206}$$

where

$$dU' = \int_{-b/2}^{b/2} d\overline{U}^{(l)}\, dz \tag{5.207}$$

Integrating equation (5.204) with respect to z, then it can be shown that,

$$\begin{aligned} dU' = \sum_{s=1}^{m} \Big\{ & \left(d\underline{\delta}_o^s\right)^t \Big[\left(\underline{B}_o^s\right)^t + \left(\underline{G}_m^s\right)^t \underline{A}_m^t \Big] \underline{\sigma}_{(o)} + \left(d\underline{\delta}_b^s\right)^t \Big[\left(\underline{G}_w^s\right)^t \underline{A}_w^t \Big] \underline{\sigma}_{(o)} \\ & + \left(d\underline{\delta}_o^s\right)' \Big[\left(\underline{G}_m^s\right)' \underline{A}_\theta^t \Big] \underline{\sigma}_{(1)} + \left(d\underline{\delta}_b^s\right)' \Big[\left(\underline{B}_b^s\right)' + \left(\underline{G}_\theta^s\right)' \underline{A}_m^t \Big] \underline{\sigma}_{(1)} + \left(d\underline{\delta}_o^s\right)' \Big[\left(\underline{G}_m^s\right)' \underline{A}_\psi^t \Big] \underline{\sigma}_{\psi o} \\ & + \left(d\underline{\delta}_\psi^s\right)' \Big[\left(\underline{B}_\psi^s\right)' + \left(\underline{G}_\psi^s\right)' \underline{A}_m^t \Big] \underline{\sigma}_{\psi o} + \left(d\underline{\delta}_b^s\right)' \Big[\left(\underline{G}_\theta^s\right)' \underline{A}_\psi^t \Big] \underline{\sigma}_{\psi 1} + \left(d\underline{\delta}_\psi^s\right)' \Big[\left(\underline{G}_\psi^s\right)' \underline{A}_\theta^t \Big] \underline{\sigma}_{\psi 1} \\ & + \left(d\underline{\delta}_b^s\right)' \Big[\left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t \Big] \underline{\sigma}_{(2)} + \left(d\underline{\delta}_\psi^s\right)' \Big[\left(\underline{G}_\psi^s\right)' \underline{A}_\psi^t \Big] \underline{\sigma}_{\psi\psi} + \left(d\underline{\delta}_\psi^s\right)' \left(\underline{B}_\gamma^s\right)' \underline{\tau} \Big\} \end{aligned} \tag{5.208}$$

5.9.4 Generalized Equations of Equilibrium

Notice that the equivalent nodal loading is assumed to have the same variation of the work done by actual loadings. The details of different types of loading can be found in Razzaq (2003). Hence, the variation of the work done by actual loads can be represented by the variation of the work done by equivalent nodal loads, as follows:

$$dW = \sum_{s=1}^{m}\left[\left(d\underline{\delta}_o^s\right)^t \underline{F}_o^s + \left(d\underline{\delta}_b^s\right)^t \underline{F}_b^s + \left(d\underline{\delta}_\psi^s\right)^t \underline{F}_\psi^s\right] \tag{5.209}$$

with

$$\underline{F}_o^s = \left\{\left(F_x^s\right)_1 \left(F_y^s\right)_1 \ldots\ldots \left(F_x^s\right)_n \left(F_y^s\right)_1\right\} \tag{5.210}$$

which represents the in-plane loading,

$$\underline{F}_b^s = \left\{\left(F_z^s\right)_1 M_1^s \quad \left(F_z^s\right)_2 M_2^s \ldots\ldots \left(F_z^s\right)_n M_n^s\right\} \tag{5.211}$$

which represents one part of the out-of-plane loading

$$\underline{F}_\psi^s = \left\{\left(M_x^s\right)_1\left(M_y^s\right)_1 \ldots\ldots\ldots\ldots \left(M_x^s\right)_n\left(M_y^s\right)_n\right\} \tag{5.212}$$

which represents the other part of the out-of-plane loading.
From the principle of virtual work due to a variation of displacement:

$$d\chi = dU - dW = 0 \tag{5.213}$$

Substituting from (5.205), (5.209) into (5.213) then after rearranging terms:

$$\begin{aligned}
&\sum_{s=1}^{m}\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left(d\underline{\delta}_o^s\right)^t\left[\left(\underline{B}_o^s\right)^t + \left(\underline{G}_m^s\right)^t \underline{A}_m^t\right]\underline{\sigma}^{(l)} + \left[\left(\underline{G}_m^s\right)^t \underline{A}_\theta^t\right]\left(-z\underline{\sigma}^{(l)}\right)\right.\\
&+\left[\left(\underline{G}_m^s\right)^t \underline{A}_\psi^t\right]\left(f_\psi \underline{\sigma}^{(l)}\right)\Big\} + \left(d\underline{\delta}_b^s\right)^t\left\{\left[\left(\underline{G}_w^s\right)^t \underline{A}_w^t\right]\underline{\sigma}^{(l)}\right.\\
&+\left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t\right]\left(-z\underline{\sigma}^{(l)}\right) + \left[\left(\underline{G}_\theta^s\right)^t \underline{A}_\psi^t\right]\left(-zf_\psi \underline{\sigma}^{(l)}\right)\\
&+\left[\left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t\right]\left(z^2\underline{\sigma}^{(l)}\right)\Big\} + \left(d\underline{\delta}_\psi^s\right)^t\left\{\left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\underline{\tau}^{(l)}\right.\\
&+\left[\left(\underline{B}_\psi^s\right)^t + \left(\underline{G}_\psi^s\right)^t \underline{A}_m^t\right]\left(f_\psi \underline{\sigma}^{(l)}\right) + \left[\left(\underline{G}_\psi^s\right)^t \underline{A}_\theta^t\right]\left(-zf_\psi \underline{\sigma}^{(l)}\right)\\
&\left.\left.+\left[\left(\underline{G}_\psi^s\right)^t \underline{A}_\psi^t\right]\left(f_\psi^2 \underline{\sigma}^{(l)}\right)\right\}\right\} dxdydz - \sum_{s=1}^{m}\left[\left(d\underline{\delta}_o^s \underline{F}_o^s + \left(d\underline{\delta}_b^s\right)^t \underline{F}_b^s + \left(d\underline{\delta}_\psi^s\right)^t \underline{F}_\psi^s\right)^t\right] = 0
\end{aligned} \tag{5.214}$$

The variations represent small arbitrary changes, hence coefficients of $d\underline{\delta}_o^s, d\underline{\delta}_b^s, d\underline{\delta}_\psi^s$ will vanish, i.e.

$$\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left[\left(\underline{B}_o^s\right)^t+\left(\underline{G}_m^s\right)^t\underline{A}_m^t\right]\underline{\sigma}^{(l)}+\left[\left(\underline{G}_m^s\right)^t\underline{A}_\theta^t\right]\left(-z\underline{\sigma}^{(l)}\right)\right.$$

$$\left.+\left[\left(\underline{G}_m^s\right)^t\underline{A}_\psi^t\right]\left(f_\psi\underline{\sigma}^{(l)}\right)\right\}dxdydz-\underline{F}_o^s=0 \qquad (5.215)$$

$$\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left[\left(\underline{G}_w^s\right)^t\underline{A}_w^t\right]\underline{\sigma}^{(l)}+\left[\left(\underline{B}_b^s\right)^t+\left(\underline{G}_\theta^s\right)^t\underline{A}_m^t\right]\left(-z\underline{\sigma}^{(l)}\right)+\left[\left(\underline{G}_\theta^s\right)^t\underline{A}_\psi^t\right]\left(-zf_\psi\underline{\sigma}^{(l)}\right)\right.$$

$$\left.+\left[\left(\underline{G}_\theta^s\right)^t\underline{A}_\theta^t\right]\left(z^2\underline{\sigma}^{(l)}\right)\right\}dxdydz-\underline{F}_b^s=0 \qquad (5.216)$$

$$\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left(\underline{B}_\gamma^s\right)^t f_\gamma(z)\underline{\tau}^{(l)}+\left[\left(\underline{B}_\psi^s\right)^t+\left(\underline{G}_\psi^s\right)^t\underline{A}_m^t\right]\left(f_\psi\underline{\sigma}^{(l)}\right)+\left[\left(\underline{G}_\psi^s\right)^t\underline{A}_\theta^t\right]\left(-zf_\psi\underline{\sigma}^{(l)}\right)\right.$$

$$\left.+\left[\left(\underline{G}_\psi^s\right)^t\underline{A}_\psi^t\right]\left(f_\psi^2\underline{\sigma}^{(l)}\right)\right\}dxdydz-\underline{F}_\psi^s=0 \qquad (5.217)$$

for s =1,2, …. ,m.

Equations (5.215), (5.216), (5.217) represent the generalized equations of equilibrium.

If an approximate solution is employed, then equations (5.215), (5.216), (5.217) will lead to Residual force vectors defined as follows:

$$\underline{R}_o^s=\underline{F}_o^s-\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left[\left(\underline{B}_o^s\right)^t+\left(\underline{G}_m^s\right)^t\underline{A}_m^t\right]\underline{\sigma}^{(l)}+\left[\left(\underline{G}_m^s\right)^t\underline{A}_\theta^t\right]\left(-z\underline{\sigma}^{(l)}\right)\right.$$

$$\left.+\left[\left(\underline{G}_m^s\right)^t\underline{A}_\psi^t\right]\left(f_\psi\underline{\sigma}^{(l)}\right)\right\}dxdydz \qquad (5.218)$$

$$\underline{R}_b^s=\underline{F}_b^s-\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\left\{\left[\left(\underline{G}_w^s\right)^t\underline{A}_w^t\right]\underline{\sigma}^{(l)}+\left[\left(\underline{B}_b^s\right)^t+\left(\underline{G}_\theta^s\right)^t\underline{A}_m^t\right]\left(-z\underline{\sigma}^{(l)}\right)+\left[\left(\underline{G}_\theta^s\right)^t\underline{A}_\psi^t\right]\left(-zf_\psi\underline{\sigma}^{(l)}\right)\right.$$

$$\left.+\left[\left(\underline{G}_\theta^s\right)^t\underline{A}_\theta^t\right]\left(z^2\underline{\sigma}^{(l)}\right)\right\}dxdydz \qquad (5.219)$$

$$\underline{R}_{\psi}^{s} = \underline{F}_{\psi}^{s} - \sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left\{ \left(\underline{B}_{\gamma}^{s}\right)^{t} f_{\gamma}(z)\underline{\tau}^{(l)} + \left[\left(\underline{B}_{\psi}^{s}\right)^{t} + \left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{m}^{t}\right]\left(f_{\psi}\,\underline{\sigma}^{(l)}\right) + \left[\left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{\theta}^{t}\right]\left(-z f_{\psi}\,\underline{\sigma}^{(l)}\right) \right.$$

$$\left. + \left[\left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{\psi}^{t}\right]\left(f_{\psi}^{2}\,\underline{\sigma}^{(l)}\right) \right\} dxdydz \qquad (5.220)$$

for s =1,2, …. ,m.

When calculating Residual forces, equations (5.218), (5.219), (5.220) are integrated analytically with respect to z, leading to:

$$\underline{R}_{o}^{s} = \underline{F}_{o}^{s} - \sum_{strips} \iint_{strip} \left\{ \left[\left(\underline{B}_{o}^{s}\right)^{t} + \left(\underline{G}_{m}^{s}\right)^{t} \underline{A}_{m}^{t}\right]\underline{\sigma}_{(o)} + \left[\left(\underline{G}_{m}^{s}\right)^{t} \underline{A}_{\theta}^{t}\right]\underline{\sigma}_{(1)} \right.$$

$$\left. + \left[\left(\underline{G}_{m}^{s}\right)^{t} \underline{A}_{\psi}^{t}\right]\underline{\sigma}_{\psi o} \right\} dxdy \qquad (5.221)$$

$$\underline{R}_{b}^{s} = \underline{F}_{b}^{s} - \sum_{strips} \iint_{strip} \left\{ \left[\left(\underline{G}_{w}^{s}\right)^{t} \underline{A}_{w}^{t}\right]\underline{\sigma}_{(o)} + \left[\left(\underline{B}_{b}^{s}\right)^{t} + \left(\underline{G}_{\theta}^{s}\right)^{t} \underline{A}_{m}^{t}\right]\underline{\sigma}_{(1)} + \left[\left(\underline{G}_{\theta}^{s}\right)^{t} \underline{A}_{\psi}^{t}\right]\underline{\sigma}_{\psi 1} \right.$$

$$\left. + \left[\left(\underline{G}_{\theta}^{s}\right)^{t} \underline{A}_{\theta}^{t}\right]\underline{\sigma}_{(2)} \right\} dxdy \qquad (5.222)$$

$$\underline{R}_{\psi}^{s} = \underline{F}_{\psi}^{s} - \sum_{strips} \iint_{strip} \left\{ \left(\underline{B}_{\gamma}^{s}\right)^{t} \underline{\tilde{\tau}} + \left[\left(\underline{B}_{\psi}^{s}\right)^{t} + \left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{m}^{t}\right]\underline{\sigma}_{\psi o} + \left[\left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{\theta}^{t}\right]\underline{\sigma}_{\psi 1} \right.$$

$$\left. + \left[\left(\underline{G}_{\psi}^{s}\right)^{t} \underline{A}_{\psi}^{t}\right]\underline{\sigma}_{\psi\psi} \right\} dxdy \qquad (5.223)$$

for s =1,2, …. ,m.

5.11 Linearization of Equations of Equilibrium

Equations (5.221), (5.222), (5.223) represent the Residual forces due to an approximate solution. If the stress is incremented by $\left(\Delta\underline{\sigma}^{(l)}, \Delta\underline{\tau}^{(l)}\right)$ such that the residuals vanish, then it can be deduced from equations (5.218), (5.219), (5.220) that:

$$0 = \underline{F}_{o}^{s} - \sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left\{ \left[\left(\underline{B}_{o}^{s}\right)^{t} + \left(\underline{G}_{m}^{s}\right)^{t}\left(\underline{A}_{m}^{t} + \Delta\underline{A}_{m}^{t}\right)\right]\left[\underline{\sigma}^{(l)} + \Delta\underline{\sigma}^{(l)}\right] \right.$$

$$\left. + \left[\left(\underline{G}_{m}^{s}\right)^{t}\left(\underline{A}_{\theta}^{t} + \Delta\underline{A}_{\theta}^{t}\right)\right](-z)\left[\underline{\sigma}^{(l)} + \Delta\underline{\sigma}^{(l)}\right] + \left[\left(\underline{G}_{m}^{s}\right)^{t}\left(\underline{A}_{\psi}^{t} + \Delta\underline{A}_{\psi}^{t}\right)\right](f_{\psi})\left[\underline{\sigma}^{(l)} + \Delta\underline{\sigma}^{(l)}\right] \right\} dxdydz$$

(5.224)

$$0 = \underline{F}_b^s - \sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left\{ \left[\left(\underline{G}_w^s \right)^t \left(\underline{A}_w^t + \Delta \underline{A}_w^t \right) \right] \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right] \right.$$

$$+ \left[\left(\underline{B}_b^s \right)^t + \left(\underline{G}_\theta^s \right)^t \left(\underline{A}_m^t + \Delta \underline{A}_m^t \right) \right] (-z) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right] + \left[\left(\underline{G}_\theta^s \right)^t \left(\underline{A}_\psi^t + \Delta \underline{A}_\psi^t \right) \right] (-z f_\psi) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right]$$

$$\left. + \left[\left(\underline{G}_\theta^s \right)^t \left(\underline{A}_\theta^t + \Delta \underline{A}_\theta^t \right) \right] (z^2) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right] \right\} dx dy dz \qquad (5.225)$$

$$0 = \underline{F}_\psi^s - \sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left\{ \left(\underline{B}_\gamma^s \right)^t f_\gamma(z) \left[\underline{\tau}^{(l)} + \Delta \underline{\tau}^{(l)} \right] \right.$$

$$+ \left[\left(\underline{B}_\psi^s \right)^t + \left(\underline{G}_\psi^s \right)^t \left(\underline{A}_m^t + \Delta \underline{A}_m^t \right) \right] (f_\psi) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right] + \left[\left(\underline{G}_\psi^s \right)^t \left(\underline{A}_\theta^t + \Delta \underline{A}_\theta^t \right) \right] (-z f_\psi) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right]$$

$$\left. + \left[\left(\underline{G}_\psi^s \right)^t \left(\underline{A}_\psi^t + \Delta \underline{A}_\psi^t \right) \right] (f_\psi^2) \left[\underline{\sigma}^{(l)} + \Delta \underline{\sigma}^{(l)} \right] \right\} dx dy dz \qquad (5.226)$$

(i) Simplification of equation (5.224)

Ignoring the terms of $\Delta \underline{A} \bullet \Delta \underline{\sigma}$, then it can be deduced from equations (5.224), (5.218) that:

$$\sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left[\left\{ \left[\left(\underline{B}_o^s \right)^t + \left(\underline{G}_m^s \right)^t \underline{A}_m^t \right] + \left[\left(\underline{G}_m^s \right)^t \underline{A}_\theta^t \right] (-z) + \left[\left(\underline{G}_m^s \right)^t \underline{A}_\psi^t \right] (f_\psi) \right\} \Delta \underline{\sigma}^{(l)} \right.$$

$$\left. + \left\{ \left(\underline{G}_m^s \right)^t \Delta \underline{A}_m^t + \left(\underline{G}_m^s \right)^t \Delta \underline{A}_\theta^t (-z) + \left(\underline{G}_m^s \right)^t \Delta \underline{A}_\psi^t (f_\psi) \right\} \underline{\sigma}^{(l)} \right] dx dy dz = \underline{R}_o^s \qquad (5.227)$$

The following integration with respect to z involved in equation (5.227):

$$\int_{-h/2}^{h/2} f(z) \Delta \underline{\sigma}^{(l)} dz = \int_{-h/2}^{h/2} f(z) \underline{D}^{(l)} \sum_{r=1}^{m} \left[\left(\underline{B}_o^r \Delta \underline{\delta}_o^r + \underline{A}_m \underline{G}_m^r \Delta \underline{\delta}_o^r + \underline{A}_w \underline{G}_w^r \Delta \underline{\delta}_b^r \right) \right.$$

$$- z \left(\underline{B}_b^r \Delta \underline{\delta}_b^r + \underline{A}_m \underline{G}_\theta^r \Delta \underline{\delta}_b^r + \underline{A}_\theta \underline{G}_m^r \Delta \underline{\delta}_o^r \right) + f_\psi(z) \left(\underline{B}_\psi^r \Delta \underline{\delta}_\psi^r + \underline{A}_\psi \underline{G}_m^r \Delta \underline{\delta}_o^r + \underline{A}_m \underline{G}_\psi^r \Delta \underline{\delta}_\psi^r \right)$$

$$\left. - z f_\psi(z) \left(\underline{A}_\psi \underline{G}_\theta^r \Delta \underline{\delta}_b^r + \underline{A}_\theta \underline{G}_\psi^r \Delta \underline{\delta}_\psi^r \right) + z^2 \underline{A}_\theta \underline{G}_\theta^r \Delta \underline{\delta}_b^r + f_\psi^2(z) \underline{A}_\psi \underline{G}_\psi^r \Delta \underline{\delta}_\psi^r \right] dz \qquad (5.228)$$

which can be simplified as follows:

$$\int_{-h/2}^{h/2} f(z)\Delta\underline{\sigma}^{(l)}dz = \sum_{r=1}^{m}\left\{\left[\left(\int_{-h/2}^{h/2} f(z)\underline{D}^{(l)}dz\right)\left[\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right] - \left(\int_{-h/2}^{h/2} zf(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\theta\underline{G}_m^r\right]\right.\right.$$
$$+\left(\int_{-h/2}^{h/2} f_\psi(z)f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\psi\underline{G}_m^r\right]\Delta\delta_o^r + \left[\left(\int_{-h/2}^{h/2} f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_w\underline{G}_w^r\right]\right.$$
$$-\left(\int_{-h/2}^{h/2} zf(z)\underline{D}^{(l)}dz\right)\left[\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right] - \left(\int_{-h/2}^{h/2} zf_\psi(z)f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\psi\underline{G}_\theta^r\right]$$
$$+\left(\int_{-h/2}^{h/2} z^2 f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\theta\underline{G}_\theta^r\right]\Delta\delta_b^r + \left[\left(\int_{-h/2}^{h/2} f_\psi(z)f(z)\underline{D}^{(l)}dz\right)\left[\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right]\right.$$
$$\left.\left.+\left(\int_{-h/2}^{h/2}\left(-zf_\psi(z)\right)f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\theta\underline{G}_\psi^r\right] + \left(\int_{-h/2}^{h/2} f_\psi^2(z)f(z)\underline{D}^{(l)}dz\right)\left[\underline{A}_\psi\underline{G}_\psi^r\right]\right]\Delta\delta_\psi^r\right\} \qquad (5.229)$$

Putting $f(z) = 1, -z, f_\psi(z)$ in equation (5.229), it can be shown that:

$$\int_{-h/2}^{h/2}\Delta\underline{\sigma}^{(l)}dz = \sum_{r=1}^{m}\left\{\left[\underline{D}_{(o)}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right) - \underline{D}_{(1)}\left(\underline{A}_\theta\underline{G}_m^r\right) + \underline{D}_{\psi o}\left(\underline{A}_\psi\underline{G}_m^r\right)\right]\Delta\underline{\delta}_o^r\right.$$
$$+\left[\underline{D}_{(o)}\left(\underline{A}_w\underline{G}_w^r\right) - \underline{D}_{(1)}\left(\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right) - \underline{D}_{\psi 1}\left(\underline{A}_\psi\underline{G}_\theta^r\right) + \underline{D}_{(2)}\left(\underline{A}_\theta\underline{G}_\theta^r\right)\right]\Delta\underline{\delta}_b^r$$
$$\left.+\left[\underline{D}_{\psi o}\left(\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right) - \underline{D}_{\psi 1}\underline{A}_\theta\underline{G}_\psi^r + \underline{D}_{\psi\psi}\underline{A}_\psi\underline{G}_\psi^r\right]\Delta\underline{\delta}_\psi^r\right\} \qquad (5.230)$$

$$\int_{-h/2}^{h/2}(-z)\Delta\underline{\sigma}^{(l)}dz = \sum_{r=1}^{m}\left\{\left[-\underline{D}_{(1)}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right) + \underline{D}_{(2)}\left(\underline{A}_\theta\underline{G}_m^r\right) - \underline{D}_{\psi 1}\left(\underline{A}_\psi\underline{G}_m^r\right)\right]\Delta\underline{\delta}_o^r\right.$$
$$+\left[-\underline{D}_{(1)}\left(\underline{A}_w\underline{G}_w^r\right) + \underline{D}_{(2)}\left(\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right) + \underline{D}_{\psi 2}\left(\underline{A}_w\underline{G}_\theta^r\right) - \underline{D}_{(3)}\left(\underline{A}_\theta\underline{G}_\theta^r\right)\right]\Delta\underline{\delta}_b^r$$
$$\left.+\left[-\underline{D}_{\psi 1}\left(\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right) + \underline{D}_{\psi 2}\left(\underline{A}_\theta\underline{G}_\psi^r\right) - \underline{D}_{\psi\psi 1}\left(\underline{A}_\psi\underline{G}_\psi^r\right)\Delta\underline{\delta}_\psi^r\right]\right\} \qquad (5.231)$$

$$\int_{-h/2}^{h/2} f_\psi(z)\Delta\underline{\sigma}^{(l)}dz = \sum_{r=1}^{m}\left\{\left[\underline{D}_{\psi o}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right) - \underline{D}_{\psi 1}\left(\underline{A}_\theta\underline{G}_m^r\right) + \underline{D}_{\psi\psi}\left(\underline{A}_\psi\underline{G}_m^r\right)\right]\Delta\underline{\delta}_o^r\right.$$
$$+\left[\underline{D}_{\psi o}\left(\underline{A}_w\underline{G}_w^r\right) - \underline{D}_{\psi 1}\left(\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right) - \underline{D}_{\psi\psi 1}\left(\underline{A}_\psi\underline{G}_\theta^r\right) + \underline{D}_{\psi 2}\left(\underline{A}_\theta\underline{G}_\theta^r\right)\right]\Delta\underline{\delta}_b^r$$
$$\left.+\left[\underline{D}_{\psi\psi}\left(\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right) - \underline{D}_{\psi\psi 1}\left(\underline{A}_\theta\underline{G}_\psi^r\right) + \underline{D}_{\psi\psi\psi}\left(\underline{A}_\psi\underline{G}_\psi^r\right)\right]\Delta\underline{\delta}_\psi^r\right\} \qquad (5.232)$$

Notice also that:

$$(1)\quad \int_{-h/2}^{h/2} \Delta \underline{A}_m^t \underline{\sigma}^{(l)} dz = \Delta \underline{A}_m^t \underline{\sigma}_{(o)} \equiv \underline{S}_{(o)} \Delta \underline{\theta}_m = \sum_{r=1}^{m} \underline{S}_{(o)} \underline{G}_m^r \Delta \underline{\delta}_o^r \tag{5.233}$$

$$\text{where} \qquad \underline{S}_{(o)} = \begin{bmatrix} \sigma_x^\circ & 0 & \tau_{xy}^\circ & 0 \\ 0 & \sigma_x^\circ & 0 & \tau_{xy}^\circ \\ \tau_{xy}^\circ & 0 & \sigma_y^\circ & 0 \\ 0 & \tau_{xy}^\circ & 0 & \sigma_y^\circ \end{bmatrix} \tag{5.234}$$

and $\sigma_x^\circ, \sigma_y^\circ, \tau_{xy}^\circ$ are the components of $\underline{\sigma}_{(o)}$

$$(2)\quad \int_{-h/2}^{h/2} \Delta \underline{A}_\theta^t (-z) \underline{\sigma}^{(l)} dz = \Delta \underline{A}_\theta^t \underline{\sigma}_{(1)} \equiv \underline{S}_{(1)} \Delta \underline{\theta}_\theta = \sum_{r=1}^{m} \underline{S}_{(1)} \underline{G}_\theta^r \Delta \underline{\delta}_b^r \tag{5.235}$$

$$\text{where} \qquad \underline{S}_{(1)} = \begin{bmatrix} \sigma_x^1 & 0 & \tau_{xy}^1 & 0 \\ 0 & \sigma_x^1 & 0 & \tau_{xy}^1 \\ \tau_{xy}^1 & 0 & \sigma_y^1 & 0 \\ 0 & \tau_{xy}^1 & 0 & \sigma_y^1 \end{bmatrix} \tag{5.236}$$

and $\sigma_x^1, \sigma_y^1, \tau_{xy}^1$ are the components of $\underline{\sigma}_{(1)}$

$$(3)\quad \int_{-h/2}^{h/2} \Delta \underline{A}_\psi^t f_\psi(z) \underline{\sigma}^{(l)} dz = \Delta \underline{A}_\psi^t \underline{\sigma}_{\psi o} \equiv \underline{S}_{\psi o} \Delta \underline{\theta}_\psi = \sum_{r=1}^{m} \underline{S}_{\psi o} \underline{G}_\psi^r \Delta \underline{\delta}_\psi^r \tag{5.237}$$

where $\underline{S}_{\psi o}$ is defined in a way similar to $\underline{S}_{(o)}$, $\underline{S}_{(1)}$ but with the component of $\underline{\sigma}_{\psi o}$ instead.

For the analysis of symmetric composites, it can be shown in *Appendix A* that $\underline{D}_{(1)} = \underline{D}_{(3)} = \underline{D}_{\psi o} = \underline{D}_{\psi 2} = \underline{D}_{\psi\psi 1} = \underline{D}_{\psi\psi\psi} = 0$. Substituting from equations (5.230), (5.231), (5.232), (5.233), (5.235), and (5.237) into (5.227), then equation (5.227) can be written as follows:

$$\sum_{strips}^{m} \sum_{r=1} \iint_{strip} \left[\left[\left(\underline{B}_o^s \right)' + \left(\underline{G}_m^s \right)' \underline{A}_m^t \right] \left\{ \left[\underline{D}_{(o)} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r \right) \right] \Delta \underline{\delta}_o^r \right.$$

$$+ \left[\underline{D}_{(o)} \underline{A}_w \underline{G}_w^r - \underline{D}_{\psi 1} \underline{A}_\psi \underline{G}_\theta^r + \underline{D}_{(2)} \underline{A}_\theta \underline{G}_\theta^r \right] \Delta \underline{\delta}_b^r + \left[- \underline{D}_{\psi 1} \underline{A}_\theta \underline{G}_\psi^r + \underline{D}_{\psi\psi} \underline{A}_\psi \underline{G}_\psi^r \right] \Delta \underline{\delta}_\psi^r \Big\}$$

$$+ \left[\left(\underline{G}_m^s \right)' \underline{A}_\theta^t \right] \left\{ \left[\underline{D}_{(2)} \underline{A}_\theta \underline{G}_m^r - \underline{D}_{\psi 1} \underline{A}_\psi \underline{G}_m^r \right] \Delta \underline{\delta}_o^r + \left[\underline{D}_{(2)} \left(\underline{B}_b^r + \underline{A}_m \underline{G}_\theta^r \right) \right] \Delta \underline{\delta}_b^r \right.$$

$$+ \left[\left(\underline{G}_m^s \right)' \underline{A}_\psi^t \right] \left\{ \left[- \underline{D}_{\psi 1} \underline{A}_\theta \underline{G}_m^r + \underline{D}_{\psi\psi} \underline{A}_\psi \underline{G}_m^r \right] \Delta \underline{\delta}_o^r + \left[- \underline{D}_{\psi 1} \left(\underline{B}_b^r + \underline{A}_m \underline{G}_\theta^r \right) \right] \Delta \underline{\delta}_b^r \right.$$

$$+\left[\underline{D}_{\psi\psi}\left(\underline{B}_{\psi}^{r}+\underline{A}_{m}\underline{G}_{\psi}^{r}\right)\right]\Delta\underline{\delta}_{\psi}^{r}\Big\}+\left(\underline{G}_{m}^{s}\right)'\underline{S}_{(0)}\underline{G}_{m}^{r}\Delta\underline{\delta}_{o}^{r}+\left(\underline{G}_{m}^{s}\right)'\underline{S}_{(1)}\underline{G}_{\theta}^{r}\Delta\underline{\delta}_{b}^{r}$$

$$+\left(\underline{G}_{m}^{s}\right)'\underline{S}_{\psi o}\underline{G}_{\psi}^{r}\Delta\underline{\delta}_{\psi}^{r}\Big]\,dxdy=\underline{R}_{o}^{s} \tag{5.238}$$

which can be rewritten in the following matrix form:

$$\sum_{strips}\sum_{r=1}^{m}\left[\left(\underline{K}_{oo}^{sr}+\underset{oo}{\underline{K}_{\sigma}^{sr}}\right)\Delta\underline{\delta}_{o}^{r}+\left(\underline{K}_{ob}^{sr}+\underset{ob}{\underline{K}_{\sigma}^{sr}}\right)\Delta\underline{\delta}_{b}^{r}+\left(\underline{K}_{o\psi}^{sr}+\underset{o\psi}{\underline{K}_{\sigma}^{sr}}\right)\Delta\underline{\delta}_{\psi}^{r}\right]=\underline{R}_{o}^{s} \tag{5.239}$$

$s = 1,2, \ldots\ldots ,m$

where for a finite strip:

$$\underline{K}_{oo}^{sr}=\iint_{strip}\Big\{\left(\underline{B}_{o}^{s}\right)'+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{m}^{t}\left[\underline{D}_{(o)}\left(\underline{B}_{o}^{r}+\underline{A}_{m}\underline{G}_{m}^{r}\right)\right]$$

$$+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\theta}^{t}\left[\underline{D}_{(2)}\underline{A}_{\theta}\underline{G}_{m}^{r}-\underline{D}_{\psi 1}\underline{A}_{\psi}\underline{G}_{m}^{r}\right]+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\psi}^{t}\left[-\underline{D}_{\psi 1}\underline{A}_{\theta}\underline{G}_{m}^{r}+\underline{D}_{\psi\psi}\underline{A}_{\psi}\underline{G}_{m}^{r}\right]\Big\}dxdy \tag{5.240}$$

$$\underline{K}_{ob}^{sr}=\iint_{strip}\Big\{\left(\underline{B}_{o}^{s}\right)'+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{m}^{t}\left[\underline{D}_{(o)}\underline{A}_{w}\underline{G}_{w}^{r}-\underline{D}_{\psi 1}\underline{A}_{\psi}\underline{G}_{\theta}^{r}+\underline{D}_{(2)}\underline{A}_{\theta}\underline{G}_{\theta}^{r}\right]$$

$$+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\theta}^{t}\left[\underline{D}_{(2)}\left(\underline{B}_{b}^{r}+\underline{A}_{m}\underline{G}_{\theta}^{r}\right)\right]+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\psi}^{t}\left[-\underline{D}_{\psi 1}\left(\underline{B}_{b}^{r}+\underline{A}_{m}\underline{G}_{\theta}^{r}\right)\right]\Big\}dxdy \tag{5.241}$$

$$\underline{K}_{o\psi}^{sr}=\iint_{strip}\Big\{\left[\left(\underline{B}_{o}^{s}\right)'+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{m}^{t}\right]\left[-\underline{D}_{\psi 1}\underline{A}_{\theta}\underline{G}_{\psi}^{r}+\underline{D}_{\psi\psi}\underline{A}_{\psi}\underline{G}_{\psi}^{r}\right]$$

$$+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\theta}^{t}\left[-\underline{D}_{\psi 1}\left(\underline{B}_{\psi}^{r}+\underline{A}_{m}\underline{G}_{\psi}^{r}\right)\right]+\left(\underline{G}_{m}^{s}\right)'\underline{A}_{\psi}^{t}\left[\underline{D}_{\psi\psi}\left(\underline{B}_{\psi}^{r}+\underline{A}_{m}\underline{G}_{\psi}^{r}\right)\right]\Big\}dxdy$$

(5.242)

$$\underset{oo}{\underline{K}_{\sigma}^{sr}}=\iint_{strip}\left(\underline{G}_{m}^{s}\right)'\underline{S}_{(o)}\underline{G}_{m}^{r}\,dxdy \tag{5.243}$$

$$\underset{ob}{\underline{K}_{\sigma}^{sr}}=\iint_{strip}\left(\underline{G}_{m}^{s}\right)'\underline{S}_{(1)}\underline{G}_{\theta}^{r}\,dxdy \tag{5.244}$$

$$\underset{o\psi}{\underline{K}_{\sigma}^{sr}}=\iint_{strip}\left(\underline{G}_{m}^{s}\right)'\underline{S}_{\psi o}\underline{G}_{\psi}^{r}\,dxdy \tag{5.245}$$

(ii) Simplification of equation (5.225)

Ignoring the terms of $\Delta\underline{A} \bullet \Delta\underline{\sigma}$, then it can be simplified as follows:

$$\sum_{strips}\int_{-h/2}^{h/2}\iint_{strip}\Big[\Big\{\Big(\underline{G}_w^s\Big)^t\underline{A}_w^t\Big]+\Big[\Big(\underline{B}_b^s\Big)^t+\Big(\underline{G}_\theta^s\Big)^t\underline{A}_m^t\Big](-z)+\Big(\underline{G}_\theta^s\Big)^t\underline{A}_\psi^t(-zf_\psi)$$
$$+\Big(\underline{G}_\theta^s\Big)^t\underline{A}_\theta^t(z^2)\Big\}\Delta\underline{\sigma}^{(l)}+\Big\{\Big(\underline{G}_w^s\Big)^t\Delta\underline{A}_w^t+\Big(\underline{G}_\theta^s\Big)^t\Delta\underline{A}_m^t(-z)+\Big(\underline{G}_\theta^s\Big)^t\Delta\underline{A}_\psi^t(-zf_\psi)$$
$$+\Big(\underline{G}_\theta^s\Big)^t\Delta\underline{A}_\theta^t(z^2)\Big\}\underline{\sigma}^{(l)}\Big]dxdydz=\underline{R}_b^s \qquad (5.246)$$

Notice that $\int_{-h/2}^{h/2}\Delta\underline{\sigma}^{(l)}\,dz$ is as defined by equation (5.230) and $\int_{-h/2}^{h/2}(-z)\Delta\underline{\sigma}^{(l)}\,dz$ is as defined by equation (5.231). New terms are required and are derived as follows:

$$\int_{-h/2}^{h/2}z^2\Delta\underline{\sigma}^{(l)}dz=\sum_{r=1}^{m}\Big\{\Big[\underline{D}_{(2)}\Big(\underline{B}_o^r+\underline{A}_m\underline{G}_m^r\Big)-\underline{D}_{(3)}\Big(\underline{A}_\theta\underline{G}_m^r\Big)+\underline{D}_{\psi 2}\Big(\underline{A}_\psi\underline{G}_m^r\Big)\Big]\Delta\underline{\delta}_o^r$$
$$+\Big[\underline{D}_{(2)}\underline{A}_w\underline{G}_w^r-\underline{D}_{(3)}\Big(\underline{B}_b^r+\underline{A}_m\underline{G}_\theta^r\Big)-\underline{D}_{\psi 3}\underline{A}_\psi\underline{G}_\theta^r+\underline{D}_{(4)}\underline{A}_\theta\underline{G}_\theta^r\Big]\Delta\underline{\delta}_b^r$$
$$+\Big[\underline{D}_{\psi 2}\Big(\underline{B}_\psi^r+\underline{A}_m\underline{G}_\psi^r\Big)-\underline{D}_{\psi 3}\underline{A}_\theta\underline{G}_\psi^r+\underline{D}_{\psi\psi 2}\underline{A}_\psi\underline{G}_\psi^r\Big]\Delta\underline{\delta}_\psi^r\Big\} \qquad (5.247)$$

$$\int_{-h/2}^{h/2}(-zf_\psi)\Delta\underline{\sigma}^{(l)}dz=\sum_{r=1}^{m}\Big\{\Big[-\underline{D}_{\psi 1}\Big(\underline{B}_o^r+\underline{A}_m\underline{G}_m^r\Big)+\underline{D}_{\psi 2}\Big(\underline{A}_\theta\underline{G}_m^r\Big)-\underline{D}_{\psi\psi 1}\Big(\underline{A}_\psi\underline{G}_m^r\Big)\Big]\Delta\underline{\delta}_o^r$$
$$+\Big[-\underline{D}_{\psi 1}\underline{A}_w\underline{G}_w^r+\underline{D}_{\psi 2}\Big(\underline{B}_b^r+\underline{A}_m\underline{G}_\theta^r\Big)+\underline{D}_{\psi\psi 2}\Big(\underline{A}_\psi\underline{G}_\theta^r\Big)-\underline{D}_{\psi 3}\Big(\underline{A}_\theta\underline{G}_\theta^r\Big)\Big]\Delta\underline{\delta}_b^r$$
$$+\Big[-\underline{D}_{\psi\psi 1}\Big(\underline{B}_\psi^r+\underline{A}_m\underline{G}_\psi^r\Big)+\underline{D}_{\psi\psi 2}\Big(\underline{A}_\theta\underline{G}_\psi^r\Big)-\underline{D}_{\psi\psi\psi 1}\Big(\underline{A}_\psi\underline{G}_\psi^r\Big)\Big]\Delta\underline{\delta}_\psi^r\Big\} \qquad (5.248)$$

with $$\underline{D}_{\psi\psi\psi m}\equiv\int_{-h/2}^{h/2}Z^n f_\psi^3(z)\underline{D}^{(l)}\,dz \qquad (5.249)$$

Notice also that, it can be proved by matrix manipulations:

(1) $$\int_{-h/2}^{h/2}\Delta\underline{A}_w^t\underline{\sigma}^{(l)}dz=\Delta\underline{A}_w^t\underline{\sigma}_{(o)}\equiv\underline{S}_w\Delta\underline{\theta}_w=\sum_{r=1}^{m}\underline{S}_w\underline{G}_w^r\Delta\underline{\delta}_b^r \qquad (5.250)$$

where

$$\underline{S}_w\equiv\begin{bmatrix}\sigma_x^\circ & \tau_{xy}^\circ\\ \tau_{xy}^\circ & \sigma_y^\circ\end{bmatrix} \qquad (5.251)$$

(2) $$\int_{-h/2}^{h/2} \Delta \underline{A}_m^t(-z)\underline{\sigma}^{(l)}dz = \Delta \underline{A}_m^t \underline{\sigma}_{(1)} \equiv \underline{S}_{(1)}\Delta\underline{\theta}_m = \sum_{r=1}^{m} \underline{S}_{(1)}\underline{G}_m^r \Delta\underline{\delta}_o^r \tag{5.252}$$

(3) $$\int_{-h/2}^{h/2} \Delta \underline{A}_\psi^t(-zf_\psi)\underline{\sigma}^{(l)}dz = \Delta \underline{A}_\psi^t \underline{\sigma}_{\psi 1} \equiv \underline{S}_{\psi 1}\Delta\underline{\theta}_\psi = \sum_{r=1}^{m} \underline{S}_{\psi 1}\underline{G}_\psi^r \Delta\underline{\delta}_\psi^r \tag{5.253}$$

where $\underline{S}_{\psi 1}$ is formed from the components of $\underline{\sigma}_{\psi 1}$ in a way similar to equation (5.234).

(4) $$\int_{-h/2}^{h/2} \Delta \underline{A}_\theta^t(z^2)\underline{\sigma}^{(l)}dz = \Delta \underline{A}_\theta^t \underline{\sigma}_2 \equiv \underline{S}_{(2)}\Delta\underline{\theta}_\theta = \sum_{r=1}^{m} \underline{S}_{(2)}\underline{G}_\theta^r \Delta\underline{\delta}_b^r \tag{5.254}$$

where $\underline{S}_{(2)}$ is formed from the components of $\underline{\sigma}_{(2)}$ in a way similar to $\underline{S}_{(o)}$ of equation (5.234).

Similarly, for the analysis of symmetric composites, it can be shown in *Appendix A* that $\underline{D}_{(1)} = \underline{D}_{(3)} = \underline{D}_{\psi o} = \underline{D}_{\psi 2} = \underline{D}_{\psi\psi 1} = \underline{D}_{\psi\psi\psi} = 0$.
Hence, equation (5.246) can be written after integration with respect to z as follows:

$$\begin{aligned}
&\sum_{strips}\sum_{r=1}^{m}\iint_{strip}\Big[\left(\underline{G}_w^s\right)^t \underline{A}_w^t\Big\{\Big[\underline{D}_{(o)}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right)\Big]\Delta\underline{\delta}_o^r \\
&+\Big[\underline{D}_{(0)}\underline{A}_w\underline{G}_w^r - \underline{D}_{\psi 1}\underline{A}_\psi\underline{G}_\theta^r + \underline{D}_{(2)}\underline{A}_\theta\underline{G}_\theta^r\Big]\Delta\underline{\delta}_b^r \\
&+\Big[-\underline{D}_{\psi 1}\underline{A}_\theta\underline{G}_\psi^r + \underline{D}_{\psi\psi}\underline{A}_\psi\underline{G}_\psi^r\Big]\Delta\underline{\delta}_\psi^r\Big\} + \Big[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t\underline{A}_m^t\Big] \\
&\Big\{\Big[\underline{D}_{(2)}\underline{A}_\theta\underline{G}_m^r - \underline{D}_{\psi 1}\underline{A}_\psi\underline{G}_m^r\Big]\Delta\underline{\delta}_o^r + \Big[\underline{D}_{(2)}\left(\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right) + \Big[-\underline{D}_{\psi 1}\left(\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right)\Big]\Delta\underline{\delta}_\psi^r\Big\} \\
&+\left(\underline{G}_\theta^s\right)^t\underline{A}_\psi^t\Big\{\Big[-\underline{D}_{\psi 1}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right)\Big]\Delta\underline{\delta}_o^r + \Big[-\underline{D}_{\psi 1}\underline{A}_w\underline{G}_w^r + \underline{D}_{\psi\psi 2}\underline{A}_\psi\underline{G}_\theta^r - \underline{D}_{\psi 3}\underline{A}_\theta\underline{G}_\theta^r\Big]\Delta\underline{\delta}_b^r \\
&+\Big[\underline{D}_{\psi\psi 2}\underline{A}_\theta\underline{G}_\psi^r - \underline{D}_{\psi\psi\psi 1}\underline{A}_\psi\underline{G}_\psi^r\Big]\Delta\underline{\delta}_\psi^r\Big\} + \left(\underline{G}_\theta^s\right)^t\underline{A}_\theta^t\Big\{\Big[\underline{D}_{(2)}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right) \\
&+\Big[\underline{D}_{(2)}\underline{A}_w\underline{G}_w^r - \underline{D}_{\psi 3}\underline{A}_\psi\underline{G}_\theta^r + \underline{D}_{(4)}\underline{A}_\theta\underline{G}_\theta^r\Big]\Delta\underline{\delta}_b^r \\
&+\Big[-\underline{D}_{\psi 3}\underline{A}_\theta\underline{G}_\psi^r + \underline{D}_{\psi\psi 2}\underline{A}_\psi\underline{G}_\psi^r\Big]\Delta\underline{\delta}_\psi^r\Big\} + \left(\underline{G}_w^s\right)^t\underline{S}_w\underline{G}_w^r\Delta\underline{\delta}_b^r \\
&+\left(\underline{G}_\theta^s\right)^t\underline{S}_{(1)}\underline{G}_m^r\Delta\underline{\delta}_o^r + \left(\underline{G}_\theta^s\right)^t\underline{S}_{(2)}\underline{G}_\theta^r\Delta\underline{\delta}_b^r + \left(\underline{G}_\theta^s\right)^t\underline{S}_{\psi 1}\underline{G}_\psi^r\Delta\underline{\delta}_\psi^r\Big]dxdy = \underline{R}_b^s
\end{aligned} \tag{5.255}$$

which can be rewritten in the following matrix form:

$$\sum_{strips}\sum_{r=1}^{m}\left[\left(\underline{K}_{bo}^{sr} + \underline{K}_{\sigma\, bo}^{sr}\right)\Delta\underline{\delta}_o^r + \left(\underline{K}_{bb}^{sr} + \underline{K}_{\sigma\, bb}^{sr}\right)\Delta\underline{\delta}_b^r + \left(\underline{K}_{b\psi}^{sr} + \underline{K}_{\sigma\, b\psi}^{sr}\right)\Delta\underline{\delta}_\psi^r\right] = \underline{R}_b^s \tag{5.256}$$

$s = 1,2, \ldots\ldots, m$

where for a strip:

$$\underline{K}_{bo}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_w^s\right)^t \underline{A}_w^t \left[\underline{D}_{(o)} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r\right)\right] + \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t \right] \left[\underline{D}_{(2)} \underline{A}_\theta \underline{G}_m^r - \underline{D}_{\psi 1} \underline{A}_\psi \underline{G}_m^r\right] \right.$$

$$\left. + \left(\underline{G}_\theta^s\right)^t \underline{A}_\psi^t \left[-\underline{D}_{\psi 1} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r\right)\right] + \left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t \left[\underline{D}_{(2)} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r\right)\right] \right\} dxdy \qquad (5.257)$$

$$\underline{K}_{bb}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_w^s\right)^t \underline{A}_w^t \left[\underline{D}_{(o)} \underline{A}_w \underline{G}_w^r - \underline{D}_{\psi 1} \underline{A}_\psi \underline{G}_\theta^r + \underline{D}_{(2)} \underline{A}_\theta \underline{G}_\theta^r \right] \right.$$

$$+ \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t \right] \left[\underline{D}_{(2)} \left(\underline{B}_b^r + \underline{A}_m \underline{G}_\theta^r\right)\right] + \left(\underline{G}_\theta^s\right)^t \underline{A}_\psi^t \left[-\underline{D}_{\psi 1} \underline{A}_w \underline{G}_w^r + \underline{D}_{\psi\psi 2} \underline{A}_\psi \underline{G}_\theta^r - \underline{D}_{\psi 3} \underline{A}_\theta \underline{G}_\theta^r\right]$$

$$\left. + \left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t \left[\underline{D}_{(2)} \underline{A}_w \underline{G}_w^r - \underline{D}_{\psi 3} \underline{A}_\psi \underline{G}_\theta^r + \underline{D}_{(4)} \underline{A}_\theta \underline{G}_\theta^r\right] \right\} dxdy \qquad (5.258)$$

$$\underline{K}_{b\psi}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_w^s\right)^t \underline{A}_w^t \left[-\underline{D}_{\psi 1} \underline{A}_\theta \underline{G}_\psi^r + \underline{D}_{\psi\psi} \underline{A}_\psi \underline{G}_\psi^r\right] + \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t \right] \left[-\underline{D}_{\psi 1} \left(\underline{B}_\psi^r + \underline{A}_m \underline{G}_\psi^r\right)\right] \right.$$

$$\left. + \left(\underline{G}_\theta^s\right)^t \underline{A}_\psi^t \left[\underline{D}_{\psi\psi 2} \underline{A}_\theta \underline{G}_\psi^r - \underline{D}_{\psi\psi\psi 1} \underline{A}_\psi \underline{G}_\psi^r\right] + \left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t \left[-\underline{D}_{\psi 3} \underline{A}_\theta \underline{G}_\psi^r + \underline{D}_{\psi\psi 2} \underline{A}_\psi \underline{G}_\psi^r\right] \right\} \qquad (5.259)$$

$$\underline{K}_{\substack{\sigma \\ bo}}^{sr} = \iint_{strip} \left(\underline{G}_\theta^s\right)^t \underline{S}_{(1)} \underline{G}_m^r \, dxdy \qquad (5.260)$$

$$\underline{K}_{\substack{\sigma \\ bb}}^{sr} = \iint_{strip} \left[\left(\underline{G}_w^s\right)^t \underline{S}_w \underline{G}_w^r + \left(\underline{G}_\theta^s\right)^t \underline{S}_{(2)} \underline{G}_\theta^r \right] dxdy \qquad (5.261)$$

$$\underline{K}_{\substack{\sigma \\ b\psi}}^{sr} = \iint_{strip} \left(\underline{G}_\theta^s\right)^t \underline{S}_{\psi 1} \underline{G}_\psi^r \, dxdy \qquad (5.262)$$

(iii) Simplification of equation (5.226)

Ignoring the terms containing $\Delta\underline{A} \bullet \Delta\underline{\sigma}$, then equation (5.226) can be simplified as follows:

$$\sum_{strips} \int_{-h/2}^{h/2} \iint_{strip} \left[\left\{ \left(\underline{B}_\gamma^s\right)^t f_\gamma(z) \Delta\underline{\tau}^{(l)} + \left\{ \left[\left(\underline{B}_\psi^s\right)^t + \left(\underline{G}_\psi^s\right)^t \underline{A}_m^t \right] \left(f_\psi(z)\right) + \left[\left(\underline{G}_\psi^s\right)^t \underline{A}_\theta^t \left(-z f_\psi\right)\right] \right.\right.$$

$$+ \left[\left(\underline{G}_\psi^s\right)^t \underline{A}_\psi^t \left(f_\psi^2\right)\right] \left.\right\} \Delta\underline{\sigma}^{(l)} + \left\{ \left[\left(\underline{G}_\psi^s\right)^t \Delta\underline{A}_m^t f_\psi(z)\right] + \left[\left(\underline{G}_\psi^s\right)^t \Delta\underline{A}_\theta^t \left(-z f_\psi\right)\right] \right.$$

$$\left.\left. + \left[\left(\underline{G}_\psi^s\right)^t \Delta\underline{A}_\psi^t \left(f_\psi^2\right)\right] \right\} \underline{\sigma}^{(l)} \right] dxdydz = \underline{R}_\psi^s \qquad (5.263)$$

Notice that:

$\int_{-h/2}^{h/2} f_\psi(z)\,\Delta\underline{\sigma}^{(l)}\,dz$ is defined by equation (5.232)

$\int_{-h/2}^{h/2} \left(-zf_\psi\right)\Delta\underline{\sigma}^{(l)}\,dz$ is defined by equation (5.248)

Using equation (5.229), it can be deduced that:

$$\int_{-h/2}^{h/2} f_\psi^2(z)\Delta\underline{\sigma}^{(l)}dz = \sum_{r=1}^{m}\left\{\left[\underline{D}_{\psi\psi}\left(\underline{B}_o^r + \underline{A}_m\underline{G}_m^r\right) - \underline{D}_{\psi\psi 1}\left(\underline{A}_\theta\underline{G}_m^r\right) + \underline{D}_{\psi\psi\psi}\left(\underline{A}_\psi\underline{G}_m^r\right)\right]\Delta\underline{\delta}_o^r\right.$$

$$+\left[\underline{D}_{\psi\psi}\underline{A}_w\underline{G}_w^r - \underline{D}_{\psi\psi 1}\left(\underline{B}_b^r + \underline{A}_m\underline{G}_\theta^r\right) - \underline{D}_{\psi\psi\psi 1}\underline{A}_\psi\underline{G}_\theta^r + \underline{D}_{\psi\psi 2}\underline{A}_\theta\underline{G}_\theta^r\right]\Delta\underline{\delta}_b^r$$

$$\left.+\left[\underline{D}_{\psi\psi\psi}\left(\underline{B}_\psi^r + \underline{A}_m\underline{G}_\psi^r\right) - \underline{D}_{\psi\psi\psi 1}\underline{A}_\theta\underline{G}_\psi^r + \underline{D}_{\psi\psi\psi\psi}\underline{A}_\psi\underline{G}_\psi^r\right]\Delta\underline{\delta}_\psi^r\right. \tag{5.264}$$

where

$$\underline{D}_{\psi\psi\psi\psi} \equiv \int_{-h/2}^{h/2} f_\psi^4(z)\,\underline{D}^{(l)}\,dz \tag{5.265}$$

Notice also that:

$$\int_{-h/2}^{h/2}\left(\underline{B}_\gamma^s\right)' f_\gamma(z)\,\Delta\underline{\tau}^{(l)}\,dz = \sum_{r=1}^{m}\int_{-h/2}^{h/2}\left(\underline{B}_\gamma^s\right)' f_\gamma^2\,\underline{\mu}^{(l)}(z)\,\underline{B}_\gamma^r\,\Delta\underline{\delta}_\psi^r\,dz$$

$$= \sum_{r=1}^{m}\left(\underline{B}_\gamma^s\right)'\underline{\mu}_{\gamma\gamma}\,\underline{B}_\gamma^r\,\Delta\underline{\delta}_\psi^r\,dz \tag{5.266}$$

By matrix manipulation, it can also be proved that:

(1) $$\int_{-h/2}^{h/2}\Delta\underline{A}_m^t\,f_\psi(z)\,\underline{\sigma}^{(l)}dz = \Delta\underline{A}_m^t\,\underline{\sigma}_{\psi o} \equiv \underline{S}_{\psi o}\,\Delta\underline{\theta}_m = \sum_{r=1}^{m}\underline{S}_{\psi o}\,\underline{G}_m^r\,\Delta\underline{\delta}_o^r \tag{5.267}$$

where $\underline{\sigma}_{\psi o} = \int_{-h/2}^{h/2} f_\psi(z)\,\underline{\sigma}^{(l)}\,dz$, and $\underline{S}_{\psi o}$ is formed from the components of $\underline{\sigma}_{\psi o}$ in a way similar to the formation of $\underline{S}_{(o)}$ as given by equation (5.234).

