

Cranfield University

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The Numerical Modelling of Elastomers

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Abstract

This thesis reports on review and research work carried out on the numerical analysis of elastomers. The two numerical techniques investigated for this purpose are the finite and boundary element methods. The finite element method is studied so that existing theory is used to develop a finite element code both to review the finite element method as applied to the stress analysis of elastomers and to provide a comparison of results and numerical approach with the boundary element method.

The research work reported on in this thesis covers the application of the boundary element method to the stress analysis of elastomers. To this end a simplified regularization approach is discussed for the removal of strong and hypersingularities generated in the system of non-linear boundary integral equations. The necessary programming details for the implementation of the boundary element method are discussed based on the code developed for this research.

Both the finite and boundary element codes developed for this research use the Mooney-Rivlin material model as the strain energy based constitutive stress strain function. For validation purposes four test cases are investigated. These are the uniaxial patch test, pressurized thick walled cylinder, centrifugal loading of a rotating disk and the J-Integral evaluation for a centrally cracked plate. For the patch test and pressurized cylinder, both plane stress and strain have been investigated. For the centrifugal loading and centrally cracked plate test cases only plane stress has been investigated. For each test case the equivalent results for an equivalent FEM program mesh have been presented.

The test results included in this thesis prove that the FE and BE derivations detailed in this work are correct. Specifically the simplified domain integral singular and hyper-singular regularization approach was shown to lead to accurate results for the test cases detailed. Various algorithm findings specific to the BEM implementation of the theory are also discussed.

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Nomenclature

Acronym

a	Crack half length
BIE	Boundary integral equation
CPU	Central processor unit
CPV	Cauchy principal value
$DBIE$	Derivative boundary integral equation
$\underline{D}$	Incremental stress/strain constitutive tensor
E	Modulus of elasticity
FEM	Finite element method
G	Crack energy release rate
$\underline{G}$	Deformation gradient matrix
$\underline{g}$	Displacement derivative matrix
J	Jacobian determinant of right Cauchy-Green tensor
K	Bulk modulus
$\underline{K}$	Linear incremental stiffness matrix
PK	Piola Kirchhoff stress
p	Modified Poisson's ratio
$\underline{p}$	Displacement vector
$\underline{C}$	Right Cauchy-Green tensor

Y	Stress intensity factor
r	Field to source point distance

Greek

ν	Poisson's ratio
ρ	Hydrostatic pressure
Ω	System domain
Γ	System boundary
Ω'	Domain excluding vanishing region centered on BIE source point
Γ'	Boundary excluding vanishing region centered on BIE source point
μ	Shear modulus
$\underline{\underline{\rho}}$	Hydrostatic pressure constitutive vector
ϕ	Strain energy per unit volume
Φ	Absolute strain energy

Subscripts

a_c	Critical crack half length
$\underline{\underline{B}}_{nl}$	Non-linear strain matrix
C_1, C_2	Mooney-Rivlin material model coefficients
G_c	Critical strain energy release rate
I_1	First right Cauchy-Green strain invariant
I_2	Second right Cauchy-Green strain invariant
I_3	Third right Cauchy-Green strain invariant
J_r	J integral
$K_{I,II,III}$	Stress intensity, mode I, II or III
$\underline{\underline{\varepsilon}}_a$	Almansi's strain
$\underline{\underline{\varepsilon}}_G$	Green strain

$\underline{N}_\rho$	Hydrostatic pressure shape function
ρ_i	Thick walled cylinder internal pressure
$(T_\alpha^{nl})_i$	Tractions defined with direction cosines at the field point
$\bar{T}_\alpha^{nl}$	Tractions defined with direction cosines at the source point
$\underline{K}_\rho$	Hydrostatic pressure 'stiffness' matrix
U_α	BIE domain loading domain integral term
$E_{\alpha\delta}$	DBIE domain loading domain integral term
ρ_m	Material density
Ω_ε	Limiting Cauchy integration domain
Γ_ε	Limiting Cauchy integration boundary

Chapter 1

Introduction

1.1 General introduction

Elastomers represent a unique set of materials that have many desirable engineering properties. Not least of which is their low cost and availability. In general though, the properties of elastomers include flexibility, resilience and long service life. Elastomers can have excellent damping and energy absorption characteristics and have the ability to seal against moisture, heat and pressure whilst being non-toxic. From a production point of view elastomers can easily be molded into almost any shape. With all these useful characteristics it is not surprising that elastomers have found many engineering applications. These include solid propellant, biomechanics and medical/dental, automotive and packaging applications to name but a few.

The wide use of elastomers today and the modern trend to optimise designs and materials for cost and performance have led to the development of numerical modelling tools to use with elastomers. The most popular numerical approach, and the one with most industrial use, the FEM, has also been developed for modelling elastomers. In recent years in an effort to cut down on computational cost and time a similar numerical technique, namely the BEM, has been developed. However the application of the BEM to modelling elastomers has not been as fully researched as the FEM.

The development of BEM code to use when modelling elastomers is subject to the same difficulties any other numerical approach would have due to the nature of elastomers. The unique properties of elastomers are due to their tangled, long chain hydrocarbon molecular structure. Without going into chemical detail the bulk properties of elastomers display the following four characteristics which must be built in and allowed for as founding assumptions for any numerical approach

modelling elastomers. These bulk properties are:

1. They are capable of finite elastic strains.
2. They have non-linear load extension behaviours.
3. Their properties are time and temperature dependent (viscoelastic).
4. They are nearly incompressible.

The near incompressibility of elastomers is manifest by a Poisson's ratio of approaching 0.5, which in conventional constitutive formulations results in a near division by zero occurrence. Hence in addition to finite strain measures, the numerical modelling of elastomers requires entirely unique and non-linear constitutive formulations. This thesis therefore seeks to investigate the numerical modelling of elastomers and overcome any mathematical complications that prevent the further application of the emerging BEM with this material.

1.2 Thesis objectives

The following list summarizes the broad aims of the research carried out.

1. To carry out a literature review of basic and current theory in the field.
2. Using current state of the art, a working finite element program for the stress analysis of elastomers will be developed. This program will then duplicate any test case results obtained by newer approaches, and in so doing will provide validation by a well understood and independent numerical technique.
3. It will be seen that the derivation of the non-linear BIE equations for use with elastomers will generate mathematical singularities. These mathematical singularities will require some form of regularization to make them tractable if any working BEM code is to be developed.
4. The main task of this research, tying in all other areas of work, will be the development of a working BEM program implemented in FORTRAN 77 code. This will require the creation of a defined mesh and boundary condition data structure and associated mesh generator program, also implemented in FORTRAN 77.

5. In order to validate the BEM program, in addition to duplicating the FEM program results, some test cases will have to be investigated. Hence, for each test case an analytical or equivalent direct numerical solution will have to be derived in order to generate some results data for comparison purposes.
6. In writing the thesis, the aim will be to present a unified and stand alone account of the research carried out. Much of the literature in this field is highly mathematical, particularly the regularization singularity removal material, hence a more readable and thorough account of the research would be of more use to practicing engineers.

1.3 Thesis structure

The second chapter of the thesis is the literature review. This chapter is broken down into four sections. The first section covers the various material models developed to characterise the non-linear constitutive relationship between stress and strain for elastomers. The second and third sections relate to the two numerical techniques implemented in this research. These being the finite and boundary element methods. It will be seen that the amount of literature on the application of the boundary element method to the numerical analysis of elastomers is fairly limited. The final section on boundary integral equation singularities is the most extensive of the literature review and most relevant to the new work developed in this thesis. The numerical singularities found in this application are typical of the type of singularities encountered in current boundary element method research in many applications. Hence there are many publications on this subject.

The numerical analysis of elastomers is inherently a non-linear process. This is because of the dual sources of non-linearity found in elastomers. Namely the non-linear geometric and material properties of elastomers. The next two chapters of the thesis therefore cover the required theory needed to start working on the application of numerical techniques to the stress analysis of elastomers. The first of the two chapters covers finite strain kinematics. Due to the large elastic deformations elastomers are capable of sustaining the normal mathematical infinitesimal assumption of the problem geometry remaining substantially unaltered before and after loading is no longer valid. Therefore new stress and strain measures are required which reference either deformed or undeformed coordinates and capture the non-linear geometric behaviour. The second of these two chapters covers the non-linear constitutive stress strain modelling of rubber. From first principals the Mooney-Rivlin material model is derived and the process by which its coefficients are curve fitted from test data is discussed.

The next two chapters cover the finite and boundary element methods respectively. For both chapters initially a brief overview of the numerical technique is given followed by an in depth derivation of the respective methods for application to the stress analysis of elastomers. For the finite element method some discussion of the solution algorithm is given, though not in such a detailed way as for the boundary element method. For the boundary element method the basic iterative and then the incremental modification to the algorithm is discussed along with numerical techniques specific to the BEM. It is in this chapter that the original thesis contribution is given for the simplified strong and hyper-singular regularization approach together with the isoparametric interpolation of the domain cell stresses.

The important task of validating the developed numerical code for the two techniques requires reliable analytical solutions. In the literature there are few useful discussions or derivations of analytical solutions for the four test cases run in this research. Therefore new semi-analytical or direct numerical solutions and algorithms are derived for validation purposes. The four test cases discussed are the uniaxial patch test, pressurized thick walled cylinder, centrifugal domain loading of a rotating disk and the J-Integral evaluation of a centrally cracked plate. The second two of these three test cases provide non-uniform stress fields for a more demanding test of the developed theory and code. These test case derivations represent additional original work, including the fracture mechanics test case, where the use of the strain energy model to evaluate the total strain energy of the cracked elastomer is original. In addition the agreement of the BEM and FEM as two independent numerical stress analysis techniques also provides another level of validation. The final two chapters present the results and conclusions drawn from the research.

Included in the thesis are several appendices which whilst not discussing new work nevertheless cover essential material that the author has found it necessary to either use directly or understand in order to proceed with the research. It is up to the reader whether they wish to read these appendices which may cover material present in standard texts. However, one feature of the literature on this subject is the lack of a unified source and approach for much of the material from first principles to state of the art. An attempt has been made in this thesis to present an as unified and stand alone source as possible, whilst still enabling the reader to go directly to the new material should they wish.

Chapter 2

Literature review

2.1 Material models

2.1.1 Introduction

It can be shown that the non-linear relationship between stress and strain can be obtained from the partial derivative of the internal strain energy with respect to strain. In addition the tangential modulus between stress and strain can be obtained as the second derivative of the internal strain energy with respect to strain [17]. These two relationships are fundamental in the implementation of the finite and boundary element methods for hyperelastic materials. Therefore the derivation of a strain energy function in terms of deformation that obeys the key assumption of isotropic incompressibility is essential.

Historically, two approaches have been developed in the derivations of strain energy functions. The first being the statistical thermodynamic approach, where the microscopic molecular structure of the material is taken into account (see references [53, 45]). The second approach, and the one on which most work has been done, is the phenomenological approach where the material is treated as a continuum. Most of the early work on strain energy formulations is based on the second approach, with the deformation expressed in terms of the strain invariants. A key early example of such work includes work done by Rivlin (see references [42, 41]), which expresses the strain energy as a polynomial expansion in terms of the strain invariants. A useful and reduced form of the Rivlin polynomial was proposed by Mooney [32], in which a two term polynomial expansion assuming absolute incompressibility. The other form of strain energy function based on the phenomenological approach expresses the deformation in terms of the principal extension ratios. These models

have the simple advantage that published data tends to be presented using stretch ratios. Important examples of such models include Valanis-Landel [55] and Ogden [35] material models.

2.1.2 Review of recent work

It can be seen from the literature that there are many material models, all of which share certain features. One of these being the requirement to calibrate the material constants from test data. This is alluded to by Boyce [3] who points out that although the Neo-Hookean and Mooney-Rivlin material models only require one and two calibrated constants respectively, their ability to represent accurately even modestly large strains is poor. Better models exist but require the evaluation of more material constants. Boyce therefore investigates the Gent and Altuda-Boyce material models concluding that despite these two models only requiring two calibrated constants they nevertheless successfully model three dimensional finite strain behaviour.

Another common feature of the various material models is that their forms are either functions of the strain invariants or the principal extension ratios. Work by Davies et.al. [11], as well as Yeoh [56] indicates that for strain invariant based models the strain energy derivative with respect to the second strain invariant is negligible in comparison to the strain energy derivative with respect to the first strain invariant. Consequently, the second strain invariant is ignored in some material models. Yeoh demonstrates that such a model, of third order polynomial form, successfully predicts all the principal stress/strain features in different modes of deformation (uniaxial tension, simple shear and uniaxial compression).

Obviously with so many forms of material model the question arises as to which model is the most accurate and efficient? Charlton et.al. [8] states that for larger and more complex strains, higher order terms in the Rivlin polynomial need to be included. Ogden [35] also investigated the accuracy of three different near incompressible material models, these being the Mooney-Rivlin, Ogden and Valanis-Landel models, using Treloar uniaxial test data for constant calibration purposes. From the three models, the Valanis-Landel formulation provides the best correspondence between the theoretical and the experimental test data, for a variety of deformation modes.

Work by Charton et.al. [8] discusses the important area of test data generation. It is stated that using methods proposed by Turner et.al [54], that multi-axial curve fitting data can be generated from a limited number of simpler strictly performed tests. With this or any other test data quality checks are a necessary part of the validation process. Charton et.al. discuss test data quality checks, specifically the

Drucker stability criterion as employed by the commercially available ABAQUS software. Methods of coping with instability once detected can include adding or subtracting data points according to perceived data trends, data biasing or lower order curve fits. There are various sources of error and convergence instability that can occur in finite element models utilizing finite strain energy models. These include truncation errors, iterative build up errors and inaccuracies inherent in the model discretization. To identify non-material model sources of error and instability, use can be made of the condition number test. The appropriate use of restraints can be used to lower this value where necessary.

2.1.3 Conclusions

It can be concluded from the literature review for material models that there are numerous forms of strain energy function to choose from. However essentially there are five types of material model, which can be broken down as follows. Statistically based thermodynamic models and phenomenological models, the latter are either strain invariant or principal extension ratio based, and assume incompressibility or near incompressibility. If near incompressibility is assumed then a deviatoric/volumetric split is implemented in the strain energy function. Adequate validation is necessary to ensure that the chosen material model once calibrated from test data in one strain mode can successfully be applied to applications in other strain modes. Care must be taken to ensure that the strain range used does not exceed the strain accuracy capability of the model. Therefore, it is essential that real test data is obtained for all the relevant modes and ranges of deformation.

It can also be concluded from the literature review on material models that even though the correct material model with accurately calibrated constants is used, it is still possible for model instability to occur due to other factors. Therefore, there are other considerations which will effect model convergence and accuracy in addition to the choice of material model, such as mesh discretization, rounding and iteration errors.

2.2 Finite element modelling of elastomeric materials

2.2.1 Introduction

The requirement to use structurally optimised elastomeric materials in an increasing range of applications has led to the development of finite element formulations for elastomers over the past fifty or so years. For elastomers, the main difficulty with the conventional finite element formulation using only the displacement field parameter is the determination of the hydrostatic pressure related to the volumetric part of the strain [57].

An early example of a finite element formulation for modelling finite strains in elastomers was given by Lindley [29]. His work describes the use of triangular elements to discretize a rubber sheet enabling the total strain energy of the sheet to be evaluated by the finite element method. Subsequently an iterative procedure was employed to move all the model nodes so as to minimise the strain energy of the sheet. Initial displacement estimates for the iterative procedure were obtained from small strain linear elasticity. The boundary conditions were applied to the model by means of prescribed displacements.

More recent developments have resulted in three different approaches that have been evolved to deal with the hydrostatic pressure related to the volumetric part of the strain. The displacement method with reduced integration [30], the penalty type formulation [34] and the mixed displacement, pressure field parameter method [46]. The most popular and developed of these methods is the mixed field parameter approach which allows for full and near incompressibility. However the interpolations used are limited by instability in the mixed patch test. This requires a stabilizing procedure, which usually involves the use of interpolations of a lower order for the pressure compared to the displacement field parameter [57].

2.2.2 Review of recent work

From the literature review, the basic mixed field parameter finite element formulation is discussed by Canga et.al. [7] and Basar [2] and consists of forming a set of non-linear simultaneous equations. These equations can be assembled in the usual finite element manner into a form compatible with the Newton-Raphson iterative solution procedure.

For each iteration the tangent stiffness element equations are evaluated from the relevant constitutive model. For example, Chen et.al. [9] describe the tangent

stiffness moduli matrix formulation based on the Rivlin polynomial material model. Holzapfel [26] and, Basar et.al [2] utilise the Ogden material model to form the constitutive relationship. With the latter work describing how the Ogden model can be reformulated from a function in terms of principal stretches to one in terms of the strain invariants. Work by Kaliske and Rothert [27] suggests that generally constitutive material models in terms of strain invariants allow for simpler and more efficient formulations. By way of demonstration Kaliske and Rothert perform finite element analyses based on constitutive models utilizing the Neo-Hookian, Mooney-Rivlin, Swanson [51], Yeoh [56], Arruda-Boyce [3] and statistically derived Kilian material models. With regard to constitutive equations based on the Ogden model Basar and Itskov state that the material model calibration is more complex as the calibration process itself requires an iterative non-linear solution procedure.

An alternative element equation formulation is proposed by Souza Neto, Peric, Dutko and Owen [33]. They point out that the element equations for low order elements can lead to more efficient and less CPU intensive finite element solutions. However, low order elements are very poor at dealing with incompressible materials. Their work therefore, is based on formulating low order element equations utilizing a multiplicative deviatoric/volumetric split in strain energy. In addition a modified deformation gradient is used. In fact most of the material models used in constitutive modelling utilise this deviatoric/volumetric strain split in order to model near incompressibility. For example, in a related but slightly different application, Holzapfel [26] uses a similar split in material response into volumetric and viscoelastic parts for slow rate viscoelastic finite strain elastomers.

As stated before, the non-linear equations formed after the element equation assembly are solved by means of the iterative Newton-Raphson procedure. Work by Canga and Becker [7] states that most finite element codes using the Newton-Raphson scheme utilise a Gauss solver for each solution of the current iteration element equation set. It is pointed out that for most practically sized models this can be a very CPU intensive process, though robust and reliable. They propose a variant of the conjugate gradient method (Newton-Raphson scheme) that is more efficient.

2.2.3 Conclusions

The mixed method of finite element modelling of elastomers is the most developed formulation and will be used as the basis for research purposes in this thesis. Essentially, approximate solutions for the displacement and pressure field parameter trial solutions are obtained from the solutions of the two corresponding sets of non-linear simultaneous equations via the Newton-Raphson iterative procedure. The

non-linear element equations for the displacement and pressure field parameters are formulated using the minimised variational approach. Thus generating the tangent moduli relating the 'out of balance' forces and pressures to the force and pressure increments [10]. Clearly the formulations involve the non-linear tangent stiffness terms and constitutive equations which utilise the strain energy material model types previously discussed in the literature review.

2.3 Boundary element modelling of elastomeric materials

2.3.1 Introduction

The BEM is a relatively new technique and although successfully applied to many engineering problems has had limited application with finite elasticity and elastomers. Early work in the field includes a paper by Phan-Tien [37]. This paper covers the application of the boundary element method to finite elasticity through the addition of non-linear domain terms to the standard boundary integral equation. The domain terms require the evaluation of deformation gradients inside the integration cells.

2.3.2 Review of recent work

The BEM is a relatively new procedure and until recently has had few applications to finite elasticity of incompressible materials. However, a recent example of the BEM in a related area is given by Bu [5] who used the BEM to analyse seismic loading of foundations in undrained saturated clays. The clay is considered as an incompressible material.

It is well documented that the BEM has the merit of reducing the dimensionality of a problem by a single order of magnitude with the added advantage of increased solution accuracy over other numerical techniques such as the finite element method. It is pointed out by Polyzos et.al. [39] however, that these advantages are compromised by the additional requirement in the case of incompressible hyperelastic materials (and other non-linear type analyses) to evaluate non-linear domain integrals.

Two recent papers by Al-Gahtari and Altero [18], and Polizzotto [38] give the most up-to-date and relevant work on the application of the BEM to stress analysis of hyperelastic and incompressible materials. Polizzotto discusses the Galerkin boundary element formulation with regard to elastomers. The generation of the non-linear domain integrals is mentioned with the additional problem of singular integrals. These are generated by the direct derivative boundary integral equations in the evaluation of the displacement gradients.

Al-Gahtari and Altero expand on this subject giving some detail on the evaluation of singular and hypersingular integrals by means of integration by parts and the use of the divergence theorem. Al-Gahtari and Altero also give details of an iterative

solution procedure assuming a constant 1st PK stress distribution within the non-linear domain terms present in each boundary integral equation.

2.3.3 Conclusions

From the limited amount of literature specific to the application of the boundary element method to the analysis of elastomers the following can be concluded. The non-linear system of boundary integral equations is generated by the addition of non-linear domain terms. These domain terms involve evaluating deformation gradients internally to the domain cells by direct derivative boundary integral equations. It will be required to regularize strong singular and hypersingular integrals generated in the derivative boundary integral equations. In the boundary integral method implemented for elastomers there will be two options for the evaluation of the domain integrals. The non-linear stress that is required to be calculated internally for each domain cell can either be assumed to be interpolated by means of 1st or 2nd order functions, or assumed constant throughout the domain cell. For the purposes of this research the non-linear stress (1st PK stress) will use 2nd order interpolation within the domain cell.

2.4 BIE singularities

2.4.1 Introduction

It has already been explained that in the evaluation of BIEs singular integrals arise. In general, from a review of the relevant papers it can be stated that because the boundary element method uses single field solutions of the governing equations some mathematical singularities arise independently of the physical problem. In the case of hyperelastic materials hypersingular integrals are generated due to the use of DBIE. In addition strong, weakly and nearly singular integrals can affect the accuracy and convergence of a non-linear boundary element problem.

In summary, it can be stated that the boundary element method accuracy is largely dependent on the accuracy with which various integrals are evaluated. Therefore the proper consideration and evaluation of singular integrals is among the most frequently discussed topics in boundary element research. The process by which the various singularities are removed so that it is possible to evaluate the integral is known as regularization.

2.4.2 Review of recent work

A important starting point to this research is to appreciate the various singularities that can occur in the evaluation of BIEs. A useful discussion of the various singular integrals is given by Martin et.al. [31], where the CPV and Hadamard Finite part integrals are defined. It is explained that these integrals are also referred to as strong and hypersingular integrals respectively. The primary purpose of the paper however, is to discuss the continuity requirements of the density function within the integral expression such that the integral exists (has a finite value). It is stated in the paper that the integral will exist if the field function (i.e. the displacements and tractions in the case of stress analysis) should have a Holder-continuous first derivative. Note that Sladek et.al. [52] also give the definition of strongly singular integrals as follows. They state that owing to the singular behaviour of the integrand a small region in the neighbourhood of the singular point must be excluded from the integration domain with the limit taken as the volume of this region tending to zero. If this limit exists and is independent of the shape of the exclusion zone, then the integral is known as weakly singular. If the limit exists only when the exclusion zone is symmetric around the singular point, the singularity is strong and the integral exists in the CPV sense. A singularity higher than CPV is called hypersingular.

In addition to strong and hypersingular integrals a paper by Sladek et.al. [49] discusses weak logarithmic singularities and nearly strongly and hypersingular integrals

arising when the field point is close to but not in the element of integration. It is mentioned that the weak logarithmic singularity can be treated numerically by special Gauss quadrature [50]. It is also mentioned that nearly singular and hypersingular integrals are not regularized as would be the case with strong and hypersingular integrals. Rather these integrals are tackled by one of two approaches. The first being the use of p- and or h- strategy discretization to enhance the computational effort in combination with error analysis [25]. The second approach, and the one discussed in their paper, is to improve the behaviour of the integrands by appropriate transformations of the integration variables. Using integration by parts in combination with polynomial interpolations, the paper shows that the evaluation of both weakly logarithmically and nearly strongly singular integrals can yield extremely high accuracy of integration without increasing computational effort. Papers also discussing this second approach to nearly strongly singular integral evaluation are also given by Sladek et.al in references [48] and [47] for the cases of 2D and 3D BEM formulations respectively.

Another paper dealing with near singular integrals by the second approach is given by Granados et.al [19] where the Taylor series expansion of the singular kernel function (discussed later in this section) is generalised by allowing the field and source point coordinates to attain complex values. It is stated in this paper that near singular integrals have the property that when transformed into the complex plane then the complex part is non-zero but small. Hence, the integral exists but cannot be evaluated by standard numerical techniques. These integrals are evaluated by synthetic division of the numerator by the denominator in a procedure known as regularization in the complex plane. The result is two integrals, the first integral is a polynomial and can be evaluated using standard quadrature. The second integral is singular though simple, and hence can be evaluated analytically (much like the CPV term). For BEM elasticity problems the singular kernels are put in terms of a numerator and denominator by expressing the radius vector (distance between the source and field point) in the complex plane.

Having discussed the types of singularity and the special cases of logarithmic and nearly singular integrals, it remains to discuss the various regularization strategies. Tanaks et.al. [52] define regularization in the following way. A regularization procedure is required to cancel the divergent terms and identify that integrals exist in the ordinary sense (the CPV or Hadamard finite parts are finite and bounded). This work also includes a means of strong singularity regularization that has been used in the research given in this thesis. Specifically, Bui [6] discusses the use of rigid body displacement to evaluate the free term coefficients in the BIE (the free terms are the CPV integrals, see equation 6.35). This work led to Rizzo et.al. [43] using the rigid body motion solution to obtain BIEs free of CPV integrals (free terms), an idea also used in the research reported in this thesis.

It will be seen that the stresses, and therefore the field gradient BIEs are required. In the finite element method these gradients would be obtained by means of shape function derivative interpolation. However, Gray et.al [20] points out that the use of C^0 continuous elements gives rise to discontinuous results at element junctions, and hence degrades the accuracy of the field function derivatives. If a derivative BIE is used, then greater accuracy is achieved from the direct boundary solution. However, the derivative BIEs give rise to domain integral strong and hypersingular integrals. An important series of papers by Guiggiani et.al. discusses these non-linear domain terms. Indeed Guiggiani [22] states that in order to solve non-linear problems, the BIEs have to be augmented with domain terms as functions of stress and strain. In an extension of his earlier work Guiggiani extends the same principles from strongly singular CPV integrals to hypersingular domain integrals [23]. Here he further states that BIEs with hypersingular kernels arise when the gradient (normal derivative) of a classical BIE equation is taken. He further states that these equations involve the derivatives of already strongly singular kernels (which is most definitely the case with hyperelastic materials).

Guiggiani suggests many researchers in the field avoid direct computation of the CPV and hypersingular integrals by pure analytical integration or by using known elementary solutions of the BIE such as rigid body motion (mentioned earlier). Work by Rizzo and Shippy [44] combined integral equations of two different problems, namely the original problem and a 'companion' problem so obtaining a new BIE free of CPV integrals. However Guiggiani suggests this and similar approaches considerably increases the analytical complexity of the BEM formulations as two different sets of kernel functions are required.

The initial work on the direct evaluation of CPV integrals was reported by Guiggiani and Castalini [24], where general contour CPV integrals are reduced to regular ones prior to the use of standard Gaussian quadrature for their evaluation. In [23] some discussion of the limiting process is given for the CPV definition, this being essential for the direct CPV integral evaluation approach. Key to all the work done by Guiggiani et.al. is the conversion of the singular kernel function into polar intrinsic coordinates by means of a Taylor series expansion of the field point relative to the source point. With a first term Taylor series expansion being used for CPV integrals [22] and a two term Taylor series expansion for hypersingular integrals [44]. Thus, the domain integral containing the singularity is split into two terms by means of subtraction and addition of the singular integral, with the first term being cancelled out by subtraction when singular and the second integral term being made regular by the conversion to intrinsic polar coordinates. As a further bonus, not only has the direct approach been based on rigorous mathematics but has been found to be totally insensitive to the mesh pattern around the singularity.

2.4.3 Conclusions

It can be stated that there are four classes of singularity that arise in boundary element formulations. Namely strong, hypersingular, logarithmic and nearly singular integrals. Alternative names for strong and hypersingular singularities are the Cauchy principal value and Hadamard finite value integrals respectively. Cauchy principal value integrals can be defined by the exclusion of a small symmetric (about the singular point) region from the integration in the neighbourhood of the singularity with the limit taken as the volume of this region tending to zero.

The process by which the strong and hypersingular integrals are made finite and bounded is known as regularization. For the cases of logarithmic and nearly strong and hypersingular integrals a regularization process is not used but one of two differing approaches is employed to evaluate the integral terms. The first approach is a mesh based p- and h- discretization scheme. The second uses integration by parts followed by polynomial interpolation.

One approach to the regularization of CPV integrals (strong singularities) is the use of rigid body displacement to help evaluate the CPV integral terms (free terms). A general approach is given in the literature for the evaluation of both strong and hypersingular integrals generated in domain terms commonly found in non-linear boundary element formulations. The method consists of subtracting and then adding the singular part of the integral. Thus generating two integral terms, the first of which is cancelled by subtraction when singular, and the second rendered calculable by standard Gauss quadrature by means of a Taylor series expansion of the field point in terms of polar coordinates.

It was found, in general that much of the literature in the field of regularization was very theoretical and highly mathematical and generally beyond most practicing engineers. Therefore a simplified approach to this problem similar to that given in this thesis is a very useful contribution to the advancement of the boundary element method.

Chapter 3

Finite deformation kinematics

3.1 Overview

Finite strain, or large deformation continuum mechanics has the additional complication over infinitesimal mechanics that the reference co-ordinates of the governing geometry change after deformation. This is not the case with small strain behaviour where the key assumption is that the deformations are so small that the reference co-ordinates do not change during deformation.

This automatically complicates the continuum mechanics because not only does the deformation become non-linear but the question arises as to whether the stress strain measures used refer to the deformed or undeformed co-ordinates? However, just as with small strain behaviour the state of deformation can be neatly summarized by means of principal stretches/strains which can then be incorporated in the finite deformation stress strain measure functions.

The chapter begins by defining the deformation gradient and the displacement derivative tensors which in turn are used in the definitions of the Cauchy-Green left and right tensors. Then, based on the small strain assumption a finite strain measure is defined which provides the basis for the Green's strain and Almansi's strain measures.

By using stretch ratios it is shown that both the Green's strain and Almansi's strain measures have the same principal stretches (eigenvalues) but different vectors of directional cosines (eigenvectors). With the Green's strain direction cosines referring to undeformed co-ordinates and Almansi's strain direction cosines referring to deformed co-ordinates.

Mention is made of the polar decomposition theorem, which decomposes the defor-

mation gradient tensor into a set of stretches followed by a rigid body rotation, or vice versa. To do this the rotation matrix is defined in terms of a matrix consisting of the three principal stretch eigen vectors. Similarly the stretch matrix (prior or after deformation) is defined in terms of stretch ratio principal values and vectors.

Prior to defining the various stress measures the deformed and undeformed infinitesimal areas are related by means of the deformation gradient matrix. The first two stress measures defined are the engineering and the Cauchy or true stress. The former defines the stress vector in terms of the initial infinitesimal force and area vectors. Whilst the latter defines the stress vector in terms of post deformation infinitesimal force and area vectors.

Finally the finite deformation stress measures are defined. These stress measures give the total non-linear stress due to a given state of deformation. They are the 1st and 2nd PK stresses. The 1st PK stress defines the total stress vector in terms of the post deformation infinitesimal force vector and pre deformation area vector. The 2nd PK stress defines the total stress vector in terms of the infinitesimal force vector and area vector prior to deformation. In practice the Cauchy stress and the 2nd PK stress are equivalent, but the 2nd PK stress allows the non-linear geometrical relationship between stress and deformation to be expressed in terms of the deformation gradient matrix. A useful expression is also given for the 1st stress in terms of the 2nd PK stress.

3.2 Deformation matrices

Consider an infinitesimal length before deformation, defined by the following Cartesian vector:

$$\Delta \vec{S}_0 = \Delta x_0 \hat{i} + \Delta y_0 \hat{j} + \Delta z_0 \hat{k} \quad (3.1)$$

and after deformation the infinitesimal length becomes:

$$\Delta \vec{S} = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k} \quad (3.2)$$

Therefore a deformation vector can be defined such that:

$$\begin{aligned} \Delta \vec{q} &= \Delta \vec{S} - \Delta \vec{S}_0 \\ \Delta \vec{q} &\equiv \Delta u \hat{i} + \Delta v \hat{j} + \Delta w \hat{k} \end{aligned} \quad (3.3)$$

The previous equations can be put in matrix notation as follows:

$$\Delta \underline{S}_0 = \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \\ \Delta z_0 \end{bmatrix} \quad \Delta \underline{S} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \quad \Delta \underline{q} = \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} \quad (3.4)$$

$$\Delta \underline{S} = \Delta \underline{S}_0 + \Delta \underline{q} \quad (3.5)$$

Using the chain rule of partial differentiation and differential lengths:

$$\begin{aligned} dx &= \frac{\partial x}{\partial x_0} dx_0 + \frac{\partial x}{\partial y_0} dy_0 + \frac{\partial x}{\partial z_0} dz_0 \\ dy &= \frac{\partial y}{\partial x_0} dx_0 + \frac{\partial y}{\partial y_0} dy_0 + \frac{\partial y}{\partial z_0} dz_0 \\ dz &= \frac{\partial z}{\partial x_0} dx_0 + \frac{\partial z}{\partial y_0} dy_0 + \frac{\partial z}{\partial z_0} dz_0 \end{aligned} \quad (3.6)$$

which can be put in matrix form as:

$$d\underline{S} = \underline{G} d\underline{S}_0 \quad (3.7)$$

where:

$$\underline{G} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix} \quad (3.8)$$

where $\underline{G}$ is the deformation gradient matrix. From equations 3.5 and 3.6 it can be seen that:

$$\begin{aligned} dx &= \left(1 + \frac{\partial u}{\partial x_0}\right) dx_0 + \frac{\partial u}{\partial y_0} dy_0 + \frac{\partial u}{\partial z_0} dz_0 \\ dy &= \frac{\partial v}{\partial x_0} dx_0 + \left(1 + \frac{\partial v}{\partial y_0}\right) dy_0 + \frac{\partial v}{\partial z_0} dz_0 \\ dz &= \frac{\partial w}{\partial x_0} dx_0 + \frac{\partial w}{\partial y_0} dy_0 + \left(1 + \frac{\partial w}{\partial z_0}\right) dz_0 \end{aligned} \quad (3.9)$$

$$d\underline{S} = (\underline{I} + \underline{g}) d\underline{S}_0 \quad (3.10)$$

with:

$$\underline{g} = \begin{bmatrix} \frac{\partial u}{\partial x_0} & \frac{\partial u}{\partial y_0} & \frac{\partial u}{\partial z_0} \\ \frac{\partial v}{\partial x_0} & \frac{\partial v}{\partial y_0} & \frac{\partial v}{\partial z_0} \\ \frac{\partial w}{\partial x_0} & \frac{\partial w}{\partial y_0} & \frac{\partial w}{\partial z_0} \end{bmatrix} \quad (3.11)$$

which is the displacement derivative matrix. Two tensors are defined from the deformation gradient, the first being the right Cauchy-Green tensor:

$$\begin{aligned} \underline{C} &= \underline{G}^t \underline{G} \\ \underline{C} &= (\underline{I} + \underline{g}^t) (\underline{g} + \underline{I}) \\ \underline{C} &= \underline{I} + (\underline{g} + \underline{g}^t) + \underline{g}^t \underline{g} \end{aligned} \quad (3.12)$$

And the second being the left Cauchy-Green tensor:

$$\begin{aligned} \underline{b} &= \underline{G} \underline{G}^t \\ \underline{b} &\equiv \underline{I} + (\underline{g} + \underline{g}^t) + \underline{g} \underline{g}^t \end{aligned} \quad (3.13)$$

3.3 Large strain representation

In the derivation of two of the large strain measures the following basic large strain measure is used. Defining the small strain as follows:

$$\varepsilon_s = \frac{dS - dS_0}{dS_0} \quad (3.14)$$

Then using a purely algebraic manipulation:

$$\begin{aligned} \varepsilon_s &= \frac{(dS - dS_0)^t (dS + dS_0)}{dS_0^t (dS + dS_0)} \\ \varepsilon_s &= \frac{\overline{dS}^2 - \overline{dS_0}^2}{\overline{dS_0}^2 (2 + \varepsilon_s)} \end{aligned} \quad (3.15)$$

and making a finite strain approximation equation 3.15 can be converted into a finite strain measure by assuming that $\varepsilon_s \ll 2$:

$$\varepsilon_f = \frac{\overline{dS}^2 - \overline{dS_0}^2}{2\overline{dS_0}^2} \quad (3.16)$$

This basic finite strain measure is then used in the definition of the Green's strain and the Almansi's finite strain measures.

3.3.1 True or logarithmic strain

In addition to the Green's and Almansi's finite strain measures there exists a finite strain measure which is the work conjugate of the true or Cauchy stress, which is known as the true or logarithmic strain. The logarithmic strain is defined as follows:

$$\varepsilon_l = \int_{l_0}^l \frac{dl}{l} = \ln \left(\frac{l}{l_0} \right) \quad (3.17)$$

The relationship between the logarithmic strain and the nominal strain can be derived using the following definition of the nominal strain:

$$\varepsilon_n = \frac{l - l_0}{l_0} \equiv \frac{l}{l_0} - 1 \quad (3.18)$$

Therefore the nominal and logarithmic strain can be related as follows:

$$\varepsilon_l = \ln(\varepsilon_n + 1) \quad (3.19)$$

This relationship is useful when inputting test data, which is often specified in nominal values, into commercial software such as ABAQUS which requires true stress and strain values. The true or Cauchy stress in terms of nominal values is given in section 3.8.1.

3.3.2 Green's strain measure

Expressing the numerator of equation 3.16 as follows:

$$\begin{aligned} d\underline{S}^t d\underline{S} - d\underline{S}_0^t d\underline{S}_0 &= d\underline{S}_0^t \underline{G}^t \underline{G} d\underline{S}_0 - d\underline{S}_0^t \underline{I} d\underline{S}_0 \\ d\underline{S}^t d\underline{S} - d\underline{S}_0^t d\underline{S}_0 &= d\underline{S}_0^t [\underline{G}^t \underline{G} - \underline{I}] d\underline{S}_0 \end{aligned} \quad (3.20)$$

From comparison of equations 3.16 and 3.20 the definition of Green's strain can then be written as:

$$\begin{aligned} \underline{\varepsilon}_G &= \frac{1}{2} (\underline{G}^t \underline{G} - \underline{I}) \\ \underline{\varepsilon}_G &= \frac{1}{2} (\underline{C} - \underline{I}) \end{aligned} \quad (3.21)$$

It can be seen that Green's strain relates to the undeformed co-ordinates.

3.3.3 Almansi's strain measure

Noticing that:

$$d\underline{S}_0 = \underline{G}^{-1} d\underline{S} \quad (3.22)$$

Then expressing the numerator in equation 3.16 in terms of deformed co-ordinates:

$$d\underline{S}^t d\underline{S} - d\underline{S}_0^t d\underline{S}_0 = d\underline{S}^t \underline{I} d\underline{S} - d\underline{S}^t (\underline{G}^{-1})^t \underline{G}^{-1} d\underline{S} \quad (3.23)$$

Noticing that:

$$(\underline{G} \underline{G}^t)^{-1} \equiv (\underline{G}^{-1})^t \underline{G}^{-1} \quad (3.24)$$

Therefore:

$$d\underline{S}^t d\underline{S} - d\underline{S}_0^t d\underline{S}_0 = d\underline{S}^t \left[\underline{I} - (\underline{G} \underline{G}^t)^{-1} \right] d\underline{S} \quad (3.25)$$

From comparison of equations 3.16 and 3.25 the definition of Almansi's strain can then be written as:

$$\begin{aligned} \underline{\varepsilon}_a &= \frac{1}{2} \left[\underline{I} - (\underline{G} \underline{G}^t)^{-1} \right] \\ \underline{\varepsilon}_a &= \frac{1}{2} [\underline{I} - \underline{b}^{-1}] \end{aligned} \quad (3.26)$$

Therefore it can be seen that Almansi's strain relates to the deformed co-ordinates. It is possible to define a modified deformation gradient matrix in terms of a displacement derivative matrix referenced to deformed co-ordinates. Therefore defining:

$$\underline{\overline{G}} \equiv \underline{G}^{-1} \quad (3.27)$$

and:

$$\underline{\overline{G}} \equiv \underline{I} + \underline{\overline{g}}$$

where:

$$\underline{\overline{g}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \quad (3.28)$$

Therefore:

$$\underline{\varepsilon}_a = \frac{1}{2} \left[\underline{I} - \underline{\overline{G}}^t \underline{\overline{G}} \right] = \frac{1}{2} \left[\underline{\overline{g}}^t + \underline{\overline{g}} + \underline{\overline{g}}^t \underline{\overline{g}} \right] \quad (3.29)$$

3.4 Stretch Ratios

Consider an infinitesimal length vector before deformation.

$$d\vec{S}_0 = dx_0 \hat{i} + dy_0 \hat{j} + dz_0 \hat{k} \quad (3.30)$$

A unit vector parallel to the undeformed vector can also be defined such that:

$$\hat{S}_0 = \frac{d\vec{S}_0}{dS_0} = l_s^0 \hat{i} + m_s^0 \hat{j} + n_s^0 \hat{k} \quad (3.31)$$

where:

$$dS_0 = \sqrt{d\bar{x}_0^2 + d\bar{y}_0^2 + d\bar{z}_0^2} \quad (3.32)$$

and:

$$l_s^0 = \frac{dx_0}{dS_0} \quad m_s^0 = \frac{dy_0}{dS_0} \quad n_s^0 = \frac{dz_0}{dS_0} \quad (3.33)$$

Hence:

$$d\vec{S}_0 = \hat{S}_0 dS_0 \quad (3.34)$$

which can be written in matrix form as follows:

$$d\underline{S}_0 = dS_0 \underline{n}_s^0 \quad (3.35)$$

where:

$$\underline{n}_s^0 = \begin{bmatrix} l_s^0 \\ m_s^0 \\ n_s^0 \end{bmatrix} \quad (3.36)$$

Similarly for an infinitesimal length vector after deformation.

$$d\vec{S} = \hat{S} dS \equiv dS \left(l_s \hat{i} + m_s \hat{j} + n_s \hat{k} \right) \quad (3.37)$$

where:

$$dS = \sqrt{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2} \quad (3.38)$$

$$l_s = \frac{dx}{dS} \quad m_s = \frac{dy}{dS} \quad n_s = \frac{dz}{dS} \quad (3.39)$$

which can be written in matrix form as follows:

$$d\underline{S} = dS \underline{n}_s \quad (3.40)$$

where:

$$\underline{n}_s = \begin{bmatrix} l_s \\ m_s \\ n_s \end{bmatrix} \quad (3.41)$$

The stretch ratio measure is as follows:

$$\lambda_s = \frac{dS}{dS_0} \quad (3.42)$$

$$\lambda_s^2 = \frac{d\underline{S}^t d\underline{S}}{d\underline{S}_0^t d\underline{S}_0} = 1 + \frac{d\underline{S}^t d\underline{S} - d\underline{S}_0^t d\underline{S}_0}{d\underline{S}_0^t d\underline{S}_0} \quad (3.43)$$

But from equations 3.20 and 3.21 it can be deduced that:

$$\lambda_s^2 = 1 + \frac{2d\underline{S}_0^t \underline{\varepsilon}_G d\underline{S}_0}{d\underline{S}_0^t d\underline{S}_0} \quad (3.44)$$

since:

$$\underline{n}_s^0 = \frac{d\underline{S}_0}{(d\underline{S}_0^t d\underline{S}_0)^{1/2}} \quad (3.45)$$

therefore:

$$\lambda_s^2 = 1 + 2\underline{n}_s^0 \underline{\varepsilon}_G \underline{n}_s^0 \quad (3.46)$$

Using equation 3.16 the component of strain in the λ_s direction can be defined directly as follows:

$$\varepsilon_s = \frac{d\underline{S}^2 - d\underline{S}_0^2}{2d\underline{S}_0^t d\underline{S}_0} \equiv \frac{1}{2} (\lambda_s^2 - 1) \quad (3.47)$$

therefore:

$$\lambda_s^2 = 1 + 2\varepsilon_s \quad (3.48)$$

Noticing that with $d\underline{S} = \underline{G}d\underline{S}_0$:

$$\lambda_s^2 = \frac{d\underline{S}_0^t \underline{G}^t \underline{G} d\underline{S}_0}{d\underline{S}_0^2} \equiv (\underline{n}_s^0)^t \underline{G}^t \underline{G} \underline{n}_s^0 \quad (3.49)$$

Defining $\frac{1}{\lambda_s^2} = \frac{d\underline{S}_0^2}{d\underline{S}^2}$ and using $d\underline{S}_0 = \underline{G}^{-1}d\underline{S}$ then it can be deduced that:

$$\frac{1}{\lambda_s^2} = \frac{d\underline{S}^t (\underline{G}^{-1})^t \underline{G}^{-1} d\underline{S}}{d\underline{S}^2} = \underline{n}_s^t (\underline{G}\underline{G}^t) \underline{n}_s \quad (3.50)$$

hence:

$$\lambda_s^2 = (\underline{n}_s^0)^t \underline{C} \underline{n}_s^0 \quad (3.51)$$

and:

$$\frac{1}{\lambda_s^2} = \underline{n}_s^t \underline{b}^{-1} \underline{n}_s \quad (3.52)$$

3.5 Principal stretches

The eigenvalue problem for the matrix can be summarized as follows:

$$\underline{C} \underline{n}^0 = \alpha \underline{n}^0 \quad (3.53)$$

The problem therefore is to find $\alpha_1, \alpha_2, \alpha_3$ and the corresponding eigenvectors $\underline{n}_1^0, \underline{n}_2^0, \underline{n}_3^0$, then from the orthogonality of the eigenvectors:

$$(\underline{n}_i^0)^t \underline{C} \underline{n}_i^0 = \alpha (\underline{n}_i^0)^t \underline{n}_i^0 = \alpha \delta_{ij} \quad (3.54)$$

for $i = j$ then:

$$\alpha = (\underline{n}_i^0)^t \underline{C} \underline{n}_i^0 \quad (3.55)$$

Comparing equation 3.55 with equation 3.51 then it can be seen that $\alpha = \lambda_i^2$ and the principal stretches satisfy the following equation:

$$\begin{aligned} \underline{C} \underline{n}^0 &= \lambda^2 \underline{n}^0 \\ (\underline{C} - \lambda^2 \underline{I}) \underline{n}^0 &= 0 \end{aligned} \quad (3.56)$$

For the non-trivial solutions:

$$|\underline{C} - \lambda^2 \underline{I}| = 0 \quad (3.57)$$

leading to:

$$\lambda^6 - I_1 \lambda^4 + I_2 \lambda^2 - I_3 = 0 \quad (3.58)$$

where:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \equiv C_{11} + C_{22} + C_{33} \quad (3.59)$$

i.e.

$$I_1 = \text{Trace}(\underline{C})$$

and:

$$I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \quad (3.60)$$

$$\begin{aligned} I_2 &= \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} + \begin{vmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{vmatrix} + \begin{vmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{vmatrix} \\ I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{31} & C_{33} \end{vmatrix} \end{aligned} \quad (3.61)$$

which can be written as:

$$I_3 = |\underline{C}|$$

noticing that:

$$\underline{n} = \frac{d\underline{S}}{dS} = \underline{G} \frac{d\underline{S}_0}{dS} \quad (3.62)$$

with $dS = dS_0\lambda$ therefore:

$$\underline{n} = \frac{1}{\lambda} \underline{G} n^0 \quad (3.63)$$

and:

$$\underline{n}^0 = \lambda \underline{G}^{-1} \underline{n} \quad (3.64)$$

Substituting into equation 3.56 with $\underline{C} = \underline{G}^t \underline{G}$, therefore:

$$\begin{aligned} (\underline{G}^t \underline{G} - \lambda^2 \underline{I}) \lambda \underline{G}^{-1} \underline{n} &= 0 \\ \lambda \underline{G}^t \underline{n} - \lambda^3 \underline{G}^{-1} \underline{n} &= 0 \end{aligned} \quad (3.65)$$

Multiplying by $\frac{1}{\lambda} \underline{G}$ and assuming $\lambda \neq 0$ then:

$$(\underline{G} \underline{G}^t - \lambda^2 \underline{I}) \underline{n} = 0 \quad (3.66)$$

i.e.

$$(\underline{b} - \lambda^2 \underline{I}) \underline{n} = 0$$

Therefore the eigenvalues of $\underline{b}$ are the same as those of $\underline{C}$, namely $\lambda_1^2, \lambda_2^2, \lambda_3^2$, but the eigenvectors of $\underline{b}$ are $\underline{n}_1, \underline{n}_2, \underline{n}_3$. It can also be proved that:

$$\begin{aligned} I_1 &= \text{Trace}(\underline{C}) = \text{Trace}(\underline{b}) \\ I_2(\underline{C}) &= I_2(\underline{b}) \\ I_3 &= |\underline{C}| = |\underline{b}| \end{aligned} \quad (3.67)$$

3.5.1 Representing the right Green tensor in terms of eigenvalues

Using the definition of the eigenvalue problem given by equation 3.56 then a matrix of eigenvectors can be defined as follows:

$$\underline{Q}(\underline{n}^0) = [\underline{n}_1^0, \underline{n}_2^0, \underline{n}_3^0] \quad (3.68)$$

and:

$$\underline{Q}^t(\underline{n}^0) = \begin{bmatrix} (\underline{n}_1^0)^t \\ (\underline{n}_2^0)^t \\ (\underline{n}_3^0)^t \end{bmatrix} = \begin{bmatrix} l_1^0 & m_1^0 & n_1^0 \\ l_2^0 & m_2^0 & n_2^0 \\ l_3^0 & m_3^0 & n_3^0 \end{bmatrix} \quad (3.69)$$

with:

$$Diag(\lambda^2) = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad (3.70)$$

hence:

$$Diag(\lambda^2) \underline{Q}^t(\underline{n}^0) = \begin{bmatrix} \lambda_1^2 (\underline{n}_1^0)^t \\ \lambda_2^2 (\underline{n}_2^0)^t \\ \lambda_3^2 (\underline{n}_3^0)^t \end{bmatrix} \quad (3.71)$$

and:

$$\underline{Q}(\underline{n}^0) Diag(\lambda^2) \underline{Q}^t(\underline{n}^0) = \lambda_1^2 \underline{n}_1^0 (\underline{n}_1^0)^t + \lambda_2^2 \underline{n}_2^0 (\underline{n}_2^0)^t + \lambda_3^2 \underline{n}_3^0 (\underline{n}_3^0)^t \quad (3.72)$$

hence it can be deduced from 3.72 that:

$$\underline{C} = \underline{G}^t \underline{G} = \underline{Q}(\underline{n}^0) Diag(\lambda^2) \underline{Q}^t(\underline{n}^0) \quad (3.73)$$

3.5.2 Representing left Green tensor in terms of eigenvalues

From the eigenvalue problem given by equation 3.66 a matrix of eigenvectors can be defined as follows:

$$\underline{Q}(\underline{n}) = [\underline{n}_1, \underline{n}_2, \underline{n}_3] \quad (3.74)$$

$$\underline{Q}^t(\underline{n}) = \begin{bmatrix} \underline{n}_1^t \\ \underline{n}_2^t \\ \underline{n}_3^t \end{bmatrix} \quad (3.75)$$

And in a similar manner to section 3.5.1 it can be deduced that:

$$\underline{b} = \underline{G} \underline{G}^t = \underline{Q}(\underline{n}) Diag(\lambda^2) \underline{Q}^t(\underline{n}) \quad (3.76)$$

3.6 The polar decomposition theorem

This theorem is based on decomposing the deformation gradient $\underline{G}$ into a set of stretches followed by rigid rotation. To do this a stretch matrix $\underline{U}$ and rotation matrix $\underline{R}$ are defined. The rotation matrix $\underline{R}$ defines the movement from $(\underline{n}_1^0, \underline{n}_2^0, \underline{n}_3^0)$ to $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ and is defined as follows:

$$\underline{R} = \underline{Q}(\underline{n}) \underline{Q}^t(\underline{n}^0) \quad (3.77)$$

Given that $\underline{Q}(\underline{n}), \underline{Q}(\underline{n}^0)$ have the following properties:

$$\begin{aligned}\underline{Q}^{-1}(\underline{n}) &= \underline{Q}^t(\underline{n}) \\ \underline{Q}^{-1}(\underline{n}^0) &= \underline{Q}^t(\underline{n}^0)\end{aligned}\tag{3.78}$$

therefore:

$$\begin{aligned}\underline{R}^{-1} &= (\underline{Q}^t(\underline{n}^0))^{-1} \underline{Q}^{-1}(\underline{n}) \\ \underline{R}^t &= \underline{Q}(\underline{n}^0) \underline{Q}^t(\underline{n})\end{aligned}\tag{3.79}$$

i.e.

$$\underline{R}^{-1} = \underline{R}^t\tag{3.80}$$

3.6.1 Decomposition of deformation gradient matrix into a rotation followed by a stretch

The $\underline{U}$ matrix is defined such that $\underline{G} = \underline{R}\underline{U}$, hence:

$$\begin{aligned}\underline{G}^t \underline{G} &= \underline{U}^t \underline{R}^t \underline{R} \underline{U} \\ \underline{G}^t \underline{G} &= \underline{U}^t \underline{U} \equiv \underline{C}\end{aligned}\tag{3.81}$$

which leads to the following results:

$$\underline{U}^t \underline{U} = \underline{Q}(\underline{n}^0) \text{Diag}(\lambda^2) \underline{Q}^t(\underline{n}^0)$$

$\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigenvalues of $\underline{U}^t \underline{U}$ with eigenvectors $\underline{n}_1^0, \underline{n}_2^0, \underline{n}_3^0$.

