

A Closed-Loop Output Error Approach for Physics Informed Trajectory Inference using Online Data

Adolfo Perrusquía, *Member, IEEE*, Weisi Guo, *Senior Member, IEEE*

Abstract—Whilst autonomous systems can be used for a variety of beneficial applications, they can also be used for malicious intentions and it is mandatory to disrupt them before they act. So, an accurate trajectory inference algorithm is required for monitoring purposes that allows to take appropriate countermeasures. This paper presents a closed-loop output error approach for trajectory inference of a class of linear systems. The approach combines the main advantages of state estimation and parameter identification algorithms in a complementary fashion using online data and an estimated model, which is constructed by the state and parameter estimates, that inform about the physics of the system to infer the followed noise-free trajectory. Exact model matching and estimation error cases are analysed. A composite update rule based on a least-squares rule is also proposed to improve robustness and parameter and state convergence. The stability and convergence of the proposed approaches are assessed via Lyapunov stability theory under the fulfilment of a persistent excitation condition. Simulation studies are carried out to validate the proposed approaches.

Index Terms—Trajectory inference, Physics Informed model, Closed-loop Output error, Least-squares composite rule, States measurements, Parameter identification, Excitation signal.

I. INTRODUCTION

The use of autonomous systems technologies have become relevant in the past years due to its high impact and benefits in a variety of beneficial applications. A clear example is the use of drones in applications related to agriculture, inspection, maintenance, and delivery, to name a few; or humanoid robots for social and medical robotics applications. However, potential threats in those applications are also increasing due to misuse of the autonomous system, e.g., disruption and terrorist attacks. Therefore, it is mandatory to design an efficient, flexible, and accurate trajectory inference algorithm for successful monitoring, fast alarm, and to apply appropriate countermeasures. In this sense, trajectory inference is important to predict where an autonomous system might be to quantify risk, or where it was between snapshots of position data to understand its mission profile or intention.

Trajectory and state estimation is a well-known problem that has been addressed by different authors in the last decades [1], [2]. Its aim is to infer the trajectory that follows a system using only previous and current measurements [3] from its states and a prior knowledge of its physics [4]. The destination or desired trajectory that follows the system can be known or unknown.

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A. Perrusquía and Weisi Guo are with the School of Aerospace, Transport and Manufacturing, Cranfield University, Bedford, UK (e-mail: Adolfo.Perrusquia-Guzman@cranfield.ac.uk).

Kalman filter [5], [6] and its variants is one of the most important algorithms, and widely used in industry, for trajectory and state estimation based on Gaussian distribution assumption [7], [8] at the prior model and sensor noise. The algorithm merges two sources of information: prior knowledge of the dynamics of the system and the posterior estimation such that the mean between these two state estimation almost matches with the exact mean of the posterior distribution. However, it is well known [9] that the Kalman filter algorithm and any of its variants can diverge fast if the prior model has a bad design, that is, the parameters of the model do not exhibit the real physics of the system. So, one main issue may arise when there exists parameter uncertainty or the dynamic model structure is unknown. Particle filter [10], [11] can overcome this issue by using only sample points and a proposed distribution which is not necessarily Gaussian. However, the selection of the distribution is not trivial and the number of samples that requires the algorithm is large which translates into a high computational cost.

Machine learning has been used for trajectory inference using measurement data [12], [13] with interesting results. Physics informed neural networks [14] uses prior knowledge of the physics of the system with a neural network to infer the trajectory. In this case, the prior knowledge stabilizes the performance of the neural network. One main drawback of this technique is that requires that the physics to be normalized to avoid instability which is difficult to achieve for multi-agent systems and if the upper bound of the dynamic model is unknown. Other approaches use recurrent neural networks [15]–[17] to add memory and take into account previous states. However, in general this kind of neural networks are not able to capture the physics of the system and the trajectory inference is not accurate.

Recent studies use output feedback reinforcement learning [18], [19] to estimate the optimal control policy of unknown linear and nonlinear systems using a linear quadratic regulator approach [20]–[22]. This particular approach gives a parameterization of the states in terms of the output measurements and the control input. So, the algorithm is able to infer the physics of the system and consequently estimate the states if the parameterization fulfils a persistency of excitation condition [23], [24]. However, this approach can only be used in a control perspective, that is, to find a stabilizing controller and not for identification purposes. Other approaches combines the capabilities of reinforcement learning and Gaussian processes to learn the kernel functions that approximate the dynamics of the unknown system [25]. However, this approach needs access to the control policy that is usually hidden in autonomous

applications [26], [27].

Since the key issue of the aforementioned approaches lies in the design of a physics informed model, then identification algorithms have to been used to estimate the parameters of the system [28]. The least-squares (LS) and gradient rules [29] are the most popular methods for system identification which only require offline or online data to estimate the parameters of the system. There exists an extensive literature that use variants of LS [2], [30]–[32] and gradient algorithms to estimate the parameters of either dynamical systems [33], cost functions [34], [35], and control policies [36], [37]. However, if the measurements of the states and control input are noisy then biased estimates are obtained [38]. Therefore, the final physics informed model will not have accurate trajectory inference results and in consequence, the threat detection will fail.

Other approaches, used mainly for parameter identification of robot manipulators [39]–[41], are the Closed-Loop Input Error (CLIE) [42], [43] and the Closed-Loop Output Error (CLOE) algorithms [44]. The idea behind these algorithms is based on a parallel model structure in which the robot and its model are simultaneously controlled by a stabilizing controller tuned with the same gains. The parameters of the model are estimated through an identification algorithm like the LS or gradient, which is driven by the input error (CLIE) or output error (CLOE) [39], [42]. These algorithms and specially the CLOE algorithm use well-tuned filters to reduce noise measurements which can produce biased estimates as outcome [45]. In addition, the algorithms are not theoretically proved despite they obtained informative results [46]. Furthermore, there are no studies showing how the previous CLIE and CLOE algorithms are affected by bounded estimation errors.

In view of the above, a trade-off between states and parameters identification can be recognized. Whilst a state estimation requires an almost accurate prior dynamic model to achieve accurate state inference, an identification algorithm requires smooth and noise-free state measurements to obtain unbiased estimates. Furthermore, there no exists a theoretical approach that combines both approaches effectively and verify its closed-loop stability and boundedness of the inferred trajectory.

Motivated by the above comments, this paper reports a CLOE approach for physics informed trajectory inference which combines the main advantages of state estimation and parameter identification algorithms. This approach is verified theoretically using Lyapunov stability theory. The main contributions of this work are the following:

- The approach combines the advantages of state estimation and identification algorithms for trajectory inference. The approach is supported by rigorous stability proofs under zero and non-zero estimation error cases.
- The physics of the system is inferred by the estimated model constructed by the estimated states and the parameter estimates.
- The regressor matrix used in the proposed CLOE algorithm is noise free and consequently the estimated states are also noise free and the estimates are unbiased.
- A composite update law based on the LS algorithm is proposed to improve parameter convergence. A complete

stability and convergence proof supports the proposed composite rule.

- The approach holds for known and unknown desired reference/destination either for set-point, time-varying, and stabilization tasks.

The contributions of this work with respect to previous approaches for trajectory inference are the following:

- 1) The estimated model of the proposed approach defines a physics informed model for trajectory inference that is noise free.
- 2) Knowledge of the control input and gains are not required since they are lumped in the parameter estimates.
- 3) Parameter estimates convergence is rigorously verified by considering persistency of excitation conditions under zero and non-zero bounded estimation error conditions.
- 4) The LS composite rule is supported by a rigorous stability proof which enables to take the advantages of both gradient and least-squares algorithms in one simple rule for stability and fast convergence. This result fills the gap on the convergence analysis of composite rules using LS algorithms.
- 5) The algorithm is simple to apply since only a gain matrix (one gain for each parameter estimate) needs tuning for the CLOE algorithm. In addition, the presented approach does not require initial parameter estimates assumption.

The paper outline is as follows: Section II presents the problem formulation where the main assumptions and constraints are clearly stated. Section III develops the CLOE algorithm for the class of linear systems used in this paper which includes the theoretic analysis using Lyapunov stability theory of two scenarios: model matching and modelling error. Section IV introduces the CLOE algorithm based on a composite update law using a least-squares algorithm for convergence and robustness improvement. Section V reports simulations studies using a 4-DOF robot and a F-16 aircraft dynamics under known and unknown reference scenarios. Conclusions and further work are reported in Section VI.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$ denote the spaces of natural numbers, real numbers, positive real numbers, real n -vectors, and real $n \times m$ -matrices, respectively; $I \in \mathbb{R}^{n \times n}$ denotes an identity matrix; $\mathcal{L}_\infty$ denotes the space of bounded signals, $\mathcal{L}_2$ denotes the space of square integrable functions, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denotes the minimum and maximum eigenvalues of matrix A , respectively; the norms $\|A\| = \sqrt{\lambda_{\max}(A^\top A)}$ and $\|x\|$ stand for the induced matrix and vector Euclidean norms, respectively; $A > B$ means that $A - B$ is positive definite; where $x \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$ and $n, m \in \mathbb{N}$.