(2) $$\int_{-h/2}^{h/2}\Delta\underline{A}_\theta^t\left(-zf_\psi\right)\underline{\sigma}^{(l)}dz = \Delta\underline{A}_\theta^t\,\underline{\sigma}_{\psi 1} \equiv \underline{S}_{\psi 1}\,\Delta\underline{\theta}_\theta = \sum_{r=1}^{m}\underline{S}_{\psi 1}\,\underline{G}_\theta^r\,\Delta\underline{\delta}_b^r \tag{5.268}$$

where $\underline{\sigma}_{\psi 1} = \int_{-h/2}^{h/2} (-z f_\psi)\underline{\sigma}^{(l)}\, dz$, and $\underline{S}_{\psi 1}$ is formed from the components of $\underline{\sigma}_{\psi 1}$ in a way similar to the formation of $\underline{S}_{(o)}$ as given by equation (5.234).

(3) $$\int_{-h/2}^{h/2} \Delta\underline{A}^t_\psi f^2_\psi(z)\underline{\sigma}^{(l)} dz = \Delta\underline{A}^t_\psi \underline{\sigma}_{\psi\psi} \equiv \underline{S}_{\psi\psi}\,\Delta\underline{\theta}_\psi = \sum_{r=1}^{m} \underline{S}_{\psi\psi}\,\underline{G}^r_\psi\,\Delta\underline{\delta}^r_\psi \qquad (5.269)$$

where $$\underline{\sigma}_{\psi\psi} = \int_{-h/2}^{h/2} f^2_\psi\,\underline{\sigma}^{(l)}\, dz \equiv \left\{\sigma^{\psi\psi}_x \quad \sigma^{\psi\psi}_y \quad \tau^{\psi\psi}_{xy}\right\} \qquad (5.270)$$

and

$$\underline{S}_{\psi\psi} = \begin{bmatrix} \sigma^{\psi\psi}_x & 0 & \tau^{\psi\psi}_{xy} & 0 \\ 0 & \sigma^{\psi\psi}_x & 0 & \tau^{\psi\psi}_{xy} \\ \tau^{\psi\psi}_{xy} & 0 & \sigma^{\psi\psi}_y & 0 \\ 0 & \tau^{\psi\psi}_{xy} & 0 & \sigma^{\psi\psi}_y \end{bmatrix} \qquad (5.271)$$

Similarly, for the analysis of symmetric composites, it can be shown in *Appendix A* that $\underline{D}_{(1)} = \underline{D}_{(3)} = \underline{D}_{\psi o} = \underline{D}_{\psi 2} = \underline{D}_{\psi\psi 1} = \underline{D}_{\psi\psi\psi} = 0$.

Substituting the relevant integrations with respect to z into equation (5.263), it can be rewritten as follows:

$$\begin{aligned}
&\sum_{strips}\sum_{r=1}^{m}\iint_{strip}\Big[\left(\underline{B}^s_\gamma\right)^{\prime}\underline{\mu}_{\gamma\gamma}\underline{B}^r_\gamma\Delta\underline{\delta}^r_\psi + \left[\left(\underline{B}^s_\psi\right)^{\prime} + \left(\underline{G}^s_\psi\right)^{\prime}\underline{A}^t_m\right]\Big\{\left[-\underline{D}_{\psi 1}\underline{A}_\theta\underline{G}^r_m + \underline{D}_{\psi\psi}\underline{A}_\psi\underline{G}^r_m\right]\Delta\underline{\delta}^r_o \\
&+\left[-\underline{D}_{\psi 1}\left(\underline{B}^r_b + \underline{A}_m\underline{G}^r_\theta\right)\right]\Delta\underline{\delta}^r_b + \left[\underline{D}_{\psi\psi}\left(\underline{B}^r_\psi + \underline{A}_m\underline{G}^r_\psi\right)\right]\Delta\underline{\delta}^r_\psi\Big\} \\
&+\left[\left(\underline{G}^s_\psi\right)^{\prime}\underline{A}^t_\theta\right]\Big\{\left[-\underline{D}_{\psi 1}\left(\underline{B}^r_o + \underline{A}_m\underline{G}^r_m\right)\right]\Delta\underline{\delta}^r_o + \left[-\underline{D}_{\psi 1}\underline{A}_w\underline{G}^r_w + \underline{D}_{\psi\psi 2}\underline{A}_\psi\underline{G}^r_\theta - \underline{D}_{\psi 3}\underline{A}_\theta\underline{G}^r_\theta\right]\Delta\underline{\delta}^r_b \\
&+\left[\underline{D}_{\psi\psi 2}\underline{A}_\theta\underline{G}^r_\psi - \underline{D}_{\psi\psi\psi 1}\underline{A}_\psi\underline{G}^r_\psi\right]\Delta\underline{\delta}^r_\psi\Big\} + \left(\underline{G}^s_\psi\right)^{\prime}\underline{A}^t_\psi\Big\{\left[\underline{D}_{\psi\psi}\left(\underline{B}^r_o + \underline{A}_m\underline{G}^r_m\right)\right]\Delta\underline{\delta}^r_o \\
&+\left[\underline{D}_{\psi\psi}\underline{A}_w\underline{G}^r_w - \underline{D}_{\psi\psi\psi 1}\underline{A}_\psi\underline{G}^r_\theta + \underline{D}_{\psi\psi 2}\underline{A}_\theta\underline{G}^r_\theta\right]\Delta\underline{\delta}^r_b + \left[-\underline{D}_{\psi\psi\psi 1}\underline{A}_\theta\underline{G}^r_\psi + \underline{D}_{\psi\psi\psi\psi}\underline{A}_\psi\underline{G}^r_\psi\right]\Delta\underline{\delta}^r_\psi\Big\} \\
&+\left(\underline{G}^s_\psi\right)^{\prime}\underline{S}_{\psi o}\underline{G}^r_m\Delta\underline{\delta}^r_o + \left(\underline{G}^s_\psi\right)^{\prime}\underline{S}_{\psi 1}\underline{G}^r_\theta\Delta\underline{\delta}^r_b + \left(\underline{G}^s_\psi\right)^{\prime}\underline{S}_{\psi\psi}\underline{G}^r_\psi\Delta\underline{\delta}^r_\psi\Big]\, dxdy = \underline{R}^s_\psi
\end{aligned} \qquad (5.272)$$

for $s = 1,2, \ldots\ldots, m$

Equation (5.272) can be rewritten in the following matrix form:

$$\sum_{strips}\sum_{r=1}^{m}\left[\left(\underline{K}^{sr}_{\psi o} + \underline{K}^{sr}_{\sigma\,\psi o}\right)\Delta\underline{\delta}^r_o + \left(\underline{K}^{sr}_{\psi b} + \underline{K}^{sr}_{\sigma\,\psi b}\right)\Delta\underline{\delta}^r_b + \left(\underline{K}^{sr}_{\psi\psi} + \underline{K}^{sr}_{\sigma\,\psi\psi}\right)\Delta\underline{\delta}^r_\psi\right] = \underline{R}^s_\psi \qquad (5.273)$$

for $s = 1,2, \ldots\ldots, m$, where for a finite strip:

$$\underline{K}^{sr}_{\psi o} = \iint_{strip} \{ \left[\left(\underline{B}^s_{\psi} \right)' + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_m \right] \left[-\underline{D}_{\psi 1} \underline{A}_{\theta} \underline{G}^r_m + \underline{D}_{\psi\psi} \underline{A}_{\psi} \underline{G}^r_m \right] + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_{\theta} \left[-\underline{D}_{\psi 1} \left(\underline{B}^r_o + \underline{A}_m \underline{G}^r_m \right) \right]$$

$$+ \left(\underline{G}^s_{\psi} \right)^t \underline{A}^t_{\psi} \left[\underline{D}_{\psi\psi} \left(\underline{B}^r_o + \underline{A}_m \underline{G}^r_m \right) \right] \} dxdy \qquad (5.274)$$

$$\underline{K}^{sr}_{\psi b} = \iint_{strip} \{ \left[\left(\underline{B}^s_{\psi} \right)' + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_m \right] \left[-\underline{D}_{\psi 1} \left(\underline{B}^r_b + \underline{A}_m \underline{G}^r_{\theta} \right) \right] + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_{\theta} \left[-\underline{D}_{\psi 1} \underline{A}_w \underline{G}^r_w \right.$$

$$\left. + \underline{D}_{\psi\psi 2} \underline{A}_{\psi} \underline{G}^r_{\theta} - \underline{D}_{\psi 3} \underline{A}_{\theta} \underline{G}^r_{\theta} \right] + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_{\psi} \left[\underline{D}_{\psi\psi} \underline{A}_w \underline{G}^r_w - \underline{D}_{\psi\psi\psi 1} \underline{A}_{\psi} \underline{G}^r_{\theta} + \underline{D}_{\psi\psi 2} \underline{A}_{\theta} \underline{G}^r_{\theta} \right] \} dxdy \qquad (5.275)$$

$$\underline{K}^{sr}_{\psi\psi} = \iint_{strip} \{ \left[\left(\underline{B}^s_{\psi} \right)' + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_m \right] \left[\underline{D}_{\psi\psi} \left(\underline{B}^r_{\psi} + \underline{A}_m \underline{G}^r_{\psi} \right) \right] + \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_{\theta} \left[\underline{D}_{\psi\psi 2} \underline{A}_{\theta} \underline{G}^r_{\psi} - \underline{D}_{\psi\psi\psi 1} \underline{A}_{\psi} \underline{G}^r_{\psi} \right]$$

$$+ \left(\underline{G}^s_{\psi} \right)' \underline{A}^t_{\theta} \left[-\underline{D}_{\psi\psi\psi 1} \underline{A}_{\theta} \underline{G}^r_{\psi} - \underline{D}_{\psi\psi\psi\psi} \underline{A}_{\psi} \underline{G}^r_{\psi} \right] \} dxdy + \underline{K}^{sr}_{\gamma\gamma} \qquad (5.276)$$

$$\underline{K}^{sr}_{\gamma\gamma} = \iint_{strip} \left(\underline{B}^s_{\gamma} \right)' \underline{\mu}_{\gamma\gamma} \underline{B}^r_{\gamma} \, dxdy \qquad (5.277)$$

$$\underline{K}^{sr}_{\substack{\sigma \\ \psi o}} = \iint_{strip} \left(\underline{G}^s_{\psi} \right)' \underline{S}_{\psi o} \underline{G}^r_m \, dxdy \qquad (5.278)$$

$$\underline{K}^{sr}_{\substack{\sigma \\ \psi b}} = \iint_{strip} \left(\underline{G}^s_{\psi} \right)' \underline{S}_{\psi 1} \underline{G}^r_{\theta} \, dxdy \qquad (5.279)$$

$$\underline{K}^{sr}_{\substack{\sigma \\ \psi\psi}} = \iint_{strip} \left(\underline{G}^s_{\psi} \right)' \underline{S}_{\psi\psi} \underline{G}^r_{\psi} \, dxdy \qquad (5.280)$$

(iv) Assembled Equations

Equations (5.239), (5.256), (5.273) can be assembled in one matrix equation as follows:

$$\sum_{strips} \sum_{r=1}^{m} \left[\left(\underline{K}^{sr} + \underline{K}^{sr}_{\sigma} \right) \right] \Delta \underline{\delta}^r_o = \underline{R}^s \qquad (5.281)$$

for $s = 1,2, \ldots\ldots, m$

where for a strip:

$$\underline{K}^{sr} = \begin{bmatrix} \underline{K}^{sr}_{oo} & \underline{K}^{sr}_{ob} & \underline{K}^{sr}_{o\psi} \\ \underline{K}^{sr}_{bo} & \underline{K}^{sr}_{bb} & \underline{K}^{sr}_{b\psi} \\ \underline{K}^{sr}_{\psi o} & \underline{K}^{sr}_{\psi b} & \underline{K}^{sr}_{\psi\psi} \end{bmatrix} \tag{5.282}$$

Similarly,

$$\underline{K}^{sr}_{\sigma} = \begin{bmatrix} \underline{K}^{sr}_{\sigma\ oo} & \underline{K}^{sr}_{\sigma\ ob} & \underline{K}^{sr}_{\sigma\ o\psi} \\ \underline{K}^{sr}_{\sigma\ bo} & \underline{K}^{sr}_{\sigma\ bb} & \underline{K}^{sr}_{\sigma\ b\psi} \\ \underline{K}^{sr}_{\sigma\ \psi o} & \underline{K}^{sr}_{\sigma\ \psi b} & \underline{K}^{sr}_{\sigma\ \psi\psi} \end{bmatrix} \tag{5.283}$$

$$\Delta\underline{\delta}^{r} = \begin{bmatrix} \Delta\underline{\delta}^{r}_{o} \\ \Delta\underline{\delta}^{r}_{b} \\ \Delta\underline{\delta}^{r}_{\psi} \end{bmatrix} \tag{5.284a}$$

$$\underline{R}^{s} = \begin{bmatrix} \underline{R}^{s}_{o} \\ \underline{R}^{s}_{b} \\ \underline{R}^{s}_{\psi} \end{bmatrix} \tag{5.284b}$$

which can be assembled for all values of *s,r* in a way similar to that of the *Mindlin* theory as mentioned in chapter 4.

5.12 Special Cases of Large Deflection

5.12.1 Case Without ψ_x, ψ_y effect

This is the case when we consider the terms containing

$$\left(\frac{\partial \psi_x}{\partial x}\right)^2, \left(\frac{\partial \psi_x}{\partial y}\right)^2, \frac{\partial \psi_x}{\partial x}\frac{\partial \psi_x}{\partial y}, \ldots\ldots\ldots$$

are negligible i.e. equations (5.41)-(5.43) can be simplified as follows:

$$\underline{\varepsilon}_L = \underline{\varepsilon}_m + \underline{\varepsilon}_w - z\underline{\hat{\varepsilon}}_{m\theta} + z^2\underline{\hat{\varepsilon}}_{\theta} \tag{5.285}$$

where

$$\underline{\varepsilon}_m = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u^\circ}{\partial x}\right)^2 + \left(\frac{\partial v^\circ}{\partial x}\right)^2 \\ \left(\frac{\partial u^\circ}{\partial y}\right)^2 + \left(\frac{\partial v^\circ}{\partial y}\right)^2 \\ 2\left(\frac{\partial u^\circ}{\partial x}\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}\frac{\partial v^\circ}{\partial y}\right) \end{bmatrix} \equiv \frac{1}{2}\underline{A}_m\underline{\theta}_m \tag{5.286}$$

$$\underline{\varepsilon}_w = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial w}{\partial y}\right)^2 \\ 2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \end{bmatrix} \equiv \frac{1}{2}\underline{A}_w\underline{\theta}_w \tag{5.287}$$

$$\hat{\underline{\varepsilon}}_\theta = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\ \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \\ 2\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial y^2}\right) \end{bmatrix} \equiv \frac{1}{2}\underline{A}_\theta\underline{\theta}_\theta \tag{5.288}$$

$$\hat{\underline{\varepsilon}}_{m\theta}(x,y) = \begin{bmatrix} \frac{\partial u^\circ}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v^\circ}{\partial x}\frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial u^\circ}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial v^\circ}{\partial y}\frac{\partial^2 w}{\partial y^2} \\ \frac{\partial u^\circ}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial u^\circ}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v^\circ}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial v^\circ}{\partial y}\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

$$\equiv \underline{A}_m\underline{\theta}_\theta = \underline{A}_\theta\underline{\theta}_m = \frac{1}{2}\left(\underline{A}_m\underline{\theta}_\theta + \underline{A}_\theta\underline{\theta}_m\right) \tag{5.289}$$

This means that in previous derivations we cancel the terms containing $\underline{A}_\psi$, $\underline{G}^s_\psi$, $\underline{G}^r_\psi$. Hence, it can be deduced that (for symmetric composites):

$$\underline{K}^{sr}_{oo} = \iint_{strip}\left\{\left[\left(\underline{B}^s_o\right)' + \left(\underline{G}^s_m\right)'\underline{A}^t_m\right]\left[\underline{D}_{(o)}\left(\underline{B}^r_o + \underline{A}_m\underline{G}^r_m\right)\right] + \left(\underline{G}^s_m\right)'\underline{A}^t_\theta\left[\underline{D}_{(2)}\underline{A}_\theta\underline{G}^r_m\right]\right\}dxdy \tag{5.290}$$

$$\underline{K}_{ob}^{sr} = \iint_{strip} \left\{ \left[\left(\underline{B}_o^s\right)' + \left(\underline{G}_m^s\right)' \underline{A}_m^t \right] \left[\underline{D}_{(o)} \underline{A}_w \underline{G}_w^r + \underline{D}_{(2)} \underline{A}_\theta \underline{G}_\theta^r \right] \right.$$
$$\left. + \left(\underline{G}_m^s\right)' \underline{A}_\theta^t \left[\underline{D}_{(2)} \left(\underline{B}_b^r + \underline{A}_m \underline{G}_\theta^r \right) \right] \right\} dxdy \tag{5.291}$$

$$\underline{K}_{o\psi}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_m^s\right)' \underline{A}_\theta \left[-\underline{D}_{\psi 1} \underline{B}_\psi^r \right] \right\} dxdy \tag{5.292}$$

$$\underline{K}_{\substack{\sigma \\ oo}}^{sr} = \iint_{strip} \left(\underline{G}_m^s\right)' \underline{S}_{(o)} \underline{G}_m^r \, dxdy \tag{5.293}$$

$$\underline{K}_{\substack{\sigma \\ ob}}^{sr} = \iint_{strip} \left(\underline{G}_m^s\right)' \underline{S}_{(1)} \underline{G}_\theta^r \, dxdy \tag{5.294}$$

$$\underline{K}_{\substack{\sigma \\ o\psi}}^{sr} = \underline{0} \tag{5.295}$$

$$\underline{K}_{bo}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_w^s\right)' \underline{A}_w^t \left[\underline{D}_{(o)} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r \right) \right] + \left[\left(\underline{B}_b^s\right)' + \left(\underline{G}_\theta^s\right)' \underline{A}_m^t \right] \left[\underline{D}_{(2)} \underline{A}_\theta \underline{G}_m^r \right] \right.$$
$$\left. + \left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t \left[\underline{D}_{(2)} \left(\underline{B}_o^r + \underline{A}_m \underline{G}_m^r \right) \right] \right\} dxdy \tag{5.296}$$

$$\underline{K}_{bb}^{sr} = \iint_{strip} \left\{ \left(\underline{G}_w^s\right)^t \underline{A}_w^t \left[\underline{D}_{(o)} \underline{A}_w \underline{G}_w^r + \underline{D}_{(2)} \underline{A}_\theta \underline{G}_\theta^r \right] + \left[\left(\underline{B}_b^s\right)^t + \left(\underline{G}_\theta^s\right)^t \underline{A}_m^t \right] \underline{D}_{(2)} \left(\underline{B}_b^r + \underline{A}_m \underline{G}_\theta^r \right) \right.$$
$$\left. \left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t \left[\underline{D}_{(2)} \underline{A}_w \underline{G}_w^r + \underline{D}_{(4)} \underline{A}_\theta \underline{G}_\theta^r \right] \right\} dxdy \tag{5.297}$$

$$\underline{K}_{b\psi}^{sr} = \iint_{strip} \left\{ \left[\left(\underline{B}_b^s\right)' + \left(\underline{G}_\theta^s\right)' \underline{A}_m^t \right] \left[-\underline{D}_{\psi 1} \underline{B}_\psi^r \right] + \left(\underline{G}_\theta^s\right)' \underline{A}_\theta^t \left[\underline{D}_{\psi 2} \underline{B}_\psi^r \right] \right\} dxdy \tag{5.298}$$

$$\underline{K}_{\substack{\sigma \\ bo}}^{sr} = \iint_{strip} \left(\underline{G}_\theta^s\right)' \underline{S}_{(1)} \underline{G}_m^r \, dxdy \tag{5.299}$$

$$\underline{K}_{\substack{\sigma \\ bb}}^{sr} = \iint_{strip} \left[\left(\underline{G}_w^s\right)' \underline{S}_w \underline{G}_m^r + \left(\underline{G}_\theta^s\right)' \underline{S}_{(2)} \underline{G}_\theta^r \right] dxdy \tag{5.300}$$

$$\underline{K}_{\substack{\sigma \\ b\psi}}^{sr} = \underline{0} \tag{5.301}$$

$$\underline{K}_{\psi o}^{sr} = \iint_{strip} \left\{ \left(\underline{B}_\psi^s\right)' \left[-\underline{D}_{\psi 1} \underline{A}_\theta \underline{G}_m^r \right] \right\} dxdy \tag{5.302}$$

$$\underline{K}^{sr}_{\psi b} = \iint_{strip} \left\{ \left(\underline{B}^{s}_{\psi}\right)^{t} \left[-\underline{D}_{\psi 1} \left(\underline{B}^{r}_{b} + \underline{A}_{m} \underline{G}^{r}_{\theta} \right) \right] \right\} dxdy \tag{5.303}$$

$$\underline{K}^{sr}_{\psi\psi} = \iint_{strip} \left(\underline{B}^{s}_{\psi}\right)^{t} \underline{D}_{\psi\psi} \underline{B}^{r}_{\psi} \, dxdy + \underline{K}^{sr}_{\gamma\gamma} \tag{5.304}$$

$$\underline{K}^{sr}_{\gamma\gamma} = \iint_{strip} \left(\underline{B}^{s}_{\gamma}\right)^{t} \underline{\mu}_{\gamma\gamma} \underline{B}^{r}_{\gamma} \, dxdy \tag{5.305}$$

$$\underline{K}^{sr}_{\sigma\ \psi o} = \underline{K}^{sr}_{\sigma\ \psi b} = \underline{K}^{sr}_{\sigma\ \psi\psi} = \underline{0} \tag{5.306}$$

5.12.2 Case of Large Deflection due to Out-of-Plane Bending (w) only

In this case we only consider the effect of w on $\underline{\varepsilon}_{L}$, i.e. equations (5.38)-(5.40) are simplified as follows:

$$\begin{aligned} \varepsilon^{L}_{x} &= \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2} \\ \varepsilon^{L}_{y} &= \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2} \\ \gamma^{L}_{xy} &= \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned} \tag{5.307}$$

$$\text{i.e} \qquad \underline{\varepsilon}_{L} \equiv \underline{\varepsilon}_{w} = \frac{1}{2} \underline{A}_{w} \underline{G}_{w} \tag{5.308}$$

Hence, all terms containing $\underline{A}_{m}$, $\underline{G}_{m}$, $\underline{A}_{\theta}$, $\underline{G}_{\theta}$ in the equations from the previous derivations are negligible, i.e. the stiffness matrices terms can be reduced as follows:

$$\underline{K}^{sr}_{oo} = \iint_{strip} \left(\underline{B}^{s}_{o}\right)^{t} \underline{D}_{(o)} \underline{B}^{r}_{o} \, dxdy \tag{5.309}$$

$$\underline{K}^{sr}_{ob} = \iint_{strip} \left\{ \left(\underline{B}^{s}_{o}\right)^{t} \left[\underline{D}_{(o)} \underline{A}_{w} \underline{G}^{r}_{w} \right] \right\} dxdy \tag{5.310}$$

$$\underline{K}^{sr}_{o\psi} = \underline{K}^{sr}_{\sigma\ \psi o} = \underline{K}^{sr}_{\sigma\ \psi b} = \underline{K}^{sr}_{\sigma\ \psi\psi} = \underline{0} \tag{5.311}$$

$$\underline{K}^{sr}_{bo} = \iint_{strip} \left(\underline{G}^{s}_{w}\right)^{t} \underline{A}^{t}_{w} \underline{D}_{(o)} \underline{B}^{r}_{o} \, dxdy \tag{5.312}$$

$$\underline{K}^{sr}_{bb} = \iint_{strip} \left\{ \left(\underline{G}^{s}_{w}\right)^{t} \underline{A}^{t}_{w} \left[\underline{D}_{(o)} \underline{A}_{w} \underline{G}^{r}_{w} \right] + \left(\underline{B}^{s}_{b}\right)^{t} \left[\underline{D}_{(2)} \underline{B}^{r}_{b} \right] \right\} dxdy \tag{5.313}$$

$$\underline{K}^{sr}_{b\psi} = \iint_{strip} \left\{ \left(\underline{B}^{s}_{b}\right)^{t} \left[-\underline{D}_{\psi 1} \underline{B}^{r}_{\psi} \right] \right\} dxdy \tag{5.314}$$

$$\underline{K}^{sr}_{\substack{\sigma \\ bo}} = \underline{K}^{sr}_{\substack{\sigma \\ \psi o}} = \underline{K}^{sr}_{\psi o} = \underline{0} \tag{5.315}$$

$$\underline{K}^{sr}_{\substack{\sigma \\ bb}} = \iint_{strip} \left(\underline{G}^{s}_{w}\right)^{t} \underline{S}_{w} \underline{G}^{r}_{w} \, dxdy \tag{5.316}$$

$$\underline{K}^{sr}_{\psi\psi} = \iint_{strip} \left(\underline{B}^{s}_{\psi}\right)^{t} \underline{D}_{\psi\psi} \underline{B}^{r}_{\psi} \, dxdy + \underline{K}^{sr}_{\gamma\gamma} \tag{5.317}$$

$$\underline{K}^{sr}_{\gamma\gamma} = \iint_{strip} \left(\underline{B}^{s}_{\gamma}\right)^{t} \underline{\mu}_{\gamma\gamma} \underline{B}^{r}_{\gamma} \, dxdy \tag{5.318}$$

$$\underline{K}^{sr}_{\psi b} = \iint_{strip} \left\{ \left(\underline{B}^{s}_{\psi}\right)^{t} \left[-\underline{D}_{\psi 1} \underline{B}^{r}_{b} \right] \right\} dxdy \tag{5.319}$$

$$\underline{K}^{sr}_{\substack{\sigma \\ \psi o}} = \underline{K}^{sr}_{\substack{\sigma \\ \psi b}} = \underline{K}^{sr}_{\substack{\sigma \\ \psi\psi}} = \underline{0} \tag{5.320}$$

Notice that, in this case:

$$\underline{K}^{sr}_{\sigma} = \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K}^{sr}_{\substack{\sigma \\ bb}} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \tag{5.321}$$

CHAPTER 6

6. Progressive Failure Analysis

6.1 Introduction

This chapter begins with the summary of the faceted shell finite strip elements which are employed in the analysis of shell structures such as box structures and stiffened plates, as discussed by Razzaq, (2003). A linear and non-linear static stress analysis algorithm, based on the new derived equations given in chapters 4 and 5 will be presented. Finally, a progressive failure methodology implemented in the finite strip analysis will also be explained.

6.2 Facetted Shell Finite strip elements

6.2.1 Local and Global Axes

Each component (plate) forming a box structure or a stiffened plate may have different individual local axes (defined here as x_L, y_L, z_L axes) and it is advantageous to use just one unique global system of axes for the whole structure, such as that shown in figure 6.1. For finite strip analysis, the y-axis will remain the same for all axes systems, i.e. the local y-axis is assumed parallel to the global y-axis. As an example, a folded plate may have three different local axes and may be as shown in figure 6.2.

Local axes for a finite strip element are defined in terms of the two end points of the strip such that:

- The local x axis is from the first end to the second end.
- The local y-axis is in the midplane of the strip, and normal to the x-axis.
- The local z-axis is normal to the x_L - y_L plane (the midplane $z_L = 0$), across the thickness.

If (x_1, y_1, z_1), (x_2, y_2, z_2) are the global coordinates of the two end points of the strip on the x_L - z_L plane, i.e. with $y_1 = y_2 = 0$, then the angle β between the local x-axis and the global x-axis can be defined, as shown in figure 6.1, hence

$$\cos\beta = \frac{x_2 - x_1}{L}, \quad \sin\beta = \frac{z_2 - z_1}{L} \tag{6.1}$$

$$\text{where } L = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \tag{6.2}$$

which represents the length of the finite strip in the x_L – direction. Hence, the directional cosines (l_1, m_1, n_1) of the local x-axis with respect to the global axes can be defined as follows:

$$l_1 = \cos\beta, m_1 = 0, n_1 = \sin\beta \tag{6.3}$$

and from geometry, the directional cosines, (l_3, m_3, n_3) of the local z-axis are:

$$l_3 = -\sin\beta, m_3 = 0, n_3 = \cos\beta \tag{6.4}$$

Also, the directional cosines (l_2, m_2, n_2) of the local y-axis are:

$$l_2 = 0, m_2 = 1, n_2 = 0 \tag{6.5}$$

Hence, the rotation matrix $\underline{R}$ of the local axes can be defined as the following:

$$\underline{R} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{6.6}$$

6.2.2 Rotation of a vector

Notice that if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in the global x, y, z directions, and $\hat{i}_L, \hat{j}_L, \hat{k}_L$ are unit vectors in the local x_L, y_L, z_L directions then it can be deduced that:

$$\left.\begin{aligned} \hat{i}_L &= l_1\hat{i} + m_1\hat{j} + n_1\hat{k} \\ \hat{j}_L &= l_2\hat{i} + m_2\hat{j} + n_2\hat{k} \equiv \hat{j} \\ \hat{k}_L &= l_3\hat{i} + m_3\hat{j} + n_3\hat{k} \end{aligned}\right\} \tag{6.7}$$

which can be written in a matrix form as follows:

$$\begin{bmatrix} \hat{i}_L \\ \hat{j}_L \\ \hat{k}_L \end{bmatrix} = \underline{R} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{6.8}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \underline{R}^{t} \begin{bmatrix} \hat{i}_L \\ \hat{j}_L \\ \hat{k}_L \end{bmatrix} \tag{6.9}$$

$$\text{i.e.} \quad \underline{R}^{-1} = \underline{R}^{t} \tag{6.10}$$

A Cartesian vector $\vec{V}$ can be represented by its local components $\left(v_x^L, v_y^L, v_z^L\right)$ or global components $\left(v_x, v_y, v_z\right)$, i.e.

$$\vec{V} = v_x^L \underline{\hat{i}}_L + v_y^L \underline{\hat{j}}_L + v_z^L \underline{\hat{k}}_L \equiv v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \tag{6.11}$$

which can also be represented in a matrix form as follows

$$V = \begin{bmatrix} v_x^L & v_y^L & v_z^L \end{bmatrix} \begin{bmatrix} \hat{i}_L \\ \hat{j}_L \\ \hat{k}_L \end{bmatrix} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \equiv \begin{bmatrix} v_x^L & v_y^L & v_z^L \end{bmatrix} \underline{R} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{6.12}$$

Hence, it can be deduced that:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \underline{R}^{t} \begin{bmatrix} v_x^L \\ v_y^L \\ v_z^L \end{bmatrix} \tag{6.13}$$

Similarly it can be proved that:

$$\begin{bmatrix} v_x^L \\ v_y^L \\ v_z^L \end{bmatrix} = \underline{R} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{6.14}$$

6.2.3 Degrees of Freedom and Rotation Matrix for Mindlin-Type elements

For Mindlin-type elements there are five degrees of freedom per node and per *y* term or harmonic, such as shown in figure 6.3. For the *i th* node and the *r th y* term or harmonic, the nodal local displacement components can be written in terms of corresponding global components as follows:

$$\begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \end{bmatrix}_{Local} = \underline{R} \begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \end{bmatrix}_{Global} \tag{6.15}$$

For the facetted version of this element, an additional degree of freedom θ_z is required. In order to avoid singularity in the element stiffness matrices, it is essential to distinguish between co-planar nodes and nodes at the intersection of plates, such as shown in figure 6.4.

- For co-planar nodes:
  We assume that $\theta_z = 0$ and θ_x , θ_y will always be considered in the directions of the local *x, y*-axes.i.e. for a co-planar node *j*:

$$\begin{bmatrix} (\theta_x^r)_j \\ (\theta_y^r)_j \\ (\theta_z^r)_j \end{bmatrix}_{Local} = \underline{I} \begin{bmatrix} (\theta_x^r)_j \\ (\theta_y^r)_j \\ (\theta_z^r)_j \end{bmatrix}_{Global} , (\theta_z^r)_j = 0 \tag{6.16}$$

- For node *i* at the intersection of two plates:

  The global θ_z may not be equal to zero and we use:

$$\begin{bmatrix} (\theta_x^r)_i \\ (\theta_y^r)_i \\ (\theta_z^r)_i \end{bmatrix}_{Local} = \underline{R} \begin{bmatrix} (\theta_x^r)_i \\ (\theta_y^r)_i \\ (\theta_z^r)_i \end{bmatrix}_{Global} \tag{6.17}$$

Hence, the rotation, matrix of the element at any node *i* and *y* value *r* can be defined such that:

$$\begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \\ (\theta_x^r)_i \\ (\theta_y^r)_i \\ (\theta_z^r)_i \end{bmatrix}_{Local} = \underline{R}_i^M \begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \\ (\theta_x^r)_i \\ (\theta_y^r)_i \\ (\theta_z^r)_i \end{bmatrix}_{Global} \tag{6.18}$$

where:

- For co-planar node j:

$$\underline{R}_j^M = \begin{bmatrix} \underline{R} & \underline{0} \\ 0 & \underline{I} \end{bmatrix} \tag{6.19}$$

- For node i at the intersection of two plates:

$$\underline{R}_i^M = \begin{bmatrix} \underline{R} & \underline{0} \\ 0 & \underline{R} \end{bmatrix} \tag{6.20}$$

Hence, for all the components of an n-node finite strip element at the *rth* y term or harmonic, we can deduce that:

$$\underset{6n\times 1}{\underline{\delta}^r_{Local}} = \underset{6n\times 6n}{\mathfrak{R}} \; \underset{6n\times 1}{\underline{\delta}^r_{Global}} \tag{6.21}$$

where $\mathfrak{R}$ is the element rotation matrix defined as follows:

$$\mathfrak{R} = \begin{bmatrix} \underline{R}_1^M & \underline{R}_2^M \cdots & \underline{R}_i^M \cdots & \underline{R}_n^M \end{bmatrix} \tag{6.22}$$

An additional boundary condition should be implemented for every co-planar node j:

$$\left(\theta_z^r\right)_j = 0, \text{ for } r = 1,2,3,\ldots,m \tag{6.23}$$

6.2.4 Degrees of Freedom and Rotation Matrix for Reissner-Type elements

From chapter 5 it is clear that the interpolation equations of Reissner-type element are based upon six degress of freedom per node i and per y term r; u_i^r, v_i^r, w_i^r, $w_{i,x}^r$, $\left(\psi_x^r\right)_i$, $\left(\psi_x^r\right)_i$, with their directions are as shown in figure 6.5. Similarly, for facetted elements additional degrees of freedom $\left(\psi_z^r\right)_i$ is needed with the following considerations:

- For co-planar nodes:
  We consider $\left(\psi_z^r\right)_i = 0$ at all y terms, and ψ_x, ψ_y will always be considered in the directions of the local x, y-axes.i.e. for a co-planar node j:

$$\begin{bmatrix} \left(\psi_x^r\right)_j \\ \left(\psi_y^r\right)_j \\ \left(\psi_z^r\right)_j \end{bmatrix}_{Local} = \underline{I} \begin{bmatrix} \left(\psi_x^r\right)_j \\ \left(\psi_y^r\right)_j \\ \left(\psi_z^r\right)_j \end{bmatrix}_{Global}, \left(\psi_z^r\right)_j = 0 \tag{6.24}$$

- For node i at the intersection of two plates:
  The global ψ_z may not be equal to zero and we use the following equation:

$$\begin{bmatrix} (\psi_x^r)_i \\ (\psi_y^r)_i \\ (\psi_z^r)_i \end{bmatrix}_{Local} = \underline{R} \begin{bmatrix} (\psi_x^r)_i \\ (\psi_y^r)_i \\ (\psi_z^r)_i \end{bmatrix}_{Global} \tag{6.25}$$

Hence, the rotation, matrix of the *Reissner*-type element at any node i and y value r can be defined such that:

$$\begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \\ w_{i,x}^r \\ (\psi_x^r)_i \\ (\psi_y^r)_i \\ (\psi_z^r)_i \end{bmatrix}_{Local} = \underline{R}_i^H \begin{bmatrix} u_i^r \\ v_i^r \\ w_i^r \\ w_{i,x}^r \\ (\psi_x^r)_i \\ (\psi_y^r)_i \\ (\psi_z^r)_i \end{bmatrix}_{Global} \tag{6.26}$$

where for a co-planar node j:

$$\underline{R}_j^H = \begin{bmatrix} \underline{R} & 0 & \underline{0} \\ 0 & 1 & 0 \\ \underline{0} & 0 & \underline{I} \end{bmatrix} \tag{6.27}$$

and for node i at the intersection of two plates:

$$\underline{R}_i^H = \begin{bmatrix} \underline{R} & 0 & \underline{0} \\ 0 & 1 & 0 \\ \underline{0} & 0 & \underline{R} \end{bmatrix} \tag{6.28}$$

Finally, for all the components of an n-node finite strip element at the *rth* y term or harmonic, we can deduce that:

$$\underset{7n\times 1}{\underline{\delta}_{Local}^r} = \underset{7n\times 7n}{\mathfrak{R}} \underset{7n\times 1}{\underline{\delta}_{Global}^r} \tag{6.29}$$

where $\mathfrak{R}$ is the element rotation matrix defined as follows:

$$\mathfrak{R} = \begin{bmatrix} \underline{R}_1^H & \underline{R}_2^H \cdots & \underline{R}_i^H \cdots & \underline{R}_n^H \end{bmatrix} \tag{6.30}$$

An additional boundary condition should be implemented for every co-planar node j:

$$(\psi_z^r)_j = 0, \text{ for } r = 1,2,3, \ldots, m \tag{6.31}$$

From the derivations of the stiffness matrices in the previous chapters, it can be deduced that:

$$\underline{K}^{sr}_{Global} = \Re^{t}\ \underline{K}^{sr}_{Local}\ \Re \tag{6.32}$$

and

$$\underline{K}^{sr}_{\sigma\ Global} = \Re^{t}\ \underline{K}^{sr}_{\sigma\ Local}\ \Re \tag{6.33}$$

6.3 Progressive Failure Methodology

6.3.1 Non-Linear Static Analysis

A nonlinear static analysis is performed to account for the geometrically nonlinear behaviour in the progressive failure analysis. The vectors and matrices in the generalized equations of equilibrium derived in the previous chapters are evaluated for every finite strip before they are assembled to form structural matrix equation for static analysis.

For nonlinear static analysis, the following equation of equilibrium is used:

$$(\underline{K}_o + \underline{K}_\sigma)\Delta\underline{\delta}_o = \Delta\underline{F} \tag{6.34}$$

where $\underline{K}_o$ represents $\underline{K}$ which contains both linear and non-linear B matrices terms and $\underline{K}_\sigma$ represents the large deflection element matrices derived in the previous chapters.

The analysis begins by considering only small deflection at the first iteration by using the standard FEM assembly rules, and can be can be represented as follows:

$$(\underline{K}_o)\underline{\delta}_o = \underline{F} \tag{6.35}$$

For linear static analysis, only the linear part of the $\underline{B}$ matrices is computed. Equation (6.35) above can be solved to obtain nodal displacement vectors using the standard Gauss elimination procedure. In this work three solvers are used namely, ordinary solver, banded solver and frontal solver. Having obtained the nodal displacement vectors $\underline{\delta}_o$ and corresponding stresses at Gaussian quadrature points, the residual vector as derived earlier in the previous chapters will be computed.

For the example of *Mindlin* element:

$$\underline{R}_{(o)} \equiv \begin{bmatrix} \underline{R}_o \\ \\ \underline{R}_b \end{bmatrix}_{(o)} \tag{6.35}$$

where

$$(\underline{R}_o)_{(o)} = \underline{F}_o - \sum_{strips}\iint_{strips}\left[\underline{B}_o^t(\underline{\delta}_o)\underline{\sigma}_{(0)} + \underline{B}_{mm}^t(\underline{\delta}_o)\underline{\sigma}_{(0)} + \underline{B}_{\theta n}^t(\underline{\delta}_o)\underline{\sigma}_{(1)}\right]dxdy \tag{6.36}$$

and

$$\underline{R}_b = \underline{F}_b - \sum_{strip}\iint_{strip}\left\{\underline{B}_\gamma^t(\underline{\delta}_o)\underline{\tilde{\tau}} + \underline{B}_{ww}^t(\underline{\delta}_o)\underline{\sigma}_{(0)} + \underline{B}_b^t(\underline{\delta}_o)\underline{\sigma}_{(1)} + \underline{B}_{m\theta}^t(\underline{\delta}_o)\underline{\sigma}_{(1)} + \underline{B}_{\theta\theta}^t(\underline{\delta}_o)\underline{\sigma}_{(2)}\right\}dxdy \tag{6.37}$$

By defining $\underline{\delta}_1 = \underline{\delta}_o + \Delta\underline{\delta}_1$, such that equation (6.34) is satisfied, then

$$\left(\underline{K}_o(\underline{\delta}_0) + \underline{K}_\sigma(\underline{\sigma}_0)\right)(\Delta\underline{\delta}_1) = \underline{R}_{(o)} \tag{6.38}$$

After t iterations $\underline{\delta}_{t-1}$ is obtained and the corresponding stresses and strains can be computed. Then the residual vector can be updated as follows:

$$\underline{R}_{(t-1)} \equiv \begin{bmatrix} \underline{R}_o \\ \\ \underline{R}_b \end{bmatrix}_{(t-1)} \tag{6.39}$$

where

$$(\underline{R}_o)_{(t-1)} = \underline{F}_o - \sum_{strips}\iint_{strips}\left[\underline{B}_o^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(0)} + \underline{B}_{mm}^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(0)} + \underline{B}_{\theta m}^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(1)}\right]dxdy \tag{6.40}$$

and

$$(\underline{R}_b)_{(t-1)} = \underline{F}_b - \sum_{strip}\iint_{strip}\left\{\underline{B}_\gamma^t(\underline{\delta}_{(t-1)})\underline{\tilde{\tau}} + \underline{B}_{ww}^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(0)} + \underline{B}_b^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(1)} + \underline{B}_{m\theta}^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(1)} + \underline{B}_{\theta\theta}^t(\underline{\delta}_{(t-1)})\underline{\sigma}_{(2)}\right\}dxdy \tag{6.41}$$

we assume that $\underline{\delta}_1 = \underline{\delta}_{(t-1)} + \Delta\underline{\delta}_1$ such that

$$\left(\underline{K}_o(\underline{\delta}_{(t-1)}) + \underline{K}_\sigma(\underline{\sigma}_{(t-1)})\right)(\Delta\underline{\delta}_t) = \underline{R}_{(t-1)} \tag{6.42}$$

which can be solved in $\Delta\underline{\delta}_t$ and a new residual vector is defined, and so on.

Iterations are then carried out until convergence is achieved by using the following criterion:

$$\frac{\Delta\underline{\delta}_t^t\,\Delta\underline{\delta}_t}{\underline{\delta}_t^t\,\underline{\delta}_t} \leq \text{ a permissible error}$$

A similar procedure can be employed for *Reissner* element.

6.3.2 Progressive Failure Algorithm

A progressive failure methodology is developed for predicting the failure of laminate composite plates and shells under geometrically non-linear deformations. It is developed based on the progressive failure concept disscused previously in section 3.5.The progressive failure analysis uses *Mindlin*-type, *Reissner*-type and *Spline* (*Reissner*-type) elements.

An algorithm for a progressive failure analysis which has been developed and is illustrated in figure 6.6. The analysis begins with the description of the finite strip model such as the boundary and loading conditions, and material properties. The second step is to compute the stiffness matrices for each strip and to assemble them to form global stiffness matrix. Next, nodal stresses at each ply, each node, each y value and each z-point are calculated and transformed into the material axes. The value of the transformed nodal stresses at the middle layer of each ply at each y value are used with one of the failure criteria discussed in chapter 3 to determine whether any failures have occurred at each load increment. If failures are detected, i.e. if the transformed stresses exceeded the respective failure criteria, a reduction in the value of the material properties will take place using a material degradation model which will be discussed in the next section. Equilibrium will then be re-established by evaluating the residual vector by using equation (6.39). If convergence is achieved after establishing equilibrium, then the applied load or displacement is increased. On the other hand, if a converged solution is not attained, the non-linear analysis at the same load increment will be repeated until a converged solution is found. The process is repeated until the catastrophic failure of the structure is reached.

6.3.2.1 Damage Modelling

In this research, a material degradation model based upon a total element or strip reduction method is implemented (Spottswood and Palazotto, 2001). For each node at any y value of each ply (middle surface) that fails, the particular strip is reduced using a stiffness reduction factor given by the following equation:

$$\text{Stiffnes Reduction Factor}(SRF) = 1.0 - \frac{\text{No. of Occurance of Failure}}{(\text{No. of Nodes} \times \text{No. of } y \text{ terms})} \qquad (6.43)$$

For a two-noded strip which consists of two y terms, the overall stiffness of that particular strip is reduced by 25% if one of the nodes at one of the y terms has failed. Likewise, the stiffness reduction continues to occur as more points in a particular strip fail. If all the points in a particular strip satisfy the respective failure criterion, then the strip is assumed to have failed completely and the value of the stiffness reduction factor (*SRF*) equals zero.

In the damage modelling implemented in this work, the material properties of a composite laminates can be slowly degraded over subsequent non-linear load increments or can be reduced to zero if a catastrophic failure occurs at an early stage of the non-linear analysis, depending on the value of the applied load or displacement. The properties are assumed to be the value of the SRF multiply by the value of the original integrated elastic modulus matrix with respect to plate axes, as defined

previously by equation (3.24). Hence, from equations (4.86) and (4.69) respectively, the material properties are degraded according to:

$$\left(\underline{D}_{(n)}\right)_{new} = \left(\underline{D}_{(n)}\right)_{original} \times SRF = \sum_{l=1}^{nl}\left[\frac{Z(U)^{n+1} - Z(L)^{n+1}}{(n+1)}\right]\underline{D}^{(l)} \times SRF \tag{6.44}$$

$$\underline{\sigma}^{(l)} = \underline{D}^{(l)} \times SRF \times \underline{\varepsilon} \tag{6.45}$$

This adjustment accounts for the material non-linearity associated with a progressive failure analysis included within a non-linear finite strip analysis.

CHAPTER 7

7. Review of Programming Package

7.1 Introduction

In this research, new equations for non-linear static analysis for plates and shells have been derived for *Mindlin-type* and *Reissner-type* finite strip elements as discussed in chapters 4 and 5 respectively. A versatile modular programming package based on the newly derived equations has been developed to carry out linear and non-linear progressive failure analysis of plates and facetted shells using *Mindlin-type*, *Hermitan-type Reissner*, and *Spline-type Reissner* finite strip elements. The package was coded in FORTRAN 77 and contains both the plate version and the facetted shell version. It is capable of undertaking the finite strip progressive failure analysis of plates and shells made of composite materials.

The programming package has been divided into three main components according to the three different types of elements been considered namely:

i) *Mindlin-type* finite strip elements component.
ii) *Hermitian-type Reissner* finite strip elements component.
iii) *Spline-type Reissner* finite strip elements component.

Each of the component can perform linear and non-linear progressive failure analysis using ordinary, banded, or frontal solver as shown in figure 7.1.Also, each component consists of several groups of modules which are responsible for specific tasks as illustrated in figure 7.2. The modules can easily be linked together and can be run on a personal computer. All the components use the same basic modules, but the difference is in the elements and shape functions subroutines, and other additional subroutines depending on the type of element used, as will be explained later in this chapter.

7.2 Input Data Module

An input data file is required in order to model plates or shells and is nearly the same for all the three types of elements. A sample of an input file for each type of elements can be found in *Appendix B*. The information in the data file will be read by the input data subroutines which contain the following basic data:

1) **Mesh Data** which generates finite strip mesh for both a plate version and a facetted shell version. This includes parameters to define the number of nodes and harmonics as well as the number of strips.

2) **Material Properties** which defines whether the material is Isotropic or Composite.

3) **Boundary Conditions** which define the type of restrains or forced displacements applied to the structure in any of its degrees of freedom.

4) **Loading Data** which specifies the equivalent nodal loading vectors applied to the structure in any of its degrees of freedom. It consists of distributed loadings, concentrated loadings, and line loadings.

5) **Analysis Type** which allows the user to choose either to perform linear static or non-linear static analysis.

6) **Output Data** which displays the displacements and stresses results at selected nodes.

7) **Failure Data** which contains information on the failure criteria and failure strength parameters that are used for the progressive failure analysis.

7.3 Load Module

This module has both the plate version and facetted shell version for the equivalent nodal loading vector calculation. It has subroutines to calculate equivalent nodal loading vectors for four different types of loadings:

i) **Distributed loading.** A type of loading that involves a double integral over each finite strip.

ii) **Concentrated loading.** A concentrated load acting at a given node and a given y position.

iii) **Line loading (x).** A line loading defined in terms of load per unit length, along a line x = constant, which involves an integration with respect to y along that line.

iv) **Line loading (y).** A line loading defined in terms of load per unit length, along a line y = constant, which involves an integration with respect to x along that line.