$\lambda_1, \lambda_2, \lambda_3$ will be the eigenvalues of $\underline{U}$ thus:

$$(\underline{U} - \lambda \underline{I}) \underline{n}^0 = 0\tag{3.82}$$

hence:

$$\underline{U} = \underline{Q}(\underline{n}^0) \text{Diag}(\lambda) \underline{Q}^t(\underline{n}^0)\tag{3.83}$$

$\underline{U}$ is a positive definite matrix. Substituting from 3.83 and 3.77 into 3.81 gives:

$$\begin{aligned}\underline{G} &= \underline{R}\underline{U} = \underline{Q}(\underline{n}) \underline{Q}^t(\underline{n}^0) \underline{Q}(\underline{n}^0) \text{Diag}(\lambda) \underline{Q}^t(\underline{n}^0) \\ \underline{G} &= \underline{Q}(\underline{n}) \text{Diag}(\lambda) \underline{Q}^t(\underline{n}^0) \\ \underline{G} &= \lambda_1 \underline{n}_1 \underline{n}_1^{ot} + \lambda_2 \underline{n}_2 \underline{n}_2^{ot} + \lambda_3 \underline{n}_3 \underline{n}_3^{ot}\end{aligned}\tag{3.84}$$

3.6.2 Decomposition of deformation gradient matrix into a stretch followed by a rotation

Defining a positive symmetric matrix $\underline{V}$ such that:

$$\underline{G} = \underline{V}\underline{R} \quad (3.85)$$

hence:

$$\underline{G}\underline{G}^t = \underline{b} = \underline{V}\underline{R}\underline{R}^t\underline{V} \quad (3.86)$$

$$\underline{G}\underline{G}^t = \underline{V}\underline{V}^t \quad (3.87)$$

Where $\underline{V}\underline{V}^t$ has the same eigenvalues as $\underline{b}$, with:

$$\underline{V}\underline{V}^t = \underline{Q}(\underline{n}) \text{Diag}(\lambda^2) \underline{Q}^t(\underline{n}) \quad (3.88)$$

$$\underline{V} = \underline{Q}(\underline{n}) \text{Diag}(\lambda) \underline{Q}^t(\underline{n}) \quad (3.89)$$

$$\underline{G} = \underline{V}\underline{R} = \underline{Q}(\underline{n}) \text{Diag}(\lambda) \underline{Q}^t(\underline{n}^0) \quad (3.90)$$

3.6.3 Green's strain in terms of principal stretches

The following can be deduced from equations 3.21, 3.81 and 3.83:

$$\begin{aligned} \underline{\varepsilon}_G &= \frac{1}{2} [\underline{G}^t\underline{G} - \underline{I}] \\ \underline{\varepsilon}_G &= \frac{1}{2} [\underline{U}^t\underline{U} - \underline{Q}(\underline{n}^0) \underline{Q}^t(\underline{n}^0)] \\ \underline{\varepsilon}_G &= \frac{1}{2} [\underline{Q}(\underline{n}^0) \text{Diag}(\lambda^2) \underline{Q}^t(\underline{n}^0) - \underline{Q}(\underline{n}^0) \underline{I} \underline{Q}^t(\underline{n}^0)] \end{aligned} \quad (3.91)$$

hence:

$$\underline{\varepsilon}_G = \underline{Q}(\underline{n}^0) \text{Diag}\left(\frac{\lambda^2 - 1}{2}\right) \underline{Q}^t(\underline{n}^0) \quad (3.92)$$

This implies that ε_G has the same eigenvectors as $\underline{C}$, with:

$$(\underline{\varepsilon}_G)_i = \frac{\lambda_i^2 - 1}{2}, \quad i = 1, 2, 3 \quad (3.93)$$

3.6.4 Almansi's strain in terms of principal stretches

From equation 3.26 the expression for Almansi's strain can be written as follows:

$$\varepsilon_a = \frac{1}{2} \left[\underline{I} - (\underline{G}\underline{G}^t)^{-1} \right] \quad (3.94)$$

Noticing that from equation 3.88:

$$\underline{G}\underline{G}^t = \underline{Q}(\underline{n}) \text{Diag}(\lambda^2) \underline{Q}^t(\underline{n}) \quad (3.95)$$

therefore:

$$\begin{aligned} (\underline{G}\underline{G}^t)^{-1} &= (\underline{Q}^t(\underline{n}))^{-1} \text{Diag}^{-1}(\lambda^2) (\underline{Q}(\underline{n}))^{-1} \\ (\underline{G}\underline{G}^t)^{-1} &= \underline{Q}(\underline{n}) \text{Diag}\left(\frac{1}{\lambda^2}\right) \underline{Q}^t(\underline{n}) \end{aligned} \quad (3.96)$$

$$\begin{aligned} \varepsilon_a &= \frac{1}{2} \left[\underline{Q}(\underline{n}) \underline{I} \underline{Q}^t(\underline{n}) - \underline{Q}(\underline{n}) \text{Diag}\left(\frac{1}{\lambda^2}\right) \underline{Q}^t(\underline{n}) \right] \\ \varepsilon_a &= \underline{Q}(\underline{n}) \text{Diag}\left(\frac{\lambda^2 - 1}{2\lambda^2}\right) \underline{Q}^t(\underline{n}) \end{aligned} \quad (3.97)$$

Hence Almansi's strain has the same eigenvectors as $\underline{b}$, with the following principal values.

$$(\varepsilon_a)_i = \frac{\lambda_i^2 - 1}{2\lambda_i^2} \quad i = 1, 2, 3 \quad (3.98)$$

3.7 Deformed and infinitesimal areas

In the final section of this chapter various stress measures will be derived which reference either the deformed or un-deformed areas. It is essential to these derivations to be able to relate the deformed and un-deformed areas, which in turn rely on the following mathematical theorems.

Theorem I:

$$\begin{aligned}\frac{\partial \phi}{\partial \xi} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \eta}\end{aligned}\tag{3.99}$$

similarly:

$$\begin{aligned}\frac{\partial \varphi}{\partial \xi} &= \frac{\partial \varphi}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \varphi}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial \xi} \\ \frac{\partial \varphi}{\partial \eta} &= \frac{\partial \varphi}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \varphi}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial \eta}\end{aligned}\tag{3.100}$$

with:

$$\begin{aligned}|\underline{J}\left(\frac{\phi, \varphi}{\xi, \eta}\right)| &= \left| \begin{array}{cc} \frac{\partial \phi}{\partial \xi} & \frac{\partial \varphi}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} & \frac{\partial \varphi}{\partial \eta} \end{array} \right| \\ |\underline{J}\left(\frac{\phi, \varphi}{\xi, \eta}\right)| &= \frac{\partial \phi}{\partial \xi} \frac{\partial \varphi}{\partial \eta} - \frac{\partial \phi}{\partial \eta} \frac{\partial \varphi}{\partial \xi}\end{aligned}\tag{3.101}$$

Substituting from 3.99 and 3.100 into 3.101 and rearranging it can be proved that:

$$\begin{aligned}|\underline{J}\left(\frac{\phi, \varphi}{\xi, \eta}\right)| &= |\underline{J}\left(\frac{\phi, \varphi}{x, y}\right)| |\underline{J}\left(\frac{x, y}{\xi, \eta}\right)| + |\underline{J}\left(\frac{\phi, \varphi}{y, z}\right)| |\underline{J}\left(\frac{y, z}{\xi, \eta}\right)| + \\ &\quad |\underline{J}\left(\frac{\phi, \varphi}{z, x}\right)| |\underline{J}\left(\frac{z, x}{\xi, \eta}\right)|\end{aligned}\tag{3.102}$$

Hence the following equations can be deduced:

$$\begin{aligned}|\underline{J}\left(\frac{y, z}{\xi, \eta}\right)| &= |\underline{J}\left(\frac{y, z}{y_0, z_0}\right)| |\underline{J}\left(\frac{y_0, z_0}{\xi, \eta}\right)| + |\underline{J}\left(\frac{y, z}{z_0, x_0}\right)| |\underline{J}\left(\frac{z_0, x_0}{\xi, \eta}\right)| + \\ &\quad |\underline{J}\left(\frac{y, z}{x_0, y_0}\right)| |\underline{J}\left(\frac{x_0, y_0}{\xi, \eta}\right)|\end{aligned}\tag{3.103}$$

$$\begin{aligned}|\underline{J}\left(\frac{z, x}{\xi, \eta}\right)| &= |\underline{J}\left(\frac{z, x}{y_0, z_0}\right)| |\underline{J}\left(\frac{y_0, z_0}{\xi, \eta}\right)| + |\underline{J}\left(\frac{z, x}{z_0, x_0}\right)| |\underline{J}\left(\frac{z_0, x_0}{\xi, \eta}\right)| + \\ &\quad |\underline{J}\left(\frac{z, x}{x_0, y_0}\right)| |\underline{J}\left(\frac{x_0, y_0}{\xi, \eta}\right)|\end{aligned}\tag{3.104}$$

$$\begin{aligned} \left| \underline{J} \begin{pmatrix} x, y \\ \xi, \eta \end{pmatrix} \right| = & \left| \underline{J} \begin{pmatrix} x, y \\ y_0, z_0 \end{pmatrix} \right| \left| \underline{J} \begin{pmatrix} y_0, z_0 \\ \xi, \eta \end{pmatrix} \right| + \left| \underline{J} \begin{pmatrix} x, y \\ z_0, x_0 \end{pmatrix} \right| \left| \underline{J} \begin{pmatrix} z_0, x_0 \\ \xi, \eta \end{pmatrix} \right| + \\ & \left| \underline{J} \begin{pmatrix} x, y \\ x_0, y_0 \end{pmatrix} \right| \left| \underline{J} \begin{pmatrix} x_0, y_0 \\ \xi, \eta \end{pmatrix} \right| \end{aligned} \quad (3.105)$$

Noticing from the definition $\underline{G}$ of the matrix:

$$\underline{G}^{-1} = \frac{1}{|\underline{G}|} \text{Cof}(\underline{G}^t) \quad (3.106)$$

where $\text{Cof}(\underline{G})$ is the cofactor matrix of $\underline{G}$, therefore:

$$\begin{aligned} \text{Cof}(\underline{G}) &= \begin{bmatrix} \left| \underline{J} \begin{pmatrix} y, z \\ y_0, z_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} y, z \\ z_0, x_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} y, z \\ x_0, y_0 \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z, x \\ y_0, z_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} z, x \\ z_0, x_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} z, x \\ x_0, y_0 \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x, y \\ y_0, z_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} x, y \\ z_0, x_0 \end{pmatrix} \right| & \left| \underline{J} \begin{pmatrix} x, y \\ x_0, y_0 \end{pmatrix} \right| \end{bmatrix} \\ \text{Cof}(\underline{G}) &= |\underline{G}| (\underline{G}^t)^{-1} \end{aligned} \quad (3.107)$$

Combining 3.103 to 3.105 and using 3.106 and 3.107 it can be proved that:

$$\begin{bmatrix} \left| \underline{J} \begin{pmatrix} y, z \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z, x \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x, y \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} = |\underline{G}| (\underline{G}^t)^{-1} \begin{bmatrix} \left| \underline{J} \begin{pmatrix} y_0, z_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z_0, x_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x_0, y_0 \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} \quad (3.108)$$

$$\begin{bmatrix} \left| \underline{J} \begin{pmatrix} y_0, z_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z_0, x_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x_0, y_0 \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} = \frac{1}{|\underline{G}|} \underline{G}^t \begin{bmatrix} \left| \underline{J} \begin{pmatrix} y, z \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z, x \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x, y \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} \quad (3.109)$$

3.7.1 Infinitesimal area before deformation

Using curvilinear co-ordinates ξ, η to define a surface with:

$$a(x_0, y_0, z_0) \equiv a(\xi, \eta)$$

$$b(\xi + d\xi, \eta)$$

$$c(\xi, \eta + d\eta)$$

The infinitesimal vectors $\vec{ab}, \vec{ac}$ can be defined as follows:

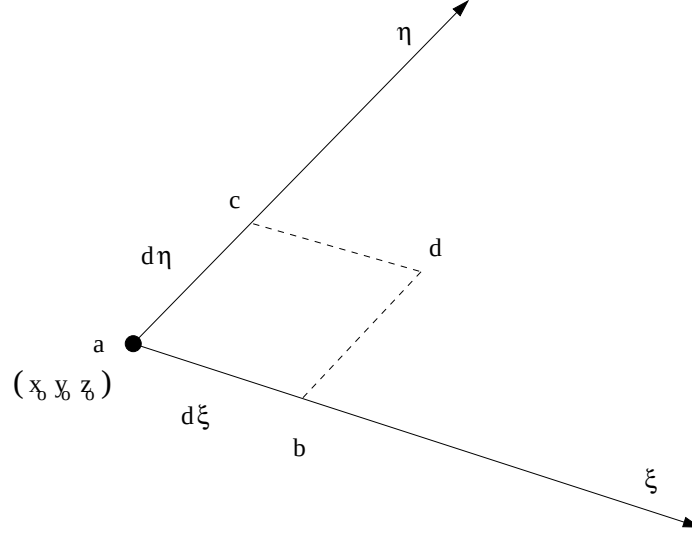


Figure 3.1: Curve linear intrinsic coordinate mapping.

$$\vec{ab} = d\vec{\xi} = \left(\frac{\partial x_0}{\partial \xi} \hat{i} + \frac{\partial y_0}{\partial \xi} \hat{j} + \frac{\partial z_0}{\partial \xi} \hat{k} \right) d\xi \quad (3.110)$$

$$\vec{ac} = d\vec{\eta} = \left(\frac{\partial x_0}{\partial \eta} \hat{i} + \frac{\partial y_0}{\partial \eta} \hat{j} + \frac{\partial z_0}{\partial \eta} \hat{k} \right) d\eta \quad (3.111)$$

The infinitesimal vector of area can be defined from:

$$\begin{aligned} d\vec{A}_0 &= d\vec{\xi} \wedge d\vec{\eta} \\ d\vec{A}_0 &= d\vec{A}_x^0 \hat{i} + d\vec{A}_y^0 \hat{j} + d\vec{A}_z^0 \hat{k} \end{aligned} \quad (3.112)$$

Hence it can be deduced that:

$$d\vec{A}_0 = \begin{bmatrix} dA_x^0 \\ dA_y^0 \\ dA_z^0 \end{bmatrix} = \begin{bmatrix} \left| \underline{J} \begin{pmatrix} y_0, z_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z_0, x_0 \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x_0, y_0 \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} d\xi d\eta \quad (3.113)$$

3.7.2 Infinitesimal area after deformation

Using a similar approach it can deduced that:

$$d\vec{A} = \begin{bmatrix} dA_x \\ dA_y \\ dA_z \end{bmatrix} = \begin{bmatrix} \left| \underline{J} \begin{pmatrix} y, z \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} z, x \\ \xi, \eta \end{pmatrix} \right| \\ \left| \underline{J} \begin{pmatrix} x, y \\ \xi, \eta \end{pmatrix} \right| \end{bmatrix} d\xi d\eta \quad (3.114)$$

Comparing equations 3.108 and 3.109 it can be deduced that:

$$d\underline{A}_0 = \frac{1}{|\underline{G}|} \underline{G}^t d\underline{A} \quad (3.115)$$

$$\begin{aligned} d\underline{A} &= |\underline{G}| (\underline{G}^t)^{-1} d\underline{A}_0 \\ d\underline{A} &= |\underline{G}| \underline{\overline{G}}^t d\underline{A}_0 \end{aligned} \quad (3.116)$$

3.8 Stress measures

If the effect of deformation on a infinitesimal area is ignored then the engineering stress vector can be defined as:

$$\vec{S}_0 = \frac{d\vec{F}_0}{dA_0} \quad (3.117)$$

From which the three stress vectors can be defined:

$$\begin{aligned} \vec{S}_x^0 &= \sigma_x^0 \hat{i} + \tau_{xy}^0 \hat{j} + \tau_{xz}^0 \hat{k} \\ \vec{S}_y^0 &= \tau_{yx}^0 \hat{i} + \sigma_y^0 \hat{j} + \tau_{yz}^0 \hat{k} \\ \vec{S}_z^0 &= \tau_{zx}^0 \hat{i} + \tau_{zy}^0 \hat{j} + \sigma_z^0 \hat{k} \end{aligned} \quad (3.118)$$

and:

$$\underline{S}_0 = l_0 \underline{S}_x^0 + m_0 \underline{S}_y^0 + n_0 \underline{S}_z^0 \quad (3.119)$$

where l_0, m_0, n_0 are the directional cosines of the normal to dA_0 , then the components of the engineering stress matrix can be defined as:

$$[\sigma^0] = \begin{bmatrix} \sigma_x^0 & \tau_{xy}^0 & \tau_{xz}^0 \\ \tau_{yx}^0 & \sigma_y^0 & \tau_{yz}^0 \\ \tau_{zx}^0 & \tau_{zy}^0 & \sigma_z^0 \end{bmatrix} \quad (3.120)$$

The traction components with respect to undeformed surfaces can be written as:

$$\begin{aligned} T_x^0 &= l_0 \sigma_x^0 + m_0 \tau_{yx}^0 + n_0 \tau_{zx}^0 \\ T_y^0 &= l_0 \tau_{xy}^0 + m_0 \sigma_y^0 + n_0 \tau_{zy}^0 \\ T_z^0 &= l_0 \tau_{xz}^0 + m_0 \tau_{yz}^0 + n_0 \sigma_z^0 \end{aligned} \quad (3.121)$$

3.8.1 Cauchy (Natural or true) stress

The Cauchy stress vector is defined with respect to the deformed infinitesimal area, thus:

$$\vec{S} = \frac{d\vec{F}}{dA} \quad (3.122)$$

The three Cauchy stress vectors can be defined using the infinitesimal areas normal to the x, y and z axes respectively:

$$\begin{aligned} \vec{S}_x &= \frac{d\vec{F}_x}{dA_x} = \sigma_x \hat{i} + \tau_{xy} \hat{j} + \tau_{xz} \hat{k} \\ \vec{S}_y &= \frac{d\vec{F}_y}{dA_y} = \tau_{yx} \hat{i} + \sigma_y \hat{j} + \tau_{yz} \hat{k} \\ \vec{S}_z &= \frac{d\vec{F}_z}{dA_z} = \tau_{zx} \hat{i} + \tau_{zy} \hat{j} + \sigma_z \hat{k} \end{aligned} \quad (3.123)$$

From which the true or Cauchy stress matrix can be defined.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (3.124)$$

which is a symmetric matrix. The traction vector can be written as follows, where $\underline{n}$ is the vector of directional cosines referred to deformed co-ordinates.

$$\underline{T} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = [\sigma]^t \underline{n} \quad (3.125)$$

It can also be deduced that the Cauchy stress vector can be obtained from:

$$\underline{S} = l\underline{S}_x + m\underline{S}_y + n\underline{S}_z \quad (3.126)$$

The true or Cauchy stress can be expressed in terms of nominal values using the following derivation. Assuming total incompressibility the deformed area A can be expressed in terms of the undeformed area A_0 as follows:

$$A = A_0 \left(\frac{l_0}{l} \right) \quad (3.127)$$

Substituting this expression into the equation for the true stress:

$$\sigma = \frac{F}{A} \equiv \frac{F}{A_0} \left(\frac{l}{l_0} \right) \equiv \sigma_n \left(\frac{l}{l_0} \right) \quad (3.128)$$

Therefore using the definition of the nominal strain given by equation 3.18 the following relationship between the Cauchy and nominal stress measures can be obtained:

$$\sigma = \sigma_n (1 + \varepsilon_n) \quad (3.129)$$

3.8.2 Reference stress or 1st PK stress

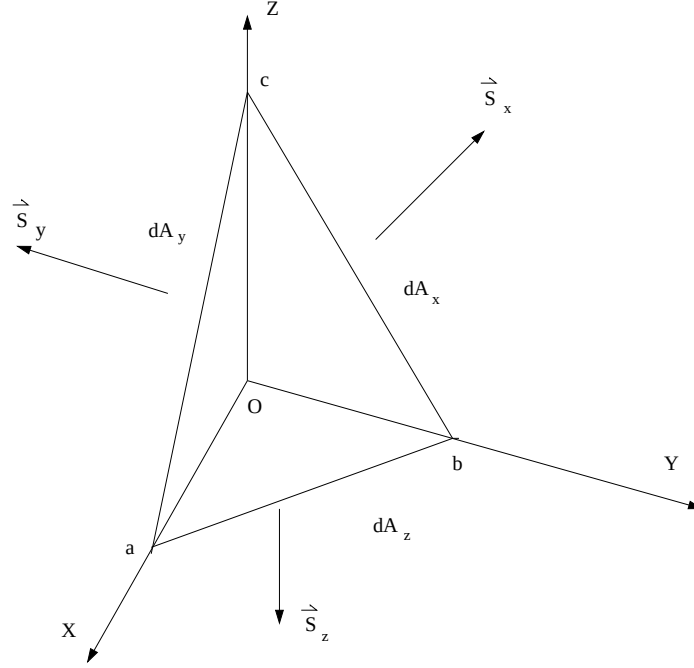


Figure 3.2: Internal stress/force equilibrium at a point.

The infinitesimal force vector $d\mathbf{F}$ acting on an infinitesimal area dA , where:

$$d\vec{A} = dA_x \hat{i} + dA_y \hat{j} + dA_z \hat{k} \quad (3.130)$$

can be obtained from the equilibrium of volume $oabc$, hence it can be deduced from equation 3.122 and 3.126 that:

$$\begin{aligned} d\mathbf{F} &= \underline{S} dA = (\underline{S}_x + m\underline{S}_y + n\underline{S}_z) dA \\ d\mathbf{F} &= \underline{S}_x dA_x + \underline{S}_y dA_y + \underline{S}_z dA_z \end{aligned} \quad (3.131)$$

Or in matrix form:

$$\begin{aligned} d\mathbf{F} &= [\underline{S}_x \underline{S}_y \underline{S}_z] d\mathbf{A} \\ d\mathbf{F} &= [\sigma]^t d\mathbf{A} = [\sigma] d\mathbf{A} \end{aligned} \quad (3.132)$$

Substituting from equation 3.116 into 3.132 then:

$$d\mathbf{F} = [\sigma] \mid \underline{G} \mid (\underline{G}^t)^{-1} d\mathbf{A}_0 \quad (3.133)$$

The 1st PK stress matrix $[\rho]$ is then defined such that:

$$d\mathbf{F} = [\rho] d\mathbf{A}_0 \quad (3.134)$$

hence:

$$[\rho] = J [\sigma] (\underline{G}^t)^{-1} \quad (3.135)$$

with:

$$J = | \underline{G} |$$

This stress matrix and its transpose is the Lagrange stress matrix:

$$\begin{aligned} [\sigma^*] &= [\rho]^t \\ [\sigma^*] &= J (\underline{G}^{-1}) [\sigma] \end{aligned} \quad (3.136)$$

3.8.3 The 2^{nd} PK stress

This is defined based on a modified infinitesimal force. Thus defining:

$$d\underline{x} = \underline{G} d\underline{x}_0 \quad (3.137)$$

And assuming the same equation applies to $d\underline{F}$ then:

$$d\underline{F} = \underline{G} d\underline{F}_0 \quad (3.138)$$

Substituting into equation 3.132:

$$\begin{aligned} d\underline{F} &= [\sigma] d\underline{A} \\ \underline{G} d\underline{F}_0 &= [\sigma] J (\underline{G}^t)^{-1} d\underline{A}_0 \\ d\underline{F}_0 &= J \underline{G}^{-1} [\sigma] (\underline{G}^{-1})^t d\underline{A}_0 \end{aligned} \quad (3.139)$$

The 2^{nd} PK stress matrix $[S]$ is defined such that:

$$d\underline{F}_0 = [S] d\underline{A}_0 \equiv [S]^t d\underline{A}_0 \quad (3.140)$$

where:

$$[S] = J \underline{G}^{-1} [\sigma] (\underline{G}^{-1})^t \quad (3.141)$$

Notice also that:

$$[\rho] = \underline{G} [S] \quad (3.142)$$

$$[\sigma] = \frac{1}{J} \underline{G} [S] \underline{G}^t \quad (3.143)$$

For small strain considerations $[S] \cong [\sigma^0]$.

Chapter 4

Finite strain constitutive modelling

4.1 Mooney-Rivlin material model

Constitutive hyperelastic models are essentially higher order elastic formulations based on functions of the total strains or stretches. The key assumption in the formulation of these models is that the material is isotropic. This is manifest by the uniform distribution of strain energy into the material during deformation. Therefore if it is possible to define a function for the strain energy density which is independent of any arbitrary choice of co-ordinate system then this function will by default enforce isotropy.

If the strain energy per unit volume can be expressed as ϕ , then the change in strain energy can be expressed as:

$$\delta\phi = \underline{\sigma} : \delta\underline{\varepsilon} = \frac{\partial\phi}{\partial\underline{\varepsilon}} : \delta\underline{\varepsilon} \quad (4.1)$$

Therefore the total strain energy density at any point in the material can be obtained from:

$$\phi = \int \delta\phi = \int \underline{\sigma} : \delta\underline{\varepsilon} \quad (4.2)$$

The incremental form of the stress strain relationship can then be obtained from:

$$\delta\underline{\sigma} = \frac{\partial^2\phi}{\partial\underline{\varepsilon}\partial\underline{\varepsilon}} : \delta\underline{\varepsilon} \quad (4.3)$$

Therefore the objective of this derivation is to express the strain energy using some function of the strain (or stretches) which is independent of the chosen co-ordinate system. It was shown in chapter two that the strain invariants are independent of the chosen co-ordinate system and can be expressed as functions of the principal stretches $\lambda_1, \lambda_2, \lambda_3$. It can further be deduced that it is possible to have the chosen co-ordinate system axes aligned with the principal axes with two of the axes parallel but opposite in direction to two of the principal axes. Therefore in order to always obtain a positive strain energy value the strain energy function should be based on the square of the principal stretches $\lambda_1^2, \lambda_2^2, \lambda_3^2$.

It was also shown in chapter two that the principal stretches squared $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigenvalues of the left and right Cauchy-Green tensors $\underline{b}$ and $\underline{C}$ respectively. The non-trivial solutions (eigenvalues) are obtained from the following equation:

$$|(\underline{C} - \lambda_p^2 \underline{I})| = |(\underline{b} - \lambda_p^2 \underline{I})| = 0 \quad (4.4)$$

which leads to the following cubic equation:

$$\lambda_p^6 - I_1 \lambda_p^4 + I_2 \lambda_p^2 - I_3 = 0 \quad (4.5)$$

The coefficients of 4.5 are the strain invariants. Expanding out 4.5 the following expressions for the strain invariants are obtained:

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_1 &= C_{11} + C_{22} + C_{33} \\ I_1 &= \text{trace}(\underline{C}) = \text{trace}(\underline{b}) \\ I_1 &= \underline{I} : \underline{C} = \underline{I} : \underline{b} \end{aligned} \quad (4.6)$$

$$\begin{aligned} I_2 &= \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \\ I_2 &= C_{11}C_{22} + C_{22}C_{33} + C_{33}C_{11} - C_{12}C_{21} - C_{23}C_{32} - C_{13}C_{31} \\ I_2 &= \frac{1}{2} (I_1^2 - \text{trace}(\underline{C}^2)) = \frac{1}{2} (I_1^2 - \text{trace}(\underline{b}^2)) \end{aligned} \quad (4.7)$$

$$\begin{aligned} I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2 \\ I_3 &= C_{11}C_{22}C_{33} + 2C_{12}C_{23}C_{13} - C_{11}C_{23}C_{32} - C_{22}C_{13}C_{31} - C_{33}C_{12}C_{21} \\ I_3 &= J^2 = |\underline{C}| = |\underline{b}| = \frac{1}{3} (\text{trace}(\underline{C}^3) - I_1 \text{trace}(\underline{C}^2) - I_2 \text{trace}(\underline{C})) \end{aligned} \quad (4.8)$$

where:

$$J = |\underline{C}| = I_3^{\frac{1}{2}}$$

Apart from isotropy the strain energy function is always required to be equal to or greater than zero. Since the values of the principal stretches will be equal to unity at zero deformation then I_1, I_2, I_3 will take the values 3, 3 and 1 respectively and the following strain energy expression can be deduced:

$$\phi = \sum_{p,q,r} C_{pqr} (I_1 - 3)^p (I_2 - 3)^q (I_3 - 1)^r \quad (4.9)$$

where:

$$C_{000} = 0$$

For incompressible materials $\lambda_1 \lambda_2 \lambda_3 = 1$, hence:

$$\phi = \sum_{p,q} C_{p,q} (I_1 - 3)^p (I_2 - 3)^q \quad (4.10)$$

where:

$$C_{00} = 0$$

Two truncated forms of 4.10 are commonly used in the analysis of rubber which are obtained by including two or one material coefficients for the Mooney-Rivlin and Neo-Hookian material models respectively.

Mooney-Rivlin:

$$\begin{aligned} \phi &= C_{10} (I_1 - 3) + C_{01} (I_2 - 3) \\ \phi &\equiv C_1 (I_1 - 3) + C_2 (I_2 - 3) \end{aligned} \quad (4.11)$$

Neo-Hookian:

$$\begin{aligned} \phi &= C_{10} (I_1 - 3) \\ \phi &\equiv C_1 (I_1 - 3) \end{aligned} \quad (4.12)$$

All real rubber like materials will have some compressibility and the numerical analysis of rubber can be simplified if some allowance for compressibility is introduced. This is achieved by making use of the bulk modulus K . Consider the 'mean value theorem':

$$F(a, b, c) = a + b + c - 3(abc)^{\frac{1}{3}} \quad (4.13)$$

This function will always be positive as long as $a \neq b \neq c$. Therefore this function can be used as the basis for a strain energy function:

$$\phi_r/c = \lambda_1^r + \lambda_2^r + \lambda_3^r - 3(\lambda_1 \lambda_2 \lambda_3)^{\frac{r}{3}} \quad (4.14)$$

Considering 4.14 with $r = 2$ the compressible form of the Neo-Hookien material model can be obtained:

$$\phi/c = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3(\lambda_1 \lambda_2 \lambda_3)^{\frac{2}{3}}$$

$$\phi/c = I_1 - 3I_3^{\frac{1}{3}} \quad (4.15)$$

By combining 4.15 with a similar expression where $r = -2$ a compressible form of the Mooney-Rivlin material model can be obtained:

$$\begin{aligned} \phi &= c_1 \Phi_2 + c_2' \Phi_{-2} = c_1 \left(I_1 - 3I_3^{\frac{1}{3}} \right) + c_2' \left(\frac{I_2}{I_3} - 3I_3^{-\frac{1}{3}} \right) \\ \phi &= c_1 \left(I_1 - 3I_3^{\frac{1}{3}} \right) + c_2 \left(I_2 - 3I_3^{\frac{2}{3}} \right) \end{aligned} \quad (4.16)$$

As equation 4.16 stands it will take the value zero when $\lambda_1 = \lambda_2 = \lambda_3$. To overcome this drawback a compressible term Φ_b involving the bulk modulus K can be added, such that:

$$\phi_b = K (J - 1)^2 / 2 \quad (4.17)$$

It is possible to split the strain energy function into volumetric and deviatoric terms by defining modified stretches as follows:

$$\bar{\lambda} = J^{-\frac{1}{3}} \lambda$$

such that:

$$\bar{J} = \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 = 1 \quad (4.18)$$

It can be shown that the various modified parameters are as follows:

$$\bar{\underline{G}} = J^{-\frac{1}{3}} \underline{G} = I_3^{-\frac{1}{6}} \underline{G} \quad (4.19)$$

$$\bar{\underline{C}} = \bar{\underline{G}}^t \bar{\underline{G}} = J^{-\frac{2}{3}} \underline{C} = I_3^{-\frac{1}{3}} \underline{C} \quad (4.20)$$

$$\bar{\underline{b}} = \bar{\underline{G}} \bar{\underline{G}}^t = J^{-\frac{2}{3}} \underline{b} = I_3^{-\frac{1}{3}} \underline{b} \quad (4.21)$$

The modified Mooney-Rivlin material model can then be written as:

$$\phi = \phi_d + \phi_b = C_1 (\bar{I}_1 - 3) + C_2 (\bar{I}_2 - 3) + \frac{1}{2} K (J - 1)^2 \quad (4.22)$$

Thus for uniform compression the first two terms become zero as $\bar{I}_1$ and $\bar{I}_2$ are equal to 3, hence all the strain energy is accounted for as volumetric strain energy.

4.2 Stress matrix based on the Mooney-Rivlin material model

The Mooney-Rivlin material model was derived in equation 4.22 as:

$$\phi = C_1 (\bar{I}_1 - 3) + C_2 (\bar{I}_2 - 3) + \frac{1}{2} K (J - 1)^2 \quad (4.23)$$

therefore differentiating 4.23 then:

$$\frac{\partial \phi}{\partial \underline{C}} = C_1 \left(\frac{\partial \bar{I}_1}{\partial \underline{C}} \right) + C_2 \left(\frac{\partial \bar{I}_2}{\partial \underline{C}} \right) + K (J - 1) \frac{\partial J}{\partial \underline{C}} \quad (4.24)$$

Substituting for the partial differentials (see appendix A) A.26, A.32 and A.37 into 4.24 gives:

$$\begin{aligned} \frac{\partial \phi}{\partial \underline{C}} &= C_1 I_3^{-\frac{1}{3}} \left(\underline{I} - \frac{1}{3} I_1 \underline{C}^{-1} \right) + C_2 I_3^{-\frac{2}{3}} \left(I_1 \underline{I} - \underline{C} - \frac{2}{3} I_2 \underline{C}^{-1} \right) + K (J - 1) \frac{1}{2} I_3^{\frac{1}{2}} \underline{C}^{-1} \\ \frac{\partial \phi}{\partial \underline{C}} &= \left(C_1 I_3^{-\frac{1}{3}} + C_2 I_3^{-\frac{2}{3}} I_1 \right) \underline{I} - C_2 I_3^{-\frac{2}{3}} \underline{C} - \left(\frac{1}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{2}{3} C_2 I_2 I_3^{-\frac{2}{3}} \right) \underline{C}^{-1} + \\ &\quad \frac{1}{2} K J (J - 1) \underline{C}^{-1} \end{aligned} \quad (4.25)$$

Since $\underline{\varepsilon}_G = \frac{1}{2} (\underline{C} - \underline{I})$ it can be deduced that:

$$\frac{\partial \phi}{\partial \underline{\varepsilon}_G} = 2 \frac{\partial \phi}{\partial \underline{C}} \quad (4.26)$$

hence:

$$\begin{aligned} \underline{S} &= \frac{\partial \phi}{\partial \underline{\varepsilon}_G} = 2 \frac{\partial \phi}{\partial \underline{C}} \\ \underline{S} &= B_1 \underline{I} + B_2 \underline{C} + B_3 \underline{C}^{-1} + K J (J - 1) \underline{C}^{-1} \end{aligned} \quad (4.27)$$

where:

$$B_1 = 2C_1 I_3^{-\frac{1}{3}} + 2C_2 I_3^{-\frac{2}{3}} I_1 \quad (4.28)$$

$$B_2 = -2C_2 I_3^{-\frac{2}{3}} \quad (4.29)$$

$$B_3 = - \left(\frac{2}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_2 I_3^{-\frac{2}{3}} \right) \quad (4.30)$$

It is further possible to simplify 4.27 by using the definition of the bulk modulus K as follows:

$$\rho = -\frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \quad (4.31)$$

$$K = -\frac{\rho}{\varepsilon_v} \quad (4.32)$$

where:

$$\varepsilon_v = \frac{dV - dV_0}{dV_0} = J - 1 \quad (4.33)$$

therefore:

$$\rho = -K\varepsilon_v = -K(J - 1) \quad (4.34)$$

The first three terms of 4.27 are the deviatoric terms. Consider the special case of uniform pressure. For this case the following can be deduced:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \quad (4.35)$$

$$\begin{aligned} I_1 &= 3\lambda^2 \\ I_2 &= 3\lambda^4 \\ I_3 &= \lambda^6 \end{aligned} \quad (4.36)$$

$$\phi = \frac{1}{2}K(J - 1)^2 \quad (4.37)$$

therefore:

$$\begin{aligned} \frac{\partial \phi}{\partial I_3} &= \frac{1}{2}K(J - 1)I_3^{-\frac{1}{2}} \\ \frac{\partial \phi}{\partial \underline{C}} &= \frac{1}{2}KJ(J - 1)\underline{C}^{-1} \end{aligned} \quad (4.38)$$

with $\underline{C} = \lambda^2 \underline{I}$, $J = \lambda^3$ and $\underline{C}^{-1} = \lambda^{-2} \underline{I}$ then:

$$\frac{\partial \phi}{\partial \underline{C}} = \frac{1}{2}K(J - 1)\lambda^3\lambda^{-2}\underline{I} \quad (4.39)$$

hence:

$$\underline{S} = 2\frac{\partial U}{\partial \underline{C}} = K\lambda(J - 1)\underline{I} \quad (4.40)$$

$$\underline{\sigma} = \frac{1}{J}\underline{G}\underline{S}\underline{G}^t \quad (4.41)$$

For this special case $\underline{G} = \lambda \underline{I}$, hence:

$$\underline{\sigma} = \frac{1}{J}\lambda^2 \underline{S} = K(J - 1)\underline{I} \quad (4.42)$$

substituting 4.32 into 4.42 then:

$$\underline{\sigma} = K\varepsilon_v \underline{I} \equiv -\rho \underline{I} \quad (4.43)$$

Using the definition 4.33 then:

$$K (J - 1) = -\rho \quad (4.44)$$

which is a repeat of 4.34 and hence it can be seen that the last term in 4.27 is due to the hydrostatic pressure, and the remaining terms are due to the deviatoric stress tensor. Thus 4.27 can be written as:

$$\underline{S} = B_1 \underline{I} + B_2 \underline{C} + B_3 \underline{C}^{-1} - \rho J \underline{C}^{-1} \quad (4.45)$$

4.2.1 Mooney-Rivlin material model calibration

For the finite and boundary element programs developed so far the Mooney-Rivlin material model has been used. The following derivation for the analytical solution to a plane stress uniaxial tension problem with one of the in plane surfaces restrained [1] provides the calibrating function in terms of the Mooney-Rivlin constants C_1 and C_2 to use with the special case of unrestrained uniaxial tension test data. Complete incompressibility has been assumed. Therefore substituting into the incompressible form of the Mooney-Rivlin equation 4.11 for the strain invariants gives:

$$\phi = C_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2 \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3 \right) \quad (4.46)$$

For the case of restrained plane stress uniaxial tension the change in strain energy can simply be expressed in terms of the work done by external forces:

$$d\phi = f_1 d\lambda_1 \quad (4.47)$$

where f_1 is the force acting on the undeformed area normal to the tensile direction. Equation 4.47 can be written in variational form as:

$$d\phi = \left(\frac{\partial \phi}{\partial \lambda_1} \right) d\lambda_1 \quad (4.48)$$

Therefore partially differentiating 4.46 with respect to λ_1 gives:

$$\frac{\partial \phi}{\partial \lambda_1} = 2C_1 \lambda_1 - 2C_2 \frac{1}{\lambda_1^3} \quad (4.49)$$

The true or Cauchy stress in the tensile direction can be expressed as:

$$t_1 = \frac{f_1}{\lambda_2 \lambda_3} \quad (4.50)$$

since $\lambda_2\lambda_3$ is the deformed area normal to the tensile direction. However, because of the condition of incompressibility it can be deduced that $\lambda_2\lambda_3 = 1/\lambda_1$. Hence substituting for f_1 from 4.47 and 4.48 then the true stress in the tensile direction can be expressed as:

$$t_1 = 2 \left[C_1 \lambda_1^2 - C_2 \frac{1}{\lambda_1^2} \right] \quad (4.51)$$

similarly:

$$t_2 = 2 \left[C_1 \lambda_2^2 - C_2 \frac{1}{\lambda_2^2} \right] \quad (4.52)$$

For plane strain however where $t_3 \neq 0$ the previous method cannot be applied and a compressive hydrostatic pressure (negative stress) term must be added to form the principal stress and extension ratio expression.

$$t_i = 2 \left[C_1 \lambda_i^2 - C_2 \frac{1}{\lambda_i^2} \right] - \rho \quad (4.53)$$

Thus for simple uniaxial tension and complete incompressibility $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. For this special case $t_2 = t_3 = 0$ therefore an expression for the hydrostatic pressure can be obtained as:

$$\rho = 2 \left[C_1 \frac{1}{\lambda} - C_2 \lambda \right] \quad (4.54)$$

Therefore substituting 4.54 into 4.53 the expression for the true stress in the tensile direction can be obtained as:

$$t = 2 \left(\lambda^2 - \frac{1}{\lambda} \right) \left[C_1 + \frac{C_2}{\lambda} \right] \quad (4.55)$$

Equation 4.55 is then the calibrating function in terms of C_1 and C_2 to use with the test data. Thus writing 4.55 in terms of C_1 and C_2 for the i^{th} stress and strain data point gives:

$$t_i = A_i C_1 + B_i C_2 \quad (4.56)$$

where:

$$A_i = 2 \left(\lambda_i^2 - \frac{1}{\lambda_i} \right) \quad B_i = 2 \left(\lambda_i - \frac{1}{\lambda_i^2} \right)$$

Two equations can be obtained from 4.56 for the two unknowns C_1 and C_2 by taking a least squares approach to the error between the empirical test data and the analytical expression given by 4.56, therefore:

$$Error = E = \sum_{i=1}^N [A_i C_1 + B_i C_2 - t_i]^2 \quad (4.57)$$

therefore:

$$\frac{\partial E}{\partial C_1} = \sum_{i=1}^N 2 [A_i C_1 + B_i C_2 - t_i] A_i = 0 \quad (4.58)$$

$$\frac{\partial E}{\partial C_2} = \sum_{i=1}^N 2 [A_i C_1 + B_i C_2 - t_i] B_i = 0 \quad (4.59)$$

Putting equations 4.58 and 4.59 in matrix format gives:

$$\sum_{i=1}^N \begin{bmatrix} A_i^2 & A_i B_i \\ A_i B_i & B_i^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} A_i t_i \\ B_i t_i \end{bmatrix} \quad (4.60)$$

Thus Cramer's rule can be used to solve for the material constants C_1 and C_2 :

$$C_1 = \frac{\begin{vmatrix} \sum A_i t_i & \sum A_i B_i \\ \sum B_i t_i & \sum B_i^2 \end{vmatrix}}{\begin{vmatrix} \sum A_i^2 & \sum A_i B_i \\ \sum A_i B_i & \sum B_i^2 \end{vmatrix}} = \frac{\sum A_i t_i \cdot \sum B_i^2 - \sum B_i t_i \cdot \sum A_i B_i}{\sum A_i^2 \cdot \sum B_i^2 - (\sum A_i B_i)^2} \quad (4.61)$$

$$C_2 = \frac{\begin{vmatrix} \sum A_i^2 & \sum A_i t_i \\ \sum A_i B_i & \sum B_i t_i \end{vmatrix}}{\begin{vmatrix} \sum A_i^2 & \sum A_i B_i \\ \sum A_i B_i & \sum B_i^2 \end{vmatrix}} = \frac{\sum A_i^2 \cdot \sum B_i t_i - \sum A_i t_i \cdot \sum A_i B_i}{\sum A_i^2 \cdot \sum B_i^2 - (\sum A_i B_i)^2} \quad (4.62)$$

4.2.2 Modulus of elasticity and bulk modulus based on Mooney-Rivlin coefficients

If the nominal stress form of equation 4.55 is taken the following expression is obtained:

$$f = 2 \left(\lambda - \frac{1}{\lambda^2} \right) \left[C_1 + \frac{C_2}{\lambda} \right] \quad (4.63)$$

Expanding this expression in terms of strains instead of extension ratios then it can be deduced that:

$$f = 2 \frac{[(C_1(1 + \varepsilon) + C_2) ((1 + \varepsilon)^3 - 1)]}{(1 + \varepsilon)^3} \quad (4.64)$$

Ignoring the higher power terms of ε it can be deduced that:

$$f = 6\varepsilon (C_1 + C_2) \quad (4.65)$$

From the definition of the Elastic modulus it can be deduced that:

$$E = \frac{f}{\varepsilon} = 6 (C_1 + C_2) \quad (4.66)$$

It can be shown that the bulk modulus of a linear elastic isotropic material, K , can be defined in terms of elastic modulus E and Poisson's ratio ν as follows:

$$K = \frac{E}{3(1 - 2\nu)} \quad (4.67)$$

Hence for small deflections the bulk modulus for an elastomer can be defined as follows:

$$K = \frac{2 (C_1 + C_2)}{(1 - 2\nu)} \quad (4.68)$$

Chapter 5

Non-linear finite element modelling of rubber

5.1 The finite element method, an overview

The formation of the element equations in the finite element method is based on the principle of virtual work. This states that for a virtual variation in field parameter values the variation in internal strain energy equals the variation in work done on the system by external forces and tractions [10].

$$\delta\Phi = \delta W \tag{5.1}$$

Having derived expressions for the variations in internal strain energy and work a system of simultaneous equations can be formed by introducing a trial function for the field parameter. Because this is a finite element formulation the trial function is formulated such that it satisfies the boundary conditions. The unknown coefficients (nodal displacements) in the field parameter trial function can then be found from the differentiated virtual work expression by equating the resulting expression to zero.

This has the effect of minimizing the virtual work for infinitesimal variations in the field parameter (displacement). This is an example of the Rayleigh-Ritz method of solving partial differential equations applied to elasticity problems.

5.2 The finite element method applied to rubber, an overview

When forming the element equations for rubber two sources of non-linearity are introduced due to the ability of rubber to undergo finite elastic deformations. These are geometric and material non-linearity. The presence of geometric non-linearity dictates that stress and strain measures should be used which are accurate for finite deformations. In this derivation the stress and strain measures used are the 2nd PK stress and Green's strain respectively. Note the 2nd PK stress and Green's strain refer to undeformed or reference coordinates

The material non-linearity requires the use of an appropriate material model to form the constitutive relationship between stress and strain. The constitutive relationship in this model is based on the Mooney-Rivlin expression for internal strain energy. The compressible form of the Mooney-Rivlin strain energy expression is of the following form (see equation 4.22):

$$\phi = C_1 (\bar{I}_1 - 3) + C_2 (\bar{I}_2 - 3) + \frac{1}{2} K (J - 1)^2 \quad (5.2)$$

The first two terms in the expression account for the deviatoric strain energy and the third term accounts for the volumetric strain energy. It will be shown that this volumetric term introduces pressure field parameters into what is known as the "mixed Lagrangian finite element formulation". The constants C_1 , C_2 and K (bulk modulus) in 5.2 are material constants and are evaluated by comparison between test data and analytical solutions for specific types of strain.

5.3 Green's strain representation

The first step in forming the virtual work expression is to express the Green's strain variation in terms of nodal displacements. The Green's strain-displacement equations are of the form:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right) \end{aligned} \quad (5.3)$$

which can be written as the sum of small and large strain vectors:

$$\underline{\varepsilon}_G = \underline{\varepsilon}_s + \underline{\varepsilon}_l \quad (5.4)$$

$$\underline{\varepsilon}_s = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \quad (5.5)$$

$$\underline{\varepsilon}_l = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \end{bmatrix} \quad (5.6)$$

The small strain vector $\underline{\varepsilon}_s$ can be related to the nodal displacements by means of the $\underline{B}$ matrix.

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \dots & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (5.7)$$

The Cartesian shape function derivatives are obtained from intrinsic shape function derivatives as follows:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \underline{J}^{-1} \left(\frac{x, y}{\xi, \eta} \right) \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \quad (5.8)$$

where:

$$\underline{J} \left(\frac{x, y}{\xi, \eta} \right) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (5.9)$$

The above relationship enables the arbitrary form of the discretized elements in Cartesian space to have generic shape function expressions in intrinsic space. The large strain vector $\underline{\varepsilon}_l$ can be broken up into a matrix and a vector of Cartesian derivatives as follows:

$$\underline{\varepsilon}_l = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2} \underline{A} \underline{\theta} \quad (5.10)$$

The $\underline{\theta}$ vector can be derived in terms of nodal displacements as follows:

$$\underline{\theta} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \sum_{i=1}^n \begin{bmatrix} u_i \frac{\partial N_i}{\partial x} \\ v_i \frac{\partial N_i}{\partial x} \\ u_i \frac{\partial N_i}{\partial y} \\ v_i \frac{\partial N_i}{\partial y} \end{bmatrix} \equiv \underline{L}(x, y) \underline{p} \quad (5.11)$$

where:

$$\underline{L}(x, y) = \begin{bmatrix} -- & \frac{\partial N_i}{\partial x} & 0 & -- \\ -- & 0 & \frac{\partial N_i}{\partial x} & -- \\ -- & \frac{\partial N_i}{\partial y} & 0 & -- \\ -- & 0 & \frac{\partial N_i}{\partial y} & -- \end{bmatrix} \quad (5.12)$$

The equation for Green's strain, equation 5.4, can now be expressed as follows:

$$\underline{\varepsilon}_G = \underline{\varepsilon}_s + \underline{\varepsilon}_l = \left[\underline{H} + \frac{1}{2} \underline{A} \right] \underline{\theta} \quad (5.13)$$

where:

$$\underline{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad (5.14)$$

The variation in Green's strain can be expressed in vector form as follows:

$$\delta \underline{\varepsilon}_G = \delta \underline{\varepsilon}_s + \frac{1}{2} \delta \underline{A} \underline{\theta} + \frac{1}{2} \underline{A} \delta \underline{\theta} + O(\delta \underline{\theta}^2) \quad (5.15)$$

It can be shown that:

$$\delta \underline{A} \underline{\theta} = \underline{A} \delta \underline{\theta} \quad (5.16)$$

then:

$$\begin{aligned} \delta \underline{\varepsilon}_G &= \delta \underline{\varepsilon}_s + \underline{A} \delta \underline{\theta} + O(\delta \underline{\theta}^2) \\ \delta \underline{\varepsilon}_G &= [\underline{H} + \underline{A}] \delta \underline{\theta} + O(\delta \underline{\theta}^2) \\ \delta \underline{\varepsilon}_G &= [\underline{H} + \underline{A}] \underline{L} \delta \underline{p} + O(\delta \underline{\theta}^2) \end{aligned} \quad (5.17)$$

Equation 5.17 can be further rearranged to give:

$$\begin{aligned} \delta \underline{\varepsilon}_G &= (\underline{B} + \underline{A} \underline{L}) \delta \underline{p} + O(\delta \underline{\theta}^2) \\ \delta \underline{\varepsilon}_G &= \underline{B}_{nl} \delta \underline{p} + O(\delta \underline{\theta}^2) \end{aligned} \quad (5.18)$$

For small virtual displacements the last term in 5.18 becomes negligible:

$$\delta \underline{\varepsilon}_G = \underline{B}_{nl} \delta \underline{p} \quad (5.19)$$

where:

$$\underline{B}_{nl} = (\underline{B} + \underline{A} \underline{L}) \quad (5.20)$$

Thus the variation in Green's strain can now be expressed in terms of nodal coordinates.