II. PROBLEM FORMULATION

Consider the following second-order linear time-invariant system [47],

$$\ddot{x} = A_1 \dot{x} + A_0 x + B_0 u + D_0, \quad (1)$$

where $x, \dot{x} \in \mathbb{R}^n$ define the generalized coordinates and its time derivative, respectively; $u \in \mathbb{R}^n$ is the control input, $A_0, A_1, B_0 \in \mathbb{R}^{n \times n}$ and $D_0 \in \mathbb{R}^n$ are the plant matrices

that will be considered unknown. System (1) stands to many mechanical, electrical, and hydraulic systems where A_1 defines a dissipative matrix coefficients (e.g., Coulomb friction, dampers, resistors), A_0 models compliance terms (e.g., linear and rotational springs, capacitors), B_0 defines the gains of the control input also associated to the inverse of an inertial matrix, and D_0 models disturbances or gravitational terms.

Assumption 1: The system (1) is controllable and only noisy measurements of the generalized coordinates and their derivatives are available, that is, we have access to full states measurements.

Assumption 2: The control input has an unknown linear structure which does not compromise the closed-loop linearity of (1).

Assume that (1) is controlled by a stabilizing control law of the form

$$u = K_1(x^d - x) + K_2(\dot{x}^d - \dot{x}) + K_3, \quad (2)$$

where $K_1, K_2 \in \mathbb{R}^{n \times n}$ are feedback control gains, $K_3 \in \mathbb{R}^n$ defines a compensation term that reduces the steady-state error [48], and $x^d \in \mathbb{R}^n$ is a known desired reference. So, the closed-loop system between (1) and (2) is

$$\begin{aligned} \ddot{x} &= (A_1 - B_0 K_2) \dot{x} + (A_0 - B_0 K_1) x + D_0 \\ &\quad + B_0 (K_1 x^d + K_2 \dot{x}^d + K_3) \\ &= -A \dot{x} - B x + C_1 x^d + C_2 \dot{x}^d + D \triangleq \Phi^\top \Theta, \end{aligned} \quad (3)$$

where $A = B_0 K_2 - A_1$, $B = B_0 K_1 - A_0$, $C_1 = B_0 K_1$, $C_2 = B_0 K_2$, and $D = D_0 + B_0 K_3$. Here $\Phi = \Phi(x, \dot{x}, x^d, \dot{x}^d) \in \mathbb{R}^{p \times n}$ is a regressor matrix and $\Theta \in \mathbb{R}^p$ denotes the parameters vector which are defined as

$$\Phi = \begin{bmatrix} -I \otimes \dot{x} \\ -I \otimes x \\ I \otimes x^d \\ I \otimes \dot{x}^d \\ I \end{bmatrix}, \quad \Theta = \begin{bmatrix} \text{vec}(A) \\ \text{vec}(B) \\ \text{vec}(C_1) \\ \text{vec}(C_2) \\ D \end{bmatrix}. \quad (4)$$

Remark 1: The matrices A and B have positive eigenvalues, that is, $\lambda(A) > 0$ and $\lambda(B) > 0$.

The following definition is required for the theoretical results presented throughout this paper.

Definition 1: [24], [49] The equilibrium point $x^* = 0$ of (1) is said to be uniformly ultimately bounded (UUB) if there exists a compact set $\mathcal{S} \subset \mathbb{R}^n$ so that for all $x_0 \in \mathcal{S}$ there exists a bound $\mathcal{B}$ and a time $\mathcal{T}(\mathcal{B}, x_0)$ such that $\|x(t) - x^*\| \leq \mathcal{B}$ for all $t \geq t_0 + \mathcal{T}$.

The closed-loop trajectories (3) are UUB and can be easily verified by choosing the next Lyapunov function

$$V = \frac{1}{2} \left[\left(\dot{x} + \frac{1}{2} A x \right)^\top \left(\dot{x} + \frac{1}{2} A x \right) + x^\top \left(\frac{1}{2} A^2 + B^\top \right) x \right]$$

which its time-derivative along the closed-loop trajectories (42) is

$$\begin{aligned} \dot{V} &= -\frac{1}{2} \dot{x}^\top A \dot{x} + \dot{x}^\top \eta - \frac{1}{2} x^\top A B x + \frac{1}{2} x^\top A \eta \\ &\leq -\frac{1}{2} \lambda_{\min}(A) \|\dot{x}\|^2 - \frac{1}{2} \lambda_{\min}(AB) \|x\|^2 \\ &\quad + \left(\|\dot{x}\| + \frac{1}{2} \lambda_{\max}(A) \|x\| \right) \|\eta\|. \end{aligned}$$

where $\eta = C_1 x^d + C_2 \dot{x}^d + D$. The above inequality can be written as

$$\begin{aligned} \dot{V} &\leq -\xi^\top \mathcal{A} \xi + \|\eta\| \mathcal{B}^\top \xi \\ &\leq -\lambda_{\min}(\mathcal{A}) \|\xi\| \left[\|\xi\| - \frac{\|\mathcal{B}\| \|\eta\|}{\lambda_{\min}(\mathcal{A})} \right] \end{aligned}$$

where $\xi = [x^\top, \dot{x}^\top]^\top$, $\mathcal{A} = \frac{1}{2} \text{diag}\{\lambda_{\min}(A), \lambda_{\min}(AB)\}$, and $\mathcal{B} = [1, \frac{1}{2} \lambda_{\max}(A)]^\top$. In consequence, $\dot{V} < 0$ as long as

$$\|\xi\| > \frac{\|\mathcal{B}\| \|\eta\|}{\lambda_{\min}(\mathcal{A})} \triangleq \mu_0. \quad (5)$$

So, if the control gains K_1 , K_2 and K_3 are chosen such that (5) is satisfied then the closed-loop trajectories (3) converge to a small compact set $\mathcal{S}_\xi$ of radius μ_0 , i.e., $\|\xi\| \leq \mu_0$ and therefore, the trajectories of (3) are UUB.

Our goal is to estimate the hidden physics of the system from the online data measurements to infer the trajectory.

III. CLOSED-LOOP OUTPUT ERROR (CLOE)

The proposed CLOE algorithm is depicted in Fig. 1. An estimated model of the unknown system is constructed such that the output error between the real measurements and the estimated states feed an identification algorithm that subsequently updates the estimated model. The proposed algorithm can be regarded as a combination between a state observer and a parameter identification algorithm which aims to take their advantages to estimate the physics of the system from online measurements and without any prior knowledge.

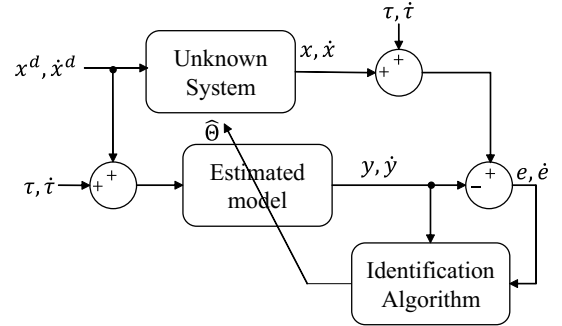


Fig. 1. Block diagram of the proposed closed-loop output approach

An user-defined excitation signal $\tau \in \mathcal{C}^1 \subset \mathbb{R}$ and its derivative $\dot{\tau} \in \mathbb{R}$ are added to the generalized coordinates measurements to guarantee parameter and state estimates convergence. Let

$$\begin{aligned} x_\tau &= x + \tau \\ \dot{x}_\tau &= \dot{x} + \dot{\tau}. \end{aligned} \quad (6)$$

Remark 2: The real closed-loop system (3) does not have the excitation signal τ in the desired reference. This signal is used only for identification purposes which can be subtracted to obtain the real signal, that is, $x = x_\tau - \tau$.