7.4 Element Stiffness Matrix Module

This module calculates each of the stiffness terms in all the sub-matrices of the stiffness matrices $\underline{K}^{sr}$ and $\underline{K}_{\sigma}^{sr}$ for each type of element in the finite strip mesh. The sub-matrices are as derived earlier in chapters 4 and 5.They are then added together by a subroutine called STIFF so as to obtain the stiffness terms for the whole element. The matrices of the elements are then assembled together by a subroutine called ASSEMBLE which is stored in the solver module. This module consists of a plate version and a facetted shell version for each type of elements. It also consists of subroutines to compute the integrated $\underline{D}$ matrices, linear and non-linear $\underline{B}$ matrices, A matrices, $\underline{S}$ matrices, and $\underline{G}$ matrices, as defined earlier in the previous chapters. This module is subdivided into three sub-modules as will be explained in the next sections.

7.4.1 Mindlin-type Element Module

This module generates the stiffness matrix of a *Mindlin*-type element. It employs *Lagrangian* interpolation subroutines from the shape function module, where in this case a subroutine called LAGRANGE has been used. The link between the main subroutines can be shown in the flowchart in figures (7.3)-(7.8). The $\underline{B}$ matrices

subroutines which are used in the generation of the element stiffness matrix are summarized in the following table:

Subroutine Name	Matrix Computed
BMATRIXO(s)	$\left(\underline{B}_o^s\right)^t$
BMATRIXO(r)	$\underline{B}_o^r$
BMATRIXB(s)	$\left(\underline{B}_b^s\right)^t$
BMATRIXB(r)	$\underline{B}_b^r$
BMATRIXMM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_m^t$
BMATRIXMM(r)	$\underline{A}_m \underline{G}_m^r$
BMATRIXTM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_\theta^t$
BMATRIXTM(r)	$\underline{A}_\theta \underline{G}_m^r$
BMATRIXMT(r)	$\underline{A}_m \underline{G}_\theta^r$
BMATRIXMT(s)	$\left(\underline{G}_\theta^s\right)^t \underline{A}_m^t$
BMATRIXTT(s)	$\left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t$
BMATRIXTT(r)	$\underline{A}_\theta \underline{G}_\theta^r$
BMATRIXWW(r)	$\underline{A}_w \underline{G}_w^r$
BMATRIXWW(s)	$\left(\underline{G}_w^s\right)^t \underline{A}_w^t$
BMATRIXG(s)	$\left(\underline{B}_\gamma^s\right)^t$
BMATRIXG(r)	$\underline{B}_\gamma^r$

7.4.2 Reissner-type (Hermitian) Module

This module generates the stiffness matrix of a Reissner (Hermitian)-type element. It employs Hermitian interpolation subroutines from the shape function module, where in this case subroutines called HERMITE, DHERMITE and D2HERMITE have been used. The link between the main subroutines can be shown in the flowchart in figures (7.9)-(7.19). The $\underline{B}$ matrices subroutines which are used in the generation of the element stiffness matrix are summarized in the following table:

Subroutine Name	Matrix Computed
BMATRIXO(s)	$\left(\underline{B}_o^s\right)^t$
BMATRIXO(r)	$\underline{B}_o^r$
BMATRIXB(s)	$\left(\underline{B}_b^s\right)^t$
BMATRIXB(r)	$\underline{B}_b^r$
BMATRIXP(s)	$\left(\underline{B}_\psi^s\right)^t$
BMATRIXP(r)	$\underline{B}_\psi^r$
BMATRIXMM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_m^t$
BMATRIXMM(r)	$\underline{A}_m \underline{G}_m^r$
BMATRIXTM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_\theta^t$
BMATRIXTM(r)	$\underline{A}_\theta \underline{G}_m^r$
BMATRIXPM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_\psi^t$
BMATRIXPM(r)	$\underline{A}_\psi \underline{G}_m^r$
BMATRIXMT(r)	$\underline{A}_m \underline{G}_\theta^r$
BMATRIXTM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_\theta^t$
BMATRIXWW(r)	$\underline{A}_w \underline{G}_w^r$
BMATRIXPM(s)	$\left(\underline{G}_m^s\right)^t \underline{A}_\psi^t$
BMATRIXPT(r)	$\underline{A}_\psi \underline{G}_\theta^r$
BMATRIXTT(s)	$\left(\underline{G}_\theta^s\right)^t \underline{A}_\theta^t$
BMATRIXTT(r)	$\underline{A}_\theta \underline{G}_\theta^r$
BMATRIXTP(r)	$\underline{A}_\theta \underline{G}_\psi^r$
BMATRIXPP(r)	$\underline{A}_\psi \underline{G}_\psi^r$
BMATRIXMP(r)	$\underline{A}_m \underline{G}_\psi^r$

7.4.2 Reissner-type (Spline) Module

This module employs the same subroutines as the *Reissner (Hermitian)*-type but using spline shape functions instead of the Hermitian shape functions. This means that HERMITE, DHERMITE and D2HERMITE hermitian shape functions subroutines are replaced by SPLINES, DSPLINES, and D2SPLINES shape functions subroutines in the shape functions module.

7.5 Internal Force Vector (FIVECTOR) Module

This module contains subroutines to generate the internal force vector for each type of finite strip elements. It also contains subroutines to compute all the integrated stresses, such as derived earlier in chapters 4 and 5. The link between main subroutines can be self explained in figures 7.20 and 7.21 for *Mindlin* and *Reissner (Hermitian)* elements respectively. Similarly, for *Reissner (Spline)*-type the only difference is in the shape functions subroutines.

7.6 Shape Functions Modules

All the one-dimensional shape functions required for interpolation along x and y directions can be found in the shape functions modules. The choice of a shape function module depends on the type of element used.

7.6.1 Mindlin-Type Element

A ***Lagrangian* shape function module** is employed for *Mindlin-type* element. This module consists of *Langangian* (polynomial) shape functions and their derivatives for use in the interpolation of all displacements parameters along both the x and y directions, which has been discussed in detail in chapter 4.

7.6.2 Reissner (Hermitian)-Type Element

A ***Hermitian* interpolation shape functions module** is used for *Reissner (Hermitian)*-Type element. It contains the *Hermitian* shape functions, their first and second order derivatives, and their numerical integrations. For *Reissner(Hermitian)-type* elements, *Hermitian* interpolations can be used for the lateral deflection w along both x and y directions whereas, *Lagrangian* interpolation are used for all other displacement parameters.

7.6.3 Reissner (Spline)-Type Element

***Spline-w* interpolation shape function module** and ***spline-all* interpolation shape function module** are used for *Reissner(spline)-type* elements. The *spline-w* interpolation shape function module contains all the subroutines in the *Hermitian* shape functions module, but with some additional subroutines which defines the *spline-type* shape functions, their derivatives, and their numerical integrations. It provides *spline-type* interpolation for the lateral deflection w along the x direction. The available polynomial interpolation along the y direction is *spline-type*

interpolation for w and *Langangian* interpolation for all other displacement parameters.

On the other hand, the *spline-all* interpolation shape function module is similar to the *spline-w* shape function module and provides *spline-type* interpolation for the lateral deflection w along both x and y directions and all other displacement components along y direction.

7.7 Solver Module

This module contains the non-linear progressive failure analysis master program and its flowchart and subroutines are self-explained in figure 7.22. It applies an iterative algorithm which has been discussed previously in section 6.3. The solver module contains the following basic procedure:

1) **Assembly of the element stiffness matrix**. A subroutine is used to call the element stiffness matrix generator and all the elements are assembled to form a global stiffness matrix. For a facetted shell element, each element matrix is rotated with respect to the global axes of the structure before they are assembled into one global stiffness matrix for the whole structure.

2) **Reduction of stiffness matrix**. A subroutine was built in order to eliminate the rows and columns corresponding to the constrained boundary conditions applied to the structure. Then, the reduced stiffness matrix and loading vector are obtained.

3) **Solving reduced equations.** A subroutine which employs the Gauss elimination method is used to solve the reduced equations

4) **Expansion of reduced equations**. A subroutine was programmed to expand the reduced solution to the global nodal displacement vector.

5) **Decoupling**. A subroutine is used to decouple the global nodal displacement vectors into sub-vectors for every y term or harmonic.

7.8 Failure Assessment Module

This module consists of subroutines to detect lamina failures and to assess the damage state of each of the three types of element during the progressive failure analysis. It contains the subroutines of the three failure theories i.e. *Tsai-Wu*, *Tsai-hill* and *Hoffman*, and a subroutine for failure assessments called FAIL ASSESS, such that as shown in figure 7.23. This module also prints out damage progression at every iteration, load increment, layer, and strip which will then be stored in the output file.

In the progressive failure strategy implemented in this work, such as discussed in chapter 6, the failure criteria are evaluated before a converged non-linear solution is achieved using the nodal stresses which have been transformed into the material axis.

If failures are detected, then the material properties for that node, harmonic, and ply in a particular element are degraded according to the damage model using equations (6.44) and (6.45). These two equations are evaluated in the integrated $\underline{D}$ matrices subroutines which are used to calculate the stiffness matrices of every strip or element.

Once a converged non-linear solution is achieved the non-linear analysis continues with the next load or prescribed displacement increment with the laminate having the new reduced stiffness as a result of the previous failures in a particular region in the laminate. This failure progression takes place until the stiffness of each of the element is reduced to zero, i.e. the total or ultimate failure of the laminate. The computational procedure for the progressive failure methodology just described is presented in figure 7.24 for each individual element.

7.9 Stress/Strain and Output Module

This module calculates the infinitesimal and finite strains at Gauss quadrature points and the strains averaged at nodes and prints out results for *Mindlin-type*, *Reissner-type*, and *Spline-type Reissner* elements. Figures 7.25 and 7.26 show the main subroutines involved in the stress and strains computation. Form the flowcharts the subroutines called LNSTRAINN and NLSTRAINN are responsible for the calculations of infinitesimal and finite strain components respectively, which are averaged at nodes. On the other hand, subroutines LNSTRAING and NLSTAING are used to calculate the infinitesimal and finite strains at Gauss quadrature points. Subroutine LNSTRAINGG is a special subroutine which is used to compute the transverse shear strains at reduced Gauss points. The stress/strain and output module also consists of subroutines to produce output results such as follows:

1) Global nodal displacements for the degrees of freedom at every y-term.

2) Global nodal forces and reactions for the degrees of freedom at every y-term.

3) Nodal stresses and strains at every y-term, and z-points (surface) for each layer.

7.9 Matrix Operations Module

This module consists of subroutines to compute all the required matrix operations in the finite strip analysis such as:

1) matrix transpose subroutine
2) matrix multiplication subroutines
3) matrix by vector subroutine
4) matrix tri-product subroutine
5) rotation matrices used for facetted shell structures

7.10 Running the Package

The programming package contains several modules, as discussed in the previous sections. It has been built in such a way that only relevant modules are to be linked together in order to compile and run the program, depending on the type of element used. As a result, the executable program has become small enough to run on normal PC's. Figures (7.27-7.29) show the linking structures for *Mindlin-type*, *Reissner (Hermitian)-type*, and *Reissner (Spline)-type* finite strip elements respectively.

CHAPTER 8

8. Results and discussions

8.1 Introduction

A finite strip programming package has been developed has been developed based on the new derivations of finite strip methods in chapters, as described in chapter 7. In order to validate the new finite strip methods and to test the capability of the newly developed package, a number of validation procedures using different case studies will be demonstrated in this chapter. The validation has been performed by comparing the results of the progressive damage analysis with corresponding results from a commercial finite element solver package.

In the validation procedure, the commercial package PATRAN has been used as a pre- and post-processor for generating finite element models and for displaying contour plots. The commercial package ABAQUS has been used as the solver in the finite element analysis. All the progressive damage analyses using ABAQUS have been achieved by linking the non-linear analysis with a user-written subroutine coded in FORTRAN language. The progressive damage methodology using ABAQUS is briefly explained in APPENDIX C. On the other hand, the finite strip meshes have been generated by a built-in one-dimensional mesh generator, which are much simpler to create than the commercial package finite element meshes.

After each validation, parametric studies were then carried out on the same test cases using the finite strip program to study the effect of thickness and stacking sequence on the structural behaviour for each type of element. Due to time constraints, a reasonable number of test cases have been selected for the validation and parametric studies.

8.2 Progressive Damage Analysis of Cantilever Plate

The analysis of cantilever plate, as shown in figure 8.1, with different types of loading has been carried out in order to assess the accuracy of the progressive damage methodology. Three types of load cases were considered namely:

1) **Tensile load** in terms of a uniform line tensile force.
2) **In-plane bending**, induced by a uniform in-plane shear force.
3) **Out-of-plane bending**, induced by a uniform out-of-plane line shear force.

Each type of load case is shown schematically in figures 8.2-8.4

The results of each case study were validated against ABAQUS. Tsai-Wu criterion was used for the failure criteria in this validation. A 12-ply cantilever plate made of Carbon/Epoxy has been considered in this case. The geometry and boundary conditions are shown schematically in figure 8.1.The thickness of each ply, the stacking sequence and the lamina material properties for the laminate are given in table 8.1. Figure 8.5 shows the finite element mesh (16 x 8) and boundary conditions of the ABAQUS model. The type of element used for ABAQUS is the 4-noded quadrilateral with full in-plane integration points called, S4. For the purpose of

validation, it is sufficient to use six 3-noded finite strip elements in the finite strip meshes.

8.2.1 Non-linear Static Analysis of Cantilever Plate under Tension

A non-linear static analysis has been carried out for the cantilever plate under tension, which is shown schematically in figure 8.2. A line force of intensity 2.0×10^6 N/m was applied axially (x-direction) . The analysis was carried out for Mindlin-type, Reissner (Hermitian)-type and Reissner (Spline)-type elements and their structural response before and after failure assessments were compared with ABAQUS using S4 element. Figure 8.6 shows the ABAQUS fringe plot and deformation with failure assessment.

Load-deflection curves were obtained which compares the three types of finite strip elements with ABAQUS finite element results. The displacements were measured at the free end of the plate. By referring to figure 8.7, all the non-linear static analysis (without damage assessment) results agree very well with ABAQUS finite element results. Likewise, it can be seen in figure 8.8 that all the progressive failure results agree reasonably well with the results from ABAQUS although there is slight difference between them in the upper region of the curves. This may be due to the use of different implementations of damage modelling between the finite strip and finite element. However, it is interesting to see that finite strip analysis can save a great deal of time in both modelling and running time as compared to the finite element analysis.

8.2.1.1 Effect of thickness on the deformation of cantilever plate under tension

The effect of thickness variation on the structural response of the cantilever plate under tension has been examined using 12-layer Carbon-Epoxy laminate, with equal thickness, h and with the following notations:

T1 : plate thickness of 0.03 m.

T2: plate thickness of 0.06 m

T3: plate thickness of 0.12 m

T4: plate thickness of 0.24 m

The plate is subjected to a line force of intensity 3.5×10^6N/m. Figure 8.9 and figure 8.10 show the plot of load increment versus axial displacement of non-linear analysis (without damage assessment) and with damage assessment respectively for laminate thickness of 0.03 metres for both cases.

The behaviour of the laminate (Mindlin-type element) for different thicknesses using Tsai-Hill failure criterion can be illustrated in figure 8.11, whereas figure 8.12 shows the behaviour of the laminate using different failure criteria for Mindlin-type element subjected to an axial load intensity of 3.5×10^6 N/m. Table 8.2 summarises the maximum axial deflections of each type of element under axial load of 3.5×10^6 N/m, using the three different failure criteria, i.e. Tsai-Hill, Tsai-Wu and Hoffman. It is clear from the table that the results correlate well with each other. From figure 8.11,

clear from the table that the results correlate well with each other. From figure 8.11, we can see that the laminate with thicknesses of T1 and T2 have failed to reach the maximum applied load. On the other hand, the thicker plates T3 and T4 were able to carry the maximum applied load but are much stiffer than the T1 and T2 plates. Hence, it can be concluded in this investigation that the laminate becomes stiffer as the thickness is increased. It is also clear that with the reduced integration schemes, Mindlin finite strip elements behave well for thin and thick plate.

8.2.1.2 Effect of fibre orientation on the deformation of cantilever plate under tension

The effect of fibre orientation on the deflections of the laminate was also investigated which consist of four layers of Carbon-Epoxy lamina with equal thickness, h and with the following orientations and notations:

SQ-1: $[0^0/90^0/90^0/0^0]$: Symmetrical cross-ply

SQ-2: $[0^0/90^0/0^0/90^0]$: Skew-symmetrical cross-ply

SQ-3: $[45^0/-45^0/-45^0/45^0]$: Symmetrical angle-ply

SQ-4: $[45^0/-45^0/45^0/-45^0]$: Skew-symmetrical angle-ply

Figures 8.13-8.16 show the structural response of cantilever plate under tension using Tsai-Wu failure Criterion for each type of fibre orientations under tensile force of intensity 5.0×10^6 N/m. With reference to figure 8.17, it can be seen that the cross-ply laminates show much higher strength than the angle-ply laminates. This is due to the fact that in the cross-ply laminates the zero degree plies (in-line with the applied force) tend to resist the tension force much better than the 45 or -45 degrees plies in the angle-ply laminates. Table 8.3 shows the results of the maximum deflections under tensile force of intensity 5.0×10^6 N/m using various failure criteria and different types of elements.

8.2.2 Non-linear Static Analysis of Cantilever Plate under In-plane-Bending

A non-linear static analysis has been carried out for the cantilever plate under in-plane-bending, which is shown schematically in figure 8.3. A line shear force of intensity 1.3×10^5 N/m was applied in y-direction. The analysis was carried out for Mindlin-type, Reissner (Hermitian)-type and Reissner (Spline)-type elements and their structural response with damage assessments were compared with ABAQUS using S4 element. Figure 8.18 shows the ABAQUS plot of Von Mises stress at the lowest surface of the cantilever plate under in-plane-bending with failure assessment.

Load-deflection curves were obtained which compares the three types of finite strip elements with ABAQUS finite element results. The displacements were measured at the free end of the plate. By referring to figures 8.19 and 8.20, it can be noticed that all the progressive failure results from finite strip agree reasonably well with the results from ABAQUS.

8.2.2.1 Effect of fibre orientation on the deformation of cantilever plate under in-plane-bending

The effect of fibre orientation on the deflections of the Carbon-Epoxy laminate was also investigated using four layers of Carbon-Epoxy laminae with equal thickness, h. The laminate is subjected to a line shear force of intensity 4.0×10^5 N/m and having the same fibre orientations and notation as in the case of tension. Figures (8.21)-(8.24) show the structural response of cantilever plate under in-plane bending using Tsai-Wu failure Criterion for each type of fibre orientations. With reference to figure 8.25, it can be observed that the cross-ply laminates is stiffer than the angle-ply laminates. Table 8.4 shows the results of the maximum deflections attained by each type of element using various failure criteria and in general, the results are in good agreement among each other.

8.2.3 Non-linear Static Analysis of Cantilever Plate under Out-of-Plane-Bending

A non-linear static analysis has been carried out for the cantilever plate under out-of-plane-bending, which is shown schematically in figure 8.4. A line out-of-plane force of intensity 8.0×10^3 N/m was applied in z-direction. Progressive damage analysis was carried out for Mindlin-type, and its structural response was compared with ABAQUS using S4 element. Figure 8.26 shows the ABAQUS fringe plot and deformation with failure assessment.

Load-deflection curves were obtained which compares the finite strip using Mindlin-type element with ABAQUS finite element results for both with and without damage assessments. By referring to figure 8.27 it can be noticed that all the progressive failure results from finite strip agree reasonably well with the results from ABAQUS. On the other hand, the non-linear static analysis results without damage consideration were also very good.

8.2.3.1 Effect of fibre orientation on the deformation of cantilever plate under out-of-plane-bending

The effect of fibre orientation on the structural behaviour of the laminate under out-of-plane bending was also investigated.The following fibre orientations were tested:

(1) $[0^0/90^0/90^0/0^0]$: Symmetrical cross-ply

(2) $[0^0/30^0/30^0/0^0]$: Symmetrical angle-ply

(3) $[45^0/-45^0/-45^0/45^0]$: Skew-symmetrical angle-ply

(4) $[30^0/-30^0/-30^0/30^0]$: Skew-symmetrical angle-ply

The cantilever plate was subjected to an out-of-plane force of intensity 1.0×10^4N/m at its free end. It can be noted from figures 8.28-8.31 that in general the load-increment versus deflection curves of Mindlin, Reissner (Hermitian) and Reissner (Spline) elements follow the same pattern. It can be observed that the most of the results for Reissner (Hermitian) element are slightly lower than those of two other elements. This may due to the different types of interpolations for the lateral deflection w used by each elements.

8.3 Progressive Damage Analysis of Square Plate

The accuracy of the present finite strip methods is further demonstrated considering a square plate under a uniform distributed load. In this case, a four-layer [0/90/90/0] square cross-ply laminated plate subjected to a uniform distributed transverse loading has been considered. The plate is assumed to be clamped along all four edges and is subjected to a uniform load intensity of 13.8 kPa. The laminate material properties and dimensions are shown in Table 8.5. Figure 8.32 show the mesh (30×30) and boundary conditions of the ABAQUS finite element model.

Non-linear progressive damage analysis has been performed using the present finite strip analysis (Mindlin element) and the results were compared with the results from ABAQUS and existing experimental results (Zaghloul and Kennedy, 1975). The ABAQUS plot of Von Mises stress at the lowest surface of a square plate under uniform pressure is illustrated in figure 8.33.

In figure 8.34, the centre deflection of the square plate under uniform distributed load was presented. From the figure, the finite strip results obtained using Mindlin element are in better agreement with the experimental results than those obtained using the ABAQUS finite element analysis. The maximum difference in the results between the finite strip analyses and the experimental is only about 10 percent. In contrast, 17 percent difference in the results has been observed between ABAQUS and the experimental results. The difference may be attributed to the inaccurate modelling of the clamped experimental boundary conditions, which may result in a small deviation on the deflections of the laminate.

8.4 Progressive damage analysis of Rectangular plate

A two-layer rectangular plate, with antisymmetric cross ply [0/90], simply supported at two sides only under uniformly distributed transverse load has been analyzed in this case. The schematic drawing of the rectangular plate including the boundary conditions is shown in figure 8.35, which have the same dimension and material properties as those used by (Reddy, 1989) as follows:

A=1.5 in

B = 9.0in

Thickness of plate, h =0.04 in

E_{11}=20 msi, E_{22} = 1.4 msi,

$G_{12} = G_{13} = G_{23}$ = 0.7 msi,

X_t = 309.0 ksi, X_c =159.0 ksi, Y_t=11.6 ksi, Y_c = 29.0 ksi, S = 23.2 ksi

For this test case 12 finite strip elements (Mindlin-type) were used to model the rectangular plate. The results of the maximum non-dimensional central deflection under uniformly distributed transverse loads are given in Table 8.6, and the results obtained using finite elements (Reddy, 1989) and finite strips (Zhang et al.,2003) are also recorded in the same table as comparison. It is interesting to see that the results of

the present finite strip analysis are in excellent agreement with the results obtained by both researchers mentioned above.

8.5 Progressive Damage of Trapezoidal Panel

In this section a progressive damage analysis of trapezoidal cantilever panel has been undertaken. The developed finite strip program is capable of modelling trapezoidal plate directly by using its built-in mesh generator. The schematic drawing of the trapezoidal panel under study is show in figure 8.36. In this case, one side of the panel is clamped while the other is free where

Length of the panel, L = 2.0 m

Width of the clamped edge, B1 = 1.0 m

Width of the loaded edge, B2 = 0.8 m

Two types of load cases have been studied which are:

4) **Tensile load** in terms of a uniform line tensile force.

5) **In-plane bending**, induced by a uniform in-plane shear force.

Each type of load case can be shown schematically in figures 8.37 and 8.38.

The results of each case study were then validated against the finite element results using ABAQUS. Again, Tsai-Wu failure theory was used for the failure criteria in this validation. A 12-ply cantilever plate made of Carbon/Epoxy has been considered in this case. The thickness of each ply, the stacking sequence and the lamina material properties for the laminate are identical to the previous case study.

8.5.1 Trapezoidal Cantilever Panel under Tension

In this section, a non-linear progressive damage analysis has been carried out for the trapezoidal plate under tension, with an applied displacement of 0.0012 m in x-direction. The finite strip analysis was carried out for Mindlin-type, and Reissner (Hermitian and Spline) type elements and their structural responses with damage assessments were compared with ABAQUS finite element analysis. Figure 8.39 shows the ABAQUS plot of Von-Mises stress at the lowest surface for trapezoidal panel under tension.

Load-deflection curves were obtained which compares the results of finite strip analysis with ABAQUS finite element results. By referring to figure 8.40 it can be noticed that the progressive failure results from finite strip agree reasonably well with the results from ABAQUS. The ABAQUS plot of damage zone of layer one in the trapezoidal panel under tension at final displacement increment can be viewed in figure 8.41. The damage contour has been plotted using the ABAQUS/Viewer post-processor.

Next, the progressive finite strip analysis was carried out to investigate the effect of varying the width of the side, B2 of the panel on the load, under an applied displacement of 0.001m in x-direction. Tsai-Hill failure criterion has been considered

in this case. From the load versus width ratio (B2/B1) curves in figure 8.42 we can conclude that increase in the width ratio is proportional to the increase in the loads.

8.5.2 Trapezoidal Cantilever Panel under In-Plane bending

A non-linear progressive damage analysis has been carried out for the trapezoidal plate under in-plane bending, with an applied displacement of 0.003 m in y-direction. The finite strip analysis was carried out using Mindlin-type, and Reissner (Hermitian and Spline) type elements and their structural responses were compared with that of ABAQUS. Figure 8.43 shows the ABAQUS plot of Von Mises stress at the lowest surface of trapezoidal panel under tension.

Load-deflection curves were obtained which compares the results between the finite strip analysis and ABAQUS finite element results. By referring to figure 8.44 we can see that all the progressive failure results from finite strip analysis and the finite element analysis are in good agreement with each other. The ABAQUS plot of damage zone of layer one in the trapezoidal panel under tension at final displacement increment can be viewed in figure 8.45. The damage contour has been plotted using the ABAQUS/Viewer post-processor.

Next, the progressive finite strip analysis was carried out to investigate the effect of varying the width of the side, B2 of the panel on the load, under an applied displacement of 0.001m in y-direction. Tsai-Hill failure criterion has been considered in this case. Figure 8.46 shows the relationship between load and width ratio (B2/B1) of the panel under in-plane bending.

8.6 Progressive Damage Analysis of Stiffened Plate

A progressive damage analysis of a stiffened plate is considered in this case in order to demonstrate the ability of the newly derived finite strip methods to deal with folded plates problems. In this test case, a simple stiffened plate, which is shown schematically in figure 8.47 is considered, where

Length in x-direction L = 2 m,

Width in y-direction, B =1 m,

Height of the stiffener in z-direction S = 0.5 m.

The plates consist of 12-ply Carbon-Epoxy composite material whose material properties and laminate configuration are identical to that of the cantilever plate (Section 8.2).

The stiffened plate has been tested using two types of load cases:

1) Case with distributed loading (figure 8.48) on the upper surface, along $-z$ - direction, with intensity of 2.0×10^5 N/m^2.

2) Case with out-of-plane line loading (figure 8.49), with load intensity of 2.0×10^6 N/m.

For validation purposes, 24 finite strip elements were used to model the plate whereas 6 elements were used for the stiffener for both test cases. On the other hand, finite element mesh and boundary conditions were modelled using the ABAQUS S4 shell element. Figures 8.50 and 8.53 show the mesh and boundary conditions for cases with distributed loading and line loading respectively.

8.6.1 Stiffened Plate under distributed loading

The finite strip analysis has been carried out using Mindlin-type and Reissner (Spline) type elements.The results for the load increment versus the out-of-plane deflections, w at the mid-point of the stiffened plate, i.e. at (L/2, B/2) were compared with the results obtained from ABAQUS. Tsai-Wu criterion was used for the failure criteria in this validation. As can be seen in figure 8.51 a good agreement between the finite strip and finite element results has been achieved. Figure 8.52 shows the ABAQUS plot of lateral displacement for stiffened plate under uniform distributed load.

8.6.2 Stiffened Plate under Uniform line loading

Load increment versus the out-of-plane deflections, w at the mid-point of the stiffened plate, i.e. at (L/2, B/2) were compared with the results obtained from ABAQUS. Again, Tsai-Wu criterion was used for the failure criteria in this validation. As can be seen in figure 8.54, the structural response of the finite strip analysis using Mindlin-type and Reissner (Spline) type elements and the finite element using ABAQUS are in good agreement with each other. Figures 8.53 and 8.55 show the finite element model and the ABAQUS plot of lateral displacement for stiffened plate under uniform line loading respectively.

8.7 Progressive Damage Analysis of Stiffened Panel

In order to prove that the theory and the newly developed program are able to solve more sophisticated and bigger problems, a blade stiffened panel has been considered. Progressive damage analysis has been performed on a blade stiffened panel and its structural response under a uniform distributed load was compared with finite element analysis using ABAQUS. The geometry and boundary conditions of the stiffened plate such as shown in figure 8.56 where

Length in x-direction L = 4 m,

Width in y-direction, B =3 m,

Height of the two stiffeners in z-direction S = 0.75 m.

The stiffened panel consists of 12-ply Carbon-Epoxy composite material whose material properties and laminate configuration are identical to that of the stiffened plate. The stiffened panel was modelled by using 12 elements for the plate and 2 elements for every stiffener. The stiffened panel was subjected to a uniformly distributed load of intensity 5.25×10^4 N/m^2 on the top surface of the panel such as shown in figure 8.57. The finite element which was modelled in PATRAN can be seen in figure 8.58.

The load increment versus the out-of-plane central deflections at (L/2, B/2) in the x-y plane curve has been plotted for both the finite strip analysis and the finite element analysis. With reference to figure 8.59 the structural response of both the finite strip and the finite element of ABAQUS were almost identical to each other. Figure 8.60 shows the ABAQUS plot of Von Mises stress at the lowest surface for stiffened panel under uniform distributed load.

CHAPTER 9

9. Conclusions and Future Research

9.1 Conclusions

The work presented in this thesis contributes to the development of finite strip methods capable of simulating the damage or failure of composite laminates via a non-linear progressive damage analysis. Most of the research objectives defined earlier have been successfully achieved by the author. The main original contributions can be summarized as follows:

- A new finite strip method based on the tangential stiffness matrix was developed using the new concept of polynomial and spline-type finite strip elements, with Mindlin and Reissner plate-bending theories for composite plates and shells.

- A progressive damage methodology and algorithm for composite laminates was successfully developed for the new finite strip methods using stress-based failure criteria.

- A finite strip analysis package which is capable of performing a non-linear progressive damage analysis for composite plates and facetted shells, has been successfully developed with *Mindlin* and *Reissner* plate bending elements.

Validation of the developed finite strip package has been successfully carried out by comparing the results with the finite element analysis using ABAQUS and with some published experimental results. Some important observations have been found which can be summarized as follows:

- A significant reduction in the modelling and effort as a result of only one-dimensional mesh required to model plates and faceted shells by using the package built-in mesh generator.

- The built-in mesh generator in the finite strip package allows the modelling of trapezoidal and stiffened plates in a more efficient way as compared with the finite element modelling.

- The validation results have proven that the newly developed elements and progressive damage procedure are very accurate for ordinary and stiffened plates for wide range of thicknesses, load cases, boundary conditions, number of layers and stacking sequences.

- The newly derived non-linear theory based on the tangential approach has led to convergence at high loads.

9.2 Future Research

As far as the research in the field of failure of composites is concerned, there has been a strong demand on the subject of progressive damage analysis of composite laminated structures under various loading and boundary conditions. The research interest in this field is growing rapidly as more composite materials are being used as substitutes to conventional metallic structures. There are several recommendations that should be made in order to improve the current work, which can be summarized as follows:

- An extension of the current progressive damage methodology to include failure criteria to predict failure by delamination.

- A damage analysis of composite laminated plates and shells for fatigue and non-linear dynamic analysis.

- Additional experimental validation, to decide the actual maximum load carrying capacity of composite laminates.

- Design formulae for predicting the ultimate strength of composite plates and shells.

REFERENCES

References

1. ABAQUS. *ABAQUS User's manual*, Vol. 1-3, Version 6.4. Pawtucket, RI:Hibbitt, Karlsson & Sorensen, 2003.

2. Akhras, G., Cheung, M.S. & Li, W. Geometrically nonlinear finite strip analysis of laminated composite plates. *Composites Part B*, 29B, 489-495, 1998.

3. Ambur, D.R., Jaunky, N., Hilburger, M.W. & Davila, C.G. Progressive failure analyses of compression-loaded composite curved panels with and without cutouts. *Composite Structures*, 65, 143-155, 2004a.

4. Ambur, D.R., Jaunky, N. & Hilburger, M.W. Progressive failure studies of stiffened panels subjected to shear loading. *Composite Structures*, 65, 129-142, 2004b.

5. Chang, F.K. & Chang, K.Y. A progressive damage model for laminated composites containing stress concentrations. *Journal of Composite Materials*, 21, 834-855, 1987.

6. Chang, F.K. & Lessard, L.B. Damage tolerance of laminated composites containing an open hole and subjected to compressive loadings: Part I. Analysis. *Journal of Composite Materials*, 25, 2-43, 1991.

7. Cheung, M.S. & Cheung, Y.K. Natural vibrations of thin flat-walled structures with different boundary conditions. *J. Sound Vibration*, 18, 325-37, 1971.

8. Cheung, M.S., Akhras, G. & Li, W. Progressive failure analysis of composite plates by the finite strip method. *Computer Methods in Applied Mechanics & Engineering*, 124, 49-61, 1995.

9. Cheung, Y.K. The finite strip method in the analysis of elastic plates with two opposite simply supported ends. *Proc. Inst. Civil Engng.*, 40, 1-7, 1968.

10. Cheung, Y.K. & Fan, S.C. Static analysis of right box girder bridges by spline finite strip method. *Proc. Inst. Civil Engng.*, 75, 311-23, 1983.

11. Cuntze, R.G. & Freund, A. The predictive capability of failure mode concept-based strength criteria for multidirectional laminates. *Composites Science & Technology*, 64, 343-377, 2004.

12. Dawe, D.J. & Wang, S. Postbuckling analysis of thin rectangular laminated plates by spline FSM. *Thin-Walled Structures*, 30, 159-79, 1998.

13. Dawe, D.J. Use of the finite strip method in predicting the behaviour of composite laminated structures. *Composite Structures*, 57, 11-36, 2002.

14. El-Zafrany, A.M. & Cookson, R.A. An explicit formula for the generalized Hermitian problem. *Communications in Applied Numerical Methods*, 1, 85-91, 1985.

15. El-Zafrany, A.M. *Finite Element Methods. MSc lecture notes,* Cranfield University, 2002.

16. El-Zafrany, A.M. *Finite Element Programming. MSc lecture notes,* Cranfield University, 2002.

17. Fan, S.C. & Cheung, Y.K. Flexural free vibrations of rectangular plates with complex support conditions. *J. Sound Vibration*, 93, 81-94, 1984.

18. Flesher, N.D. & Herakovich, C.T. Predicting delamination in composite structures. *Composites Science & Technology*, 2005.

19. Gdoutos, E.E., Pilakoutas, K. & Rodopoulos, C.A. *Failure Analysis of Industrial Composite Materials*. McGraw-Hill, New York, 2000.

20. Goyal, V.K., Jaunky, N.R., Johnson, E.R. & Ambur, D.R. Intralaminar and interlaminar progressive failure analyses of composite panels with circular cutouts. *Composite Structures*, 64, 91-105, 2004.

21. Graves-Smith, T.R. & Sridharan, S. A finite strip method for the buckling of plat structures under arbitrary loading. *International Journal of Mechanical Sciences*, 20, 685-93, 1978a.

22. Graves-Smith, T.R. & Sridharan, S. A finite strip method for the post-locally-buckled analysis of plate structures. *International Journal of Mechanical Sciences*, 20, 833-42, 1978b.

23. Gummadi, L.N.B. & Palazotto, A.N. Progressive failure analysis of composite cylindrical shells considering large rotations. *Composite Part B*, 29B, 547-563, 1998.

24. Hancock, G.J., Davids, A.J., Key, P.W., Lau, S.C.W. & Rasmussen, K.J.R. Recent developments in the buckling and non-linear analysis of thin-walled structural members. *Thin-Walled Structures*, 9, 309-38, 1990.

25. Hegaze, M.M.E., *Finite element analysis of fatigue damage of composite laminated structures*. PhD Thesis, Cranfield University, 2002.

26. Hill, R. A theory of the yielding and plastic flow of anisotropic metals. *Proceedings of the Royal Society of London*, Series A, 193, 281-297, 1948.

27. Hinton, E.,Razzaque, A., Zienkiewicz, O.C., Davies, J. D. A simple finite element solution for plates of homogeneous, sandwich and cellular construction. *Proc. Inst. Civil Engs.*, 59, 43-65, 1975.

28. Hoffman, O. The brittle strength of orthotropic materials. *Journal of Composite Materials*, 1, 200-206, 1967.

29. Kelly, G. & Hallstrom, S. Strength and failure mechanisms of composite laminates subject to localised transverse loading. *Composite Structures*, 2004.

30. Kermanidis, T.B. & Labeas, G.N. Static and stability analysis of composite plates by a semi-analytical method. *Computers & Structures*, 57, 673-679, 1995.

31. Kim, Y., Davalos, J.F. & Barbero, E.J. Progressive failure analysis of laminated composite beams. *Journal of Composite Materials*, 30, 536, 1996.

32. Kong, J. & Cheung, Y.K. A generalized spline finite strip for the analysis of plates. *Thin-Walled Structures*, 22, 181-202, 1995.

33. Lam, S.S.E. & Zou, G.P. Higher-order shear deformable finite strip for the flexure analysis of composite laminates. *Engineering Structures*, 23, 198-206, 2001.

34. Lau, S.C. & Hancock, G.J. Buckling of thin flat-walled structures by a spline finite strip method. *Thin-Walled Structures*, 4, 265-94, 1986.

35. Lee, J.D. Three dimensional finite element analysis of damage accumulation in composite laminate. *Computers & Structures*, 15, 335-350, 1982.

36. Lessard, L.B. & Chang, F.K. Damage tolerance of laminated composites containing an open hole and subjected to compressive loadings: Part II. Experiment. *Journal of Composite Materials*, 25, 44-64, 1991.

37. Loja, M.A.R., Soares, C.M.M. & Soares, C.A.M. Modelling and design of adaptive structures using B-spline strip models. *Composite Structures*, 57, 245-251, 2002.

38. Loughlan, J. The influence of bend-twist coupling on the shear buckling response of thin laminated composite plates. *Thin–Walled Structures*, 34, 97-114, 1999.

39. Loughlan, J. *Fibre Reinforced Plastic. MSc lecture notes,* Cranfield University, 2000.

40. Loughlan, J. The shear buckling behaviour of thin composite plates with particular reference to the effects of bend-twist coupling. *International Journal of Mechanical Sciences*, 43, 771-792, 2001.

41. Loughlan, J. The influence of mechanical couplings on the compressive stability of anti-symmetric angle-ply laminates. *Composite Structures*, 57, 473-482, 2002.

42. Mindlin, R.D. Influence of rotary inertia and shear in flexural motions of isotropic elastic plates. *Journal of Applied Mechanics*, 18, 31-38, 1951.

43. Naik, G.N. & Murty, A.V.K. A failure mechanism based approach for design of composite laminates. *Composite Structures*, 45, 71-80, 1999.

44. Ochoa, O.O. & Engblom, J.J. Analysis of failure in composites. *Composites Science & Technology*, 28, 87-102, 1987.

45. Padhi, G.S., Shenoi, R.A., Moy, S.S.J. & Hawkins, G.L. Progressive failure and ultimate collapse of laminated composite plates in bending. *Composite Structures*, 40, 277-291, 1998. First ply failure ABAQUS

46. Petyt, M. Finite strip analysis of flat skin-stringer structures. *J. Sound Vibration*, 54, 537-47, 1977.

47. Plank, R.J. & Wittrick, W.H. Buckling under combined loading of thin, flat-walled structures by a complex finite strip method. *Int. J. Numer. Methods Engng.*, 8, 323-39, 1974.

48. Prusty, B.G., Ray, C. & Satsangi, S.K. First ply failure analysis of stiffened panels-a finite element approach. *Composite Structures*, 51, 73-81, 2001.

49. Przemieniecki, J.S. Finite element structural analysis of local instability. *AIAA Journal*, 11, 33-9, 1973.

50. Razzaq, R.J. *Nonlinear static and dynamic analysis of composite layered plates and shells using finite strip methods*. PhD Thesis, Cranfield University, 2003.

51. Reddy, J.N. A refined shear deformation theory for the analysis of laminated plates. *NASA Contractor Report 3955*, 1986.

52. Reddy, J.N. On refined computational models of composite laminates. *Int. J. Numer. Methods Engng.*, 27, 361-382, 1989.

53. Reddy, J.N. *Mechanics of Laminated Composite Plates and Shells*. 2nd edition, CRC Press, New York, 2004.

54. Reddy, Y.S. & Reddy, J.N. Linear and non-linear failure analysis of composite laminates with transverse shear. *Composites Science & Technology*, 44, 227-255, 1992.

55. Reddy, Y.S.N., Moorthy, C.M.D & Reddy, J.N. Non-linear progressive failure analysis of laminated composite plates. *Int. J. Non-Linear Mechanics*, 30, 629-649, 1995.

56. Reissner, E. The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics*, 12, 69-76, 1945.

57. Roufaeil, O.L. & Tran-Cong, T. Finite strip elements for laminated composite plates with transverse shear strain discontinuities. *Composite Structures*, 56, 249-258, 2002.

58. Sheikh, A.H. & Mukhopadhyay, M. Geometric nonlinear analysis of stiffened plates by the spline finite strip method. *Computers & Structures*, 76, 765-785, 2000.

59. Singh, S.B. & Kumar, A. Postbuckling response and strength of laminates under combined in-plane loads. *Composites Science & Technology*, 59, 727-736, 1999.

60. Sleight, D.W., Knight, N.F. & Wang, J.T. Evaluation of a progressive failure analysis methodology for laminated composite structures. *AIAA Paper No. 97-1187*, 2257-72, 1997.

61. Sleight, D.W. Progressive failure analysis methodology for laminated composite structures. *NASA/TP-1999-209107*, 1999.

62. Smith, C.S. Bending, buckling and vibration of orthotropic plate-beam structures. *J. Ship. Res.*, 12, 249-68, 1968.

63. Soares, C.M.M., Soares, C.A.M., Correia, V.M.F. & Loja, M.A.R. Higher-order B-spline models for laminated composite structures with integrated sensors and actuators. *Composite Structures*, 54, 267-274, 2001.

64. Spottswood, S.M. & Palazotto, A.N. Progressive failure analysis of a composite shell. *Composite Structures*, 53, 117-131, 2001.

65. Tan, S.C. A progressive failure model for composite laminates containing openings. *Journal of Composite Materials*, 25, 556-77, 1991. thermal residual strength.

66. Tsai, S.W. Strength characteristics of composite materials, *NASA CR-224*, 1965.

67. Tsai, S.W. & Wu, E.M. A general theory of strength for anisotropic materials. *Journal of Composite Materials*, 5, 58-80, 1971.

68. Tsai, S.W. A survey of macroscopic failure criteria for composite materials. *Journal of Reinforced Plastics & Composites*, 3, 40-63, 1984.

69. Turvey, G.J. & Wittrick, W.H. The influence of orthotropy on the stability of some multi-plate structures in compression. *Aero Quart*, 24, 1-8, 1973.

70. Wang, S. & Dawe, D.J. The use of spline functions in calculating the natural frequencies of anisotropic rectangular laminates. In: Marshall IH, editor. *Composite Structures-4*, London: Elsevier, p. 1.447-60, 1987.

71. Wang, S. & Zhang, Y. Vibration analysis of rectangular composite laminated plates using layerwise B-spline finite strip method. *Composite Structures*, 68, 349-358, 2005.

72. Wang, W.J., Tseng, Y.P. & Lin, K.J. Stability of laminated plates using finite strip method based on higher-order plate theory. *Composite Structures*, 34, 65-76, 1996.

73. Wittrick, W.H. A unified approach to the initial buckling of stiffened panels in compression. *Aero Quart*, 12, 265-83, 1968a.

74. Wittrick, W.H. General sinusoidal stiffness matrices for buckling and vibration analysis of thin-walled structures. *International Journal of Mechanical Sciences*, 10, 946-66, 1968b.

75. Zaghloul, S.A. & Kennedy, J.B. Nonlinear behaviour of symmetrically laminated plates. *Journal of Applied Mechanics*, 42, 234-236, 1975.

76. Zhang, Y., Wang, S. & Petersson, B. Large deflection analysis of composite laminates. *Journal of Materials Processing Technology*, 138, 34-40, 2003.

77. Zhong, W.X., Cheung, Y.K. & Li, Y. The precise finite strip method. *Computers & Structures*, 69, 773-783, 1998.

78. Zienkiewicz, O.C. & Taylor, R.L. *The finite element method.* 5th edition, Butterworth & Heinemann, Oxford, 2000.

TABLES

Tables

Table 8.1. Material Properties of Cabon-Epoxy plates and shells

Parameter	Value
Longitudinal Modulus E_{11} [GPa]	134.75
Transverse Modulus E_{22} [GPa]	8.24
Shear Modulus in x-y plane G_{12} [GPa]	7.0
Shear Modulus in y-z plane G_{23} [GPa]	7.0
Shear Modulus in z-x plane G_{31} [GPa]	7.0
Major Poisson's Ratio v_{12}	0.325
No. of layers	12
Thickness of each layer [m]	0.0025
Stacking sequence	$[-45/0/45]_{2s}$
X_t – longitudinal (fibre direction) tensile failure strength. [Mpa]	1500.0
Y_t – transverse (matrix direction) tensile failure strength. [Mpa]	50.0
X_c - longitudinal (fibre direction) compressive failure strength. [Mpa]	1200.0
Y_c - transverse (matrix direction) compressive failure strength. [Mpa]	250.0
S – in-plane shear failure strength[Mpa]	70.0

Table 8.2.Maximum axial displacement (m) of a C/E cantilever plate under tension with different thickness variation.

Laminate Thickness h(m)	Failure Theories	Element Type		
		Mindlin	Reissner (Hermitian)	Reissner (Spline)
0.03	Tsai-Hill	2.05E-03	2.03E-03	1.92E-03
	Tsai-Wu	1.91E-03	1.93E-03	1.91E-03
	Hoffman	2.02E-03	2.07E-03	1.90E-03
0.06	Tsai-Hill	2.07E-03	2.07E-03	1.88E-03
	Tsai-Wu	2.08E-03	1.95E-03	1.90E-03
	Hoffman	2.05E-03	1.90E-03	1.94E-03
0.12	Tsai-Hill	9.40E-04	9.40E-04	9.40E-04
	Tsai-Wu	9.43E-04	9.43E-04	9.40E-04
	Hoffman	9.43E-04	9.43E-04	9.40E-04
0.24	Tsai-Hill	4.69E-04	4.68E-04	4.69E-04
	Tsai-Wu	4.69E-04	4.68E-04	4.69E-04
	Hoffman	4.69E-04	4.68E-04	4.69E-04

Table 8.3.Maximum axial displacement (m) of a C/E cantilever plate under tension with different stacking sequences.

Stacking Sequence	Failure Theories	Element Type		
		Mindlin	Reissner (Hermitian)	Reissner (Spline)
[0°/90°/90°/0°]	Tsai-Hill	1.48E-03	1.48E-03	1.48E-03
	Tsai-Wu	1.48E-03	1.48E-03	1.48E-03
	Hoffman	1.48E-03	1.48E-03	1.48E-03
[0°/90°/0°/90°]	Tsai-Hill	1.50E-03	1.48E-03	1.48E-03
	Tsai-Wu	1.49E-03	1.48E-03	1.48E-03
	Hoffman	1.50E-03	1.48E-03	1.48E-03
[45°/-45°/-45°/45°]	Tsai-Hill	1.01E-03	1.06E-03	1.01E-03
	Tsai-Wu	9.01E-04	9.20E-04	9.10E-04
	Hoffman	9.09E-04	9.20E-04	9.10E-04
[45°/-45°/45°/-45°]	Tsai-Hill	1.05E-03	1.06E-03	1.01E-03
	Tsai-Wu	9.39E-04	9.20E-04	9.10E-04
	Hoffman	9.39E-04	9.20E-04	1.01E-03

Table 8.4. .Maximum y-displacement (m) of a C/E cantilever plate under in-plane-bending with various stacking sequences.

Stacking Sequence	Failure Theories	Element Type		
		Mindlin	Reissner (Hermitian)	Reissner (Spline)
[0°/90°/90°/0°]	Tsai-Hill	3.37E-03	3.17E-03	3.10E-03
	Tsai-Wu	3.37E-03	3.17E-03	3.04E-03
	Hoffman	3.37E-03	3.17E-03	3.04E-03
[0°/90°/0°/90°]	Tsai-Hill	3.38E-03	3.17E-03	3.10E-03
	Tsai-Wu	3.38E-03	3.17E-03	3.04E-03
	Hoffman	3.38E-03	3.17E-03	3.04E-03
[45°/-45°/-45°/45°]	Tsai-Hill	9.23E-03	8.43E-03	9.05E-03
	Tsai-Wu	1.07E-02	9.44E-03	1.02E-02
	Hoffman	1.07E-02	1.10E-02	1.10E-02
[45°/-45°/45°/-45°]	Tsai-Hill	9.75E-03	8.43E-03	9.05E-03
	Tsai-Wu	9.31E-03	9.44E-03	1.02E-02
	Hoffman	9.31E-03	9.44E-03	1.02E-02

Table 8.5. Material properties and dimensions of square plate.

Parameter	Value
Longitudinal Modulus E_{11} [GPa]	12.61
Transverse Modulus E_{22} [GPa]	12.61
Shear Modulus in x-y plane G_{12} [GPa]	2.15
Shear Modulus in y-z plane G_{23} [GPa]	2.15
Shear Modulus in z-x plane G_{31} [GPa]	2.15
Major Poisson's Ratio v_{12}	0.24
Length and width of plate [mm]	304.8
Thickness of each layer [mm]	0.6
X_t – longitudinal (fibre direction) tensile failure strength.[Mpa]	1316.94
Y_t – transverse (matrix direction) tensile failure strength.[Mpa]	42.75
X_c - longitudinal (fibre direction) compressive failure strength.[Mpa]	1220.0
Y_c - transverse (matrix direction) compressive failure strength.. [Mpa]	168.20
S – in-plane shear failure strength.[Mpa]	48.30

Table 8.6. Non-dimensional out-of-plane deflections of rectangular plate under uniformly distributed loads.

