

Generalized Time-to-Go Inversion Guidance with Impact Time and Angle Constraints

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A major challenge in homing guidance is that accurately predicting the time-to-go for missiles is generally difficult, which limits the practical application of a certain class of guidance laws with impact-time constraints or cooperative capabilities. To address this issue, a generalized method referred to as the time-to-go inversion guidance is presented, where the time-to-go is first specified as an explicit function and then the corresponding guidance law is inversely derived. By employing this new method, a variety of guidance laws and their globally accurate time-to-go solutions can be developed, regardless of whether the impact-angle constraint is considered or not. Different from all existing studies, the proposed method addresses the problem of time-to-go estimation without relying on any linearizing approximations or numerical iterations. In order to enhance the practicality of the proposed method, it is augmented with bias commands to result in either the impact time control guidance law or the impact time and angle control guidance law. Numerical simulation results demonstrate the higher accuracy, larger feasible solution region, and lower computational demand of the proposed guidance laws even for the missiles with non-constant speeds.

I. Introduction

In recent years, advanced homing guidance laws with stringent impact constraints have attracted considerable attention due to their superior attack effectiveness against modern defensive targets. For instance, impact time control guidance (ITCG) laws can be utilized to develop a cooperative strategy to saturate the target with multiple missiles, while impact angle control guidance (IACG) laws can improve warhead lethality by approaching the target from a desired direction. Sometimes, the desired impact time and angle need to be satisfied simultaneously to further increase

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the attack effectiveness, resulting in the development of impact time and angle control guidance (ITACG) laws.

The concept of time-to-go, which is the remaining flight time for the missile to capture the target, is crucial in developing advanced guidance laws with impact constraints. For the ITCG problem, an intuitive approach to achieve the impact-time constraint is to augment the proportional navigation guidance (PNG) law [1] with an additional bias term. Such a bias term is specifically designed to ensure that the error between the estimated and desired time-to-go converges to zero before the final collision. This approach was first introduced in [2] to derive a suboptimal ITCG law under a linearized guidance model. Thereafter, a more rigorous ITCG law was proposed in [3] based on generalized time-to-go estimations. Expanding on the works of [2, 3], numerous PN-based ITCG laws were then developed in [4–6], where the convergence of the time-to-go error was realized using different dynamic models. Besides the PNG augmentation approach, the ITCG problem has also been addressed by combining the time-to-go estimation with nonlinear control techniques. Based upon the time-to-go estimations in [2, 3], the sliding-mode control technique of [7], the finite-time control method of [8], and the barrier Lyapunov approach of [9] have all demonstrated their effectiveness in designing ITCG and cooperative guidance laws.

In view of the above works, the precision of time-to-go estimation directly determines the success of ITCG laws. However, accurately estimating the time-to-go is difficult because of the nonlinear nature of the guidance model. The time-to-go estimation in [2–9] were approximately derived by assuming that the heading error of the missile is sufficiently small. This estimation method will cause inevitable errors and degrade the accuracy of the corresponding ITCG laws. In order to tackle this issue, considerable research efforts have been dedicated to improving the precision of time-to-go estimation. Based on Gaussian hypergeometric and beta functions, the exact time-to-go for the pure PNG law was derived in [10–13]. Neural network techniques were also applied in [14, 15] to map the current states with the exact time-to-go. The major benefit of the estimation methods in [10–15] is that they do not rely on the small heading-error assumption; whereas, they may require more computational efforts for online updates or comprehensive datasets for training neural networks. In addition, exact time-to-go solutions were also obtained by guiding the missile to track a specific polynomial trajectory [16]. Recently, variable navigation gains were incorporated into the PNG law to generate globally precise time-to-go solutions [17, 18]. Although the time-to-go estimation in [16–18] are analytical and accurate, they were tailored to the conventional PNG law or specific polynomial trajectories, which may limit the flexibility in guidance law design. Overall, it is worthwhile to further explore a generalized time-to-go solution applicable to diverse guidance laws.

Apart from the aforementioned studies, ITCG laws have also been developed without relying on the estimated time-to-go. For example, the desired impact time was successfully achieved via the polynomial shaping approach in [19, 20], in which the target range or lead-angle profile was modeled as a polynomial with undetermined coefficients. Additionally, several ITCG laws were designed in [21–23] by guiding the missile to track a desired lead angle dependent on the current states of the guidance system. In [24], a three-dimensional ITCG law was proposed based on a sliding

surface that does not require the time-to-go estimation.

For the IACG problem, the guidance law design has been carried out in the literature using various techniques, such as the optimal control [25, 26], polynomial shaping [27, 28], state-dependent Riccati equation [29], and prescribed performance control [30]. Despite the progress of the above studies, estimating the time-to-go for an IACG law remains a challenging task in practice. For most IACG laws, the desired impact angle is achieved by involving an additional term for velocity direction error feedback. Compared to guidance laws without additional terms or feedback errors, predicting the future states of the guidance system becomes more challenging under IACG laws. To estimate the time-to-go of IACG laws, the average speed of the missile and the arc length of flight paths were approximately calculated in [25]. Furthermore, the time-to-go of IACG laws was also obtained in recent works [31, 32] under the assumption of small heading errors. However, the time-to-go solutions in [25, 31, 32] will inevitably result in estimation errors, especially when the missile trajectory is highly curved.