Without loss of generality, if the new output is x_τ means that the desired reference is given by $x^d + \tau$ and satisfies the following closed-loop dynamics

$$\begin{aligned} \ddot{x}_\tau &= -A \dot{x}_\tau - B x_\tau + C_1 (x^d + \tau) + C_2 (\dot{x}^d + \dot{\tau}) + D \\ &= \Phi^\top \Theta \end{aligned} \quad (7)$$

where

$$\Phi = \Phi(x_\tau, \dot{x}_\tau, x^d, \dot{x}^d, \tau, \dot{\tau}) = \begin{bmatrix} -I \otimes \dot{x}_\tau \\ -I \otimes x_\tau \\ I \otimes (x^d + \tau) \\ I \otimes (\dot{x}^d + \dot{\tau}) \\ I \end{bmatrix}. \quad (8)$$

Define the estimated model dynamics as

$$\ddot{y} = -\hat{A}\dot{y} - \hat{B}y + \hat{C}_1(x^d + \tau) + \hat{C}_2(\dot{x}^d + \dot{\tau}) + \hat{D} \triangleq \Phi_y^\top \hat{\Theta}, \quad (9)$$

where $y \in \mathbb{R}^n$ are the states of the estimated model, $\hat{A}$, $\hat{B}$, $\hat{C}_1$, $\hat{C}_2 \in \mathbb{R}^{n \times n}$, and $\hat{D} \in \mathbb{R}^n$ are estimates of A, B, C_1, C_2 , and D , respectively. The matrix $\Phi_y = \Phi_y(y, \dot{y}, x^d, \dot{x}^d, \tau, \dot{\tau}) \in \mathbb{R}^{p \times n}$ defines the regressor matrix that depends only on the estimated model states and $\hat{\Theta} \in \mathbb{R}^p$ denotes the parameter estimates vector which are defined as

$$\Phi_y = \begin{bmatrix} -I \otimes \dot{y} \\ -I \otimes y \\ I \otimes (x^d + \tau) \\ I \otimes (\dot{x}^d + \dot{\tau}) \\ I \end{bmatrix}, \quad \hat{\Theta} = \begin{bmatrix} \text{vec}(\hat{A}) \\ \text{vec}(\hat{B}) \\ \text{vec}(\hat{C}_1) \\ \text{vec}(\hat{C}_2) \\ \hat{D} \end{bmatrix}. \quad (10)$$

Define the output error $e \in \mathbb{R}^n$ as

$$e = x_\tau - y. \quad (11)$$

A. Model matching

Consider that the proposed CLOE algorithm is able to identify exactly the physics of the unknown system. Then, the CLOE dynamics satisfies the following ODE

$$\begin{aligned} \ddot{e} &= -A\dot{x}_\tau - Bx_\tau + C_1(x^d + \tau) + C_2(\dot{x}^d + \dot{\tau}) + D \\ &\quad + \hat{A}\dot{y} + \hat{B}y - \hat{C}_1(x^d + \tau) - \hat{C}_2(\dot{x}^d + \dot{\tau}) - \hat{D} \\ &= -A\dot{e} - Be - \Phi_y^\top \tilde{\Theta}, \end{aligned} \quad (12)$$

where $\tilde{\Theta} = \hat{\Theta} - \Theta \in \mathbb{R}^p$ is the parametric error which is defined as follows

$$\tilde{\Theta} = \hat{\Theta} - \Theta = \begin{bmatrix} \text{vec}(\hat{A} - A) \\ \text{vec}(\hat{B} - B) \\ \text{vec}(\hat{C}_1 - C_1) \\ \text{vec}(\hat{C}_2 - C_2) \\ \hat{D} - D \end{bmatrix}. \quad (13)$$

The following theorem establishes the stability and parameter convergence of the proposed trajectory inference approach.

Theorem 1: Consider online full states measurements of a second-order linear time invariant system of the form (3) and the estimated model (9). If the parameters $\hat{\Theta}$ are updated as

$$\dot{\hat{\Theta}} = \dot{\tilde{\Theta}} = \Gamma \Phi_y(e + \dot{e}), \quad (14)$$

where $\Gamma \in \mathbb{R}^{p \times p}$ is a positive definite gain matrix, and $A - I > 0$ then $\tilde{\Theta}, e, \dot{e}, y, \dot{y}$ and Φ_y remain bounded and the output error e converges to zero.

Proof: Consider the following Lyapunov function

$$V = \frac{1}{2} \dot{e}^\top \dot{e} + \frac{1}{2} e^\top (B^\top + A)e + \frac{1}{2} \tilde{\Theta}^\top \Gamma^{-1} \tilde{\Theta} + e^\top \dot{e} \quad (15)$$

which can also be written as

$$V = \frac{1}{2} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^\top \begin{bmatrix} B^\top + A & I \\ I & I \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \frac{1}{2} \tilde{\Theta}^\top \Gamma^{-1} \tilde{\Theta}. \quad (16)$$

The function V is positive definite if $B^\top + A - I > 0$ or, more strictly, if $A - I > 0$. The time derivative of (15) along the CLOE dynamics (12) is

$$\begin{aligned} \dot{V} &= \dot{e}^\top (-A\dot{e} - Be - \Phi_y^\top \tilde{\Theta}) + e^\top (B^\top + A)\dot{e} \\ &\quad + e^\top (-A\dot{e} - Be - \Phi_y^\top \tilde{\Theta}) + \dot{e}^\top \dot{e} + \tilde{\Theta}^\top \Gamma^{-1} \dot{\tilde{\Theta}} \\ &= -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \tilde{\Theta}^\top (\Phi_y(e + \dot{e}) - \Gamma^{-1} \dot{\tilde{\Theta}}). \end{aligned}$$

If the update law is chosen as in (14), then $\dot{V}$ reduces to

$$\begin{aligned} \dot{V} &= -\dot{e}^\top (A - I)\dot{e} - e^\top Be \\ &= -\begin{bmatrix} e \\ \dot{e} \end{bmatrix}^\top \underbrace{\begin{bmatrix} B & 0 \\ 0 & A - I \end{bmatrix}}_{Q \in \mathbb{R}^{2n \times 2n}} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \leq -\lambda_{\min}(Q) \|E\|^2 \end{aligned} \quad (17)$$

where $E = [e^\top, \dot{e}^\top]^\top \in \mathbb{R}^{2n}$. The matrix Q is positive definite if $A - I > 0$ which matches with the positivity assumption of V in (16). From (17), boundedness of E can be concluded, that is, $E \in \mathcal{L}_\infty$ and also that $V(0) \geq V$ for any time instance t . Furthermore, boundedness of E implies boundedness of the estimated states y and $\dot{y}$; and hence the regressor matrix Φ_y is also bounded. So, $E, y, \dot{y}, \Phi_y \in \mathcal{L}_\infty$.

If we integrate both sides of (17) in a time-interval $[0, t]$ gives

$$\begin{aligned} V(t) - V(0) &\leq -\lambda_{\min}(Q) \int_0^t \|E\|^2 d\tau, \\ V(0) &\geq V(0) - V(t) \geq \lambda_{\min}(Q) \int_0^t \|E\|^2 d\tau, \end{aligned}$$

Then it follows that,

$$\int_0^t \|E\|^2 d\tau \leq \frac{V(0)}{\lambda_{\min}(Q)} < \infty. \quad (18)$$

From (18), it follows that E is an $\mathcal{L}_2$ function. Since the parametric error $\tilde{\Theta}$ and the signals within E are bounded allows concluding that $\dot{E} = [\dot{e}^\top, \ddot{e}^\top]^\top$ is also bounded, that is, $\dot{E} \in \mathcal{L}_\infty$. Barbalat's lemma is applied to conclude that E converges to zero. This completes the proof. ■

Parameter convergence is obtained if the regressor matrix Φ_y fulfils the following persistence of excitation condition [50]

Definition 2: [49] The regressor matrix $\Phi_y : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{p \times n}$ is persistently exciting (PE) if there exists $\beta_1, \beta_2, T > 0$ such that for all $t \geq 0$ the next relationship is fulfilled

$$\beta_1 I \leq L = \int_t^{t+T} \Phi_y(\tau) \Phi_y^\top(\tau) d\tau \leq \beta_2 I. \quad (19)$$

The above definition implies that the product of $\Phi_y \Phi_y^\top$ never loses rank at any window of length T . The estimated trajectory $\hat{x}$ can be recovered as

$$\hat{x} = y - \tau. \quad (20)$$

Lemma 1: [49], [51] The update rule (14) and the system defined by the following linear time-varying system

$$\begin{aligned}\dot{w} &= B(t)u \\ z &= C(t)w,\end{aligned}\quad (21)$$

are equivalent for some bounded piecewise continuous functions $B(t)$ and $C(t)$. Furthermore, the PE condition (19) is equivalent to the uniform complete observability (UCO) [24] of system (21). In addition, if u and z are bounded, then w is bounded.

Proof: System (21) and the update rule (14) are equivalent under the control feedback $u = e + \dot{e}$, $B(t) = \Gamma\Phi_y$, $C(t) = \Phi_y^\top$, and $w = \tilde{\Theta}$. Then, the Observability Grammian

$$N(t, t+T) = \int_t^{t+T} C^\top(\tau)C(\tau)d\tau,$$

is equivalent to the PE condition (19). If system (21) is UCO, then the regressor Φ_y is bounded. Furthermore, if u and z are bounded it implies that $\tilde{\Theta}$ and $\tilde{\Theta}$ are also bounded. This completes the proof. ■

The result of Lemma 1 will be used in the next section to verify boundedness of the parametric error $\tilde{\Theta}$ under bounded input u and bounded output z .

B. Modelling error case

Assume that the proposed approach contains an irreducible identification error due to measurement noise, disturbances, or lack of excitement [52]. In these terms, consider the new CLOE dynamics

$$\ddot{e} = -A\dot{e} - Be - \Phi_y^\top \tilde{\Theta} + \varepsilon \quad (22)$$

where $\varepsilon = \Phi^\top \Theta - \Phi_y^\top \hat{\Theta} \in \mathbb{R}^n$ is an irreducible and bounded modelling error $\|\varepsilon\| \leq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$.

The next theorem establishes the uniform ultimate boundedness (UUB) of the CLOE dynamics (22) and boundedness of the parameter estimates vector $\hat{\Theta}$ as long as the PE condition (19) is fulfilled.