Load q (psi)	w/h		
	Present FS method *(Mindlin)*	**Zhang et al,2003**	**Reddy,1989**
0.04	-1.037	-1.035	-1.034
0.05	-1.100	-1.100	-1.100
0.10	-1.330	-1.328	-1.327
0.25	-1.710	-1.705	-1.705
0.50	-2.077	-2.073	-2.075
0.75	-2.330	-2.330	-2.332
1.00	-2.540	-2.535	-2.532

FIGURES

Figures

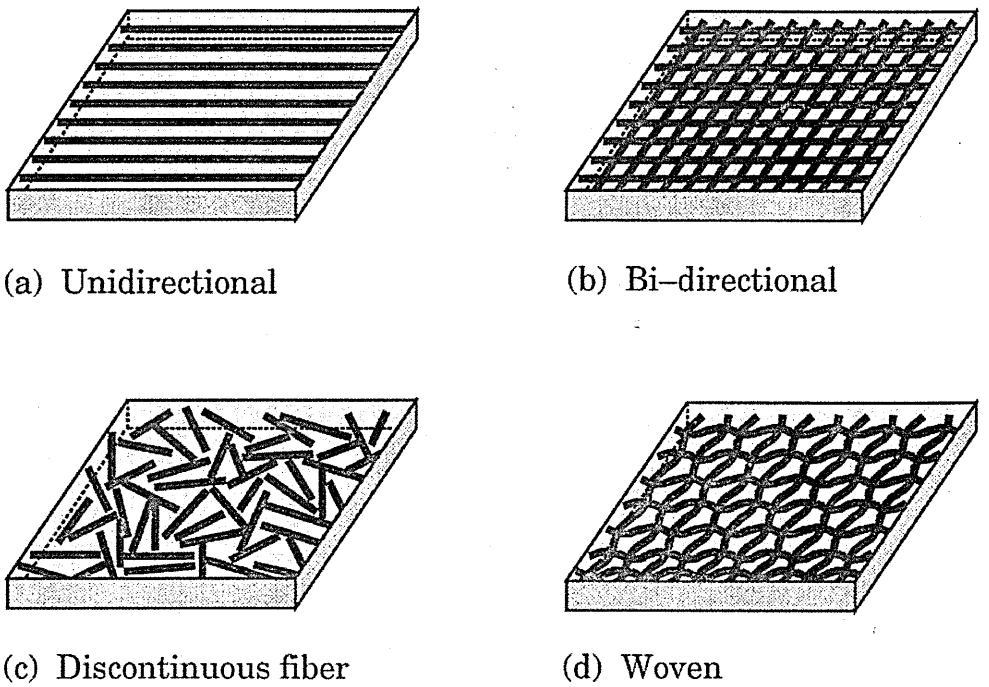

Figure 3.1 Various types of fibre-reinforced composite laminae

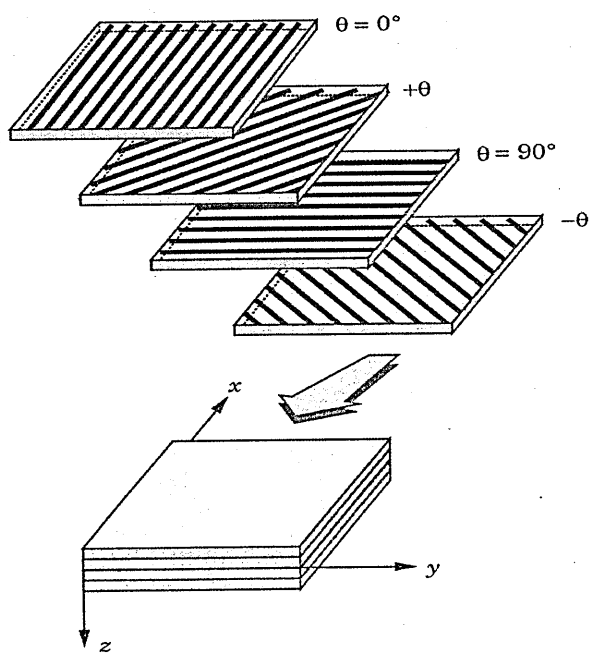

Figure 3.2. Stacking sequence of a laminate

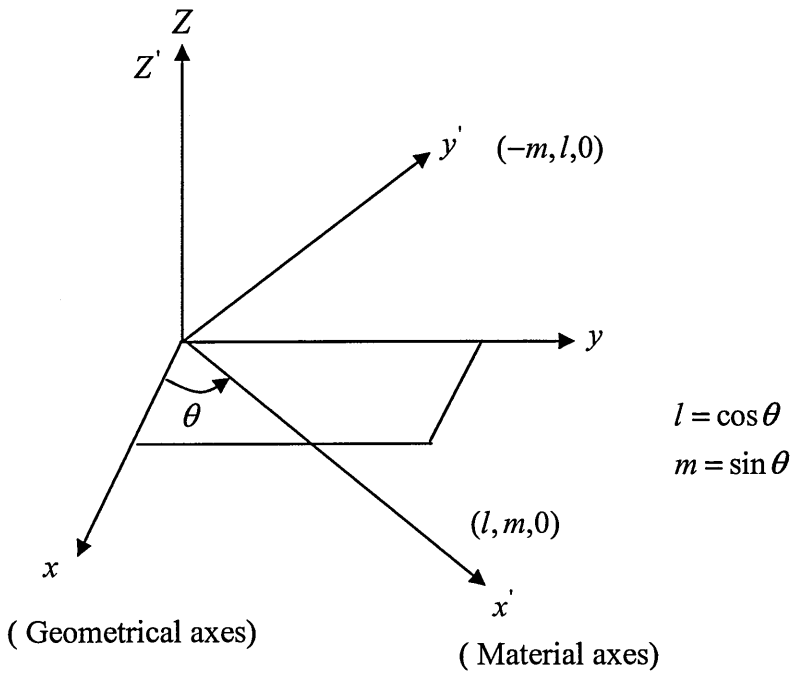

Figure 3.3. Local and material axes of lamina

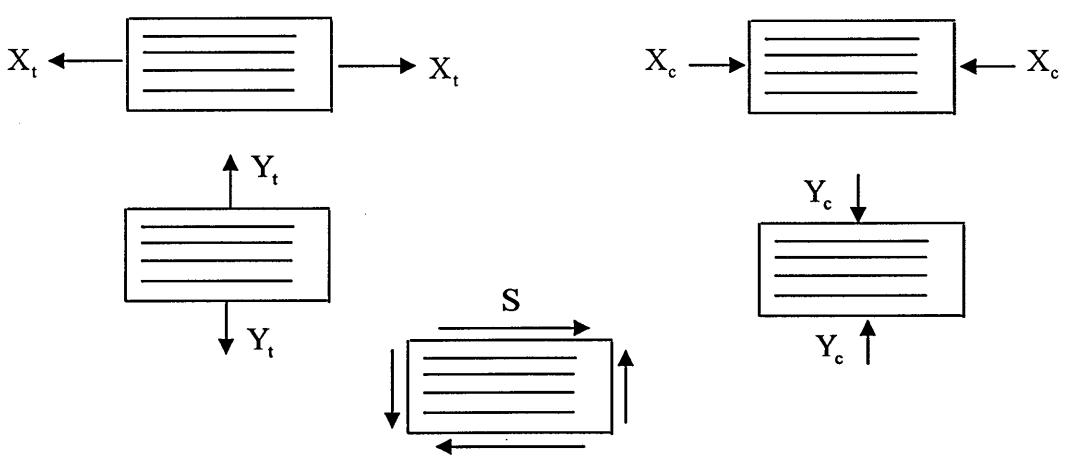

Figure 3.4. Five basic failure strengths for a specially orthotropic ply

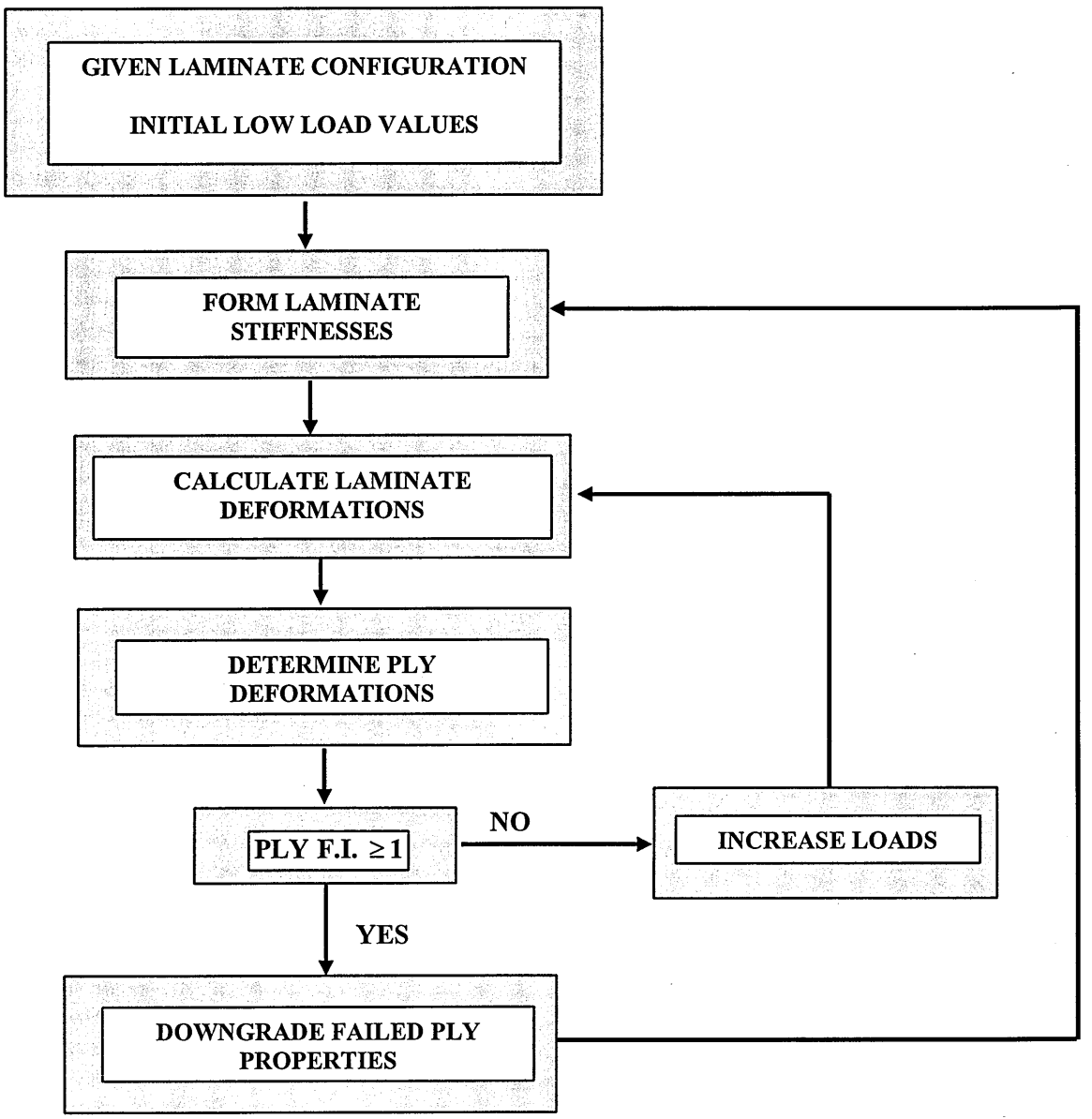

Figure 3.5. Progressive failure methodology for composite laminated plate.

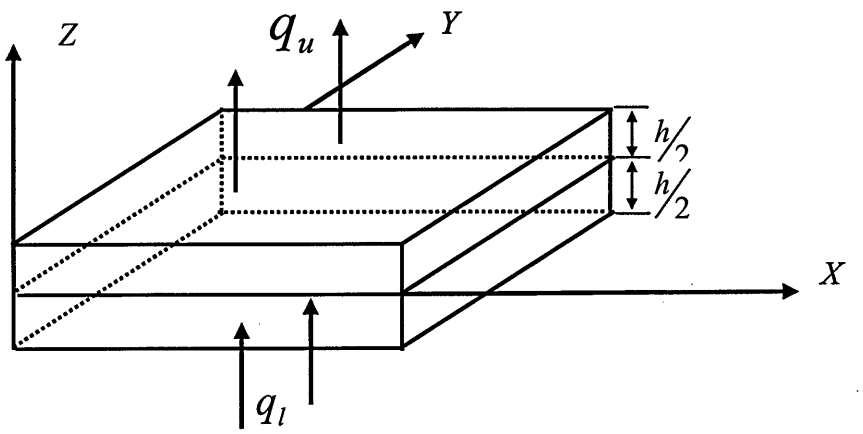

Figure 4.1 Plate reference axes

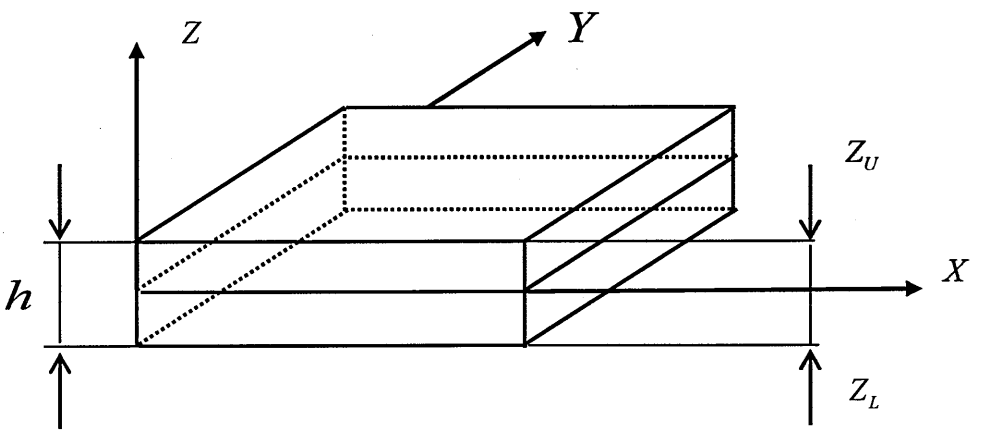

Figure 4.2. Plate with local xyz axes

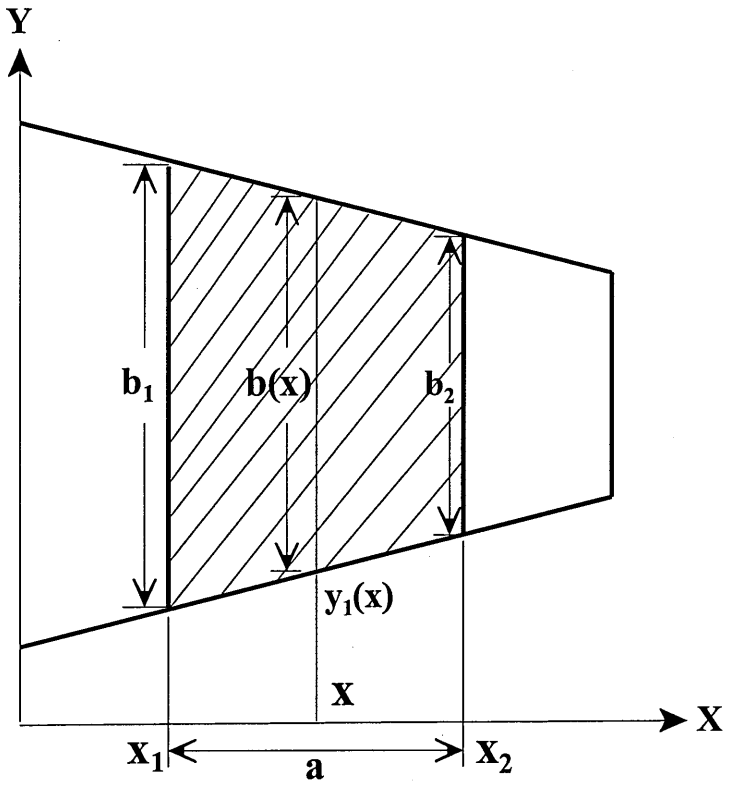

Figure 4.3. Finite strip element

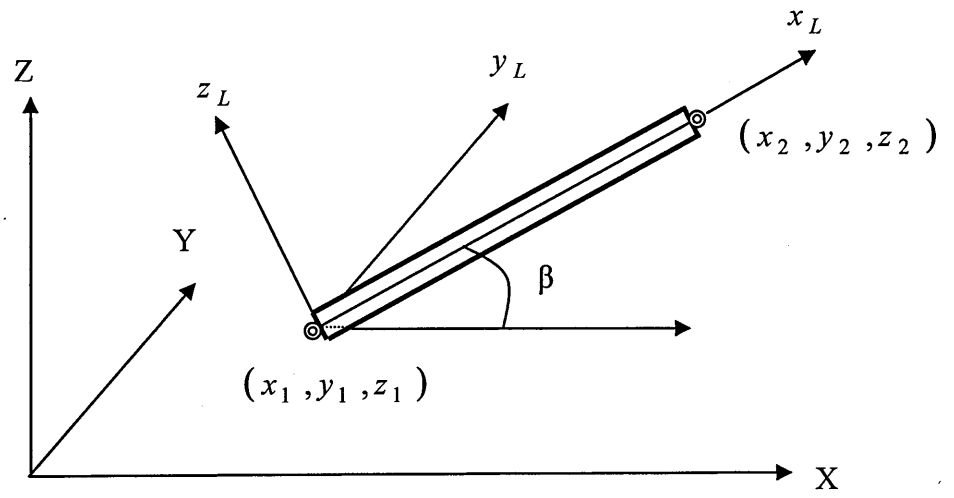

Figure 6.1. Plate local and global axis

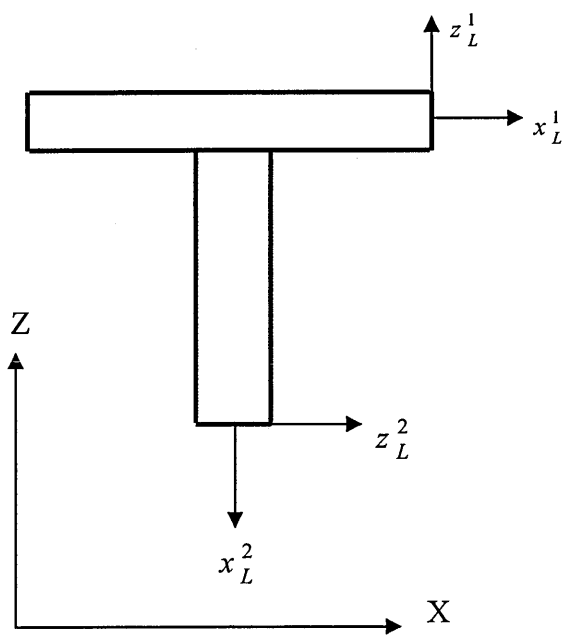

Figure 6.2. Local and global axes of a folded plate

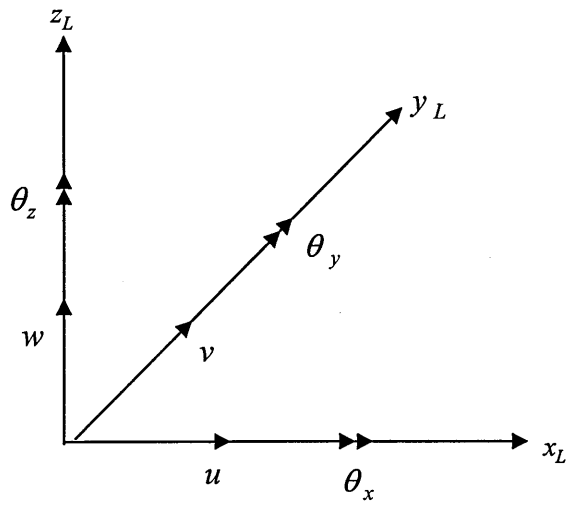

Figure 6.3. degrees of freedom for Mindlin-type finite strip element

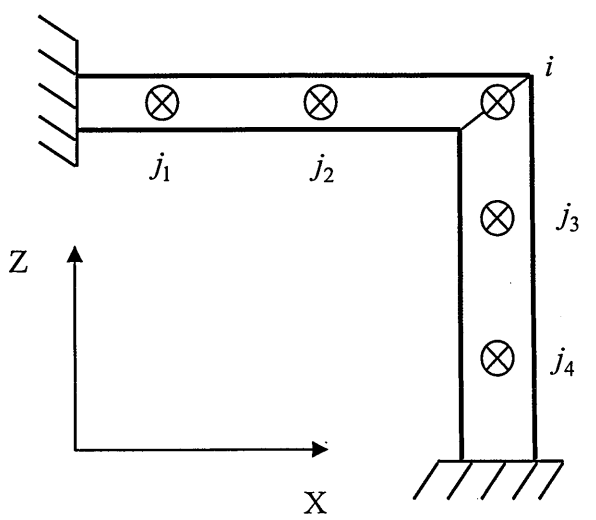

Figure 6.4. Co-planar and non-co-planar nodes

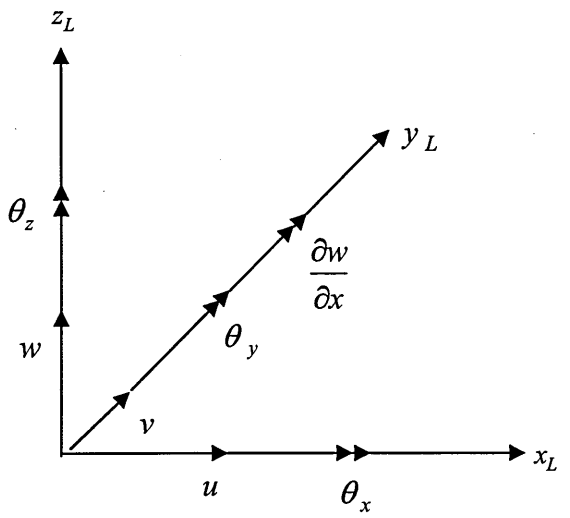

Figure 6.5. Degrees of freedom for Reissner-type finite strip element

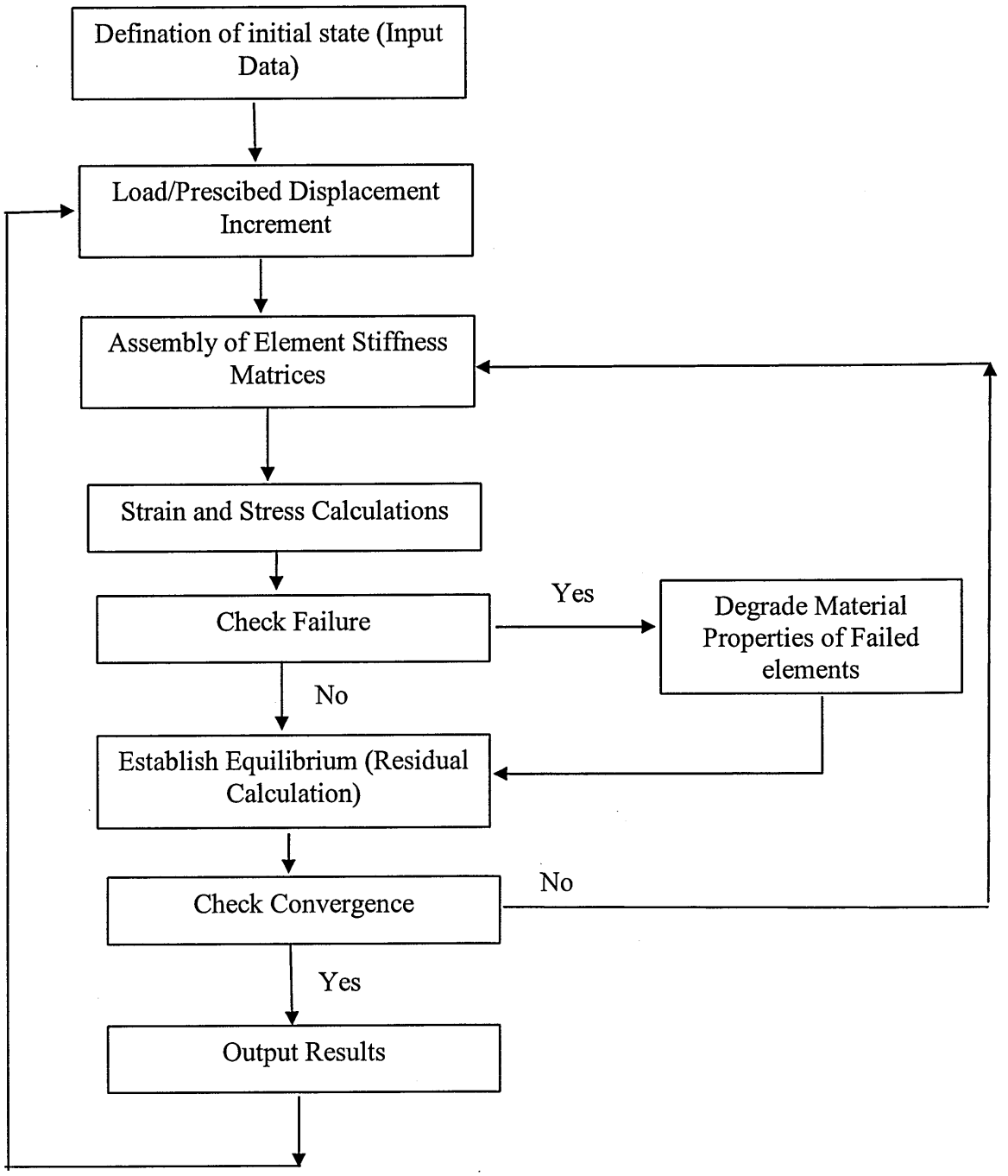

Figure 6.6. Progressive failure analysis algorithm

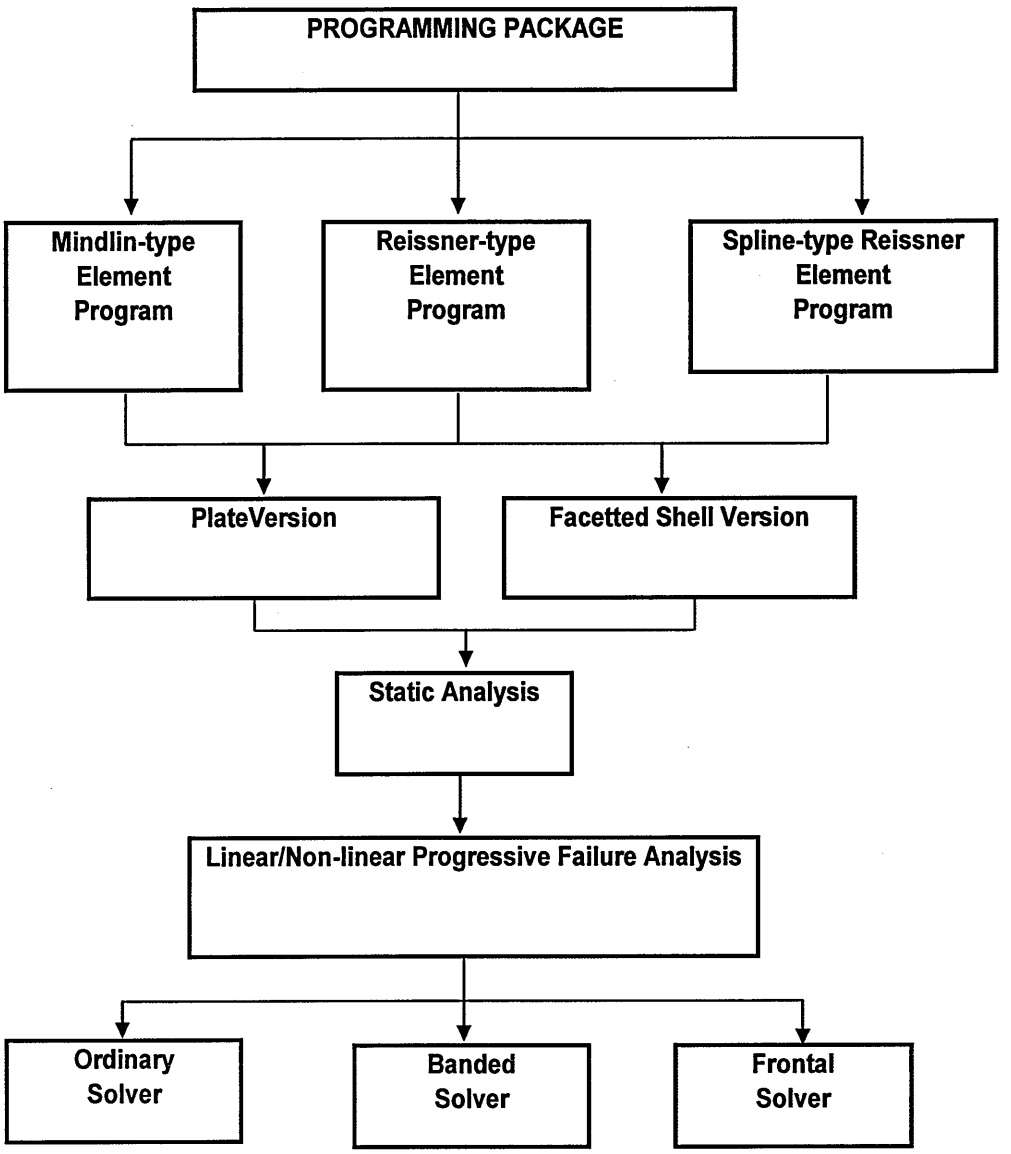

Figure 7.1.Layout of the Programming package

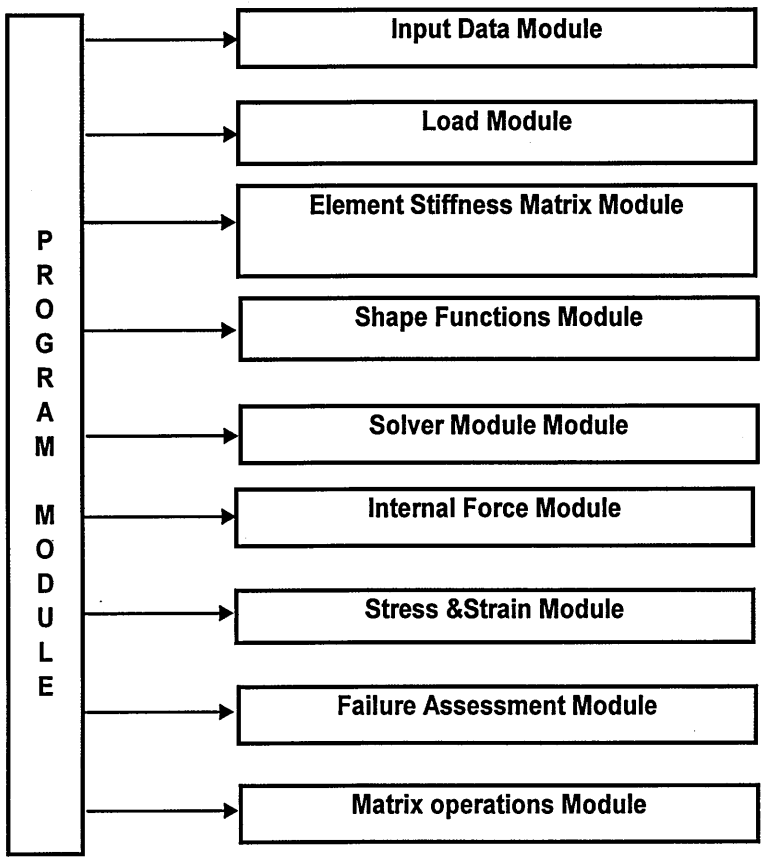

Figure 7. 2. Program basic modules

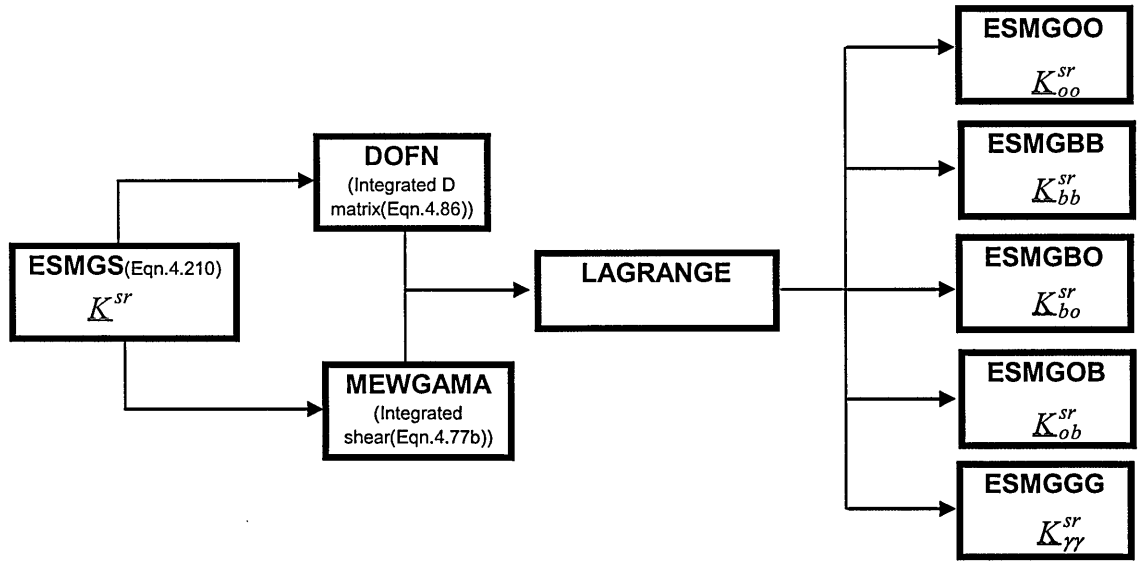

Figure 7.3. Element stiffness matrix generator $\underline{K}^{sr}$ of Mindlin-type element

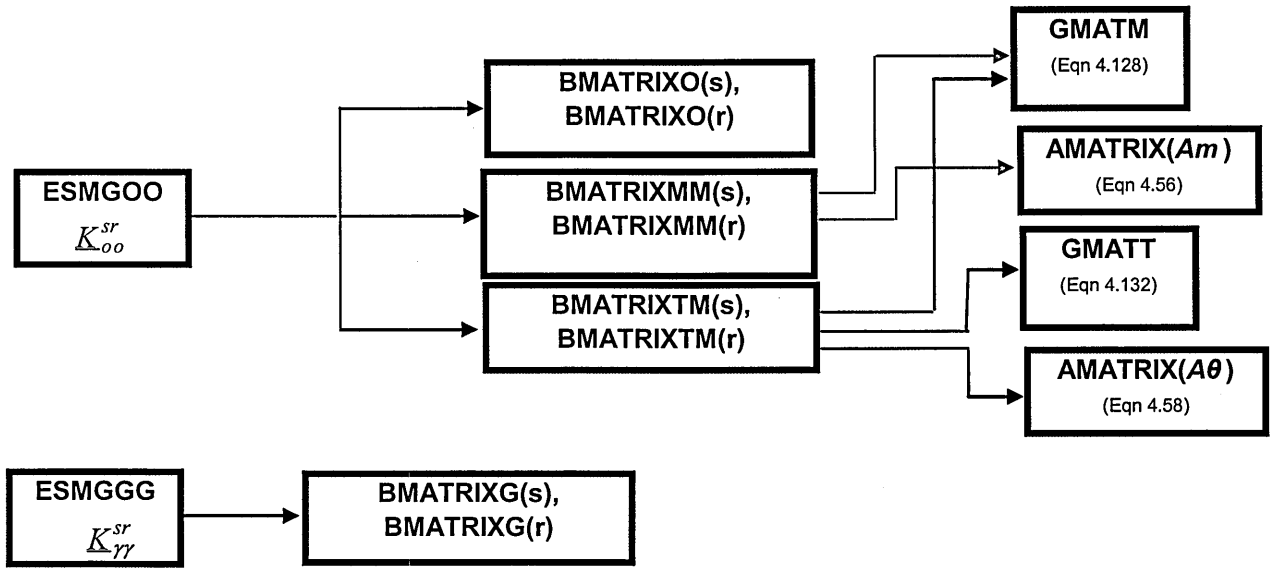

Figure 7.4.Main subroutines of Mindlin-type element for $\underline{K}_{oo}^{sr}$ and $\underline{K}_{\gamma\gamma}^{sr}$.

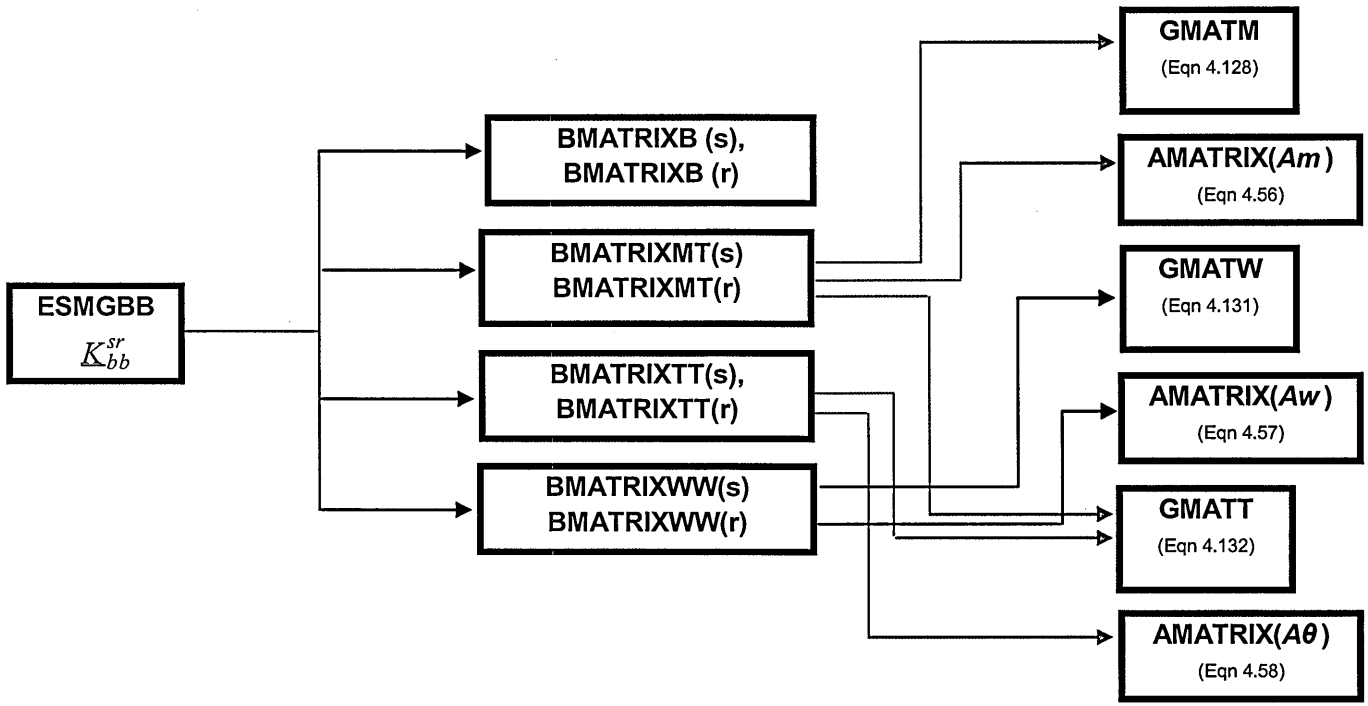

Figure 7.5.Main subroutines of Mindlin-type element for $\underline{K}_{bb}^{sr}$.

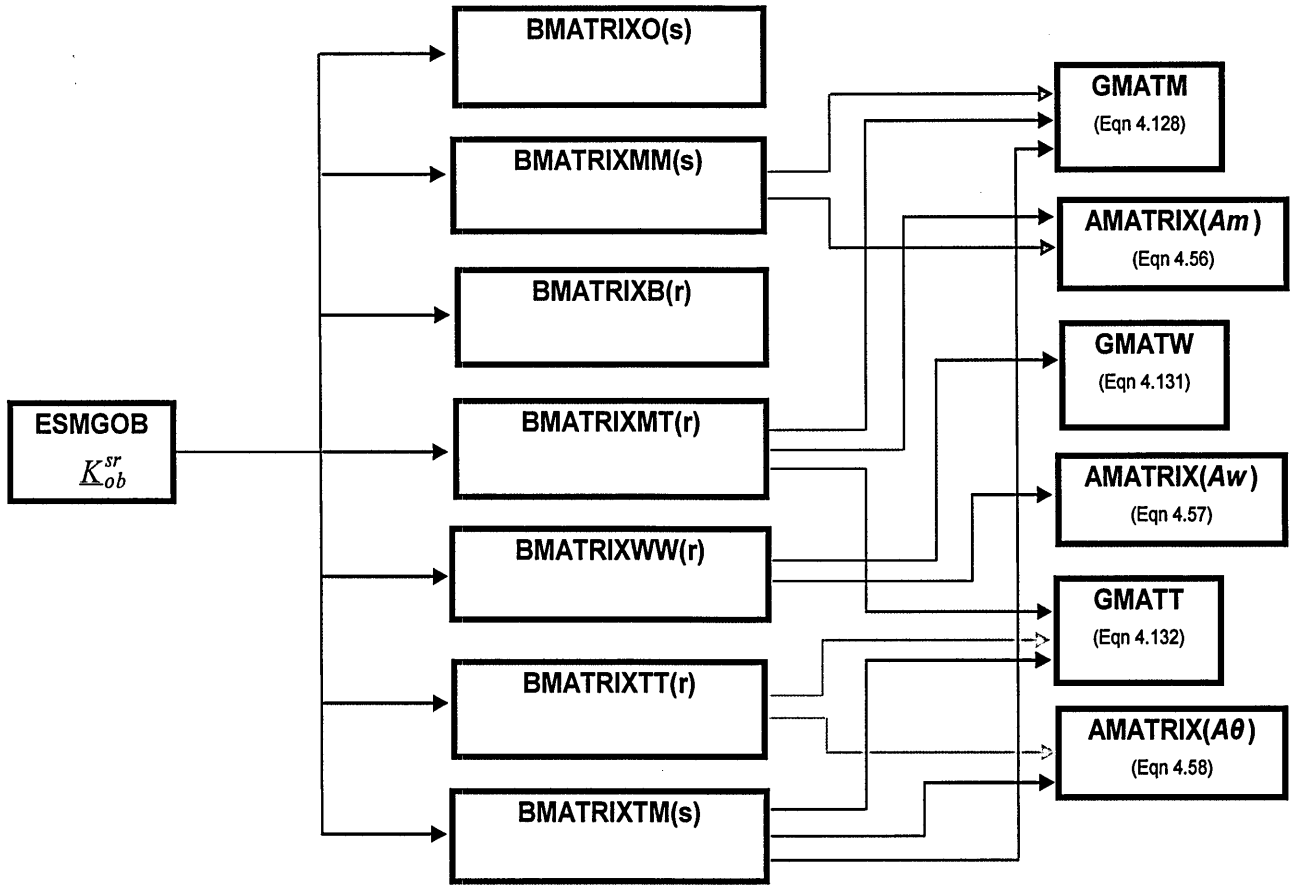

Figure 7.6.Main subroutines of Mindlin-type element for $\underline{K}_{ob}^{sr}$

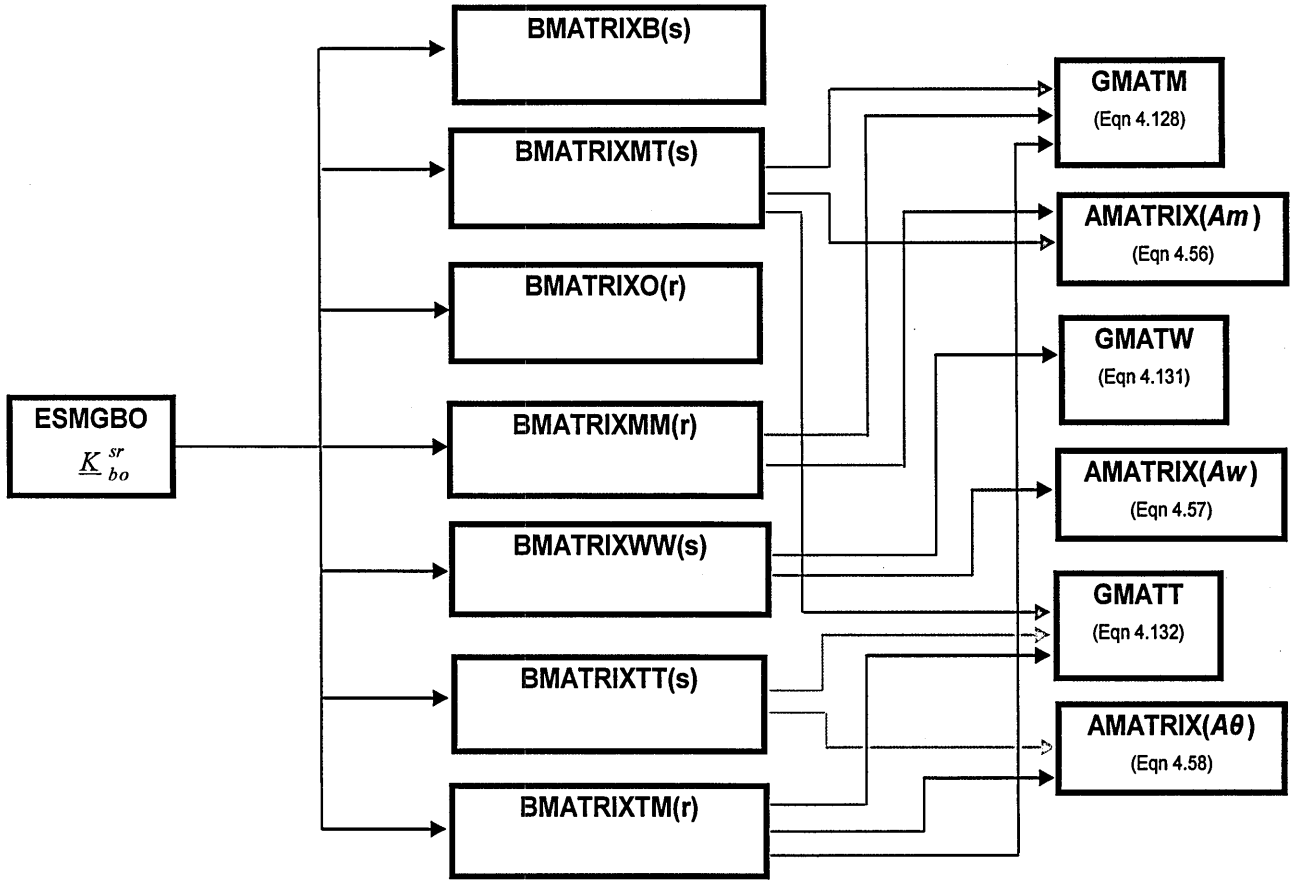

Figure 7.7.Main subroutines of Mindlin-type element for $\underline{K}_{bo}^{sr}$

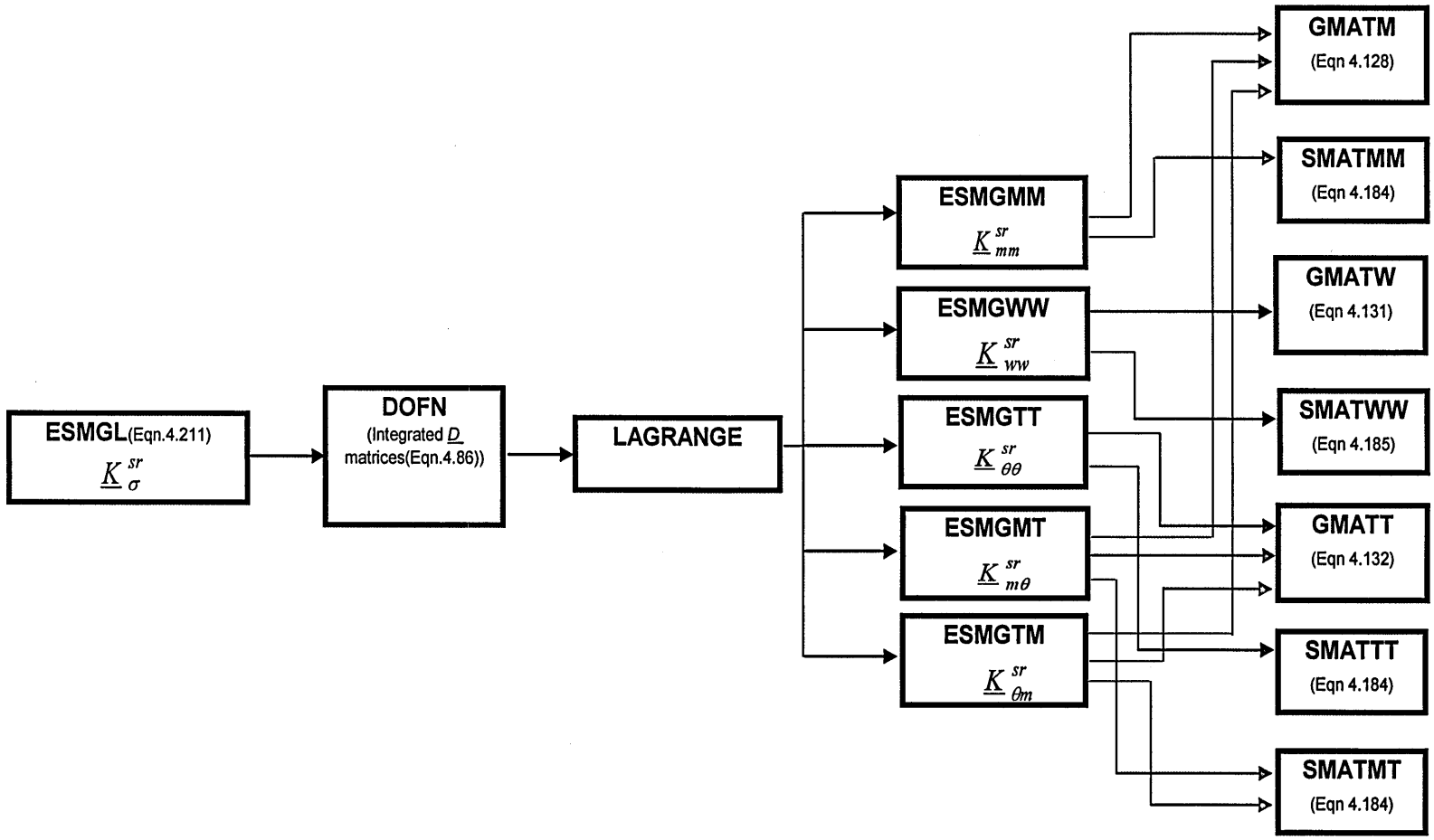

Figure 7.8.Main subroutines for $\underline{K}_{\sigma}^{sr}$ of Mindlin-type element

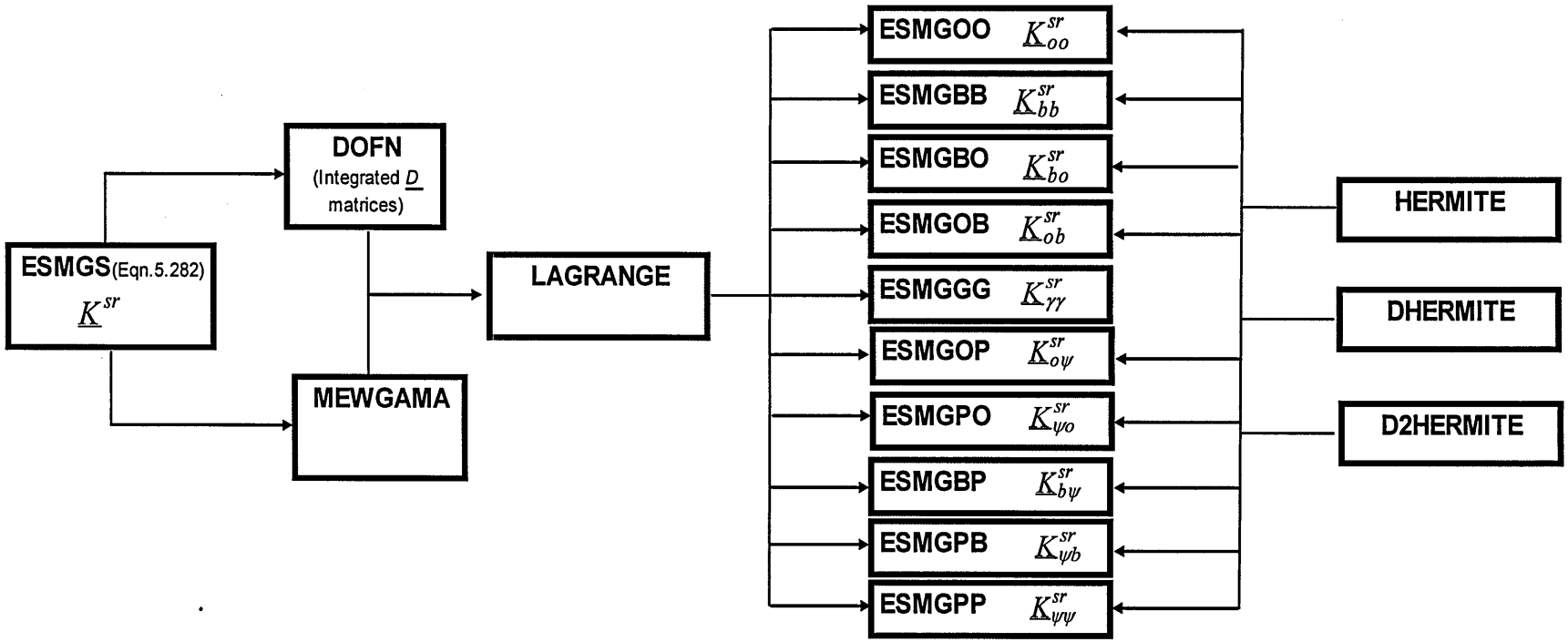

Figure 7.9. Main subroutines for $\underline{K}^{sr}$ of Reissner-type element

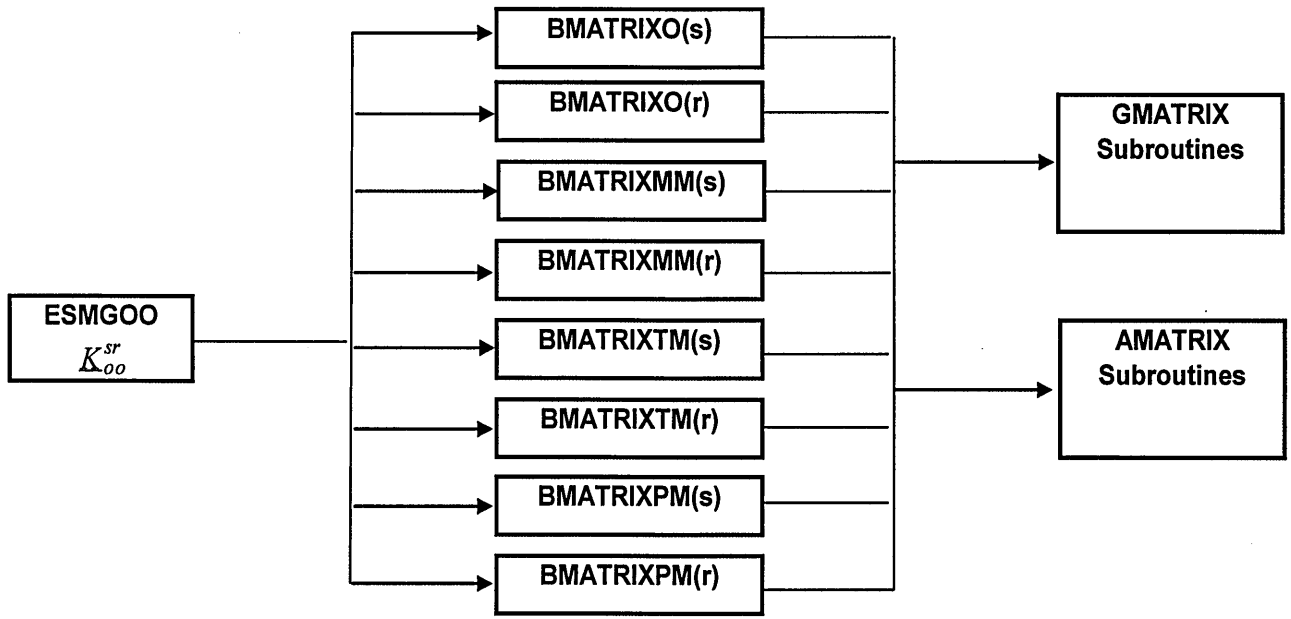

Figure 7.10. Main subroutines for $\underline{K}_{oo}^{sr}$ of Reissner-type element