5.4 Element equation formulation

The following expression for strain energy can be deduced:

$$\Phi = \int \int \int_{V_0} \delta \underline{\varepsilon}_G^t \underline{S} dV_o \quad (5.21)$$

Substituting 5.20 into 5.21 it is now possible to re-write the virtual work expression 5.1 in terms of nodal displacements:

$$\delta \underline{p}^t \int \int \int_{V_0} \underline{B}_{nl}^t \underline{S} dV_o - \delta \underline{p}^t \underline{f}_e = \delta \underline{p}^t \underline{f} \quad (5.22)$$

Equation 5.22 represents an imbalance between internal and external virtual energy. Since $\delta \underline{p}$ represents a vector of arbitrary virtual infinitesimal nodal displacements:

$$\begin{aligned} \int \int \int_{V_0} \underline{B}_{nl}^t \underline{S} dV_o - \underline{f}_e &= \underline{f} \\ \equiv q_i - q_e &= q \end{aligned} \quad (5.23)$$

where q_i represents the internal forces and q_e the external forces with q being the out of balance force. Assuming exact equilibrium the variation of 5.23 leads to:

$$\delta q = \int \int \int_{V_0} \underline{B}_{nl}^t \delta \underline{S} dV + \int \int \int_{V_0} \delta \underline{B}_{nl}^t \underline{S} dV_o = 0 \quad (5.24)$$

The evaluation of the second term in 5.24 can be achieved if it is realised that this equation is equivalent to the variational form of equation 5.21:

$$\delta \Phi = \int \int \int_{V_0} \delta \underline{\varepsilon}_G : \delta \underline{S} + \int \int \int_{V_0} \delta (\delta \underline{\varepsilon}_G) : \underline{S} dV_o \quad (5.25)$$

but it can be shown that:

$$\underline{\varepsilon}_G = \frac{1}{2} (\underline{G}^t \underline{G} - \underline{I}) = \frac{1}{2} (\underline{g} + \underline{g}^t) + \frac{1}{2} \underline{g}^t \underline{g} \quad (5.26)$$

For repeated differentiation of 5.26:

$$\delta \underline{\varepsilon}_G = \frac{1}{2} \underline{G}^t \delta \underline{g} + \frac{1}{2} \delta \underline{g}^t \underline{G} + \left(\frac{1}{2} \delta \underline{g}^t \underline{g} \right)_h \quad (5.27)$$

For infinitesimal virtual changes the last term in 5.27 becomes negligible:

$$\delta \underline{\varepsilon}_G = \frac{1}{2} \underline{G}^t \delta \underline{g} + \frac{1}{2} \delta \underline{g}^t \underline{G} \quad (5.28)$$

where $\underline{G}$, the deformation gradient equals $(\underline{g} + \underline{I})$, with $\underline{g}$ being the displacement derivative matrix.

$$\delta \underline{\varepsilon}_G = \frac{1}{2} (\underline{g}^t + \underline{I}) \delta \underline{g} + \frac{1}{2} \delta \underline{g}^t (\underline{g} + \underline{I}) \quad (5.29)$$

$$\delta (\delta \underline{\varepsilon}_G) = \delta \underline{g}^t \delta \underline{g} \quad (5.30)$$

Substitution of 5.30 into the last term of 5.25 gives.

$$\int \int \int_{V_0} \delta (\delta \underline{\varepsilon}_G) : \underline{S} dV_o = \int \int \int_{V_0} \underline{S} : \delta \underline{g}^t \delta \underline{g} dV_o \quad (5.31)$$

For a two-dimensional case 5.31 can be expanded in matrix form as.

$$\int \int \int_{V_0} \underline{S} : \delta \underline{g}^t \delta \underline{g} dV_o = \int \int \int_{V_0} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} : \begin{bmatrix} \frac{\partial \delta u}{\partial x} & \frac{\partial \delta v}{\partial x} \\ \frac{\partial \delta u}{\partial y} & \frac{\partial \delta v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial \delta u}{\partial x} & \frac{\partial \delta u}{\partial y} \\ \frac{\partial \delta v}{\partial x} & \frac{\partial \delta v}{\partial y} \end{bmatrix} dV_o \quad (5.32)$$

which can further be re-arranged as:

$$\begin{aligned} \int \int \int_{V_0} \underline{S} : \delta \underline{g}^t \delta \underline{g} dV_o &= \int \int \int_{V_0} \begin{bmatrix} \frac{\partial \delta u}{\partial x} \\ \frac{\partial \delta u}{\partial y} \\ \frac{\partial \delta v}{\partial x} \\ \frac{\partial \delta v}{\partial y} \end{bmatrix}^t \begin{bmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{22} & 0 & 0 \\ 0 & 0 & S_{11} & S_{12} \\ 0 & 0 & S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial \delta u}{\partial x} \\ \frac{\partial \delta u}{\partial y} \\ \frac{\partial \delta v}{\partial x} \\ \frac{\partial \delta v}{\partial y} \end{bmatrix} dV_o \\ \int \int \int_{V_0} \underline{S} : \delta \underline{g}^t \delta \underline{g} dV_o &= \int \int \int_{V_0} \delta \underline{\theta}^t \hat{S} \delta \underline{\theta} dV_o \end{aligned} \quad (5.33)$$

but from equation 5.11: $\delta \underline{\theta} = \underline{L} \delta \underline{p}$, therefore:

$$\int \int \int_{V_0} \underline{S} : \delta \underline{g}^t \delta \underline{g} dV_o = \delta \underline{p}^t \int \int \int_{V_0} \underline{L}^t \hat{S} \underline{L} dV_o \delta \underline{p} \quad (5.34)$$

The first term in 5.24 however involves the variation of the 2nd PK stress. In the derivation of the non-linear variational constitutive relationship between $\underline{S}$ and $\underline{\varepsilon}_G$ based on the Mooney-Rivlin strain energy function the following matrix expression is obtained (see appendix D, equations D.8 and D.9):

$$\delta \underline{S} = \underline{D}(p, \rho) \delta \underline{\varepsilon} + \underline{\varrho}(p) \delta \underline{\rho} \quad (5.35)$$

with $\underline{D}(p, \rho)$ being a matrix and $\underline{\varrho}$ being a vector. It can be seen that $\underline{D}(p, \rho)$ and $\underline{\varrho}$ are both functions of the field parameters. Also note that because of the volumetric term in the Mooney-Rivlin strain energy equation, pressure field parameters have been introduced into the formulation. Equations 5.34 and 5.35 can be substituted into 5.24 to give:

$$\delta q = \int \int \int_{V_0} \left(\underline{B}_{nl}^t [\underline{D}(p, \rho)] \underline{B}_{nl} \delta \underline{p} + \underline{B}_{nl}^t \underline{\varrho}(p) \underline{N}_p^t \delta \underline{\rho} + \underline{L}^t \hat{S} \underline{L} \delta \underline{p} \right) dV_o \quad (5.36)$$

$$\delta q = \int \int \int_{V_o} \left(\underline{B}_{nl}^t \underline{D}(p, \rho) \underline{B}_{nl} + \underline{L}^t \hat{S} \underline{L} \right) dV_o \delta \underline{p} + \int \int \int_{V_o} \underline{B}_{nl}^t \underline{\varrho}(p) \underline{N}_p^t dV_o \delta \underline{\rho} \quad (5.37)$$

$$\delta q = \underline{K} \delta \underline{p} + \underline{P} \delta \underline{\rho} \quad (5.38)$$

where:

$$\underline{K} = \int \int \int_{V_o} \left(\underline{B}_{nl}^t \underline{D}(p, \rho) \underline{B}_{nl} + \underline{L}^t \hat{S} \underline{L} \right) dV_o \quad (5.39)$$

$$\underline{P} = \int \int \int_{V_o} \underline{B}_{nl}^t \underline{\varrho}(p) \underline{N}_p^t dV_o \quad (5.40)$$

Note: the residual vector δq is derived by equating the variation in the internal strain energy to the variation in the externally applied loads, thus:

$$\begin{aligned} \delta \Phi &= \delta W \\ \int \int \int_{V_o} \delta \underline{\varepsilon}_G^t \underline{S} dV_o &= \delta \underline{p}^t \underline{F} \\ \int \int \int_{V_o} \delta \underline{p}^t \underline{B}_{nl}^t \underline{S} dV_o &= \delta \underline{p}^t \underline{F} \end{aligned}$$

Therefore the variation in load, the residual load vector is given by:

$$\delta \underline{q} = \underline{F} - \int \int \int_{V_o} \underline{B}_{nl}^t \underline{S} dV_o \quad (5.41)$$

Equation 5.38 represents an element equation based on the conservation of energy (principle of virtual work). Compliance with the conservation of energy ensures that kinematic constraints and internal and external equilibrium conditions are automatically satisfied.

The total number of element equations must equal the number of unknown field parameter coefficients (nodal values). Consequently an element equation based on a pressure-displacement relationship is required. This relationship can be stated as follows.

$$\rho = -K (J - 1) = -K \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (5.42)$$

This equation represents a partial differential equation and can be solved by use of a Galerkin Weighted-Residual approach, hence:

$$\int \int \int_{V_o} \left((J - 1) + \frac{\rho}{K} \right) \delta \rho dV_o = \delta \underline{\rho}^t \int \int \int_{V_o} \underline{N}_p \left((J - 1) + \frac{\rho}{K} \right) dV_o = 0 \quad (5.43)$$

$$\equiv \delta \underline{\rho}^t \underline{f} = 0$$

This relationship holds for any virtual pressure change $\delta \underline{\rho}$ so that $\underline{f}$ represents a lack of pressure compatibility:

$$\underline{f} = \int \int \int_{V_o} \underline{N}_\rho \left((J - 1) + \frac{\rho}{K} \right) dV_o \quad (5.44)$$

Again the variational approach is taken to obtaining the variational pressure compatibility equations. Defining the variation in pressure ρ and the coordinate Jacobian J as follows:

$$\rho_{new} = \rho_{old} + d\rho \equiv \rho + d\rho \quad (5.45)$$

$$J_{new} = J_{old} + dJ \equiv J + dJ \quad (5.46)$$

Then the variation of equation 5.44 can be written as:

$$\delta \underline{f} = \int \int \int_{V_o} \underline{N}_\rho \left[\frac{\rho + d\rho}{K} + (J - 1) + dJ \right] dV_o = 0 \quad (5.47)$$

which can be written as:

$$\int \int \int_{V_o} \underline{N}_\rho \frac{d\rho}{K} dV_o + \int \int \int_{V_o} \underline{N}_\rho dJ dV_o = -\underline{f} \quad (5.48)$$

with:

$$d\rho = \underline{N}_\rho^t d\underline{\rho} \quad (5.49)$$

$$dJ = \frac{dJ}{dI_3} \frac{\partial I_3}{\partial \underline{\varepsilon}_G} d\underline{\varepsilon}_G \equiv \frac{1}{2} I_3^{-\frac{1}{2}} 2 \frac{\partial I_3}{\partial \underline{C}} d\underline{\varepsilon}_G$$

$$dJ = \frac{1}{2} \cdot 2 I_3^{-\frac{1}{2}} I_3 \underline{C}^{-1} d\underline{\varepsilon}_G$$

It was shown from equations D.57 to D.59 in appendix D that $\underline{\rho} = -I_3^{\frac{1}{2}} \underline{C}^{-1}$, therefore:

$$dJ = -\underline{\rho}^t(p) d\underline{\varepsilon}_G = -\underline{\rho}^t(p) \underline{B}_{nl} dp \quad (5.50)$$

Substituting equation 5.49 and 5.50 into equation 5.48 gives the variation of the hydrostatic pressure imbalance, thus:

$$\left(\int \int \int_{V_o} \underline{N}_\rho \underline{N}_\rho^t dV_o \right) \frac{d\rho}{K} - \left(\int \int \int_{V_o} \underline{N}_\rho \underline{\rho}^t \underline{B}_{nl} dV_o \right) dp = \delta \underline{f} \quad (5.51)$$

which can be put in the following form:

$$\frac{\partial f}{\partial \rho} \delta \underline{\rho} + \frac{\partial f}{\partial p} \delta \underline{p} = \underline{\dot{H}} \delta \underline{\rho} - \underline{P}^t \delta \underline{p} = \delta \underline{f} = 0 \quad (5.52)$$

where:

$$\underline{\dot{H}} = \frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial p} \frac{\partial p}{\partial \rho} = \int \int \int_{V_0} \frac{1}{K} \underline{N}_\rho \underline{N}_\rho^t dV_0 \quad (5.53)$$

Note: In equation 5.51 it has been assumed that the residual vector is given by $\delta \underline{f} = -\underline{f}$ for the following reason. The definition of the pressure imbalance variation is:

$$\delta \underline{f} = \underline{f}_{new} - \underline{f}_{old}$$

Thus if $\underline{f}_{new} = 0$ then $\delta \underline{f} = -\underline{f}$, which was shown to be given by equation 5.44.

5.5 Linearisation of element equations

Equation 5.52 and 5.38 represent the governing element equations for the nodal displacement $\underline{p}$ and pressures $\underline{\rho}$. Equation 5.52 and 5.38 can be re-written as follows:

$$\delta \underline{q} = \underline{K} \delta \underline{p} + \underline{P} \delta \underline{\rho} = \frac{\partial q}{\partial p} \delta \underline{p} + \frac{\partial q}{\partial \rho} \delta \underline{\rho} = 0 \quad (5.54)$$

$$\delta \underline{f} = -\underline{P}^t \delta \underline{p} + \underline{\dot{H}} \delta \underline{\rho} = \frac{\partial f}{\partial p} \delta \underline{p} + \frac{\partial f}{\partial \rho} \delta \underline{\rho} = 0 \quad (5.55)$$

Written in the above form the application of the Rayleigh-Ritz procedure becomes apparent with a system of equations formed from the variation of the field parameters $\delta \underline{p}$ and $\delta \underline{\rho}$ extremized to minimise the virtual 'out of balance' force and pressure variations $\delta \underline{q}$ and $\delta \underline{f}$.

The non-linearity of the element equations is manifest by the fact the matrix expressions $\underline{K}$, $\underline{P}$ and $\underline{\dot{H}}$ are themselves functions of the field parameters p and ρ . Consequently an iterative solution procedure based on the Newton-Raphson algorithm is required. The Newton-Raphson algorithm is based on Taylor's series for numerical evaluation of functions, but with the higher order terms disregarded. The Taylor series is of the form:

$$f(x+h) = f(x) + h \frac{\partial f}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{h^3}{3!} \frac{\partial^3 f}{\partial x^3} + \dots \quad (5.56)$$

If the higher order terms in 5.56 are ignored the resulting equation can be re-arranged to give the following expression, which summarizes the Newton-Raphson approach:

$$\delta f = f(x + h) - f(x) = h \frac{\partial f}{\partial x} \quad (5.57)$$

Thus it can be seen that in the Newton-Raphson scheme the term δf represents the error in the non-linear expression f evaluated at the previous iteration value of the field parameter x . This system of equations can be solved for h , the field parameter increment to be used in the next iteration where a new δf error value is calculated. This process is repeated until the algorithm is said to converge when usually the ratio h^2/x^2 (the square of the field parameter increment over the square of the field parameter value) reduces below a specified value.

A slightly modified Newton-Raphson scheme can be used where the external load is split up into increments. This basically means that a separate Newton-Raphson solution is run for each load step. This enables the non-linear function f to be evaluated over a range of load/displacement points.

With reference to the rubber finite element formulation the Newton-Raphson equations can be summarized as follows:

$$\begin{bmatrix} \delta \underline{q} \\ \delta \underline{f} \end{bmatrix} = \begin{bmatrix} \underline{K} & \underline{P} \\ -\underline{P}^t & \underline{\dot{H}} \end{bmatrix} \begin{bmatrix} \delta \underline{p}_1 \\ \delta \underline{\rho}_1 \end{bmatrix} \quad (5.58)$$

$$\begin{aligned} \delta \underline{q} &= \underline{K} (\underline{p} + \delta \underline{p}_0) + \underline{P} (\underline{\rho} + \delta \underline{\rho}_0) \\ \delta \underline{f} &= -\underline{P}^t (\underline{p} + \delta \underline{p}_0) + \underline{\dot{H}} (\underline{\rho} + \delta \underline{\rho}_0) \end{aligned} \quad (5.59)$$

where $\delta \underline{p}_0, \delta \underline{\rho}_0$ are the current field parameter increments and $\delta \underline{p}_1, \delta \underline{\rho}_1$ are the next field parameter increments.

5.5.1 Element equation formulation, plane stress

It was explained in section 5.4 how two sets of non-linear simultaneous equations need to be solved when dealing with the case of plane strain. This being necessary because of the two field functions for displacement and hydrostatic pressure. However for the case of plane stress the hydrostatic pressure is not independent and can be expressed in terms of the displacement field function. Therefore with

a modified constitutive stress variation matrix $\underline{K}$ (see appendix D) only the set of non-linear simultaneous equations for the displacement field variable are required. The unknowns are then simply the nodal displacements as is the case for standard linear finite element formulations.

5.6 The evaluation of the right Cauchy-Green tensor

As stated in equation 3.12 the right Cauchy-Green tensor can be expressed as:

$$\underline{C} = \underline{G}^t \underline{G} \quad (5.60)$$

Thus for the evaluation of $\underline{C}$ it is required to find the deformation gradient matrix $\underline{G}$, whose individual elements are the derivatives of the current co-ordinates with respect to the reference co-ordinates. Again with reference to chapter three $\underline{G}$ can be expressed in terms of the displacement derivative matrix $\underline{g}$ as follows:

$$\underline{G} = (\underline{g} + \underline{I}) \quad (5.61)$$

The components of $\underline{g}$ are the derivatives of the displacement with respect to the reference co-ordinates. Therefore to evaluate $\underline{C}$ it is necessary to evaluate the displacement derivative components. For two-dimensional problems it is possible to define four of the displacement derivatives as being zero, namely:

$$\frac{\partial u}{\partial z_0} = \frac{\partial v}{\partial z_0} = \frac{\partial w}{\partial x_0} = \frac{\partial w}{\partial y_0} = 0 \quad (5.62)$$

In addition, for the case of plane strain it is possible to set the $\frac{\partial w}{\partial z_0}$ derivative to zero, hence in the deformation gradient matrix it follows from 5.61 that:

$$\frac{\partial z}{\partial z_0} = 1 \quad (5.63)$$

Therefore for the plane strain two-dimensional case it is only necessary to evaluate the four X and Y component displacement derivatives. From standard finite element practice it will be remembered that the derivative of the field parameter (displacement) is obtained by summing the products of the element node values and the corresponding shape function derivatives, (with respect to Cartesian or intrinsic co-ordinates) thus:

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \sum_{i=1}^N u_i \frac{\partial N_i(x,y)}{\partial x} \\ \frac{\partial u(\xi,\eta)}{\partial \xi} &= \sum_{i=1}^N u_i \frac{\partial N_i(\xi,\eta)}{\partial \xi} \end{aligned} \quad (5.64)$$

Hence to evaluate the displacement derivatives with respect to the reference co-ordinates the shape function derivatives with respect to the reference co-ordinates are required [16]. These derivatives can be expanded using the chain rule and intrinsic co-ordinates as follows:

$$\begin{aligned}\frac{\partial N_i}{\partial x_0} &= N_{i,x_0} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x_0} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x_0} \\ \frac{\partial N_i}{\partial y_0} &= N_{i,y_0} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y_0} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y_0}\end{aligned}\quad (5.65)$$

Equation 5.65 can be re-arranged and put in matrix format as follows:

$$\begin{bmatrix} \frac{\partial x_0}{\partial \xi} & \frac{\partial y_0}{\partial \xi} \\ \frac{\partial x_0}{\partial \eta} & \frac{\partial y_0}{\partial \eta} \end{bmatrix} \begin{bmatrix} N_{i,x_0} \\ N_{i,y_0} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}\quad (5.66)$$

The intrinsic derivatives of the shape functions in 5.66 are easily defined. And hence the reference co-ordinate derivatives in 5.66 can be defined using the isoparametric interpolation as normal:

$$\begin{aligned}\frac{\partial x_0}{\partial \xi} &= \sum_{i=1}^N (x_0)_i \frac{\partial N_i}{\partial \xi} & \frac{\partial y_0}{\partial \xi} &= \sum_{i=1}^N (y_0)_i \frac{\partial N_i}{\partial \xi} \\ \frac{\partial x_0}{\partial \eta} &= \sum_{i=1}^N (x_0)_i \frac{\partial N_i}{\partial \eta} & \frac{\partial y_0}{\partial \eta} &= \sum_{i=1}^N (y_0)_i \frac{\partial N_i}{\partial \eta}\end{aligned}\quad (5.67)$$

Hence 5.66 can be solved for the unknowns N_{i,x_0} and N_{i,y_0} as follows:

$$\begin{aligned}\begin{bmatrix} N_{i,x_0} \\ N_{i,y_0} \end{bmatrix} &= \begin{bmatrix} \frac{\partial x_0}{\partial \xi} & \frac{\partial y_0}{\partial \xi} \\ \frac{\partial x_0}{\partial \eta} & \frac{\partial y_0}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \\ \begin{bmatrix} N_{i,x_0} \\ N_{i,y_0} \end{bmatrix} &= \frac{1}{a} \begin{bmatrix} \frac{\partial y_0}{\partial \eta} & -\frac{\partial y_0}{\partial \xi} \\ -\frac{\partial x_0}{\partial \eta} & \frac{\partial x_0}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}\end{aligned}\quad (5.68)$$

where:

$$a = \left[\frac{\partial x_0}{\partial \xi} \frac{\partial y_0}{\partial \eta} - \frac{\partial y_0}{\partial \xi} \frac{\partial x_0}{\partial \eta} \right]\quad (5.69)$$

Having solved for the shape function derivatives from 5.68 it is then straight forward to obtain the displacement derivatives by isoparametric interpolation.

$$\begin{aligned}\frac{\partial u}{\partial x_0} &= \sum_{i=1}^N u_i \frac{\partial N_i}{\partial x_0} & \frac{\partial u}{\partial y_0} &= \sum_{i=1}^N u_i \frac{\partial N_i}{\partial y_0} \\ \frac{\partial v}{\partial x_0} &= \sum_{i=1}^N v_i \frac{\partial N_i}{\partial x_0} & \frac{\partial v}{\partial y_0} &= \sum_{i=1}^N v_i \frac{\partial N_i}{\partial y_0}\end{aligned}\quad (5.70)$$

For plane stress however the $\frac{\partial w}{\partial z_0}$ component is not zero and the following approach is required for its evaluation. From the definition of J as the determinant of the deformation gradient matrix $\underline{G}$ it can be deduced that:

$$J = |\underline{G}| = G_{33} [G_{11}G_{22} - G_{12}G_{21}] \quad (5.71)$$

If total incompressibility is assumed then $J = 1$, hence the G_{33} component can be solved from equation 5.71.

Chapter 6

Non-linear boundary element modelling of rubber

6.1 Introduction

In the FEM the element equations were derived starting from a variational equation formed using the principle of virtual work. Trial field parameter solutions were then placed in the variational equation to obtain a residual expression. The unknown coefficients in the trial function were then obtained by extremising the resulting residual equation. The trial functions were chosen so that the boundary conditions were satisfied before hand. In order to reduce the order of the trial solution the domain of the system was discretised into elements. This resulted in a system of element equations, one for each nodal degree of freedom.

In the BEM the governing equation used for elasticity problems is the equation of internal equilibrium. Again a trial function is used in the equation of equilibrium to give a residual expression, but this time the trial solution is chosen so that the equation of equilibrium is satisfied in the domain of the system before hand. The unknowns then become the coefficients of the trial solution when at the boundary of the system. The residual equation is then multiplied by a weighting function (the fundamental solution in the case of the Galerkin weighted residual) and integrated over the boundary. The resulting weighted residual expression is then equated to zero to solve for the unknown coefficients. In order to reduce the order of the trial solution used the boundary is discretised into elements as in the finite element method. The unknowns then become the nodal displacements and tractions on the boundary elements. Again this results in a system of BIEs, one for each nodal degree of freedom (displacement or traction) on the boundary.

6.2 Theory

6.2.1 The basic non-regularized BIE for elastomers

The starting point for the derivation of the non-linear BIE is the equation of internal equilibrium referred to undeformed co-ordinates (see appendix C). Note f_α represents the domain loading in the following equation.

$$\sum_{\beta=1}^2 \frac{\partial \sigma_{\beta\alpha}}{\partial x_\beta} + f_\alpha = 0 \quad (6.1)$$

The stress in 6.1 is the first Piola-Kirchhoff stress or total stress and can be split into linear and non-linear components such that:

$$\underline{\sigma} = \underline{\sigma}_{nl} + \underline{\sigma}_l \quad (6.2)$$

Then the equilibrium equation can now be written in terms of linear and non-linear stresses, with the non-linear stresses taking on the appearance of domain loading terms. Hence:

$$\sum_{\beta=1}^2 \frac{\partial \sigma_{\beta\alpha}^l}{\partial x_\beta} + f_\alpha^{nl} + f_\alpha = 0 \quad (6.3)$$

where:

$$f_\alpha^{nl} = \sum_{\beta=1}^2 \frac{\partial \sigma_{\beta\alpha}^{nl}}{\partial x_\beta} \quad (6.4)$$

Expanding out equation 6.3 and using the weighted residual approach to form the BIE. Thus using the two arbitrary weighting functions u^* and v^* :

$$\iint_{\Omega} \left[u^* \left(\frac{\partial \sigma_x^l}{\partial x} + \frac{\partial \tau_{yx}^l}{\partial y} + f_x^{nl} + f_x \right) + v^* \left(\frac{\partial \tau_{xy}^l}{\partial x} + \frac{\partial \sigma_y^l}{\partial y} + f_y^{nl} + f_y \right) \right] d\Omega = 0 \quad (6.5)$$

Obtaining the weak form of equation 6.5 (derivative on the field parameter reduced by an order of one) by using integration by parts gives:

$$\begin{aligned} & \oint_{\Gamma} [u^* (l\sigma_x^l + m\tau_{xy}^l) + v^* (l\tau_{xy}^l + m\sigma_y^l)] d\Gamma - \\ & \iint_{\Omega} \left[\sigma_x^l \frac{\partial u^*}{\partial x} + \tau_{yx}^l \frac{\partial u^*}{\partial y} + \tau_{xy}^l \frac{\partial v^*}{\partial x} + \sigma_y^l \frac{\partial v^*}{\partial y} \right] d\Omega + \\ & \iint_{\Omega} [u^* f_x^{nl} + v^* f_y^{nl}] d\Omega + \iint_{\Omega} [u^* f_x + v^* f_y] d\Omega = 0 \end{aligned} \quad (6.6)$$

Writing:

$$T_x^l = l\sigma_x^l + m\tau_{yx}^l \quad (6.7)$$

$$T_y^l = l\tau_{xy}^l + m\sigma_y^l \quad (6.8)$$

$$\underline{\varepsilon}^* = (\varepsilon_x^* \ \varepsilon_y^* \ \gamma_{xy}^*) \quad (6.9)$$

$$\underline{\sigma}^l = (\sigma_x^l \ \sigma_y^l \ \tau_{xy}^l) \quad (6.10)$$

where:

$$\varepsilon_x^* = \frac{\partial u^*}{\partial x} \quad \varepsilon_y^* = \frac{\partial v^*}{\partial y} \quad \gamma_{xy}^* = \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}$$

therefore equation 6.6 can be written as:

$$\begin{aligned} & - \iint_{\Omega} (\underline{\varepsilon}^*)^t \underline{\sigma}^l d\Omega + \oint_{\Gamma} (T_x^l u^* + T_y^l v^*) d\Gamma + \\ & \iint_{\Omega} (f_x^{nl} u^* + f_y^{nl} v^*) d\Omega + \iint_{\Omega} [u^* f_x + v^* f_y] d\Omega = 0 \end{aligned} \quad (6.11)$$

For the linear $\underline{\sigma}^l$ in the absence of initial stress and strain:

$$(\underline{\varepsilon}^*)^t \underline{\sigma}^l = (\underline{\varepsilon}^*)^t \underline{D}^l \underline{\varepsilon}^l = (\underline{\sigma}^*)^t \underline{\varepsilon}^l \quad (6.12)$$

where:

$$\underline{\sigma}^* = (\sigma_x^* \ \sigma_y^* \ \tau_{xy}^*)$$

Thus the first term in equation 6.11 can be re-written as:

$$\begin{aligned} & \iint_{\Omega} (\underline{\varepsilon}^*)^t \underline{\sigma}^l d\Omega = \iint_{\Omega} (\underline{\sigma}^*)^t \underline{\varepsilon}^l d\Omega \\ & \equiv \iint_{\Omega} \left[\sigma_x^* \frac{\partial u}{\partial x} + \tau_{xy}^* \frac{\partial u}{\partial y} + \tau_{xy}^* \frac{\partial v}{\partial x} + \sigma_y^* \frac{\partial v}{\partial y} \right] d\Omega \end{aligned} \quad (6.13)$$

Using integration by parts equation 6.13 can be written as:

$$\begin{aligned} & \iint_{\Omega} (\underline{\varepsilon}^*)^t \underline{\sigma}^l d\Omega = \oint_{\Gamma} (T_x^* u + T_y^* v) d\Gamma \\ & - \iint_{\Omega} \left[u \left(\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{xy}^*}{\partial y} \right) + v \left(\frac{\partial \tau_{xy}^*}{\partial y} + \frac{\partial \sigma_y^*}{\partial x} \right) \right] d\Omega \end{aligned} \quad (6.14)$$

where:

$$T_x^* = l\sigma_x^* + m\tau_{yx}^* \quad (6.15)$$

$$T_y^* = l\tau_{xy}^* + m\sigma_y^* \quad (6.16)$$

Substituting equation 6.14 into equation 6.11 gives the inverse weighted residual. The equation is an inverse weighted residual because the derivatives have been removed totally from the field parameters u and v , thus:

$$\oint_{\Gamma} (T_x^l u^* + T_y^l v^*) d\Gamma - \oint_{\Gamma} (T_x^* u + T_y^* v) d\Gamma + \iint_{\Omega} (f_x^{nl} u^* + f_y^{nl} v^*) d\Omega + \iint_{\Omega} [u^* f_x + v^* f_y] d\Omega - \iint_{\Omega} \left[u \left(\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{yx}^*}{\partial y} \right) + v \left(\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} \right) \right] d\Omega = 0 \quad (6.17)$$

Taking another set of equilibrium equations including fundamental loading terms:

$$\sum_{\beta=1}^2 \frac{\partial \sigma_{\beta\alpha}^*}{\partial x_{\beta}} + f_{\alpha}^* = 0 \quad (6.18)$$

Thus the last domain term in equation 6.17 can be written as:

$$\iint_{\Omega} \left[u \left(\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{yx}^*}{\partial y} \right) + v \left(\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} \right) \right] d\Omega = - \iint_{\Omega} [u f_x^* + v f_y^*] d\Omega \quad (6.19)$$

The fundamental loading terms are defined as follows:

$$f_x^* = e_x \delta(x - x_i, y - y_i) \quad (6.20)$$

$$f_y^* = e_y \delta(x - x_i, y - y_i) \quad (6.21)$$

where e_x and e_y are the applied force per unit thickness at the source point (x_i, y_i) . Using the following properties of the Dirac delta function:

$$\iint_{\Omega} u(x, y) \delta(x - x_i, y - y_i) d\Omega = C_i u(x_i, y_i) \quad (6.22)$$

where

$C_i = 1$ if the source point (x_i, y_i) is inside the domain Ω .

$C_i = 0$ if the source point (x_i, y_i) is outside the domain Ω .

$C_i = 0.5$ if the source point (x_i, y_i) is on a smooth continuous boundary Γ .

$C_i = \frac{\alpha_i}{2\pi}$ if the source point (x_i, y_i) is on a corner of angle α_i on the boundary Γ enclosing the domain Ω ,

hence substituting equations 6.20 and 6.21 into equation 6.19 and then putting the result into equation 6.17 gives the non-linear BIE.

$$C_i u_i e_x + C_i v_i e_y + \oint_{\Gamma} (T_x^* u + T_y^* v) d\Gamma = \oint_{\Gamma} (T_x^l u^* + T_y^l v^*) d\Gamma + \iint_{\Omega} (f_x^{nl} u^* + f_y^{nl} v^*) d\Omega + \iint_{\Omega} [u^* f_x + v^* f_y] d\Omega \quad (6.23)$$

The fundamental solution parameters T_x^*, T_y^*, u^* and v^* are functions of the distance between the source and field points. It can be shown that the fundamental solutions can be expressed in terms of kernel functions and the fundamental load components. The exact form of the kernel functions is given in appendix B.

$$u_\alpha^*(x - x_i, y - y_i) = \sum_{\beta=1}^2 G_{\alpha\beta}(x - x_i, y - y_i) e_\beta \quad (6.24)$$

$$T_x^* = F_{11}e_x + F_{12}e_y \quad (6.25)$$

$$T_y^* = F_{21}e_x + F_{22}e_y \quad (6.26)$$

where for $\alpha = 1, 2$ then $u_1 = u$ and $u_2 = v$.

Noting the arbitrary nature of the e_x and e_y components then for a single direction component equation 6.23 can be written as:

$$\begin{aligned} C_i u_{\alpha i} + \sum_{\beta=1}^2 \oint_{\Gamma} F_{\beta\alpha} u_\beta d\Gamma = \\ \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} T_\beta^l d\Gamma + \sum_{\beta=1}^2 \iint_{\Omega} G_{\beta\alpha} f_\beta^{nl} d\Omega + U_\alpha \end{aligned} \quad (6.27)$$

where the domain loading is represented by U_α :

$$U_\alpha = \sum_{\beta=1}^2 \iint_{\Omega} f_\alpha G_{\beta\alpha} d\Omega$$

A further simplification can be achieved if the non-linear domain term is integrated by parts, thus:

$$\begin{aligned} \sum_{\beta=1}^2 \iint_{\Omega} G_{\beta\alpha} f_\beta^{nl} d\Omega \equiv \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[G_{\gamma\alpha} \frac{\partial \sigma_{\gamma\beta}^{nl}}{\partial x_\beta} \right] d\Omega \equiv \\ \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} T_\beta^{nl} d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[\sigma_{\beta\gamma}^{nl} \frac{\partial G_{\gamma\alpha}}{\partial x_\beta} \right] d\Omega \end{aligned} \quad (6.28)$$

Substituting equation 6.28 into 6.27 and noting that:

$$T_x = T_x^l + T_x^{nl} = l\sigma_x + m\tau_{xy} \quad (6.29)$$

$$T_y = T_y^l + T_y^{nl} = l\tau_{xy} + m\sigma_y \quad (6.30)$$

then:

$$\begin{aligned}
& C_i u_{\alpha i} + \sum_{\beta=1}^2 \oint_{\Gamma} F_{\beta\alpha} u_{\beta} d\Gamma = \\
& \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} T_{\beta} d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[\sigma_{\beta\gamma}^{nl} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \right] d\Omega + U_{\alpha}
\end{aligned} \tag{6.31}$$

Equation 6.31 is the basic non-regularized BIE of displacement. The singularities present in this BIE as the field point becomes coincident with the source point are as follows. The first boundary integral term contains a $1/r$ strong singularity. The second boundary integral term contains a weak $\log(r)$ singularity and the final domain integral term contains both a $1/r$ strong singularity and a $1/r^2$ hyper-singularity. The $1/r^2$ hyper-singularity occurs as a consequence of the non-linear stress components which are generated by displacement derivative BIEs. These displacement derivative BIEs are discussed in section 6.2.3 of this report. Note that the domain integral U_{α} representing the domain loading, such as that found for the test case of a rotating disk, can be reduced to a boundary integral as explained in appendix G. Note:

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{6.32}$$

where (x, y) is the field point and (x_i, y_i) is the source point.

6.2.2 Displacement BIE regularization

Before the equation stated by 6.31 can be used in any computer algorithm some means of making the various singularities calculable needs to be found. The $\log(r)$ singularity in the second domain term can be dealt with by means of logarithmic quadrature (see appendix F). The $1/r$ strong singularity contained in the second boundary integral term in the $F_{\beta\alpha}$ kernel function can be removed by means of the Cauchy principal value theorem [28], which can be stated as follows:

$$\oint_{\Gamma} F_{\beta\alpha} u_{\beta} d\Gamma = \lim_{\epsilon \rightarrow 0} \left[\oint_{\Gamma - \Gamma_{\epsilon}} F_{\beta\alpha} u_{\beta} d\Gamma + \oint_{\Gamma_{\epsilon}} F_{\beta\alpha} u_{\beta} d\Gamma \right] \quad (6.33)$$

where Γ_{ϵ} represents the part of a small circle with radius ϵ center (x_i, y_i) crossing the boundary around (x_i, y_i) as shown in figure 6.1 below.

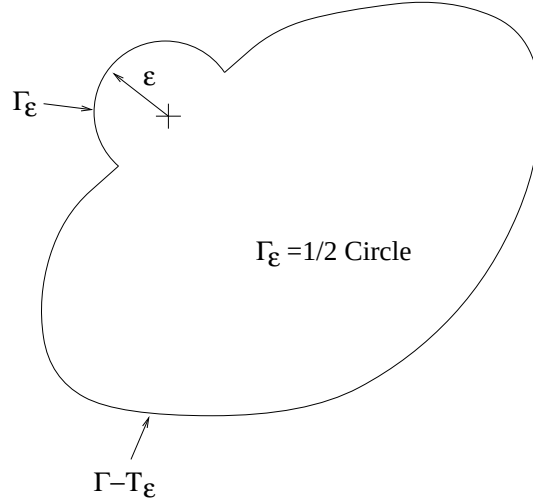


Figure 6.1: Cauchy integration region on boundary.

The following notation is used:

$$\lim_{\epsilon \rightarrow 0} \oint_{\Gamma - \Gamma_{\epsilon}} F_{\beta\alpha} u_{\beta} d\Gamma \equiv \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma \quad (6.34)$$

which means the contour integral without the Cauchy principal value, which is defined from:

$$\lim_{\epsilon \rightarrow 0} \oint_{\Gamma_{\epsilon}} F_{\beta\alpha} u_{\beta} d\Gamma \quad (6.35)$$

Using the 'mean value theorem' equation 6.35 can be written as:

$$\lim_{\epsilon \rightarrow 0} \oint_{\Gamma_{\epsilon}} F_{\beta\alpha} u_{\beta} d\Gamma \equiv u_{\beta_i} \left(\lim_{\epsilon \rightarrow 0} \oint_{\Gamma_{\epsilon}} F_{\beta\alpha} d\Gamma \right) \equiv \Upsilon_{\beta\alpha_i} u_{\beta_i} \quad (6.36)$$

where $\Upsilon_{\beta\alpha_i}$ is the Cauchy principal value defined as:

$$\Upsilon_{\beta\alpha_i} = \lim_{\epsilon \rightarrow 0} \oint_{\Gamma_\epsilon} F_{\beta\alpha} d\Gamma \quad (6.37)$$

hence:

$$\oint_{\Gamma} F_{\beta\alpha} u_{\beta} d\Gamma = \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma + \Upsilon_{\beta\alpha_i} u_{\beta_i} \quad (6.38)$$

Thus substituting equation 6.38 into equation 6.31 gives:

$$\begin{aligned} & \sum_{\beta=1}^2 [C_i \delta_{\alpha\beta} + \Upsilon_{\beta\alpha_i}] u_{\beta_i} + \sum_{\beta=1}^2 \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma = \\ & \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} T_{\beta} d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[\sigma_{\beta\gamma}^{nl} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \right] d\Omega + U_{\alpha} \end{aligned} \quad (6.39)$$

Then defining:

$$C_{\alpha\beta_i} = C_i \delta_{\alpha\beta} + \Upsilon_{\beta\alpha_i} \quad (6.40)$$

then equation 6.39 can be written as:

$$\begin{aligned} & \sum_{\beta=1}^2 C_{\alpha\beta_i} u_{\beta_i} + \sum_{\beta=1}^2 \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma = \\ & \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} T_{\beta} d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[\sigma_{\beta\gamma}^{nl} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \right] d\Omega + U_{\alpha} \end{aligned} \quad (6.41)$$

If the condition of rigid translation is applied then:

$$u_{\beta}(x, y) = C_{\beta}$$

$$T_{\beta} = f_{\beta} = \sigma_{\alpha\beta}^{nl} = 0$$

hence:

$$\begin{aligned} & \sum_{\beta=1}^2 \left(C_{\alpha\beta_i} + \oint_{\Gamma'} F_{\beta\alpha} d\Gamma \right) C_{\beta} = 0 \\ & C_{\alpha\beta_i} = - \oint_{\Gamma'} F_{\beta\alpha} d\Gamma \end{aligned} \quad (6.42)$$

The Cauchy principal value term $\Upsilon_{\beta\alpha_i}$ in equation 6.40 can be shown [12] to be zero for source points inside the problem domain or on a smooth boundary. However the Cauchy principal value for corner points is not zero and needs to be taken into account in equation 6.40. Fortunately this is easily achieved by utilizing equation 6.42 to evaluate the full modified Dirac Delta constant. The domain integral term in equation 6.41 contains a $1/r$ singularity. The singularity will not occur unless the source point is on an integration cell. The method of regularization is as follows. Applying the following theorem:

$$\iint_{\Omega} \frac{\partial G_{\alpha\beta}}{\partial x_{\delta}} f(x, y) d\Omega = \iint_{\Omega} \frac{\partial G_{\alpha\beta}}{\partial x_{\delta}} (f - f_i) d\Omega + \iint_{\Omega} \frac{\partial G_{\alpha\beta}}{\partial x_{\delta}} f_i d\Gamma \quad (6.43)$$

The second domain term in equation 6.41 can be written as:

$$\begin{aligned} & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \sigma_{\beta\gamma}^{nl} d\Omega = \\ & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega + \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} (\sigma_{\beta\gamma}^{nl})_i d\Omega \right] \end{aligned} \quad (6.44)$$

Then applying integration by parts:

$$\begin{aligned} & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \sigma_{\beta\gamma}^{nl} d\Omega = \\ & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega + \oint_{\Gamma} G_{\gamma\alpha} l_{\beta} (\sigma_{\beta\gamma}^{nl})_i d\Gamma \right] \end{aligned} \quad (6.45)$$

then defining:

$$\bar{T}_x^{nl} = l (\sigma_x^{nl})_i + m (\tau_{yx}^{nl})_i \quad (6.46)$$

$$\bar{T}_y^{nl} = l (\tau_{xy}^{nl})_i + m (\sigma_y^{nl})_i \quad (6.47)$$

These are not the same as $(T_x^{nl})_i$ and $(T_y^{nl})_i$ which are defined as follows:

$$(T_x^{nl})_i = l_i (\sigma_x^{nl})_i + m_i (\tau_{yx}^{nl})_i \quad (6.48)$$

$$(T_y^{nl})_i = l_i (\tau_{xy}^{nl})_i + m_i (\sigma_y^{nl})_i \quad (6.49)$$

Hence equation 6.45 can be written as:

$$\begin{aligned} & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \sigma_{\beta\gamma}^{nl} = \\ & \left[\sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega + \sum_{\gamma=1}^2 \oint_{\Gamma} G_{\gamma\alpha} \bar{T}_{\gamma}^{nl} d\Gamma \right] \end{aligned} \quad (6.50)$$

Enabling equation 6.41 to be written as:

$$\sum_{\beta=1}^2 C_{\alpha\beta i} u_{\beta i} + \sum_{\beta=1}^2 \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma = \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} [T_{\beta} - \bar{T}_{\beta}^{nl}] d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + U_{\alpha} \quad (6.51)$$

Equation 6.51 represents a fully regularized form of the displacement BIE (assuming the non-linear stresses in the last domain term are known). The second boundary integral term can be solved by logarithmic quadrature. The final domain term can be shown to have no singularity if the Cauchy principal value theorem is taken, hence:

$$\sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega = \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega - \Omega_{\varepsilon}} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + \iint_{\Omega_{\varepsilon}} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega \right] \quad (6.52)$$

Applying integration by parts and the mean value theorem to the last term in 6.52 then:

$$\sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega = \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega - \Omega_{\varepsilon}} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] \oint_{\Gamma_{\varepsilon}} G_{\gamma\alpha} l_{\beta} d\Gamma \right] \quad (6.53)$$

It can be shown (Appendix B) that the kernel function $G_{\gamma\alpha}$ in equation 6.53 is of the form:

$$G_{\gamma\alpha} = \frac{1}{8\pi\mu(1-p)} [[C - (3-4p) \log r] \delta_{\gamma\alpha} + l_{\gamma} l_{\alpha}] \quad (6.54)$$

Given:

$$d\Gamma = r d\theta$$

then:

$$\lim_{\varepsilon \rightarrow 0} \oint_{\varepsilon} G_{\gamma\alpha} l_{\delta} d\Gamma = \lim_{\varepsilon \rightarrow 0} \oint_{\theta_1}^{\theta_2} \frac{l_{\delta}}{8\pi\mu(1-p)} [[\varepsilon - (3-4p)\varepsilon \log \varepsilon] \delta_{\gamma\alpha} + \varepsilon l_{\gamma} l_{\alpha}] d\theta \quad (6.55)$$

where θ_1 and θ_2 are the values of θ at the starting and ending points defining Γ_ε . It can be shown using L'hospital's rule that:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \varepsilon \rightarrow 0$$

Hence it can be deduced that:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} G_{\gamma\alpha} l_\beta d\Gamma \rightarrow 0 \tag{6.56}$$

Therefore it can be seen that the domain integral in equation 6.51 has no Cauchy principal value and is regularized in the form shown by equation 6.51.

6.2.3 Regularized displacement derivative BIE

The regularized displacement derivative form of equation 6.51 will contain two $1/r^2$ hyper-singularity integral terms when the derivative is taken. The first will be for the first boundary integral term involving the kernel function derivative $\frac{\partial F_{\beta\alpha}}{\partial x_\delta}$. The second will be due to the domain integral term involving the double derivative $\frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta}$ kernel which generates the $1/r^2$ singularity. Taking the singularity due to the derivative kernel $\frac{\partial F_{\beta\alpha}}{\partial x_\delta}$ first, if equation 6.42 is substituted into equation 6.51 then the following expression is obtained:

$$\begin{aligned} & \sum_{\beta=1}^2 \oint_{\Gamma'} F_{\beta\alpha} (u_\beta - u_{\beta i}) d\Gamma = \\ & \sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} [T_\beta - \bar{T}_\beta^{nl}] d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_\beta} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + U_\alpha \end{aligned} \quad (6.57)$$

Note that the Cauchy principal value of the first boundary integral term vanishes. Differentiating equation 6.57 with respect to $x_{\delta i}$ involves the following:

$$\begin{aligned} & \frac{\partial}{\partial x_{\delta i}} [F_{\beta\alpha} (x - x_i, y - y_i) [u_\beta (x, y) - u_\beta (x_i, y_i)]] \equiv \\ & \frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}} [u_\beta - u_{\beta i}] - F_{\beta\alpha} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} \equiv -\frac{\partial F_{\beta\alpha}}{\partial x_\delta} [u_\beta - u_{\beta i}] - F_{\beta\alpha} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} \end{aligned} \quad (6.58)$$

Noticing that for a source point (x_i, y_i) on an internal domain point:

$$-\oint_{\Gamma'} F_{\beta\alpha} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} d\Gamma \equiv -\frac{\partial u_{\beta i}}{\partial x_{\delta i}} \oint_{\Gamma'} F_{\beta\alpha} d\Gamma \equiv C_{\beta\alpha i} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} \equiv \delta_{\beta\alpha} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} \quad (6.59)$$

Thus using equation 6.59 and differentiating equation 6.57 it can be deduced that:

$$\begin{aligned} & \sum_{\beta=1}^2 C_{\beta\alpha i} \frac{\partial u_{\beta i}}{\partial x_{\delta i}} = \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_\delta} (u_\beta - u_{\beta i}) d\Gamma - \\ & \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_\delta} [T_\beta - \bar{T}_\beta^{nl}] d\Gamma + \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + E_{\alpha\delta} \end{aligned} \quad (6.60)$$

where:

$$E_{\alpha\delta} = \frac{\partial U_\alpha}{\partial x_\delta}$$

which, though not obvious, is a fully regularized displacement derivative BIE. The first boundary integral term has been shown experimentally (by developed software)

to be regularized for the $1/r^2$ singularity in the $\frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}}$ kernel function as long as the problem boundary nodes and post solution BIE source points are coincident. This is achieved by the $(u_\beta - u_{\beta i})$ factor cancelling out the singularity when the source and field points are coincident. The second boundary integral in equation 6.60 has a $1/r$ singularity in the $\frac{\partial G_{\beta\alpha}}{\partial x_{\delta i}}$ kernel function and can be shown to have no Cauchy principal value and hence be regularized. Consider the simplified boundary integral:

$$\oint_{\Gamma} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} d\Gamma \quad (6.61)$$

Applying the Cauchy principal value theorem to equation 6.61:

$$\oint_{\Gamma} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} d\Gamma = \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Gamma - \Gamma_{\varepsilon}} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} d\Gamma + \oint_{\Gamma_{\varepsilon}} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} d\Gamma \right] \quad (6.62)$$

Extracting the Cauchy principal value term and assuming $r \rightarrow \varepsilon$ it can be deduced that:

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Gamma_{\varepsilon}} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} d\Gamma = \lim_{\varepsilon \rightarrow 0} \oint_{\theta_1}^{\theta_1 + 2\pi} \frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} \varepsilon d\theta \quad (6.63)$$

From equation B.76 (but taking the derivative w.r.t a field point) in appendix B:

$$\frac{\partial G_{\alpha\beta}}{\partial x_{\gamma}} = \frac{1}{8\pi\mu(1-p)r} \left[\frac{\partial r}{\partial x_{\alpha}} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_{\beta}} \delta_{\alpha\gamma} - 2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial x_{\gamma}} - [3 - 4p] \frac{\partial r}{\partial x_{\alpha}} \delta_{\alpha\beta} \right]$$

And noting:

$$\begin{aligned} x &= x_i + r \cos \theta & y &= y_i + r \sin \theta \\ \frac{\partial r}{\partial x} &\equiv l \equiv \cos \theta & \frac{\partial r}{\partial y} &\equiv m \equiv \sin \theta \end{aligned}$$

It can be shown that equation 6.63 evaluates to zero for a source point on a smooth boundary or inside the problem domain where the integration limits are $\theta = 0 \rightarrow \pi$ and $\theta = 0 \rightarrow 2\pi$ respectively. Note that by a similar proof to that given in section 6.2.2 for equation 6.56 the domain integral term in equation 6.60 can be shown to have no Cauchy principal value and therefore has its hyper-singularity removed. Consider the final domain term in equation 6.60 and apply the Cauchy principal value theorem:

$$\begin{aligned} & \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_{\delta} \partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega = \\ & \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega - \Omega_{\varepsilon}} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_{\delta} \partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega + \iint_{\Omega_{\varepsilon}} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_{\delta} \partial x_{\beta}} \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] d\Omega \right] \end{aligned}$$

(6.64)

Then just considering the last domain integral in equation 6.64 and applying the mean value theorem followed by integration by parts the following is obtained:

$$\left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] \lim_{\varepsilon \rightarrow 0} \oint_{\Gamma_\varepsilon} l_\delta \frac{\partial G_{\gamma\alpha}}{\partial x_\delta} d\Gamma \equiv \left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right] \lim_{\varepsilon \rightarrow 0} \oint_0^{2\pi} l_\delta \frac{\partial G_{\gamma\alpha}}{\partial x_\delta} \varepsilon d\theta \quad (6.65)$$

Hence when the source and field points are coincident the $1/\varepsilon$ singularity is cancelled by the ε term as well as the $\left[\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i \right]$ factor. A slightly different approach can be taken to regularizing the final domain integral term starting from the following partially regularized derivative BIE:

$$\begin{aligned} & \sum_{\beta=1}^2 C_{\alpha\beta i} \frac{\partial u_\beta}{\partial x_{\delta i}} + \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}} (u_\beta - u_{\beta i}) d\Gamma = \\ & - \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_\delta} T_\beta d\Gamma + \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \left[\sigma_{\beta\gamma}^{nl} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} \right] d\Omega + E_{\alpha\delta} \end{aligned} \quad (6.66)$$

The Cauchy principal value theorem can be applied to the final domain term resulting in:

$$\begin{aligned} & \sum_{\beta=1}^2 C_{\alpha\beta i} \frac{\partial u_\beta}{\partial x_{\delta i}} + \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}} (u_\beta - u_{\beta i}) d\Gamma = \\ & - \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_\delta} T_\beta d\Gamma + E_{\alpha\delta} \\ & \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega - \Omega_\varepsilon} \sigma_{\beta\gamma}^{nl} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} d\Omega + \iint_{\Omega_\varepsilon} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} (\sigma_{\beta\gamma}^{nl})_i d\Omega \right] \end{aligned} \quad (6.67)$$

Using the mean value theorem and integration by parts the last domain term in equation 6.67 can be simplified giving:

$$\begin{aligned} & \sum_{\beta=1}^2 C_{\alpha\beta i} \frac{\partial u_\beta}{\partial x_{\delta i}} + \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}} (u_\beta - u_{\beta i}) d\Gamma = \\ & - \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_\delta} T_\beta d\Gamma + E_{\alpha\delta} + \\ & \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega - \Omega_\varepsilon} \sigma_{\beta\gamma}^{nl} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} d\Omega + (\sigma_{\beta\gamma}^{nl})_i \oint_{\Gamma_\varepsilon} \frac{\partial G_{\gamma\alpha}}{\partial x_\beta} l_\delta d\Gamma \right] \end{aligned} \quad (6.68)$$

where:

$$l_\delta = \frac{\partial r}{\partial x_\delta} \quad (6.69)$$

Defining:

$$J_{\gamma\alpha\delta\beta} = \frac{\partial^2 G_{\gamma\alpha}}{\partial x_\delta \partial x_\beta} \quad (6.70)$$

$$T_{\gamma\alpha\delta\beta} = \frac{\partial G_{\gamma\alpha}}{\partial x_\beta} l_\delta \quad (6.71)$$

$$\iint_{\Omega'} = \iint_{\Omega - \Omega_\varepsilon} \quad (6.72)$$

enabling equation 6.68 to be written as:

$$\begin{aligned} \sum_{\beta=1}^2 C_{\alpha\beta i} \frac{\partial u_\beta}{\partial x_{\delta i}} + \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta i}} (u_\beta - u_{\beta i}) d\Gamma = \\ - \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_\delta} T_\beta d\Gamma + E_{\alpha\delta} + \\ \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega'} \sigma_{\beta\gamma}^{nl} J_{\gamma\alpha\delta\beta} d\Omega + (\sigma_{\beta\gamma}^{nl})_i \oint_{\Gamma_\varepsilon} T_{\gamma\alpha\delta\beta} d\Gamma \right] \end{aligned} \quad (6.73)$$

The domain loading domain integral $E_{\alpha\beta}$ can be reduced to a boundary integral as discussed in appendix G.