Theorem 2: Consider the CLOE dynamics (22). The parameter estimates $\hat{\Theta}$ are updated by (14) and the regressor matrix Φ_y fulfills the PE condition (19) and that $A - I > 0$. Assume there exists a bound

$$\alpha_1 = \min\{\lambda_{\min}(A - I), \lambda_{\min}(B)\} \quad (23)$$

which verifies

$$\alpha_1 > \sqrt{2}\bar{\varepsilon} + \rho \quad (24)$$

for any $\rho \in \mathbb{R}^+$. Then the trajectories of (22) are UUB with a practical bound $\mu_1 = \frac{\sqrt{2}\bar{\varepsilon}}{\alpha_1}$ and the parameter estimates $\hat{\Theta}$ remain bounded.

Proof: Consider the Lyapunov function (15). The time-derivative of V along the system trajectories (22) and the update rule (14) is

$$\begin{aligned}\dot{V} &= -\dot{e}^\top (A - I)\dot{e} - e^\top Be + (e + \dot{e})^\top \varepsilon \\ &\leq -\lambda_{\min}(A - I)\|\dot{e}\|^2 - \lambda_{\min}(B)\|e\|^2 + \|\varepsilon\|(\|e\| + \|\dot{e}\|).\end{aligned}$$

Define $\xi = [\|e\|, \|\dot{e}\|]^\top$ and $\delta = [1, 1]^\top$, then

$$\begin{aligned}\dot{V} &\leq -\alpha_1 \|\xi\|^2 + \xi^\top \delta \|\varepsilon\| \\ &\leq -\alpha_1 \|\xi\|^2 + \sqrt{2}\|\xi\|\bar{\varepsilon}.\end{aligned}$$

Notice that $\|\xi\| = \|E\|$. Consequently,

$$\dot{V} = -\alpha_1 \|E\| \left(\|E\| - \frac{\sqrt{2}\bar{\varepsilon}}{\alpha_1} \right). \quad (25)$$

Therefore $\dot{V} \leq 0$ as long as

$$\|E\| > \frac{\sqrt{2}\bar{\varepsilon}}{\alpha_1} \equiv \mu_1. \quad (26)$$

So if the states' measurements are rich enough such that the PE condition (19) and the bound (24) are satisfied, then the trajectories of system (22) are UUB and converge to a bounded set S_E of radius μ_1 , i.e., $\|E\| \leq \mu_1$.

Additionally, boundedness of E implies boundedness of $y, \dot{y}$, and Φ_y . This allows to conclude that $\|\dot{E}\|$ is also bounded and therefore,

$$z \equiv \ddot{e} + A\dot{e} + Be - \varepsilon \quad (27)$$

is also bounded. By Lemma 1 boundedness of E and z ensures boundedness of the parametric error $\tilde{\Theta}$, and hence $\hat{\Theta}$. This completes the proof. ■

Remark 3: The approach holds for nonlinear inputs u (e.g. inputs with saturations, dead zones, or discontinuous functions) [53], [54] in the sense that the CLOE algorithm is able to infer the trajectory of the real system but the final physics informed model is useless because the parameter estimates remain bounded but do not converge.

IV. LEAST-SQUARES (LS) COMPOSITE RULE

The proposed CLOE algorithm can be improved by adding a second term in the update rule (14). This additional term defines a composite update rule which is common in adaptive controllers [55]. The classic composite rules are based on either gradient or LS rules. It is well known that LS algorithms outperforms gradient algorithms in terms of convergence time as it is stated on the CLOE and inverse dynamic methods [45]. However, these kind of methods do not report stability and convergence analysis and hence, this is a gap in current LS implementations which we aim to solve in this section.

They key idea of the proposed formulation is to minimize the difference between the real and the estimated systems to accelerate convergence and reduce the modelling error. Consider the model matching case. The LS-rule aims to minimize the residual term in the CLOE dynamics (12), i.e.,

$$J = \frac{1}{2} \int_0^t \|\Phi_y^\top \tilde{\Theta}\|^2 d\tau. \quad (28)$$

The least-squares (LS) update rule is given by the next system of differential equations [56]

$$\begin{aligned}\dot{\hat{\Theta}} &= -P\Phi_y\Phi_y^\top \tilde{\Theta} \\ \dot{P} &= -P\Phi_y\Phi_y^\top P\end{aligned} \quad (29)$$

where the covariance matrix P is defined by

$$P^{-1} = \int_0^t \Phi_y(\tau)\Phi_y^\top(\tau)d\tau,$$

and satisfies

$$\dot{P}P^{-1} + P\frac{d}{dt}(P^{-1}) = 0. \quad (30)$$

So, the update rule (14) is slightly modified by the LS rule (29) as

$$\dot{\hat{\Theta}} = \dot{\tilde{\Theta}} = (\Gamma + P)\Phi_y(e + \dot{e}) - \Delta P\Phi_y\Phi_y^\top \tilde{\Theta} \quad (31)$$

where $\Delta \in \mathbb{R}^{p \times p}$ is a matrix gain. Whilst the first term of the right-hand side of (31) guarantee stability and boundedness of the parameter estimates, the second term improve parameter convergence and reduce the modelling error under the fulfillment of the PE condition (19).

Remark 4: In contrast to classical LS rules, the initial values of the covariance matrix P should satisfy $\Gamma - P > 0$ to avoid instability in the identification procedure and also P does not have to be close to Γ . A good choice for the initial value of $P(0)$ are: $\frac{1}{2}\Gamma$, $\frac{3}{4}\Gamma$ or lower initializations. The LS rule only provides of robustness but not stability. Stability is ensured with the CLOE algorithm.

The next result shows the exponential stability of the CLOE dynamics (12) under the composite update rule (31).

Theorem 3: Consider the CLOE dynamics (12). Assume that the regressor Φ_y fulfils the PE condition (19) such that P^{-1} exists for any time instance t , and that $A - I > 0$. Let the parameter estimates vector $\hat{\Theta}$ be generated by the composite rule (31) with

$$\Delta = \frac{(\Gamma + P)P^{-1}}{1 + \|(\Gamma + P)P^{-1}\|}. \quad (32)$$

Then the parametric error vector $\tilde{\Theta}$ converges exponentially to zero and therefore, $\hat{\Theta}$ converges to Θ .

Proof: Consider the following Lyapunov function

$$V = \frac{1}{2}\dot{e}^\top \dot{e} + \frac{1}{2}e^\top e + e^\top \dot{e} + \frac{1}{2}\tilde{\Theta}^\top (\Gamma + P)^{-1} \tilde{\Theta}. \quad (33)$$

From (30) it holds that

$$\frac{d}{dt}(\Gamma + P)^{-1} = -(\Gamma + P)^{-1}P\Phi_y\Phi_y^\top P(\Gamma + P)^{-1} \quad (34)$$

The time-derivative of (33) along the CLOE dynamics trajectories (12) and the composite update rule (31) is

$$\begin{aligned} \dot{V} = & -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \tilde{\Theta}^\top (\Gamma + P)^{-1} \Delta P\Phi_y\Phi_y^\top \tilde{\Theta} \\ & + \frac{1}{2}\tilde{\Theta}^\top (\Gamma + P)^{-1} P\Phi_y\Phi_y^\top P(\Gamma + P)^{-1} \tilde{\Theta}. \end{aligned}$$

Using the gain Δ defined by (32) simplifies $\dot{V}$ to

$$\begin{aligned} \dot{V} = & -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \alpha \tilde{\Theta}^\top \Phi_y\Phi_y^\top \tilde{\Theta} \\ & + \frac{1}{2}\tilde{\Theta}^\top (\Gamma + P)^{-1} P\Phi_y\Phi_y^\top P(\Gamma + P)^{-1} \tilde{\Theta}. \end{aligned}$$

where $\alpha = \frac{1}{1 + \|(\Gamma + P)P^{-1}\|} > 0$. Since $P < \Gamma$ in the sense that $\Gamma - P > 0$ is positive definite then it holds that

$$\begin{aligned} (\Gamma + P)^{-1} & \leq P^{-1}, \\ \|(\Gamma + P)^{-1}P\| & = \frac{1}{\|(\Gamma + P)P^{-1}\|} \end{aligned}$$

and hence,

$$\begin{aligned} \dot{V} = & -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \alpha \tilde{\Theta}^\top \Phi_y\Phi_y^\top \tilde{\Theta} \\ & + \frac{1}{2}\|\Phi_y^\top P(\Gamma + P)^{-1} \tilde{\Theta}\|^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality

$$\begin{aligned} \dot{V} \leq & -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \alpha \tilde{\Theta}^\top \Phi_y\Phi_y^\top \tilde{\Theta} \\ & + \frac{1}{2}\|P(\Gamma + P)^{-1}\|^2 \|\Phi_y^\top \tilde{\Theta}\|^2. \end{aligned}$$