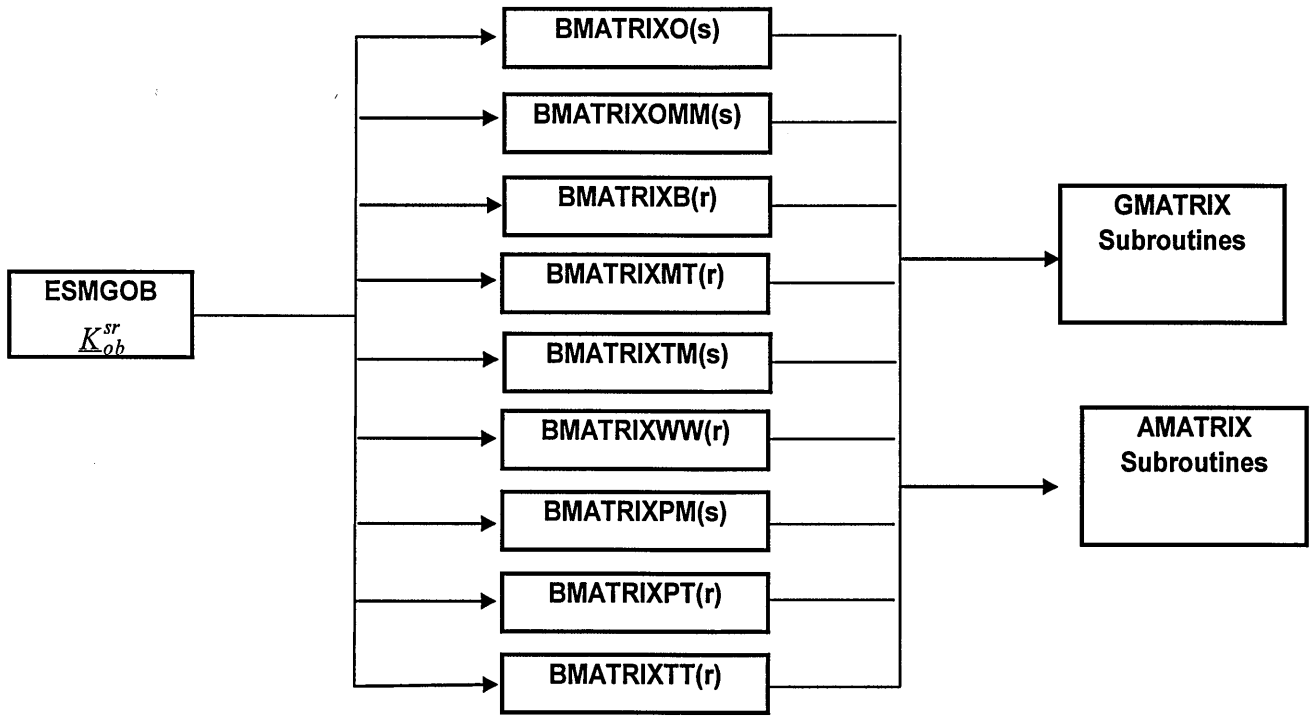

Figure 7.11. Main subroutines for $\underline{K}_{ob}^{sr}$ of Reissner-type element

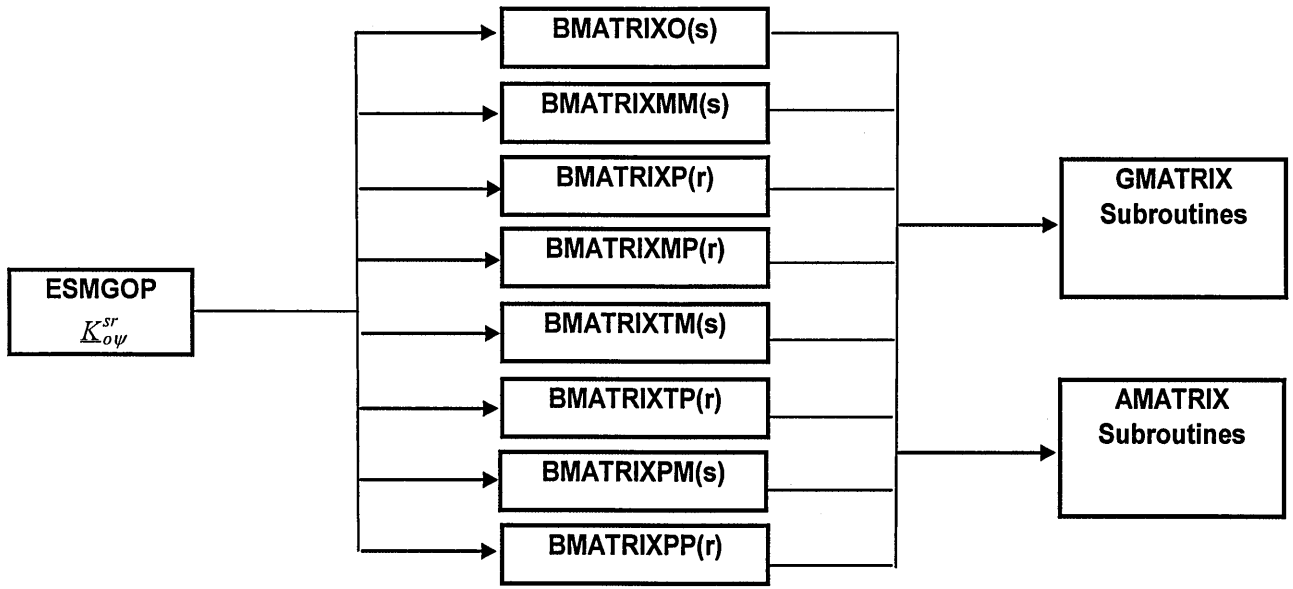

Figure 7.12. Main subroutines for $\underline{K}_{o\psi}^{sr}$ of Reissner-type element

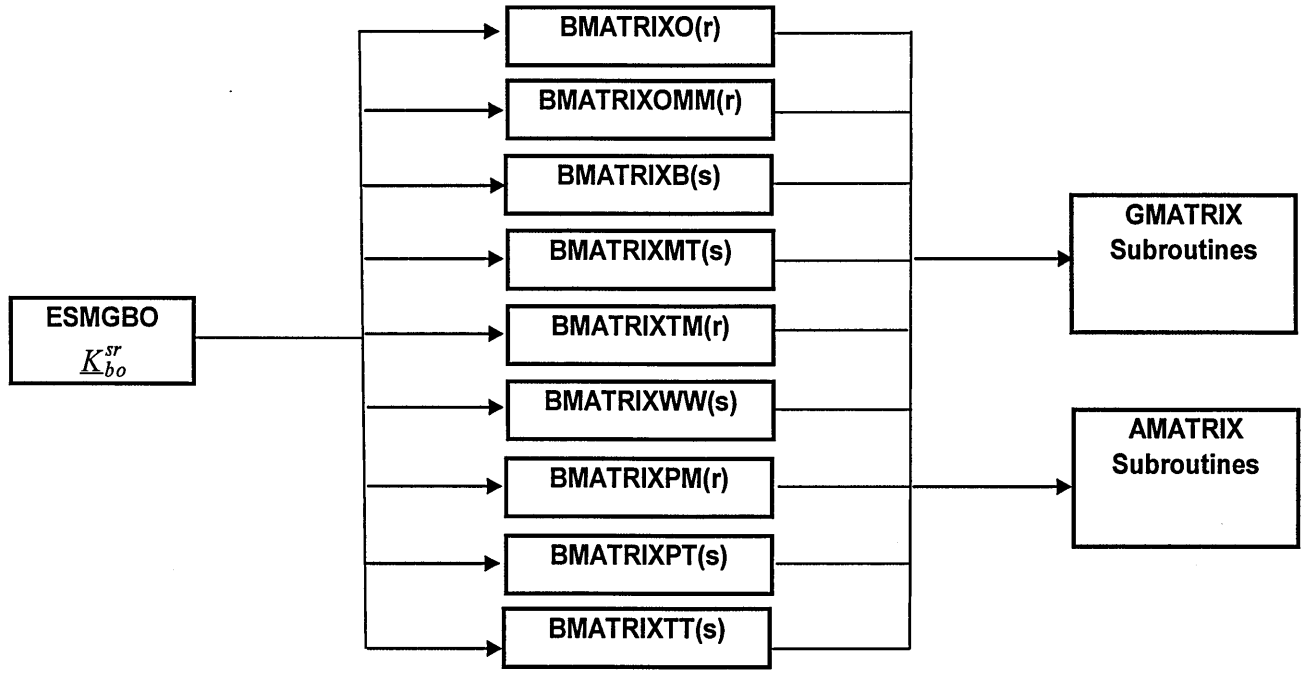

Figure 7.13. Main subroutines for $\underline{K}_{bo}^{sr}$ of Reissner-type element

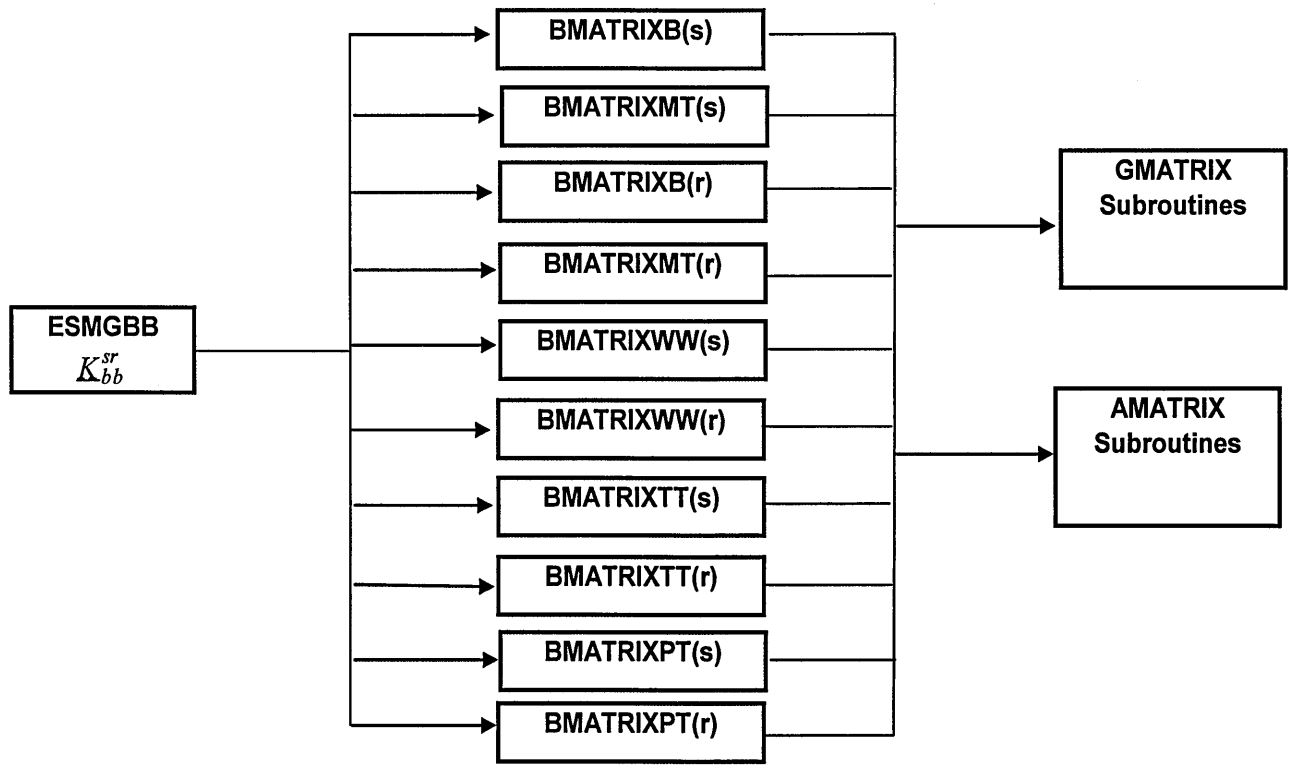

Figure 7.14. Main subroutines for $\underline{K}_{bb}^{sr}$ of Reissner-type element

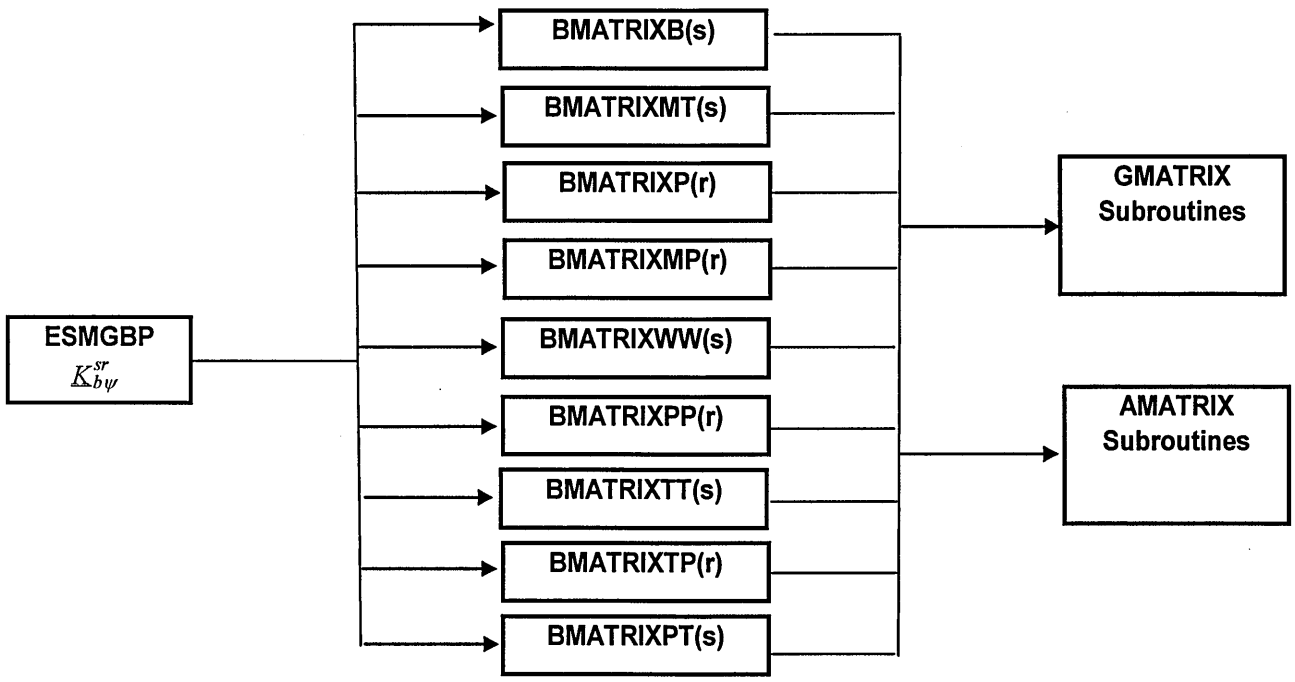

Figure 7.15. Main subroutines for $\underline{K}_{b\psi}^{sr}$ of Reissner-type element

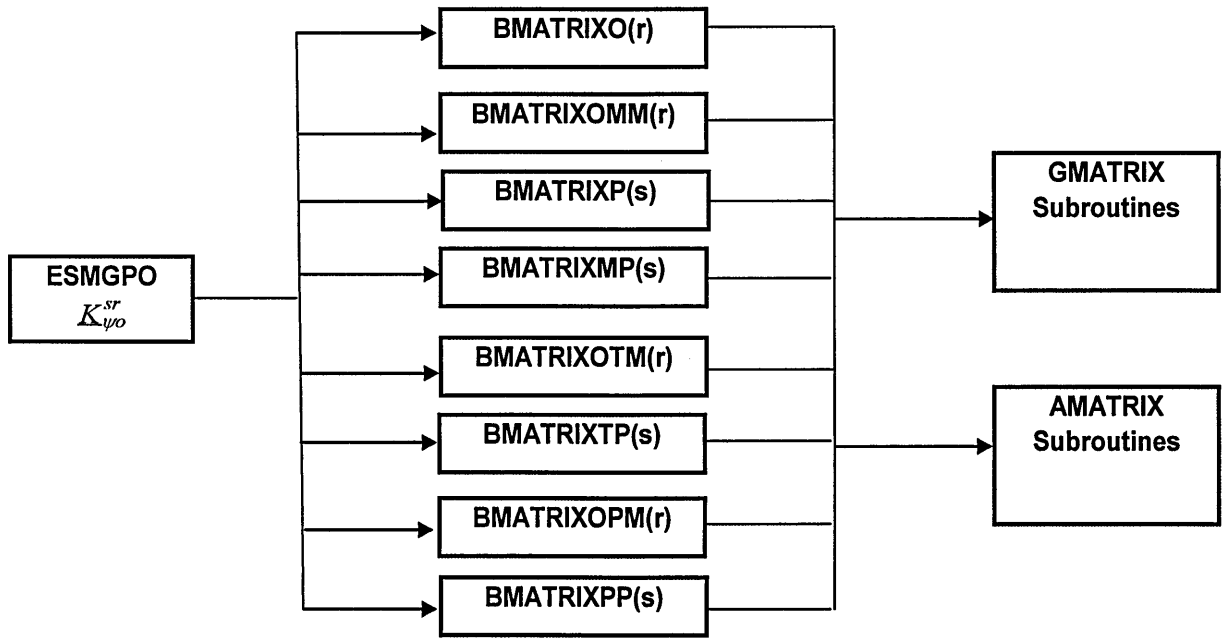

Figure 7.16. Main subroutines for $\underline{K}_{\psi o}^{sr}$ of Reissner-type element

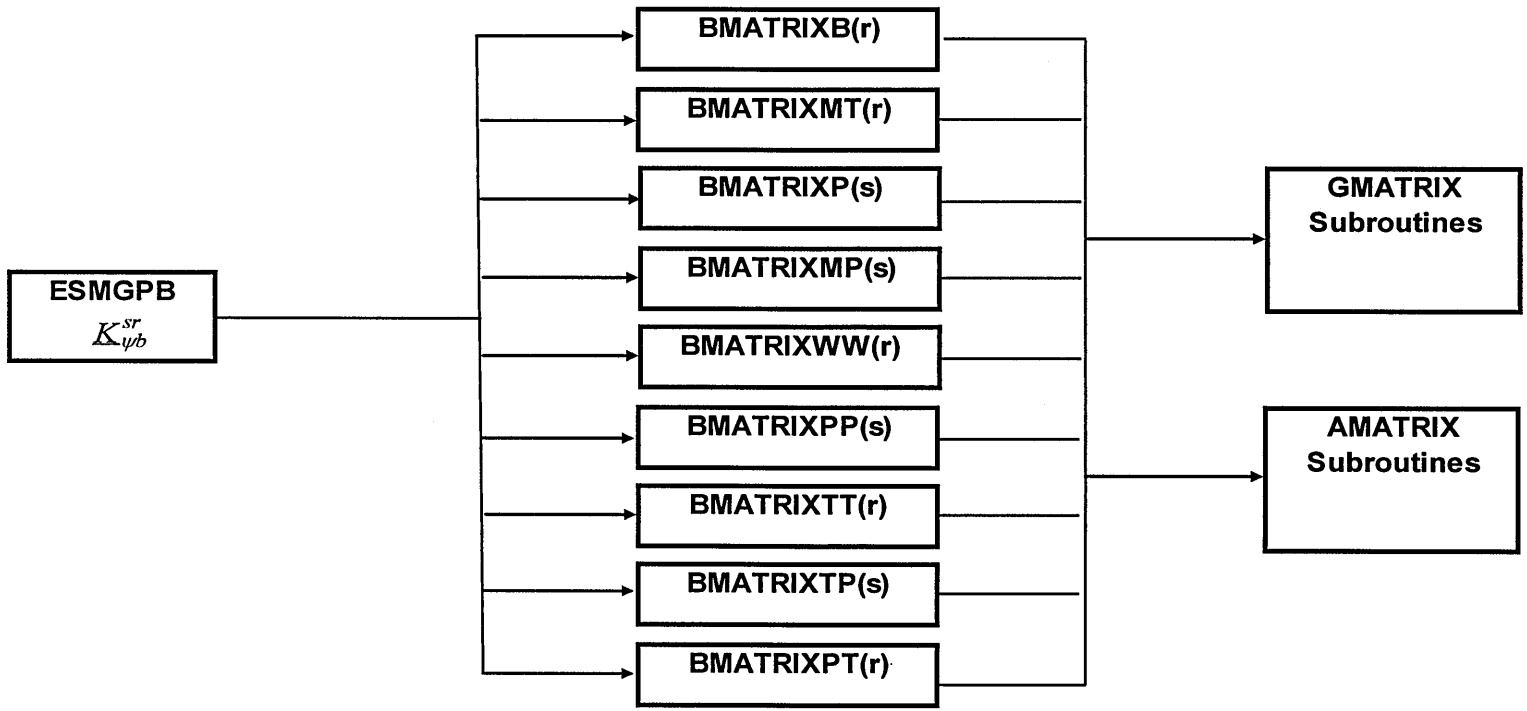

Figure 7.17. Main subroutines for $\underline{K}_{\psi b}^{sr}$ of Reissner-type element

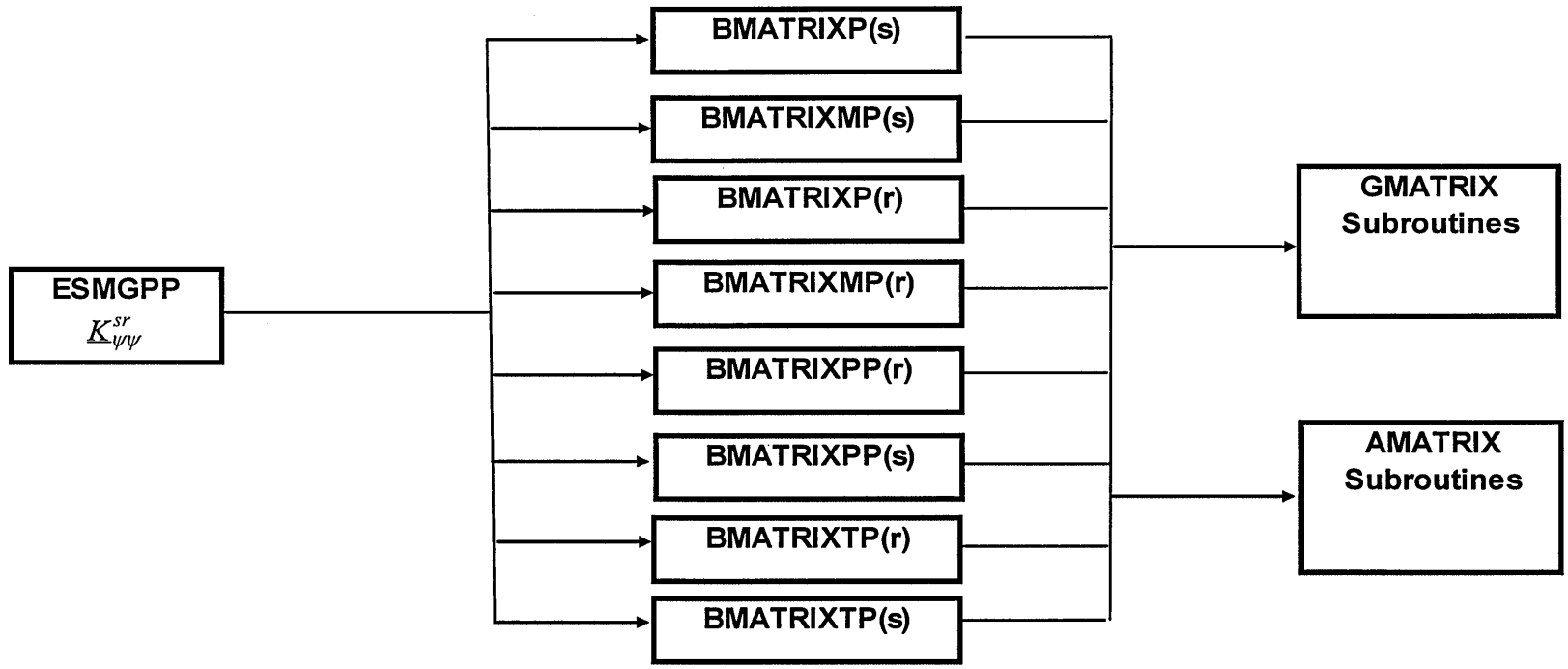

Figure 7.18. Main subroutines for $\underline{K}_{\psi\psi}^{sr}$ of Reissner-type element

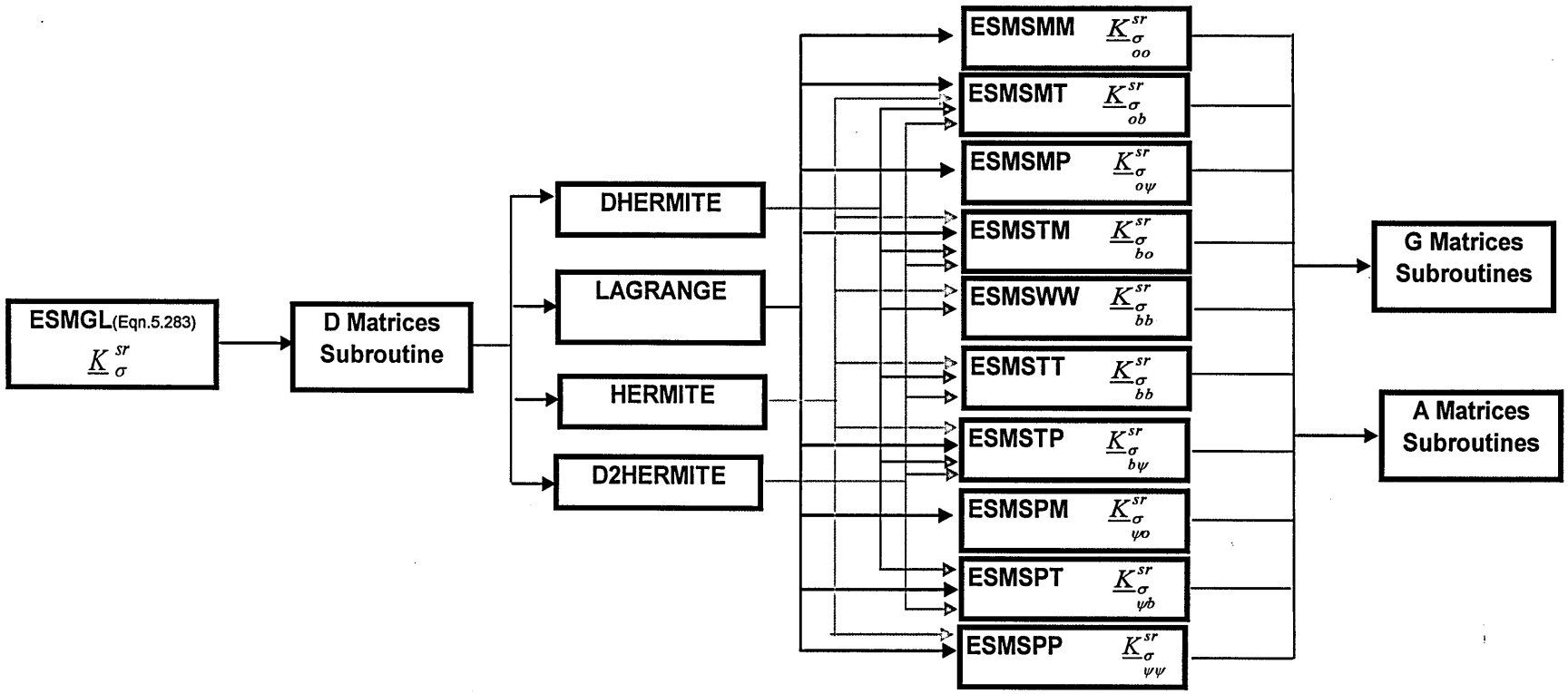

Figure 7.19. Main subroutines for $\underline{K}_{\sigma}^{sr}$ of Reissner-type element

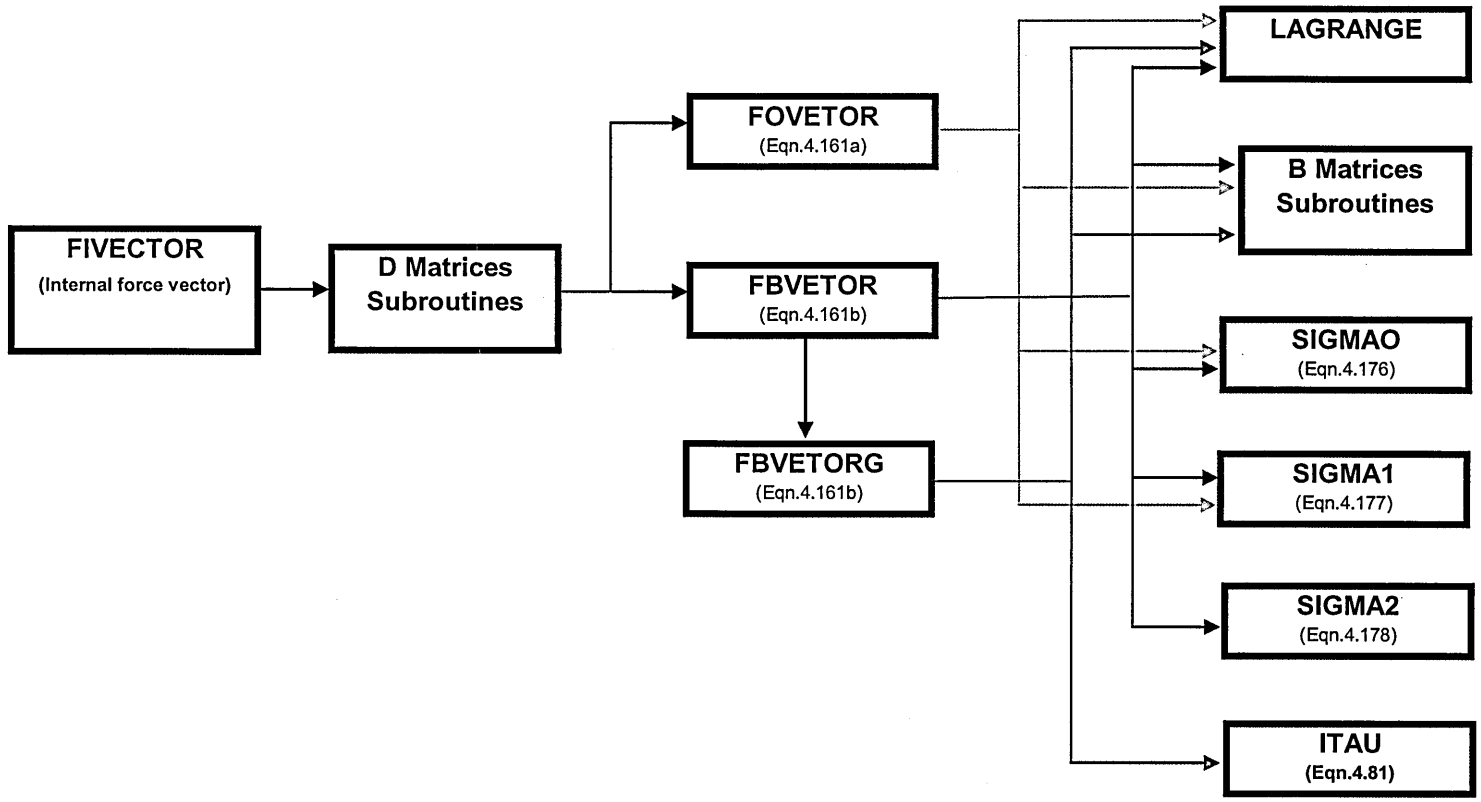

Figure 7.20. Internal force vector (FIVECTOR) module for Mindlin-type element

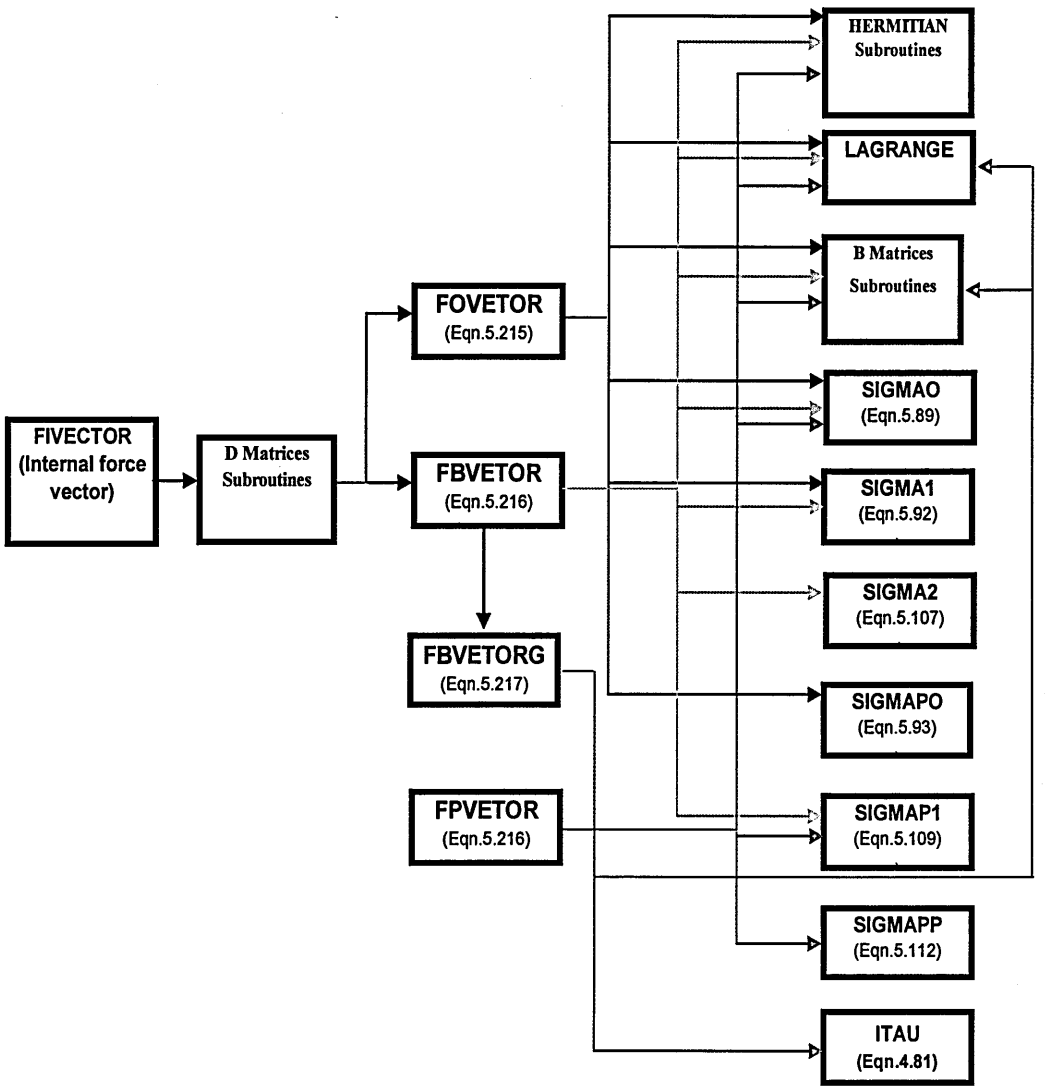

Figure 7.21. Internal force vector (FIVECTOR) module for Reissner-type element

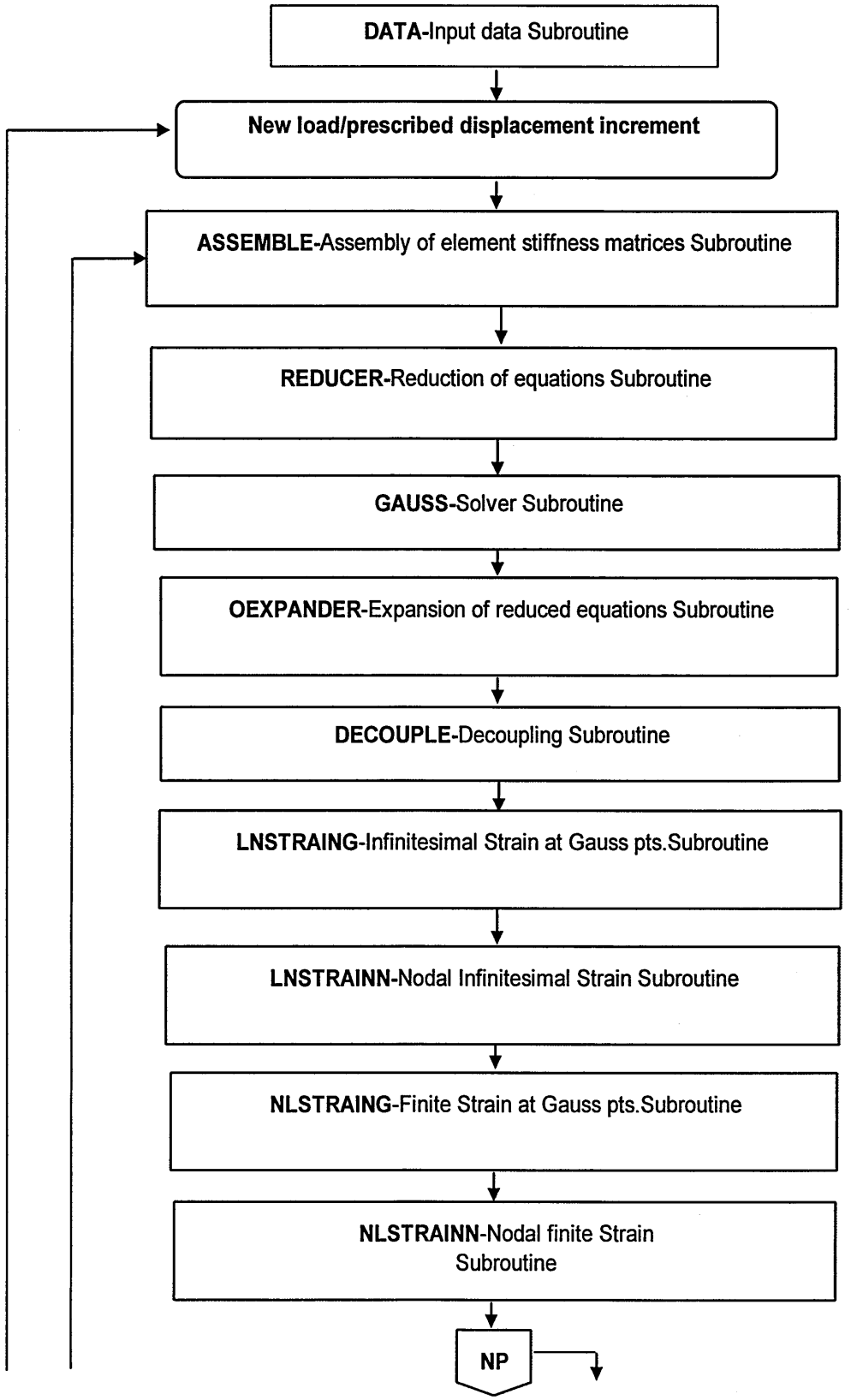

Figure 7.22. Non-linear static progressive failure algorithm (continued)