6.3 Program implementation

6.3.1 Numerical evaluation of the non-linear domain terms in the BIE

In the preceding process it will be required to evaluate the domain terms in the BIE of displacement, equation 6.51, and the displacement derivative BIE, equation 6.60. The evaluation of these terms utilises techniques borrowed from the finite element method. The domain enclosed by the boundary is discretized into domain cells. Consequently the domain terms can be evaluated using Gauss quadrature for each of the domain cells that make up the total problem domain. The process is summarized by the following two equations.

$$U_{\alpha i} = - \sum_{c=1}^{n_c} \left(\sum_{l=1}^{N_q} \sum_{m=1}^{N_q} \left[\sum_{\gamma=1}^2 \sum_{\beta=1}^2 \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} (\sigma_{\beta\gamma}^{nl}(\varepsilon, \eta)) \right] |J| W_l W_m \right) \quad (6.74)$$

$$\frac{\partial U_{\alpha i}}{\partial x_{\delta}} = - \sum_{c=1}^{n_c} \left(\sum_{l=1}^{N_q} \sum_{m=1}^{N_q} \left[\sum_{\gamma=1}^2 \sum_{\beta=1}^2 J_{\gamma\alpha\delta\beta} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] \right] |J| W_l W_m \right) \quad (6.75)$$

The non-linear stresses at each Gauss point are evaluated from the difference between the 1st PK stress and the linear stress (obtained from the product of the total strain and the linear constitutive matrix). It can be shown (see appendix E) that the free term coefficients in equation 6.68 are constants and have the following values. In table 6.1 μ is the shear modulus and p is the modified Poisson's ratio (plane stress or strain).

Table 6.1: Displacement derivative free terms.

$T_{1111} = -\frac{(5-8p)}{16\mu(1-p)}$	$T_{1211} = 0$	$T_{2111} = 0$	$T_{2211} = -\frac{(7-8p)}{16\mu(1-p)}$
$T_{1112} = 0$	$T_{1212} = \frac{1}{16\mu(1-p)}$	$T_{2112} = \frac{1}{16\mu(1-p)}$	$T_{2212} = 0$
$T_{1121} = 0$	$T_{1221} = \frac{1}{16\mu(1-p)}$	$T_{2121} = \frac{1}{16\mu(1-p)}$	$T_{2122} = 0$
$T_{1122} = -\frac{(7-8p)}{16\mu(1-p)}$	$T_{1222} = 0$	$T_{2122} = 0$	$T_{2222} = -\frac{(5-8p)}{16\mu(1-p)}$

6.3.2 Solution algorithm

It is possible to discretise equation 6.68 into the following form:

$$\Delta \underline{u} = \underline{B} \underline{u} + \underline{D} \underline{t} + \underline{g}(\Delta \underline{u}, \rho) \quad (6.76)$$

If N is the number of source points (nodes) and M the number of domain cells then the matrices $\underline{B}$ and $\underline{D}$ have a dimension $4M \times 2N$ and the vector $\underline{g}$ the dimension $4 \times M$. The dimensions of the $\underline{B}$ and $\underline{D}$ matrices are $4M \times 2N$ because there are 4 derivative components and two displacement components. The discretized form of equation 6.51 can be written as:

$$\underline{H} \underline{u} = \underline{G} \underline{t} + \underline{f}(\Delta \underline{u}, \rho) \quad (6.77)$$

Thus $\underline{H}$ and $\underline{G}$ are $2N \times 2N$ matrices and $\underline{f}$ is a $2N$ vector. By applying prescribed boundary conditions to equation 6.77 it can be re-written as:

$$\underline{A} \underline{y} = \underline{C} + \underline{f}(\Delta \underline{u}, \rho) \quad (6.78)$$

It can be seen that the BIE involves a non-linear term, which is a function of the displacement derivatives and the hydrostatic pressure. With the above equations it is possible to formulate the iterative solution procedure for non-linear boundary element analysis as follows. From equation 6.77 (the discretized system of boundary integral equations) it can be seen that non-linear terms are involved which are a function of the displacement derivatives and hydrostatic pressure. The dependence of this term on the hydrostatic pressure and the displacement derivatives stems from the term's involvement of the first Piola-Kirchhoff stress (total stress) in its derivative. These non-linear terms can be seen to augment the force/traction vector $\underline{C}$ in equation 6.78, the reduced system of BIEs after prescribed boundary conditions have been applied. Thus a solution to equation 6.78 must involve an iterative solution procedure.

As with most iterative solution procedures the first step is to solve the equivalent linear problem to obtain an initial estimate of the solution. This involves solving equation 6.78 but ignoring the domain term with the non-linear stress. The solution vector to this equation can then be used to evaluate the displacement derivatives and hydrostatic pressures to use in the full non-linear BIE (equation 6.78 including the non-linear term).

Obviously this first solution vector, and subsequent solution vectors before convergence, will not fully satisfy the full non-linear set of BIE and a residual or error value will result if the solution vector is substituted back into equation 6.78. This residual can be reduced if a new set of hydrostatic pressures and stresses are calculated based on the previous iteration results. These values can then be used to

compute new non-linear domain terms to augment the standard linear BIE force vector. The solution to this set of equations will result in an improved vector of total nodal displacements and tractions. The difference between the new solution vector and the old one gives the iterative improvement in the solution vector. This process is repeated until the ratio of the square of the solution vector increment over the square of the total solution vector results in a value less than a given permissible (see figure 6.2).

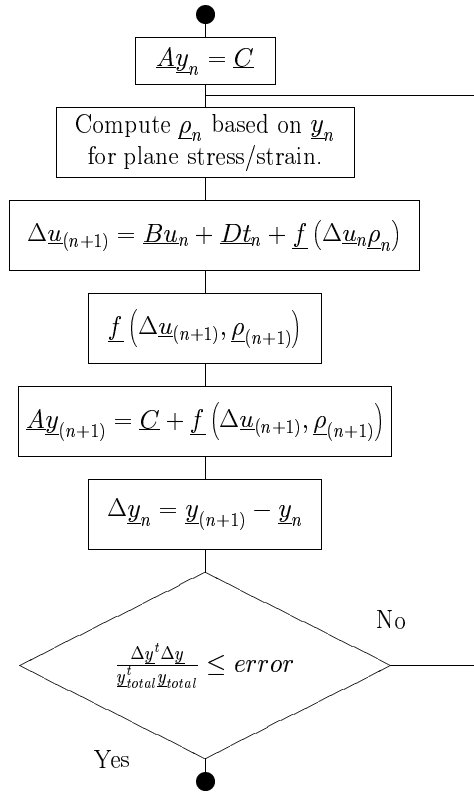


Figure 6.2: Iterative solution of non-linear boundary integral equations.

6.3.3 Incremental modification to the solution algorithm

In section 6.3.2 the basic iterative solution procedure was explained. It is the case that for non-linear problems the only way to achieve convergence of the iterative process for most load cases is to break the applied load down into load increments. This is certainly the only way to achieve convergence for the type of non-linear hyperelastic stress problems dealt with in this research, particularly for higher loads where the strains are several tens of percent. Therefore it is essential to develop an incremental version of the basic iterative solution algorithm discussed in section 6.3.2. This can be achieved with the following algorithm (see figure 6.3).

As is usual for most non-linear numerical algorithms the first iteration of the first increment is the solution of the equivalent linear problem. The next part of the algorithm is as discussed in section 6.3.2 where by the previous solution vector (tractions and displacements) is used to evaluate the displacement derivatives from the DBIE to enable the non-linear stresses to be found so as to compute the non-linear domain terms in the system of BIEs.

The significant difference however with the incremental algorithm is that for each iteration the 'old' and 'new' values of the solution vector and the non-linear stresses (at each domain Gauss point and at each source point) are stored. Therefore the total solution vector is formed from the total cumulative sum of the change in displacements and tractions at each iteration of each increment. Similarly in both the BIE and the DBIE the change in non-linear stress (the difference between the current and previous iteration values) is used in the non-linear domain term. This means that for each increment the non-linear domain term reduces for each iteration until convergence occurs. Therefore this algorithm is pseudo incremental because at each iteration of every increment the system of BIEs is still solved for the total displacements and tractions.

Note that for later increments with larger loads the problem geometry changes, therefore some additional steps are required. Because the BIE derivations are based on first 1^{st} PK stress values the prescribed tractions (surface stresses) need to be converted to 1^{st} PK values, which will start to differ from nominal values as the strain increases.

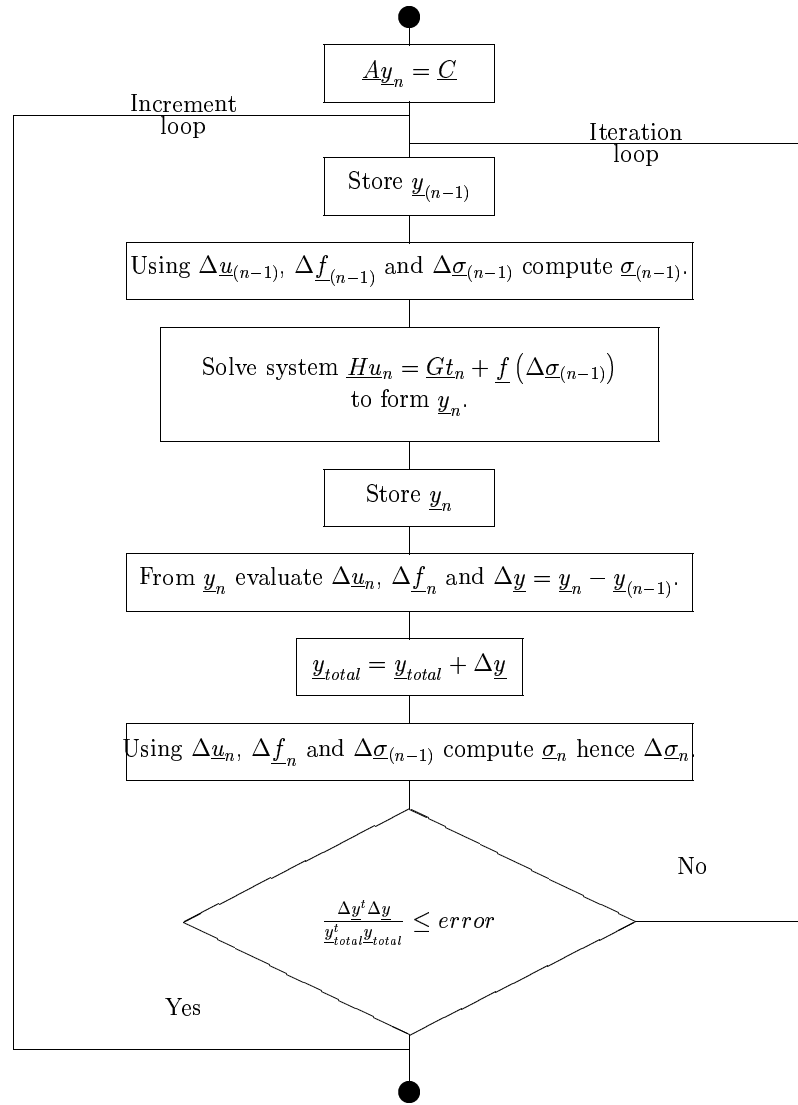


Figure 6.3: Incremental modification to iterative algorithm.

6.3.4 Displacement derivatives at corners

It is obvious that surface tractions are discontinuous at corners. For the displacement BIEs this was overcome by having coincident nodes on the corners, each one attached to one of the elements meeting at that corner. Hence in the solution to the system of simultaneous BIEs each discontinuous traction could be solved for its source point at one of the coincident boundary nodes at the corners.

However for the DBIE needed for the evaluation of stresses this is not possible. The DBIE is evaluated at the source point after the simultaneous system of displacement BIEs has been solved for the boundary tractions and displacements. Since the DBIE represented by equation 6.60 or 6.73 involves terms containing source point tractions, when the source point is on a corner the DBIEs will not be accurate.

Therefore another approach to finding the displacement derivatives at a corner, for a two dimensional problem for example, is to form four simultaneous equations in terms of the derivative components $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{dv}{dx}$ and $\frac{dv}{dy}$. Since the traction components are still discontinuous at the corner two sets of simultaneous equations are solved for points slightly offset from the corner. The two sets of displacement derivatives components are then averaged to obtain a close approximation to the corner values. Typically the corner offset in intrinsic coordinates is about 0.05 or less for this application.

The four equations to use for the simultaneous approach are taken from the definition of surface tractions and the partial intrinsic derivatives of the displacement components, hence:

$$T_x = l\sigma_x + m\tau_{yx} \quad (6.79)$$

$$T_y = l\tau_{xy} + m\sigma_y \quad (6.80)$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} \quad (6.81)$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \xi} \quad (6.82)$$

Standard constitutive relationships can be used to put equations 6.79 and 6.80 in terms of strain components. In addition equations 6.81 and 6.82 can be put in terms of element unit normal vector direction cosines by dividing through by the Jacobian determinant of the element the source point is on. Therefore for finite strain applications the linear small strain traction at the source point must be used. With these points in mind equations 6.79 to 6.82 can be re-written as follows:

$$\frac{T_x - T_x^{nl}}{\mu} = \frac{2l}{(1-2p)} \left[(1-p) \frac{\partial u}{\partial x} + p \frac{\partial v}{\partial y} \right] + m \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \quad (6.83)$$

$$\frac{T_y - T_y^{nl}}{\mu} = \frac{2m}{(1-2p)} \left[p \frac{\partial u}{\partial x} + (1-p) \frac{\partial v}{\partial y} \right] + l \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \quad (6.84)$$

$$\frac{\partial u}{\partial \xi} / J = -m \frac{\partial u}{\partial x} + l \frac{\partial u}{\partial y} \quad (6.85)$$

$$\frac{\partial v}{\partial \xi} / J = -m \frac{\partial v}{\partial x} + l \frac{\partial v}{\partial y} \quad (6.86)$$

Or in matrix form:

$$\begin{bmatrix} \frac{2(1-p)l}{(1-2p)} & m & m & \frac{2lp}{(1-2p)} \\ \frac{2mp}{(1-2p)} & l & l & \frac{2m(1-p)}{(1-2p)} \\ -m & l & 0 & 0 \\ 0 & 0 & -m & l \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{T_x - T_x^{nl}}{\mu} \\ \frac{T_y - T_y^{nl}}{\mu} \\ \frac{\partial u}{\partial \xi} / J \\ \frac{\partial v}{\partial \xi} / J \end{bmatrix} \quad (6.87)$$

This set of simultaneous equations is then solved for the two offset corner points, and the average of their derivative components taken.

6.3.5 Plane stress and strain cases

In section 6.3.2 it was explained that as part of the solution algorithm it is necessary to evaluate the in plane displacement derivatives in order to populate the right Green matrix $\underline{C}$. This is achieved using displacement derivative BIEs as explained. For two-dimensional problems the C_{13} , C_{23} , C_{31} and C_{23} components are zero. However the remaining C_{33} component value must be evaluated having considered the type of stress analysis as follows.

Two of the three types of analysis that can be carried out in two-dimensional stress analysis are plane stress and plane strain. For both these cases two problems arise, namely, how to evaluate the C_{33} component of the Right Green tensor $\underline{C}$ and how to evaluate the hydrostatic pressure?

Taking the case of plane strain first, it is obvious the C_{33} component is unity. However, the evaluation of the hydrostatic pressure is not straight forward. For the case of plane strain there is no stress boundary condition to use for the solution of the hydrostatic pressure at a point. This would also be the case for the more general three dimensional case. Therefore another set of simultaneous equations is required, covering each boundary node. The governing equation for compatibility between the state of deformation and the hydrostatic pressure is equation 5.42 already given in the finite element modelling chapter, hence:

$$\rho = -K(J - 1) \quad (6.88)$$

As discussed in the finite element chapter if a weighted residual approach is used the equation can be re-written as:

$$\sum_e \iiint \delta \rho \left[\frac{\rho}{K} + (J - 1) \right] dV = 0 \quad (6.89)$$

where the domain of the problem has been discretized into domain cells, just as with the finite element method. Isoparametric interpolation can be used to interpolate nodal values to Gauss points within the element domain using shape functions, hence:

$$\sum_e \iiint \delta \underline{\rho}^t \underline{M} \left[\frac{\rho}{K} + (J - 1) \right] dV = 0 \quad (6.90)$$

where $\underline{M}$ is the pressure interpolating shape function. For an infinitesimal change in deformation this equation can be linearised as follows:

$$\rho = \rho + \Delta\rho \quad J = J + \Delta J$$

hence:

$$\sum_e \int \int \int_e \underline{M} \left[\frac{\rho + \Delta\rho}{K} + (J + \Delta J - 1) \right] dV = 0 \quad (6.91)$$

which can be written as:

$$F + \sum_e \int \int \int_e \left[\frac{\Delta\rho}{K} + \Delta J \right] \underline{M} dV = 0 \quad (6.92)$$

where:

$$F = \sum_e \int \int \int_e \left[\frac{\rho}{K} + (J - 1) \right] \underline{M} dV \quad (6.93)$$

Again the infinitesimal change in hydrostatic pressure can be expressed in terms of nodal values using isoparametric shape functions, thus:

$$\Delta\rho = \underline{M}^t \Delta \underline{\rho} \quad (6.94)$$

Also using shape functions and the Right Cauchy-Green tensor strain invariants the infinitesimal change in the determinant of the Right Cauchy-Green tensor, ΔJ , was shown in chapter 5 to be given by equation 5.50, repeated here for clarity.

$$\Delta J = -\varrho^t \Delta \underline{\varepsilon}_G = -\varrho^t \underline{B}_{nl} \Delta \underline{\delta} \quad (6.95)$$

With $\Delta \underline{\delta}$ being the deformation change and $\underline{B}_{nl}$ the non-linear strain matrix (see equation 5.20). Therefore the linearised constitutive equation relating the state of deformation to the hydrostatic pressure can be written as follows:

$$\underline{F} + \sum_e \int \int \int_e \underline{M} \left[\underline{M}^t \frac{\Delta \rho}{K} - \underline{\varrho}^t B_{nl} \Delta \underline{\delta} \right] dV = 0 \quad (6.96)$$

This equation can be re-written as:

$$\underline{K}_\rho \Delta \underline{\rho} = \sum_e \int \int \int_e \underline{M} \underline{\varrho}^t B_{nl} dV \Delta \underline{\delta} - \underline{F} \quad (6.97)$$

where:

$$\underline{K}_\rho = \sum_e \int \int \int_e \frac{\underline{M} \underline{M}^t}{K} dV \quad (6.98)$$

The system of equations represented by equation 6.97 is then assembled and solved in exactly the same way as the finite element method to evaluate the pressure increment vector $\Delta \underline{\rho}$ for a given deformation increment vector. The whole process is repeated with each iteration of the outer displacement boundary integral equation loop.

It is worth mentioning at this stage a refinement to the algorithm required if the Mooney-Rivlin material model is used. This topic will be expanded upon in more detail in the concluding chapter of the thesis. The Mooney-Rivlin coefficients C_1 and C_2 can accurately be curve fitted from test data for the material, geometry and loading in question. However, the bulk modulus K , which has been shown in section 4.2.2 of chapter 4 to be a function of C_1 and C_2 is also a function of Poisson's ratio ν . Since elastomers have a Poisson's ratio close to 0.5 (near incompressibility) a $(1 - 2\nu)$ factor appears in the denominator of the Mooney-Rivlin derived bulk modulus expression. Hence the value of bulk modulus K is highly susceptible to the arbitrary choice of Poisson's ratio value. In the absence of compressibility data for a given elastomer an additional corrective iteration is required on the hydrostatic pressure ρ derived from the iterative solution of equation 6.97. This is due to the involvement of the bulk modulus K in the denominator of the stiffness matrix $\underline{K}_\rho$ elements. The exact form of this corrective iteration will be discussed in the conclusion of this thesis.

Plane stress analysis is less problematical. If it is assumed that the material is incompressible then the determinant of the Right-Green tensor $\underline{C}$ must be unity. Accepting this then the value of the C_{33} component can be found. In addition the

hydrostatic pressure can easily be found using the constitutive equations and the zero out of plane stress boundary condition. Alternatively the hydrostatic pressure can be eliminated algebraically from the constitutive equations for the case of plane stress as follows (see appendix D, equations repeated here for clarity).

$$S_x = B_1 (1 - C_{22}C_{33}^2) + B_2 (C_{11} - C_{22}C_{33}^3) \quad (6.99)$$

$$S_y = B_1 (1 - C_{11}C_{33}^2) + B_2 (C_{22} - C_{11}C_{33}^3) \quad (6.100)$$

$$S_{xy} = C_{12}C_{33}^2 B_1 + B_2 (C_{12} + C_{12}C_{33}^3) \quad (6.101)$$

where

$$B_1 = 2C_1 + 2C_2 I_1 \quad (6.102)$$

$$B_2 = -2C_2 \quad (6.103)$$

Chapter 7

Direct numerical solutions

7.1 Direct numerical patch test solution

The patch test is a well known verification test in numerical stress analysis. The patch test refers to the case of uniform axial load being applied to a unit cube of homogeneous isotropic material. The patch test can be further broken down into two subsets, namely plane stress and plane strain. In the former plane stress case the out of plane deformation is unrestrained, in the latter it is not. In both cases a vertical constraint is applied to allow for a state of equilibrium.

7.1.1 Plane stress and strain

For both plane stress and strain it can be shown that:

$$A = \frac{A_o}{\lambda} \quad (7.1)$$

where A is the deformed coordinate cross sectional area, A_o is the undeformed coordinate cross sectional area and λ is the axial extension ratio. Therefore for plane stress and strain it can be deduced that:

$$\sigma_x = \frac{F}{A} = \lambda \frac{F}{A_o} = \lambda \sigma_o \quad (7.2)$$

where σ_x is the true or Cauchy stress and σ_o the nominal stress. From equation 3.143 it was shown that:

$$\underline{\sigma} = \frac{1}{J} \underline{G} \underline{S} \underline{G}^t \quad (7.3)$$

where $\underline{G}$ is the deformation gradient matrix, $\underline{S}$ is the 2nd Piola-Kirchhoff stress matrix and J is the deformed to undeformed coordinate Jacobian matrix. Thus assuming complete incompressibility and substituting for the x component values it can be deduced that:

$$\sigma_x = \lambda^2 S_x \quad (7.4)$$

therefore from equation 7.2 it can be deduced that:

$$\sigma_o = \lambda S_x \quad (7.5)$$

By definition from the Mooney-Rivlin material model we have:

$$\frac{\partial \phi}{\partial \underline{\varepsilon}} = \underline{S} \quad (7.6)$$

where ϕ is the strain energy and $\underline{\varepsilon}$ is Green's strain. Since Green's strain was given in equation 3.21 as:

$$\underline{\varepsilon} = \frac{1}{2} (\underline{C} - \underline{I}) \quad (7.7)$$

It can be deduced that:

$$\underline{S} = \frac{\partial \phi}{\partial \underline{\varepsilon}} = 2 \frac{\partial \phi}{\partial \underline{C}} \quad (7.8)$$

where $\underline{C}$ is the Right-Green tensor. Therefore the general approach to deriving the analytical solution is to utilise equation 7.8 to obtain an expression for $\underline{S}$ (via the strain energy model) and then to use the result in equation 7.5, which becomes the governing equation:

$$\lambda S_x - \sigma_o = 0 \quad (7.9)$$

Thus defining equation 7.9 as $f(\lambda)$, then the Newton-Raphson equation can be written:

$$f(\lambda_o) + \Delta \lambda f'(\lambda_o) = 0 \quad (7.10)$$

Starting from an arbitrary initial value of λ_o equation 7.10 can be solved for $\Delta \lambda_o$ to arrive at a new value of λ , namely $\lambda_1 = \lambda_o + \Delta \lambda_o$. This value is substituted back into equation 7.10, a new $\Delta \lambda_1$ solved for, and hence λ_1 . This process is repeated until convergence occurs when $\frac{(\Delta \lambda_n)^2}{\lambda_n^2}$ is less than a given permissible value. This process is the same for plane stress and strain except for plane strain an outer iterative loop is required to solve for the hydrostatic pressure ρ before convergence can be achieved.

7.1.2 Plane stress

The following description is the derivation of the governing equation $f(\lambda)$ to use in the Newton-Raphson scheme for plane stress. From the definition of the strain invariants and substituting in the $\underline{C}$ components for the case of plane stress it can be deduced that:

$$I_1 = \lambda^2 + \frac{2}{\lambda} \quad (7.11)$$

$$I_2 = 2\lambda + \frac{1}{\lambda^2} \quad (7.12)$$

In appendix D on constitutive modelling it was shown that (assuming total incompressibility with $I_3 = 1$):

$$B_1 = 2C_1 + 2C_2 I_1 \equiv 2C_1 + 2C_2 \left(\lambda^2 + \frac{2}{\lambda} \right) \quad (7.13)$$

$$B_2 = -2C_2 \quad (7.14)$$

Also in appendix D for the case of plane stress, using the Mooney-Rivlin material model, it was shown in equation D.69 that:

$$\begin{aligned} S_x &= B_1 (1 - C_{22} C_{33}^2) + B_2 (C_{11} - c_{22} C_{33}^3) \\ &\equiv 2C_1 \left(1 - \frac{1}{\lambda^3} \right) + 2C_2 \left(\frac{1}{\lambda} - \frac{1}{\lambda^4} \right) \end{aligned} \quad (7.15)$$

Therefore the governing equation to be used in the Newton-Raphson process can be written as:

$$C_1 \left(\lambda - \frac{1}{\lambda^2} \right) + C_2 \left(1 - \frac{1}{\lambda^3} \right) - \frac{\sigma_o}{2} = 0 \quad (7.16)$$

7.1.3 Plane strain, incompressible

The derivation of the governing equation is similar to the plane stress derivation except additional complications arise due to the hydrostatic pressure ρ in the following expression (see equation D.33 in appendix D):

$$S_x = B_1 + B_2 \lambda^2 + \frac{(B_3 - \rho J)}{\lambda^2} \quad (7.17)$$

From the definition of the strain invariants and the plane strain boundary conditions it can be deduced that:

$$I_1 = I_2 = \lambda^2 + \frac{1}{\lambda^2} + 1 \quad (7.18)$$

The expressions for I_1 and I_2 can be substituted into equation 7.13 and the following equation:

$$B_3 = - \left(\frac{2}{3} C_1 I_1 + \frac{4}{3} C_2 I_2 \right) \quad (7.19)$$

to utilise in the general 2nd Piola-Kirchhoff stress equation 7.17. Given an estimate of the hydrostatic pressure ρ the following governing equation can be used in the standard Newton-Raphson procedure:

$$B_1 \lambda + B_2 \lambda^3 + \frac{(B_3 - \rho)}{\lambda} - \sigma_o = 0 \quad (7.20)$$

Therefore some means of calculating ρ is required once the internal Newton-Raphson process has converged on a λ . The hydrostatic pressure ρ can be defined as the negative of the mean stress as follows:

$$\rho = - \frac{(\sigma_x + \sigma_y + \sigma_z)}{3} \quad (7.21)$$

Now for the plane strain version of the patch test it can be assumed $\sigma_y = 0$. Thus using the definition of Poisson's ratio (with $\nu = 0.5$):

$$\sigma_z = \nu (\sigma_x + \sigma_y) = \nu \sigma_x \equiv \frac{1}{2} \sigma_x \quad (7.22)$$

Thus substituting for σ_x in equation 7.21 and assuming $\sigma_z = S_z$ we can conclude:

$$S_z = -\rho \quad (7.23)$$

Again, using the general 2nd PK stress constitutive equations derived in equation 4.45 of chapter 4 it can be deduced that:

$$S_z = B_1 + B_2 + B_3 - \rho J \quad (7.24)$$

Therefore the analytical solution for the plane strain case of the patch test involves a Newton-Raphson loop nested within an iterative loop for the hydrostatic pressure. The Newton-Raphson loop uses the governing equation 7.20 and the pressure iteration loop uses equations 7.24/7.23.

7.1.4 Plane strain, compressible

It was found that at higher strains there was a significant and increasing divergence between the direct numerical and the numerical results. This is due to the boundary and finite element programs allowing for compressibility, but the direct numerical solution only catering for incompressibility. Therefore a compressible direct numerical solution to the patch test is required.

The general approach is first to introduce compressibility terms into the standard constitutive parameters using the general case $\lambda_1 = \lambda_x$, $\lambda_2 = \lambda_y$ and $\lambda_3 = \lambda_z = 1$ as follows:

$$I_1 = 1 + \lambda_1^2 + \lambda_2^2$$

$$I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_1^2 \lambda_2^2$$

$$I_3 = \lambda_1^2 \lambda_2^2$$

$$J = \sqrt{I_3} = \lambda_1 \lambda_2$$

$$\overline{C}_1 = C_1 I_3^{-\frac{1}{3}}$$

$$\overline{C}_2 = C_2 I_3^{-\frac{2}{3}}$$

Having all the constitutive parameters in compressible form the two governing equations are derived so that they can be solved simultaneously by the Newton-Raphson method. The first of the governing equations is derived by equating two equations for the hydrostatic pressure. The first equated equation comes from the definition of bulk modulus K , i.e. $\rho = -K(J - 1)$. And the second equation is derived from

the Mooney-Rivlin constitutive expression for the 2nd PK stress in the global y direction, which by definition for plane strain uniaxial tension, is zero. Hence the first governing equation can be stated as:

$$f_1(\lambda_1, \lambda_2) = (B_1 + B_2\lambda_2^2) \frac{\lambda_2}{\lambda_1} + \frac{B_3}{\lambda_1\lambda_2} + K(\lambda_1\lambda_2 - 1) = 0 \quad (7.25)$$

The second governing equation is derived for the axial extension again from the Mooney-Rivlin constitutive equation but this time for the axial 2nd PK stress. Using the relationship (see equation 7.5):

$$S_x = \frac{\sigma_o}{\lambda_1} \quad (7.26)$$

Equating this expression with the expression for the 2nd PK stress and cross multiplying by λ_1 then substituting for the hydrostatic pressure ρ the second governing equations is obtained:

$$f_2(\lambda_1, \lambda_2) = B_1\lambda_1 + B_2\lambda_1^3 + \frac{B_3}{\lambda_1} + K\lambda_2(J - 1) - \sigma_o = 0 \quad (7.27)$$

Re-arranging governing equations 7.25 and 7.27 to be in terms of the constitutive B term coefficients:

$$f_1(\lambda_1, \lambda_2) = B_1 \left(\frac{\lambda_2}{\lambda_1} \right) + B_2 \left(\frac{\lambda_2^3}{\lambda_1} \right) + \frac{B_3}{\lambda_1\lambda_2} + K(\lambda_1\lambda_2 - 1) \quad (7.28)$$

$$f_2(\lambda_1, \lambda_2) = B_1\lambda_1 + B_2\lambda_1^3 + \frac{B_3}{\lambda_1} + K(\lambda_1\lambda_2^2 - \lambda_2) - \sigma_o \quad (7.29)$$

Therefore there are two non-linear governing equations in the unknowns λ_1 and λ_2 which are to be solved by the Newton-Raphson algorithm. Hence the differentials of the governing equations with respect to λ_1 and λ_2 will be required:

$$\frac{\partial f_1}{\partial \lambda_1} = \frac{\partial B_1}{\partial \lambda_1} \left(\frac{\lambda_2}{\lambda_1} \right) + \left(\frac{\partial B_3}{\partial \lambda_1} \right) / (\lambda_1\lambda_2) - B_1 \frac{\lambda_2}{\lambda_1^2} - B_2 \frac{\lambda_2^3}{\lambda_1^2} - \frac{B_3}{\lambda_1^2\lambda_2} + K\lambda_2 \quad (7.30)$$

$$\frac{\partial f_1}{\partial \lambda_2} = \frac{\partial B_1}{\partial \lambda_2} \left(\frac{\lambda_2}{\lambda_1} \right) + \left(\frac{\partial B_3}{\partial \lambda_2} \right) / (\lambda_1 + \lambda_2) + \frac{B_1}{\lambda_1} + 3B_2 \left(\frac{\lambda_2^2}{\lambda_1} \right) - \frac{B_3}{\lambda_1 \lambda_2^2} + K \lambda_1 \quad (7.31)$$

$$\frac{\partial f_2}{\partial \lambda_1} = \frac{\partial B_1}{\partial \lambda_1} \lambda_1 + \frac{\partial B_3}{\partial \lambda_1} / \lambda_1 + K (\lambda_2^2) + B_1 + 3B_2 \lambda_1^2 - \frac{B_3}{\lambda_1^2} \quad (7.32)$$

$$\frac{\partial f_2}{\partial \lambda_2} = \lambda_1 \left(\frac{\partial B_1}{\partial \lambda_2} \right) + \left(\frac{1}{\lambda_1} \right) \frac{\partial B_3}{\partial \lambda_2} + K (2\lambda_1 \lambda_2 - 1) \quad (7.33)$$

where:

$$\frac{\partial B_1}{\partial \lambda_1} = 4\overline{C}_2 \lambda_1 \quad (7.34)$$

$$\frac{\partial B_1}{\partial \lambda_2} = 4\overline{C}_2 \lambda_2 \quad (7.35)$$

$$B_2 = Constant \quad (7.36)$$

$$\frac{\partial B_3}{\partial \lambda_1} = - \left(\frac{4}{3} \overline{C}_1 + \frac{8}{3} \overline{C}_2 \right) \lambda_1 - \frac{8}{3} \overline{C}_2 \lambda_1 \lambda_2^2 \quad (7.37)$$

$$\frac{\partial B_3}{\partial \lambda_2} = - \left(\frac{4}{3} \overline{C}_1 + \frac{8}{3} \overline{C}_2 \right) \lambda_2 - \frac{8}{3} \overline{C}_2 \lambda_1^2 \lambda_2 \quad (7.38)$$

Using a first term Taylor series expansion the following pair of simultaneous equations in $\Delta \lambda_1$ and $\Delta \lambda_2$ are obtained:

$$\left(\frac{\partial f_1}{\partial \lambda_1} \right) \Delta \lambda_1 + \left(\frac{\partial f_1}{\partial \lambda_2} \right) \Delta \lambda_2 = -f_1 (\lambda_1, \lambda_2) \quad (7.39)$$

$$\left(\frac{\partial f_2}{\partial \lambda_1}\right) \Delta \lambda_1 + \left(\frac{\partial f_2}{\partial \lambda_2}\right) \Delta \lambda_2 = -f_2(\lambda_1, \lambda_2) \quad (7.40)$$

hence using Cramer's rule $\Delta \lambda_1$ and $\Delta \lambda_2$ are obtained as:

$$\Delta \lambda_1 = \left(f_2 \frac{\partial f_1}{\partial \lambda_2} - f_1 \frac{\partial f_2}{\partial \lambda_2}\right) / D \quad (7.41)$$

$$\Delta \lambda_2 = \left(f_1 \frac{\partial f_2}{\partial \lambda_1} - f_2 \frac{\partial f_1}{\partial \lambda_1}\right) / D \quad (7.42)$$

where:

$$D = \left(\frac{\partial f_1}{\partial \lambda_1}\right) \left(\frac{\partial f_2}{\partial \lambda_2}\right) - \left(\frac{\partial f_1}{\partial \lambda_2}\right) \left(\frac{\partial f_2}{\partial \lambda_1}\right) \quad (7.43)$$

The the next iterative values for $(\lambda_1)_n$ and $(\lambda_2)_n$ are obtained from $(\lambda_1)_o + \Delta \lambda_1$ and $(\lambda_2)_o + \Delta \lambda_2$ respectively. The process can be repeated until $|\Delta \lambda_1|$ and $|\Delta \lambda_2|$ are less than a permissible value.

7.2 Internally pressurized thick walled cylinder

7.2.1 Finite deformation kinematics and constitutive equations

It has already been stated that for a pressurized cylinder λ_r , λ_θ and λ_z represent principal strains. Additionally from the condition of total incompressibility λ_z can be expressed in terms of λ_r and λ_θ as follows:

$$\lambda_z^2 = \frac{1}{\lambda_r^2 \lambda_\theta^2} \quad (7.44)$$

Therefore the displacement derivative matrix $\underline{g}$ in this case can be expressed as:

$$\underline{g} = \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & 0 \\ 0 & \frac{u}{r} & 0 \\ 0 & 0 & \frac{\partial w}{\partial z} \end{bmatrix} \quad (7.45)$$

Hence the deformation gradient matrix $\underline{G}$ can be written as:

$$\underline{G} = \begin{bmatrix} \left(1 + \frac{\partial u}{\partial r}\right) & 0 & 0 \\ 0 & \left(1 + \frac{u}{r}\right) & 0 \\ 0 & 0 & \left(1 + \frac{\partial w}{\partial z}\right) \end{bmatrix} \equiv \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{bmatrix} \quad (7.46)$$

Thus for this special case the Right Cauchy-Green tensor and its inverse can be written as:

$$\underline{C} = \begin{bmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{bmatrix} \quad (7.47)$$

$$\underline{C}^{-1} = \begin{bmatrix} \lambda_r^{-2} & 0 & 0 \\ 0 & \lambda_\theta^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{bmatrix} \quad (7.48)$$

The strain invariants of the Right Cauchy-Green tensor can thus be written as:

$$I_1 = \lambda_r^2 + \lambda_\theta^2 + \frac{1}{\lambda_r^2 \lambda_\theta^2} \quad (7.49)$$

$$I_2 = \lambda_r^2 \lambda_\theta^2 + \lambda_r^2 + \lambda_\theta^2 \quad (7.50)$$

$$I_3 = 1 \quad (7.51)$$

From the constitutive derivations given in chapter 4 it can be deduced that:

$$S_r = B_1 + B_2 \lambda_r^2 (B_3 - \rho) \lambda_r^{-2} \quad (7.52)$$

$$S_\theta = B_1 + B_2 \lambda_\theta^2 (B_3 - \rho) \lambda_\theta^{-2} \quad (7.53)$$

Due to incompressibility $J = 1$. The hydrostatic pressure ρ required by equations 7.52 and 7.53 can be obtained from consideration of the plane stress and strain boundary conditions. For the case of plane stress $\sigma_z = 0$, thus:

$$\rho = B_1 C_{33} + B_2 C_{33}^2 + B_3 \quad (7.54)$$

Note: $\sigma_z = \lambda_z S_z$, since S_z is the 2^{nd} PK stress, and σ_z is the 1^{st} PK stress. In this derivation all the stress components are the 1^{st} PK stresses as these will refer to reference or Lagrangian material coordinates.

In the case of plane strain the geometry of the problem is used and the internal pressure force is assumed to be reacted against by a uniform axial stress in the cylinder wall. Hence if R_2 is the outer radius and R_1 the inner radius:

$$\sigma_z \pi (R_2^2 - R_1^2) = \pi R_1^2 \rho_i \quad (7.55)$$

therefore:

$$\sigma_z = \frac{\rho_i}{\Theta^2 - 1} \quad (7.56)$$

where $\Theta = \frac{R_2}{R_1}$. Since $\lambda_z = \frac{1}{\lambda_r \lambda_\theta}$ and using the relationship between the 1st and 2nd PK stresses:

$$\sigma_z = \lambda_z S_z = \frac{1}{\lambda_r \lambda_\theta} S_z \quad (7.57)$$

Hence, with $C_{33} = \lambda_z = 1$ it can be deduced that:

$$\rho = B_1 + B_2 + B_3 - \frac{1}{(K^2 - 1)} \quad (7.58)$$

7.2.2 Direct numerical solution

The solution algorithm can be broken down into three stages as follows:

1. Iteration on λ_r on the inside cylinder surface such that the radial stress is equal and opposite to the internal pressure ρ_i .
2. Iteration on λ_r for each data point through the radial thickness up to and including the outer surface until the equation of internal equilibrium in polar coordinates (equation H.10) is satisfied.
3. Iteration on u_o , the internal surface deformation, by repeating steps 1 and 2 until the outer radial stress is zero.

Note that steps 1 and 3 ensure that the boundary conditions are satisfied. All three steps contain two standard numerical iterative processes that are used repeatedly in the algorithm. These two techniques are the False position iterative method and the modified Euler iterative algorithm.

False position algorithm

All three steps use the same iterative algorithm, which can be summarized as follows. Consider the function $f(x) = y$. The aim is given some prescribed value y_o solve for x . Step one of the algorithm is to form the relative error:

$$D_1 = \frac{[y_o - f(x_1)]}{y_o} \quad (7.59)$$

For the first iteration only scale the independent variable x_1 by some fraction of the relative error D_1 :

$$x_2 = x_1(1 - D_1 F) \quad (7.60)$$

where:

$$F < 1$$

Using the new estimate of the independent variable x_2 , find a new relative error D_2 :

$$D_2 = \frac{[y_o - f(x_2)]}{y_o} \quad (7.61)$$

Assuming the change of relative error D from D_1 to D_2 is linear the intersection of the line joining points D_1 to D_2 at $D = 0$ will give a new value of independent variable x (see figure 7.1). Thus the equation of the line from points (x_1, D_1) to (x_2, D_2) to be solved for x is given by:

$$\frac{D - D_1}{D_2 - D_1} = \frac{x - x_1}{x_2 - x_1} \quad (7.62)$$

thus assuming $D = 0$ then:

$$x_3 = x_1 - \frac{(x_2 - x_1) D_1}{(D_1 - D_2)} \quad (7.63)$$

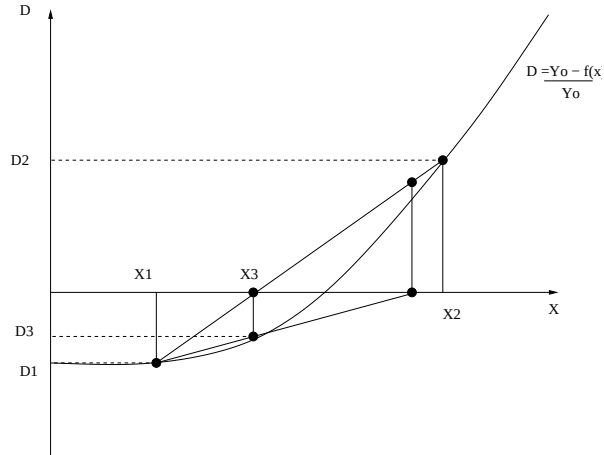


Figure 7.1: False position relative error verses independent variable x

Then with relative error D_3 for independent variable value x_3 the next independent variable value x_4 is found from the intersection of the line (x_3, D_3) to (x_1, D_1) with the axis $D = 0$ using equation 7.63. The process is repeated until D_n is less than a permissible value. The algorithm can be summarized by the flow chart shown in figure 7.2.

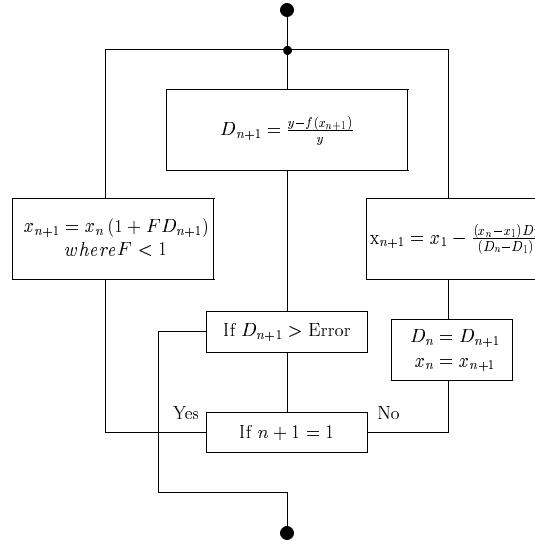


Figure 7.2: False position algorithm

Modified Euler technique

Within step 2 it is required to solve two differential equations. To do this a modified Euler approach has been used. In the standard Euler method the first term Taylor series expansion of the function is used to obtain the next iteration approximate solution to the equation:

$$y_n = y_{n-1} + \Delta x f'(x_{n-1}) \quad (7.64)$$

To reduce the cumulative build up in error the modified Euler approach assumes the average of the gradients between two points is approximately equal to the gradient of the function between the two points (see figure 7.3).

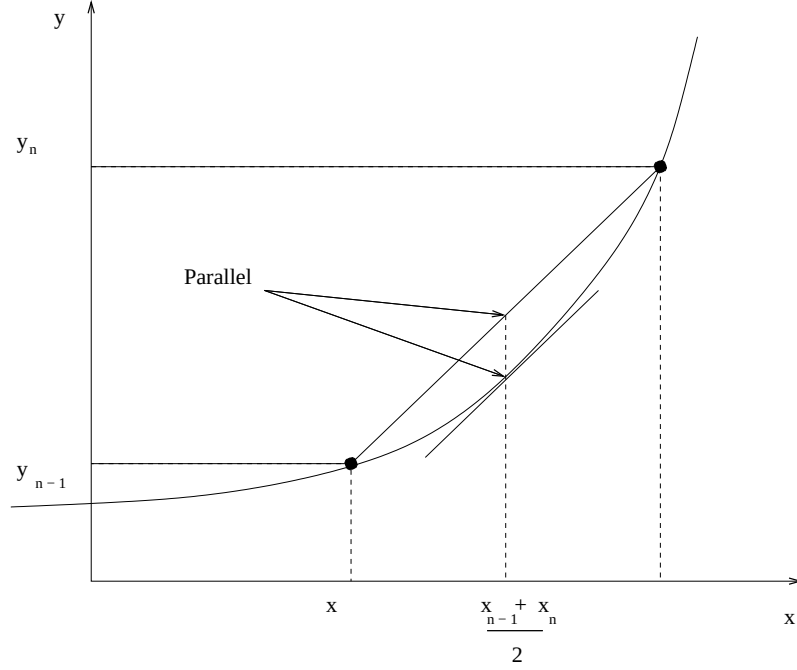


Figure 7.3: Modified Euler approach

7.2.3 Direct solution algorithm

Prior to the iterative process Lamé's solution for an internally pressurized thick walled cylinder is used to obtain initial estimates for the radial and hoop extension ratios λ_r and λ_θ (equations H.39 and H.39).

The first iteration is on λ_r for the internal radial stress using the previously discussed algorithm with $y = f(x)$ where y is the internal pressure ρ_i and $f(x)$ is the 1st PK stress in the radial direction (equation H.39 multiplied by λ_r). Hence λ_r is iterated on until the radial stress is equal and opposite to the internal pressure.

Step two of the algorithm is the iteration on λ_r until the equation of internal equilibrium is satisfied for each radial data point. Within step two the modified Euler approach previously discussed is used twice within the step two false position iteration. The first Euler application is used to solve the following differential equation:

$$u_{n+1} = u_n + \frac{du}{dr} \Delta r \quad (7.65)$$

Since the modified Euler approach is used the average of the gradients at r_{n+1} and r_n is used. Hence:

$$u_{n+1} = u_n + \left[\frac{\left(\frac{du}{dr}\right)_{n+1} + \left(\frac{du}{dr}\right)_n}{2} \right] \Delta r \quad (7.66)$$

since $\lambda_r = 1 + \frac{du}{dr}$ then:

$$\lambda_{ra} = \frac{(\lambda_{rn+1} + \lambda_{rn})}{2} \quad (7.67)$$

therefore:

$$\left(\frac{du}{dr}\right)_a = \left[\frac{\left(\frac{du}{dr}\right)_{n+1} + \left(\frac{du}{dr}\right)_n}{2} \right] = [\lambda_{ra} - 1] \quad (7.68)$$

hence:

$$\Delta u = \Delta R [\lambda_{ra} - 1] \quad (7.69)$$

$$u_{n+1} = u_n + \Delta u \quad (7.70)$$

Enabling the iterative hoop extension ratio to be found:

$$\lambda_{\theta a} = 1 + \frac{u_{n+1}}{R_{n+1}} \quad (7.71)$$

The second modified Euler application within step two is with the equation of internal equilibrium for polar coordinates which can be put in the following form:

$$\frac{d}{dr} (r\sigma_r) = \sigma_\theta \quad (7.72)$$

Instead of the false position relative error being taken at the radial point the relative error is found at the mid point between the current and the previous position data point:

$$\frac{d}{dr}(r\sigma_r) = \frac{[r_{n+1}\sigma_{rn+1} - r_n\sigma_{rn}]}{\Delta r} = \frac{\sigma_{\theta n+1} + \sigma_{\theta n}}{2} \quad (7.73)$$

Thus the relative error given by the modified Euler approach is given by:

$$D = \left[(r_{n+1}\sigma_{rn+1} - r_n\sigma_{rn}) - \Delta r \left(\frac{\sigma_{\theta n+1} + \sigma_{\theta n}}{2} \right) \right] / \Delta r \left(\frac{\sigma_{\theta n+1} + \sigma_{\theta n}}{2} \right) \quad (7.74)$$

Finally the outer iteration is on λ_θ by repeating steps 1 and 2 until the outer radial 1st PK stress is zero. Again the basic false position algorithm is used, with $y = 0$ and $f(x)$ given by the outer radial 1st PK stress as obtained from steps 1 and 2. The whole direct numerical solution is summarized in figure 7.4.

7.3 Rotating disk

The derivation and algorithm of the direct numerical solution for a rotating disk of elastomeric material are identical to that described in section 7.2 except with the following differences.

1. The first step discussed in section 7.2.3 iterates on λ_r on the inside surface of the rotating disk until the radial stress is zero (or numerically insignificant).
2. The equation of internal equilibrium 7.72 has an additional inertia loading term added such that it becomes:

$$\frac{d}{dr}(r\sigma_r) = \sigma_\theta - \rho_m \omega^2 r^2 \quad (7.75)$$

Consequently the relative error expression to use with the modified Euler approach is of the form:

$$D = \frac{\left[(r_{n+1}\sigma_{rn+1} - r_n\sigma_{rn}) - \Delta r \left[\left(\frac{\sigma_{\theta n+1} + \sigma_{\theta n}}{2} \right) - \frac{\rho_m \omega^2}{2} (r_{n+1}^2 + r_n^2) \right] \right]}{\Delta r \left[\left(\frac{\sigma_{\theta n+1} + \sigma_{\theta n}}{2} \right) - \frac{\rho_m \omega^2}{2} (r_{n+1}^2 + r_n^2) \right]} \quad (7.76)$$

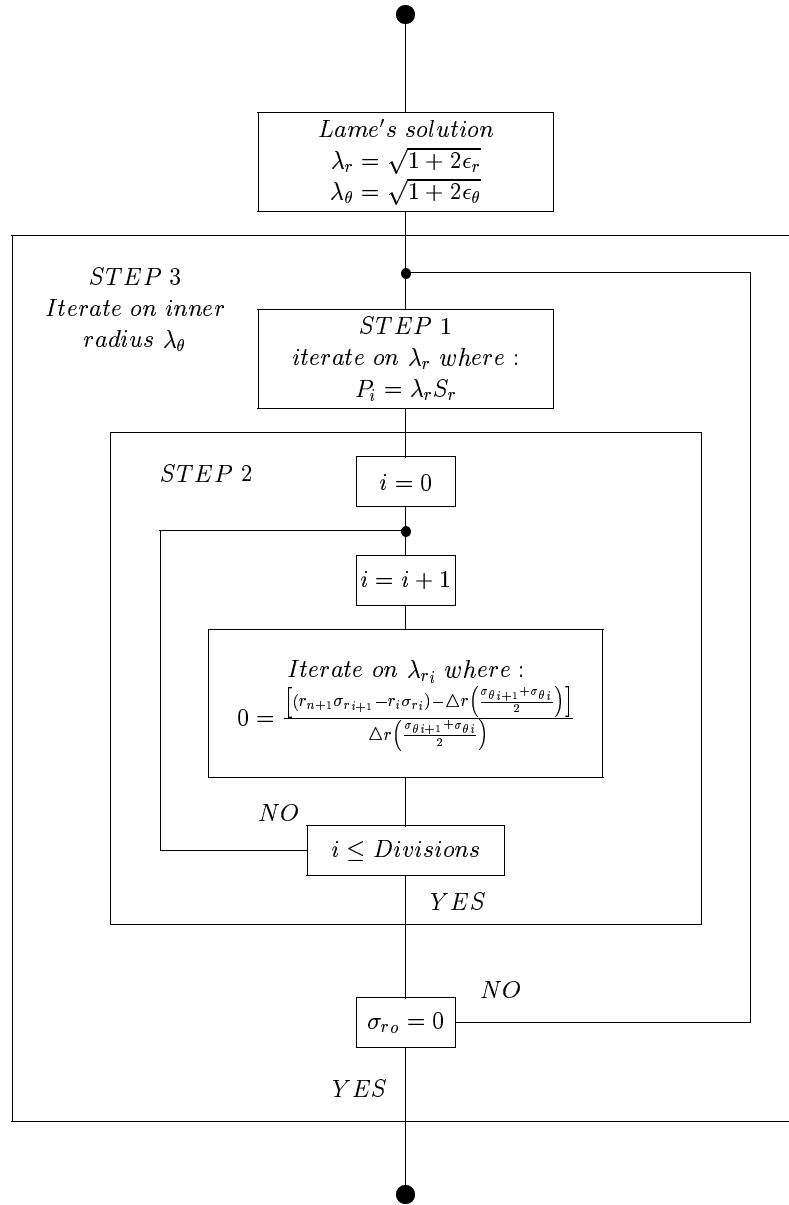


Figure 7.4: Direct numerical algorithm for the pressurized cylinder test case

Chapter 8

Case studies

Note: Unless otherwise stated all stresses quoted or plotted in this chapter are in units of $\frac{N}{cm^2}$, with units of length and deformation in cm . In addition all meshes are assumed 1 unit thick.