Since Γ is larger than P then it always hold that

$$\|P(\Gamma + P)^{-1}\|^2 \leq \alpha,$$

so,

$$\dot{V} \leq -\dot{e}^\top (A - I)\dot{e} - e^\top Be - \frac{\alpha}{2} \tilde{\Theta}^\top \Phi_y\Phi_y^\top \tilde{\Theta}.$$

$\dot{V}$ can be expressed as

$$\dot{V} = -E^\top QE - \frac{\alpha}{2} \|\Phi_y^\top \tilde{\Theta}\|^2 \quad (35)$$

The above equality verifies that both E and $\tilde{\Theta}$ are $\mathcal{L}_\infty$ functions and $V(0) \geq V$ if $A - I > 0$. The following holds from (35)

$$\begin{aligned} \dot{V} & \leq -E^\top QE \\ & \leq -\lambda_{\min}(Q)\|E\|^2. \end{aligned} \quad (36)$$

The results of Theorem 1 are used to conclude convergence of the output error to zero. Suppose that $E = 0$, then the composite rule is reduced to

$$\dot{\tilde{\Theta}} = -\Delta P\Phi_y\Phi_y^\top \tilde{\Theta}. \quad (37)$$

The term $(\Gamma + P)P^{-1}$ in Δ increases as P decreases which can destabilize the CLOE algorithm, so the normalizing term $1 + \|(\Gamma + P)P^{-1}\|$ is used to maintain the LS gain bounded. This rule is the standard least mean squares algorithm [57] which satisfies the following properties

$$\begin{aligned} \frac{d}{dt}P^{-1} & = \Phi_y\Phi_y^\top \\ \frac{d}{dt}\left(P^{-1}(t)\tilde{\Theta}(t)\right) & = (I - \Delta)\Phi_y\Phi_y^\top \tilde{\Theta}. \end{aligned} \quad (38)$$

Let $M(t) = (I - \Delta)\Phi_y\Phi_y^\top P(t)$. Integrating (38) gives the LS solution

$$\begin{aligned} P^{-1}(t) & = P^{-1}(0) + \int_0^t \Phi_y(\tau)\Phi_y^\top(\tau)d\tau \\ P^{-1}(t)\tilde{\Theta}(t) & = \exp^{\int_0^t M(\tau)d\tau} P^{-1}(0)\tilde{\Theta}(0) \\ \tilde{\Theta}(t) & = P(t) \exp^{\int_0^t M(\tau)d\tau} P^{-1}(0)\tilde{\Theta}(0), \end{aligned} \quad (39)$$

and $\tilde{\Theta}$ converges exponentially to zero under the fulfillment of the PE condition (19). Finally, convergence of the parameter estimates $\hat{\Theta}$ to their real values Θ can be concluded under a vanishing output feedback E . This completes the proof. ■

The modelling error case is an extension of the results presented in Theorem 2 and Theorem 3.

Remark 5: For the proposed method, the involved computation is dominated by the CLOE identification rule to estimate $\hat{\Theta}$. If one does the calculations of the right hand side of (14), the CLOE one has growth with $(np)^2$ due to the number

of parameters of the linear system. Thus, the complexity is given by $\mathcal{O}((np)^2)$. The same procedure is applied to update rule (31), the CLOE+LS one has growth with $n^2p^3(1+p^4)$, so the complexity is given by $\mathcal{O}(n^2p^3(1+p^4))$ with the LS term n^2p^7 dominating. It is clear that the simple CLOE algorithm in (14) requires less computational cost in comparison to the CLOE+LS algorithm in (31) and exhibits the same computational complexity to standard gradient descent rules [34]. Moreover, the CLOE+LS update rule shows a compromise between computational complexity (which grows as the number of parameters increases) and fast parameter estimates convergence due to the exponential stability nature of the LS rule.

V. SIMULATION STUDIES

The performance of the CLOE and CLOE+LS inference algorithm are assessed using a 4-DOF robot model with a gearbox and a F-16 aircraft model.

A. 4-DOF Robot

Consider a 4-DOF robot with actuator/gear train [58], [59] which satisfies the following unknown dynamic equation

$$\ddot{q} = -\dot{A}_0\dot{q} + B_0u + D_0, \quad (40)$$

where $A_0, B_0 > 0 \in \mathbb{R}^{4 \times 4}$ and $D_0 \in \mathbb{R}^4$. Assume that the robot is controlled by a hidden and stabilizing proportional-derivative (PD) control law

$$u = K_p(q^d - q) + K_d(\dot{q}^d - \dot{q}), \quad (41)$$

where $K_p, K_d \in \mathbb{R}^{4 \times 4}$ are the proportional and derivative matrices gains. If the control gains K_p and K_d are chosen large enough then the closed-loop trajectories (42) satisfies (5). Two cases are considered: known desired reference, and unknown desired reference.

1) *Known desired reference*: The closed-loop trajectories between the robot dynamics (40) and the control law (41) is

$$\ddot{q} = -A\dot{q} + B(q^d - q) + C\dot{q}^d + D, \quad (42)$$

where $A = A_0 + B_0K_d > 0$, $B = B_0K_p > 0$, $C = B_0K_d > 0$, and $D = D_0$. The estimated model has the following structure

$$\ddot{y} = -\hat{A}\dot{y} + \hat{B}(q^d + \tau - y) + \hat{C}(\dot{q}^d + \dot{\tau}) + \hat{D} = \Phi_y^\top \hat{\Theta}, \quad (43)$$

where $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$ are estimates of A, B, C , and D . The regressor $\Phi_y \in \mathbb{R}^{16 \times 4}$ and the parameter estimates vector $\hat{\Theta} \in \mathbb{R}^{16}$ are defined as

$$\Phi_y = \begin{bmatrix} -I \otimes \dot{y} \\ I \otimes (q^d + \tau - y) \\ I \otimes (\dot{q}^d + \dot{\tau}) \\ I \end{bmatrix}, \quad \hat{\Theta} = \begin{bmatrix} \text{vec}(\hat{A}) \\ \text{vec}(\hat{B}) \\ \text{vec}(\hat{C}) \\ \hat{D} \end{bmatrix}. \quad (44)$$

Notice that the regressor does not contain noisy measurements such that it avoids biased parameter estimates and hence residual error in the states' estimates.

The excitation signal vector τ is designed as a linear combination of two chaotic duffing systems of the form

$$\begin{aligned} \dot{\zeta}_{1i} &= \zeta_{2i}\omega_i\pi \\ \dot{\zeta}_{2i} &= [-0.25 + \zeta_{2i} + \zeta_{1i} - 1.05\zeta_{1i}^3 + 0.3\sin(\omega_i\pi t)]\omega_i\pi \\ \tau &= \begin{bmatrix} 0.35 & -0.25 \\ 0 & -0.25 \\ 0.1 & 0.15 \\ 0.35 & 0 \end{bmatrix} \begin{bmatrix} \zeta_{11} \\ \zeta_{12} \end{bmatrix} \quad \zeta_{1i}(0) = \zeta_{2i}(0) = 0; \\ \omega_1 &= 5 \text{ rad/s}; \omega_2 = 4 \text{ rad/s}; \quad i = 1, 2. \end{aligned} \quad (45)$$

First, consider a constant desired reference given by $q^d = [\frac{\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}]^\top$. An unknown stabilizing PD control law is used to guarantee position tracking with bounded error. Small uniform random noise $\Delta q \sim \mathcal{N}(0, \sigma^2)$ with variance $\sigma^2 = 0.1$ is added at the position and velocity measurements to model sensor noise. The simulations last 100 seconds, but we only consider the first 20 and 50 seconds of simulation time for visualization purposes. The position measurements used for the trajectory inference algorithm are shown in Fig. 2(a). Notice that these measurements are not rich enough to guarantee parameter estimates convergence. So, the excitation signals τ is added to the position and velocity measurements (see Fig. 2(b)) to guarantee both parameters and states estimates convergence.

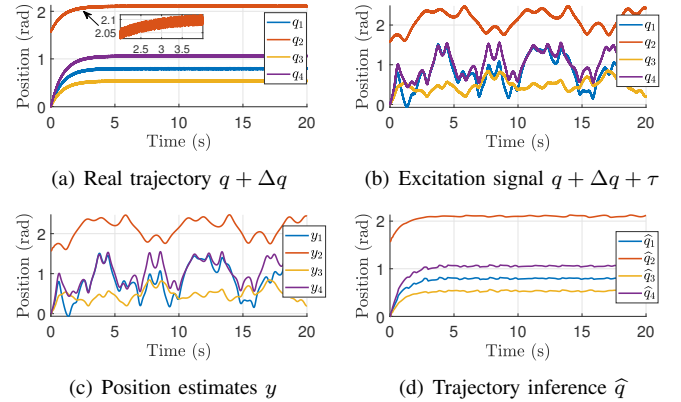


Fig. 2. Constant desired reference results

Different gain matrices Γ are tested until the best performance of the identification algorithm is achieved. The final gain matrix is set to $\Gamma = 1000I_{16}$. Fig. 2(c) shows the inference results. Notice that the obtained signals are noise-free and exhibit the same behaviour as the original input signals in Fig. 2(b). The state estimates of the real desired trajectory can be easily obtained by computing $\hat{q} = y - \tau$. The estimated trajectory results are shown in Fig. 2(d). Similarly, the inferred trajectories are noise-free and exhibit a smooth response. In this scenario, the proposed algorithm obtains good trajectory inference results with noise attenuation capabilities.