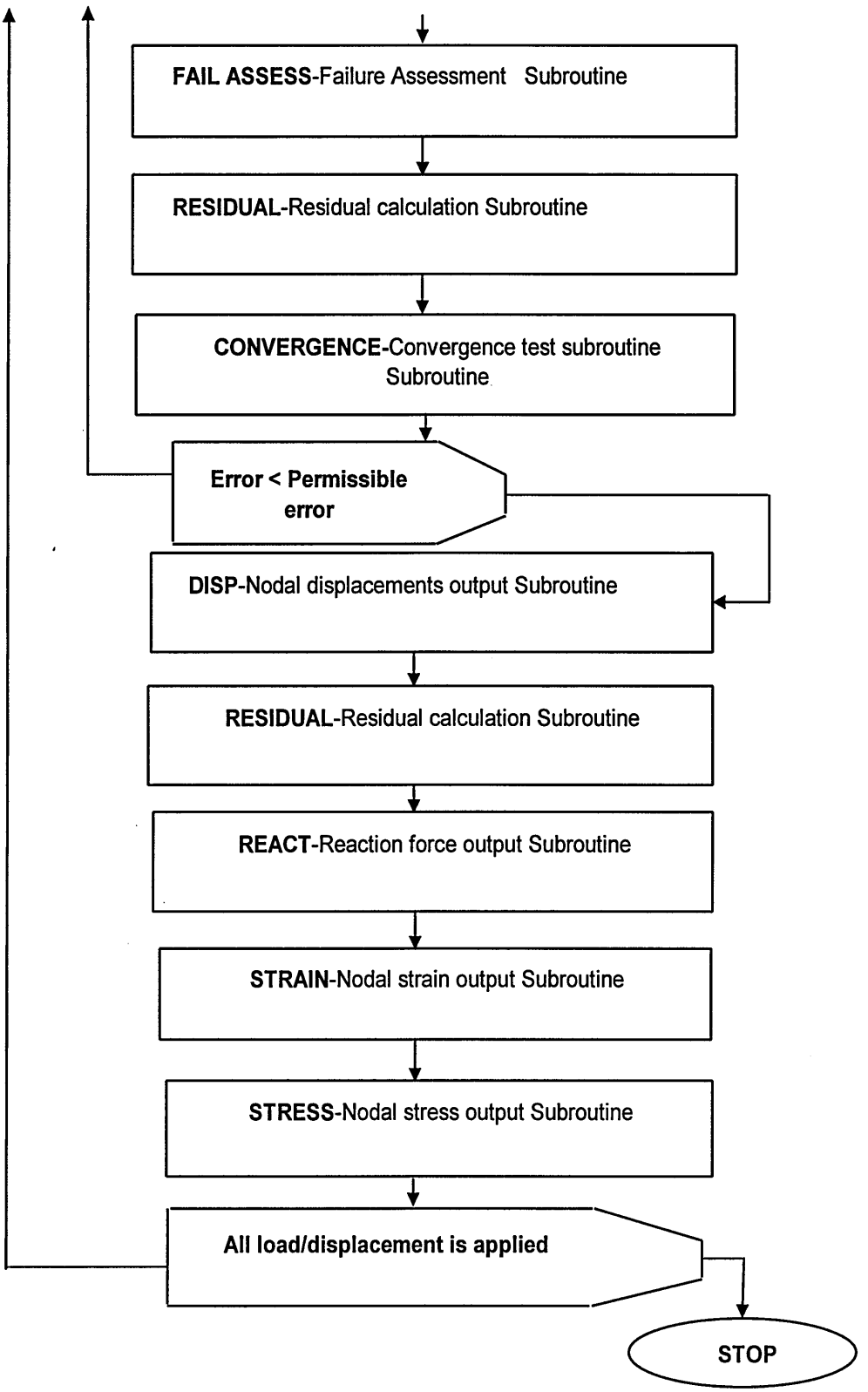

Figure 7.22 Non-linear static progressive failure algorithm

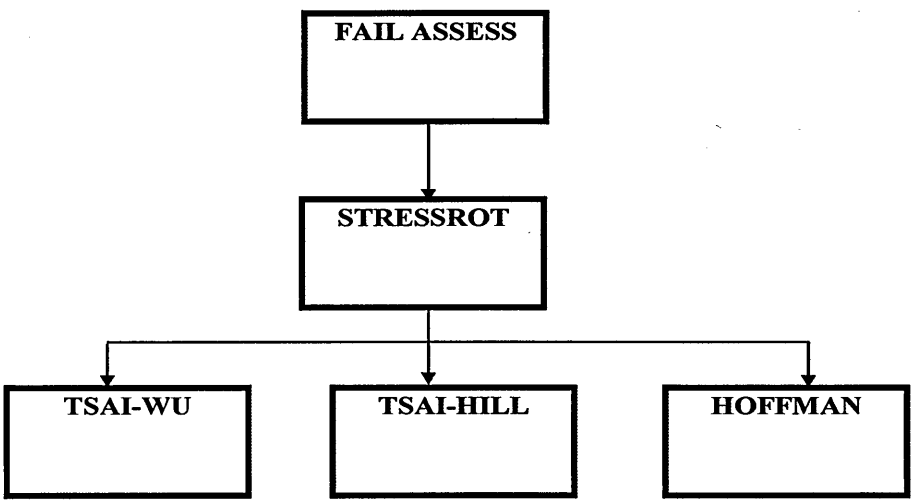

Figure 7.23 Failure Assessment Module

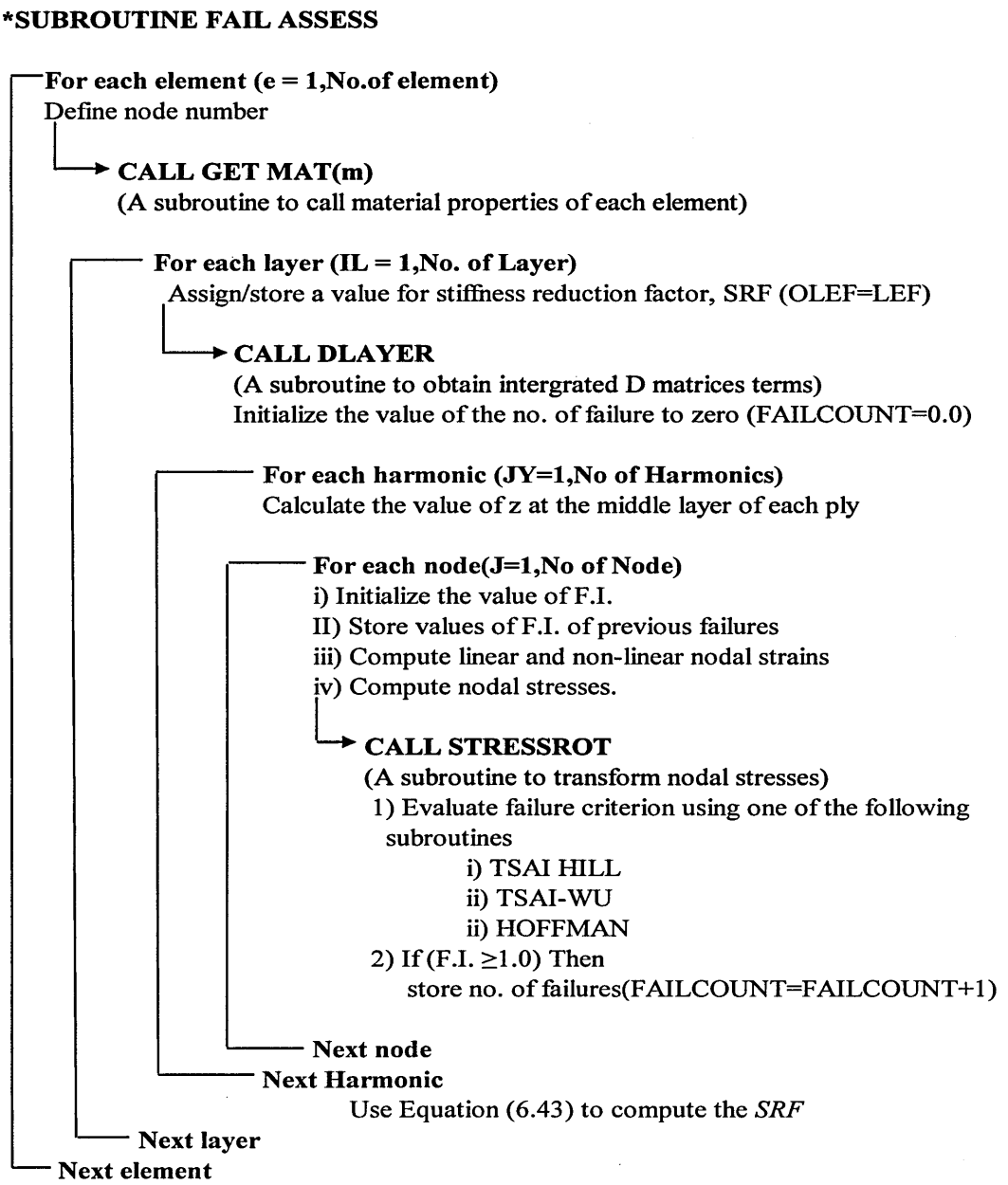

Figure 7.24. Progressive failure methodology for each individual element

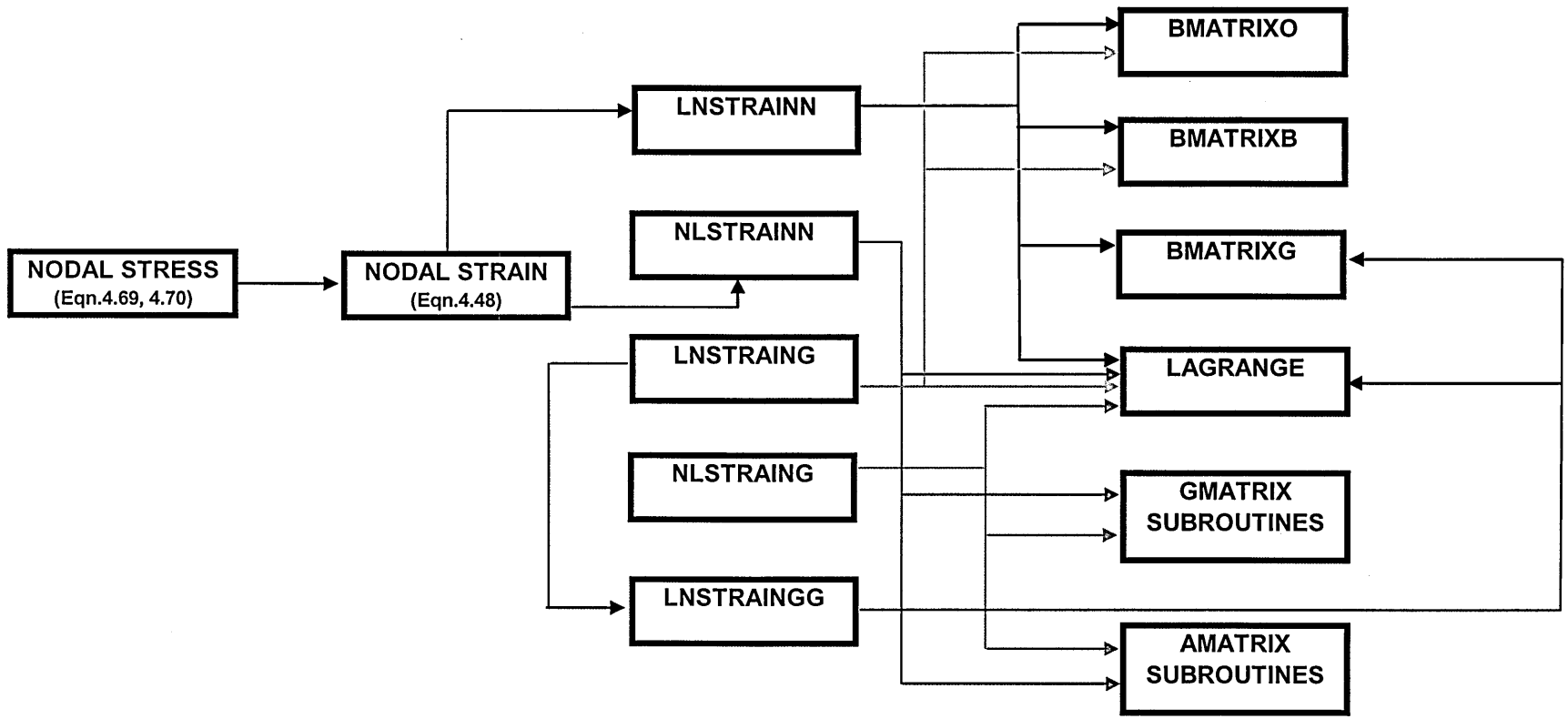

Figure 7.25. Stress and Strain Main Subroutines for Mindlin element

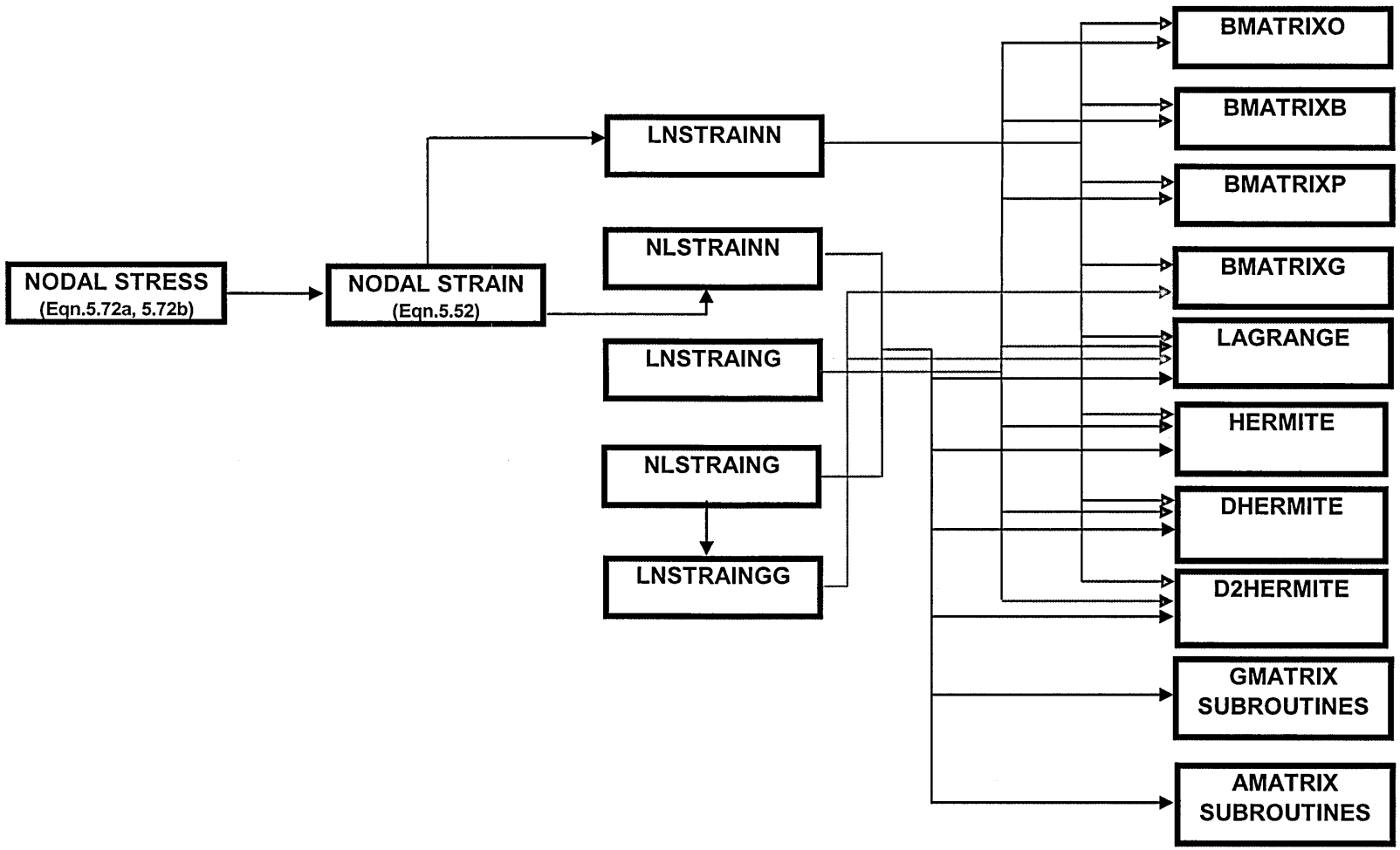

Figure 7.26. Stress and Strain Main Subroutines for Reissner element

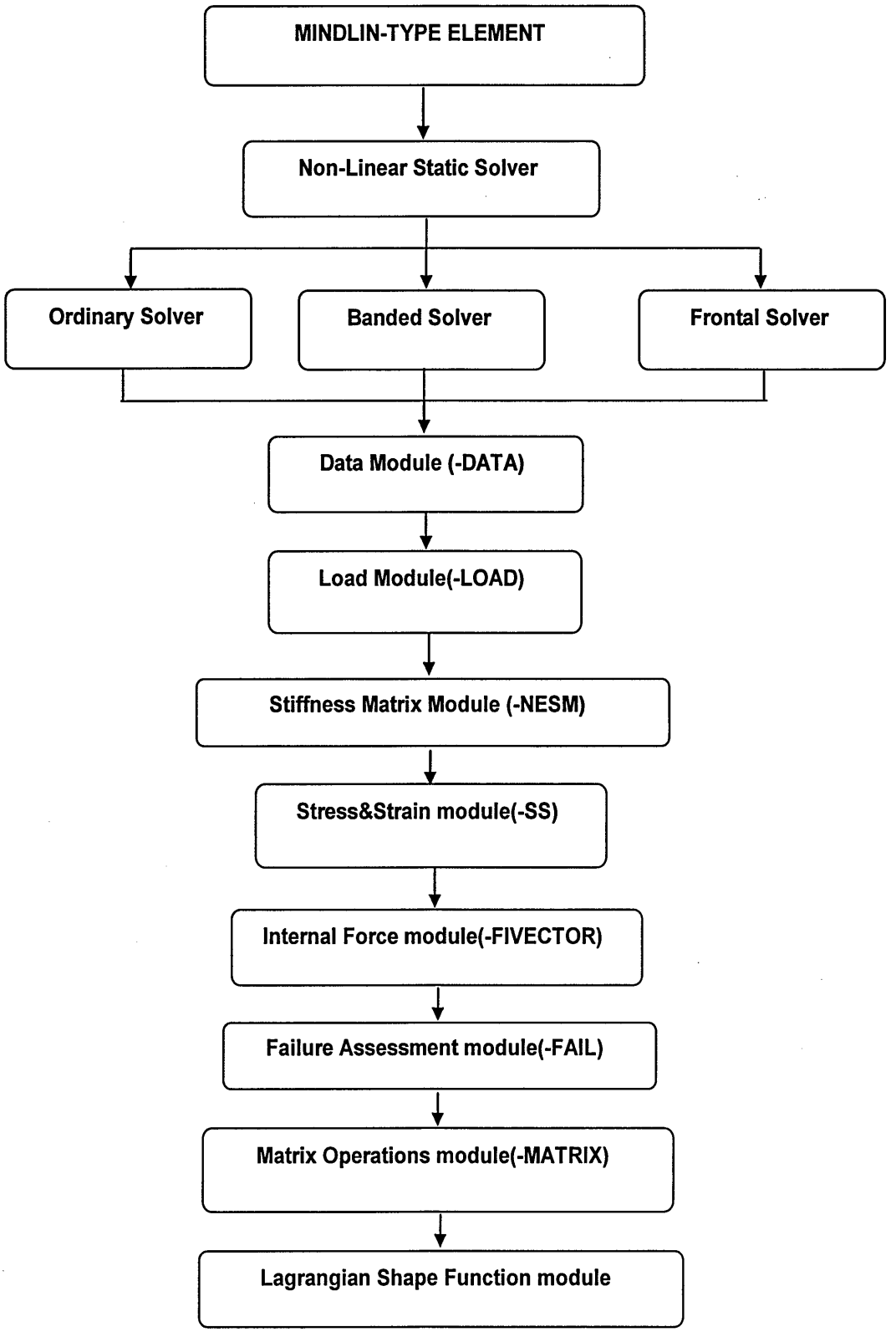

Figure 7.27. Linking structures for Mindlin-type element

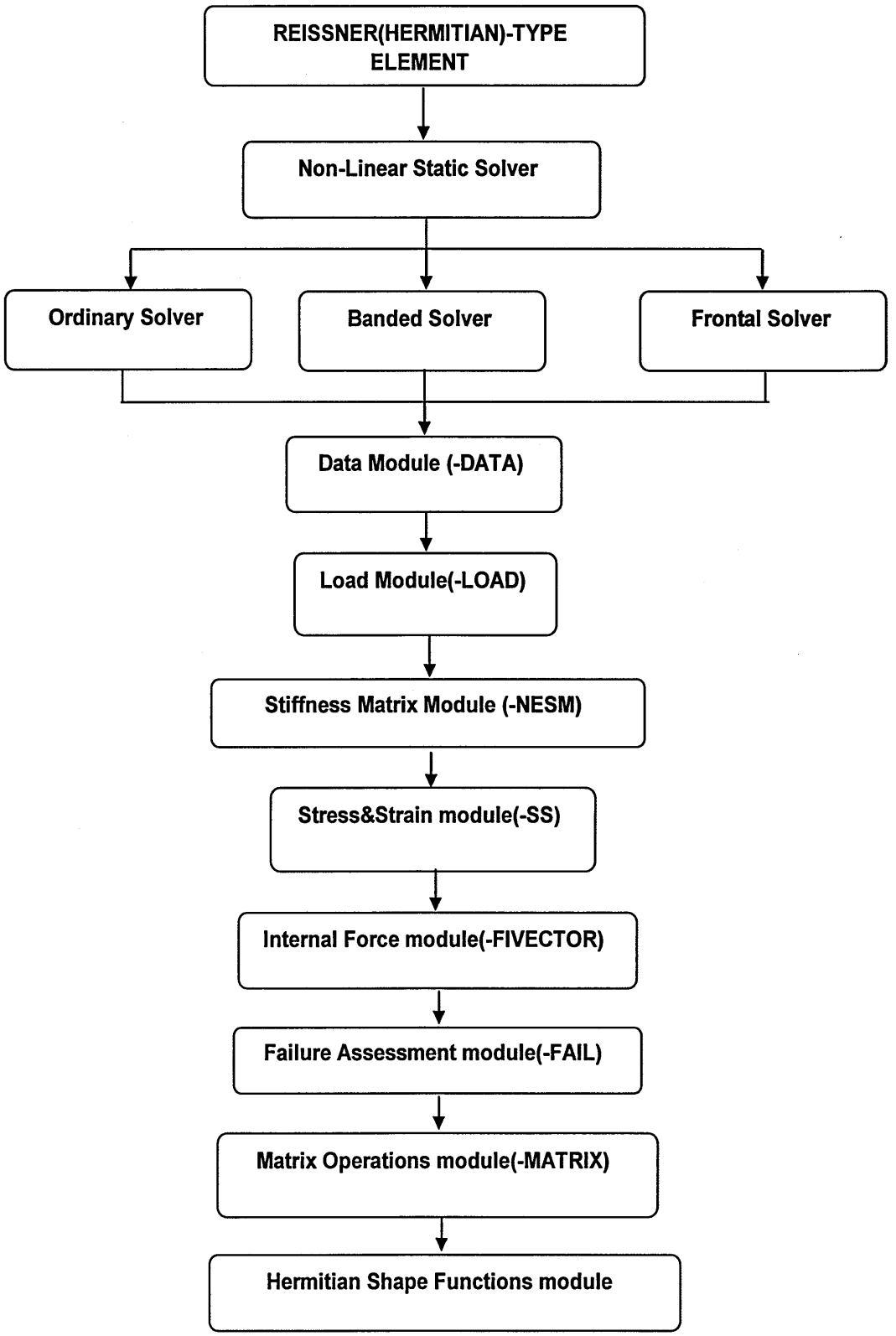

Figure 7.28. Linking structures for Reissner (Hermitian)-type element

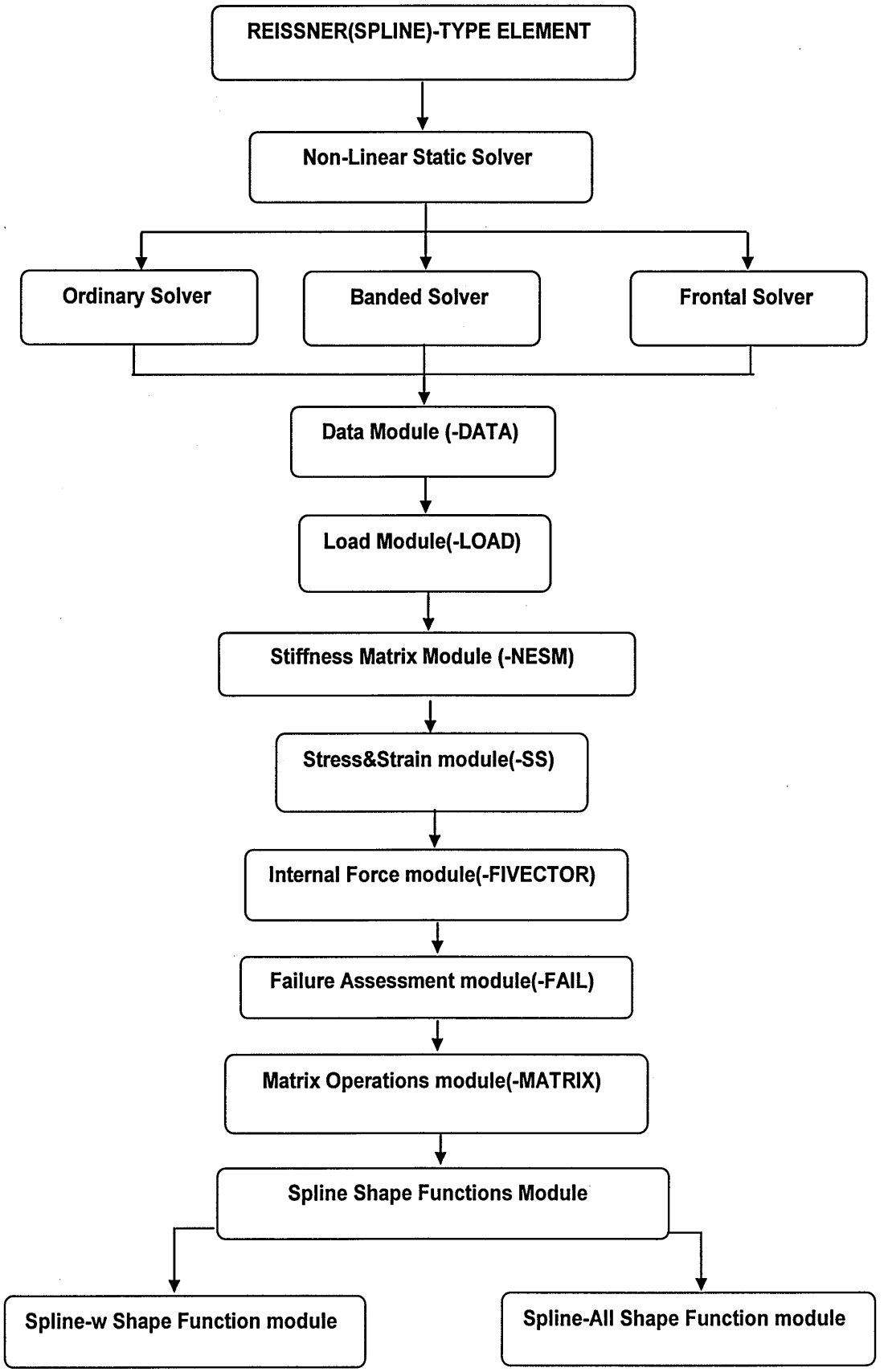

Figure 7.29. Linking structures for Reissner (Spline)-type element

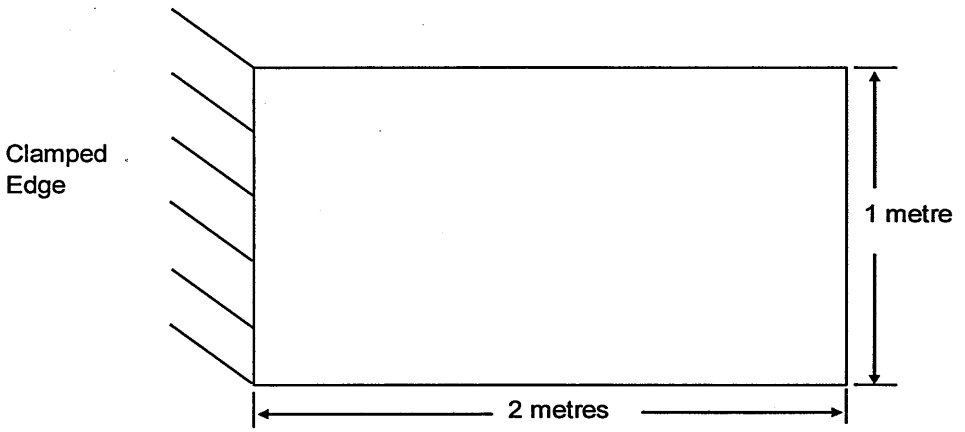

Figure 8.1. Geometry and boundary conditions of cantilever plate

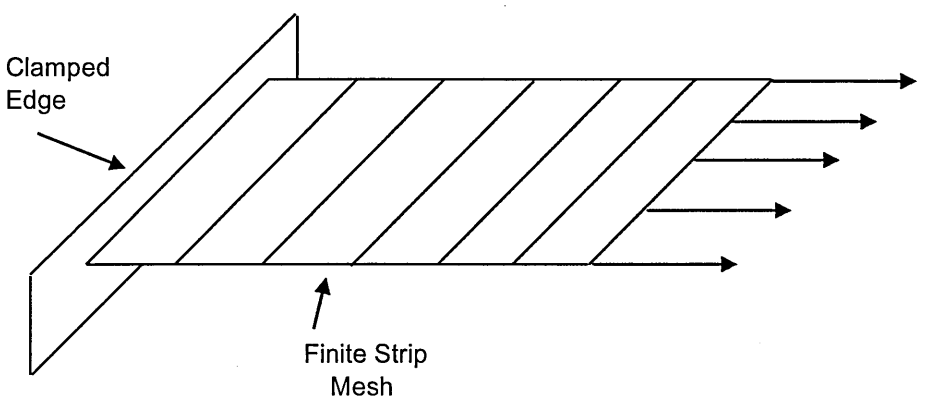

Figure 8.2.Cantilever plate under tension

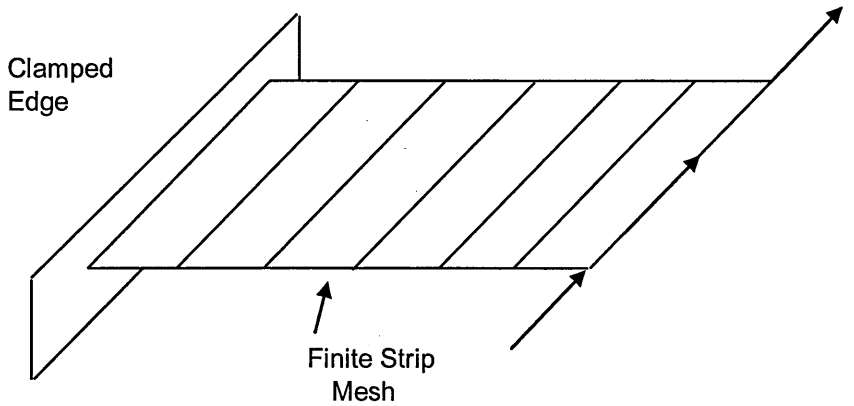

Figure 8.3.Cantilever plate under in-plane-bending

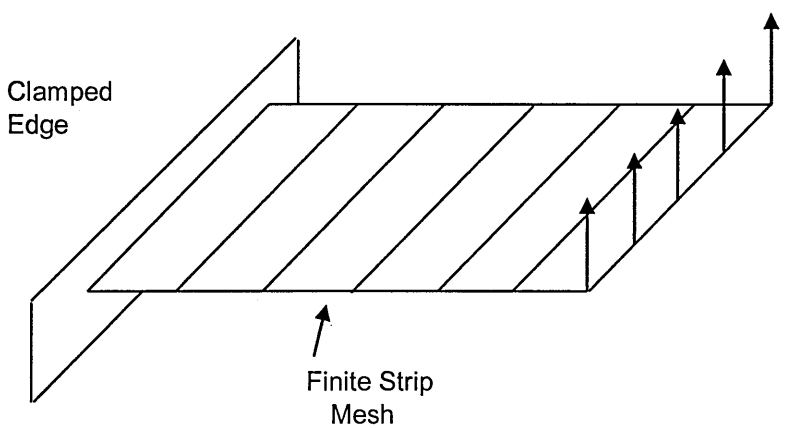

Figure 8.4.Cantilever plate under out-of-plane-bending

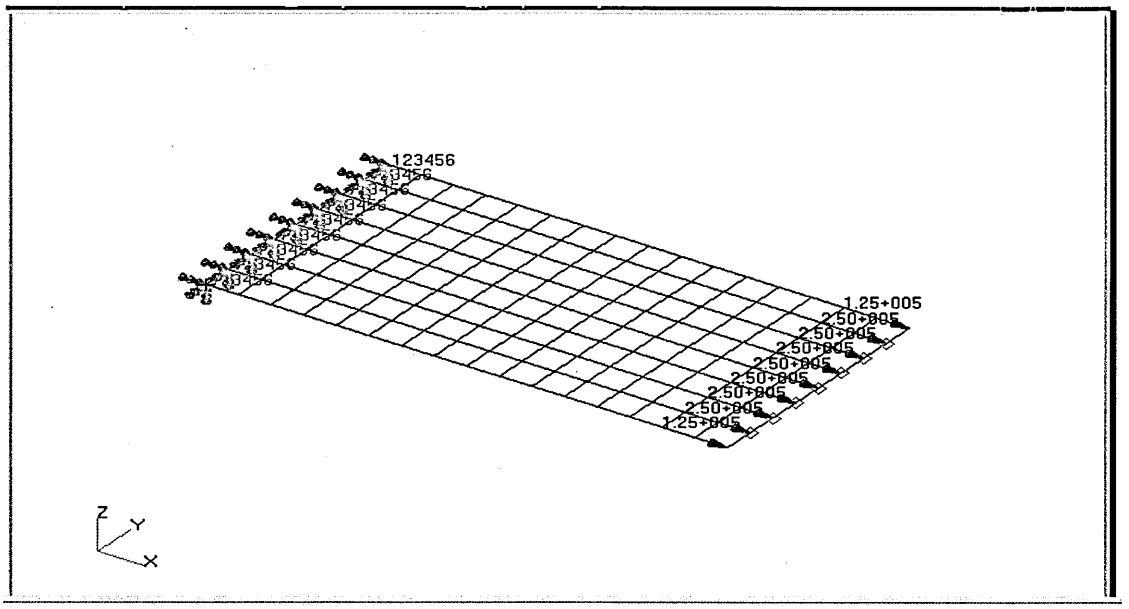

Figure 8.5. Finite element mesh and boundary conditions modelled in PATRAN.

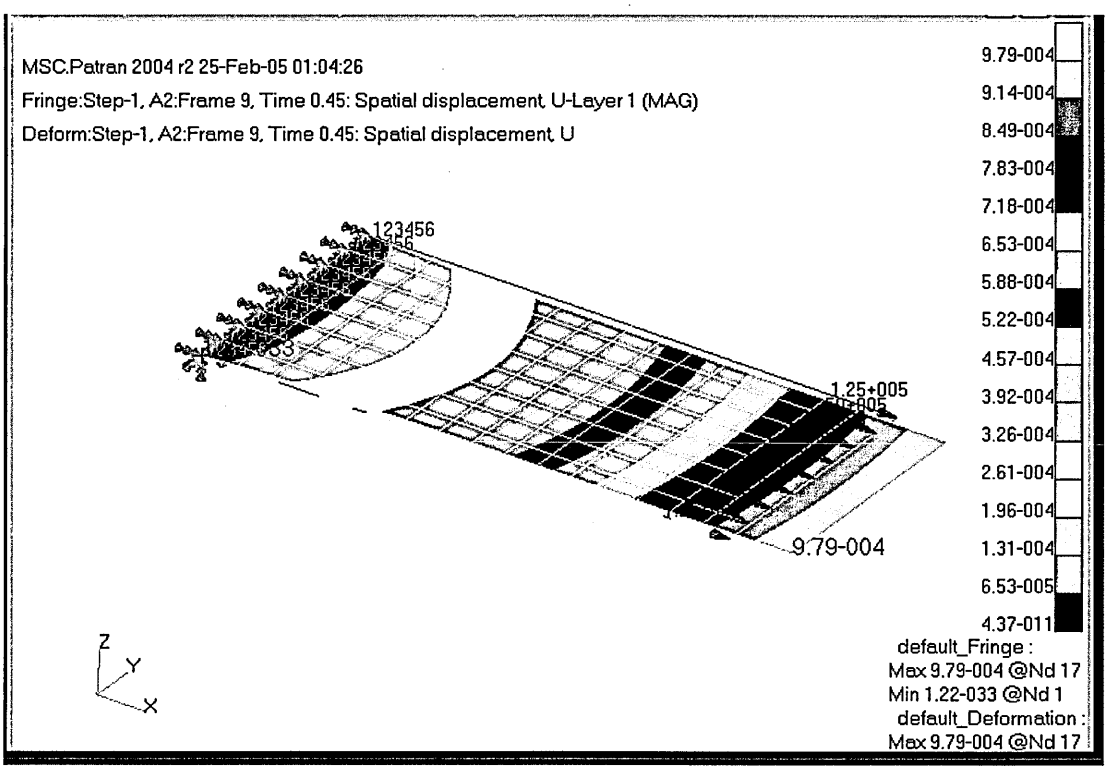

Figure 8.6. ABAQUS plot of displacement in x-direction for cantilever plate under tension.

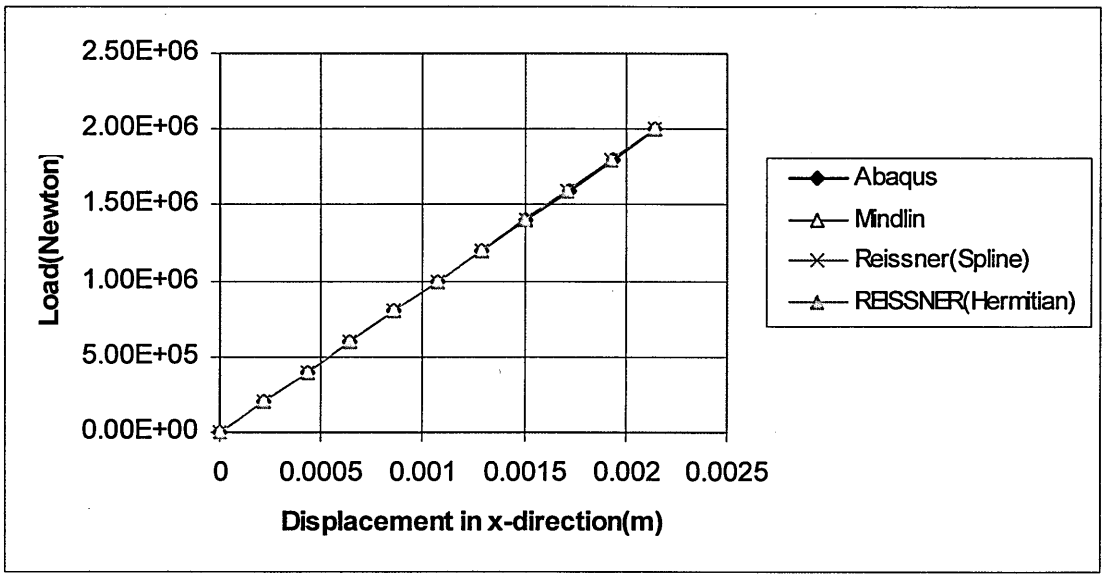

Figure 8.7. Validation of structural response of cantilever plate under tension without damage assessment.

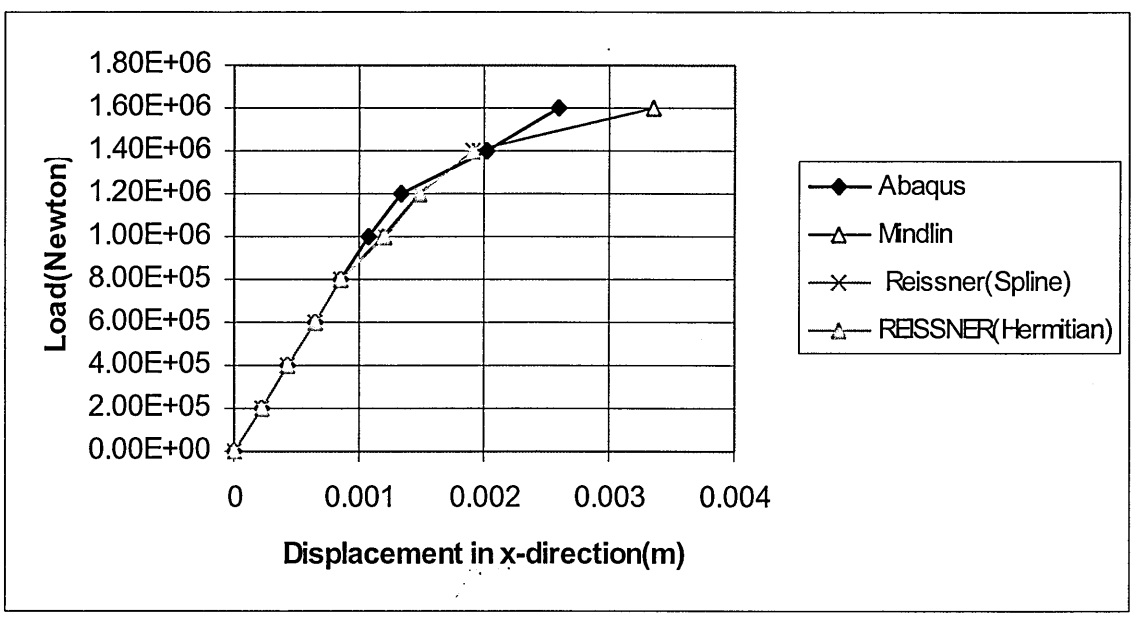

Figure 8.8. Validation of structural response of cantilever plate under tension using Tsai-Wu failure Criterion.

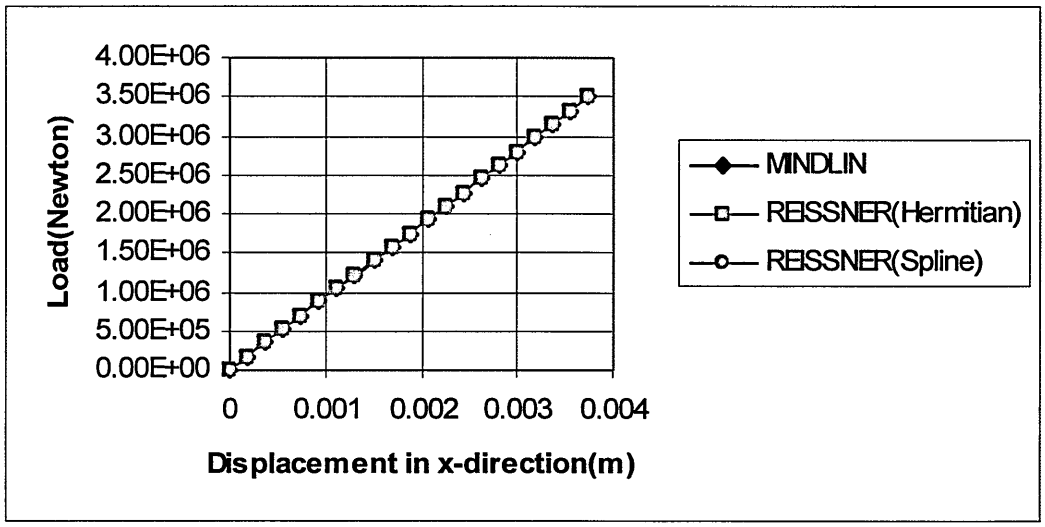

Figure 8.9. Structural response of cantilever plate under tension using without failure assessment, thickness of 0.03m

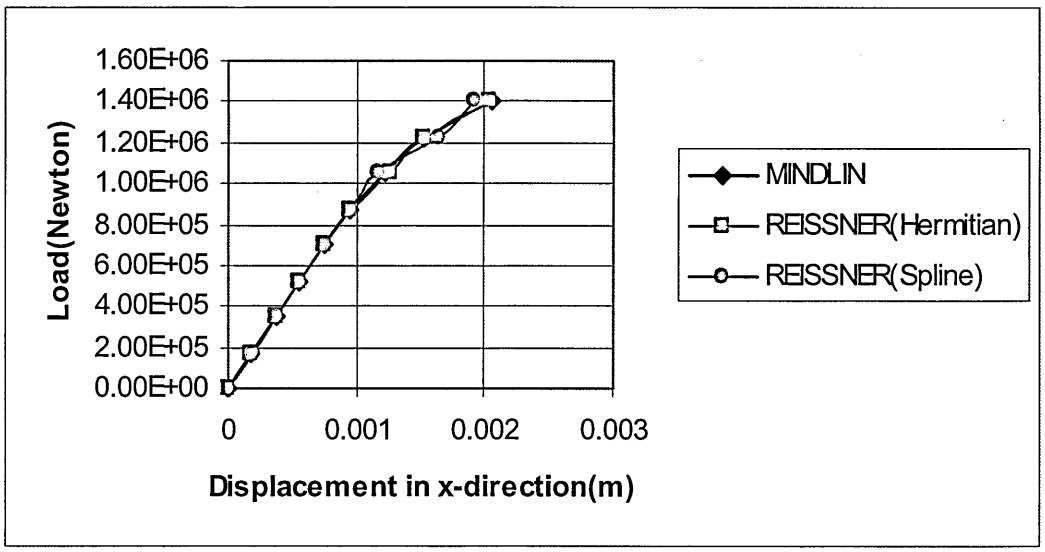

Figure 8.10. Structural response of cantilever plate under tension using Tsai-Hill failure Criterion, thickness of 0.03m.

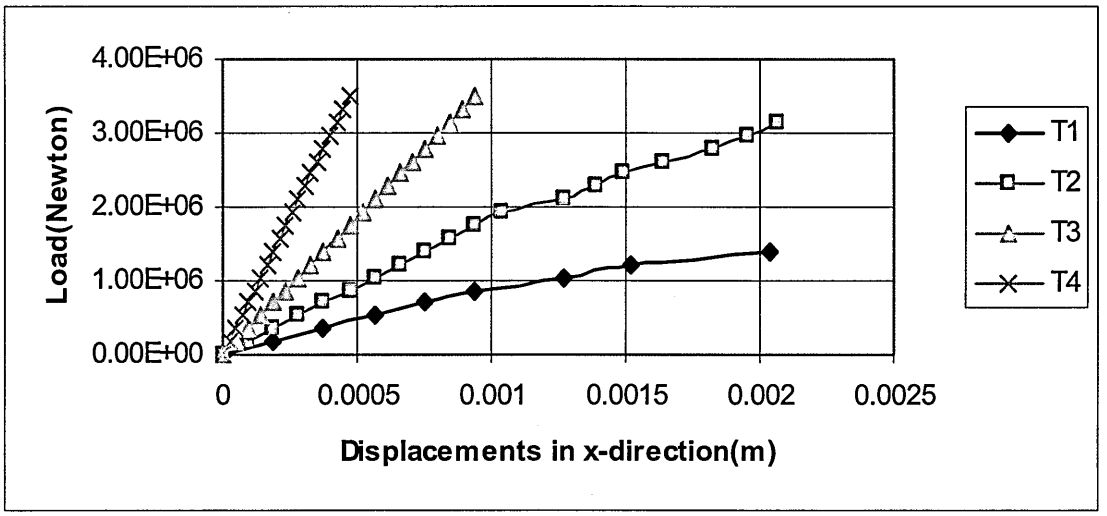

Figure 8.11.Structural response (Mindlin element) of cantilever plate under tension with various laminate thicknesses using Tsai-Hill failure Criterion.

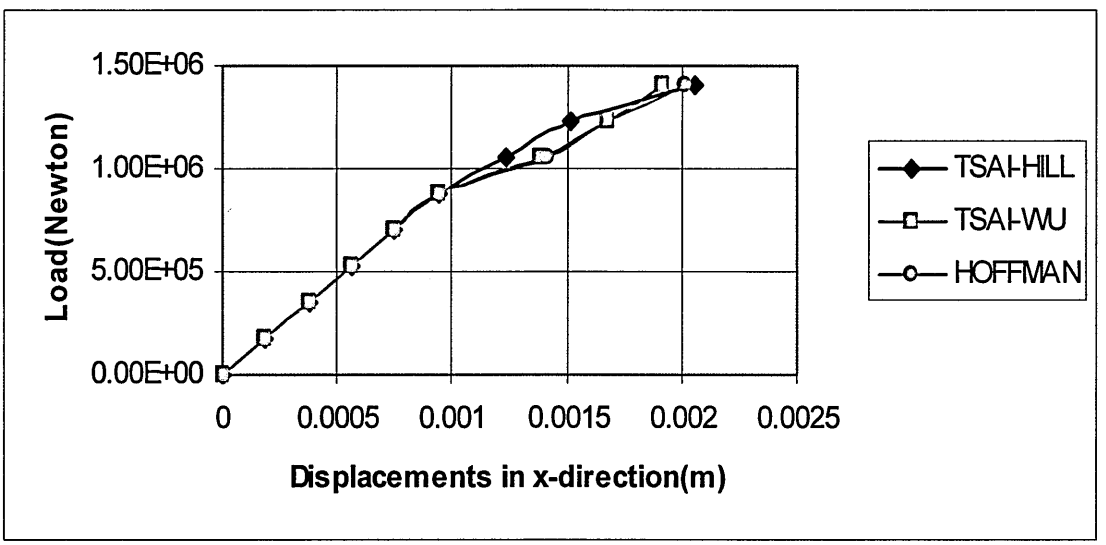

Figure 8.12. Structural response (Mindlin element) of cantilever plate under tension with thickness of 0.03m using Tsai-Hill failure Criterion.

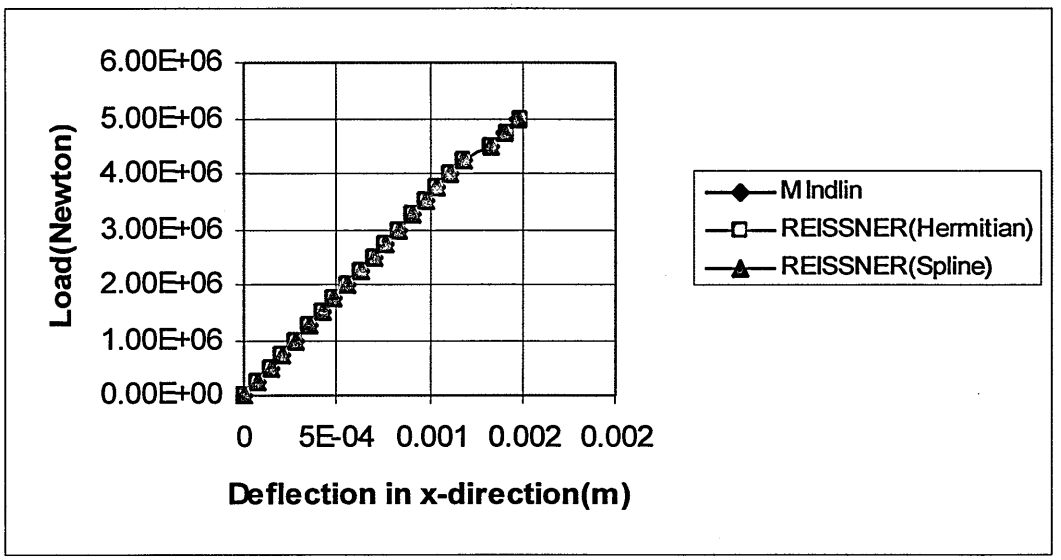

Figure 8.13. Structural response of $[0^0/90^0/90^0/0^0]$ cantilever plate under tension using Tsai-Wu failure Criterion.

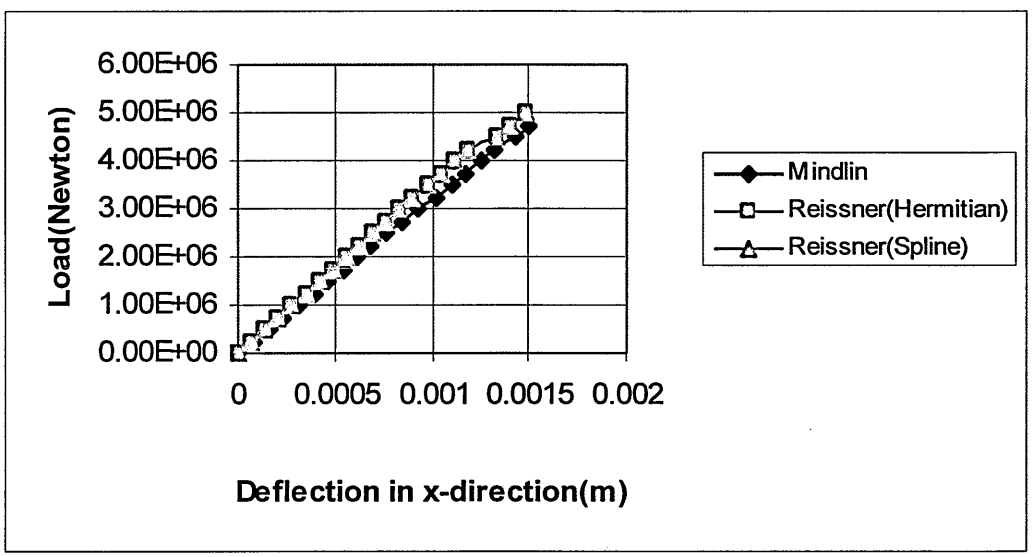

Figure 8.14. Structural response of $[0^0/90^0/0^0/90^0]$ cantilever plate under tension using Tsai-Wu failure Criterion.

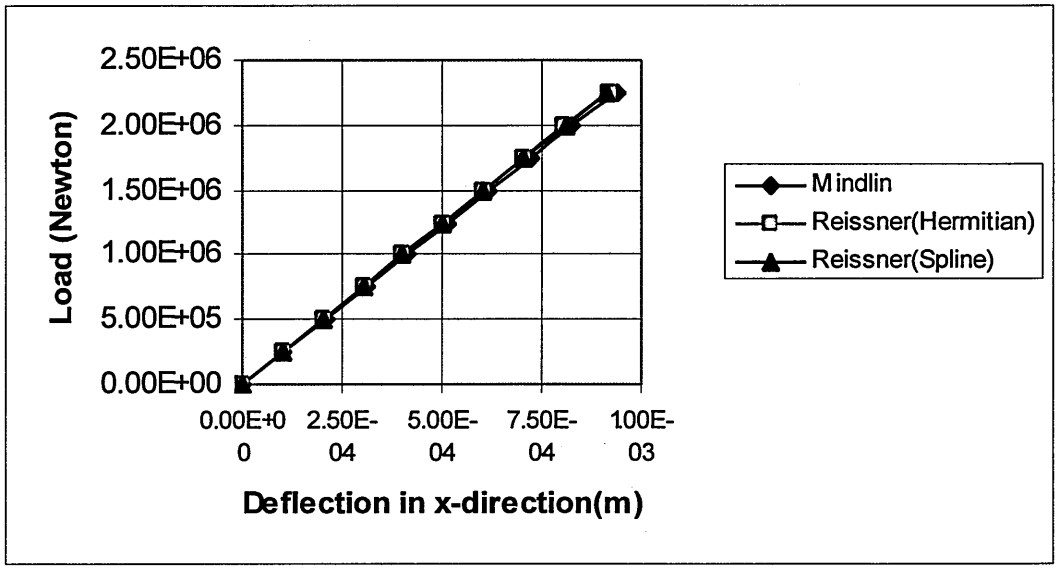

Figure 8.15. Structural response of [45^0/-45^0/-45^0/45^0] cantilever plate under tension using Tsai-Wu failure Criterion.

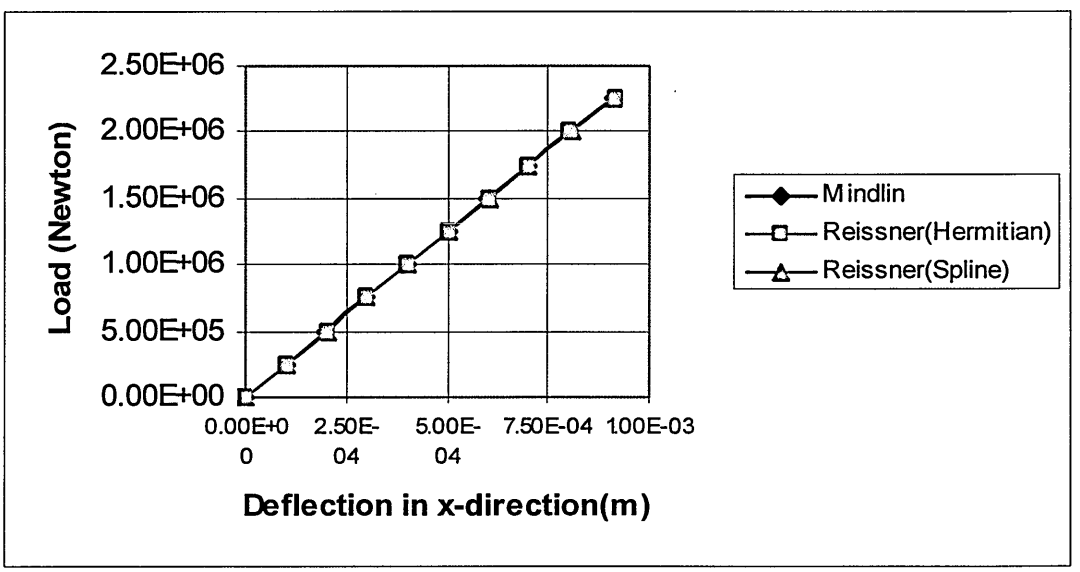

Figure 8.16. Structural response of [45^0/-45^0/45^0/-45^0] cantilever plate under tension using Tsai-Wu failure Criterion.

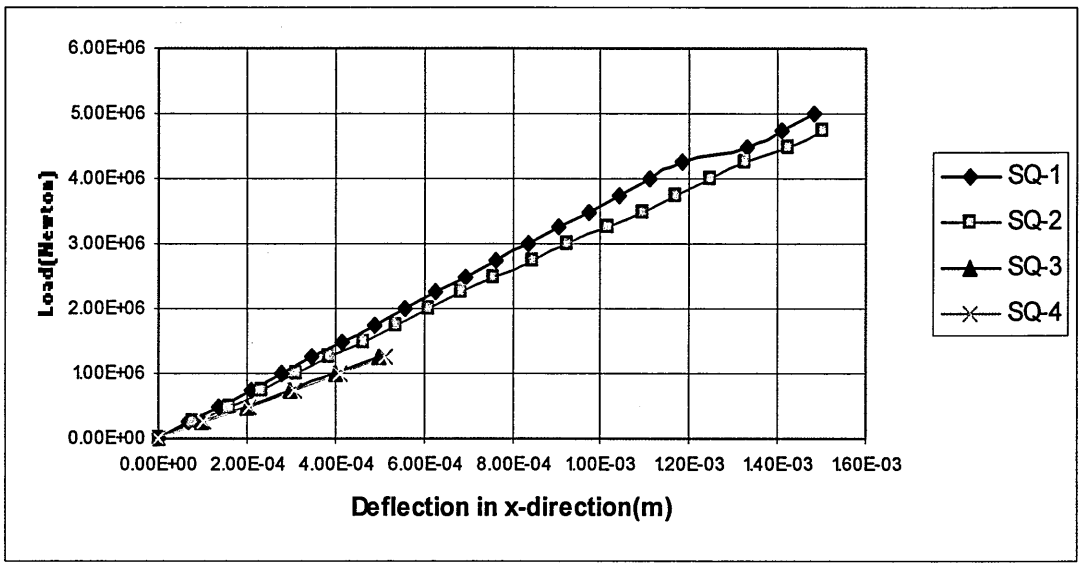

Figure 8.17. Structural response (Mindlin element) of cantilever plate under tension with various stacking sequence using Tsai-Hill failure Criterion.

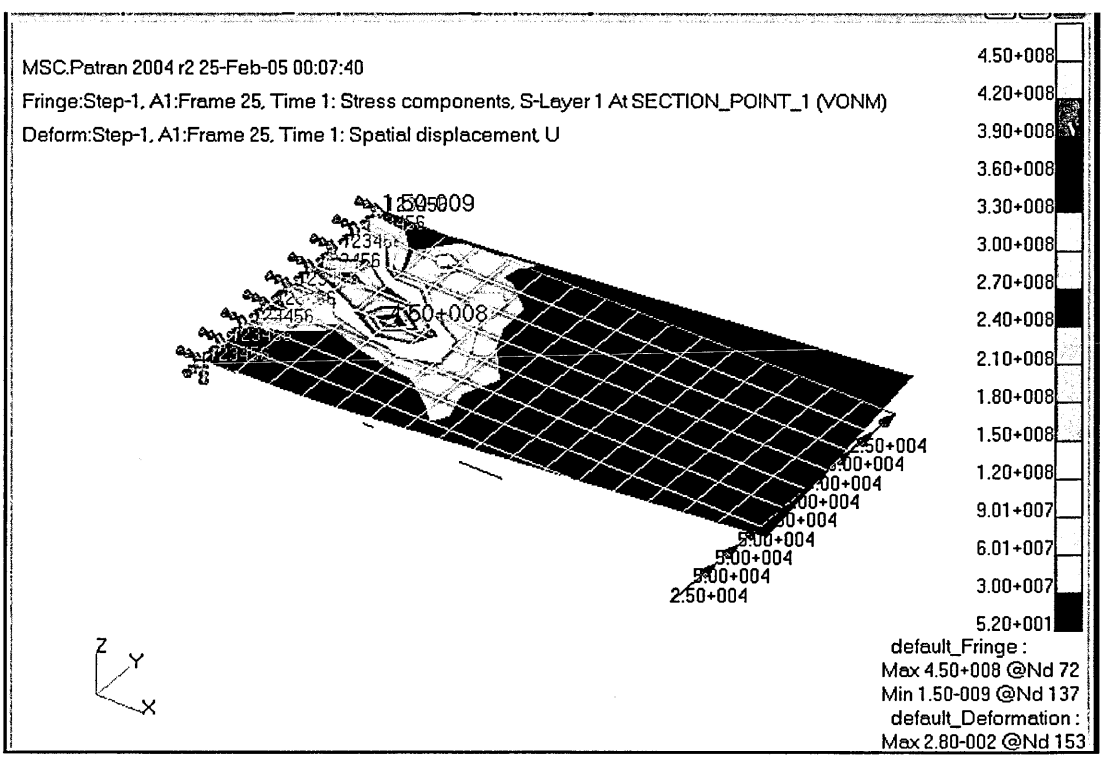

Figure 8.18. ABAQUS plot of Von Mises stress at the lowest surface of the cantilever plate under in-plane-bending.