8.1 Mooney-Rivlin curve fit

It was described in section 4.2.1 how the two Mooney-Rivlin coefficients could be curve fitted from empirical test data using the analytically derived simple tension expression (equation 4.55). In figure 8.1 the analytical expression for the stress in the test piece as a function of extension is used to plot curve fitted stress values. The empirical test data values are also plotted for reference. Important practical engineering conclusions about the use of elastomer strain energy models are demonstrated by this plot and will be discussed in the conclusion chapter of this thesis.

In figure 8.1 the stress/extension data both for Treloar [53] derived test data and Mooney-Rivlin curve fitted uniaxial tension are plotted. Therefore a comparison between the empirical test data and the analytical results utilizing curve fitted constant values can be made. The analytical expression used to plot the curve fit series is given by equation 4.55 in section 4.2.1 for varying numbers of curve fit data points. In figure 8.1 it can be seen that three curve fits were performed for 23, 11 and 10 data points respectively.

As can be seen from figure 8.1 although the curve fit expression is identical for each data fit the number of data points, and hence the amount of empirical test data used by the curve fit algorithm, dramatically affects the shape of the curve fitted curve. It can be seen that only the 23 data point curve begins to capture the 'exponential'

nature of the empirical stress/extension data, all be it inaccurately. The 10 and 11 data point curves appear to capture the initial gradient of the stress/extension curve. This is not a problem if the approximate range of extension ratios of the FEM/BEM model are known in advance as the number of data points can be chosen to give the best fit between the analytical and the numerical over the range of expected extension ratios. Thus the 11 data point curve fitted Mooney-Rivlin coefficients would provide adequate FEM/BEM results accuracy up to 150 percent strain. Theoretically for the application of the Mooney-Rivlin material model accurate results should only be obtained up to strains of 100 percent or less.

It is worth noting however that because direct numerical expressions/algorithms have been derived for the test cases used in this research then exact or close agreement between analytical and empirical data curves is not necessary. As long as the BEM and FEM programs converged and a direct numerical result set was available (all using curve fitted Mooney-Rivlin material constants) then a meaningful comparison between the direct numerical and the FEM and BEM results could be made.

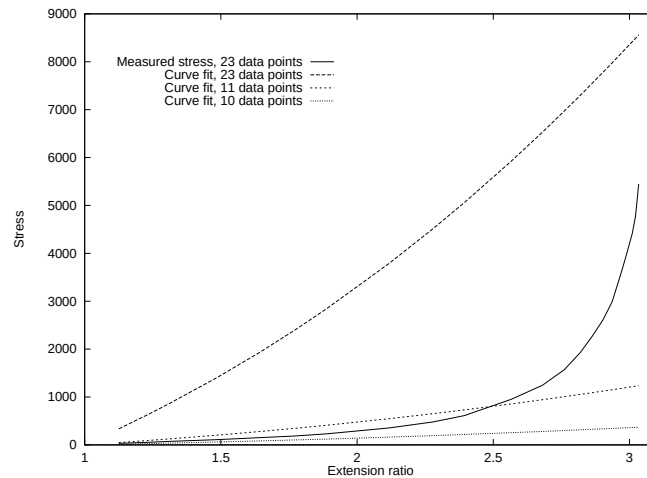


Figure 8.1: Mooney-Rivlin curve fit results

8.2 Patch test results

Two patch test cases for plane stress and plane strain were run, the results for stress versus percentage strain and load versus percentage strain are shown in figures 8.3 and 8.4 for plane stress and figures 8.5 and 8.6 for plane strain respectively. The patch test refers to a uniaxial extension of a unit cube of material, in this case an elastomer. It provides the most basic validation case for numerical stress analysis software and as in this research, is often the first validation test case for any new numerical formulation. For both plane stress and strain the mesh used was identical to that shown in figure 8.2. It consisted of eight three noded boundary elements enclosing four eight noded quadrilateral elements. The mesh was constrained along all the nodes on the left hand edge in the axial direction and restrained vertically on the bottom left hand node. Hence the model should exhibit uniform stress. The results plotted are for the deflection and stresses at nodes on the end face of the unit cube of material. FEM results for an equivalent FEM program and mesh are also plotted for comparison purposes.

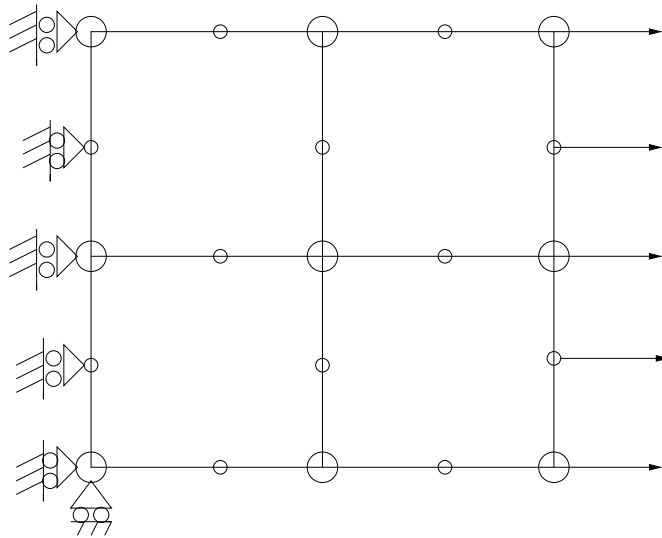


Figure 8.2: Patch test mesh geometry

As can be seen from figures 8.3, 8.4, 8.5 and 8.6 the agreement between the numerical and the direct numerical (analytical) results for the plane stress and strain patch test results is good. This is particularly the case for the FEM results, which agree exactly with the direct numerical results. The BEM results are close but do show an increasing divergence between the direct numerical and BEM results as the strain increases. It can only be stated that due to the increased numerical complexity of

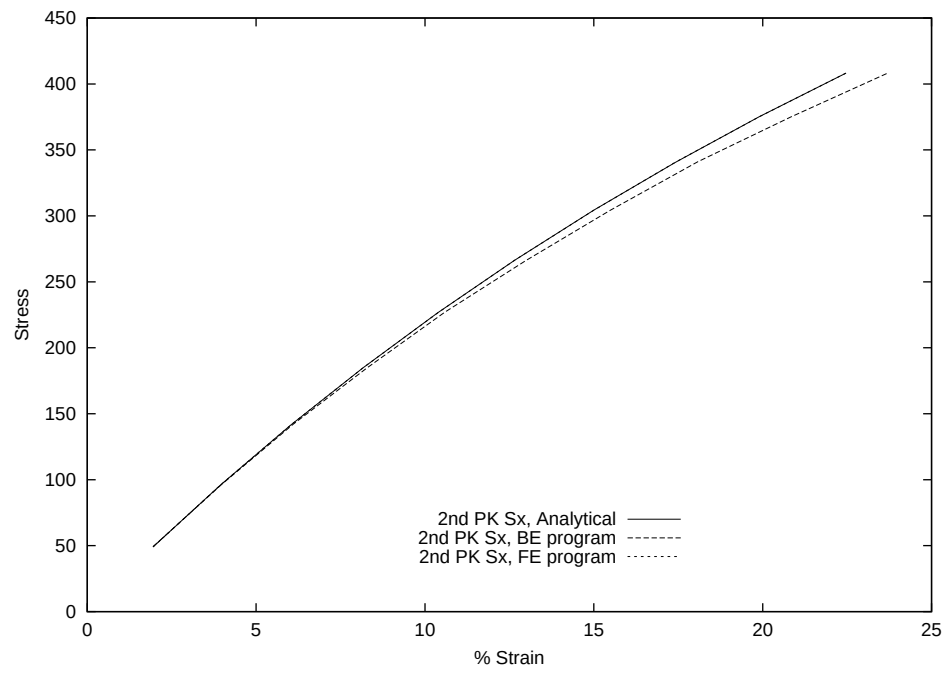


Figure 8.3: Patch test stress results, plane stress

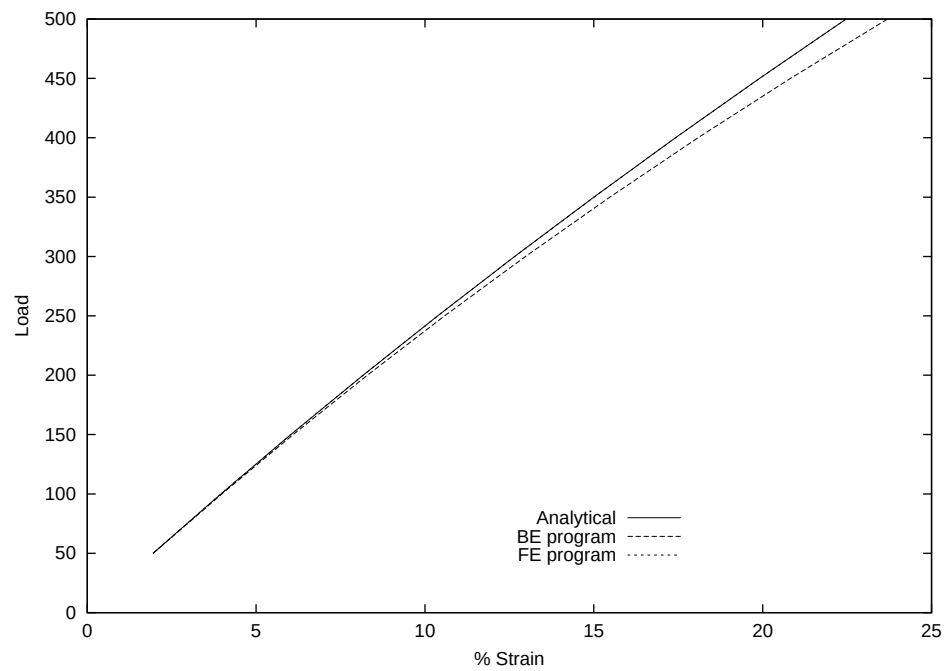


Figure 8.4: Patch test deflection results, plane strain

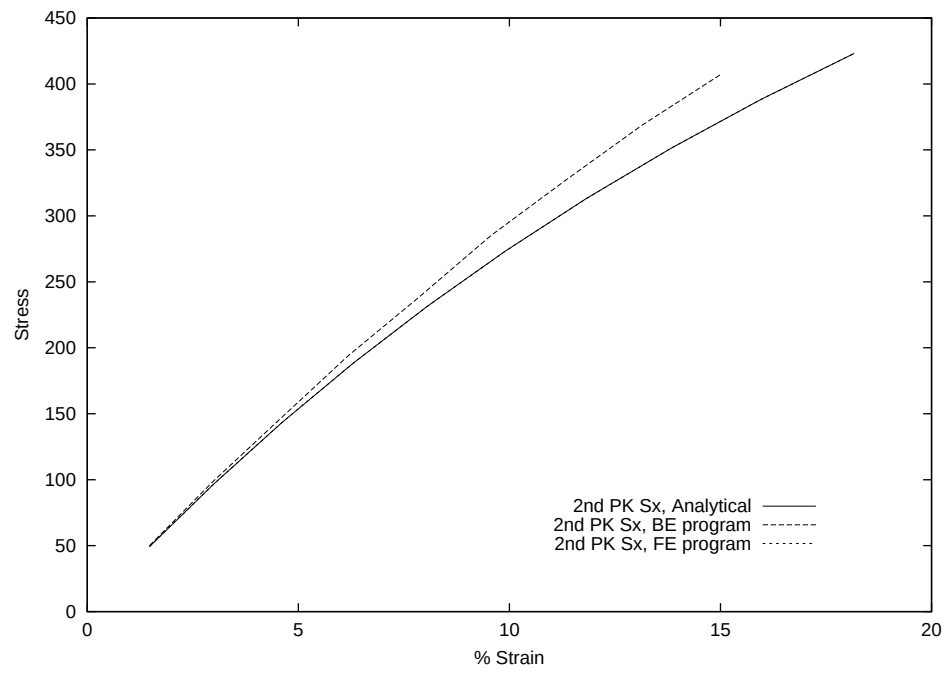


Figure 8.5: Patch test stress results, plane strain

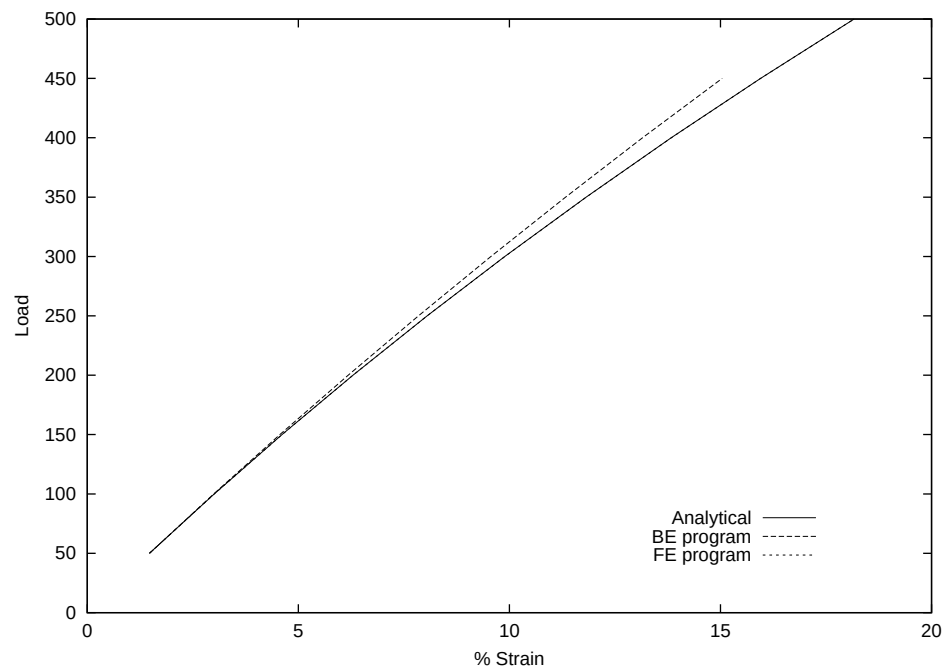


Figure 8.6: Patch test deflection results, plane strain

the BEM algorithm the divergence of results is due to digitizing and rounding errors in the BEM program. An effort was made to reduce these by using double precision for the FORTRAN 77 code. However the BEM results are sufficient to demonstrate that the algorithm and specifically the regularization approach is correct.

8.3 Pressurized thick walled cylinder

For additional validation to test the program performance for non-uniform stress fields the case of a thick walled pressurized cylinder was modelled. The cylinder was modelled in Cartesian coordinates using two planes of symmetry so that only one quarter of the cylinder was meshed. Hence the nodes along the vertical edge were restrained in the x direction and free in the y direction. Conversely nodes along the horizontal edge were restrained in the y direction and free to move in the x direction. Note that no mesh convergence using finer meshes was carried out as the results obtained with the mesh shown were accurate enough.

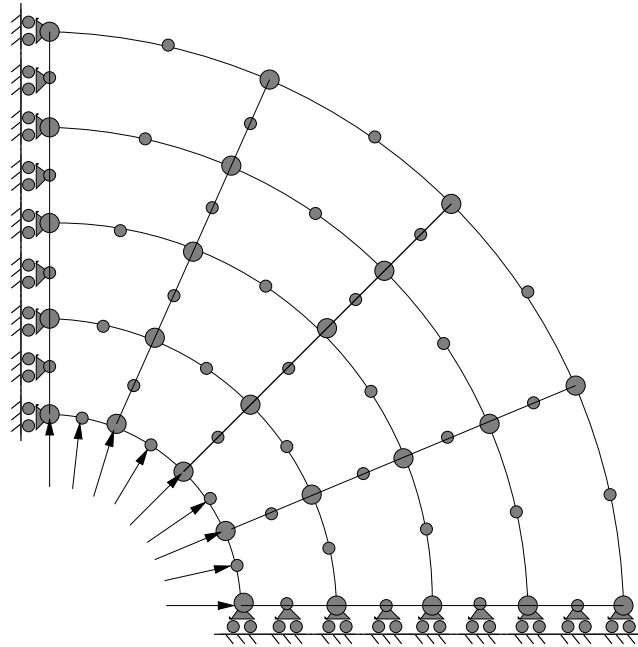


Figure 8.7: Pressurized cylinder mesh

As shown in figure 8.7 both the finite and boundary element meshes used 16 eight noded quadrilateral elements for the domain. The boundary element mesh also included 16 second order three noded boundary elements around the perimeter of the domain. For the boundary element mesh the load was applied as surface tractions to the nodes on the inner radius. For the finite element mesh the load was applied as an equivalent set of nodal point loads on the inner radius. Figures 8.8 and 8.10 show radial and hoop stress distribution results through the cylinder wall for the cases of plane stress and strain respectively. Likewise figures 8.9 and 8.11 show the radial deflection distribution through the cylinder thickness results for the cases of

plane stress and strain respectively.

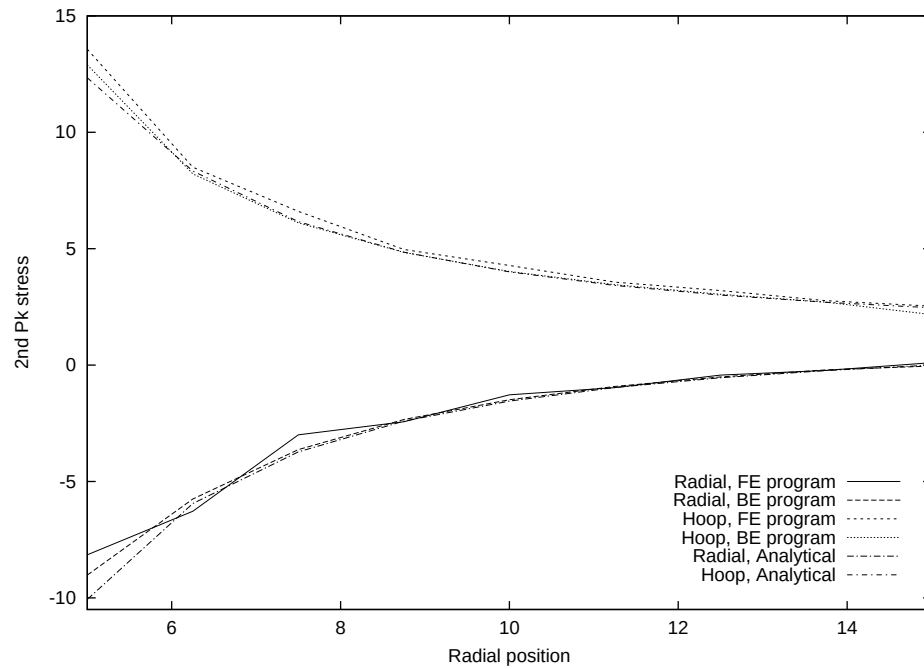


Figure 8.8: Pressurized cylinder stress plot, plane stress

For both the BEM and FEM results it can be seen that agreement between the numerical (FEM and BEM) and direct numerical results is good and that the expected shape of displacement and stress distributions is correct. Specifically, the radial stress is equal and opposite to the internal pressure on the internal radius and zero radial stress on the outer radius (see figures 8.8 and 8.10). In addition the absolute value of hoop stress at all radial positions is greater than the radial stress. Both the FEM and BEM radial displacement results in figure 8.9 show a small constant offset error compared to the direct numerical solution. Again this can be attributed to rounding and digitizing errors in the FEM and BEM algorithms.

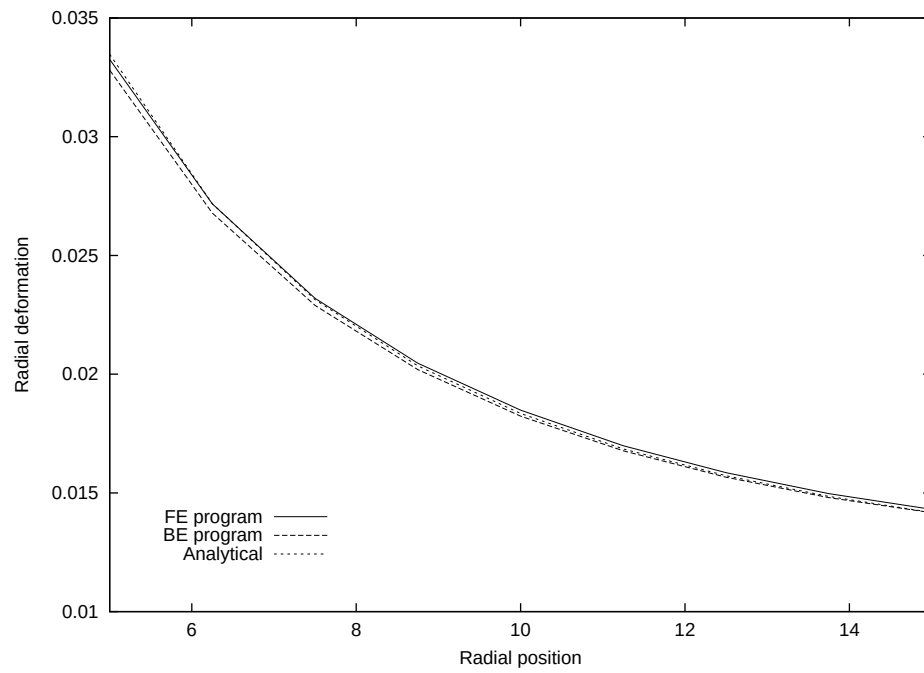


Figure 8.9: Pressurized cylinder radial deflection plot, plane stress

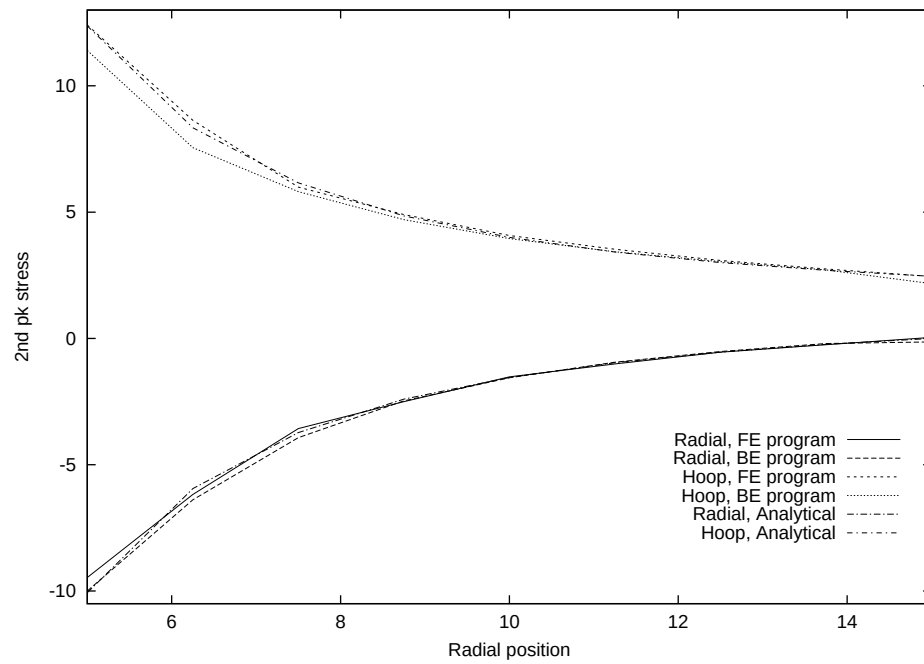


Figure 8.10: Pressurized cylinder stress plot, plane strain

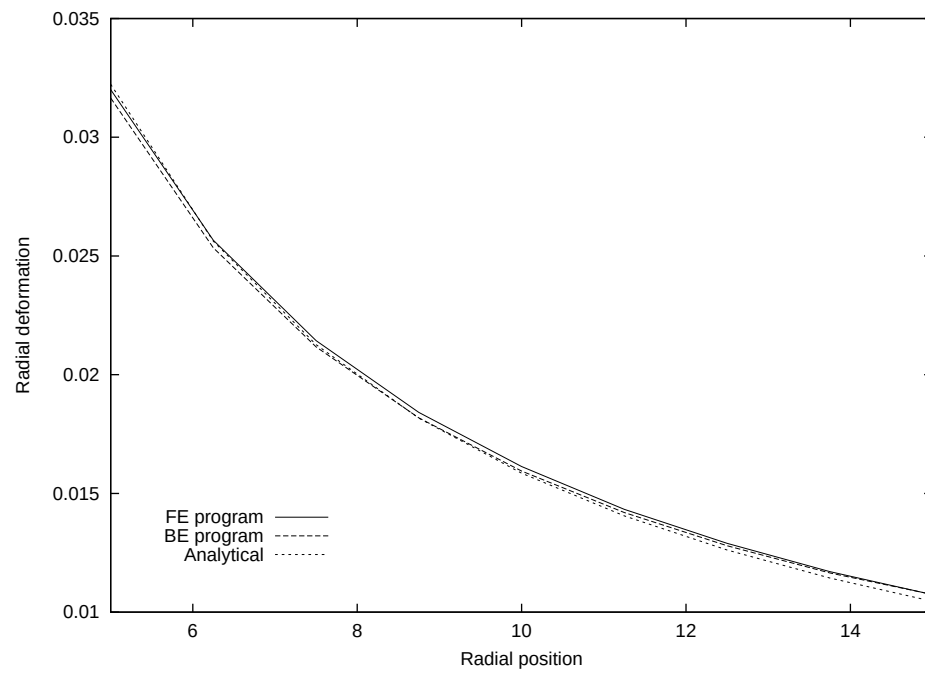


Figure 8.11: Pressurized cylinder radial deflection plot, plane strain

8.4 Rotating disk

Another example of a non-uniform stress field is that due to centrifugal loading of a rotating disk. For this case study a similar mesh geometry to the pressurized cylinder mesh was used except with twice the mesh density to obtain better accuracy. The results shown in figures 8.12 and 8.13 are for the stress and radial deflection distributions through the disk respectively (for the case of plane stress). In both plots results are given for the finite and boundary element methods as well as the direct numerical approach (analytical). The material density used was 0.1 Kg/cm^3 and the angular velocity was 0.1 radians per second. The theory required to implement centrifugal loading for a FEM model is discussed in appendix G section G.1. The theory required to implement inertia loading in the BEM is discussed in appendix G section G.2.

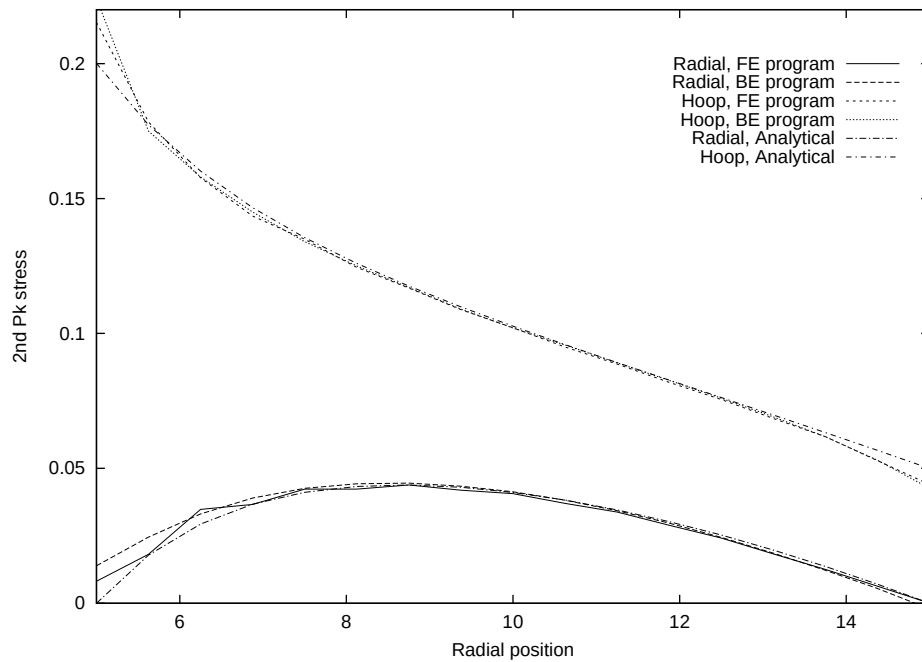


Figure 8.12: Rotating disk stress plot, plane stress

The boundary conditions for the rotating disk test case are satisfied with zero radial stress on the inner and outer radii of the disk (see figure 8.12). Again there is a small offset error for the radial deformation FEM and BEM results shown in figure 8.13 which can be attributed to digitizing effects.

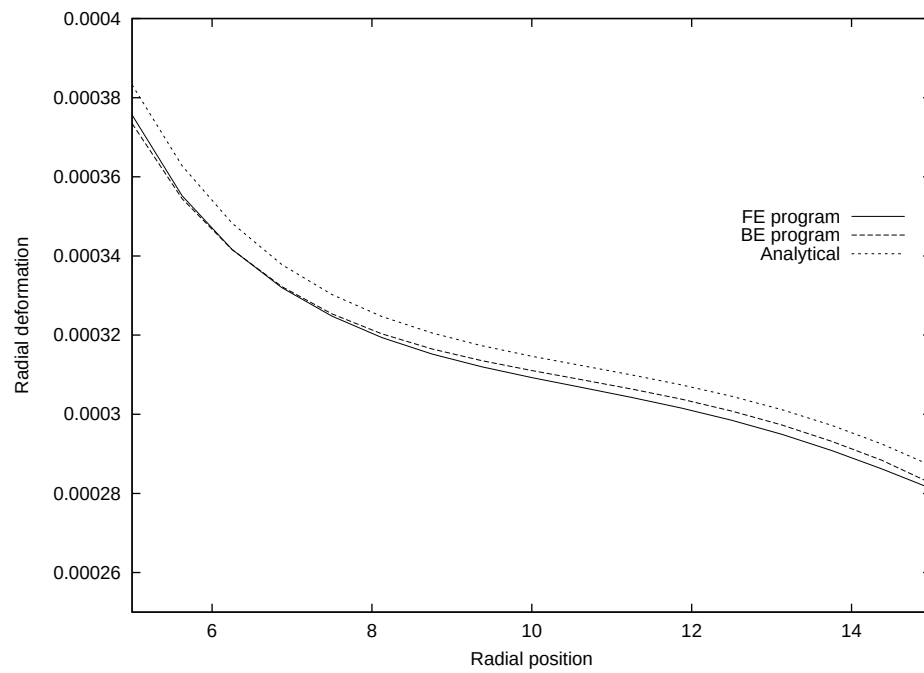


Figure 8.13: Rotating disk radial deflection plot, plane stress

8.5 Centrally cracked plate

In appendix I it was shown by equation I.24 that the negative of the rate of change of potential energy Φ_p with respect to crack length a was equivalent to the J-integral J_r crack propagation criterion. And equation I.23 showed that the rate of change of potential energy with respect to crack length was equal to the derivative of the difference between the total strain energy Φ_a and the work done by externally applied loads F with respect to crack length:

$$\frac{d\Phi_p}{da} = \frac{d}{da} (\Phi_a - F) = -\frac{d}{da} (F - \Phi_a) \quad (8.1)$$

With this definition in mind it is possible to derive a simplified approach to the evaluation of the J-integral J_r for a given crack geometry and loading as an alternative to the expression derived by Rice [40]. If it is assumed that the potential energy Φ_p varies as a quadratic with crack length a then the gradient of a line joining two points on the potential energy curve will be equal to the gradient at a point corresponding to a crack length mid way between the two crack lengths defining the straight line.

Hence for two crack lengths, one incrementally longer than the other, if the total strain energy and work done by external loads can be evaluated then using equation 8.1 the J-integral for the mid crack length can be found, thus:

$$J_r = -\frac{d\Phi_p}{da} \bigg|_{a_1 + \frac{\Delta a}{2}} \equiv -\left(\frac{\Phi_{p2} - \Phi_{p1}}{\Delta a} \right) \quad (8.2)$$

Therefore to further validate the BEM code the case study of a symmetric crack has been chosen to compare J-integral values calculated from the BEM and FEM program for an elastomeric material. The mesh geometry has two planes of symmetry, hence the mesh models one quarter of a plate with a vertically aligned crack arranged symmetrically in the middle of the plate. The plate is uniformly loaded (before deformation) in tension in the x direction as shown in figure 8.14.

This geometry has been interpreted with one quarter of the geometry meshed with all the nodes along the top edge restrained globally in the y direction and all the nodes along the left hand edge restrained horizontally in the x direction (see figure 8.15). The elements used are eight noded quadrilateral elements. The load is applied as a uniformly distributed horizontal load on the right hand edge nodes (for the BEM

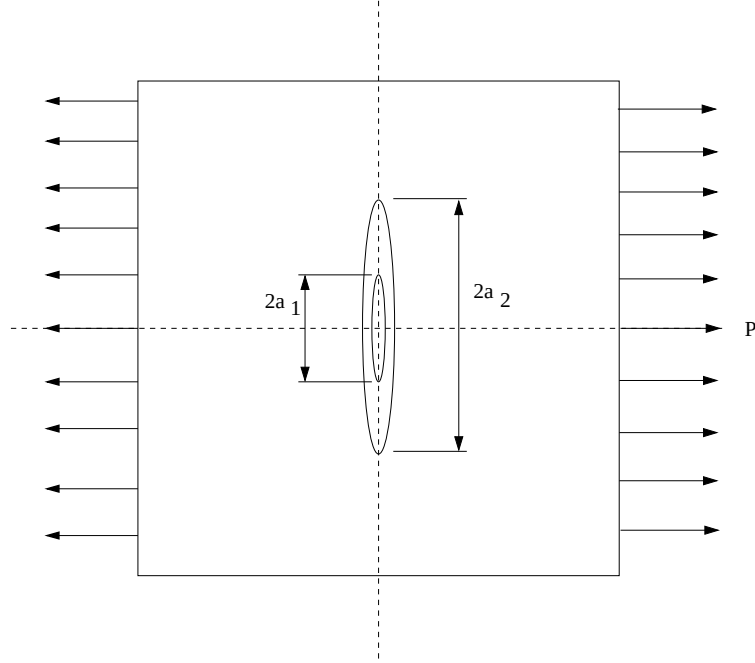


Figure 8.14: Central crack geometry

the load is applied as tractions at the nodes, for the FEM the load is applied as point forces at the nodes). Hence to model the crack the horizontal restraints per node are released from the left hand edge restrained nodes on successive program runs an element at a time. Hence the change in the difference between the strain energy and work done between each program run can be used to give the value of the J-integral at the mid side node of the released element edge. The mesh used contained 256 elements consisting of sixteen rows and columns of second order quadrilateral elements.

8.5.1 J-Integral evaluation, FEM

As shown in equation 8.1 the J-integral J_r is given as the derivative of the difference between the total strain energy Φ_a and the work done by external loads F . Hence the FEM approach is to solve for the mesh and boundary conditions for two identical meshes except the restrained left hand edge has one element edge freed up on the second mesh. The change in the difference between strain energy and externally applied load can then be found for a crack extension of one element length, and then by equation 8.2 the J-integral J_r for the crack length corresponding to the mid side node of the freed element edge. The total strain energy can be evaluated from

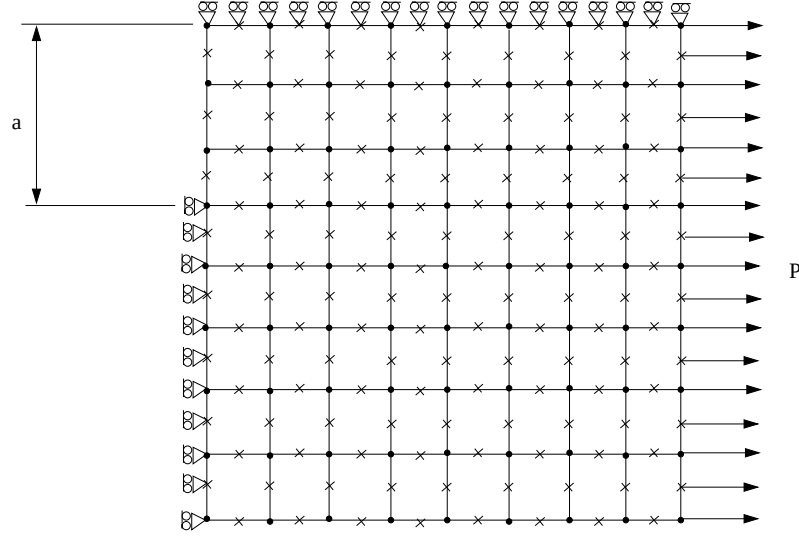


Figure 8.15: Central crack mesh

the integration over the whole domain of the Mooney-Rivlin strain energy function given by equation 4.22, hence:

$$\Phi_a = \int \int \int_{\Omega} \left[C_1 (\bar{I}_1 - 3) + C_2 (\bar{I}_2 - 3) + \frac{K}{2} (J - 1)^2 \right] d\Omega \quad (8.3)$$

The total work done by externally applied loads is simply the sum of the product of the load component and the deflection component at each node with an applied point load, hence:

$$F = \sum_i^2 \sum_j^{n_p} \left([u_i f_i]_j \right) \quad (8.4)$$

where i is the degree of freedom x and y , and j is the j^{th} node out of n_p boundary nodes in the mesh.

8.5.2 J-Integral evaluation, BEM

The J-integral J_r evaluation approach is identical to that for the FEM in that consecutive pairs of BEM program runs are made for crack lengths symmetrically

shorter and longer than the evaluation crack length. Again the difference in the total strain energy and the work done by external loads divided by the crack length extension between the pair of BEM program runs gives the J-integral value, as per equation 8.1. The total strain energy for the BEM is given by equation 8.3. However the work done by externally applied loads cannot be found by summing the products of the prescribed loads and nodal displacements because for the BEM the loads are not available. However if the units are checked for the following expression, which sums the products of the prescribed tractions (surface stresses) and the boundary node displacements, and noting the Jacobian determinant used in the numerical integration has units of length squared then it can be seen that this expression has units of work (Newton meters).

$$F = \sum_i^2 \sum_j^{n_p} \int_{\Gamma} [T_i u_i]_j d\Gamma \quad (8.5)$$

Again i is the degree of freedom for the x and y components and j is the j^{th} node out of n_p boundary nodes.

8.5.3 J-integral J_r results

In figure 8.16 the J-integral J_r values derived from the potential energy approach described in sections 8.5.1 and 8.5.2 for both the BEM and FEM respectively are given in addition to the Rice J-integral results (as discussed in appendix I.3). The results for the potential energy approach were obtained using equation 8.1 for multiple runs of the BEM and FEM programs. The mesh used for both the BEM and FEM for the potential energy approach was similar to figure 8.15, except the actual mesh density was twice that shown and in the results plot the x axis represents the full crack length $2a$. For the Rice J-integral the mesh was identical except the restraints were changed so that all the nodes on the left hand edge were restrained in the x direction and the nodes along the bottom edge were restrained in the y direction with the uniform load applied as tractions acting vertically along the top edge. By doing this, and effectively rotating the crack through 90° the global x axis was aligned with the direction of positive crack length growth, which is required by the derivation given in appendix I.3.

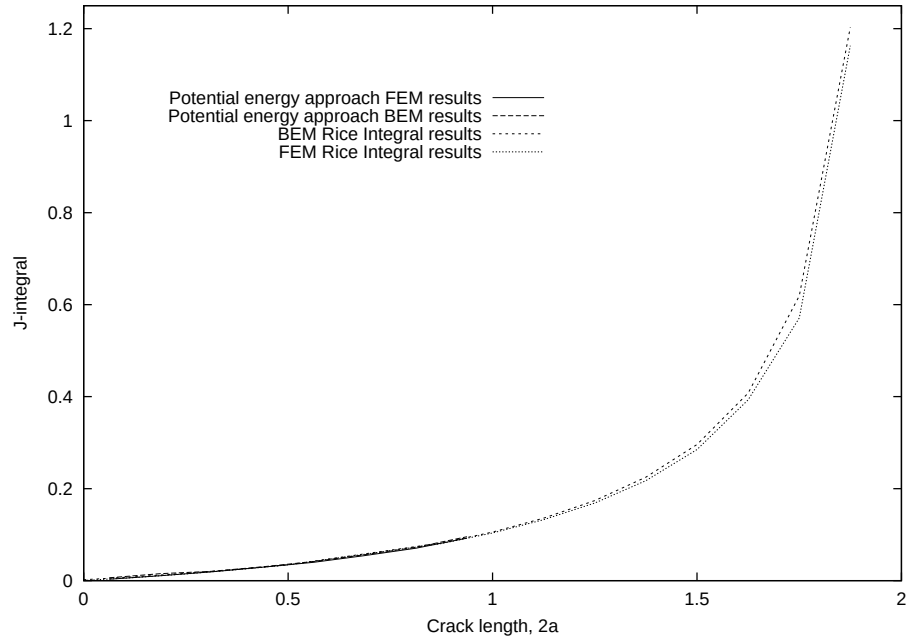


Figure 8.16: J-Integral results J_r

Due to symmetry the J-integral results for both approaches were multiplied by a factor. Taking the potential energy approach first, because only one quarter of the cracked plate was meshed the total potential energy must be multiplied by a factor of four. However because for the total crack length there are two crack tips there will

be total crack growth of $2\Delta a$, hence by equation 8.1 the J-integral value needs to be divided by two. Thus combining the two factors for the potential energy approach the resultant multiplication factor is two. Similarly for the Rice J-integral approach, because the mesh only has half the J-integral contour needed per crack tip these results also need to be factored by two.

Figure 8.16 shows four curves representing the J-Integral results for the potential energy and Rice integral approaches for the FEM and BEM program results utilizing equivalent symmetrical crack meshes. As can be seen the four curves match well and hence provide the validation that the two independently derived J-Integral approaches agree for both the BEM and FEM results. Though it is worth pointing out that the potential energy approach only gave sensible results for crack lengths up to forty five percent of the plate width. The Rice-integral approach appears to give reliable results for crack widths up to ninety percent of the plate width. These results show that the use of the Mooney-Rivlin material model directly to obtain the total strain energy in the system is valid.

Chapter 9

Conclusions

Program algorithm conclusions

In developing the non-linear BEM program several issues specific to the BEM approach became apparent. Most important of which, and something which is not commented on in the literature, is the need to iterate on the hydrostatic pressure to use when calculating the 1st PK stress at source points. As already mentioned (section 6.3.5) this is due to the arbitrary choice of Poisson's ratio resulting in an inaccurate bulk modulus (in the absence of bulk compression test data). This is not an issue for plane stress cases, but for plane strain where the hydrostatic pressure cannot be made a dependent variable it will prove impossible to achieve accurate stress values. Indeed for the more general case of three dimensional analysis this is also the case because the hydrostatic pressure cannot be made a dependent variable. The solution to this problem taken in this work was to iterate on the hydrostatic pressure at a boundary node with a prescribed traction until the calculated stress equaled the prescribed traction (surface stress). The number of iterations was stored and this value was then used as the limiting number of hydrostatic iterations on all other boundary nodes. The iteration was based on calculating the 1st PK stress in the x and y direction and then using the mean stress/hydrostatic pressure relationship to provide a new estimate of the hydrostatic pressure. The maximum number of iterations was required to be ascertained because the hydrostatic pressure iteration was found to be divergent. Hence the iteration was stopped when the correct hydrostatic pressure was achieved. Rather than when the change in pressure between iterations fell below a specified minimum.

There is a form of singularity not detailed in this thesis, namely a weak singularity. For this application it is possible for these to occur during the domain integral numerical integration if a relatively large number of Gauss integration points are

specified. If this is the case, then during the domain integral numerical integration when the source point is on the boundary and the current Gauss field point is inside a domain cell but close to the boundary, then a weak singularity will occur. It is possible to remove these weak singularities directly but the approach taken in the developed BEM program was to evaluate the required field variable (displacement derivatives in this case) on the boundary and internal domain cell nodes and then to use standard isoparametric interpolation to obtain the displacement derivative components at the internal Gauss points. If this approach was not taken and no other allowance was made for the weak singularities then the resulting inaccurate stress values at the internal Gauss points prevented the BEM program converging.

Simplified regularization approach

In chapter 6.2 the theory for the regularization of the various forms of singularity present in the non-linear BIE and DBIE were derived. Before further discussion it is worth summarizing the regularization techniques for the various singularities. The regularized BIE equation 6.51 was given as:

$$\sum_{\beta=1}^2 \oint_{\Gamma} G_{\beta\alpha} [T_{\beta} - \bar{T}_{\beta}^{nl}] d\Gamma - \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega + \sum_{\beta=1}^2 \oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma = \quad (9.1)$$

The regularized singularities present in this BIE are as follows:

1. $\frac{1}{r}$ singularities in the integral $\oint_{\Gamma'} F_{\beta\alpha} u_{\beta} d\Gamma$.
2. $\log \frac{1}{r}$ singularity in the integral $\oint_{\Gamma} G_{\beta\alpha} [T_{\beta} - \bar{T}_{\beta}^{nl}] d\Gamma$.
3. $\frac{1}{r}$ singularity in the integral $\iint_{\Omega} \frac{\partial G_{\gamma\alpha}}{\partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega$.

The first singularity is regularized by the application of the CPV theorem to the integral term. The resulting CPV term is rendered calculable by the use of rigid body motion and the properties of the Dirac Delta function. The $\log \frac{1}{r}$ singularity is dealt with using logarithmic quadrature as discussed in appendix F. The third singularity present in the domain integral is again removed by the application of

the CPV theorem and the CPV term shown to be zero for internal domain source points.

For the DBIE two slightly different forms of regularized equation were derived, equations 6.60 and 6.73:

$$\begin{aligned} \sum_{\beta=1}^2 C_{\beta\alpha_i} \frac{\partial u_{\beta_i}}{\partial x_{\delta_i}} &= \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta}} (u_{\beta} - u_{\beta_i}) d\Gamma - \\ \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_{\delta}} [T_{\beta} - \bar{T}_{\beta}^{nl}] d\Gamma &+ \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \iint_{\Omega} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_{\delta} \partial x_{\beta}} [\sigma_{\beta\gamma}^{nl} - (\sigma_{\beta\gamma}^{nl})_i] d\Omega \end{aligned} \quad (9.2)$$

$$\begin{aligned} \sum_{\beta=1}^2 C_{\alpha\beta_i} \frac{\partial u_{\beta}}{\partial x_{\delta_i}} + \sum_{\beta=1}^2 \oint_{\Gamma'} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta_i}} (u_{\beta} - u_{\beta_i}) d\Gamma &= \\ - \sum_{\beta=1}^2 \oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_{\delta}} T_{\beta} d\Gamma + \lim_{\varepsilon \rightarrow 0} \sum_{\gamma=1}^2 \sum_{\beta=1}^2 \left[\iint_{\Omega'} \sigma_{\beta\gamma}^{nl} J_{\gamma\alpha\delta\beta} d\Omega + (\sigma_{\beta\gamma}^{nl})_i \oint_{\Gamma_{\varepsilon}} T_{\gamma\alpha\delta\beta} d\Gamma \right] \end{aligned} \quad (9.3)$$

The singularities present in DBIE 9.2 and 9.3 are as follows:

1. A $\frac{1}{r^2}$ singularity in the $\oint_{\Gamma} \frac{\partial F_{\beta\alpha}}{\partial x_{\delta}} d\Gamma$ integral.
2. A $\frac{1}{r}$ singularity in the $\oint_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_{\delta}} d\Gamma$ integral.
3. A $\frac{1}{r^2}$ singularity in the $\iint_{\Omega} \frac{\partial^2 G_{\gamma\alpha}}{\partial x_{\delta} \partial x_{\beta}} d\Omega$ domain integral.

The $\frac{1}{r^2}$ singularity in the first type of singularity listed for DBIE has been shown to be regularized by the performance of the program rather than a direct mathematical proof. This is because the singularity present in the integral is of the order 0^{-2} , however the regularizing factor $(u_{\beta} - u_{\beta_i})$ present in equations 9.2 and 9.3 is only of the order 0^1 when the field and source points are coincident. This singularity is only manifest when a source point is coincident with a boundary Gauss point, when the traction kernel derivative $\frac{\partial F_{\beta\alpha}}{\partial x_{\delta}}$ has the singularity 0^{-2} . Therefore when the initial solution of the system of BIEs is made this singularity will not arise as the stresses (and hence the DBIE) are not required at the boundary Gauss points. However, if the stress at a point on the boundary which is coincident with a Gauss point is required then the $\frac{1}{r^2}$ singularity in this integral term will arise. In this event, if post

solution of the system of BIEs a BIE has been solved for the displacement at this source point then the $(u_\beta - u_{\beta_i})$ factor kicks in and cancels the hypersingularity. The $(u_\beta - u_{\beta_i})$ factor is required even when not regularizing a singularity however since it is created as a consequence of applying the rigid body result (equation 6.42) to the derivative of the $F_{\beta\alpha}(u_\beta - u_{\beta_i})$ kernel in the intermediate stage BIE equation 6.57. Thus omission of u_{β_i} will result in non-convergence.

The second $\frac{1}{r}$ singularity listed can again be shown to have no CPV value when the CPV theorem is applied. As shown by equation 6.63, the CPV term evaluates to zero when calculated at an internal source point or when the source point is on a boundary (note the integration limits are changed to $0 \rightarrow 2\pi$ and $0 \rightarrow \pi$ respectively).

The third $\frac{1}{r^2}$ hypersingularity listed can be regularized in one of two ways. The first approach is to use the CPV theorem, then the mean value theorem followed by integration by parts. This shows that the CPV term evaluates to zero, hence equation 9.2 is obtained. The second approach is to 'remove' the subtraction of the integral at the source point from equation 9.2, thus obtaining equation 6.66. With the CPV theorem applied to the last domain integral in equation 6.66, followed by the mean value theorem and integration by parts, the resulting CPV value terms are shown in appendix E to have constant values for internal domain cell Gauss points. Hence, the hypersingularity is removed and the original domain integral rendered calculable.

As can be seen from the results in chapter 8 the developed program clearly converges to finite strains for all the test cases investigated. These test cases included non-uniform stress field results for the pressurized cylinder and rotating disk. The developed FEM program results also shown do not allow for accuracy comparisons to be made but do provide a third set of independent results to prove that the BEM and direct numerical results are correct. However, it can be seen that for the patch test results the FEM program does give the best agreement with the direct numerical results. The BEM results by comparison increasingly diverge from those of the direct numerical solution as the load and strain increase. This can be explained by the increased numerical complexity and hence greater rounding or digitizing effects that can be expected from the cumulative error of real numbers being truncated as binary numbers (note double precision was used in all developed programs). It was mentioned that there are two alternative regularization approaches that can be used for the domain integral term in the DBIE. In the developed program these two approaches were both coded. However, no discernable difference in program results were observed for the two methods. The results contained in this thesis do show that the simplified regularization approach used for the non-linear BIE and DBIE equations for rubber do allow for convergence and accurate results, as will

be discussed in the further work section of this thesis. There are many alternative material models and singularity effects which could be researched further to ascertain whether the BEM can ultimately provide an efficient and accurate numerical approach to the stress analysis of elastomers.

9.1 Summary of original findings and work

The following list summarizes the original findings and work contained in the research reported in this thesis.

1. The derivation, coding and validation of a simplified domain integral singular and hyper-singular regularization approach was original.
2. The use of second order isoparametric interpolation for the 1st PK stress distribution used in the BIE and DBIE domain integral terms had not been done before.
3. Reporting on the need to iterate on the hydrostatic pressure to use in the 2nd PK stress expression at a point when using the Mooney-Rivlin strain energy function in the absence of bulk modulus compressibility data was important. This iteration only being necessary for the case of plane strain or three dimensional stress analysis.
4. For the case of plane stress two dimensional stress analysis the elimination of hydrostatic pressure from the 2nd PK stress formulation by utilizing the zero out of plane stress boundary condition was reported. The 2nd PK stress components were stated explicitly.
5. Validation was obtained using results both from the FEM and the BEM confirming that the Rice J-Integral method and the potential energy J-Integral method are entirely equivalent for evaluating the J-Integral crack propagation criterion.
6. Proof was given that it is valid to use the strain energy function (in this case the Mooney-Rivlin model) to obtain the value of the total system strain energy first integral term in the Rice J-Integral expression.
7. The originally derived and algorithm coding of direct numerical, semi-analytical test case solutions for the uniaxial patch test, pressurized cylinder and rotating disk gave an extra set of validation results.

9.2 List of publications

1. 24th World Boundary Element Conference, Sintra Portugal, June 2002. Paper presented and published entitled 'Boundary element analysis of hyper-elastic elastomeric materials'.
2. ICCES 2003, International Conference on Computational and Experimental Engineering and Sciences, Corfu Greece, July 2003. Paper presented and published entitled 'Numerical modelling of elastomers using the boundary element method'.
3. International Journal for Numerical Methods in Engineering. Paper submitted entitled 'Numerical modelling of elastomers using the boundary element method'.