Fig. 3 shows the parameter estimates convergence. Despite the parameter estimates converge, they do not converge to their real values since there are no constraints in the control gains K_p and K_d of the control input u . Furthermore, since the excitation signal τ is added at the output of the system, then its amplitude increases or decreases the real parameter values;

in consequence, the inference algorithm estimates different parameters that behaves similar to the real ones such that the output of the estimated model matches with the real system output. Additional constraints or knowledge of the control input are required to guarantee convergence to their real values. However, the final estimates gives to the physics informed model a sense in how the real-world system behaves and allows to infer the trajectory with high accuracy and noise attenuation.

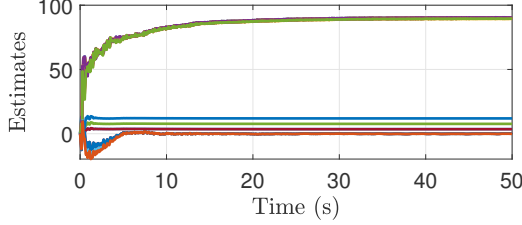


Fig. 3. Parameter estimates convergence: Regulation case

We further test the approach using the following time-varying reference

$$q^d = \begin{bmatrix} \frac{\pi}{4} + \frac{\pi}{12} \sin(\pi t) \\ \frac{2\pi}{3} + \frac{\pi}{12} \sin(0.16\pi t) \\ \frac{\pi}{6} + \frac{\pi}{4} \sin(0.2\pi t) \\ \frac{\pi}{3} + \frac{\pi}{6} \sin(0.16\pi t) \end{bmatrix}. \quad (46)$$

The same noise, excitation signal, and gain Γ are used for this case to demonstrate the robustness of the proposed technique under different references. Fig. 4 shows the inference results for the time-varying reference. Fig. 4(a) shows the noisy trajectory followed by the 4-DOF robot. Similarly to the regulation case, the addition of the excitation signal (see Fig. 4(b)) is required to guarantee parameters and states estimates convergence. Additionally, since the regressor matrix Φ_y does not contain noisy signals then the final position estimates in Fig. 4(c) and the trajectories inference results of Fig. 6(c) are noise free and smooth.

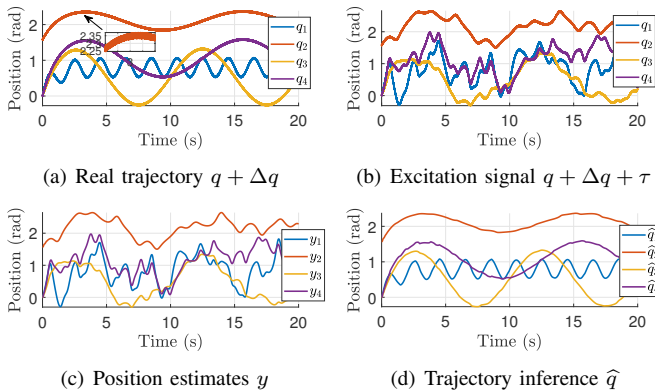


Fig. 4. Time-varying desired reference results

Fig. 5 shows the parameter estimates converge to almost the same values of the regulation case which demonstrates robustness of the algorithm against different references. Furthermore, the final estimates help to construct the physics informed

model that can be further used by different algorithms for intent prediction such as physics informed neural networks [14] and Gaussian processes [7]. This is topic for further research.

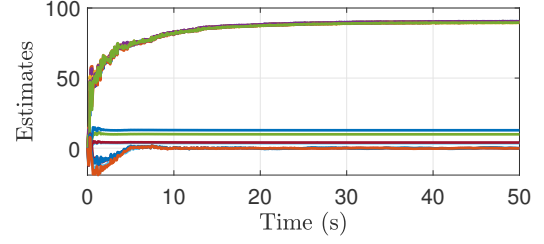


Fig. 5. Parameter estimates convergence: Tracking case

The LS composite rule is used to improve the performance of the CLOE inference algorithm. Since the CLOE update rule (14) possesses a stronger dynamics than the LS rule, then a small initial covariance matrix (in comparison with Γ) is needed to avoid unstable performances as it is stated on Remark 4. An initial covariance matrix $P(0) = 500I_{16}$ is proposed that verifies $P < \Gamma$. The results for both the regulation and tracking tasks are exhibited in Fig. 6.

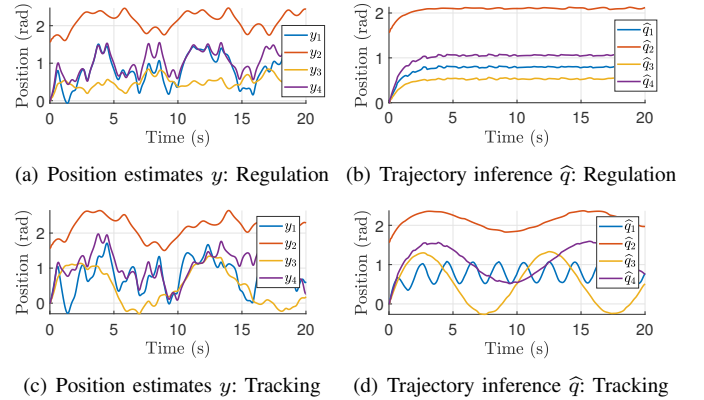


Fig. 6. CLOE+LS inference results

Fig. 6(a) and Fig. 6(c) show the position estimates results where the noise is attenuated and the position measurements with excitation signal (see Fig. 2(b) and Fig. 4(b)) are tracked accurately. The inference results for both regulation (see Fig. 6(b)) and tracking (see Fig. 6(d)) exhibit smooth and high accurate trajectories. However, these results do not show any difference with the simple CLOE algorithm. The main difference between the CLOE and the CLOE+LS is given by the parameter estimates convergence (see Fig. 7) which effectively shows an improvement in the time convergence, that is, whilst the CLOE's estimates converge in approximately 15 seconds, the CLOE+LS' estimates converge in approximately 5 seconds.

2) *Unknown desired reference*: The previous case assumes that the desired reference is known in advance. In this case, the proposed algorithm will be extended for unknown desired reference as is depicted in Fig. 8. The main difference of this approach respect to the diagram of Fig. 1 is that the

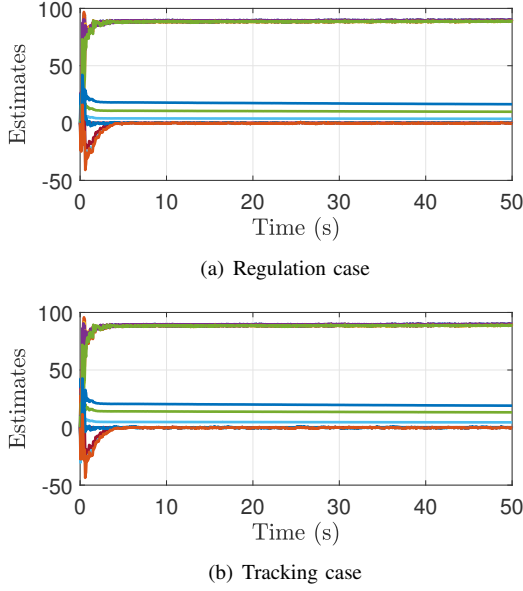


Fig. 7. Parameter estimates convergences: CLOE+LS results

estimated model uses uniquely the excitation signal τ as input and the desired reference is hidden within the parameters of the estimated model.

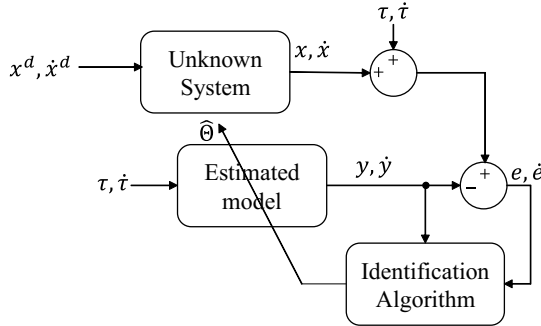


Fig. 8. Block diagram of the CLOE approach without desired reference knowledge

For this case, the estimated model has the following structure

$$\ddot{q} = -A\dot{q} + B(\tau - q) + C\dot{\tau} + D, \quad (47)$$

where A , B , and C are defined as the previous case and D is rewritten as $D = D_0 + B_0 K_p q^d + B_0 K_d \dot{q}^d$. So, the estimated model is written as follows

$$\ddot{y} = -\hat{A}\dot{y} + \hat{B}(\tau - y) + \hat{C}\dot{\tau} + \hat{D} = \Phi_y^T \hat{\Theta}, \quad (48)$$

where the regressor Φ_y is defined as

$$\Phi_y = \begin{bmatrix} -I \otimes \ddot{y} \\ I \otimes (\tau - y) \\ I \otimes \dot{\tau} \\ I \end{bmatrix}, \quad \hat{\Theta} = \begin{bmatrix} \text{vec}(\hat{A}) \\ \text{vec}(\hat{B}) \\ \text{vec}(\hat{C}) \\ \hat{D} \end{bmatrix}. \quad (49)$$

In contrast to the known desired reference case, parameter estimates convergence cannot be guaranteed for time-varying

references because the parameter estimate $\hat{D}$ becomes time-varying. The same noise, excitation signal, gain Γ , and initial covariance matrix $P(0)$ are used in these experiments to verify the robustness of the algorithm. The results for both the CLOE and CLOE+LS algorithms are exhibited in Fig. 9.