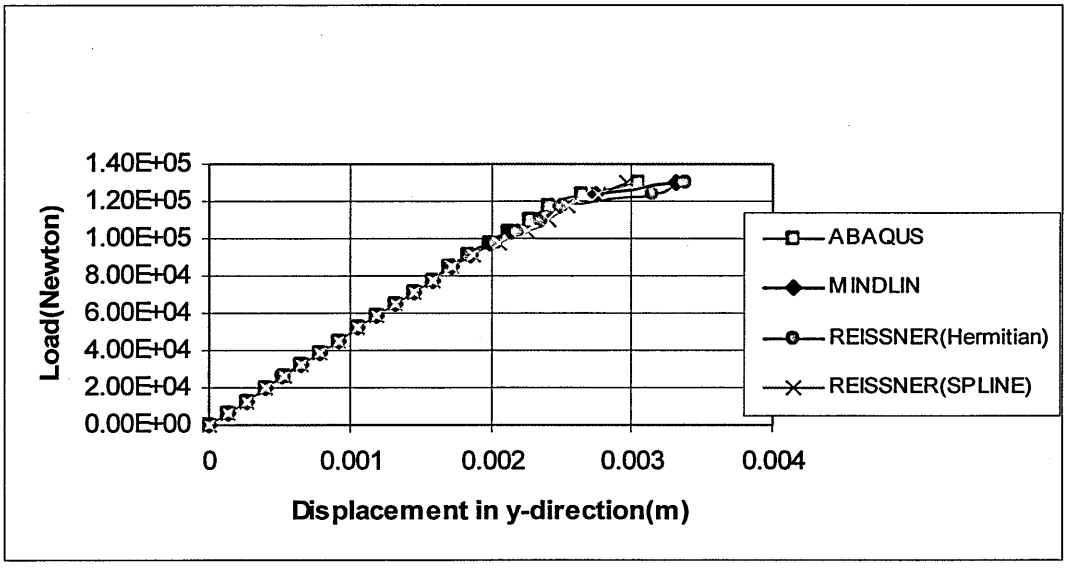

Figure 8.19. Validation of structural response of cantilever plate under in-plane-bending using Tsai-Wu failure Criterion

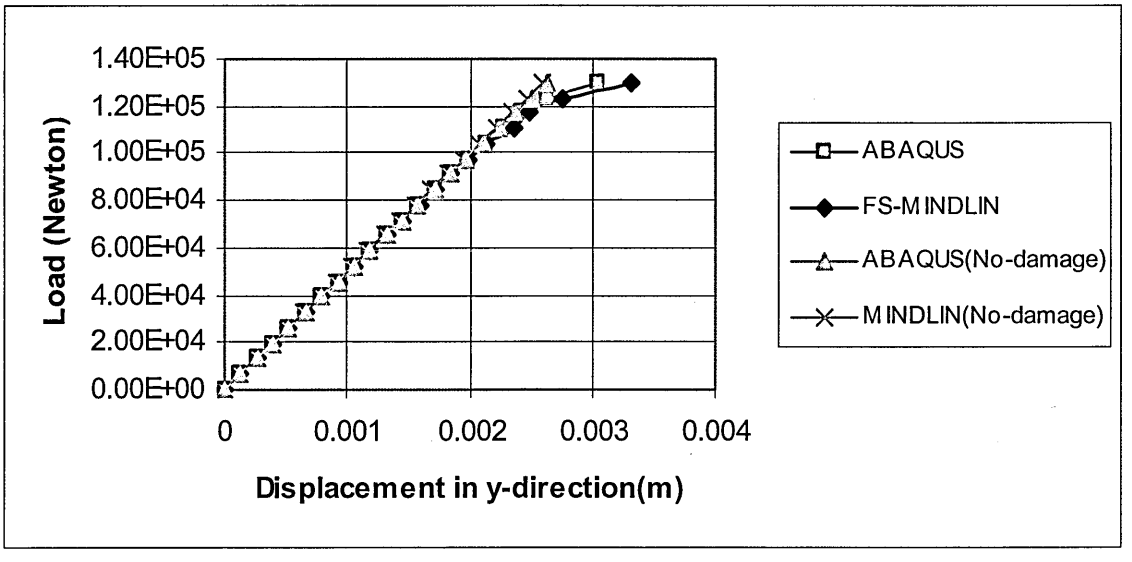

Figure 8.20. Validation of structural response of cantilever plate under in-plane-bending using Tsai-Wu failure Criterion with and without damage assessment.

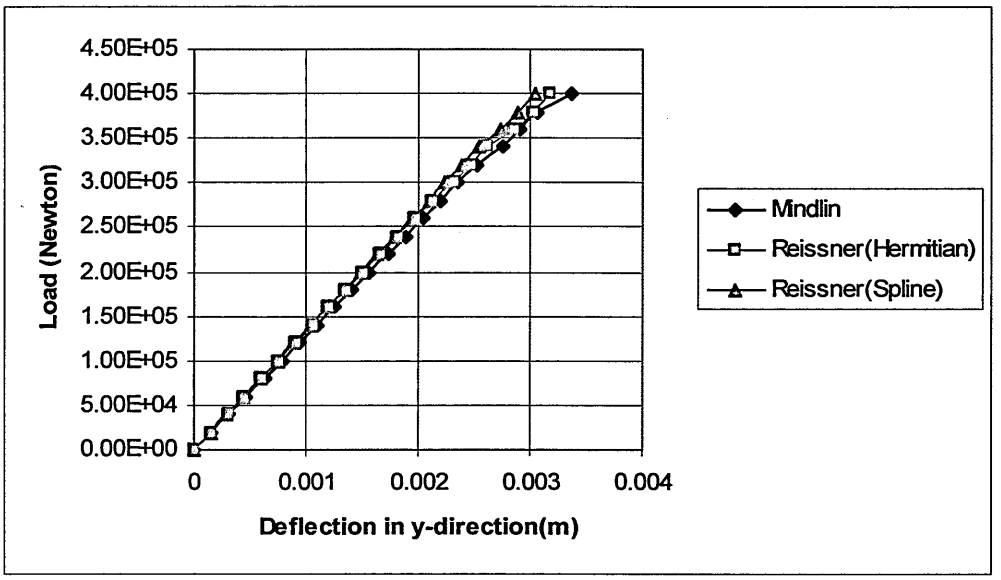

Figure 8.21. Structural response of $[0^0/90^0/90^0/0^0]$ cantilever plate under in-plane-bending using Hoffman failure Criterion.

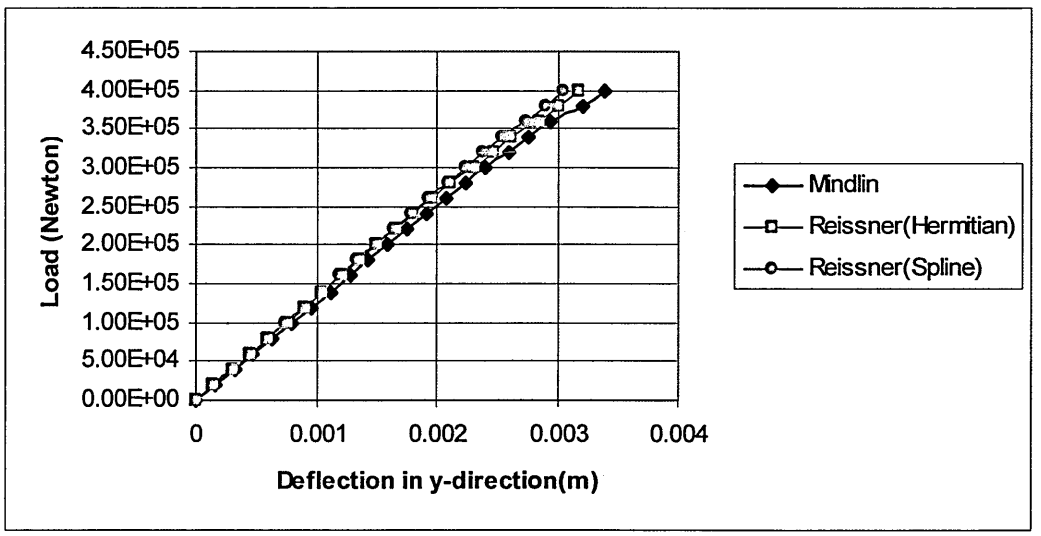

Figure 8.22. Structural response of $[0^0/90^0/0^0/90^0]$ cantilever plate under in-plane-bending using Hoffman failure Criterion.

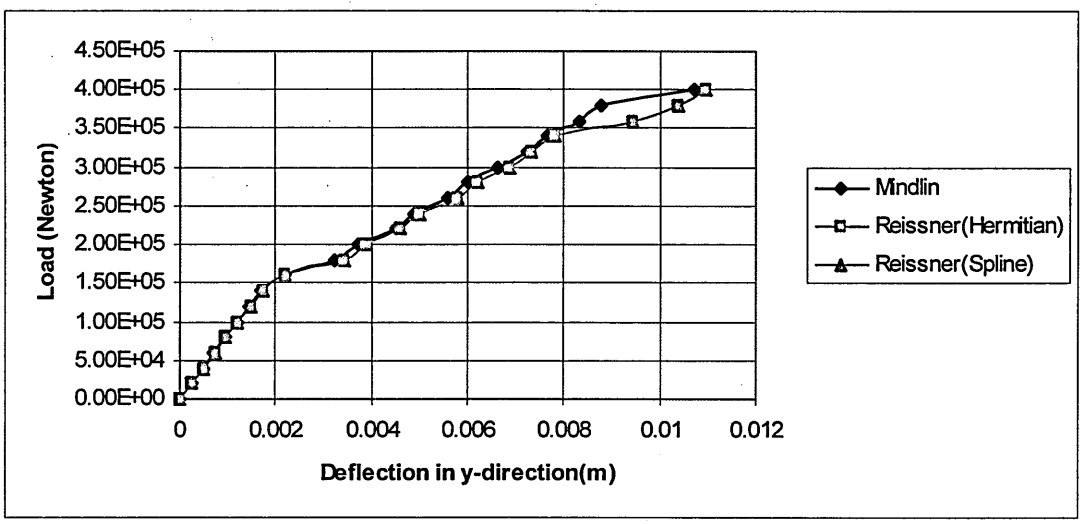

Figure 8.23. Structural response of [45^0/-45^0/-45^0/45^0] cantilever plate under in-plane-bending using Hoffman failure Criterion

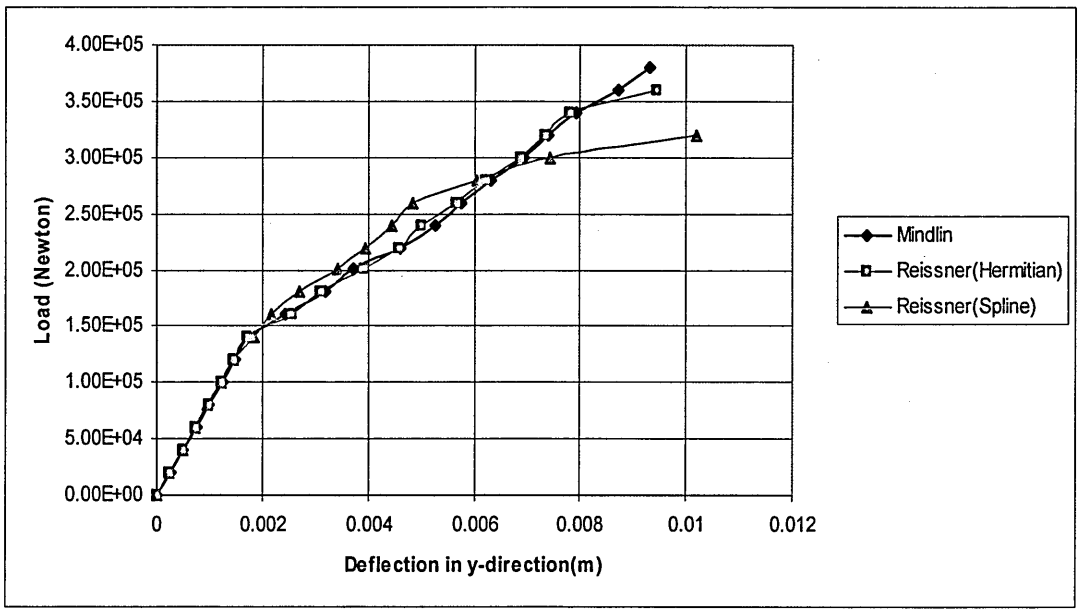

Figure 8.24. Structural response of [45^0/-45^0/45^0/-45^0] cantilever plate under in-plane-bending using Hoffman failure Criterion.

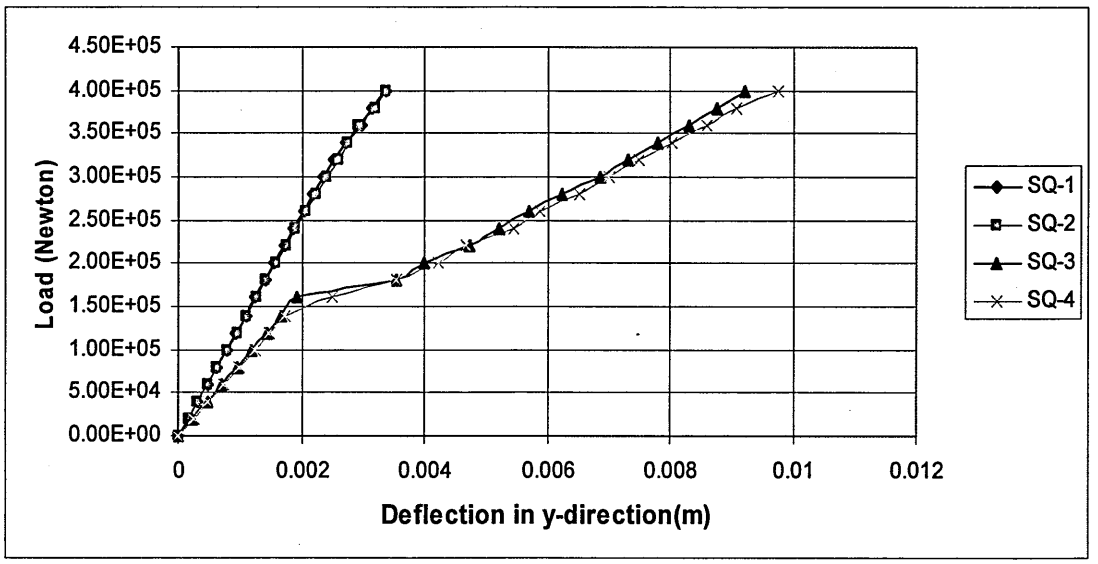

Figure 8.25. Structural response (Mindlin element) of cantilever plate under in-plane-bending with various stacking sequence using Tsai-Hill failure Criterion.

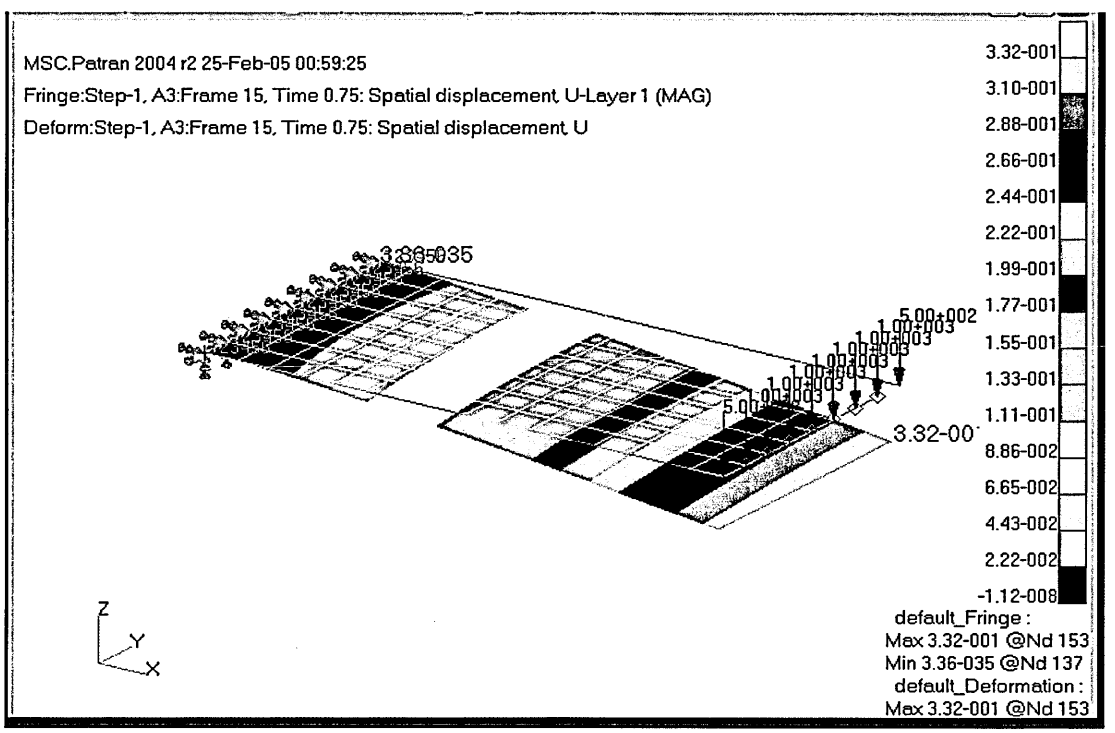

Figure 8.26. ABAQUS plot of lateral deflection for cantilever plate under out-of-plane-bending.

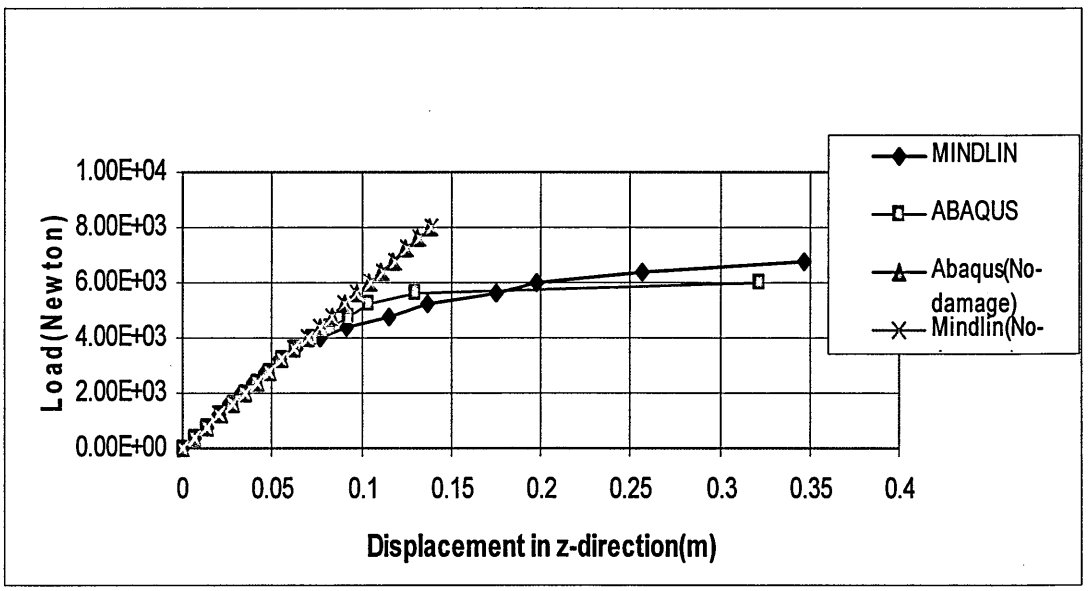

Figure 8.27. Validation of structural response of cantilever plate under out-of-plane-bending using Tsai-Wu failure Criterion with and without damage assessment.

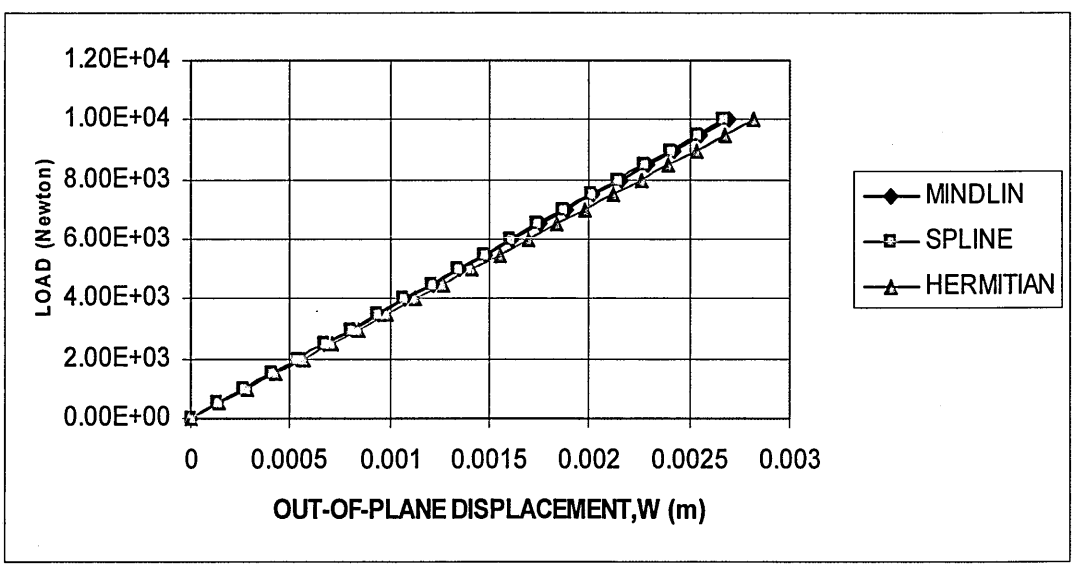

Figure 8.28. Structural response of $[0^0/90^0/90^0/0^0]$ cantilever plate under out-of-plane-bending using Tsai-Wu failure Criterion

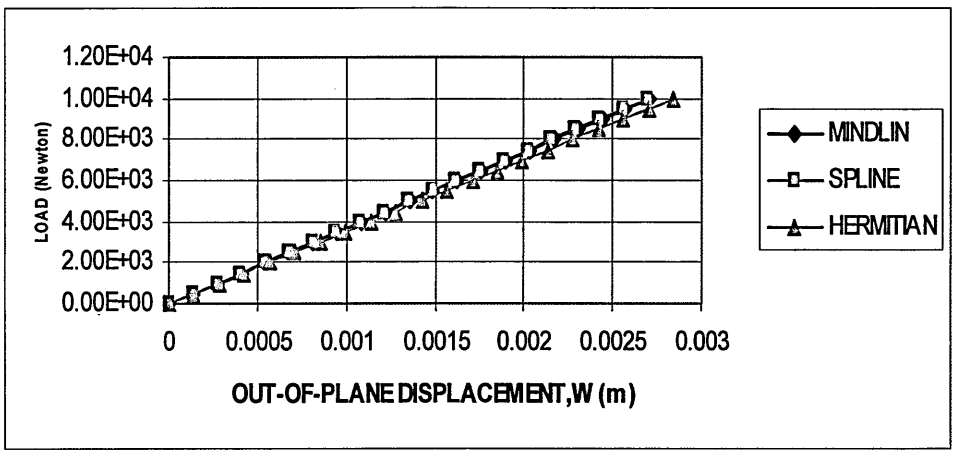

Figure 8.29. Structural response of $[0^0/30^0/30^0/0^0]$ cantilever plate under out-of-plane-bending using Tsai-Wu failure Criterion.

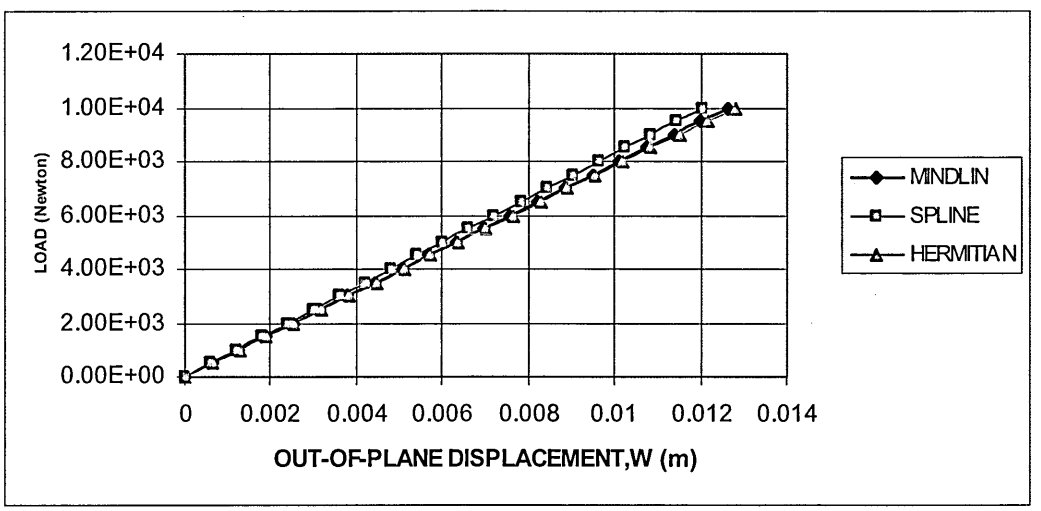

Figure 8.30. Structural response of $[45^0/-45^0/-45^0/45^0]$ cantilever plate under out-of-plane-bending using Tsai-Wu failure Criterion

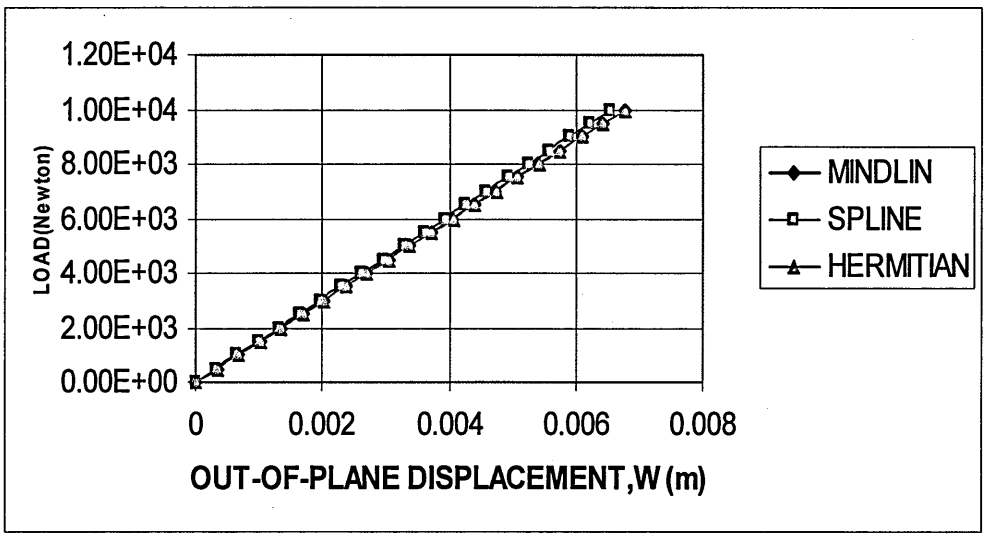

Figure 8.31. Structural response of [30^0/-30^0/-30^0/30^0] cantilever plate under out-of-plane-bending using Tsai-Wu failure Criterion.

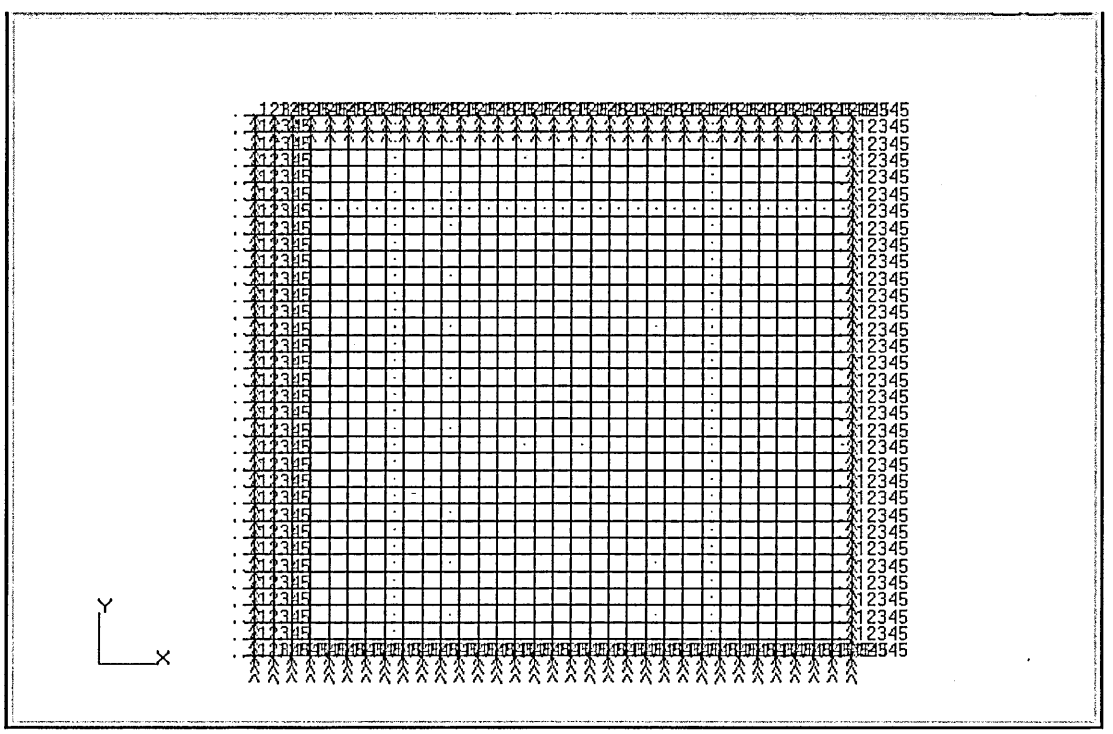

Figure 8.32. Finite element mesh and boundary conditions for a clamped square plate.

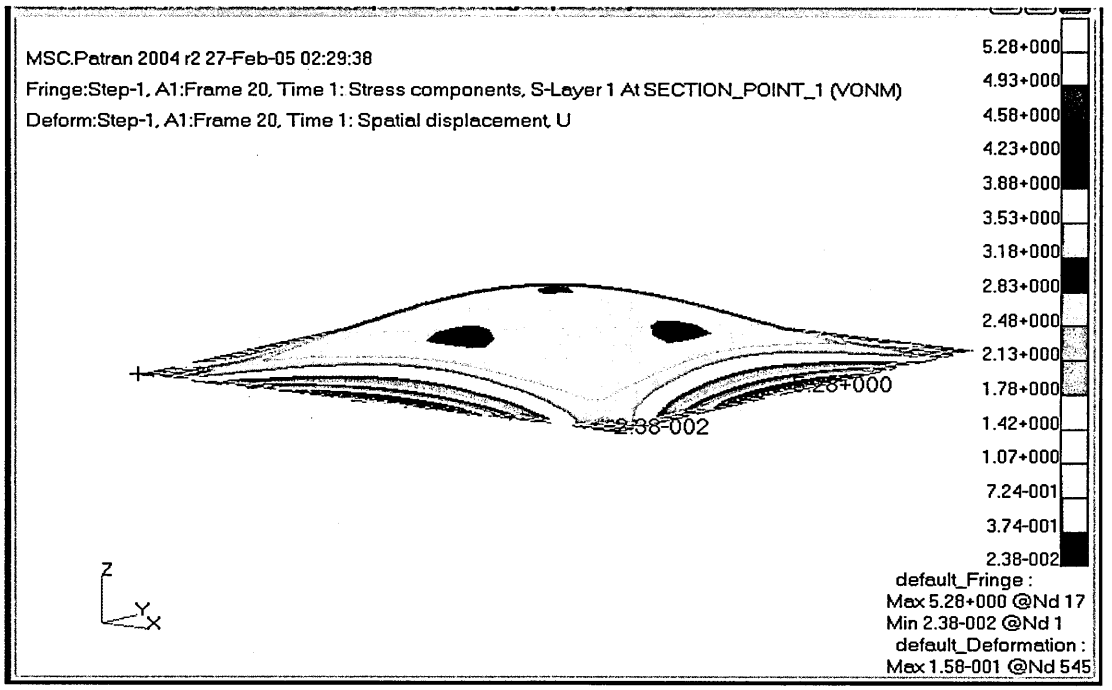

Figure 8.33. ABAQUS plot of Von Mises stress at the lowest surface of a square plate under uniform pressure

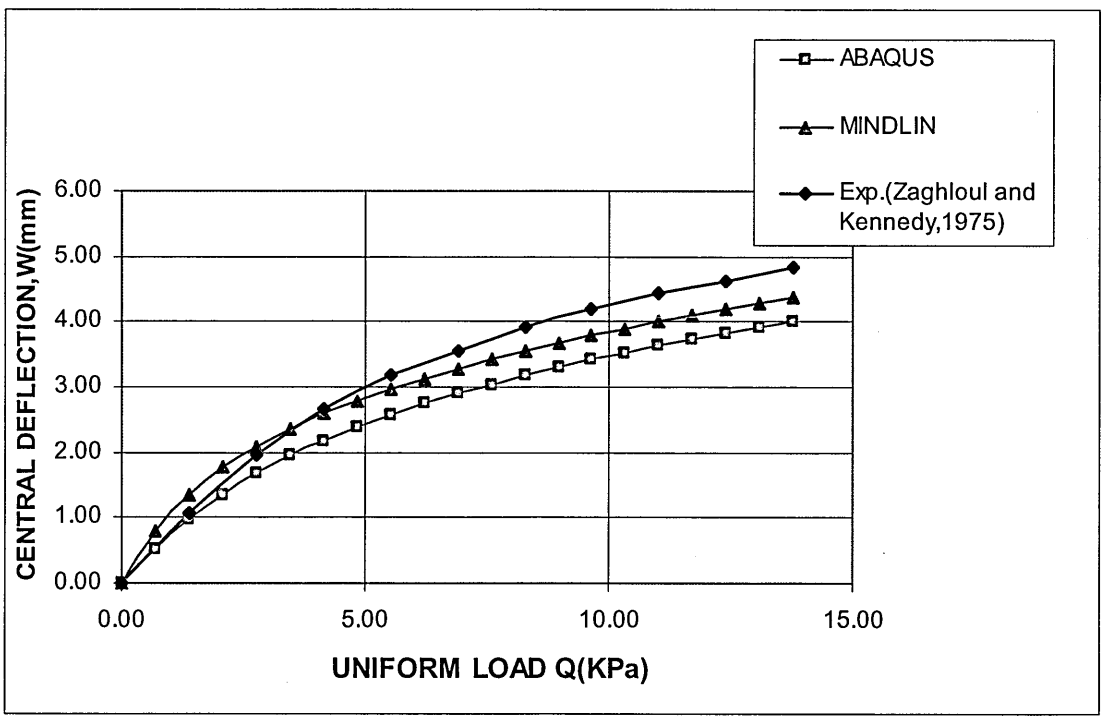

Figure 8.34. Centre deflection versus uniform transverse load for a clamped square laminate

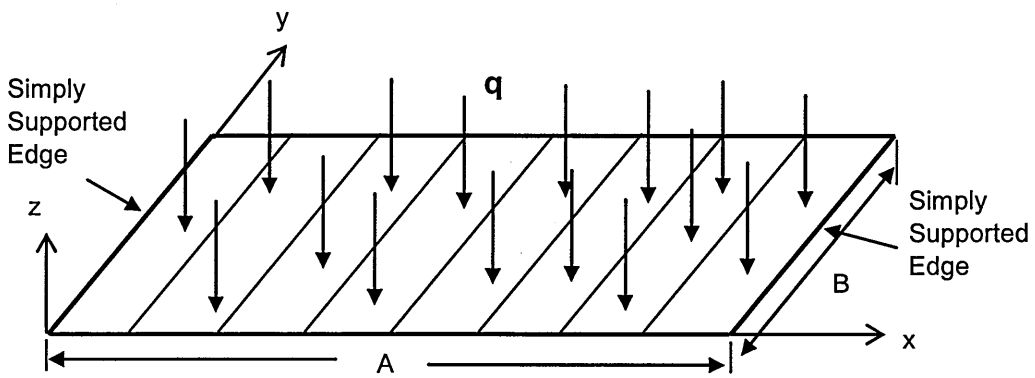

Figure 8.35. Geometry and boundary conditions of rectangular plate

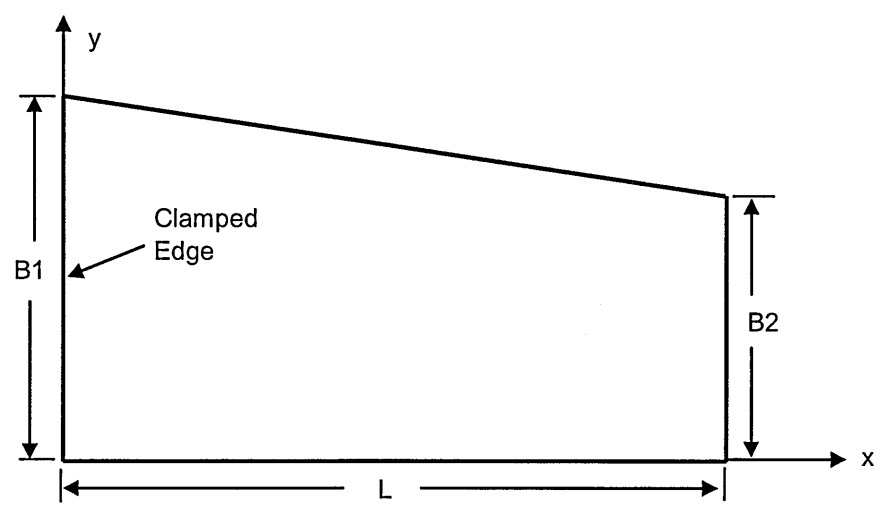

Figure 8.36. Geometry and boundary conditions the trapezoidal panel.

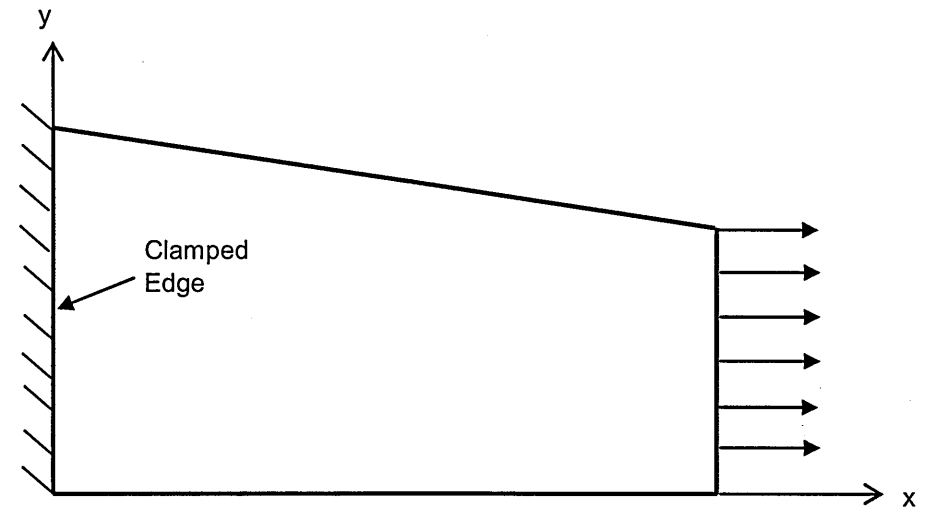

Figure 8.37. Trapezoidal panel under tension.

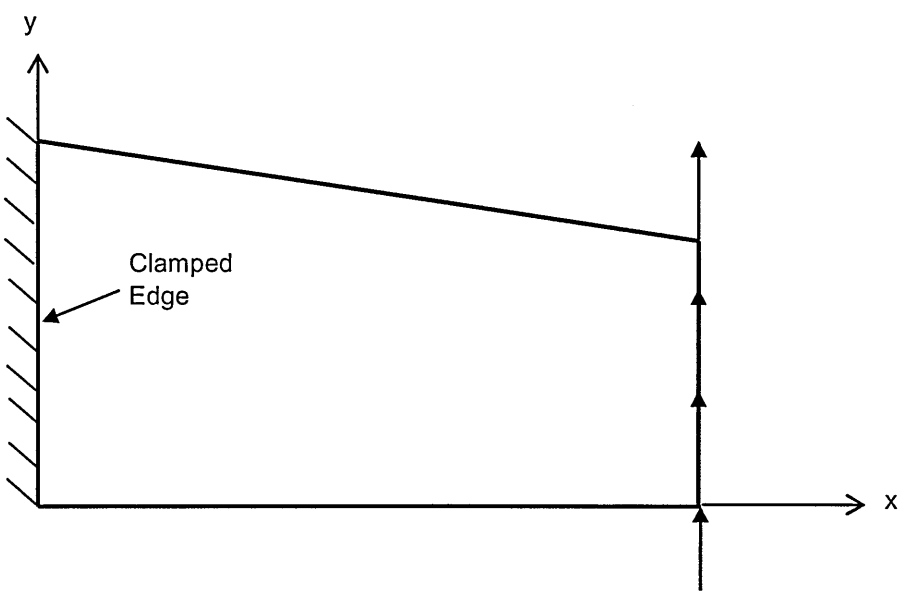

Figure 8.38. Trapezoidal panel under in-plane bending.

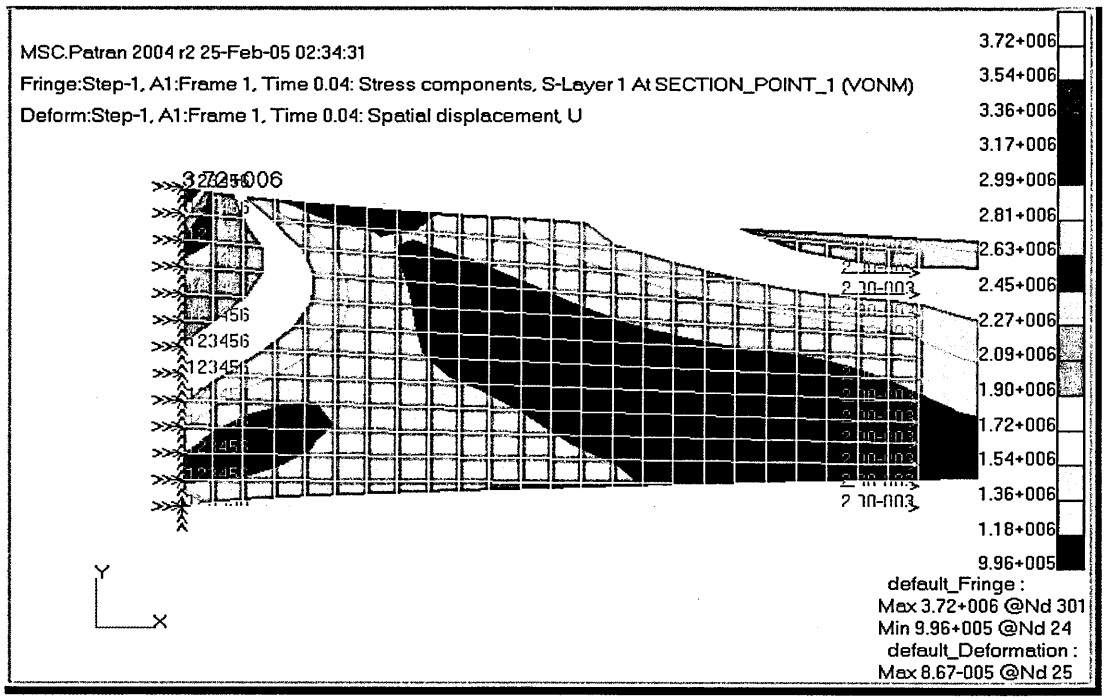

Figure 8.39. ABAQUS plot of Von-Mises stress at the lowest surface for trapezoidal panel under tension.

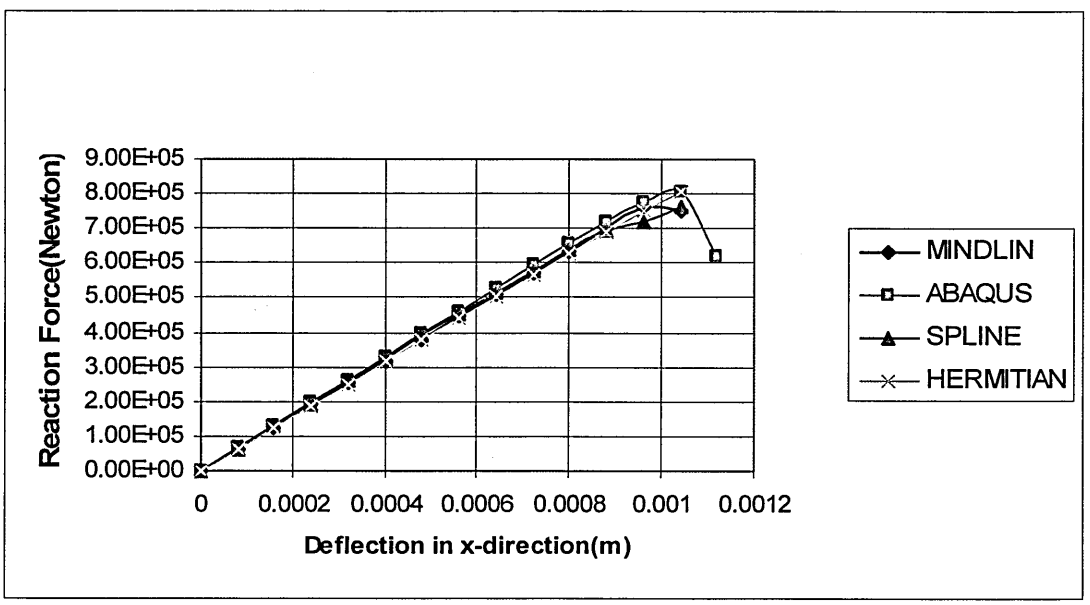

Figure 8.40. Load-deflection curves of trapezoidal panel under tension.

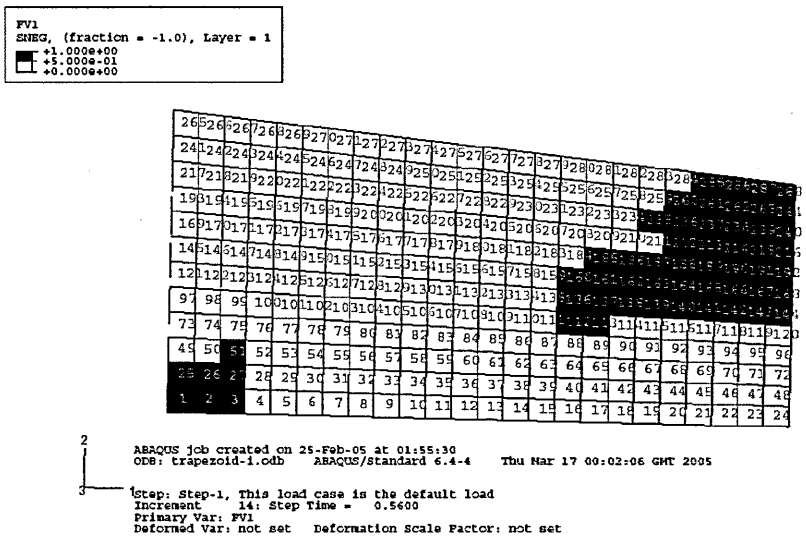

Figure 8.41. ABAQUS damage zone in layer one of trapezoidal panel under tension at final load increment.

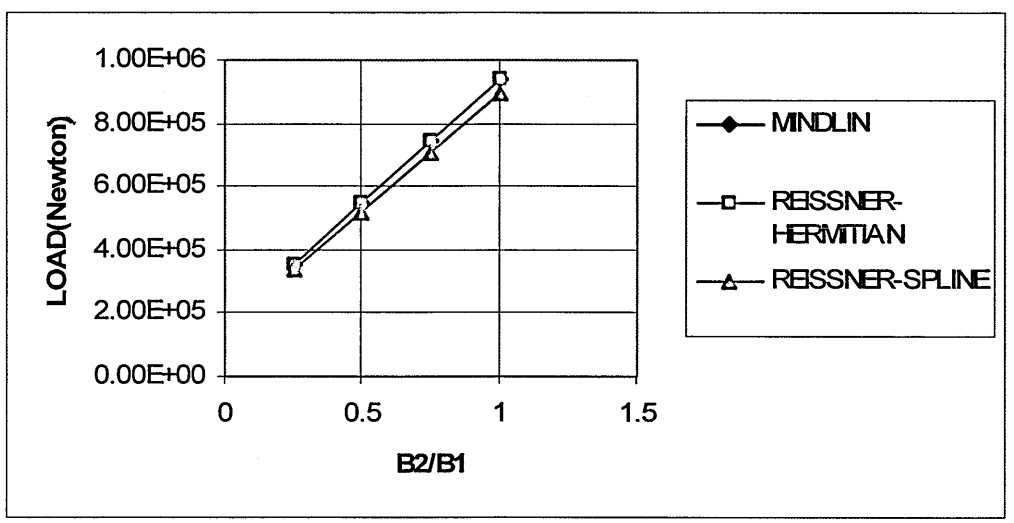

Figure 8.42. Load versus width ratio of trapezoidal panel under tension.

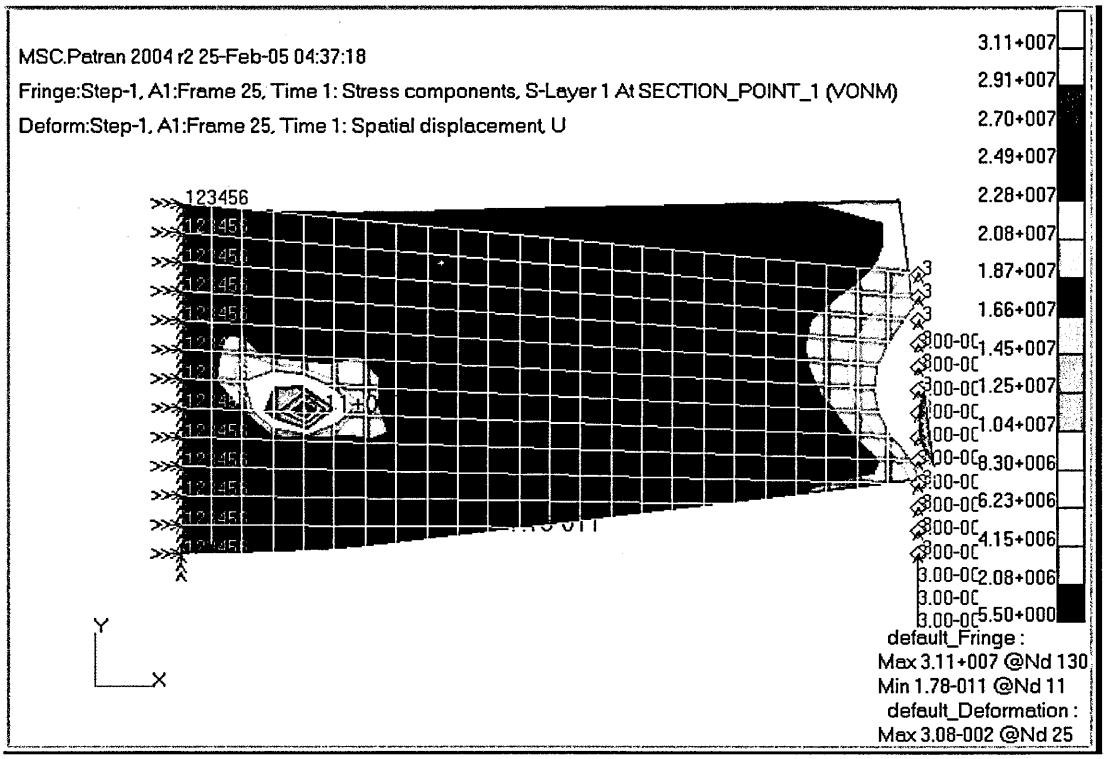

Figure 8.43. ABAQUS plot of Von Mises stress at the lowest surface of trapezoidal panel under tension.