9.3 Future work

The following list summarizes the research work that could be continued from the results and findings presented in this thesis.

1. Implementation of additional material models such as the Rivlin Polynomial $N = 2$ and Ogden material models could be attempted. This would entail the derivation of first and second partial derivatives of the strain energy models w.r.t Green's strain (twice the derivative w.r.t the right Cauchy-Green tensor $\underline{C}$ components), and deriving the corresponding direct numerical (analytical) validation solutions utilizing these material models.
2. Testing the Guiggiani [23] Taylor series expansion singular and hypersingular regularization domain integral approach for elastomers to provide an alternative approach to regularizing the domain integral singularities.
3. Additional coding to investigate the implementation of near logarithmic, singular and hyper-singular integral regularization techniques could be undertaken.
4. To expand the capability of the developed program a BEM algorithm for three dimensional problems could be developed.

References

- [1] N. Alan. *Engineering with rubber. How to design rubber components*. Rubber division of American chemical society. Hanser, New York, 1992.
- [2] Y. Basar and M. Itskov. Finite element formulation of the ogden material model with application to rubber-like shells. *Int. J. for Num. Meth. in Eng.*, 42:1279–1305, 1998.
- [3] M. C. Boyce. Direct comparison of the gent and the arruda-boyce constitutive models of rubber elasticity. *Rubber chemistry and technology*, 69:781–785, 1996.
- [4] D. Broek. *Elementary engineering fracture mechanics*. Martinus Nijhoff publishers, 1982.
- [5] S. Bu. Impedance functions of square foundations embedded in an incompressible half-space. *Engineering analysis with boundary elements*, 22:1–11, 1998.
- [6] H. D. Bui. An integral equations method for solving the problems of a plane crack of arbitrary shape. *J Mech Phys Solids*, 25:29–39, 1977.
- [7] M. E. Canga and E. B. Becker. An iterative technique for the finite element analysis of near-incompressible materials. *Comput. Meth. Appl. Mech. Engrg.*, 170:79–101, 1999.
- [8] D. J. Charlton. A review of the methods to characterise rubber elastic behaviour for use in finite element analysis. *Rubber chemistry and technology*, 67:481–503, 1994.
- [9] J. S. Chen, K. Satyamurthy, and L.R. Hirschfeld. Consistent finite element procedures for non-linear rubber elasticity with a higher order strain energy function. *Computers and Structures*, 50:715–727, 1994.
- [10] M. A. Crisfield. *Non-linear finite element analysis of solids and structures*. John Wiley and Sons, 1997.

- [11] C. K. L. Davies, D. K. De, and A. G. Thomas. Characterisation of the behaviour of rubber for engineering design purposes: 1: Stress-strain relations. *Rubber chemistry and technology*, 67:716–728, 1994.
- [12] A. El-Zafrany. Corner jump functions for two-d elasticity problems. Technical report, Cranfield University, 1990.
- [13] A. El-Zafrany. *Techniques of the boundary element method*. Ellis Horwood Ltd, 1993.
- [14] A. El-Zafrany and R. A. Cookson. Boundary element stress analysis with domain type loading. *Advances in the use of the boundary element method for stress analysis, Mech. Eng. Publ. London*, 1987.
- [15] H. L. Ewalds and R. J. H. Wanhill. *Fracture mechanics*. Edward Arnold Publishing, 1986.
- [16] J. Fine, E. B. Becker, and T. Miller. Finite element analysis and the design of rubber components. Technical report, Short course notes, Austin Texas, 1993.
- [17] I. Fried and A. R. Johnson. A note on elastic energy density functions for largely deformed compressible rubber solids. *Compt. Meth. Appl. Mech.*, 69:53, 1988.
- [18] H. J. Gahtani and N. J. Altiero. Application of the boundary element method to rubber-like elasticity. *Apply. Math. Modelling*, 20, 1996.
- [19] J. J. Granados and R. Gallego. Regularization of nearly hypersingular integrals in the boundary element method. *Engineering analysis with boundary elements*, 25:165–184, 2001.
- [20] L. J. Gray and C. S. Soucie. A hermite interpolation algorithm for hypersingular boundary integrals. *Int J Numer Meth Eng*, 36:2357–2367, 1993.
- [21] A. A. Griffith. The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc. Lond.*, A(221):163–197, 1921.
- [22] M. Guiggiani and A. Gigante. A general algorithm for multidimensional cauchy principal value integrals in the boundary element method. *ASME Journal of Applied Mechanics*, 57:906–915, 1990.
- [23] M. Guiggiani, G. Krishnasary, T.J. Rudolphi, and F.J. Rizzo. A general algorithm for the numerical solution of hypersingular boundary integral equations. *ASME Journal of Applied Mechanics*, 59:604–614, 1992.

- [24] M. Guiggiani and P. Castalini. Direct computation of cauchy principal value integrals in advanced boundary elements. *International Journal of Numerical Methods in Engineering*, 24:1711–1720, 1987.
- [25] W. Hackbusch and S. A. Sauter. On numerical cubatures of nearly singular surface integrals in bem collocation. *Computing*, 52:139–159, 1994.
- [26] G. A. Holzapfel. On large strain viscoelasticity: Continuum formulation and finite element applications to elastomeric structures. *Int. J. for Num. Meth. In Eng.*, 39:3903–3926, 1996.
- [27] M. Kaliske and H. Rothert. On the finite element implementation of rubber-like materials at finite strains. *Engineering Computations*, 14:216–232, 1997.
- [28] R. P. Kanwal. *Linear integral equations*. Academic press inc., 1971.
- [29] P. B. Lindley. Plane stress analysis of rubber at high strains using finite elements. *Journal of Strain Analysis*, 6(1):45–52, 1971.
- [30] D. S. Malkus and T. J. R. Hughes. Mixed finite element methods - reduced integration techniques: a unification of concepts. *Comp. Meth. Appl. Mech. Engng.*, 15:63–81, 1978.
- [31] P. A. Martin and F. J. Rizzo. Hypersingular integrals: how smooth must the density be? *International journal for numerical methods in engineering*, 39:687–704, 1996.
- [32] M. Mooney. A theory of large elastic deformations. *J. Apply. Phys.*, 11:582, 1940.
- [33] E. A. S. Neto, D. Peric, M. Dutko, and D. R. J. Owen. Design of simple low order finite elements for large strain analysis of nearly incompressible solids. *Int. J. Solids Structures*, 33(20-22):3277–3296, 1996.
- [34] J. T. Oden. A theory of penalty methods for finite element approximation of highly non-linear problems in continuum mechanics. *Comp. Struc.*, 8:445–449, 1978.
- [35] R. W. Ogden. Recent advances in the phenomenological theory of rubber elasticity. *Rubber chemistry and technology*, 59:361–383, 1986.
- [36] A. P. Parker. *The mechanics of fracture and fatigue: an introduction*. Royal Military College of Sci. E. and F.N.SPON, 1981.
- [37] N. Phan-Tien. Rubber like elasticity by boundary element method: Finite deformation of a circular slice. *Rheol. Acta.*, 27:230–240, 1988.

- [38] C. Polizzotto. A symmetric galerkin boundary/domain element method for finite elastic deformations. *Comput. Methods Apply. Mech. Engrg.*, 189:481–514, 2000.
- [39] D. Polyzos, S. V. Tsinopoulos, and D. E. Beskos. Static and dynamic boundary element analysis in incompressible linear elasticity. *Eur. J. Mech. A. Solids*, 17:515–536, 1998.
- [40] J. R. Rice. A path independent integral and approximate analysis of strain concentrations by notches and cracks. *Trans. ASME, J. Appl. Mech.*, 35:379–386, 1964.
- [41] R. S. Rivlin. Large elastic deformations of isotropic materials. ii: Some simple problems in cylindrical polar coordinates. *British Rubber Producer’s Research Association*, 240:509–525, 1948.
- [42] R. S. Rivlin. Large elastic deformations of isotropic materials. iv: Further developments of the general theory. *Philos. Trans. R. Soc. London*, A241:379, 1948.
- [43] F. Rizzo and D. J. Shippy. An advanced boundary integral method for three dimensional elasticity. *Int J Numer Meth Eng*, 11:1751–1768, 1977.
- [44] F. J. Rizzo and D. J. Shippy. *Developments in boundary element methods-4*. Elsevier Applied Science Publishers, Barking, Essex, England, 1986. P.K. Banerjee, J.O. Watson, A boundary element method for axisymmetric elastic bodies, pp 67-69.
- [45] S. L. Rosen. *Fundamental principles of polymeric materials for practising engineers*. Barnes and Noble Inc, NY, 1971.
- [46] T. Scharnhorst and T. H. H. Pian. Finite element analysis of rubber like materials by a mixed model. *Int. J. Num. Meth. Engng.*, 12:665–676, 1978.
- [47] V. Sladek, J. Sladek, and M. Tanaka. Some numerical aspects in computation of singular integrals in bems, in 3d. In *50th Proceedings of the JASCOME meeting*, pages 1–33. JASCOME, 1997.
- [48] V. Sladek, J. Sladek, and M. Tanaka. Optimal transformations of the integration variables in computation of singular integrals in bem. *International journal of numerical methods in engineering*, 47:1263–1283, 2000.
- [49] V. Sladek, J. Sladek, and M. Tanaka. Numerical integration of logarithmic and nearly logarithmic singularity in bems. *Applied Mathematical Modelling*, 25:901–922, 2001.

- [50] A. H. Stroud and D. Secrest. *Gaussian Quadrature Formulas*. Prentice-Hall, Englewood NJ, Cliffs, 1966.
- [51] S. R. Swanson. A constitutive model for high elongation elastic materials. transactions of asme. *J. ENG. MAT. AND TECH.*, 107:110–114, 1985.
- [52] M. Tanaks, V. Saldek, and J. Sladek. Regularisation techniques applied to boundary element methods. *ASME, Appli Mech Rev*, 47(10), October 1994.
- [53] L. R. G. Treloar. *The physics of rubber elasticity*. Oxford University Press, England, 2nd edition, 1958.
- [54] D. M. Turner and M. Brennan. The multiaxial elastic behaviour of rubber. *Plastics and Rubber Processing and Applications*, 14:183–188, 1990.
- [55] K. C. Valanis and R. F. Landel. The strain energy function of a hyperelastic material in terms of extension ratios. *J. Apply. Phys.*, 38:2997, 1967.
- [56] O. H. Yeoh. Some forms of the strain energy function for rubber. *Rubber chemistry and technology*, 66:754–771, 1993.
- [57] O. C. Zienkiewicz and R. L. Taylor. *The finite element method*. Butterworth and Heinemann, 2000.

Appendix A

Tensor theorems

It is essential to develop some mathematical theorems concerning the differentiation of vector and matrix quantities in order to proceed further with the constitutive modelling of rubber given in chapter 3. The strain invariants will also be differentiated in this appendix.

A.1 Theorem I: Differentiation of a scalar with respect to a matrix quantity

If X is a scalar quantity which is a function of $Y_{n \times n}$ then by the rules of differentiation:

$$dX = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial X}{\partial Y_{ij}} dY_{ij} \quad (\text{A.1})$$

defining:

$$\underline{A} = \frac{\partial X}{\partial \underline{Y}} \quad A_{ij} = \frac{\partial X}{\partial Y_{ij}} \quad (\text{A.2})$$

therefore:

$$dX = \sum_{i=1}^n \sum_{j=1}^n A_{ij} dY_{ij} \quad (\text{A.3})$$

If $\underline{C} = \underline{A}\underline{B}^t$ then:

$$C_{ij} = \sum_k A_{ik} (B_{kj})^t = \sum_k A_{ik} B_{jk}$$

therefore:

$$C_{ii} = \sum_k A_{ik} B_{ik} = \sum_j A_{ij} B_{ij} \equiv \text{Trace}(\underline{C})$$

hence:

$$\begin{aligned} dX &= \text{Trace} \left(\frac{\partial X}{\partial \underline{Y}} d\underline{Y}^t \right) = \frac{\partial X}{\partial \underline{Y}} : d\underline{Y} \\ dX &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial X}{\partial Y_{ij}} dY_{ij} \end{aligned}$$

Result:

if:

$$dX = \underline{A} : d\underline{Y} \quad (\text{A.4})$$

then:

$$\frac{\partial X}{\partial \underline{Y}} = \underline{A} \quad (\text{A.5})$$

A.2 Theorem II: Differentiation of a matrix quantity

If $\underline{X}$ is a matrix quantity of order $n \times n$, and $\underline{Y}$ is also a matrix quantity of order $n \times n$ with $\underline{X}$ being a function of $\underline{Y}$, then using the rules of differentiation:

$$dX_{ij} = \sum_k^n \sum_l^n \frac{\partial X_{ij}}{\partial Y_{kl}} dY_{kl} \quad (\text{A.6})$$

Defining a fourth order tensor $\underline{Z}$, such that:

$$Z_{ijkl} = \frac{\partial X_{ij}}{\partial Y_{kl}} \quad (\text{A.7})$$

then:

$$dX_{ij} = \sum_{k=1}^n \sum_{l=1}^n \frac{\partial X_{ij}}{\partial Y_{kl}} dY_{kl} \equiv \sum_{k=1}^n \sum_{l=1}^n Z_{ijkl} dY_{kl} \quad (\text{A.8})$$

and:

$$\underline{Z} = \frac{\partial \underline{X}}{\partial \underline{Y}} \equiv Z_{ijkl} = \frac{\partial X_{ij}}{\partial Y_{kl}} \quad (\text{A.9})$$

Result:

$$d\underline{X} = \underline{Z} : d\underline{Y} \quad (\text{A.10})$$

A.3 Theorem III: Differentiation of a matrix product

Let $\underline{X} = \underline{Y} \underline{Z}$ then:

$$d\underline{X} = d\underline{Y} \underline{Z} + \underline{Y} d\underline{Z} \quad (\text{A.11})$$

Or with $X_{ij} = \sum_{k=1}^n Y_{ik} Z_{kj}$ then:

$$dX_{ij} = \sum_{k=1}^n (dY_{ik} Z_{kj} + Y_{ik} dZ_{kj}) \quad (\text{A.12})$$

If $\underline{X}$, $\underline{Y}$ and $\underline{Z}$ are functions of $\underline{\phi}$ then:

$$\frac{\partial X_{ij}}{\partial \phi_{lm}} = \sum_{k=1}^n \left(\frac{\partial Y_{ik}}{\partial \phi_{lm}} Z_{kj} + Y_{ik} \frac{\partial Z_{kj}}{\partial \phi_{lm}} \right) \quad (\text{A.13})$$

and:

$$dX_{ij} = \sum_l \sum_m \sum_k \left(\frac{\partial Y_{ik}}{\partial \phi_{lm}} Z_{kj} + Y_{ik} \frac{\partial Z_{kj}}{\partial \phi_{lm}} \right) d\phi_{lm} \quad (\text{A.14})$$

Result:

$$d\underline{X} = \left(\frac{\partial \underline{Y}}{\partial \underline{\phi}} : d\underline{\phi} \right) \underline{Z} + \underline{Y} \left(\frac{\partial \underline{Z}}{\partial \underline{\phi}} : d\underline{\phi} \right) \quad (\text{A.15})$$

A.4 Theorem IV: Differentiation of the inverse of a matrix

Noticing that:

$$\underline{C} \underline{C}^{-1} = \underline{C}^{-1} \underline{C} = \underline{I}$$

then:

$$(d\underline{C}) \underline{C}^{-1} + \underline{C} (d\underline{C}^{-1}) = (d\underline{C}^{-1}) \underline{C} + \underline{C}^{-1} (d\underline{C}) = 0 \quad (\text{A.16})$$

Multiplying by $\underline{C}^{-1}$ then:

$$(d\underline{C}^{-1}) = -\underline{C}^{-1} d\underline{C} \underline{C}^{-1} \quad (\text{A.17})$$

hence:

$$dC_{ij}^{-1} = - \sum_k \sum_{k'} C_{ik}^{-1} C_{k'j}^{-1} dC_{kk'} \quad (\text{A.18})$$

Result:

$$\frac{\partial C_{ij}^{-1}}{\partial C_{lm}} = -C_{il}^{-1} C_{mj}^{-1} \quad (\text{A.19})$$

A.5 Theorem V: Differentiation of a determinant

Let:

$$\underline{A} = \text{Cofactor}(\underline{C})$$

then:

$$\underline{C}^{-1} = \underline{A}^t / |\underline{C}|$$

$$\underline{A}^t = |\underline{C}| \underline{C}^{-1}$$

post multiplying by $\underline{C}$:

$$\underline{A}^t \underline{C} = |\underline{C}|$$

therefore:

$$d|\underline{C}| = \underline{A}^t d\underline{C}$$

hence it can be deduced that:

$$d|\underline{C}| = |\underline{C}| \underline{C}^{-1} : d\underline{C}$$

Result:

$$\frac{\partial |\underline{C}|}{\partial \underline{C}} = |\underline{C}| \underline{C}^{-1} \quad (\text{A.20})$$

A.6 Differentiation of strain invariants

A.6.1 Differentiation of I_1

By definition:

$$I_1 = \text{Trace}(\underline{C}) = C_{11} + C_{22} + C_{33}$$

$$I_1 = \text{Trace}(d\underline{C}) \equiv \underline{I} : d\underline{C} \quad (\text{A.21})$$

hence:

$$dI_1 = dC_{11} + dC_{22} + dC_{33} \quad (\text{A.22})$$

hence:

$$\frac{\partial I_1}{\partial \underline{C}} = \underline{I} \equiv \frac{\partial I_1}{\partial C_{ij}} = \delta_{ij} \quad (\text{A.23})$$

Using modified invariants:

$$\bar{I}_1 = I_3^{-\frac{1}{3}} I_1 \quad (\text{A.24})$$

then:

$$\begin{aligned} d\bar{I}_1 &= \left(-\frac{1}{3} I_3^{-\frac{4}{3}} dI_3 \right) I_1 + I_3^{-\frac{1}{3}} dI_1 \\ d\bar{I}_1 &= -\frac{1}{3} I_3^{-\frac{1}{3}} I_1 \underline{C}^{-1} : d\underline{C} + I_3^{-\frac{1}{3}} \underline{I} : d\underline{C} \\ d\bar{I}_1 &= I_3^{-\frac{1}{3}} \left(\underline{I} - \frac{1}{3} I_1 \underline{C}^{-1} \right) : d\underline{C} \end{aligned} \quad (\text{A.25})$$

hence:

$$\frac{\partial \bar{I}_1}{\partial \underline{C}} = I_3^{-\frac{1}{3}} \left(\underline{I} - \frac{1}{3} I_1 \underline{C}^{-1} \right)$$

or:

$$\frac{\partial \bar{I}_1}{\partial C_{ij}} = I_3^{-\frac{1}{3}} \left(\delta_{ij} - \frac{1}{3} I_1 C_{ij}^{-1} \right) \quad (\text{A.26})$$

A.6.2 Differentiation of I_2

By definition:

$$I_2 = \frac{1}{2} (I_1^2 - \text{Trace}(\underline{C}^2)) \quad (\text{A.27})$$

hence:

$$\begin{aligned} dI_2 &= \frac{1}{2} [2I_1 dI_1 - 2\text{Trace}(\underline{C} d\underline{C})] \\ dI_2 &= I_1 (\underline{I} : d\underline{C}) - \underline{C} : d\underline{C} \\ dI_2 &= (I_1 \underline{I} - \underline{C}) : d\underline{C} \end{aligned} \quad (\text{A.28})$$

therefore:

$$\frac{\partial I_2}{\partial \underline{C}} = I_1 \underline{I} - \underline{C}$$

or:

$$\frac{\partial I_2}{\partial C_{ij}} = \delta_{ij} I_1 - C_{ij} \quad (\text{A.29})$$

using:

$$\bar{I}_2 = I_3^{-\frac{2}{3}} I_2 \quad (\text{A.30})$$

then:

$$\begin{aligned} d\bar{I}_2 &= \left(-\frac{2}{3} I_3^{-\frac{5}{3}} dI_3 \right) I_2 + I_3^{-\frac{2}{3}} dI_2 \\ d\bar{I}_2 &= -\frac{2}{3} I_3^{-\frac{2}{3}} I_2 \underline{C}^{-1} : d\underline{C} + I_3^{-\frac{2}{3}} (I_1 \underline{I} - \underline{C}) : d\underline{C} \\ d\bar{I}_2 &= I_3^{-\frac{2}{3}} \left(I_1 \underline{I} - \underline{C} - \frac{2}{3} I_2 \underline{C}^{-1} \right) : d\underline{C} \end{aligned} \quad (\text{A.31})$$

Result:

$$\frac{\partial \bar{I}_2}{\partial \underline{C}} = I_3^{-\frac{2}{3}} \left(I_1 \underline{I} - \underline{C} - \frac{2}{3} I_2 \underline{C}^{-1} \right)$$

or:

$$\frac{\partial \bar{I}_2}{\partial C_{ij}} = I_3^{-\frac{2}{3}} \left(\delta_{ij} I_1 - C_{ij} - \frac{2}{3} I_2 C_{ij}^{-1} \right) \quad (\text{A.32})$$

A.6.3 Differentiation of I_3

By definition:

$$I_3 = |\underline{C}| \tag{A.33}$$

using A.20 it can be deduced that:

$$dI_3 = I_3 \underline{C}^{-1} : d\underline{C} \tag{A.34}$$

hence:

$$\frac{\partial I_3}{\partial \underline{C}} = I_3 \underline{C}^{-1} \tag{A.35}$$

Noticing also that $\bar{I}_3 = 1$ hence $d\bar{I}_3 = 0$. Using $J = \sqrt{I_3}$, therefore:

$$\begin{aligned} dJ &= \frac{1}{2} I_3^{-\frac{1}{2}} dI_3 \\ dJ &= \frac{1}{2} I_3^{\frac{1}{2}} \underline{C}^{-1} : d\underline{C} \end{aligned} \tag{A.36}$$

Result:

$$\frac{\partial J}{\partial \underline{C}} = \frac{1}{2} I_3^{\frac{1}{2}} \underline{C}^{-1} \tag{A.37}$$

Appendix B

Kernel functions and their derivatives

In this appendix the displacement and traction kernel functions $G_{\alpha\beta}$ $F_{\alpha\beta}$ respectively are derived for two dimensional stress analysis problems. The requirement to formulate DBIE to directly compute strain components and hence stresses also dictates that expressions for $\frac{\partial G_{\alpha\beta}}{\partial x_\gamma}$, $\frac{\partial F_{\alpha\beta}}{\partial x_\gamma}$ and $\frac{\partial G_{\alpha\beta}^2}{\partial x_\delta \partial x_\gamma}$ are also evaluated.

B.1 Fundamental solution derivation

The stress strain compatibility equations can be written as:

$$\sigma_x^* = d_{11}\epsilon_x^* + d_{12}\epsilon_y^* \quad (\text{B.1})$$

$$\sigma_y^* = d_{21}\epsilon_x^* + d_{22}\epsilon_y^* \quad (\text{B.2})$$

$$\tau_{xy}^* = d_{33}\gamma_{xy}^* \quad (\text{B.3})$$

where:

$$d_{11} = d_{22} = 2\mu(1-p)/(1-2p)$$

$$d_{12} = d_{21} = 2\mu/(1-2p)$$

$$d_{33} = \mu$$

p is the modified Poisson's ratio and μ is the shear modulus. The equations of equilibrium can be stated as follows:

$$\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{xy}^*}{\partial y} + f_x^* = 0 \quad (\text{B.4})$$

$$\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} + f_y^* = 0 \quad (\text{B.5})$$

From equation B.4 and equations B.1 to B.3:

$$\nabla u^* + \frac{1}{(1-2p)} \frac{\partial}{\partial x} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) + \frac{f_x^*}{\mu} = 0 \quad (\text{B.6})$$

similarly:

$$\nabla v^* + \frac{1}{(1-2p)} \frac{\partial}{\partial y} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) + \frac{f_y^*}{\mu} = 0 \quad (\text{B.7})$$

Equations B.6 and B.7 can be re-written as:

$$\nabla u^* + \frac{1}{(1-2p)} \frac{\partial}{\partial x} (\nabla \cdot \vec{q}^*) + \frac{f_x^*}{\mu} = 0 \quad (\text{B.8})$$

$$\nabla v^* + \frac{1}{(1-2p)} \frac{\partial}{\partial y} (\nabla \cdot \vec{q}^*) + \frac{f_y^*}{\mu} = 0 \quad (\text{B.9})$$

The displacement vector can be written in terms of the 'Galerkin vector' defined as follows:

$$\vec{G}^* = G_x \hat{i} + G_y \hat{j} \quad (\text{B.10})$$

such that:

$$\vec{q}^* = \nabla^2 \vec{G}^* - \frac{1}{2(1-p)} \nabla (\nabla \cdot \vec{G}^*) \quad (\text{B.11})$$

Allowing the governing partial differential equation to be written as:

$$\nabla^2 \vec{q}^* + \frac{1}{(1-2p)} \nabla (\nabla \cdot \vec{q}^*) + \frac{\vec{f}^*}{\mu} = 0 \quad (\text{B.12})$$

Substituting equation B.11 into equation B.12 it can be shown that:

$$\nabla^4 \vec{G}^* + \frac{\vec{f}^*}{\mu} = 0 \quad (\text{B.13})$$

Defining:

$$G_x^* = g^* e_x \quad G_y^* = g^* e_y \quad (\text{B.14})$$

$$f_x^* = e_x \delta(x - x_i, y - y_i) \quad (\text{B.15})$$

$$f_y^* = e_y \delta(x - x_i, y - y_i) \quad (\text{B.16})$$

hence equation B.13 can be written as:

$$\nabla^4 g^* + \frac{\delta(x - x_i, y - y_i)}{\mu} = 0 \quad (\text{B.17})$$

Defining the function:

$$\nabla^2 g^* = \frac{w^*}{\mu} \quad (\text{B.18})$$

Allowing the governing partial differential equation to be written:

$$\nabla^2 w^* - \delta(x - x_i, y - y_i) = 0 \quad (\text{B.19})$$

The solution of equation B.19 (which is the fundamental solution) can be obtained by considering the case of 2-D thermal heat conduction. The governing equation for this case is:

$$\nabla^2 T^* + \overline{Q}(x, y)/K = 0 \quad (\text{B.20})$$

$\overline{Q}$ being the heat generated per unit area. Consider a point heat source in a infinite domain as shown in figure B.1. Thus defining the heat flux at the field point and

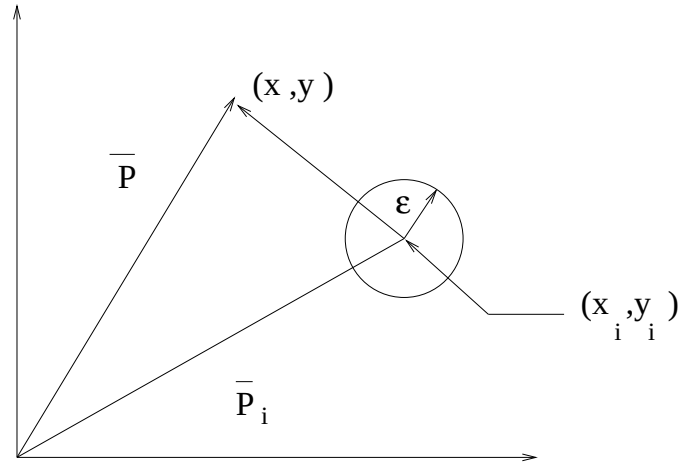


Figure B.1: Field and source points for a steady state field problem.

using the 'Delta Dirac' function:

$$\overline{Q}(x, y) = Q\delta(x - x_i, y - y_i) \quad (\text{B.21})$$

Enabling equation B.20 to be written as:

$$\nabla^2 T^* + \left(\frac{Q}{K}\right) \delta(x - x_i, y - y_i) = 0 \quad (\text{B.22})$$

Which is of the same form as equation B.19. Equation B.22 can be solved as follows. The heat flux vector of a circle of radius r centred on a source point will be:

$$\vec{g} = -K \nabla T^* \quad (\text{B.23})$$

Defining a unit vector $\hat{r}$ in the direction of $\vec{r}$ such that:

$$\hat{r} = (x - x_i)\hat{i} + (y - y_i)\hat{j} \quad (\text{B.24})$$

Hence the heat flux magnitude across the circle will be given by:

$$g = \vec{q} \cdot \hat{r} = -K \nabla T^* \cdot \hat{r} \quad (\text{B.25})$$

The heat flux magnitude across the circle can also be given by:

$$g = \frac{Q}{2\pi r} \quad (\text{B.26})$$

Hence equating equations B.25 and B.26 gives:

$$\frac{dT^*}{dr} = -\frac{Q}{2\pi K r} \quad (\text{B.27})$$

which has the solution:

$$T^* = \frac{Q}{K} \left[\frac{1}{2\pi} \log \frac{1}{r} + C \right] \quad (\text{B.28})$$

Thus in general for 2-D field problems:

$$w^* = \frac{1}{2\pi} \log \frac{1}{r} + C_1 \quad (\text{B.29})$$

thus:

$$\nabla^2 g^* = \frac{1}{2\pi\mu} \log \frac{1}{r} + C_1 \quad (\text{B.30})$$

The solution of equation B.30 giving the fundamental solution can be achieved as follows. It can be shown that for cylindrical co-ordinates:

$$\nabla^2 g^* = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad (\text{B.31})$$

In this case $\frac{\partial^2 g^*}{\partial \theta^2} = \frac{\partial^2 g^*}{\partial z^2} = 0$, hence:

$$\nabla^2 g^* = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial g^*}{\partial r} \right) = \frac{1}{2\pi\mu} \log \frac{1}{r} + C_1 \quad (\text{B.32})$$

Solving equation B.32 as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial g^*}{\partial r} \right) = \frac{1}{2\pi\mu} \log \frac{1}{r} + C_1$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial g^*}{\partial r} \right) = \frac{1}{2\pi\mu} r \log \frac{1}{r} + rC_1$$

Integrating the first term by parts with:

$$u = -\log r \quad v = \frac{r^2}{2}$$

thus:

$$r \frac{\partial g^*}{\partial r} = \frac{1}{2\pi\mu} \left[-\frac{r^2 \log r}{2} + \int \frac{r}{2} dr \right] + C_1 \frac{r^2}{2} + C_3$$

$$r \frac{\partial g^*}{\partial r} = \frac{1}{2\pi\mu} \left[-\frac{r^2 \log r}{2} + \frac{r^2}{4} dr \right] + C_1 \frac{r^2}{2} + C_3$$

assume $C_3 = 0$:

$$\frac{\partial g^*}{\partial r} = \frac{1}{2\pi\mu} \left[-\frac{r \log r}{2} + \frac{r}{4} \right] + \frac{C_1 r}{2}$$

$$\frac{\partial g^*}{\partial r} = \frac{1}{4\pi\mu} \left[r \log \frac{1}{r} + \frac{r}{2} \right] + \frac{C_1 r}{2}$$

$$g^* = \frac{1}{4\pi\mu} \left[\frac{r^2 \log \frac{1}{r}}{2} + \frac{r^2}{4} + \frac{r^2}{4} \right] + \frac{C_1 r^2}{4} + C_2$$

$$g^* = \frac{r^2}{8\pi\mu} \left[\log \frac{1}{r} + 1 + C_1 \right] + C_2 \tag{B.33}$$

Writing equation B.11 in tensorial form:

$$u_\alpha^* = \nabla^2 \vec{G}_\alpha^* - \frac{1}{2(1-p)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial G_\alpha^*}{\partial x_1} + \frac{\partial G_\alpha^*}{\partial x_2} \right) \tag{B.34}$$

where:

$$(u_1^*, u_2^*) = (u, v)$$

$$(G_1^*, G_2^*) = (G_x^*, G_y^*)$$

And using equation B.14 then:

$$u_\alpha^* = \nabla g^* e_\alpha - \frac{1}{2(1-p)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial (g^* e_x)}{\partial x_1} + \frac{\partial (g^* e_x)}{\partial x_1} \right) \tag{B.35}$$

Defining:

$$G_{\alpha\beta}(x - x_i, y - y_i) = \nabla^2 g^* \delta_{\alpha\beta} - \frac{1}{2(1-p)} \left(\frac{\partial^2 g^*}{\partial x_\alpha \partial x_\beta} \right) \quad (\text{B.36})$$

Thus equation B.35 can be written as:

$$u_\alpha^*(x - x_i, y - y_i) = G_{\alpha 1}(x - x_i, y - y_i) e_x + G_{\alpha 2}(x - x_i, y - y_i) e_y \quad (\text{B.37})$$

Therefore $G_{\alpha\beta}$ are the displacement kernel functions.

B.2 Explicit form of the displacement kernel $G_{\alpha\beta}$

Differentiating the fundamental solution equation B.33.

$$\begin{aligned} \frac{\partial g^*}{\partial x_\alpha} &= \frac{1}{8\pi\mu} \left[-2r \log r - \frac{r^2}{r} \right] \frac{\partial r}{\partial x_\alpha} + \frac{2r}{8\pi\mu} (C_1 + 1) \frac{\partial r}{\partial x_\alpha} \\ \frac{\partial g^*}{\partial x_\alpha} &= \frac{1}{8\pi\mu} \frac{\partial r}{\partial x_\alpha} [2r(C_1 + 1) - 2r \log r - r] \\ \frac{\partial g^*}{\partial x_\alpha} &= \frac{1}{8\pi\mu} \frac{\partial r}{\partial x_\alpha} [(2C_1 + 1)r - 2r \log r] \end{aligned} \quad (\text{B.38})$$

thus:

$$\frac{\partial^2 g^*}{\partial x_\beta \partial x_\alpha} = \frac{\partial}{\partial x_\beta} \left[\frac{r}{8\pi\mu} \frac{\partial r}{\partial x_\alpha} ([2C_1 + 1] - 2 \log r) \right] \quad (\text{B.39})$$

Theorem I

$$\begin{aligned} r &= ((x - x_i)^2 + (y - y_i)^2)^{\frac{1}{2}} \\ \frac{\partial r}{\partial (x_\alpha)_i} &= \frac{1}{2} \cdot \frac{1}{r} \cdot 2(x_\alpha - x_{\alpha i}) = \left(\frac{x_\alpha - x_{\alpha i}}{r} \right) \\ r \frac{\partial r}{\partial x_{\alpha i}} &= x_\alpha - x_{\alpha i} \\ \frac{\partial}{\partial x_\beta} \left(r \frac{\partial r}{\partial x_{\alpha i}} \right) &= \delta_{\alpha\beta} \end{aligned} \quad (\text{B.40})$$

Hence using theorem I, equation B.39 becomes:

$$\frac{\partial^2 g^*}{\partial x_\beta \partial x_\alpha} = \frac{1}{8\pi\mu} \left[\delta_{\alpha\beta} ([2C_1 + 1] - 2 \log r) - \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \quad (\text{B.41})$$

Substituting equations B.41 and B.30 into equation B.36 it can be shown that:

$$G_{\alpha\beta} = \frac{1}{8\pi\mu(1-p)} \left[\delta_{\alpha\beta} \left([3 - 4p] \left(\log \frac{1}{r} + C_1 \right) - 0.5 \right) + \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \quad (\text{B.42})$$

B.3 Explicit form of the displacement kernel $F_{\alpha\beta}$

Small or Cauchy strain is given by:

$$\varepsilon_{\alpha\beta}^* = \frac{1}{2} \left(\frac{\partial u_{\beta}^*}{\partial x_{\alpha}} + \frac{\partial u_{\alpha}^*}{\partial x_{\beta}} \right) \quad (\text{B.43})$$

Using equation B.37 then it can be shown that:

$$\varepsilon_{\alpha\beta}^* = A_{\alpha\beta 1}^* e_x + A_{\alpha\beta 2}^* e_y \quad (\text{B.44})$$

where:

$$A_{\alpha\beta\gamma}^* = \frac{1}{2} \left(\frac{\partial G_{\beta\alpha}}{\partial x_{\alpha}} + \frac{\partial G_{\alpha\gamma}}{\partial x_{\beta}} \right) \quad (\text{B.45})$$

Using the explicit form of $G_{\alpha\beta}$ given by equation B.42 it can be shown that:

$$A_{\alpha\beta\gamma}^* = \frac{1}{8\pi\mu(1-p)r} \left[(1-2p) \left[\frac{\partial r}{\partial x_{\alpha}} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_{\beta}} \delta_{\alpha\gamma} \right] - \frac{\partial r}{\partial x_{\gamma}} \delta_{\beta\alpha} + 2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial x_{\gamma}} \right]$$

Theorem II

Given:

$$f(x - x_i, y - y_i) \quad \varepsilon = x - x_i \quad \Gamma = y - y_i$$

then

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial f}{\partial \Gamma} \frac{\partial \Gamma}{\partial x_i}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial x} + \frac{\partial f}{\partial \Gamma} \frac{\partial \Gamma}{\partial x}$$

but:

$$\frac{\partial f}{\partial x_i} = -\frac{\partial f}{\partial \varepsilon} d\varepsilon \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \varepsilon} d\varepsilon$$

Therefore:

$$\frac{\partial f}{\partial x_i} = -\frac{\partial f}{\partial x} \quad (\text{B.46})$$

Thus changing the differentiation in equation B.45 from differentiation with respect to the field point to differentiation with respect to the source point and using theorem II (given by equation B.46) then:

$$A_{\alpha\beta\gamma}^* = \frac{-1}{8\pi\mu(1-p)r} \left[(1-2p) \left[\frac{\partial r}{\partial x_{\alpha}} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_{\beta}} \delta_{\alpha\gamma} \right] - \frac{\partial r}{\partial x_{\gamma}} \delta_{\beta\alpha} + 2 \frac{\partial r}{\partial x_{\alpha}} \frac{\partial r}{\partial x_{\beta}} \frac{\partial r}{\partial x_{\gamma}} \right]$$

$$(B.47)$$

Now

$$\sigma_{\alpha\beta} = 2\mu \left[\varepsilon_{\alpha\beta} + \frac{v}{1-2v} (\varepsilon_x + \varepsilon_y + \varepsilon_z) \delta_{\alpha\beta} \right] \quad (B.48)$$

For 2-D problems this equation can be generalized for plane stress or plane strain using a modified Poisson's ratio p .

$$p = v \quad (B.49)$$

$$p = \frac{v}{(1+v)} \quad (B.50)$$

such that:

$$\begin{aligned} \sigma_{\alpha\beta} = & \frac{2\mu}{1-2p} [(1-p)\delta_{1\beta}\varepsilon_{11} + p\delta_{1\beta}\varepsilon_{22}] \delta_{\alpha\beta} + \\ & \frac{2\mu}{1-2p} [(1-p)\delta_{2\beta}\varepsilon_{22} + p\delta_{2\beta}\varepsilon_{11}] \delta_{\alpha\beta} + \mu(1-\delta_{\alpha\beta})\varepsilon_{\alpha\beta} \end{aligned} \quad (B.51)$$

Using index notation:

$$\sigma_{\alpha\beta} = \frac{2\mu}{(1-2p)} \delta_{\alpha\beta} [(1-p)\varepsilon_{\alpha\beta} + p\varepsilon_{\gamma\gamma}(1-\delta_{\gamma\alpha})] + \mu(1-\delta_{\alpha\beta})\varepsilon_{\alpha\beta} \quad (B.52)$$

This can be expanded as:

$$\sigma_x = \frac{2\mu}{(1-2p)} [(1-p)\varepsilon_x + p\varepsilon_y] \quad (B.53)$$

$$\sigma_y = \frac{2\mu}{(1-2p)} [(1-p)\varepsilon_y + p\varepsilon_x] \quad (B.54)$$

$$\tau_{xy} = \mu\gamma_{xy} = 2\mu\varepsilon_{xy} \quad (B.55)$$

thus:

$$\sigma_x = \frac{2\mu}{(1-2p)} [(1-2p)\varepsilon_x + p(\varepsilon_x + \varepsilon_y)]$$

$$\sigma_y = \frac{2\mu}{(1-2p)} [(1-2p)\varepsilon_y + p(\varepsilon_x + \varepsilon_y)]$$

Therefore in general:

$$\sigma_{\alpha\beta} = 2\mu \left[\varepsilon_{\alpha\beta} + \frac{p}{1-2p} (\varepsilon_x + \varepsilon_y) \delta_{\alpha\beta} \right] \quad (B.56)$$

If the strains in equation B.56 are expressed using B.44 then:

$$\sigma_{\alpha\beta} = 2\mu \left[e_x \left(A_{\alpha\beta 1} + \frac{p\delta_{\alpha\beta}}{(1-2p)} [A_{111} + A_{221}] \right) + e_y \left(A_{\alpha\beta 2} + \frac{p\delta_{\alpha\beta}}{(1-2p)} [A_{112} + A_{222}] \right) \right]$$

Theorem III

$$\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = \frac{(x - x_i)^2 + (y - y_i)^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (\text{B.57})$$

Hence substituting for $A_{\alpha\beta\gamma}$ from equation B.47 and making use of theorem III (equation B.57):

$$\sigma_{\alpha\beta} = D_{\alpha\beta 1} e_x + D_{\alpha\beta 2} e_y \quad (\text{B.58})$$

where:

$$D_{\alpha\beta\gamma} = \frac{-1}{4\pi(1-p)r} \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} + (1-2p) \left(\frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - \frac{\partial r}{\partial x_\gamma} \delta_{\alpha\beta} \right) \right] \quad (\text{B.59})$$

Defining:

$$T_\alpha = \sum_{\beta=1}^2 l_\beta \sigma_{\alpha\beta} \quad (\text{B.60})$$

then:

$$T_x = l\sigma_x + m\tau_{xy} \quad (\text{B.61})$$

$$T_y = l\tau_{yx} + m\sigma_y \quad (\text{B.62})$$

Substituting equation B.58 into equations B.61 and B.62 gives:

$$T_\alpha = \sum_{\beta=1}^2 l_\beta (e_x D_{\alpha\beta 1} + e_y D_{\alpha\beta 2}) \quad (\text{B.63})$$

Expanding out equation B.63 to give:

$$T_\alpha = l(e_x D_{1\alpha 1} + e_y D_{1\alpha 2}) + m(e_x D_{2\alpha 1} + e_y D_{2\alpha 2})$$

$$\begin{aligned}
T_\alpha = \frac{-1}{4\pi(1-p)r} & \left[l \left(e_x \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial^2 r}{\partial x^2} + (1-2p) \frac{\partial r}{\partial x_\alpha} \right] + \right. \right. \\
& e_y \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} + (1-2p) \left(\frac{\partial r}{\partial x} \delta_{\alpha 2} - \frac{\partial r}{\partial y} \delta_{\alpha 1} \right) \right] \left. \right) \\
+ m & \left(e_x \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} + (1-2p) \left(\frac{\partial r}{\partial y} \delta_{\alpha 1} - \frac{\partial r}{\partial x} \delta_{\alpha 2} \right) \right] + \right. \\
& \left. \left. e_y \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial^2 r}{\partial y^2} + (1-2p) \frac{\partial r}{\partial x_\alpha} \right] \right) \right]
\end{aligned} \tag{B.64}$$

Noting that:

$$\frac{\partial r}{\partial n} = l \frac{\partial r}{\partial x} + m \frac{\partial r}{\partial y}$$

since:

$$\nabla r = \frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j}$$

$$\hat{n} = l \hat{i} + m \hat{j}$$

thus:

$$\frac{\partial \hat{r}}{\partial n} = \nabla r \cdot \hat{n} = l \frac{\partial r}{\partial x} + m \frac{\partial r}{\partial y}$$

Therefore from equation B.64 it can be deduced that:

$$T_x = F_{11}e_x + F_{12}e_y \tag{B.65}$$

$$T_y = F_{21}e_x + F_{22}e_y \tag{B.66}$$

where:

$$F_{\alpha\beta} = \frac{-1}{4\pi(1-p)r} \left[2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + (1-2p) \left(l_\beta \frac{\partial r}{\partial x_\alpha} - l_\alpha \frac{\partial r}{\partial x_\beta} + \frac{\partial r}{\partial n} \delta_{\alpha\beta} \right) \right] \tag{B.67}$$

B.4 Kernel derivatives $\frac{\partial G_{\alpha\beta}}{\partial x_\gamma}$ and $\frac{\partial^2 G_{\alpha\beta}}{\partial x_\delta \partial x_\gamma}$

The following theorems are useful in deriving the kernel derivatives. Given that:

$$r = ([x - x_i]^2 + [y - y_i]^2)^{\frac{1}{2}} \tag{B.68}$$

the following theorems can be derived.

Theorem IV

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2} ([x - x_i]^2 + [y - y_i]^2)^{-\frac{1}{2}} 2 (x - x_i) = \frac{x - x_i}{r} \\ \frac{\partial r}{\partial x_\alpha} &= \frac{x_\alpha - x_{\alpha i}}{r}\end{aligned}\tag{B.69}$$

Theorem V

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial x_\beta} \right) = \frac{\partial}{\partial x_\alpha} \left[\frac{x_\beta - x_{\beta i}}{([x - x_i]^2 + [y - y_i]^2)^{\frac{1}{2}}} \right]$$

For the case $\alpha \neq \beta$:

$$\begin{aligned}\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial x_\beta} \right) &= -\frac{1}{2} \left(\frac{(x_\beta - x_{\beta i})(x_\alpha - x_{\alpha i}) \cdot 2}{([x - x_i]^2 + [y - y_i]^2)^{\frac{3}{2}}} \right) \\ \frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial x_\beta} \right) &= -\frac{1}{r} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta}\end{aligned}$$

For the case $\alpha = \beta$:

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial x_\beta} \right) = \frac{1}{r}$$

thus:

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial x_\beta} \right) = \frac{1}{r} \left[\delta_{\alpha\beta} - \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right]\tag{B.70}$$

Theorem VI

$$\begin{aligned}\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial n} \right) &= \frac{\partial}{\partial x_\alpha} \left[l \left(\frac{x_\alpha - x_{\alpha i}}{r} \right) + m \left(\frac{x_\beta - x_{\beta i}}{r} \right) \right] \\ &= \frac{l}{r} - \left[\frac{l}{r} \left(\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\alpha} \right) + \frac{m}{r} \left(\frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\alpha} \right) \right] \\ &= \frac{l}{r} - \frac{\partial r}{\partial x_\alpha} \left(\frac{l}{r} \frac{\partial r}{\partial x_\alpha} + \frac{m}{r} \frac{\partial r}{\partial x_\beta} \right) \\ &= \frac{l}{r} - \frac{1}{r} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial n}\end{aligned}$$

thus:

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial r}{\partial n} \right) = \frac{1}{r} \left([\hat{n} \cdot l_\alpha] - \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial n} \right) \quad (\text{B.71})$$

where:

$$\hat{n} \cdot l_\alpha = l\delta_{1\alpha} + m\delta_{2\alpha} \quad (\text{B.72})$$

Theorem VII

$$\frac{\partial}{\partial x_\gamma} \left[\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] = \frac{\partial}{\partial x_\gamma} \left[\frac{(x_\alpha - x_{\alpha i})}{r} \frac{(x_\beta - x_{\beta i})}{r} \right]$$

For the case $\gamma = \alpha$:

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} \left[\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] &= \frac{1}{r^2} (x_\beta - x_{\beta i}) + \frac{(x_\alpha - x_{\alpha i})^2 (x_\beta - x_{\beta i}) \cdot 2 \cdot (-1)}{r^4} \\ &= \frac{1}{r} \left[\frac{\partial r}{\partial x_\beta} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \end{aligned}$$

Similarly for $\gamma = \beta$:

$$\frac{\partial}{\partial x_\beta} \left[\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] = \frac{1}{r} \left[\frac{\partial r}{\partial x_\alpha} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\beta} \right]$$

Therefore in general:

$$\frac{\partial}{\partial x_\gamma} \left[\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] = \frac{1}{r} \left[\frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] \quad (\text{B.73})$$

Theorem VIII

$$\begin{aligned} \frac{\partial}{\partial x_\gamma} \left[\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] &= \frac{\partial}{\partial x_\gamma} \left[\left(l \frac{\partial r}{\partial x} + m \frac{\partial r}{\partial y} \right) \frac{1}{r^2} (x_\alpha - x_{\alpha i}) (x_\beta - x_{\beta i}) \right] \\ &= \frac{\partial}{\partial x_\gamma} \left[l \frac{(x_\alpha - x_{\alpha i})^2 (x_\beta - x_{\beta i})}{r} + m \frac{(x_\beta - x_{\beta i})^2 (x_\alpha - x_{\alpha i})}{r} \right] \frac{1}{r^2} \end{aligned}$$

After some algebraic manipulation this can be shown to be:

$$\begin{aligned} \frac{\partial}{\partial x_\gamma} \left[\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] &= \frac{1}{r} \left[(\hat{n} \cdot l_\gamma) \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + \right. \\ &\quad \left. \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 3 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] \quad (\text{B.74}) \end{aligned}$$

Theorem IX

$$\begin{aligned}\frac{\partial}{\partial x_\alpha} \left(\frac{1}{r} \frac{\partial r}{\partial x_\beta} \right) &= \frac{\partial}{\partial x_\alpha} \left(\frac{x_\beta - x_{\beta i}}{r^2} \right) \\ &= \frac{\delta_{\alpha\beta}}{r^2} - 2 \frac{(x_\beta - x_{\beta i})(x_\alpha - x_{\alpha i})}{r^4}\end{aligned}$$

thus:

$$\frac{\partial}{\partial x_\alpha} \left(\frac{1}{r} \frac{\partial r}{\partial x_\beta} \right) = \frac{1}{r^2} \left[\delta_{\alpha\beta} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \quad (\text{B.75})$$

Theorem X

$$\frac{\partial}{\partial x_\delta} \left[\frac{1}{r} \frac{\partial r}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial}{\partial x_\gamma} \right] = \frac{\partial}{\partial x_\delta} \left(\frac{1}{r} \frac{\partial r}{\partial x_\alpha} \right) \cdot \frac{\partial}{\partial x_\delta} \left(\frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right)$$

Hence using theorem IX for the first term and theorem VII for the second:

$$\begin{aligned}\frac{\partial}{\partial x_\delta} \left[\frac{1}{r} \frac{\partial r}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial}{\partial x_\gamma} \right] &= \frac{1}{r^2} \left[\delta_{\delta\alpha} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} + \\ &\quad \frac{1}{r} \frac{\partial r}{\partial x_\alpha} \left[\frac{1}{r} \left(\frac{\partial r}{\partial x_\beta} \delta_{\gamma\delta} + \frac{\partial r}{\partial x_\gamma} \delta_{\beta\delta} - 2 \left[\frac{\partial r}{\partial x_\delta} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] \right) \right]\end{aligned}$$

therefore:

$$\begin{aligned}\frac{\partial}{\partial x_\delta} \left[\frac{1}{r} \frac{\partial r}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial}{\partial x_\gamma} \right] &= \frac{1}{r^2} \left[\delta_{\delta\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} + \delta_{\gamma\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + \right. \\ &\quad \left. \delta_{\beta\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\gamma} - 4 \frac{\partial r}{\partial x_\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] \quad (\text{B.76})\end{aligned}$$

Using equation B.42:

$$G_{\alpha\beta} = \frac{1}{8\pi\mu(1-p)} \left[\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - \delta_{\alpha\beta} ([3-4p] (\log r + C_1) + 0.5) \right]$$

thus:

$$\frac{\partial G_{\alpha\beta}}{\partial x_\gamma} = \frac{1}{8\pi\mu(1-p)} \frac{\partial}{\partial x_\gamma} \left(\frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - [(3-4p) (\log r + C_1) + 0.5] \delta_{\alpha\beta} \right)$$

using theorem VII:

$$\frac{\partial G_{\alpha\beta}}{\partial x_\gamma} = \frac{1}{8\pi\mu(1-p)r} \left[\frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} - \right. \\ \left. [3-4p] \delta_{\alpha\beta} \cdot \frac{(x_\gamma - x_{\gamma i})}{r} \right]$$

$$\frac{\partial G_{\alpha\beta}}{\partial x_\gamma} = \frac{1}{8\pi\mu(1-p)r} \left[\frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} - [3-4p] \frac{\partial r}{\partial x_\gamma} \delta_{\alpha\beta} \right]$$

Then using theorem II:

$$\frac{\partial G_{\alpha\beta}}{\partial x_{\gamma i}} = \frac{1}{8\pi\mu(1-p)r} \left[(3-4p) \frac{\partial r}{\partial x_\gamma} \delta_{\alpha\beta} - \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} - \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} + 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] \quad (\text{B.77})$$

Using theorem II again:

$$\frac{\partial G_{\alpha\beta}}{\partial x_\gamma} = \frac{-1}{8\pi\mu(1-p)r} \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} + (3-4p) \frac{\partial r}{\partial x_\gamma} \delta_{\alpha\beta} - \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} - \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} \right]$$

thus:

$$\frac{\partial^2 G_{\alpha\beta}}{\partial x_\delta \partial x_\gamma} = \frac{-1}{8\pi\mu(1-p)r} \frac{\partial}{\partial x_\delta} \left[2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} + (3-4p) \frac{\partial r}{\partial x_\gamma} \delta_{\alpha\beta} \right. \\ \left. - \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} - \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} \right]$$