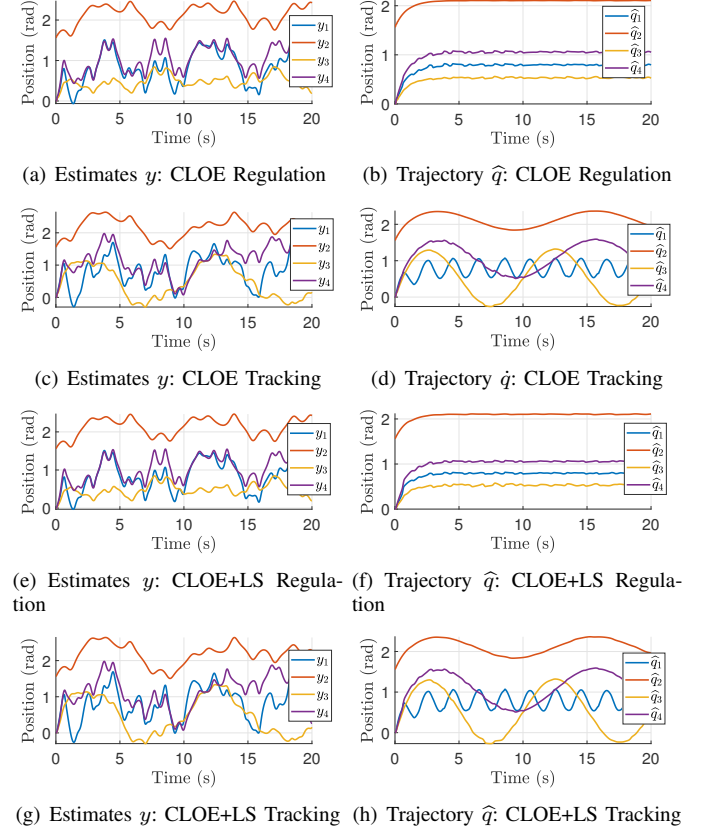


Fig. 9. CLOE and CLOE+LS Inference results

The results show that the proposed algorithms are still capable to infer the trajectory of the real system (see Figs. 9(b), 9(d), 9(f), and 9(h)) even though the desired reference is unknown. In addition, noise attenuation and smooth responses are also obtained. However, parameter estimates convergence cannot be achieved as it is shown in Fig. 10, where the estimates do not converge as in the other cases because we are dealing with time-varying parameters that do not match with the proposed formulation, that is, the final model does not provides an adequate physics informed model and therefore, it cannot be used for other intent algorithms. Nevertheless, the parameter estimates remain bounded and verifies the results of Theorem 2 and Theorem 3.

The mean of the norm of the output error and the estimation error of the last 10 seconds of simulation time were used as metrics to compare the performance of the algorithms. The metrics are defined as

$$\bar{e}_i = \frac{1}{N} \|k e_i\|, \quad \bar{q}_i = \frac{1}{N} \|k(q_i^d - \hat{q}_i)\|, \quad i = 1, \dots, 4, \quad (50)$$

where k is a scaling factor. Table I summarizes the metric results for the known and unknown desired reference cases using a scaling factor of $k = 100$. Some interesting conclusions can be obtained from these results. Both the CLOE and

TABLE I
OUTPUT AND ESTIMATION ERROR RESULTS

Method	Known q^d				Unknown q^d			
	Regulation		Tracking		Regulation		Tracking	
	$\bar{e}_i$	$\bar{q}_i$	$\bar{e}_i$	$\bar{q}_i$	$\bar{e}_i$	$\bar{q}_i$	$\bar{e}_i$	$\bar{q}_i$
CLOE	0.0281	0.0278	0.0277	0.0321	0.0281	0.0277	0.0386	0.0452
	0.0329	0.0263	0.0321	0.0295	0.0217	0.0185	0.0217	0.0227
	0.0262	0.0279	0.0258	0.0205	0.0274	0.0295	0.0390	0.0380
	0.0273	0.0279	0.0269	0.0346	0.0273	0.0278	0.0307	0.0423
CLOE+LS	0.0281	0.0278	0.0386	0.0322	0.0294	0.0291	0.0352	0.0427
	0.0330	0.0264	0.0217	0.0296	0.0232	0.0185	0.0234	0.0231
	0.0262	0.0279	0.0390	0.0205	0.0287	0.0312	0.0436	0.0416
	0.0273	0.0279	0.0307	0.0347	0.0278	0.0283	0.0283	0.0376

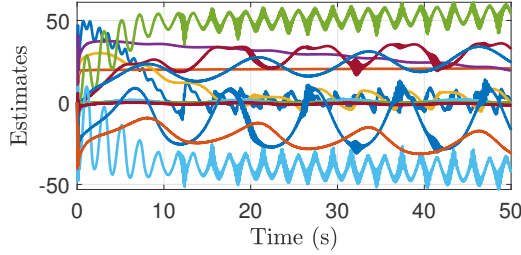


Fig. 10. Parameter estimates for unknown desired reference

the CLOE+LS have similar output and estimation error results for the known reference case where the main difference lies on the improvement of the parameter estimates convergence (see Fig. 7). On the other hand, the CLOE+LS algorithm has larger output and estimation error in comparison to the simple CLOE algorithm because by accelerating the parameter estimates convergence may lead to parameters that gives similar trajectory inference but with different accuracy. Furthermore, the excitation signal plays an important role in the accuracy results because if the CLOE+LS rule is applied using a poor excitation signal, then the estimates will have large estimation error or, in the worst case, will present parametric drift.

3) *Comparisons:* Kalman filter is used as comparison algorithm to exhibit the weakness of state estimation algorithms when the physics informed model is poor accurate.

A continuous-time Kalman filter algorithm is used where the covariance matrix associated to the modelling error is proposed as an identity matrix, that is, $Q = I_8$. The covariance matrix for the output measurements is set to $R = 0.03I_8$ to attenuate the Gaussian noise. The prior model is proposed as a stable system of the form

$$\mathcal{A} = \begin{bmatrix} 0I_4 & I_4 \\ -20I_4 & -20I_4 \end{bmatrix}.$$

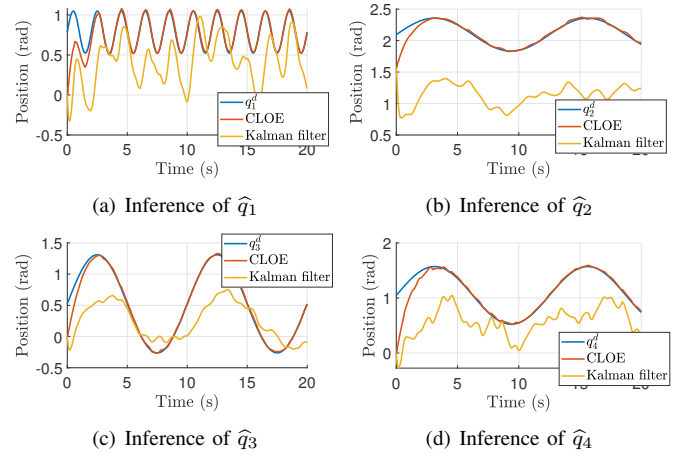


Fig. 11. Inference Comparisons between CLOE and Kalman filter algorithms

It is clear that the prior knowledge assumes a decoupled dynamics which does not exhibit the real physics of the robot manipulator. The comparisons results are exhibited in Fig. 11. Notice that Kalman filter shows noise attenuation which is one of its main advantages in state estimation problems. However, since the prior model $\mathcal{A}$ is not a physics informed model of the real system, then the trajectory inference is poor accurate with large variance and bias. In the other hand, the proposed CLOE algorithm is able to infer the desired trajectory with high accuracy by constructing a stable physics informed model.