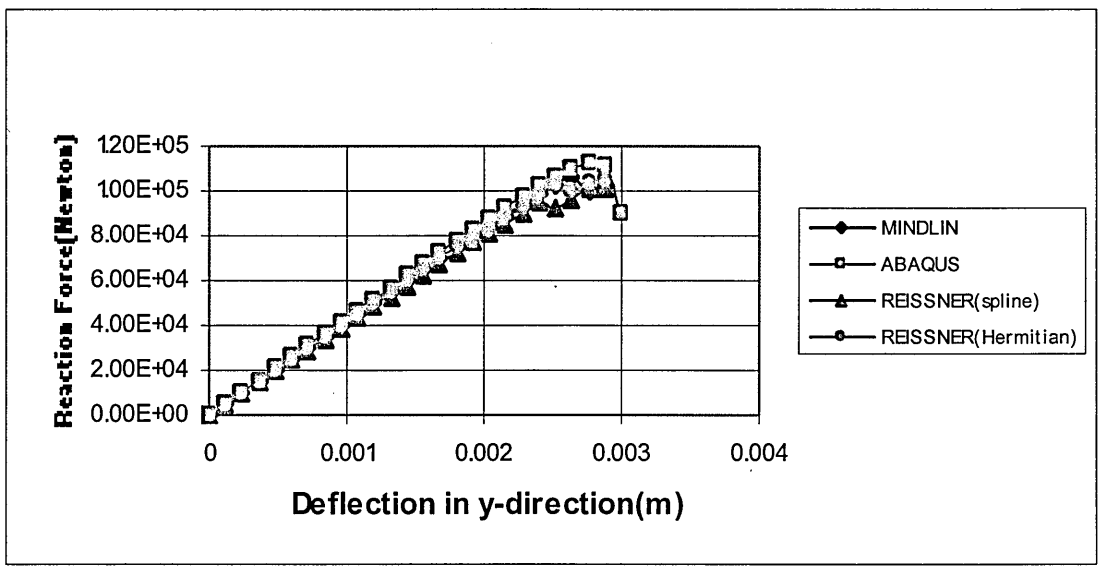

Figure 8.44. Load-deflection curves of trapezoidal panel under in-plane bending.

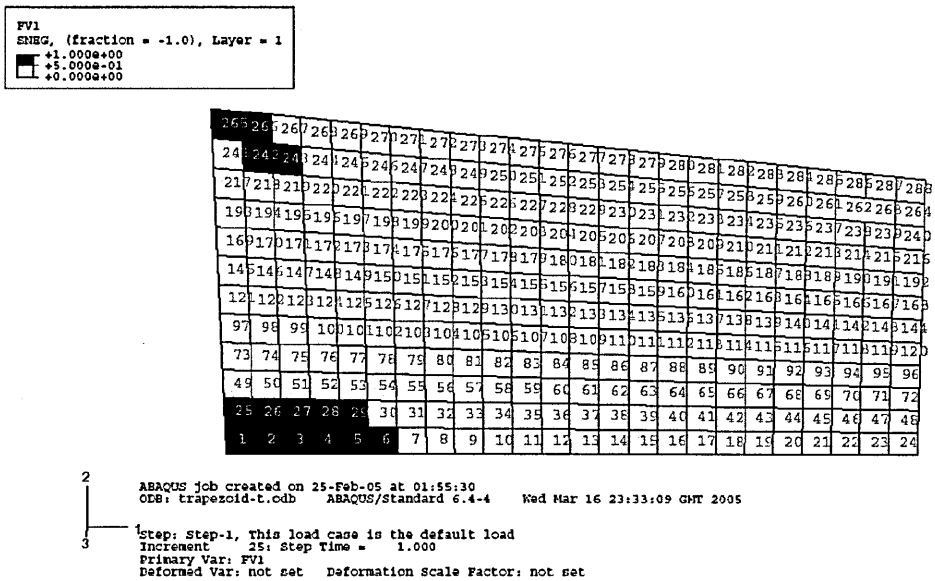

Figure 8.45. ABAQUS damage zone in layer one of trapezoidal panel under in-plane bending at final load increment.

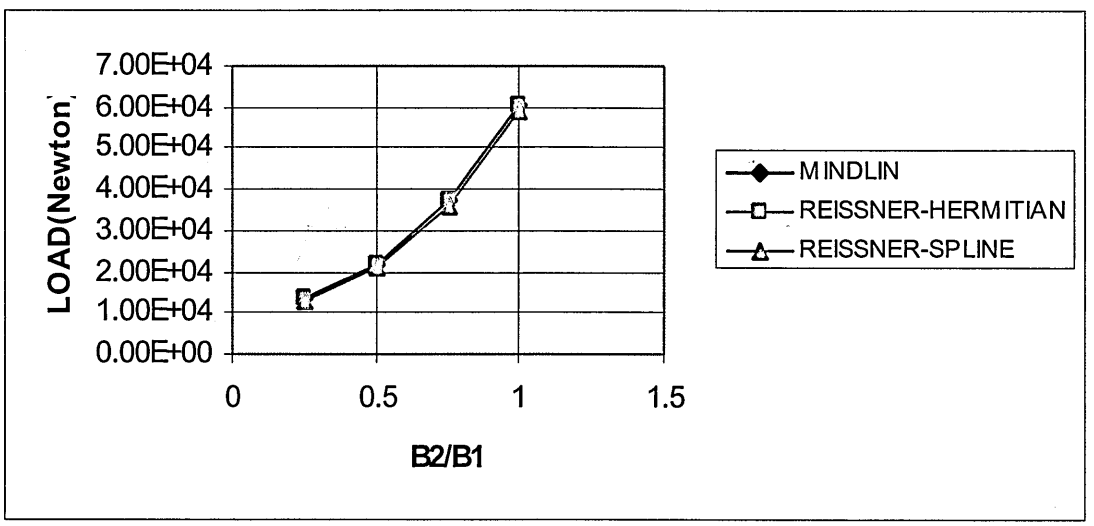

Figure 8.46. Load versus width ratio of trapezoidal panel under in-plane bending.

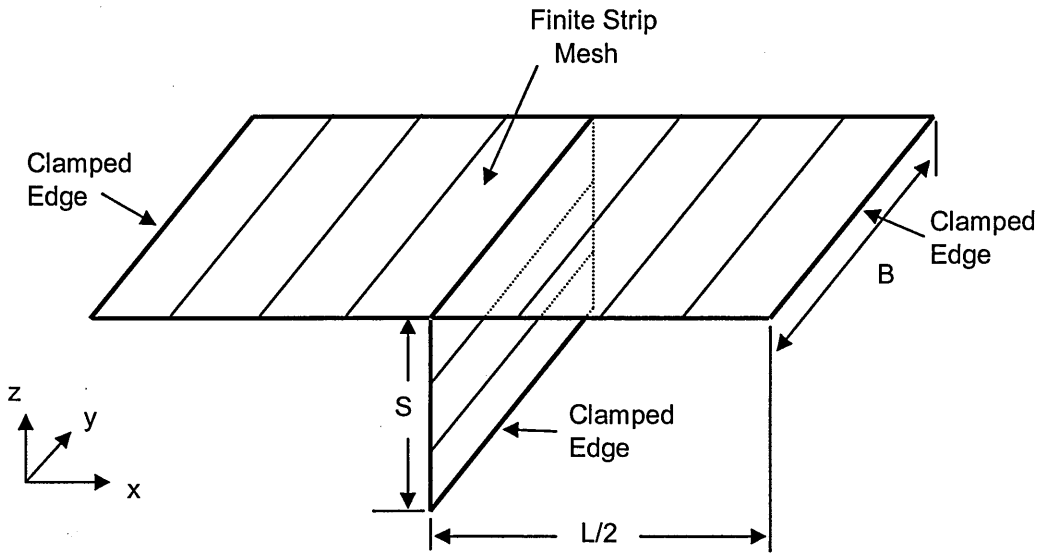

Figure 8.47. Geometry and boundary conditions of Stiffened plate.

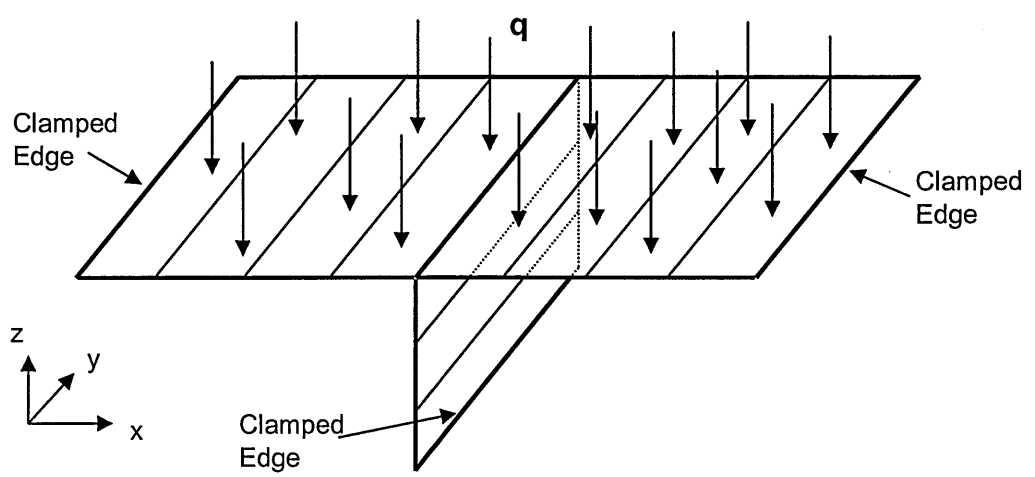

Figure 8.48. Stiffened plate under uniform distributed load.

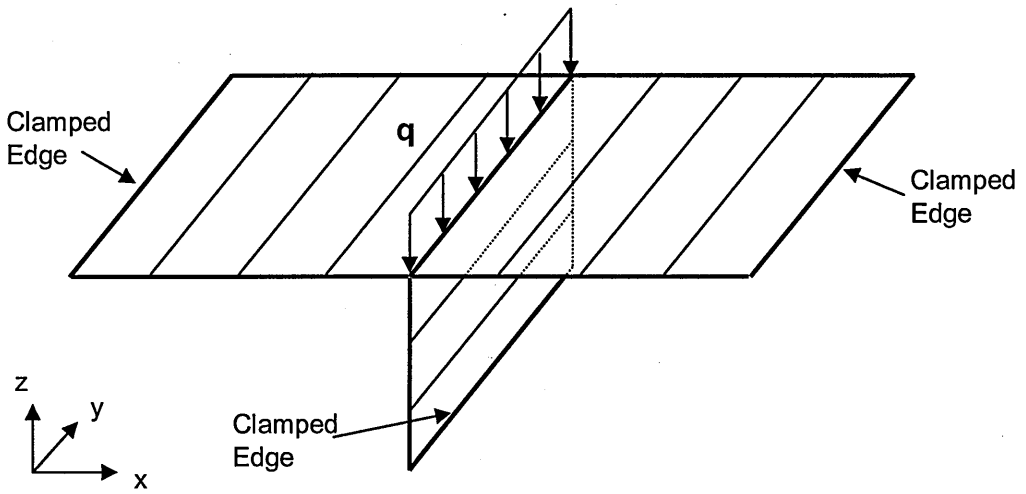

Figure 8.49. Stiffened plate under uniform line loading.

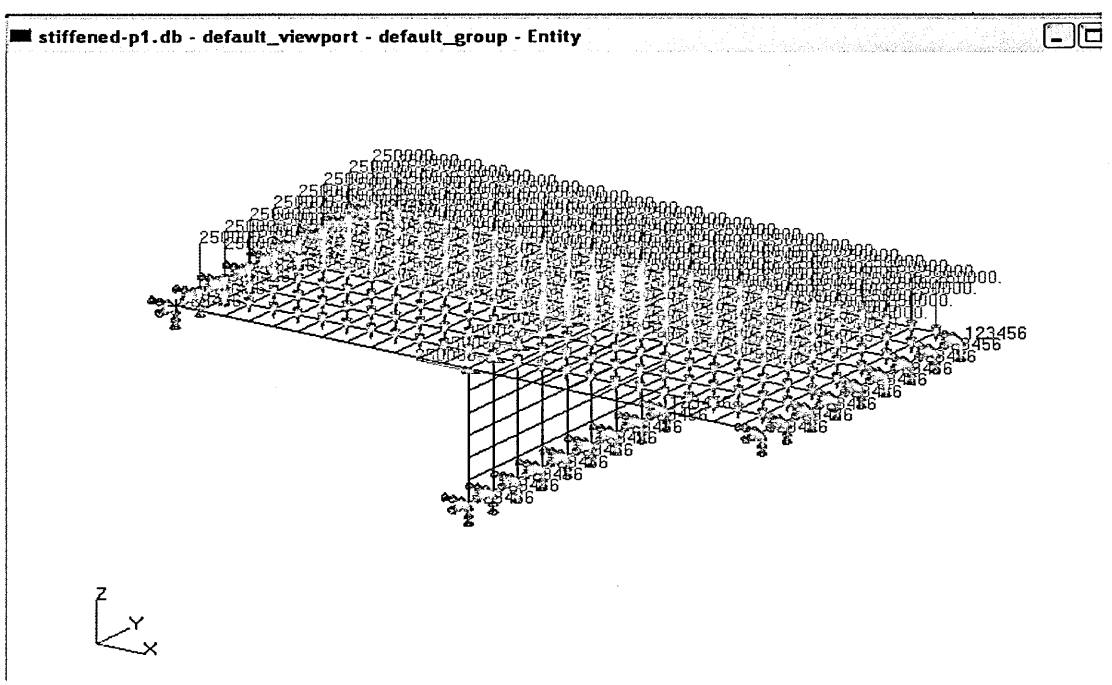

Figure 8.50. ABAQUS stiffened plate finite element mesh and boundary conditions under uniform distributed load.

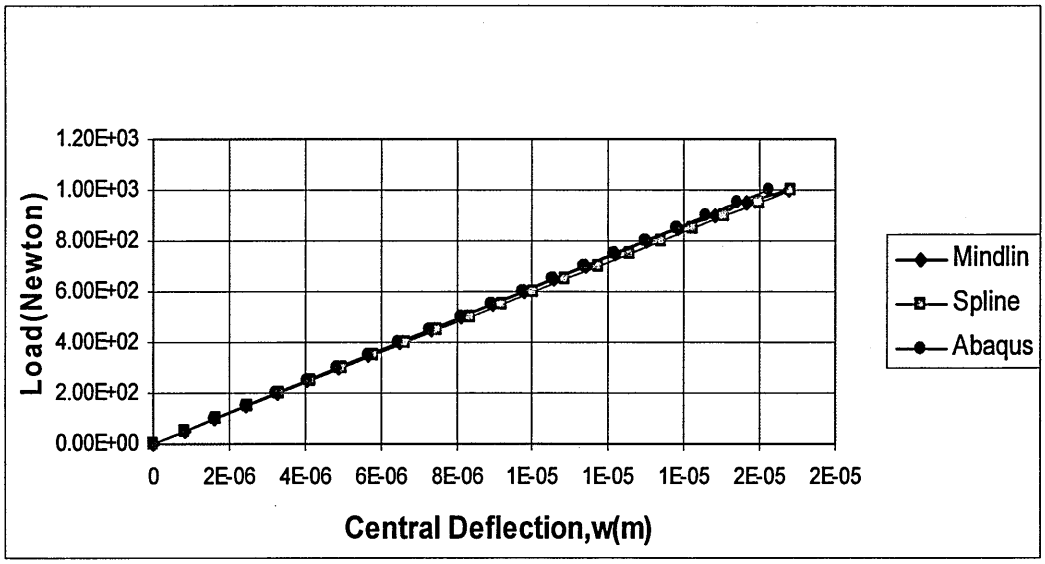

Figure 8.51. Validation of structural response of stiffened plate under uniform distributed load using Tsai-Wu failure Criterion

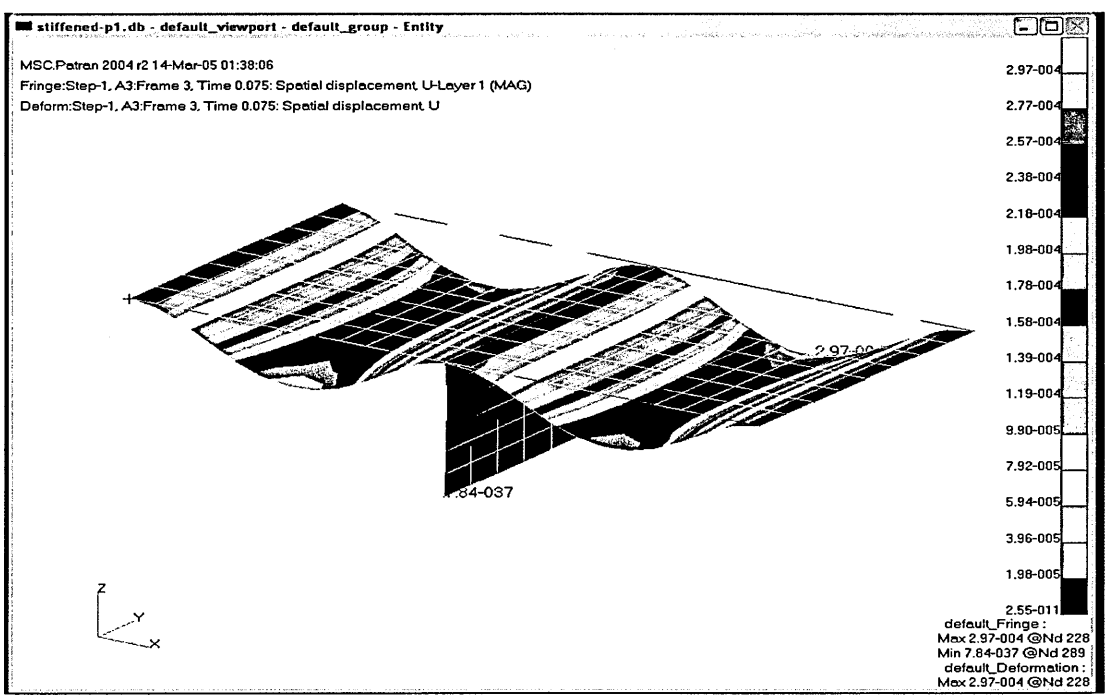

Figure 8.52. ABAQUS plot of lateral displacement for stiffened plate under uniform distributed load.

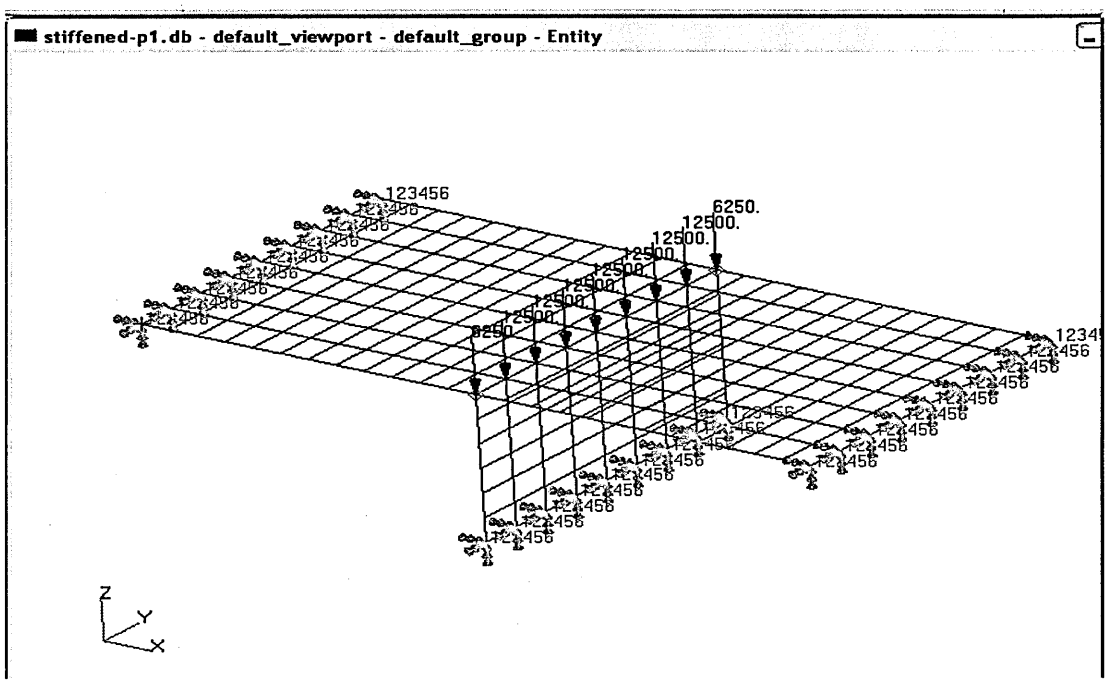

Figure 8.53. ABAQUS stiffened plate finite element mesh and boundary conditions under uniform line loading.

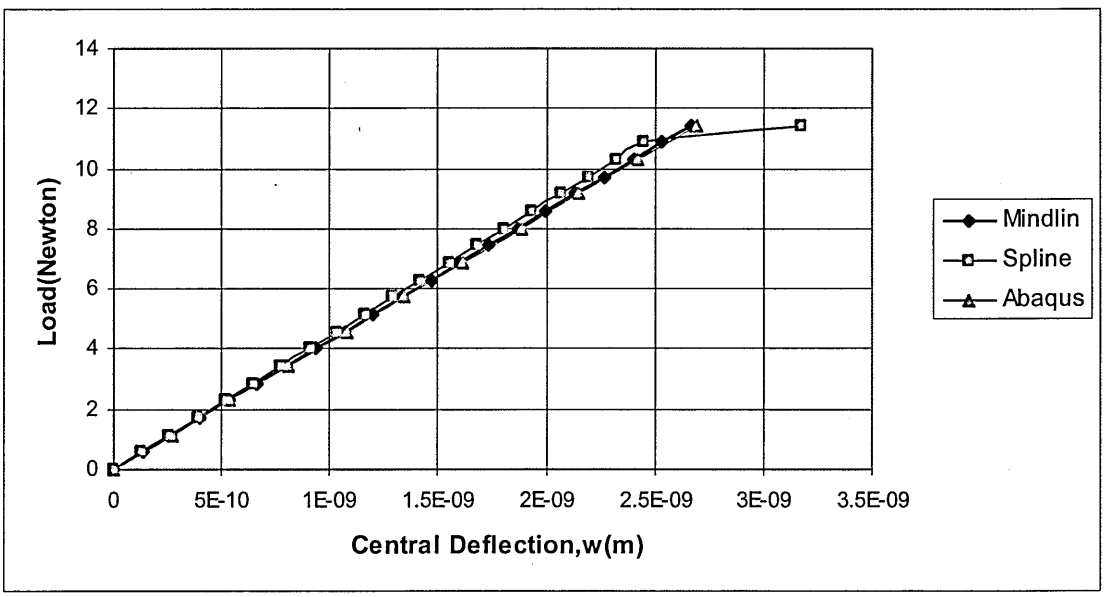

Figure 8.54. Validation of structural response of stiffened plate under uniform line loading using Tsai-Wu failure Criterion.

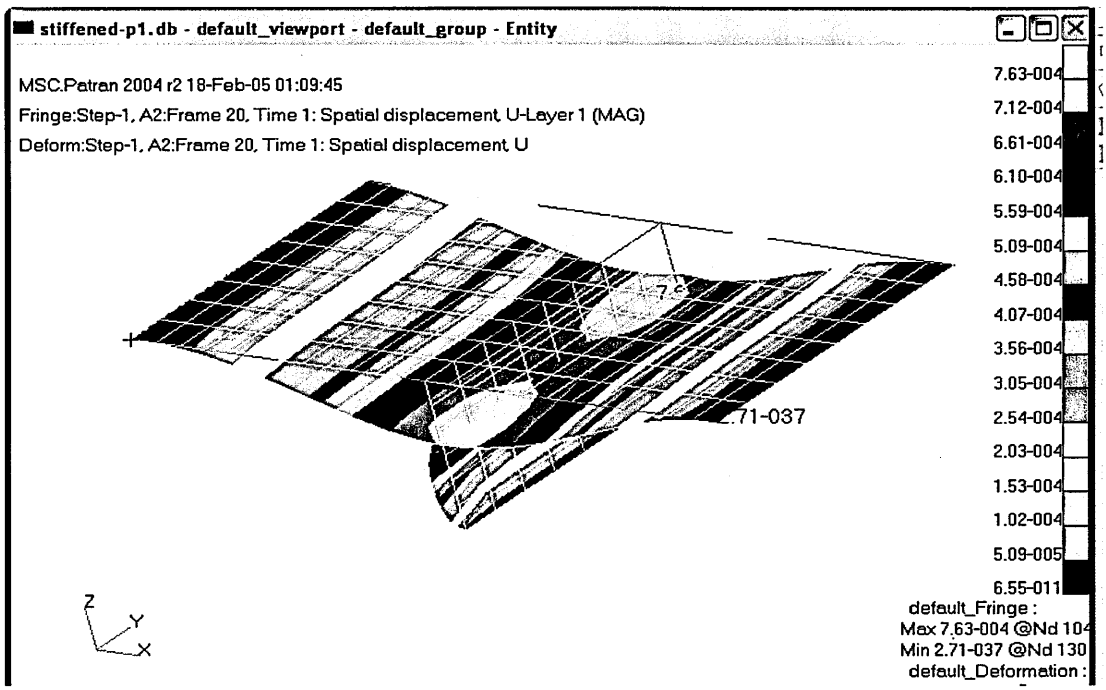

Figure 8.55. ABAQUS plot of lateral displacement for stiffened plate under uniform line loading.

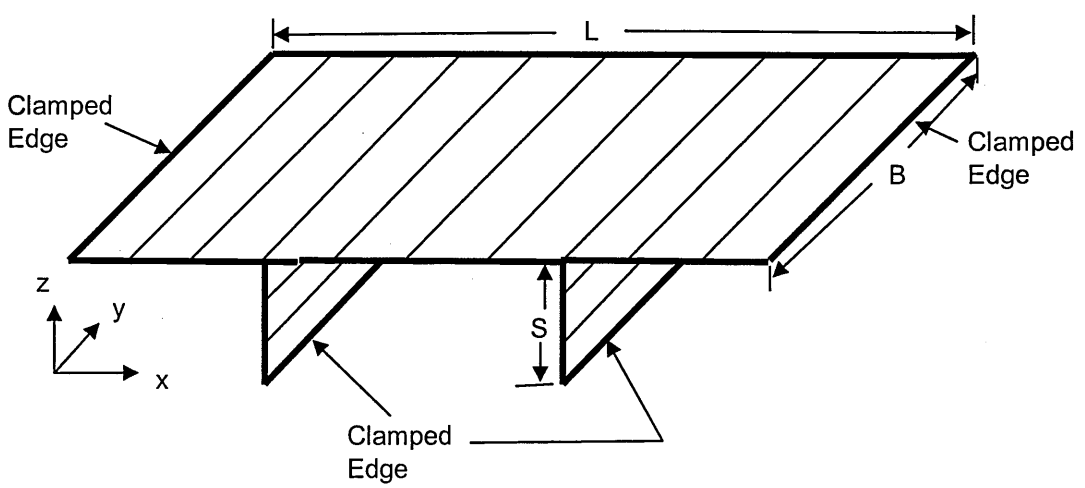

Figure 8.56. Geometry and boundary conditions of blade stiffened panel.

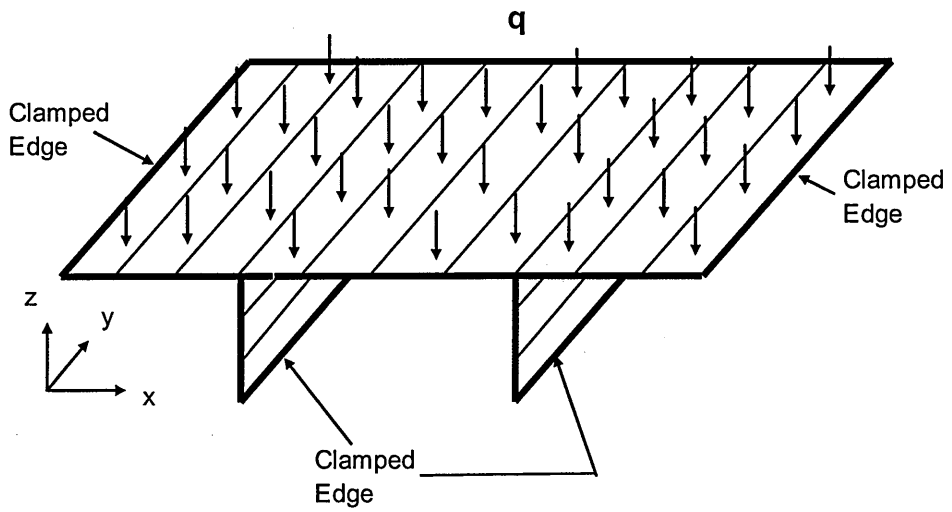

Figure 8.57. Stiffened panel under uniform distributed load.

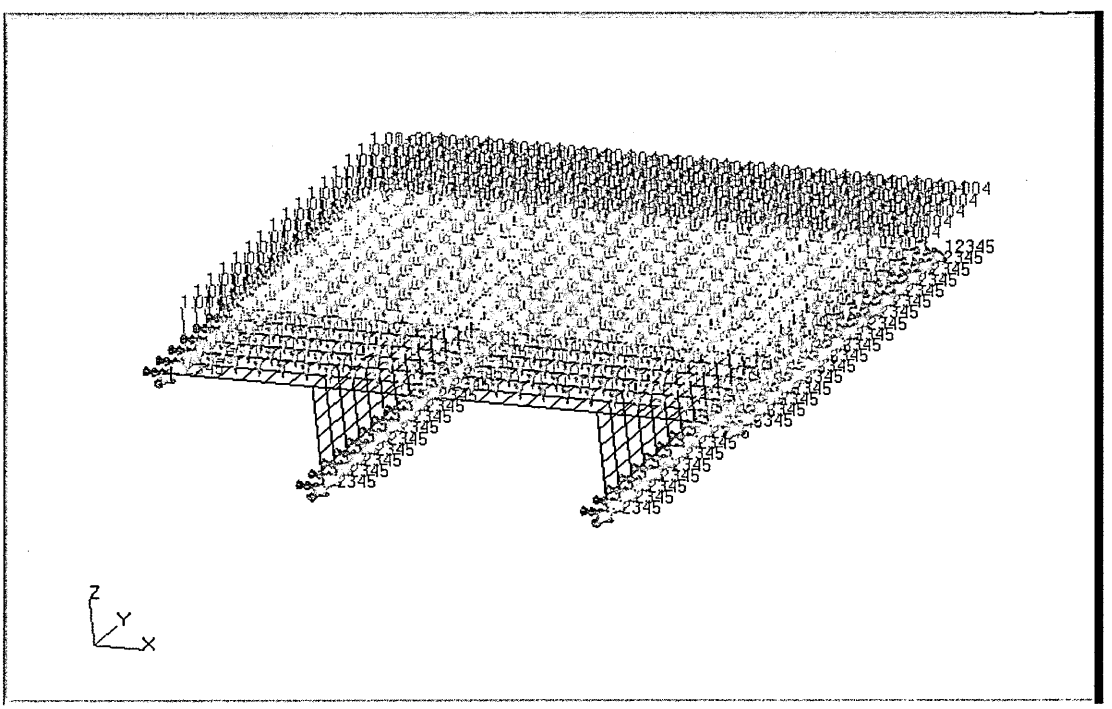

Figure 8.58. ABAQUS model of the blade stiffened panel.

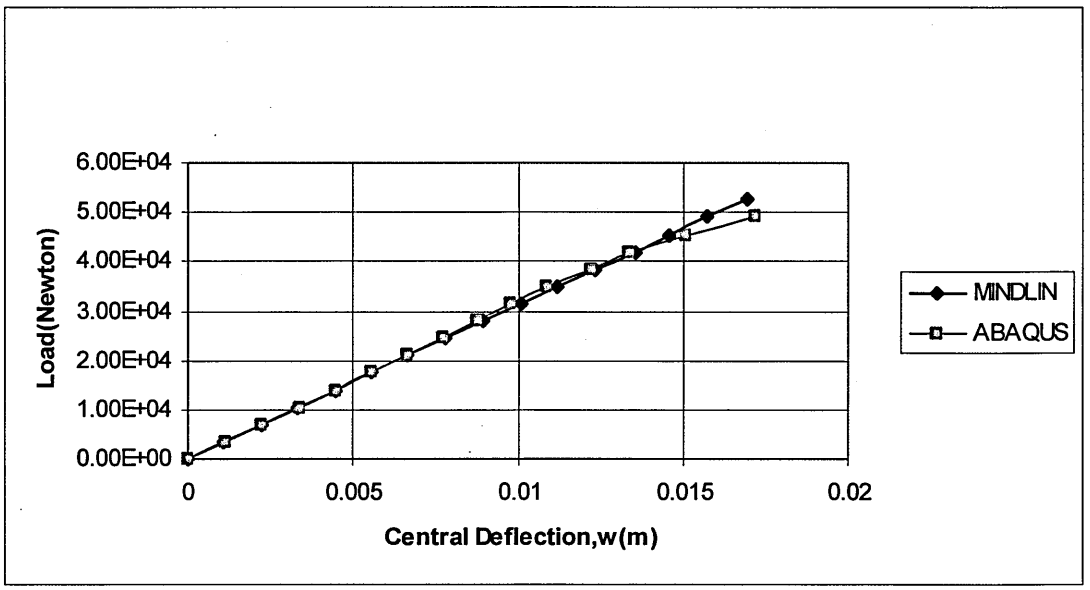

Figure 8.59. Validation of structural response of stiffened panel under uniform distributed load using Tsai-Wu failure Criterion.

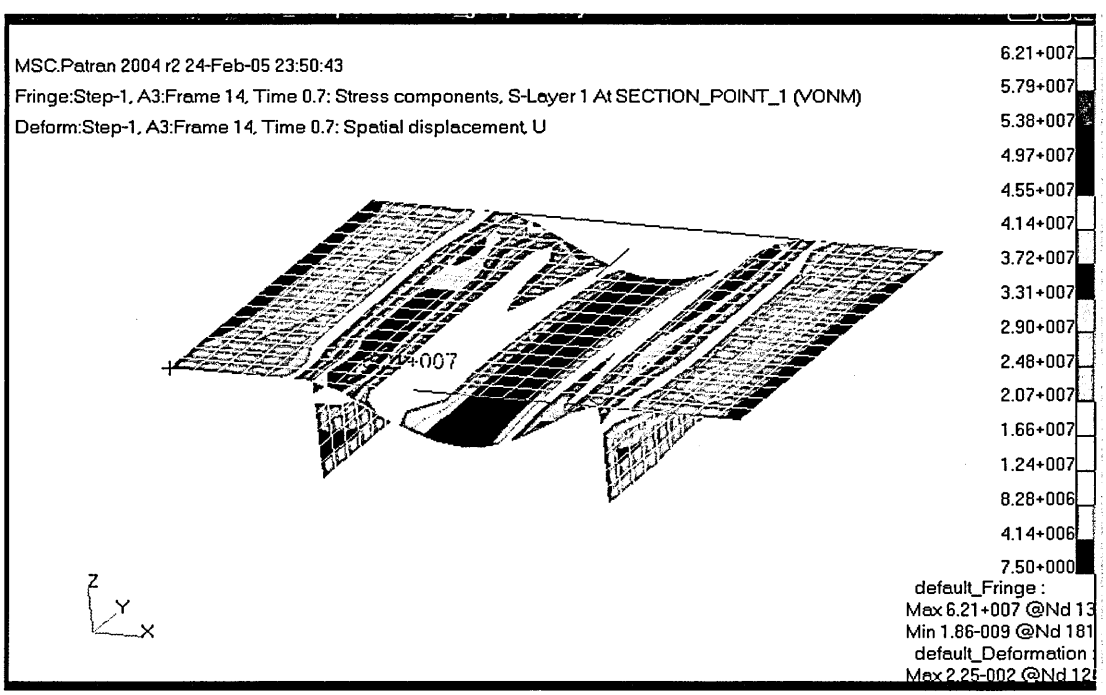

Figure 8.60. ABAQUS plot of Von Mises stress at the lowest surface for stiffened panel under uniform distributed load.

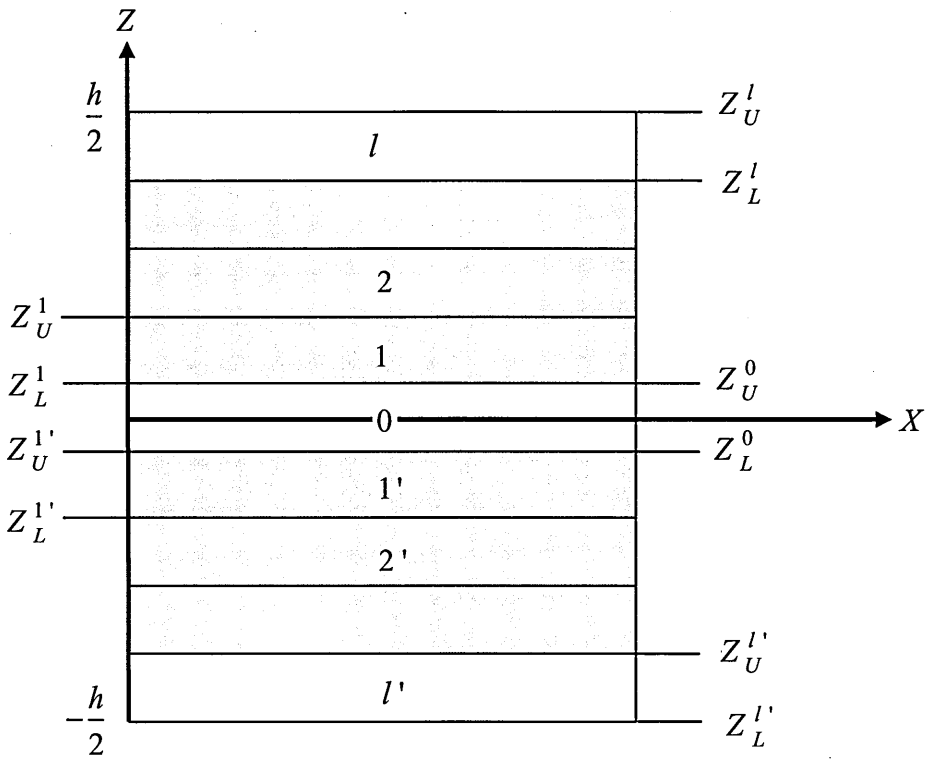

Figure A.1. Symmetric laminate configuration

APPENDIX A

Appendix A. *D* matrices for Symmetric Composite of Reissner-type Element

Consider a composite laminate such as shown in figure *A.1*. It is made up of several layers with the total thickness of h and are symmetric about plane $z = 0$, which is the mid-plane of layer 0. The total number of layers for such laminate configuration is given by $N_l = 2m+1$ where $m = 0, 1, 2, 3\ldots$

It can be clearly seen from figure *A1* that for symmetry:

$$\begin{aligned} Z_L^0 &= -Z_U^0 , \\ Z_U^{1'} &= -Z_L^1 , \\ Z_L^{1'} &= -Z_U^1 , \\ Z_L^{l'} &= -Z_U^l \\ Z_U^{l'} &= -Z_L^l \end{aligned} \tag{A.1}$$

and using the definition of the integrated $\underline{D}$ matrices (integrated analytically over the laminate thickness) discussed previously in chapters 4 and 5:

$$\underline{D}_{(2n-1)} = \int_{-h/2}^{h/2} Z^{(2n-1)}\, \underline{D}^{(l)}\, dz \tag{A.2}$$

from figure (A.1) it can be deduced that:

$$\underline{D}_{(2n-1)} = I_1 + I_0 + I_2 \tag{A.3}$$

where:

$$I_0 = \int_{Z_L^0}^{Z_U^0} Z^{(2n-1)} \underline{D}^{(l)} dz \tag{A.4}$$

$$I_1 = \int_{-h/2}^{Z_U^{1'}} Z^{(2n-1)} \underline{D}^{(l)} dz \tag{A.5}$$

$$I_2 = \int_{Z_L^1}^{h/2} Z^{(2n-1)} \underline{D}^{(l)} dz \tag{A.6}$$

From equation (A.4) it can be proven that:

$$I_0 = \int_{Z_L^0}^{Z_U^0} Z^{(2n-1)} \underline{D}^{(0)} dz = \frac{1}{2n} z^{2n} \Big|_{Z_L^0}^{Z_U^0} \underline{D}^{(0)} = 0 \tag{A.7}$$

The summation of equation (A.5) and (A.6) yields:

$$I_1 + I_2 = \int_{-h/2}^{Z_U^{1'}} Z^{(2n-1)} \underline{D}^{(l)} dz + \int_{Z_L^1}^{Z_U^{1'}} Z^{(2n-1)} \underline{D}^{(l)} dz$$

$$I_1 + I_2 = \int_{Z_L^1}^{h/2} Z^{(2n-1)} \underline{D}^{(l)} dz - \int_{Z_U^{1'}}^{-h/2} Z^{(2n-1)} \underline{D}^{(l)} dz \tag{A.8}$$

$$= \sum_{l=1}^{m} \int_{Z_L^l}^{Z_U^l} Z^{(2n-1)} \underline{D}^{(l)} dz - \int_{Z_U^{l'}}^{Z_L^{l'}} Z^{(2n-1)} \underline{D}^{(l)} dz$$

$$= \sum_{l=1}^{m} \left\{ \frac{Z^{2n}}{2n} \underline{D}^{(l)} \Big|_{Z_L^l}^{Z_U^l} - \frac{Z^{2n}}{2n} \underline{D}^{(l')} \Big|_{Z_U^{l'}}^{Z_L^{l'}} \right\} = 0 \tag{A.9}$$

where for symmetry $\underline{D}^{(l)} = \underline{D}^{(l')}$.

By substituting equation (A.7) and (A.9) into (A.3):

$$\underline{D}_{(2n-1)} = I_1 + I_0 + I_2 = 0$$

Hence,

$$\underline{D}_{(1)} = \int_{-h/2}^{h/2} Z \, \underline{D}^{(l)} dz = 0$$

$$\underline{D}_{(3)} = \int_{-h/2}^{h/2} Z^3 \, \underline{D}^{(l)} dz = 0$$

It can be proved in a similar way that:

$$\underline{D}_{\psi o} = \int_{-h/2}^{h/2} f_\psi (z) \underline{D}^{(l)} dz = 0$$

$$\underline{D}_{\psi 2} = \underline{D}_{\psi\psi 1} = \underline{D}_{\psi\psi\psi} = 0$$

where

$$f_\psi (z) = \frac{3}{2} z - \frac{2z^3}{h^2}$$

APPENDIX B

Appendix B. Sample input files

B.1 Cantilever Plate

B.1.1 Three-node Mindlin input data file

```
** Carbon/Epoxy Composite Shell
** Using Mindlin Finite Strip 3-node Element
** Non-Linear static analysis under tension
** TSAI-WU Failure Theory

Harmonics: NHAR,IHAR
4,4

Strips: NBN,NBL,NODE/I,XB(I),ZB(I)
2,1,3
1,0.0,0.0
2,2.0,0.0
1,1.0,1.0,1,1,2,6

COMPOSITE MODULE
1
134.75E9,8.24E9,0.325,7.0E9,7.0E9,7.0E9,800
12
1,-45,2.5E-3
2,  0,2.5E-3
3, 45,2.5E-3
4,-45,2.5E-3
5,  0,2.5E-3
6, 45,2.5E-3
7, 45,2.5E-3
8,  0,2.5E-3
9,-45,2.5E-3
10,45,2.5E-3
11, 0,2.5E-3
12,-45,2.5E-3

Boundary Conditions
1
0,0.0,1,1
0,0

Loads: NLINC,NLN,NLX,NLY,NDL
-10,0,1,0,0
1,-13,1,2.0e6, 0.0, 0.0, 0.0, 0.0

STATIC MODULE
0
50,1.0E-7,1.0
```

OUTPUT MODULE
0
1,13
1,1
Gauss Quadrature Data (Reduced)
2,1,2

FAILURE DATA
2,1500.0E,50.0E,1200.0E,250.0E,70.0E

B.1.2 Two-node Reissner(Hermitain) element input data file

** Carbon/Epoxy Composite Shell
** Using Mindlin Finite Strip 2-node Element
** Non-Linear static analysis under in-plane bending
** Tsai-Wu Failure Theory

Harmonics: NHAR,IHAR
4,2

Strips: NBN,NBL,NODE/I,XB(I),ZB(I)
2,1,2
1,0.0,0.0
2,2.0,0.0
1,1.0,1.0,1,1,2,12

COMPOSITE MODULE
1
134.75E9,8.24E9,0.325,7.0E9,7.0E9,7.0E9,800
12
1,-45,2.5E-3
2, 0,2.5E-3
3, 45,2.5E-3
4,-45,2.5E-3
5, 0,2.5E-3
6, 45,2.5E-3
7, 45,2.5E-3
8, 0,2.5E-3
9,-45,2.5E-3
10,45,2.5E-3
11, 0,2.5E-3
12,-45,2.5E-3

Boundary Conditions
1
0,0.0,1,1
0,0

Loads: NLINC,NLN,NLX,NLY,NDL
-20,0,1,0,0
1,-13,1,0.0,1.3e5,0.0, 0.0, 0.0, 0.0

STATIC MODULE
0
50,1.0E-7,1.0

OUTPUT MODULE
0
1,13
1,1

Gauss Quadrature Data (Reduced)
2,1,2

FAILURE DATA
2,1500.0,50.0,1200.0,250.0,70.0

B.2. Stiffened Plate

B.2.1. Two-node Reissner (Spline) element input file

** Carbon/Epoxy Composite Stiffened Plate
** Using Spline Finite Strip 2-node Element
** Non-linear static analysis of stiffend plate
** under uniform distributed load
** Tsai-Wu Failure Theory

Harmonics: NHAR,IHAR
5,3

Strips: NBN,NBL,NODE/I,XB(I),ZB(I)
4,3,2
1,0.0,0.0
2,1.0,0.0
3,2.0,0.0
4,1.0,-0.5
1,1.0,1.0,1,1,2,12
2,1.0,1.0,1,2,3,12
3,1.0,1.0,1,2,4,6

COMPOSITE MODULE
1
134.75E9,8.24E9,0.325,7.0E9,7.0E9,7.0E9,800
12

1,-45,2.5E-3
2, 0,2.5E-3
3, 45,2.5E-3
4,-45,2.5E-3
5, 0,2.5E-3
6, 45,2.5E-3
7, 45,2.5E-3
8, 0,2.5E-3
9,-45,2.5E-3
10,45,2.5E-3
11, 0,2.5E-3
12,-45,2.5E-3

Boundary Conditions
1
0,0.0,3,1,3,4
0,0

Gauss Quadrature Data (Reduced)
2,1,2

Loads: NLINC,NLN,NLX,NLY,NDL
-20,0,0,0,1
-0.1e4,-0.1e4,0.0

STATIC MODULE
0
200,1.0E-7,1.0

OUTPUT MODULE
0
2,7,19
1,1

FAILURE DATA
2,1500.0E3,50.0E3,1200.0E3,250.0E3,70.0E3

APPENDIX C

APPENDIX C. Progressive Damage Analysis Algorithm by ABAQUS

C.1 Damage Modelling of composites using user-written subroutine USDFLD

In order to simulate the progressive damage in composite laminates using the ABAQUS finite element commercial package, a user-written subroutine has to be included in the non-linear analysis by linking it with the input file before running the analysis. The user subroutine is written in FORTRAN language where it is saved as an independent file which contains the equation of Tsai-Wu failure theory.

A progressive damage scheme was developed for the ABAQUS finite element package using the USDFLD user-written subroutine (ABAQUS User Manual Version 6.4, 2003). In this procedure ABAQUS will call the USDFLD subroutine at all integration points of the elements which have the material properties defined in terms of the field variable, *FV1*. At each load increment, stresses are computed and the Tsai-Wu failure criterion is evaluated to determine the damage occurrence in the elements.

In order to simulate non-linear material properties in ABAQUS, a material degradation model has to be implemented. The elastic material properties of the laminate are made to depend on a field variable, *FV1*. For example, when the failure indices of the Tsai-Wu failure criterion has exceeded 1.0, then all the values of the moduli i.e. E_{11}, E_{22}, G_{12}, G_{23}, G_{13} and Poisson ratio (v_{12}) are reduced to zero. The values of the failure indices are not assigned directly to the field variable. Instead, they are stored as solution dependent state variable, *SDV1*. Only if the value of the failure index is greater or equal 1.0 the field variable is set equal to 1.0. After the failure index has exceeded 1.0 the user defined field variable continues to have the value of 1.0 although the stresses may be reduced significantly. This ensures that the damage material does not heal after it has become damaged. This procedure is repeated at every load increment in the non-linear analysis. Hence, with the implementation of this procedure we are able to model the non-linear material properties in the progressive damage analysis using ABAQUS. The sample USDFLD subroutine which has been coded in FORTRAN is presented in section C.2.

C.2. Sample user-written subroutine (USDFLD)

```
SUBROUTINE USDFLD(FIELD,STATEV,PNEWDT,DIRECT,T,CELENT,TIME,DTIME,
     1 CMNAME,ORNAME,NFIELD,NSTATV,NOEL,NPT,LAYER,KSPT,KSTEP,KINC,
     2 NDI,nshr,coord,jmac,jmtyp,matlayo,laccflg)
C
      INCLUDE 'ABA_PARAM.INC'
C
C     MATERIAL AND STRENGTH PARAMETERS

PARAMETER(YT=50.0D5,XC=1200.0D5,XT=1500.0D5,YC=250.0D5,S=70.0D5)
C
      CHARACTER*80 CMNAME,ORNAME
      CHARACTER*8  FLGRAY(15)
      DIMENSION
FIELD(NFIELD),STATEV(NSTATV),DIRECT(3,3),T(3,3),TIME(2),
     * coord(*),jmac(*),jmtyp(*)
      DIMENSION ARRAY(15),JARRAY(15)
C
C     INITIALIZE FAILURE FLAGS FROM STATEV.
      FAIL     = STATEV(1)

C
C     GET STRESSES FROM PREVIOUS INCREMENT
      CALL GETVRM('S',ARRAY,JARRAY,FLGRAY,jrcd,
     $     jmac, jmtyp, matlayo, laccflg)
      S11 = ARRAY(1)
      S22 = ARRAY(2)
      S12 = ARRAY(4)


C     TSAI-WU FAILURE THEORY
      IF (FAIL .LT. 1.D0) THEN

      T1=(1/XT)-(1/XC)
      T2=(1/YT)-(1/YC)
      T11=1/(XT*XC)
      T22=1/(YT*YC)
      T33=1/(S**2)
      C=T11*T22
      T12=-1/2.0*(SQRT(C))

      W=T1*S11+T2*S22+T11*S11**2+T22*S22**2
      R=T33*S12**2+2*T12*S11*S22
      FAIL=W+R
      STATEV(1)=FAIL
      END IF


C     UPDATE FIELD VARIABLES
C
      FIELD(1) = 0.D0
      IF (FAIL .GT. 1.D0) FIELD(1) = 1.D0

      RETURN
      END
```