Then using theorems X and IX:

$$\frac{\partial^2 G_{\alpha\beta}}{\partial x_\delta \partial x_\gamma} = \frac{1}{8\pi\mu(1-p)} \frac{1}{r^2} \left[8 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\delta} - 2 \delta_{\delta\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} - 2 \delta_{\gamma\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - \right. \\ \left. 2 \delta_{\beta\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\gamma} - (3-4p) \left[\delta_{\gamma\delta} - 2 \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\delta} \right] \delta_{\alpha\beta} + \right. \\ \left. \delta_{\beta\gamma} \left[\delta_{\alpha\delta} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\delta} \right] + \delta_{\alpha\gamma} \left[\delta_{\beta\delta} - 2 \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\delta} \right] \right]$$

$$\begin{aligned} \frac{\partial^2 G_{\alpha\beta}}{\partial x_\delta \partial x_\gamma} = \frac{1}{8\pi\mu(1-p)r^2} & \left[8 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\delta} - (3-4p)\delta_{\alpha\beta} \left[\delta_{\gamma\delta} - 2 \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\delta} \right] + \right. \\ & \delta_{\beta\gamma} \left[\delta_{\alpha\delta} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\delta} \right] + \delta_{\alpha\gamma} \left[\delta_{\beta\delta} - 2 \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\delta} \right] - \\ & \left. 2\delta_{\gamma\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} - 2\delta_{\gamma\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - 2\delta_{\beta\delta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\gamma} \right] \end{aligned} \quad (\text{B.78})$$

B.5 Kernel derivative $\frac{\partial F_{\alpha\beta}}{\partial x_\gamma}$

From equation B.67

$$F_{\alpha\beta} = \frac{-1}{4\pi(1-p)r} \left[2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + (1-2p) \left(l_\beta \frac{\partial r}{\partial x_\alpha} - l_\alpha \frac{\partial r}{\partial x_\beta} + \frac{\partial r}{\partial n} \delta_{\alpha\beta} \right) \right]$$

hence:

$$\begin{aligned} \frac{\partial F_{\alpha\beta}}{\partial x_{\gamma i}} = \frac{-1}{4\pi(1-p)} \frac{\partial}{\partial x_{\gamma i}} \frac{1}{r} & \left[2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + \right. \\ & \left. (1-2p) \left(l_\beta \frac{\partial r}{\partial x_\alpha} - l_\alpha \frac{\partial r}{\partial x_\beta} + \frac{\partial r}{\partial n} \delta_{\alpha\beta} \right) \right] \end{aligned}$$

Thus, using theorems IX, VIII and VI:

$$\begin{aligned} \frac{\partial F_{\alpha\beta}}{\partial x_{\gamma i}} = \frac{1}{r} \frac{\partial r}{\partial x_\gamma} F_{\alpha\beta} + \frac{-1}{4\pi(1-p)r} & \left[\frac{2}{r^2} \left[(\hat{n} \cdot l_\alpha) \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + \frac{\partial r}{\partial x_n} \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \right. \right. \\ & \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 3 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \left. \right] + \frac{(1-2p)}{r^2} \left[l_\beta \left(\delta_{\gamma\alpha} - \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\alpha} \right) - \right. \\ & \left. l_\alpha \left(\delta_{\gamma\beta} - \frac{\partial r}{\partial x_\gamma} \frac{\partial r}{\partial x_\beta} \right) + \delta_{\alpha\beta} \left(\hat{n} \cdot l_\gamma - \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\gamma} \right) \right] \left. \right] \end{aligned}$$

Finally, using theorem II:

$$\begin{aligned}
\frac{\partial F_{\alpha\beta}}{\partial x_\gamma} = \frac{1}{r} \left[\left(\frac{\partial r}{\partial x_\gamma} \right) F_{\alpha\beta} + \frac{2}{br} \left[\hat{n} \cdot l_\gamma \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \delta_{\beta\gamma} + \right. \right. \\
\left. \left. \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} \delta_{\alpha\gamma} - 3 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} \right] + \right. \\
\left. \frac{(1-2p)}{br} \left[l_\beta \delta_{\alpha\gamma} - l_\alpha \delta_{\beta\gamma} - \frac{\partial r}{\partial x_\gamma} \left(l_\beta \frac{\partial r}{\partial x_\alpha} - l_\alpha \frac{\partial r}{\partial x_\beta} \right) + \right. \right. \\
\left. \left. \delta_{\alpha\beta} \left(\hat{n} \cdot l_\gamma - \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\gamma} \right) \right] \right]
\end{aligned} \tag{B.79}$$

where

$$b = 4\pi(1-p)r$$

Appendix C

Equations of internal equilibrium

C.1 With respect to deformed co-ordinates

In this section the governing internal equations of equilibrium are derived. Initially the equations of equilibrium are derived with respect to deformed co-ordinates. Because of the complications introduced in using deformed co-ordinates with finite strains it is required to derive the internal equations of equilibrium with respect to un-deformed co-ordinates. This is covered in the second part of this section. The equations of infinitesimal equilibrium are formed using the principle of virtual work due to rigid virtual displacement as follows:

$$\iiint_V \delta \underline{q} \cdot \underline{f} dV + \iint_A \delta \underline{q} \cdot \underline{T} dA = 0 \quad (\text{C.1})$$

where

$$\delta \underline{q} = \delta u \cdot \hat{i} + \delta v \cdot \hat{j} + \delta w \cdot \hat{k}$$

$$\underline{f} = f_x \cdot \hat{i} + f_y \cdot \hat{j} + f_z \cdot \hat{k}$$

Note: $\underline{T}$ and $\underline{f}$ are based on deformed co-ordinates. The divergence theorem can be stated as:

$$\begin{aligned} \iiint_V \nabla \cdot \underline{\phi} dV &= \iint_A \underline{\phi} \cdot \hat{n} dA \\ \iint_A (l\phi + m\phi + n\phi) dA &= \iiint_V \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \right) dV \end{aligned} \quad (\text{C.2})$$

Therefore using integration by parts:

$$\iint_A \delta \underline{q} \cdot \underline{T} dA = \iiint_V \left[\frac{\partial}{\partial x} (\delta \underline{q} \cdot \underline{s}_x) + \frac{\partial}{\partial y} (\delta \underline{q} \cdot \underline{s}_y) + \frac{\partial}{\partial z} (\delta \underline{q} \cdot \underline{s}_z) \right] dV \quad (C.3)$$

If $\delta \underline{q}$ has arbitrary constant components due to a virtual rigid translation then:

$$\iint_A \delta \underline{q} \cdot \underline{T} dA = \iiint_V \delta \underline{q} \cdot \left(\frac{\partial \underline{s}_x}{\partial x} + \frac{\partial \underline{s}_y}{\partial y} + \frac{\partial \underline{s}_z}{\partial z} \right) dV \quad (C.4)$$

Sustituting C.4 into C.1 gives:

$$\iiint_V \delta \underline{q} \cdot \left(\frac{\partial \underline{s}_x}{\partial x} + \frac{\partial \underline{s}_y}{\partial y} + \frac{\partial \underline{s}_z}{\partial z} + \underline{f} \right) dV = 0 \quad (C.5)$$

Since $\underline{q}$ represents an arbitrary parameter it can be deduced that:

$$\frac{\partial \underline{s}_x}{\partial x} + \frac{\partial \underline{s}_y}{\partial y} + \frac{\partial \underline{s}_z}{\partial z} + \underline{f} = 0 \quad (C.6)$$

Which is the equation of equilibrium. This equation can be put in index notation as follows:

$$\sigma_{ij,i} + f_j = 0 \quad (C.7)$$

C.2 With respect to un-deformed co-ordinates

Equation C.7 is the internal equation of equilibrium with respect to deformed co-ordinates. To derive the same equation but with respect to un-deformed co-ordinates consider force equilibrium between internal stresses and the externally applied force vector as follows.

$$d\underline{F} = \underline{s} dA$$

$$d\underline{F} \equiv \underline{s}_x dA_x + \underline{s}_y dA_y + \underline{s}_z dA_z \quad (C.8)$$

i.e. $d\underline{F} = \underline{\sigma}_{ij} d\underline{A}$, where:

$$d\underline{A} = \begin{bmatrix} dA_x \\ dA_y \\ dA_z \end{bmatrix} \quad (C.9)$$

On the boundary the applied load must equal the product of the traction and area components, therefore:

$$d\underline{F} = \underline{T} d\underline{A} = \underline{T}^o d\underline{A}^o \quad (C.10)$$

Where $\underline{T}^0$ is a traction vector based on un-deformed co-ordinates. With $\underline{T} = \underline{s}$ then:

$$\underline{\sigma}_{ij}^t d\underline{A} = \underline{T}^o d\underline{A}^o \quad (\text{C.11})$$

From equation 3.116 it was shown that:

$$\underline{\sigma}_{ij}^t |\underline{G}| (\underline{G}^t)^{-1} d\underline{A}^o = \underline{T}^o d\underline{A}^o$$

Defining $(\underline{\sigma}_{ij}^o)^t = \underline{\sigma}_{ij}^t |\underline{G}| (\underline{G}^t)^{-1}$ or $\underline{\sigma}_{ij}^o = |\underline{G}| \underline{G}^{-1} \underline{\sigma}_{ij}^t$ then:

$$\underline{T}^o = (\underline{\sigma}_{ij}^o)^t \cdot \begin{bmatrix} l^o \\ m^o \\ n^o \end{bmatrix} \quad (\text{C.12})$$

leading to:

$$\begin{aligned} T_x^o &= l^o \sigma_x^o + m^o \tau_{yx}^o + n^o \tau_{zx}^o \\ T_y^o &= l^o \tau_{xy}^o + m^o \sigma_y^o + n^o \tau_{zy}^o \\ T_z^o &= l^o \tau_{xz}^o + m^o \tau_{yz}^o + n^o \sigma_z^o \end{aligned} \quad (\text{C.13})$$

Notice that $\underline{\sigma}_{ij}^o$ is not a symmetric matrix. The equation of equilibrium with respect to deformed co-ordinates (equation C.7) can also be written as.

$$\left[\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right] \cdot \underline{\sigma}_{ij}^t + \underline{f} = 0 \quad (\text{C.14})$$

Using the chain rule of partial differentiation it can be proved that:

$$\left[\begin{array}{c} \frac{\partial}{\partial x^o} \\ \frac{\partial}{\partial y^o} \\ \frac{\partial}{\partial z^o} \end{array} \right] = \underline{J} \left(\frac{x, y, z}{x^o, y^o, z^o} \right) \left[\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right] \quad (\text{C.15})$$

hence:

$$\left[\begin{array}{ccc} \frac{\partial}{\partial x^o} & \frac{\partial}{\partial y^o} & \frac{\partial}{\partial z^o} \end{array} \right] = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right] \underline{J}^t \quad (\text{C.16})$$

or:

$$\left[\begin{array}{ccc} \frac{\partial}{\partial x^o} & \frac{\partial}{\partial y^o} & \frac{\partial}{\partial z^o} \end{array} \right] = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right] \underline{G} \quad (\text{C.17})$$

hence:

$$\left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right] = \left[\begin{array}{ccc} \frac{\partial}{\partial x^o} & \frac{\partial}{\partial y^o} & \frac{\partial}{\partial z^o} \end{array} \right] \underline{G}^{-1} \quad (\text{C.18})$$

Noticing also that:

$$\underline{f} = \frac{d\underline{F}}{dV} = \frac{d\underline{F}}{dV^o} \cdot \frac{dV^o}{dV} \quad (\text{C.19})$$

i.e.

$$\underline{f} = \frac{1}{|\underline{G}|} \cdot \underline{f}^o \quad (\text{C.20})$$

where $\underline{f}^0$ represents domain loading intensity with respect to un-deformed co-ordinates. Substituting from C.17 and C.18 into C.14 gives.

$$\begin{aligned} \left[\frac{\partial}{\partial x^o} \quad \frac{\partial}{\partial y^o} \quad \frac{\partial}{\partial z^o} \right] \underline{G}^{-1} \underline{\sigma}_{ij} + \frac{1}{|\underline{G}|} \underline{f}^o &= \underline{0} \\ \left[\frac{\partial}{\partial x^o} \quad \frac{\partial}{\partial y^o} \quad \frac{\partial}{\partial z^o} \right] |\underline{G}| \underline{G}^{-1} \underline{\sigma}_{ij} + \underline{f}^o &= \underline{0} \end{aligned} \quad (\text{C.21})$$

Defining:

$$\underline{\sigma}_{ij}^o = |\underline{G}| \underline{G}^{-1} \underline{\sigma}_{ij} \quad (\text{C.22})$$

which is consistent with the earlier derivation, therefore:

$$\left[\frac{\partial}{\partial x^o} \quad \frac{\partial}{\partial y^o} \quad \frac{\partial}{\partial z^o} \right] \underline{\sigma}_{ij}^o + \underline{f}_i^o = 0 \quad (\text{C.23})$$

leading to:

$$\sigma_{ij,i}^o + f_j^o = 0 \quad (\text{C.24})$$

Note that:

$$\underline{\sigma}_{ij}^o = \underline{P}_{ij}^t \quad (\text{C.25})$$

which is known as Lagrange's or the 1st Piola-Kirchoff stress. This stress can be expressed in terms of the 2nd Piola-Kirchoff stress as follows.

$$\underline{\sigma}_{ij}^o = \underline{S}_{ij} \cdot \underline{G}^t \quad (\text{C.26})$$

In the derivations of the non-linear boundary integral equations the stress used in all the equations unless other wise stated is the 1st Piola-Kirchoff stress. And as such the superscripted 'o' will be omitted from the stress symbol σ_{ij}^o .

Appendix D

Explicit derivation of the stress strain constitutive tensor

Note that in the following derivations the material model used is the Mooney-Rivlin strain energy function.

D.1 General partial derivative constitutive equations

From 4.45 the form of the differential of the 2nd Piola-Kirchoff stress can be deduced as follows:

$$dS_{ij} = \sum_{kl} d_{ijkl} d\varepsilon_{kl} + \varrho_{ij} d\rho \quad (\text{D.1})$$

where:

$$d_{ijkl} = \frac{\partial S_{ij}}{\partial \varepsilon_{kl}} \equiv 2 \frac{\partial S_{ij}}{\partial C_{kl}} \quad \varrho_{ij} = \frac{\partial S_{ij}}{\partial \rho} \quad (\text{D.2})$$

using: $\underline{C} = 2\underline{\varepsilon}_G + \underline{I}$, therefore $d\underline{C} = 2d\underline{\varepsilon}_G$. Expanding out D.1 and assuming C_{12} and C_{21} are not independent:

$$dS_{ij} = \frac{\partial S_{ij}}{\partial C_{11}} dC_{11} + \frac{\partial S_{ij}}{\partial C_{22}} dC_{22} + 2 \frac{\partial S_{ij}}{\partial C_{12}} dC_{12} + \frac{\partial S_{ij}}{\partial \rho} d\rho \quad (\text{D.3})$$

with $S_{ij} = S_x, S_y, S_{xy}$, using $d\underline{C} = 2d\underline{\varepsilon}_G$ then:

$$dS_{ij} = \left(2 \frac{\partial S_{ij}}{\partial C_{11}} \right) d\varepsilon_x + \left(2 \frac{\partial S_{ij}}{\partial C_{22}} \right) d\varepsilon_y + 2 \left(2 \frac{\partial S_{ij}}{\partial C_{12}} \right) d\varepsilon_{xy} + \frac{\partial S_{ij}}{\partial \rho} d\rho \quad (\text{D.4})$$

Using engineering components it can be deduced that:

$$dS_x = \left(2 \frac{\partial S_x}{\partial C_{11}}\right) d\varepsilon_x + \left(2 \frac{\partial S_x}{\partial C_{22}}\right) d\varepsilon_y + \left(2 \frac{\partial S_x}{\partial C_{12}}\right) d\gamma_{xy} + \frac{\partial S_x}{\partial p} dp \quad (\text{D.5})$$

$$dS_y = \left(2 \frac{\partial S_y}{\partial C_{11}}\right) d\varepsilon_x + \left(2 \frac{\partial S_y}{\partial C_{22}}\right) d\varepsilon_y + \left(2 \frac{\partial S_y}{\partial C_{12}}\right) d\gamma_{xy} + \frac{\partial S_y}{\partial p} dp \quad (\text{D.6})$$

$$dS_{xy} = \left(2 \frac{\partial S_{xy}}{\partial C_{11}}\right) d\varepsilon_x + \left(2 \frac{\partial S_{xy}}{\partial C_{22}}\right) d\varepsilon_y + \left(2 \frac{\partial S_{xy}}{\partial C_{12}}\right) d\gamma_{xy} + \frac{\partial S_{xy}}{\partial \rho} d\rho \quad (\text{D.7})$$

Thus putting equations D.5 to D.7 into matrix form:

$$d\underline{S} = \underline{D}d\underline{\varepsilon} + \underline{\rho}d\rho \quad (\text{D.8})$$

where:

$$\begin{aligned} d_{11} &= 2 \frac{\partial S_x}{\partial C_{11}}, d_{12} = 2 \frac{\partial S_x}{\partial C_{22}}, d_{13} = \frac{\partial S_x}{\partial C_{12}} \\ d_{21} &= 2 \frac{\partial S_y}{\partial C_{11}}, d_{22} = 2 \frac{\partial S_y}{\partial C_{22}}, d_{23} = \frac{\partial S_y}{\partial C_{12}} \\ d_{31} &= 2 \frac{\partial S_{xy}}{\partial C_{11}}, d_{32} = 2 \frac{\partial S_{xy}}{\partial C_{22}}, d_{33} = \frac{\partial S_{xy}}{\partial C_{12}} \\ \rho_1 &= \frac{\partial S_x}{\partial \rho}, \rho_2 = \frac{\partial S_y}{\partial \rho}, \rho_3 = \frac{\partial S_{xy}}{\partial \rho} \end{aligned} \quad (\text{D.9})$$

D.2 Derivative equations (plane strain)

For the case of plane strain:

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \quad (\text{D.10})$$

given:

$$\underline{C} = 2\underline{\varepsilon}_G + \underline{I} \quad (\text{D.11})$$

hence:

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{D.12})$$

From the eigen value problem:

$$|\underline{C} - \lambda^2 \underline{I}| = \begin{vmatrix} C_{11} - \lambda^2 & C_{12} & 0 \\ C_{21} & C_{22} - \lambda^2 & 0 \\ 0 & 0 & 1 - \lambda^2 \end{vmatrix} = 0 \quad (\text{D.13})$$

leading to:

$$[\lambda^4 - (C_{11} + C_{22})\lambda^2 + C_{11}C_{22} - C_{12}C_{21}](\lambda^2 - 1) = 0 \quad (\text{D.14})$$

therefore:

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1 = C_{11} + C_{22} + 1 \quad (\text{D.15})$$

$$I_2 = (\lambda_1^2 + \lambda_2^2)\lambda_3^2 + \lambda_1^2\lambda_2^2 = C_{11}C_{22} - C_{12}C_{21} \quad (\text{D.16})$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = C_{11}C_{22} - C_{12}C_{21} \quad (\text{D.17})$$

from D.15 then:

$$\begin{aligned} \frac{\partial I_1}{\partial C_{11}} &= \frac{\partial I_1}{\partial C_{22}} = 1 \\ \frac{\partial I_1}{\partial C_{12}} &= \frac{\partial I_1}{\partial C_{21}} = 0 \end{aligned} \quad (\text{D.18})$$

with $I_3 = C_{11}C_{22} - C_{12}C_{21}$ and ignoring the symmetry of $\underline{C}$:

$$\begin{aligned} \frac{\partial I_3}{\partial C_{11}} &= C_{22}, \frac{\partial I_3}{\partial C_{22}} = C_{11} \\ \frac{\partial I_3}{\partial C_{12}} &= -C_{21}, \frac{\partial I_3}{\partial C_{21}} = -C_{12} \end{aligned} \quad (\text{D.19})$$

noticing that $I_2 = I_1 + I_3 - 1$ then:

$$\begin{aligned} \frac{\partial I_2}{\partial C_{11}} &= 1 + C_{22} \equiv I_1 - C_{11} \\ \frac{\partial I_2}{\partial C_{22}} &= 1 + C_{11} \equiv I_1 - C_{22} \\ \frac{\partial I_2}{\partial C_{12}} &= -C_{21}, \frac{\partial I_2}{\partial C_{21}} = -C_{12} \end{aligned} \quad (\text{D.20})$$

noticing that:

$$\underline{C}^{-1} = \frac{1}{I_3} \begin{bmatrix} C_{22} & -C_{12} & 0 \\ -C_{21} & C_{11} & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad (\text{D.21})$$

therefore:

$$\begin{aligned} C_{11}^{-1} &= \frac{C_{22}}{I_3}, C_{22}^{-1} = \frac{C_{11}}{I_3} \\ C_{12}^{-1} &= -\frac{C_{12}}{I_3}, C_{21}^{-1} = -\frac{C_{21}}{I_3} \end{aligned} \quad (\text{D.22})$$

It can be deduced that $\frac{\partial C_{11}^{-1}}{\partial C_{11}} = \frac{\partial C_{11}^{-1}}{\partial I_3} \frac{\partial I_3}{\partial C_{11}} = -\frac{C_{22}^2}{I_3^2}$, similarly:

$$\frac{\partial C_{11}^{-1}}{\partial C_{22}} = -\frac{C_{12}C_{21}}{I_3^2}, \frac{\partial C_{11}^{-1}}{\partial C_{12}} = \frac{C_{22}C_{21}}{I_3^2}, \frac{\partial C_{11}^{-1}}{\partial C_{21}} = -\frac{C_{12}C_{22}}{I_3^2} \quad (\text{D.23})$$

$$\begin{aligned} \frac{\partial C_{22}^{-1}}{\partial C_{11}} &= -\frac{C_{21}C_{12}}{I_3^2}, \frac{\partial C_{22}^{-1}}{\partial C_{22}} = -\frac{C_{11}^2}{I_3^2} \\ \frac{\partial C_{22}^{-1}}{\partial C_{12}} &= \frac{C_{21}C_{11}}{I_3^2}, \frac{\partial C_{22}^{-1}}{\partial C_{21}} = \frac{C_{11}C_{12}}{I_3^2} \end{aligned} \quad (\text{D.24})$$

$$\begin{aligned}\frac{\partial C_{12}^{-1}}{\partial C_{11}} &= \frac{C_{22}C_{12}}{I_3^2}, \quad \frac{\partial C_{12}^{-1}}{\partial C_{22}} = \frac{C_{12}C_{11}}{I_3^2} \\ \frac{\partial C_{22}^{-1}}{\partial C_{12}} &= -\frac{C_{11}C_{22}}{I_3^2}, \quad \frac{\partial C_{22}^{-1}}{\partial C_{21}} = -\frac{C_{12}^2}{I_3^2}\end{aligned}\tag{D.25}$$

$$\begin{aligned}\frac{\partial C_{21}^{-1}}{\partial C_{11}} &= \frac{C_{21}C_{22}}{I_3^2}, \quad \frac{\partial C_{21}^{-1}}{\partial C_{22}} = \frac{C_{11}C_{21}}{I_3^2} \\ \frac{\partial C_{21}^{-1}}{\partial C_{12}} &= -\frac{C_{21}^2}{I_3^2}, \quad \frac{\partial C_{21}^{-1}}{\partial C_{21}} = -\frac{C_{11}C_{22}}{I_3^2}\end{aligned}\tag{D.26}$$

Finding the derivative expressions in D.9 will involve evaluating the derivatives of the coefficients of B_1 , B_2 and B_3 in D.44 using the above proofs. Thus it can be shown that:

$$\begin{aligned}\frac{\partial B_1}{\partial C_{11}} &= 2C_2I_3^{-\frac{2}{3}} - \frac{C_{22}}{I_3} \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_1I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_1}{\partial C_{22}} &= 2C_2I_3^{-\frac{2}{3}} - \frac{C_{11}}{I_3} \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_1I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_1}{\partial C_{12}} &= \frac{C_{21}}{I_3} \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_1I_3^{-\frac{2}{3}} \right)\end{aligned}\tag{D.27}$$

$$\begin{aligned}\frac{\partial B_2}{\partial C_{11}} &= \frac{C_{22}}{I_3} \left(\frac{4}{3}C_2I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_2}{\partial C_{22}} &= \frac{C_{11}}{I_3} \left(\frac{4}{3}C_2I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_2}{\partial C_{12}} &= -\frac{C_{21}}{I_3} \left(\frac{4}{3}C_2I_3^{-\frac{2}{3}} \right)\end{aligned}\tag{D.28}$$

$$\begin{aligned}\frac{\partial B_3}{\partial C_{11}} &= \frac{C_{22}}{I_3} \left[\frac{2}{9}C_1I_1I_3^{-\frac{1}{3}} + \frac{8}{9}C_2I_2I_3^{-\frac{2}{3}} - \frac{4}{3}C_2I_3^{\frac{1}{3}} \right] - \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_3}{\partial C_{22}} &= \frac{C_{11}}{I_3} \left[\frac{2}{9}C_1I_1I_3^{-\frac{1}{3}} + \frac{8}{9}C_2I_2I_3^{-\frac{2}{3}} - \frac{4}{3}C_2I_3^{\frac{1}{3}} \right] - \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_3^{-\frac{2}{3}} \right) \\ \frac{\partial B_3}{\partial C_{12}} &= -\frac{C_{22}}{I_3} \left[\frac{2}{9}C_1I_1I_3^{-\frac{1}{3}} + \frac{8}{9}C_2I_2I_3^{-\frac{2}{3}} - \frac{4}{3}C_2I_3^{\frac{1}{3}} - \frac{4}{3}C_2I_3^{\frac{1}{3}} \right]\end{aligned}\tag{D.29}$$

If it is noticed that $C_{22} = I_1 - C_{11} - 1$ then:

$$-\frac{4}{3}C_2I_3^{-\frac{2}{3}}C_{22} = -\frac{4}{3}C_2I_3^{-\frac{2}{3}}(I_1 - C_{11} - 1)$$

and it can be deduced that:

$$\begin{aligned}\frac{\partial B_3}{\partial C_{11}} &= \left(\frac{4}{3}C_2I_3^{\frac{1}{3}} \right) \frac{C_{11}}{I_3} - \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_1I_3^{-\frac{2}{3}} \right) + \\ &\quad \frac{C_{22}}{I_3} \left(\frac{2}{9}C_1I_1I_3^{-\frac{1}{3}} + \frac{8}{9}C_2I_2I_3^{-\frac{2}{3}} \right)\end{aligned}\tag{D.30}$$

Similarly $C_{11} = I_1 - C_{22} - 1$ then:

$$\begin{aligned}\frac{\partial B_3}{\partial C_{22}} &= \left(\frac{4}{3}C_2I_3^{\frac{1}{3}} \right) \frac{C_{22}}{I_3} - \left(\frac{2}{3}C_1I_3^{-\frac{1}{3}} + \frac{4}{3}C_2I_1I_3^{-\frac{2}{3}} \right) + \\ &\quad \frac{C_{11}}{I_3} \left(\frac{2}{9}C_1I_1I_3^{-\frac{1}{3}} + \frac{8}{9}C_2I_2I_3^{-\frac{2}{3}} \right)\end{aligned}\tag{D.31}$$

whilst from $C_{12} = C_{21}$:

$$\frac{\partial B_3}{\partial C_{12}} = \left(\frac{4}{3} C_2 I_3^{\frac{1}{3}} \right) \frac{C_{12}}{I_3} - \frac{C_{21}}{I_3} \left(\frac{2}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{8}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right) \quad (\text{D.32})$$

D.3 Explicit stress strain equations (plane strain)

From equation 4.45 it can be deduced that:

$$S_x = B_1 + C_{11} B_2 + C_{11}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) \quad (\text{D.33})$$

therefore:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{ij}} &= \frac{\partial B_1}{\partial C_{ij}} + (\delta_{li} \delta_{lj}) B_2 + C_{11} \frac{\partial B_2}{\partial C_{ij}} - C_{li}^{-1} C_{jl}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) + \\ &\quad C_{11}^{-1} \left(\frac{\partial B_3}{\partial C_{ij}} - \frac{1}{2} \rho I_3^{-\frac{1}{2}} \frac{\partial I_3}{\partial C_{ij}} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial S_x}{\partial C_{11}} &= B_2 - C_{11}^{-1} C_{11}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) + \frac{\partial B_1}{\partial C_{11}} + C_{11} \frac{\partial B_2}{\partial C_{11}} + \\ &\quad C_{11}^{-1} \left(\frac{\partial B_3}{\partial C_{11}} - \frac{1}{2} \rho I_3^{\frac{1}{2}} \frac{C_{22}}{I_3} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial S_x}{\partial C_{11}} &= \frac{C_{22}^2}{I_3^2} \left[\frac{2}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_2 I_3^{-\frac{2}{3}} \right] + \frac{C_{22}^2}{I_3^2} \rho I_3^{\frac{1}{2}} - \\ &\quad \frac{C_{22}}{I_3} \left(\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right) + C_{11} \left[\frac{C_{22}}{I_3} \frac{4}{3} C_2 I_3^{-\frac{2}{3}} \right] - \frac{1}{2} \frac{C_{22}^2}{I_3^2} \rho I_3^{\frac{1}{2}} + \\ &\quad \frac{C_{22}}{I_3} \left[\frac{4}{3} C_2 I_3^{\frac{1}{3}} - \left(\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right) \right] \\ &\quad \left[\frac{C_{22}}{I_3} \left(\frac{2}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{8}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right) \right] \end{aligned}$$

therefore:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{11}} &= \frac{C_{22}^2}{I_3^2} \left(\frac{8}{9} C_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_1 I_3^{-\frac{2}{3}} \right) - 2 \frac{C_{22}}{I_3} \left[\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right] \\ &\quad + \frac{8}{3} \frac{C_{11} C_{22}}{I_3^2} C_2 I_3^{\frac{1}{3}} + \frac{1}{2} \rho \frac{C_{22}^2}{I_3^2} I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.34})$$

similarly:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{22}} &= \frac{\partial B_1}{\partial C_{22}} + C_{11} \frac{\partial B_2}{\partial C_{22}} + C_{11}^{-1} \left(\frac{\partial B_3}{\partial C_{22}} - \frac{1}{2} \rho I_3^{-\frac{1}{2}} \frac{\partial I_3}{\partial C_{22}} \right) - \\ &\quad C_{12}^{-1} C_{21}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) \end{aligned} \quad (\text{D.35})$$

hence it can be deduced that:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{22}} &= 2C_2 I_3^{-\frac{2}{3}} + \frac{4}{3} C_2 \left(\frac{C_{11}^2 + C_{22}^2}{I_3} \right) I_3^{-\frac{2}{3}} - \\ &\left(\frac{C_{11} + C_{22}}{I_3} \right) \left(\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right) \\ &- \frac{1}{I_3} \left(\frac{2}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right) + \\ &\frac{C_{11} C_{22}}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] + \frac{C_{12}^2 - I_3}{2I_3^2} \rho I_3^{\frac{1}{2}} \end{aligned} \quad (D.36)$$

It can also be proved by direct differentiation:

$$\frac{\partial S_x}{\partial C_{22}} = \frac{\partial S_y}{\partial C_{11}} \quad (D.37)$$

similarly:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{12}} &= \frac{\partial B_1}{\partial C_{12}} + C_{11} \frac{\partial B_2}{\partial C_{12}} - C_{11}^{-1} C_{21}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) + \\ &C_{11}^{-1} \left(\frac{\partial B_3}{\partial C_{12}} - \frac{1}{2} \rho I_3^{-\frac{1}{2}} \frac{\partial I_3}{\partial C_{12}} \right) \end{aligned} \quad (D.38)$$

From which it can be deduced that:

$$\begin{aligned} \frac{\partial S_x}{\partial C_{12}} &= \frac{\partial S_x}{\partial C_{21}} = \frac{C_{12}(C_{22} - C_{11})}{I_3} \frac{4}{3} C_2 I_3^{-\frac{2}{3}} + \frac{C_{12}}{I_3} \left[\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right] \\ &- \frac{C_{12} C_{22}}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] - \frac{1}{2} \frac{C_{22} C_{12}}{I_3^2} \rho I_3^{\frac{1}{2}} \end{aligned} \quad (D.39)$$

therefore with reference to equation D.9 it can be deduced that:

$$\begin{aligned} d_{11} &= 2 \frac{\partial S_x}{\partial C_{11}} = \frac{16}{3} \left(\frac{C_{11} C_{22}}{I_3^2} \right) C_2 I_3^{\frac{1}{3}} - \frac{8}{3} \left(C_{22} I_3^{-\frac{4}{3}} \right) (C_1 + 2C_2 \bar{I}_1) \\ &+ \frac{8}{9} \left(\frac{C_{22}^2}{I_3^2} \right) (2C_1 \bar{I}_1 + 5C_2 \bar{I}_2) + \rho \left(\frac{C_{22}^2}{I_3^2} \right) I_3^{\frac{1}{2}} \end{aligned} \quad (D.40)$$

$$\begin{aligned} d_{12} &= 2 \frac{\partial S_x}{\partial C_{22}} = 4C_2 I_3^{-\frac{2}{3}} + \frac{8}{3} \left(\frac{C_{11}^2 + C_{22}^2}{I_3^2} \right) C_2 I_3^{\frac{1}{3}} - \\ &\frac{4}{3} (C_{11} + C_{22}) I_3^{-\frac{4}{3}} (C_1 + 2C_2 \bar{I}_1) - \frac{4}{3} \frac{1}{I_3} (C_1 \bar{I}_1 + 2C_2 \bar{I}_2) + \\ &\frac{8}{9} \left(\frac{C_{11} C_{22}}{I_3^2} \right) (2C_1 \bar{I}_1 + 5C_2 \bar{I}_2) + \rho \left(\frac{C_{12}^2 - I_3}{I_3^2} \right) I_3^{\frac{1}{2}} \end{aligned} \quad (D.41)$$

$$\begin{aligned} d_{13} &= 2 \frac{\partial S_x}{\partial C_{12}} = \frac{8}{3} \frac{C_{12}(C_{22} - C_{11})}{I_3^2} C_2 I_3^{\frac{1}{3}} + \frac{4}{3} \left(C_{12} I_3^{-\frac{4}{3}} \right) (C_1 + 2C_2 \bar{I}_1) \\ &- \frac{8}{9} \left(\frac{C_{12} C_{22}}{I_3^2} \right) (2C_1 \bar{I}_1 + 5C_2 \bar{I}_2) - \rho \left(\frac{C_{12} C_{22}}{I_3^2} \right) I_3^{\frac{1}{2}} \end{aligned} \quad (D.42)$$

From equation D.44 it can be deduced that:

$$S_y = B_1 + C_{22} B_2 + C_{22}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) \quad (D.43)$$

therefore:

$$\frac{\partial S_y}{\partial C_{22}} = \frac{\partial B_1}{\partial C_{22}} + B_2 + C_{22} \frac{\partial B_2}{\partial C_{22}} + \left(\frac{\partial B_3}{\partial C_{22}} - \frac{1}{2} \rho I_3^{-\frac{1}{2}} \frac{\partial I_3}{\partial C_{22}} \right) \frac{C_{11}}{I_3} - \left(B_3 - \rho I_3^{\frac{1}{2}} \right) \left(\frac{C_{11}}{I_3} \right)^2 \quad (\text{D.44})$$

hence:

$$\begin{aligned} \frac{\partial S_y}{\partial C_{22}} &= \frac{C_{11}^2}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] - 2 \frac{C_{11}}{I_3} \left[\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right] \\ &+ \frac{8}{3} C_2 \left(\frac{C_{11} C_{22}}{I_3^2} \right) I_3^{\frac{1}{3}} + \frac{1}{2} \rho \frac{C_{11}^2}{I_3^2} I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.45})$$

similarly it can be proved that:

$$\begin{aligned} \frac{\partial S_y}{\partial C_{12}} &= \frac{4}{3} \frac{C_{12}(C_{11}-C_{22})}{I_3} C_2 I_3^{-\frac{2}{3}} + \frac{C_{12}}{I_3} \left[\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right] \\ &- \frac{C_{11} C_{12}}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] - \frac{1}{2} \frac{C_{11} C_{12}}{I_3^2} \rho I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.46})$$

therefore with reference to equation D.9 it can be deduced that:

$$\begin{aligned} d_{22} &= 2 \frac{\partial S_y}{\partial C_{22}} = \frac{16}{3} \left(\frac{C_{11} C_{22}}{I_3^2} \right) C_2 I_3^{\frac{1}{3}} - \frac{8}{3} \left(C_{11} I_3^{-\frac{4}{3}} \right) (C_1 + 2 C_2 \bar{I}_1) \\ &+ \frac{8}{9} \left(\frac{C_{11}^2}{I_3^2} \right) (2 C_1 \bar{I}_1 + 5 C_2 \bar{I}_2) + \rho \left(\frac{C_{11}^2}{I_3^2} \right) I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.47})$$

$$\begin{aligned} d_{23} &= 2 \frac{\partial S_y}{\partial C_{12}} = \frac{8}{3} \frac{C_{12}(C_{11}-C_{22})}{I_3^2} C_2 I_3^{\frac{1}{3}} + \frac{4}{3} \left(C_{12} I_3^{-\frac{4}{3}} \right) (C_1 + 2 C_2 \bar{I}_1) \\ &- \frac{8}{9} \left(\frac{C_{11} C_{12}}{I_3^2} \right) (2 C_1 \bar{I}_1 + 5 C_2 \bar{I}_2) - \rho \left(\frac{C_{11} C_{12}}{I_3^2} \right) I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.48})$$

$$d_{21} = 2 \frac{\partial S_y}{\partial C_{11}} = d_{12} \quad (\text{D.49})$$

From equation D.44 it can be deduced that:

$$S_{xy} = C_{12} B_2 + C_{12}^{-1} \left(B_3 - \rho I_3^{\frac{1}{2}} \right) \quad (\text{D.50})$$

therefore:

$$\begin{aligned} \frac{\partial S_{xy}}{\partial C_{11}} &= \frac{4}{3} \frac{C_{12}(C_{22}-C_{11})}{I_3} C_2 I_3^{-\frac{2}{3}} + \frac{C_{12}}{I_3} \left[\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right] \\ &- \frac{C_{22} C_{12}}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] - \frac{1}{2} \frac{C_{22} C_{12}}{I_3^2} \rho I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.51})$$

$$\begin{aligned} \frac{\partial S_{xy}}{\partial C_{22}} &= \frac{4}{3} \frac{C_{12}(C_{11}-C_{22})}{I_3} C_2 I_3^{-\frac{2}{3}} + \frac{C_{12}}{I_3} \left(\frac{2}{3} C_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_1 I_3^{-\frac{2}{3}} \right) \\ &- \frac{C_{11} C_{12}}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] - \frac{1}{2} \frac{C_{11} C_{12}}{I_3^2} \rho I_3^{\frac{1}{2}} \end{aligned} \quad (\text{D.52})$$

$$\begin{aligned}
\frac{\partial S_{xy}}{\partial C_{12}} &= B_2 + C_{12} \frac{\partial B_2}{\partial C_{12}} + \left(B_3 - \rho I_2^{\frac{1}{2}} \right) \frac{C_{11} C_{22}}{I_3^2} - \frac{C_{12}}{I_3} \left[\frac{\partial B_3}{\partial C_{12}} - \frac{1}{2} \rho I_3^{-\frac{1}{2}} \frac{\partial I_3}{\partial C_{12}} \right] \\
\frac{\partial S_{xy}}{\partial C_{12}} &= -2C_2 I_3^{-\frac{2}{3}} - \frac{C_{12}^2}{I_3} \left(\frac{4}{3} C_2 I_3^{-\frac{2}{3}} \right) + \\
&\quad \left[\frac{2}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_2 I_3^{-\frac{2}{3}} + \rho I_3^{\frac{1}{2}} \right] \frac{C_{11} C_{22}}{I_3^2} \\
&\quad + \frac{C_{12}}{I_3} \left(\frac{C_{12}}{I_3} \right) \left[\frac{2}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{8}{9} C_2 I_2 I_3^{-\frac{2}{3}} - \frac{4}{3} C_2 I_3^{\frac{1}{3}} \right] - \frac{1}{2} \frac{C_{12}}{I_3} \frac{C_{12}}{I_3} I_3^{\frac{1}{2}} \rho
\end{aligned}$$

hence:

$$\begin{aligned}
\frac{\partial S_{xy}}{\partial C_{12}} &= -2C_2 I_3^{-\frac{2}{3}} - \frac{8}{3} \frac{C_{12}^2}{I_3} C_2 I_3^{-\frac{2}{3}} + \frac{C_{12}^2}{I_3^2} \left[\frac{8}{9} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{20}{9} C_2 I_2 I_3^{-\frac{2}{3}} \right] \\
&\quad + \frac{1}{I_3} \left[\frac{2}{3} C_1 I_1 I_3^{-\frac{1}{3}} + \frac{4}{3} C_2 I_2 I_3^{-\frac{2}{3}} \right] + \frac{(I_3 + C_{11} C_{22})}{2I_3^2} \rho I_3^{\frac{1}{2}}
\end{aligned} \tag{D.53}$$

therefore with reference to equation D.9 it can be deduced that:

$$\begin{aligned}
d_{33} &= 2 \frac{\partial S_{xy}}{\partial C_{12}} = -4C_2 I_3^{-\frac{2}{3}} - \frac{16}{3} \left(\frac{C_{12}^2}{I_3^2} \right) C_2 I_3^{\frac{1}{3}} + \frac{4}{3} \frac{1}{I_3} (C_1 \bar{I}_1 + 2C_2 \bar{I}_2) \\
&\quad + \frac{8}{9} \left(\frac{C_{12}^2}{I_3^2} \right) (2C_1 \bar{I}_1 + 5C_2 \bar{I}_2) + \rho \frac{(I_3 + C_{11} C_{22})}{I_3^2} I_3^{\frac{1}{2}}
\end{aligned} \tag{D.54}$$

$$d_{31} = 2 \frac{\partial S_{xy}}{\partial C_{11}} = d_{13} \tag{D.55}$$

$$d_{32} = 2 \frac{\partial S_{xy}}{\partial C_{22}} = d_{23} \tag{D.56}$$

$$\varrho_1 = \frac{\partial S_x}{\partial p} = -C_{22} I_3^{-1/2} \equiv -\frac{C_{22}}{J} \tag{D.57}$$

$$\varrho_2 = \frac{\partial S_y}{\partial p} = -C_{11} I_3^{-1/2} \equiv -\frac{C_{11}}{J} \tag{D.58}$$

$$\varrho_3 = \frac{\partial S_x}{\partial p} = C_{12} I_3^{-1/2} \equiv \frac{C_{12}}{J} \tag{D.59}$$

D.4 Explicit stress strain equations (plane stress)

The condition for plane stress is that the out of plane stress in the Cartesian z direction is zero. Equation D.44 can be written for the out of plane stress as follows:

$$S_z = B_1 + B_2 C_{33} + (B_3 - \rho J) C_{33}^{-1} \tag{D.60}$$

Hence this boundary condition can be used to express the hydrostatic pressure ρ , contained in the term $(B_3 - \rho J)$, as follows:

$$(B_3 - \rho J) = -C_{33} [B_1 + B_2 C_{33}] \tag{D.61}$$

In addition if the elastomer is treated as incompressible then it can be stated that:

$$I_3 = |\underline{C}| = 1 = C_{33} (C_{11}C_{22} - C_{12}^2) \quad (\text{D.62})$$

For plane stress the Right Green Cauchy tensor is given by:

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \quad (\text{D.63})$$

hence it can be deduced that:

$$C_{33} = \frac{1}{C_{11}C_{22} - C_{12}^2} \quad (\text{D.64})$$

$$C_{33} = C_{33}^{-1} \quad (\text{D.65})$$

$$C_{11}^{-1} = C_{22}C_{33} \quad (\text{D.66})$$

$$C_{22}^{-1} = C_{11}C_{33} \quad (\text{D.67})$$

$$C_{12}^{-1} = -C_{12}C_{33} \quad (\text{D.68})$$

With these results it is possible to re-write equations D.33, D.43 and D.50 as follows:

$$S_x = B_1 (1 - C_{22}C_{33}^2) + B_2 (C_{11} - C_{22}C_{33}^3) \quad (\text{D.69})$$

$$S_y = B_1 (1 - C_{11}C_{33}^2) + B_2 (C_{22} - C_{11}C_{33}^3) \quad (\text{D.70})$$

$$S_{xy} = C_{12}C_{33}^2B_1 + B_2 (C_{12} + C_{12}C_{33}^3) \quad (\text{D.71})$$

Since $I_3 = 1$ and $I_1 = C_{11} + C_{22} + C_{33}$ it can be deduced that:

$$B_1 = 2C_1 + 2C_2I_1 \quad (\text{D.72})$$

$$B_2 = -2C_2 \quad (\text{D.73})$$

therefore:

$$\frac{\partial B_1}{\partial C_{11}} = 2C_2 \frac{\partial I_1}{\partial C_{11}} = 2C_2 \left[1 + \frac{\partial C_{33}}{\partial C_{11}} \right] \quad (\text{D.74})$$

$$\frac{\partial B_1}{\partial C_{22}} = 2C_2 \frac{\partial I_1}{\partial C_{22}} = 2C_2 \left[1 + \frac{\partial C_{33}}{\partial C_{22}} \right] \quad (\text{D.75})$$

$$\frac{\partial B_1}{\partial C_{12}} = 2C_2 \frac{\partial I_1}{\partial C_{12}} = 2C_2 \frac{\partial C_{33}}{\partial C_{22}} \quad (\text{D.76})$$

Using the quotient differential rule on equation D.64 it can be deduced that:

$$\frac{\partial C_{33}}{\partial C_{11}} = -C_{22}C_{33}^2 \quad (\text{D.77})$$

$$\frac{\partial C_{33}}{\partial C_{11}} = -C_{11} C_{33}^2 \quad (\text{D.78})$$

$$\frac{\partial C_{33}}{\partial C_{12}} = 2C_{12} C_{33}^2 \quad (\text{D.79})$$

Therefore substituting equations D.74 to D.76 into equations D.77 to D.79 gives:

$$\frac{\partial B_1}{\partial C_{11}} = 2C_2 [1 - C_{22} C_{33}^2] \quad (\text{D.80})$$

$$\frac{\partial B_1}{\partial C_{22}} = 2C_2 [1 - C_{11} C_{33}^2] \quad (\text{D.81})$$

$$\frac{\partial B_1}{\partial C_{12}} = 4C_2 C_{12} C_{33}^2 \quad (\text{D.82})$$

Using the previously derived identities the 2nd Piola-Kirchhoff stress components D.69 to D.71 can be differentiated with respect to C_{11} , C_{22} and C_{12} to give the following results:

$$\frac{\partial S_x}{\partial C_{11}} = 4C_1 C_{22}^2 C_{33}^3 + 4C_2 C_{22} C_{33}^2 [C_{22} C_{33} (I_1 - C_{33}) - 1] \quad (\text{D.83})$$

$$\frac{\partial S_x}{\partial C_{22}} = 2C_1 C_{33}^2 (2C_{11} C_{22} C_{33} - 1) + 2C_2 [1 + 2(1 - C_{11} C_{22} C_{33}) C_{33}^2 (C_{33} - I_1)] \quad (\text{D.84})$$

$$\frac{\partial S_x}{\partial C_{12}} = -8C_1 C_{12} C_{22} C_{33}^3 + 4C_2 C_{12} C_{33}^2 [1 + 2C_{22} C_{33} (C_{33} - I_1)] \quad (\text{D.85})$$

$$\frac{\partial S_y}{\partial C_{11}} = 2C_1 C_{33}^2 (2C_{11} C_{22} C_{33} - 1) + 2C_2 [1 + 2C_{33}^2 (C_{11} C_{22} C_{33} - 1) (I_1 - C_{33})] \quad (\text{D.86})$$

$$\frac{\partial S_y}{\partial C_{22}} = 4C_1 C_{11}^2 C_{33}^2 + 4C_2 C_{11} C_{33}^2 [C_{11} C_{33} (I_1 - C_{33}) - 1] \quad (\text{D.87})$$

$$\frac{\partial S_y}{\partial C_{12}} = -8C_1 C_{12} C_{11} C_{33}^2 + 4C_2 C_{12} C_{33}^2 [1 + 2C_{11} C_{33} (C_{33} - I_1)] \quad (\text{D.88})$$

$$\frac{\partial S_{xy}}{\partial C_{11}} = -4C_1 C_{12} C_{22} C_{33}^3 + 2C_2 C_{12} C_{33}^2 [1 + 2C_{22} C_{33} (C_{33} - I_1)] \quad (\text{D.89})$$

$$\frac{\partial S_{xy}}{\partial C_{22}} = -4C_1 C_{12} C_{11} C_{33}^3 + 2C_2 C_{12} C_{33}^2 [1 - 2C_{11} C_{33} (I_1 - C_{33})] \quad (\text{D.90})$$

$$\frac{\partial S_{xy}}{\partial C_{12}} = 2C_1 C_{33}^2 (1 + 4C_{12}^2 C_{33}) + 2C_2 [4C_{12}^2 C_{33}^3 (I_1 - C_{33}) + C_{33}^2 (I_1 - 1) - 1] \quad (\text{D.91})$$

Appendix E

Cauchy principal value free terms

In section 6.3 the free term coefficients for equation 6.68 are given in table 6.1. These constants are the Cauchy principal value terms of the expanded displacement derivative BIE domain integrals. The fact that these terms are rendered finite and definite by the application of the Cauchy principal value theorem, integration by parts and the mean value theorem has the effect of removing the $\frac{1}{r^2}$ hypersingularity from the domain integral. It can be shown that (see appendix B):

$$\frac{\partial G_{\alpha\beta}}{\partial x_\gamma} = -\frac{1}{8\pi\mu(1-p)r} \frac{1}{r} \left\{ \begin{array}{l} 2\frac{\partial r}{\partial x_\alpha}\frac{\partial r}{\partial x_\beta}\frac{\partial r}{\partial x_\gamma} + (3-4p)\frac{\partial r}{\partial x_\gamma}\delta_{\alpha\beta} \\ -\frac{\partial r}{\partial x_\alpha}\delta_{\beta\gamma} - \frac{\partial r}{\partial x_\beta}\delta_{\alpha\gamma} \end{array} \right\} \quad (\text{E.1})$$

In equation 6.73 the following free term integral appeared:

$$\oint_{\Gamma_\varepsilon} T_{\gamma\alpha\delta\beta} d\Gamma \quad (\text{E.2})$$

where:

$$T_{\gamma\alpha\delta\beta} = \frac{\partial G_{\gamma\alpha}}{\partial x_\beta} l_\delta \quad (\text{E.3})$$

Over Γ_ε , $\frac{\partial r}{\partial x} = \cos \theta$, $\frac{\partial r}{\partial y} = \sin \theta$ and $d\Gamma_\varepsilon = r d\theta$, therefore:

$$\begin{aligned} \frac{\partial G_{11}}{\partial x} &= -\frac{1}{8\pi\mu(1-p)r} \frac{\partial r}{\partial x} \left[(1-4p) + 2 \left(\frac{\partial r}{\partial x} \right)^2 \right] \\ &\equiv -\frac{1}{8\pi\mu(1-p)r} \cos \theta [(1-4p) + 2 \cos^2 \theta] \\ &\equiv -\frac{1}{8\pi\mu(1-p)r} \cos \theta [(1-4p) + \cos 2\theta] \end{aligned} \quad (\text{E.4})$$

similarly:

$$\frac{\partial G_{11}}{\partial y} = -\frac{1}{8\pi\mu(1-p)r} \sin \theta [(1-4p) + \cos 2\theta] \quad (\text{E.5})$$

and:

$$\begin{aligned}\frac{\partial G_{12}}{\partial x} &= \frac{\partial G_{21}}{\partial x} = -\frac{1}{8\pi\mu(1-p)r} \left[\frac{\partial r}{\partial y} \right] \left[-1 + 2 \left(\frac{\partial r}{\partial x} \right)^2 \right] \\ &\equiv -\frac{1}{8\pi\mu(1-p)r} \sin \theta [\cos 2\theta]\end{aligned}\quad (\text{E.6})$$

$$\begin{aligned}\frac{\partial G_{12}}{\partial y} &= \frac{\partial G_{21}}{\partial y} = -\frac{1}{8\pi\mu(1-p)r} \frac{\partial r}{\partial x} \left[-1 + 2 \left(\frac{\partial r}{\partial y} \right)^2 \right] \\ &\equiv -\frac{1}{8\pi\mu(1-p)r} \cos \theta [-\cos 2\theta]\end{aligned}\quad (\text{E.7})$$

$$\begin{aligned}\frac{\partial G_{22}}{\partial x} &= -\frac{1}{8\pi\mu(1-p)r} \frac{\partial r}{\partial x} \left[(3 - 4p) + 2 \left(\frac{\partial r}{\partial y} \right)^2 \right] \\ &\equiv -\frac{1}{8\pi\mu(1-p)r} \cos \theta [(4 - 4p) + \cos 2\theta]\end{aligned}\quad (\text{E.8})$$

$$\frac{\partial G_{22}}{\partial y} = -\frac{1}{8\pi\mu(1-p)r} \sin \theta [(2 - 4p) - \cos 2\theta] \quad (\text{E.9})$$