B. F-16 aircraft dynamics

The proposed approach can be extended for state-space models. In particular, we are interested in stabilization problems which can be seen as a special case of the unknown desired reference. Consider a F-16 aircraft dynamics [60] given by the next state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 \\ 0.8225 & -1.07741 & -0.17555 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u\end{aligned}\quad (51)$$

Assume (51) be controlled by a general feedback feedforward control law of the form

$$u = K(x^d - x) - B^\dagger(Ax^d - \dot{x}^d), \quad (52)$$

where $K \in \mathbb{R}^{1 \times 3}$ control gain and $B^\dagger$ stands to the Moore-Penrose pseudoinverse of matrix B . Substituting the control law (52) in (51) gives

$$\begin{aligned}\dot{x} &= Ax + BK(x^d - x) - Ax^d + \dot{x}^d \\ &= -A_K(x^d - x) + \dot{x}^d = \Phi^\top \Theta,\end{aligned}\quad (53)$$

where $A_K = A - BK$ and

$$\Phi = \begin{bmatrix} -I \otimes (x^d - x) \\ \text{diag}\{\dot{x}^d\} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \text{vec}(A_K) \\ \vec{1} \end{bmatrix}, \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For this system, a sum of exponential sinusoidal functions are used as excitation signal. The signal τ is added to the states measurements to guarantee parameters and states estimates convergence. Therefore, the estimated model has the next structure

$$\dot{y} = -\hat{A}_K(x^d + \tau - y) + \dot{x}^d + \dot{\tau} = \Phi_y^\top \hat{\Theta} + \dot{x}^d + \dot{\tau}, \quad (54)$$

where

$$\Phi_y = -I \otimes (x^d + \tau - y), \quad \hat{\Theta} = \text{vec}(\hat{A}_K).$$

Several gain matrices Γ are tested until the best performance of the inference algorithm is achieved. The final gain is set to $\Gamma = 1000I_9$ and the initial covariance matrix is $P(0) = 500I_9$ to fulfil Remark 4. The results of the CLOE and CLOE+LS algorithms are exhibited in Fig. 12. Similarly to the 4-DOF robot case, the state measurements have noise (see Fig. 12(a)) and the exponential excitation signal τ is added to guarantee parameter and states convergence (see Fig. 12(b)). The state estimates and the trajectory inference results are shown in Fig. 12(c)-12(f) which clearly exhibits good state estimation and noise attenuation.

On the other hand, parameter estimates convergence is achieved by adding the excitation signal at the output measurement, but since the inference problem lacks of constraints (in terms of the control input, dynamic model structure, desired reference, control gains, etc.), then there exists multiple parameters that achieve the same closed-loop performance. This fact can be seen in Fig. 13 where different parameter estimates are obtained using the either the CLOE or CLOE+LS algorithms. Notice that the composite LS rule accelerates parameter convergence as in Fig. 7. The output and estimation error results are depicted in Table II using a scaling factor of $k = 100$.

Notice that the parameter estimates obtained from the CLOE+LS algorithm give a higher estimation error $\bar{q}_i$ than the CLOE algorithm. This means that different parameter

TABLE II
OUTPUT AND ESTIMATION ERROR RESULTS

CLOE		CLOE+LS	
$\bar{e}_i$	$\bar{q}_i$	$\bar{e}_i$	$\bar{q}_i$
0.0032	8.6662×10^{-5}	0.0032	1.0467×10^{-4}
0.0032	8.6815×10^{-5}	0.0032	7.6844×10^{-5}
0.0032	8.6451×10^{-5}	0.0032	1.2576×10^{-4}

estimates can give good trajectory inference but with different accuracies. Knowledge of the control input may be a solution of the multiple parameter estimates issue in order to maintain the parameter estimates bounded. Other solution implies to not use the additional excitation signal τ by ensuring that the states measurements fulfil the PE condition (19).

VI. CONCLUSIONS

This paper presents a trajectory inference algorithm based on a closed-loop output error approach using only online data measurements. The proposed approach combines the main advantages of an identification algorithm and a state observer to infer the physics of the real system and consequently infer the trajectory given an online measurement. The regressor matrix of the algorithm only use the measurements of the estimated states instead of the real measurements to avoid biased estimates. A composite update rule based on a least-squares method is also proposed. A rigorous stability and convergence analysis is assessed using Lyapunov stability theory under the fulfilment of a persistency of exciting condition. Numerical simulations are carried out in a 4-DOF robot and F-16 aircraft dynamics to support the proposed approach.

The main disadvantages and limitations of the proposed approach can be summarized as follows: i) the linearity assumption of the control input that helps to lump together the parameters of the linear system with the controller gains, ii) the design of an adequate excitation signal for parameter estimates convergence, and iii) different real-world scenarios do not meet the full states measurements assumption.

Regarding future work, physics informed neural networks and machine learning can be used as an alternative tools to enhance the CLOE approach. Moreover, reinforcement learning gives an interesting research area to optimize the output error by seeking the best parameter estimates of an estimated model. Additionally, complementary algorithms will be analysed for intention inference problems based on the proposed physics informed algorithm.

Another concern for future work lies on the generalization of the approach for nonlinear systems and constrained systems. In addition, the design of the excitation signal is still an open problem for any identification and state estimation algorithms. Furthermore, the proposed approach assumes that the output contains all the state so, future work will address outputs with partial knowledge of the state vector.

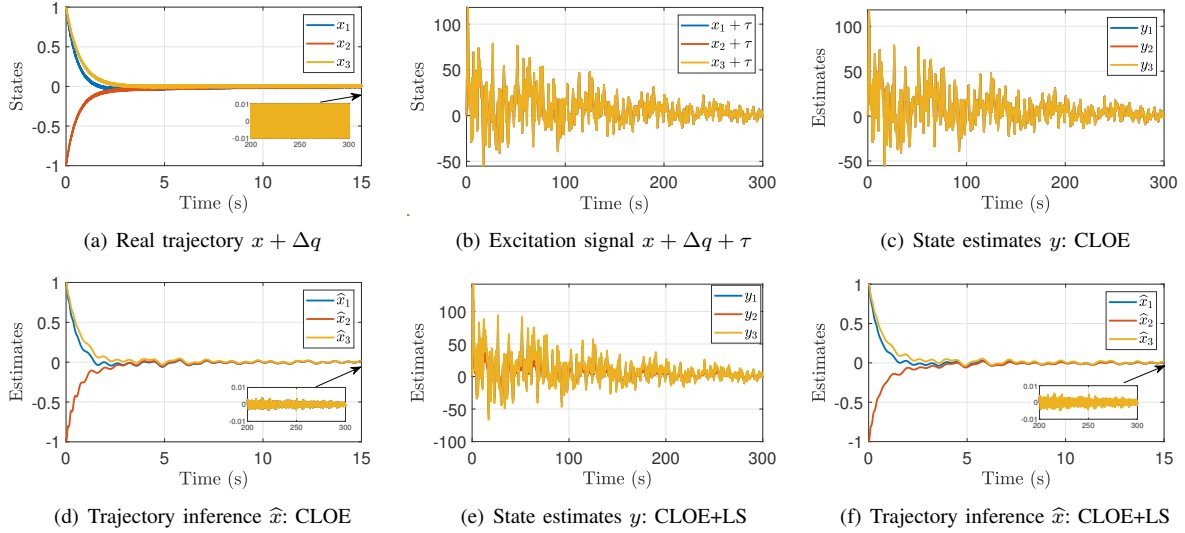


Fig. 12. Trajectory inference results: F-16 aircraft system

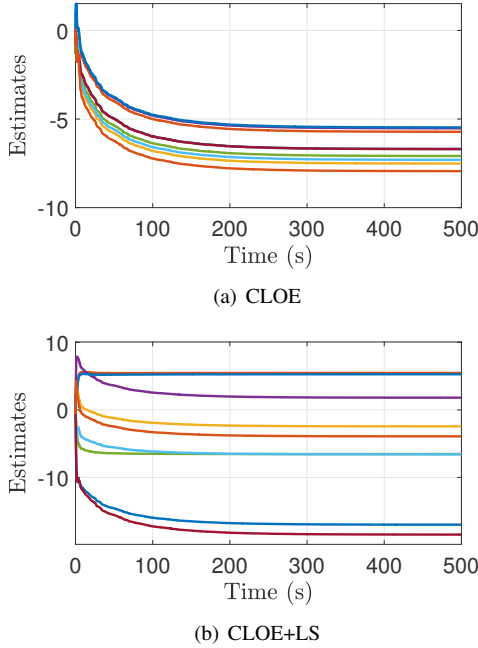


Fig. 13. Parameter estimates convergence results

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Adolfo Perrusquía received the B.Eng. degree in Mechatronic Engineering from the National Polytechnic Institute (UPIITA-IPN) in 2014, and the M.S. and Ph.D. degrees, both in Automatic Control from the Automatic Control Department at the CINVESTAV-IPN in 2016 and 2020, respectively. He is currently a research fellow at the School of Aerospace, Transport and Manufacturing, Cranfield University, and a UK IC Postdoctoral Research Fellow. He is a member of the IEEE Computational Intelligence Society. His main research of interest

focuses on robotics, mechanisms, machine learning, reinforcement learning, nonlinear control, system modeling and system identification.



Weisi Guo (S07, M11, SM17) received his MEng, MA, and Ph.D. degrees from the University of Cambridge, UK. He is Chair Professor of Human Machine Intelligence at Cranfield University. He has published over 180 papers and is PI on a number of molecular communication research grants. His research has won him several international awards. He was a Turing Fellow at the Alan Turing Institute.



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Perrusquía, Adolfo; Guo, Weisi

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