Therefore from equation E.4:

$$\begin{aligned}\frac{\partial G_{11}}{\partial x} \frac{\partial r}{\partial x} &= -\frac{1}{8\pi\mu(1-p)r} \cos^2 \theta [(2 - 4p) + \cos 2\theta] \\ &= -\frac{1}{8\pi\mu(1-p)r} \left(\frac{1+\cos 2\theta}{2} \right) [2(1 - 2p) + \cos 2\theta] \\ &= -\frac{1}{8\pi\mu(1-p)r} \left\{ (1 - 2p)(1 + \cos 2\theta) + \frac{1}{2} \cos 2\theta + \frac{1}{2} \cos^2 2\theta \right\} \\ &= -\frac{1}{8\pi\mu(1-p)r} \left\{ (1 - 2p) + \left(\frac{3}{2} - 2p \right) \cos 2\theta + \frac{1+\cos 4\theta}{4} \right\} \\ &= -\frac{1}{8\pi\mu(1-p)r} \left\{ \frac{5-8p}{4} + \left(\frac{3}{2} - 2p \right) \cos 2\theta + \frac{1}{4} \cos 4\theta \right\}\end{aligned}\quad (\text{E.10})$$

hence:

$$\begin{aligned}T_{1111} &= \oint_{\Gamma_\varepsilon} \frac{\partial G_{11}}{\partial x} \frac{\partial r}{\partial x} d\Gamma_\varepsilon \\ &= -\frac{1}{8\pi\mu(1-p)} \left(\frac{5-8p}{4} \right) 2\pi \\ &= -\frac{(5-8p)}{16\mu(1-p)}\end{aligned}\quad (\text{E.11})$$

From equation E.4:

$$\begin{aligned}\frac{\partial G_{11}}{\partial x} \frac{\partial r}{\partial y} &= -\frac{1}{8\pi\mu(1-p)r} \cos \theta \sin \theta [(2 - 4p) + \cos 2\theta] \\ &= -\frac{1}{8\pi\mu(1-p)r} \left[(1 - 2p) \sin 2\theta + \frac{1}{4} \sin 4\theta \right]\end{aligned}\quad (\text{E.12})$$

therefore:

$$T_{1112} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{11}}{\partial x} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = 0 \quad (\text{E.13})$$

From equation E.6:

$$\frac{\partial G_{12}}{\partial x} \frac{\partial r}{\partial x} = -\frac{1}{8\pi\mu(1-p)r} \sin \theta \cos \theta [\cos 2\theta] \quad (\text{E.14})$$

therefore:

$$T_{1211} = T_{2111} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{12}}{\partial x} \frac{\partial r}{\partial x} d\Gamma_\varepsilon = 0 \quad (\text{E.15})$$

From equation E.6:

$$\begin{aligned} \frac{\partial G_{12}}{\partial x} \frac{\partial r}{\partial y} &= -\frac{1}{8\pi\mu(1-p)r} \sin^2 \theta \cos 2\theta \\ &= -\frac{1}{8\pi\mu(1-p)r} \left(\frac{1-\cos 2\theta}{2} \right) \cos 2\theta \\ &= -\frac{1}{8\pi\mu(1-p)r} \left[\frac{\cos 2\theta}{2} - \frac{\cos^2 2\theta}{2} \right] \\ &= -\frac{1}{8\pi\mu(1-p)r} \left[\frac{\cos 2\theta}{2} - \left(\frac{1+\cos 4\theta}{2} \right) \right] \end{aligned} \quad (\text{E.16})$$

therefore:

$$T_{1212} = T_{2112} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{12}}{\partial x} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = \frac{1}{16\mu(1-p)} \quad (\text{E.17})$$

From equation E.8:

$$\begin{aligned} \frac{\partial G_{22}}{\partial x} \frac{\partial r}{\partial x} &= -\frac{1}{8\pi\mu(1-p)r} \cos^2 \theta [(4-4p) - \cos 2\theta] \\ &= -\frac{1}{16\pi\mu(1-p)r} (1 + \cos 2\theta) [(4-4p) - \cos 2\theta] \\ &= -\frac{1}{16\pi\mu(1-p)r} \left\{ (4-4p) + (4-4p) \cos 2\theta - \cos 2\theta - \frac{1+\cos 4\theta}{2} \right\} \\ &= -\frac{1}{16\pi\mu(1-p)r} \left\{ \frac{7-8p}{2} + (3-4p) \cos 2\theta - \frac{1}{2} \cos 4\theta \right\} \end{aligned} \quad (\text{E.18})$$

therefore:

$$T_{2211} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{22}}{\partial x} \frac{\partial r}{\partial x} d\Gamma_\varepsilon = -\frac{7-8p}{16\mu(1-p)} \quad (\text{E.19})$$

From equation E.8:

$$\frac{\partial G_{22}}{\partial x} \frac{\partial r}{\partial y} = -\frac{1}{8\pi\mu(1-p)r} \sin \theta \cos \theta [(4-4p) - \cos 2\theta] \quad (\text{E.20})$$

therefore:

$$T_{2212} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{22}}{\partial x} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = 0 \quad (\text{E.21})$$

Similarly:

$$T_{1121} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{11}}{\partial y} \frac{\partial r}{\partial x} d\Gamma_\varepsilon = 0 \quad (\text{E.22})$$

From equation E.5:

$$\frac{\partial G_{11}}{\partial y} \frac{\partial r}{\partial y} = -\frac{1}{16\pi\mu(1-p)r} \{(4-4p) - (4-4p)\cos 2\theta + \cos 2\theta\} \quad (\text{E.23})$$

therefore:

$$T_{1122} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{11}}{\partial y} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = -\frac{(7-8p)}{16\mu(1-p)} \quad (\text{E.24})$$

From equation E.7:

$$\frac{\partial G_{12}}{\partial y} \frac{\partial r}{\partial y} = -\frac{1}{8\pi\mu(1-p)} \sin \theta \cos \theta (\cos 2\theta) \quad (\text{E.25})$$

therefore:

$$T_{1222} = T_{2122} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{12}}{\partial y} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = 0 \quad (\text{E.26})$$

From equation E.7:

$$\begin{aligned} \frac{\partial G_{12}}{\partial y} \frac{\partial r}{\partial x} &= \frac{1}{8\pi\mu(1-p)} \cos^2 \theta \cos 2\theta \\ &= \frac{1}{8\pi\mu(1-p)} \left(\frac{1+\cos 2\theta}{2} \right) \cos 2\theta \\ &= \frac{1}{16\pi\mu(1-p)} \left[\frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{2} \right] \\ &= \frac{1}{16\pi\mu(1-p)} \left[\frac{\cos 2\theta}{2} + \left(\frac{1+\cos 4\theta}{4} \right) \right] \end{aligned} \quad (\text{E.27})$$

therefore:

$$T_{1221} = T_{2121} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{12}}{\partial y} \frac{\partial r}{\partial x} d\Gamma_\varepsilon = \frac{1}{16\mu(1-p)} \quad (\text{E.28})$$

From equation E.9:

$$\begin{aligned} \frac{\partial G_{22}}{\partial y} \frac{\partial r}{\partial y} &= -\frac{1}{8\pi\mu(1-p)} \sin^2 \theta [(2-4p) - \cos 2\theta] \\ &= -\frac{1}{16\pi\mu(1-p)} [(2-4p)(1 - \cos 2\theta) - \cos 2\theta + \cos^2 2\theta] \end{aligned} \quad (\text{E.29})$$

therefore:

$$T_{2222} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{22}}{\partial y} \frac{\partial r}{\partial y} d\Gamma_\varepsilon = -\frac{5-8p}{16\mu(1-p)} \quad (\text{E.30})$$

similarly:

$$T_{2221} = \oint_{\Gamma_\varepsilon} \frac{\partial G_{22}}{\partial y} \frac{\partial r}{\partial x} d\Gamma_\varepsilon = 0 \quad (\text{E.31})$$

Appendix F

Singular integration for $\log \frac{1}{r}$ singularities

It is stated that the $\log r$ singularity in the second term of the basic BIE equation 6.31 can be dealt with by means of logarithmic quadrature. The method for the regularization process starts by splitting the intrinsic co-ordinate space into two parts as follows [13]:

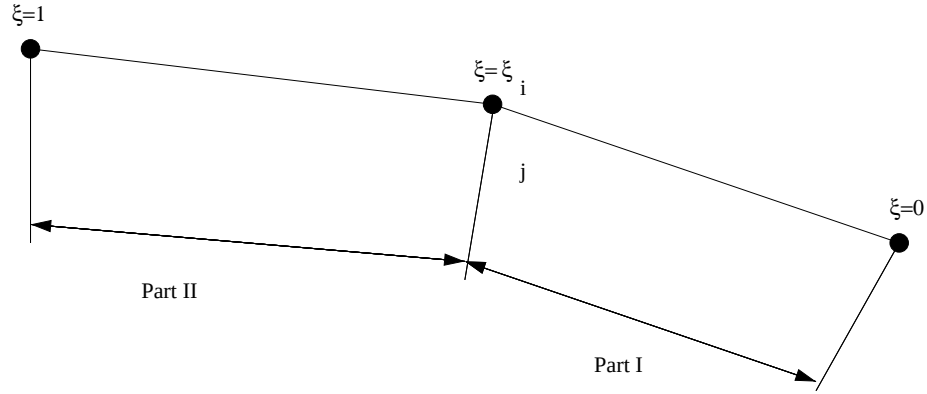


Figure F.1: Three noded element with two integration zones.

$$I = \int_0^1 \left[F_1 \log \frac{1}{r} \right] d\xi \quad (\text{F.1})$$

where:

$$r = ([x - x_i]^2 + [y - y_i]^2)^{\frac{1}{2}}$$

$$F_1 = F_1(x - x_i, y - y_i)$$

Let $(x_i, y_i) \rightarrow (x_j, y_j)$ for the j^{th} node of the e^{th} element. Hence for integration in Part I (see figure F.1) where the intrinsic co-ordinates vary from $\xi = 0$ to $\xi = \xi_j = \frac{j-1}{n-1}$, define:

$$\frac{1}{r} = \frac{\xi_j}{\xi_j - \xi} \cdot \frac{\xi_j - \xi}{\xi_j r} \quad (F.2)$$

therefore:

$$\log \frac{1}{r} = \log \frac{\xi_j}{\xi_j - \xi} + \log \frac{\xi_j - \xi}{\xi_j r} \quad (F.3)$$

Let:

$$\phi_1 = \frac{\xi_j - \xi}{\xi_j} \quad (F.4)$$

thus:

$$\xi \rightarrow 0 \quad \phi_1 = 1$$

$$\xi \rightarrow \xi_j \quad \phi_1 = 0$$

Therefore:

$$d\phi_1 = \frac{\partial \phi_1}{\partial \xi} d\xi \quad (F.5)$$

thus:

$$d\phi_1 = -\frac{d\xi}{\xi_j} \quad (F.6)$$

Enabling the integral to be transformed as follows:

$$\begin{aligned} & \int_0^{\xi_j} F_1(\xi) \log \frac{1}{r} d\xi = \\ & \int_{\xi=0}^{\xi_j} F_1(\xi) \log \frac{\xi_j - \xi}{\xi_j r} d\xi + \int_{\xi=0}^{\xi_j} F_1(\xi) \log \frac{\xi_j}{\xi_j - \xi} d\xi \equiv \\ & \int_{\xi=0}^{\xi_j} F_1(\xi) \log \frac{1}{\eta_1} d\xi + \int_0^1 F_1(\phi_1) \log \frac{1}{\phi_1} d\phi_1 \end{aligned} \quad (F.7)$$

where:

$$\eta_1 = \frac{\xi_j r}{\xi_j - \xi} = \frac{r}{\phi_1} \quad (F.8)$$

Note that in the second term of the last integral term of equation F.7 the limits of the integration have been swapped to remove the negative sign.

For integration in the Part II zone, from $\xi = \xi_j$ to $\xi = 1$, define:

$$\frac{1}{r} = \frac{1 - \xi_j}{\xi - \xi_j} \cdot \frac{\xi - \xi_j}{(1 - \xi_j) r} \quad (\text{F.9})$$

thus:

$$\log \frac{1}{r} = \log \frac{1 - \xi_j}{\xi - \xi_j} + \log \frac{\xi - \xi_j}{(1 - \xi_j) r} \quad (\text{F.10})$$

let:

$$\phi_2 = \frac{\xi - \xi_j}{1 - \xi_j} \quad d\phi_2 = \frac{d\xi}{1 - \xi_j}$$

thus:

$$\begin{aligned} \xi &\rightarrow \xi_j & \phi_2 &= 0 \\ \xi &\rightarrow 1 & \phi_2 &= 1 \end{aligned} \quad (\text{F.11})$$

therefore:

$$\begin{aligned} &\int_{\xi_j}^1 F_1(\xi) \log \frac{1}{r} d\xi = \\ &\int_{\xi_j}^1 F_1(\xi) \log \frac{1}{\eta_2} d\xi + \int_0^1 F_1(\phi_2) \log \frac{1}{\phi_2} (1 - \xi_j) d\phi_2 \end{aligned} \quad (\text{F.12})$$

where:

$$\eta_2 = \frac{r}{\phi_2}$$

Therefore the overall integral can be written as:

$$I = \int_0^1 \left[F_1(\xi) \log \frac{1}{r} \right] d\xi \equiv I_0 + I_1 + I_2 \quad (\text{F.13})$$

where:

$$I_0 = \int_0^1 \left[F_1(\xi) \log \frac{1}{r} \right] d\xi \quad (\text{F.14})$$

$$I_1 = \int_0^1 F_1(\phi_1) \log \frac{1}{\phi_1} \xi_j d\phi_1 \quad (\text{F.15})$$

$$I_2 = \int_0^1 F_1(\phi_2) \log \frac{1}{\phi_2} (1 - \xi_j) d\phi_2 \quad (\text{F.16})$$

where:

$$\eta = \frac{\xi_j r}{(\xi_j - \xi)} \quad \xi < \xi_j$$

$$\eta = \frac{(1 - \xi_j) r}{(\xi - \xi_j)} \quad \xi > \xi_j$$

$$\phi_1 = \frac{\xi_j - \xi}{\xi_j} \quad \xi = \xi_j (1 - \phi_1)$$

$$\phi_2 = \frac{\xi - \xi_j}{1 - \xi_j} \quad \xi = \xi_j + (1 - \xi_j) \phi_2$$

I_0 can be evaluated by the usual Gaussian quadrature. I_1 and I_2 can be evaluated using logarithmic quadrature from:

$$\int_0^1 \eta \log \left(\frac{1}{\eta} \right) d\eta = \sum_{s=1}^{N_s} F(b_s) W_s \quad (\text{F.17})$$

Note: For $j = 1$ then $I_1 = 0$, Part II only. Similarly $j = n$ then $I_2 = 0$, Part I only. For I_0 , as $\eta \rightarrow 0$ then $r \rightarrow 0$, so no singularity in this term and hence standard quadrature can be used.

Appendix G

Inertia loading

G.1 Inertia loading in FEM

In order to model rotational loading for the FEM an equivalent nodal loading vector needs to be derived. This is done utilizing Alembert's principle which assumes a dynamic system moving at constant velocity can be reduced to a static system if a virtual inertial force equal and opposite to the driving force is applied. Consider a point mass m within an element domain which is subject to a constant acceleration a . With Alembert's principle the accelerating force can be balanced by an equal and opposite inertial force:

$$F = -ma \tag{G.1}$$

The work done by the driving force $\underline{F}$ moving with a infinitesimal displacement vector $\delta \underline{q}$ can therefore be written as:

$$\delta W = \int \underline{F}^t \delta \underline{q} dV \equiv - \iint \rho_m a \delta \underline{q} h dx dy \tag{G.2}$$

where h is the element thickness and ρ_m the material density. In the case of centrifugal loading the accelerating force is due to the centripetal acceleration directed inwards towards the origin of rotation, which can be evaluated as follows:

$$a = \rho_m r \omega^2 \tag{G.3}$$

where r is the radius to the center of rotation and ω is the angular velocity. Hence the centrifugal acceleration can be resolved into x and y components (relative to the origin of rotation) and used to evaluate the equivalent nodal loading vector as follows:

$$\delta W = - \iint (a_x \delta u + a_y \delta v) \rho_m h dx dy \quad (G.4)$$

Which for a discretized element domain can be written as:

$$\delta W = \sum_{i=1}^n [(F_x)_i \delta u_i + (F_y)_i \delta v_i] \quad (G.5)$$

where:

$$(F_x)_i = - \int_0^1 \int_0^1 \rho_m a_x N_i |J| h d\xi d\eta$$

$$(F_y)_i = - \int_0^1 \int_0^1 \rho_m a_y N_i |J| h d\xi d\eta$$

where N_i and $|J|$ are the isoparametric shape functions and Jacobian determinant evaluated at the intrinsic ξ and η coordinates. The double integrals in equations G.4 and G.5 can be evaluated from Gauss quadrature as follows (note Q is the number of quadrature points):

$$(F_x)_i = - \sum_{r=1}^Q \sum_{s=1}^Q \rho_m h a_x N_i (\xi_r \eta_s) |J(\xi_r \eta_s)| W_r W_s \quad (G.6)$$

$$(F_y)_i = - \sum_{r=1}^Q \sum_{s=1}^Q \rho_m h a_y N_i (\xi_r \eta_s) |J(\xi_r \eta_s)| W_r W_s \quad (G.7)$$

G.2 Inertial loading in BEM

Rotational inertia loading is modelled in the BEM as follows. If an engineering component, which is plane symmetric with respect to the x-y plane, is rotating with a uniform angular speed ω , around an axis parallel to the z axis and intersecting the x-y plane at (x_o, y_o) , then body forces are generated. The effect of this on the BIE and DBIE equation is to add domain terms of the following form:

$$U_\alpha = \iint_{\Omega} \sum_{\beta=1}^2 (f_\beta G_{\beta\alpha}) \, dx dy \quad (\text{G.8})$$

$$E_{\alpha\delta} = \frac{\partial U_\alpha}{\partial x_\delta} = \iint_{\Omega} \sum_{\beta=1}^2 \left(f_\beta \frac{\partial G_{\beta\alpha}}{\partial x_\delta} \right) \, dx dy \quad (\text{G.9})$$

where:

$$f_x = \rho_m (x - x_o) \omega^2 \quad (\text{G.10})$$

$$f_y = \rho_m (y - y_o) \omega^2 \quad (\text{G.11})$$

with equations G.8 for the BIE and equation G.9 for the DBIE respectively. Defining the kernel function as follows:

$$G_{\alpha\beta} = \frac{1}{8\pi\mu(1-p)} \left[\left((3-4p) \log \frac{1}{r} - \left(\frac{7-8p}{2} \right) \right) \delta_{\alpha\beta} + \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right] \quad (\text{G.12})$$

From the definition of the kernel function G.12 given by equation B.36 in appendix B equation G.8 can be written as follows:

$$U_\alpha = \iint \left[f_\alpha \nabla^2 g^* - \frac{1}{2(1-p)} \sum_{\beta=1}^2 f_\beta \frac{\partial^2 g^*}{\partial x_\alpha \partial x_\beta} \right] \, dx dy \quad (\text{G.13})$$

Starting using integration by parts on equation G.13 these domain terms can be reduced to boundary integrals [14]:

$$\begin{aligned}
U_\alpha = & \oint_\Gamma \left[f_\alpha \frac{\partial g^*}{\partial n} - \frac{1}{2(1-p)} \sum_\beta f_\beta l_\alpha \frac{\partial g^*}{\partial x_\beta} \right] d\Gamma - \\
& \oint_\Gamma \left[g^* \frac{\partial f_\alpha}{\partial n} - \frac{1}{2(1-p)} \sum_\beta g^* l_\beta \frac{\partial f_\beta}{\partial x_\alpha} \right] d\Gamma + \\
& \iint_\Omega g^* \left[\nabla^2 f_\alpha - \frac{1}{2(1-p)} \sum_\beta \frac{\partial^2 f_\beta}{\partial x_\alpha \partial x_\beta} \right] dx dy
\end{aligned} \tag{G.14}$$

For the case of inertial loading it can be deduced that:

$$\nabla^2 f_x = 0 \quad , \quad \frac{\partial^2 f_\beta}{\partial x_\alpha \partial x_\beta} = 0$$

therefore:

$$U_\alpha = \sum_\beta \left(\oint_\Gamma \varphi_{\alpha\beta}^* f_\beta d\Gamma \right) - f'_\alpha \oint_\Gamma g^* d\Gamma \tag{G.15}$$

where:

$$f'_\alpha = \frac{\partial f_\alpha}{\partial n} - \frac{1}{2(1-p)} \frac{\partial f_n}{\partial x_\alpha} \quad , \quad f_n = l f_x + m f_y$$

$$\varphi_{\alpha\beta}^* = \frac{\partial g^*}{\partial n} \delta_{\alpha\beta} - \frac{1}{2(1-p)} \frac{\partial g^*}{\partial x_\beta}$$

Similarly for the domain loading term of the DBIE it can be shown that:

$$E_{\alpha\delta} \equiv \frac{\partial U_\alpha}{\partial x_\delta} \equiv \left[\begin{array}{c} \frac{\partial U}{\partial \bar{x}} \\ \frac{\partial U}{\partial \bar{y}} \\ \frac{\partial V}{\partial \bar{x}} \\ \frac{\partial V}{\partial \bar{y}} \end{array} \right] \tag{G.16}$$

where:

$$E_{\alpha\delta} = - \sum_{e=1}^{n_e} \int \left[\underline{\varphi}^* \underline{f} - \underline{\omega}^* \underline{\tilde{f}} \right] dr \quad (\text{G.17})$$

$$\underline{\varphi}^* = \begin{bmatrix} \frac{\partial \varphi_{11}^*}{\partial x} & \frac{\partial \varphi_{12}^*}{\partial x} \\ \frac{\partial \varphi_{21}^*}{\partial x} & \frac{\partial \varphi_{22}^*}{\partial x} \\ \frac{\partial \varphi_{11}^*}{\partial y} & \frac{\partial \varphi_{12}^*}{\partial y} \\ \frac{\partial \varphi_{21}^*}{\partial y} & \frac{\partial \varphi_{22}^*}{\partial y} \end{bmatrix} \quad (\text{G.18})$$

$$\underline{\omega}^* = \begin{bmatrix} \frac{\partial g^*}{\partial x} & 0 \\ \frac{\partial g^*}{\partial y} & 0 \\ 0 & \frac{\partial g^*}{\partial x} \\ 0 & \frac{\partial g^*}{\partial y} \end{bmatrix} \quad (\text{G.19})$$

$$\begin{aligned} \frac{\partial \varphi^*}{\partial x_\gamma} = - \frac{1}{8\pi\mu(1-p)} & \left[\left(\frac{2 \log r + 1}{2} \right) [2(1-p)l_\alpha \delta_{\alpha\beta} - l_\alpha \delta_{\beta\gamma}] + \right. \\ & \left. \left[2(1-p) \frac{\partial r}{\partial n} \delta_{\alpha\beta} - l_\alpha \frac{\partial r}{\partial x_\beta} \right] \frac{\partial r}{\partial x_\gamma} \right] \end{aligned} \quad (\text{G.20})$$

$$\underline{\tilde{f}} = \begin{bmatrix} \frac{\partial f_x}{\partial n} - \frac{1}{2(1-p)} \frac{\partial f_n}{\partial x} \\ \frac{\partial f_y}{\partial n} - \frac{1}{2(1-p)} \frac{\partial f_n}{\partial y} \end{bmatrix} \quad (\text{G.21})$$

Appendix H

Pressurized cylinder direct numerical solution, review of governing equations

H.1 Strain components in cylindrical coordinates

Consider the deformed infinitesimal element of material in cylindrical coo-ordinates shown in figure H.1.

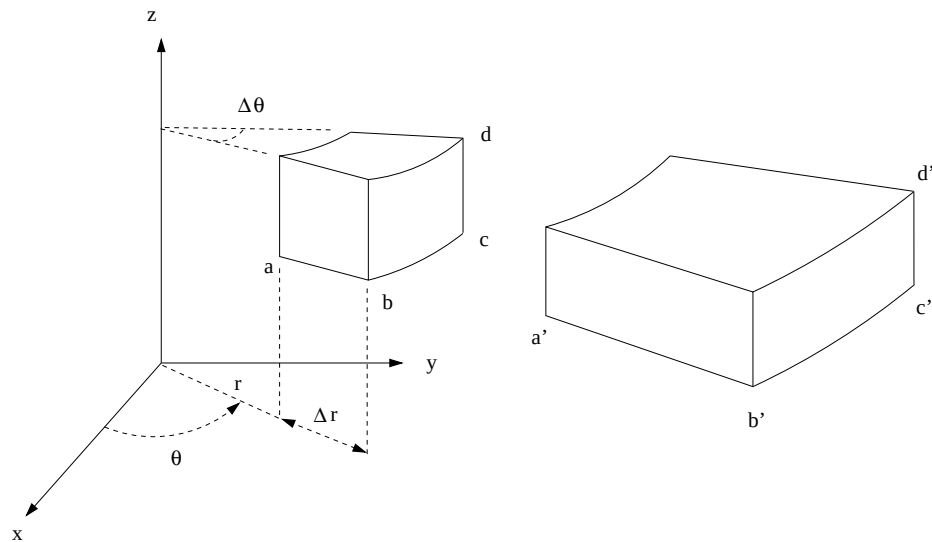


Figure H.1: Cylindrical coordinate infinitesimal element of material

The strain components can be defined from the following equations:

$$\epsilon_r = \lim_{\overline{ab} \rightarrow 0} \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} \equiv \lim_{\Delta r \rightarrow 0} \frac{(r + \Delta r) - \Delta r}{\Delta r} \equiv \frac{du}{dr} \quad (\text{H.1})$$

$$\epsilon_\theta = \lim_{\overline{bc} \rightarrow 0} \frac{\overline{b'c'} - \overline{bc}}{\overline{bc}} \equiv \lim_{\Delta \theta \rightarrow 0} \frac{(r + u)\Delta \theta - r\Delta \theta}{r\Delta \theta} \equiv \frac{u}{r} \quad (\text{H.2})$$

$$\epsilon_\theta = \lim_{\overline{cd} \rightarrow 0} \frac{\overline{c'd'} - \overline{cd}}{\overline{cd}} \equiv \lim_{\Delta z \rightarrow 0} \frac{(\Delta z + \Delta w)\Delta z}{\Delta z} \equiv \frac{\partial w}{\partial z} \quad (\text{H.3})$$

therefore:

$$\lambda_r = (1 + u') \quad (\text{H.4})$$

$$\lambda_\theta = \left(1 + \frac{u}{r}\right) \quad (\text{H.5})$$

$$\lambda_z = \left(1 + \frac{\partial w}{\partial z}\right) \quad (\text{H.6})$$

H.2 Equation of internal equilibrium in cylindrical coordinates

Consider equilibrium of forces acting on the infinitesimal element shown in figure H.2, hence:

$$\frac{\partial(\Delta F_r)}{\partial r} \Delta r - (\Delta F_\theta) \Delta \theta + f_r \Delta r \Delta \theta \Delta z \quad (\text{H.7})$$

where

$$\Delta F r = \sigma_r r \Delta \theta \Delta z \quad (\text{H.8})$$

$$\Delta F \theta = \sigma_\theta \Delta r \Delta z \quad (\text{H.9})$$

Therefore the equation of equilibrium in cylindrical coordinates can be written as:

$$\frac{1}{r} \frac{d(r\sigma_r)}{dr} - \frac{\sigma_\theta}{r} + f_r = 0 \quad (\text{H.10})$$

where f_r represents body forces.

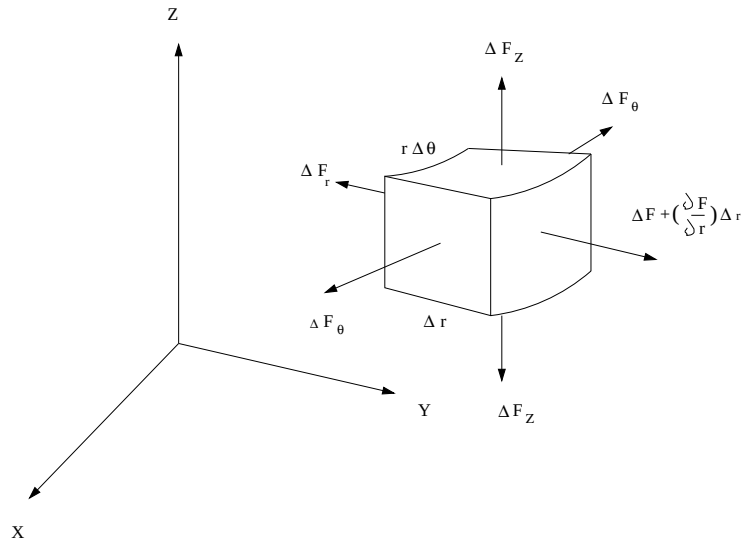


Figure H.2: Forces acting on a infinitesimal element of material in cylindrical coordinates

H.3 Lamé's solution for a pressurized thick walled cylinder

Given:

$$\epsilon_{\theta} = \frac{u}{r} \quad \epsilon_r = \frac{du}{dr} \equiv \frac{(r\epsilon_{\theta})}{dr} \quad (\text{H.11})$$

from Hooke's law:

$$\epsilon_r = \frac{1}{E} [\sigma_r - \nu (\sigma_{\theta} + \sigma_z)] \quad (\text{H.12})$$

$$\epsilon_{\theta} = \frac{1}{E} [\sigma_{\theta} - \nu (\sigma_z + \sigma_r)] \quad (\text{H.13})$$

therefore:

$$\sigma_r - \nu (\sigma_{\theta} + \sigma_z) = \frac{d}{dr} [r (\sigma_{\theta} - \nu (\sigma_z + \sigma_r))] \quad (\text{H.14})$$

Carrying out the differentiation and re-arranging gives:

$$(\sigma_r - \sigma_{\theta}) (1 + \nu) = r \frac{d}{dr} (\sigma_{\theta} - \nu (\sigma_z + \sigma_r)) \quad (\text{H.15})$$

from Hooke's law:

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_r + \sigma_{\theta})] \quad (\text{H.16})$$

it can be deduced that:

$$\frac{d\sigma_z}{dr} = \frac{d}{dr} [\nu (\sigma_r + \sigma_{\theta})] \quad (\text{H.17})$$

substituting equation H.17 into equation H.15 gives:

$$(\sigma_r - \sigma_\theta) = r \frac{d}{dr} [\sigma_\theta(1 - \nu) - \nu\sigma_r] \quad (\text{H.18})$$

The equation of equilibrium in cylindrical coordinates can be written as:

$$(\sigma_r - \sigma_\theta) = -r \frac{d\sigma_r}{dr} \quad (\text{H.19})$$

by comparison of equations H.18 and H.19 it can be deduced that:

$$-\sigma_r = \sigma_\theta(1 - \nu) - \nu\sigma_r + \text{Constant} \quad (\text{H.20})$$

therefore:

$$(\sigma_r + \sigma_\theta)(1 - \nu) = \text{Constant} = 2A \quad (\text{H.21})$$

Substituting for σ_θ in equation H.19 gives:

$$2A - 2\sigma_r = r \frac{d\sigma_r}{dr} \quad (\text{H.22})$$

which can be solved to give:

$$Ar^2 = r^2\sigma_r + B \quad (\text{H.23})$$

thus:

$$\sigma_r = A - \frac{B}{r^2} \quad (\text{H.24})$$

Substituting equation H.24 into equation H.22 it can be deduced that:

$$\sigma_{\theta} = A + \frac{B}{r^2} \quad (\text{H.25})$$

If R_1 is the inner radius and R_2 is the outer radius of the cylinder, with ρ as the internal pressure then the boundary conditions can be stated as:

$$\sigma_r = -\rho \quad r \rightarrow R_1 \quad (\text{H.26})$$

$$\sigma_r = 0.0 \quad r \rightarrow R_2 \quad (\text{H.27})$$

Using the boundary conditions represented by equations H.26 and H.27 and equation H.24 the constants can be solved for to give:

$$A = \frac{\rho}{\Theta^2 - 1} \quad (\text{H.28})$$

$$B = \frac{\rho R_2^2}{\Theta^2 - 1} \quad (\text{H.29})$$

where:

$$\Theta = \frac{R_2}{R_1} \quad (\text{H.30})$$

Therefore the stress distribution through the thickness of the pressurized thick walled cylinder made from a linear homogeneous material are given by the following two equations for radial and hoop stresses respectively:

$$\sigma_r = \frac{\rho}{\Theta^2 - 1} \left(1 - \frac{R_2^2}{r^2} \right) \quad (\text{H.31})$$

$$\sigma_{\theta} = \frac{\rho}{\Theta^2 - 1} \left(1 + \frac{R_2^2}{r^2} \right) \quad (\text{H.32})$$

H.4 Lamé's solution with respect to elastomers

For the direct numerical solution of the non-linear pressurized rubber cylinder the linear case of Lamé's solution with some modification will be used for the initial conditions of the iterative solution procedure. Hence for the two test cases considered (plane stress and strain) and using Lamé's solution:

$$\sigma_z = 0 \quad (\text{H.33})$$

$$\sigma_z = \nu (\sigma_r + \sigma_\theta) \equiv \frac{\rho}{\Theta^2 - 1} \quad (\text{H.34})$$

For an incompressible material such as rubber the following can be stated:

$$\mu = \frac{E}{2(1 + \nu)} \equiv \frac{E}{3} \quad (\text{H.35})$$

and using the Mooney-Rivlin material model it can be shown that:

$$\mu = 2 (C_1 + C_2) \quad (\text{H.36})$$

It is also the case that in cylindrical coordinates the strain components ϵ_r , ϵ_θ and ϵ_z are also the principal strain components. Hence using Cauchy's definition of finite strain ϵ_r and ϵ_θ can be defined as follows:

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{2} \left(\frac{u}{r} \right)^2 \quad (\text{H.37})$$

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2} \left(\frac{du}{dr} \right)^2 \quad (\text{H.38})$$

Using equations H.37 and H.38 it can be deduced that:

$$\lambda_\theta = \left(1 + \frac{u}{r}\right) = \sqrt{1 + 2\epsilon_\theta} \quad (\text{H.39})$$

$$\lambda_r = \left(1 + \frac{\mathrm{d}u}{\mathrm{d}r}\right) = \sqrt{1 + 2\epsilon_r} \quad (\text{H.40})$$

Appendix I

Fracture mechanics

I.1 Crack propagation criterion

It is well known that cracked components will fail when subject to a critical load. To understand why a component fails at a critical load and why the component does not fail at a load below the critical despite the very high stresses at the crack tip an appreciation of the energy changes due to incremental crack extension is required. Griffith [21] proposed the two following conditions for the extension of an existing crack:

1. The stress ahead of the crack must exceed a critical value.
2. The total energy of the system must be reduced by the extension of the crack.

To illustrate this a Griffith approach can be used [36] by considering a remotely clamped crack free plate subject to a uniform stress (see figure I.1). By introducing a crack into this plate the total system energy can be broken down into strain energy and crack surface energy. Considering the strain energy first, as a crack of length $2a$ is introduced into the plate there will be a relaxation of the material above and below the crack, and hence some strain energy will be released. To approximate the strain energy released in this way it can be assumed that the relaxed zone of the material is in the form of a triangle of height $2a$. The relaxed volume of material with thickness t can be approximated as:

$$\beta a^2 t \tag{I.1}$$

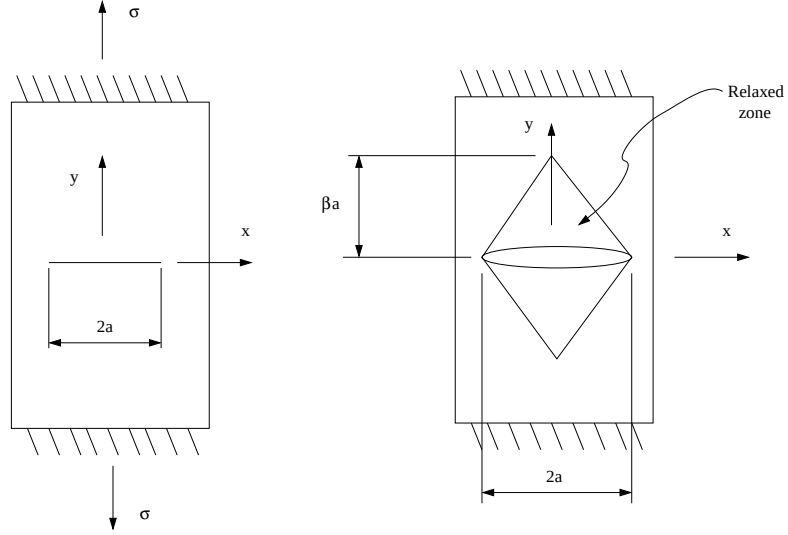


Figure I.1: Fixed end cracked plate

Thus the strain energy released per unit thickness for simple tension is given by:

$$\phi = \frac{\sigma^2}{2E} \beta a^2 \quad (\text{I.2})$$

This last result is in good agreement with the accurate solution obtained by Griffith for plane stress:

$$\phi = \frac{\sigma^2}{2E} \pi a^2 \quad (\text{I.3})$$

The partial derivative of the strain energy with respect to crack extension is therefore:

$$\frac{\partial \phi}{\partial a} = \frac{\sigma^2}{E} \pi a \quad (\text{I.4})$$

A similar pair of expressions can be derived for plane strain conditions:

$$\phi = \frac{\sigma^2}{2E} \pi a^2 (1 - \nu^2)$$

$$\frac{\partial \phi}{\partial a} = \frac{\sigma^2 \pi a}{E} (1 - v^2) \quad (\text{I.5})$$

The second of the two components that make up the total potential energy is the surface energy of the crack, W , the required energy to produce extension of the crack tip. If it is assumed that the energy required to produce equal increments of crack tip extension remains constant as the crack grows then the surface energy W of the crack will increase linearly with increasing crack length. The point of instability can therefore be described as the stationary point of the total energy curve, beyond this point the strain energy released during an incremental crack extension exceeds the energy absorbed creating a new crack surface (see figure I.2). Defining the strain energy release rate per crack tip, G , for an incremental crack extension as follows:

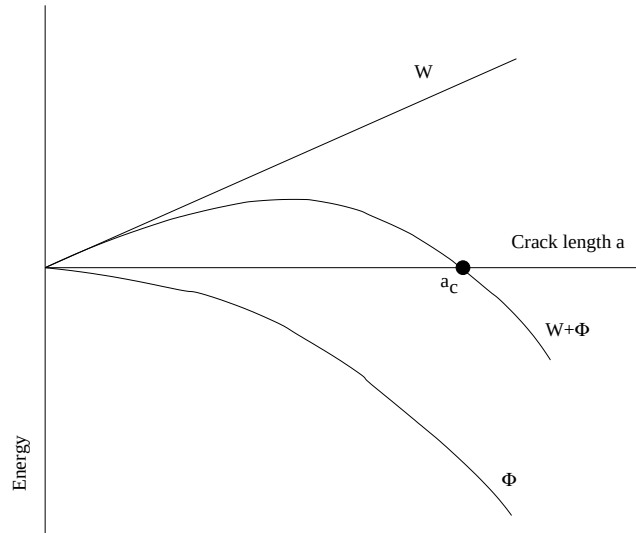


Figure I.2: Energy balance criterion

$$G = \frac{\partial \phi}{\partial a} \quad (\text{I.6})$$

Thus with reference to equations I.4 and I.5:

$$G = \frac{\pi}{E} \sigma^2 a \quad (\text{I.7})$$

$$G = \frac{\pi}{E} \sigma^2 a (1 - v^2) \quad (\text{I.8})$$

where equations I.7 and I.8 are for plane stress and strain respectively. The energy absorbed during an incremental crack extension is usually assigned the symbol R . Therefore the threshold condition for unstable crack growth can be expressed as:

$$\frac{\partial \phi}{\partial a} = \frac{\partial W}{\partial a} \quad \text{OR} \quad G = R \quad (\text{I.9})$$

Since R , the energy absorbed during an incremental crack extension, has a constant value, the critical strain energy release rate G_c at the point of instability can be defined as:

$$G_c = \frac{\sigma_c^2 \pi a}{E} \quad (\text{I.10})$$

$$G_c = \frac{\sigma_c^2 \pi a}{E} (1 - v^2) \quad (\text{I.11})$$

Equations I.10 and I.11 (plane stress and strain respectively) can be rearranged to have material properties on the left and geometrical and loading properties on the right:

$$(G_c E) = \sigma_c (\pi a)^{0.5} \quad (\text{I.12})$$

$$(G_c E) = \sigma_c (\pi a)^{0.5} (1 - v^2)^{0.5} \quad (\text{I.13})$$

The expression on the right of equations I.12 and I.13 can be shown to be a measure of the crack tip stress field singularity [4] and is given the symbol K_I . This parameter will also have a critical value above which the crack will propagate. The critical value of stress intensity has the symbol K_{IC} .

$$K_{IC} = \sigma_C (\pi a)^{0.5} \quad (\text{I.14})$$

Equations I.12, I.13 and I.14 are for the special case of a cracked plate with fixed ends subject to a uniform stress. For more general cases of loading and crack geometry a coefficient Y , or stress intensity factor, can be obtained for the form of loading and crack geometry of interest. It will be shown that another measure of the ease of crack propagation, the J-integral, is constant for any contour surrounding a crack tip and for a linearly elastic material the J-integral equals the strain energy release rate G . Thus with reference to equations I.11 and I.13 it can be seen that:

$$K_I = (JE)^{0.5} \quad (\text{I.15})$$

$$K_I = \left(\frac{JE}{1 - v^2} \right)^{0.5} \quad (\text{I.16})$$

Note: The suffix I after the stress intensity symbol refers to the first of the three modes of crack tip deformation. The three modes of crack tip deformation are as follows:

1. Opening mode, having symmetry about the (x, y) and (x, z) planes.
2. Sliding or shear mode, having anti-symmetry about the (x, z) plane, and symmetry about the (x, y) plane.
3. Tearing mode, having anti-symmetry about the (x, y) and (x, z) planes.

I.2 The J-Integral

A similar approach to the Griffith energy concept of crack propagation is the J-Integral concept first introduced by Rice [40]. Consider the following energy constituents in the total energy of a cracked plate [15].

$$\Phi = \Phi_o + \Phi_a + \Phi_\gamma - F \quad (\text{I.17})$$

where:

- Φ_O is the elastic energy of the loaded un-cracked plate (a constant).
- Φ_a is the change in the elastic energy caused by introducing the crack in the plate.
- Φ_γ the change in the elastic surface energy caused by the formation of the crack surfaces.
- F The work done on the system by external loads.

As with the Griffith approach a crack in a component will propagate as soon as the total system energy Φ stops increasing with crack length. Thus the point of instability will occur when:

$$\frac{d\Phi}{da} \leq 0 \quad (\text{I.18})$$

With reference to equation I.17 and taking note that the elastic energy of the loaded plate is constant, then equation I.18 gives:

$$\frac{d}{da} (\Phi_a + \Phi_\gamma - F) \leq 0 \quad (\text{I.19})$$

Re-arranging equation I.20:

$$\frac{d}{da} (F - \Phi_a) \geq \frac{d\Phi_\gamma}{da} \quad (\text{I.20})$$

The left hand side of equation I.20 represents the energy available for crack extension. This energy being the difference between the energy supplied by external loads and the increase in elastic strain energy. The right hand side of the inequality represents the elastic surface energy of the crack surfaces, which is the energy needed for the crack to grow. Thus equation I.20 is another statement of the point of instability when the crack starts to propagate. And as such is equivalent to equation I.9 defined in terms of the energy release rate G and the crack resistance R . For elastic behaviour (though not necessarily linear) it is possible to define a non-linear equivalent of G , namely J_r :

$$J_r = \frac{d}{da} (F - \Phi_a) \quad (\text{I.21})$$

returning to the energy balance equation I.17 it is possible to define the potential energy of the system Φ_p as follows:

$$\Phi_p = \Phi_o + \Phi_a - F \quad \Phi = \Phi_p + \Phi_\gamma \quad (\text{I.22})$$

Since Φ_o is constant then differentiation of Φ_p yields:

$$\frac{d\Phi_p}{da} = \frac{d}{da} (\Phi_a - F) = -\frac{d}{da} (F - \Phi_a) \quad (\text{I.23})$$

By comparing equation I.23 with I.21 it can be deduced that:

$$J_r = -\frac{d\Phi_p}{da} \quad (\text{I.24})$$

It was shown in equation I.20 that $\frac{d}{da} (F - \Phi_a)$ represents the energy available for crack propagation, and hence $-\frac{d\Phi_p}{da}$ represents a release of energy to overcome the crack propagation resistance energy $\frac{d\Phi_\gamma}{da}$ for an increase in the crack surface by da . The main use of the J-Integral is its numerical evaluation for a contour surrounding a crack tip, which if found to be positive indicates the crack in question will propagate.

I.3 Rice J-Integral evaluation

The alternative way to evaluate the J-integral other than the potential energy approach previously discussed is to integrate the expression derived in the following section over a boundary contour enclosing the crack tip. Consider a cracked body of unit thickness as shown in figure I.3. The body has a perimeter Γ_o and a surface A . Traction $\underline{T}$ acts on a part S_o of the perimeter and performs external work ΔF . Parts of the body undergo a displacement represented as a displacement vector $\underline{u}$.

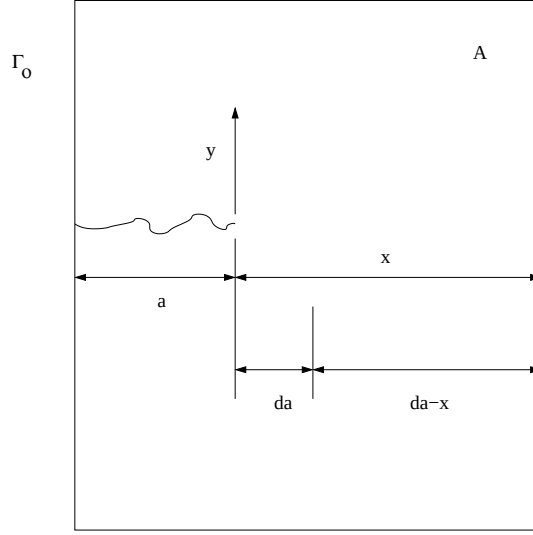


Figure I.3: Rice J-integral crack geometry

Restating equation I.17:

$$\Phi = \Phi_o + \Phi_a + \Phi_\gamma - F \quad (\text{I.25})$$

And defining the change in system energy due to crack increment growth of Δa :

$$\Delta \Phi = \Delta \Phi_a + \Delta \Phi_\gamma - \Delta F \quad (\text{I.26})$$

Defining the incremental change in potential energy as:

$$\Delta \Phi_p = \Delta \Phi_a - \Delta F \quad (\text{I.27})$$

Substituting equation I.27 into equation I.26 it can be seen that:

$$\Delta \Phi = \Delta \Phi_p + \Delta \Phi_\gamma \quad (\text{I.28})$$

Equating equation I.28 with equation I.26 it can be deduced that:

$$\Delta \Phi_p = \Delta \Phi_a - \Delta F \quad (\text{I.29})$$

Taking the limit $\Delta a \rightarrow 0$ then:

$$d\Phi_p = d\Phi_a - dF \quad (\text{I.30})$$

Integrating equation I.30 results in the constant Φ_{01} , the energy content of the body before the load increment ΔF was applied, hence:

$$\Phi_p = \Phi_a - F + \Phi_{01} \quad (\text{I.31})$$

In equation I.31 the term $\Phi_a + \Phi_{01}$ represents the total strain energy contained in the body. The total strain energy content can be expressed in terms of the strain energy density Φ as follows:

$$\Phi_a + \Phi_{01} = \iint_A \Phi dx dy \quad (\text{I.32})$$

The work done by externally applied loads can be expressed as:

$$F = \int_{\Gamma_o} \underline{T}^t \underline{u} ds \quad (\text{I.33})$$

where $\underline{u}$ is the displacement vector. Substituting equation I.31 gives:

$$\Phi_p = \iint_A \Phi dx dy - \int_{\Gamma_o} \underline{T}^t \underline{u} ds \quad (\text{I.34})$$

Differentiating equation I.34 with respect to crack length a :

$$\frac{d\Phi_p}{da} = \iint_A \frac{\partial \Phi}{\partial a} dx dy - \int_{\Gamma_o} \underline{T}^t \frac{\partial \underline{u}}{\partial a} ds \quad (\text{I.35})$$

Equation I.35 is the J-integral expression giving the change in potential energy per unit crack extension. The sign convention of the equation can be modified with reference to figure I.3 if the co-ordinate system origin is taken as the crack tip. If the perimeter boundary Γ_0 is fixed then it can be seen that $da = -dx$ and $\frac{d}{da} = -\frac{d}{dx}$, in other words the positive x axis direction is in the direction of crack growth, thus:

$$\frac{d\Phi_p}{da} = - \iint_A \frac{\partial \Phi}{\partial x} dx dy + \int_{\Gamma_o} \underline{T}^t \frac{\partial \underline{u}}{\partial x} ds \quad (\text{I.36})$$

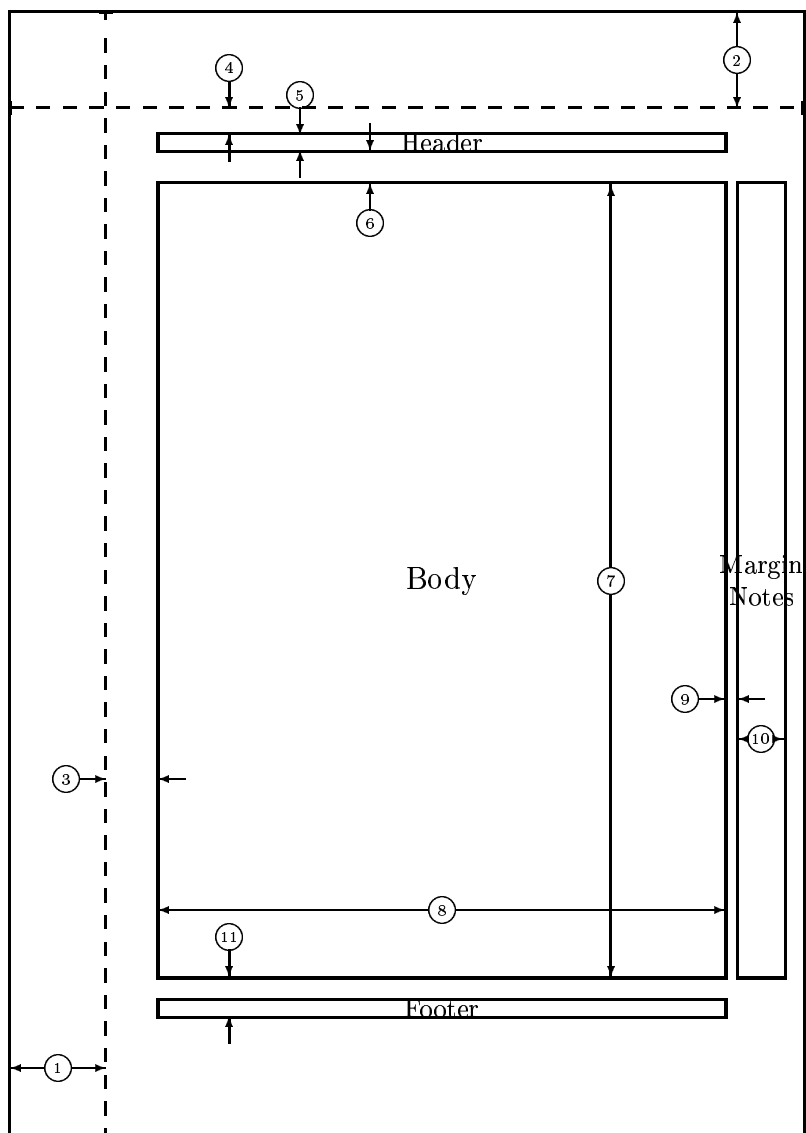
Using integration by parts and equating equation I.36 with equation I.24 it can be seen that the Rice J-integral expression is given by:

$$J_r = \int_{\Gamma_o} \Phi dy - \int_{\Gamma_o} \underline{T}^t \frac{\partial \underline{u}}{\partial x} ds \quad (\text{I.37})$$

Expanding out the traction and displacement derivative components equation I.37 can be written as:

$$J_r = \int_{\Gamma_o} \Phi dy - \int_{\Gamma_o} \left(T_x \frac{\partial u}{\partial x} + T_y \frac{\partial v}{\partial x} \right) ds \quad (\text{I.38})$$

It can be shown that for a closed contour the value of the J-integral expression given by equation I.38 is zero [15]. However, for a contour enclosing a crack tip this expression will always be greater than zero. For this research the evaluation of this expression was simplified because the total strain energy Φ could be found directly from the Mooney-Rivlin strain energy expression 4.23. Also the displacement derivatives in the second term of equation I.38 were already evaluated during the non-linear solution algorithm.



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