For the ITACG problem, in general, a guidance law can be designed by incorporating a bias term into an IACG law to nullify the time-to-go error. By employing this approach, the approximate time-to-go solutions from [25, 31, 32] have been applied to develop various ITACG laws or impact-angle-constrained cooperative guidance laws [31–35]. These ITACG laws have intuitive and analytical forms; however, inaccurate time-to-go estimation may hinder their practical implementation. On the other hand, ITACG laws were developed in the literature [36–45] without relying on the time-to-go estimation. In [36–39], the missile was directed to follow a polynomial of system states, where the selection of polynomial coefficients was accomplished through numerical iterative schemes. In [40–43], the desired impact time and angle were simultaneously achieved using geometrical approaches, including the virtual target, circle involute, Bezier spiral, and sinusoidal trajectories. In addition to the above polynomial shaping and geometrical approaches, the ITACG problem was also investigated in [44, 45] for highly nonlinear guidance geometry by utilizing look-angle tracking approach and nonlinear error dynamics. Nevertheless, compared to the intuitive guidance laws in [31–35] with command augmentation, the ITACG laws of [36–45] may necessitate more computational efforts or lack clear physical meanings.

To address the shortcomings of inaccurate time-to-go estimation, this paper proposes a new generalized method for guidance law design, where the time-to-go of the missile is first specified as an explicit function and then the corresponding guidance law is inversely derived. Furthermore, most aforementioned ITCG and ITACG laws were developed under the assumption that the missile speed remains constant during the entire homing phase. These guidance laws could fail to attack the target in practice, because the speed of actual missiles is freely varying subject to thrust and drag. Until now, there are only a few works that can satisfy the impact-time constraint under the missile speed variation [14, 15, 17, 19, 32, 35, 39, 41, 42]. In this paper, by specifying the distance-to-go as explicit functions, the missile speed variation is also taken into consideration for ITCG and ITACG laws design. The main contributions of this paper are:

1) A new generalized method, referred to as the time-to-go inversion guidance, is developed without relying on any linearizing approximations or numerical iterative algorithms. Different from all existing methods, the proposed approach is a unified tool for deriving various guidance laws with globally-exact time-to-go solutions, making it suitable for ITCG law design.

2) The proposed time-to-go inversion guidance is further enhanced by accommodating the impact-angle constraint, resulting in an effective guidance framework for addressing the IACG problem. For the first time, our new method is able to generate various IACG laws with accurate and analytical time-to-go solutions, which significantly facilitates the design of ITACG laws.

3) Several ITCG and ITACG laws, applicable to both constant-speed and varying-speed missiles, are further developed by integrating our time-to-go inversion guidance with the command augmentation approach. In comparison with similar guidance laws in recent literature, our guidance laws are shown to achieve more precise impact-time control for a wider feasible region with less control effort.

The rest of this paper is organized as follows. The lateral guidance geometry and objectives of this work are introduced in Sec. II. The time-to-go inversion guidance concept is described in Secs. III and IV, without and with considering the impact-angle constraint, respectively. In Secs. V and VI, several ITCG and ITACG laws are developed and numerical comparative study results are presented.

II. Problem Formulation

A. Lateral Guidance Model

Figure 1 depicts the lateral guidance geometry of a missile M relative to a stationary target T , where $OX_I Y_I$ is the inertial reference frame. The speed of the missile is denoted by V . The range and the LOS angle between the missile and target are represented by R and λ , respectively. The symbols γ and σ denote the lateral flight-path angle and the lead angle of the missile, respectively. Without loss of generality, the heading error of the missile is defined as the absolute value of the lead angle in this paper.

The lateral guidance design model is described as

$$\dot{R} = -V \cos \sigma \quad (1a)$$

$$\dot{\lambda} = -\frac{V \sin \sigma}{R} \quad (1b)$$

$$\dot{\gamma} = \frac{A}{V} \quad (1c)$$

$$\dot{\sigma} = \dot{\gamma} - \dot{\lambda} = \frac{A}{V} + \frac{V \sin \sigma}{R} \quad (1d)$$

where A is the lateral control acceleration acting on the missile perpendicular to the flight direction, which is a typical

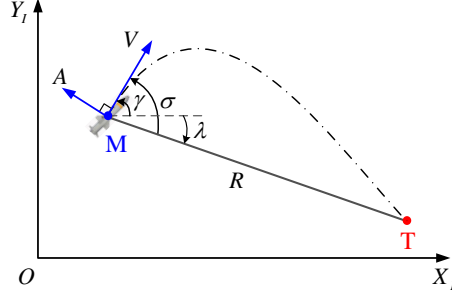


Fig. 1 Lateral guidance geometry.

characteristic of most skid-to-turn (STT) missiles. All the guidance objectives in this paper need to be achieved only by adjusting the flight direction of the missile, not by actively controlling the missile's speed. In addition, the time-to-go of the missile is defined as $t_{go} = t_f - t$, where t_f is the final time.

B. Motivations and Objectives

1) The accurate time-to-go estimation for homing missiles is worthy of further investigation, especially in the presence of large heading errors. Exact time-to-go solutions have been developed in the literature using Gaussian hypergeometric or beta functions [10–13], neural networks techniques [14, 15], special polynomial trajectories [16], and PNG laws with time-varying gains [17, 18]. Unlike all previous works that estimate the time-to-go of a certain guidance law, *the first objective of this paper* is to develop a generalized guidance method. This method not only provides accurate time-to-go solutions but also offers a general tool through which various new guidance laws can be designed. Furthermore, all time-to-go solutions within our proposed guidance framework have simple analytical forms, allowing for convenient onboard implementation.

2) Solving the time-to-go for an IACG law is indeed more challenging, even when the missile speed is assumed to be constant. This is because most IACG laws incorporate an additional feedback term to adjust the impact angle, which makes it more difficult to accurately predict the future states of the guidance system. This challenging issue has been addressed in [25, 31, 32] based on linearizing approximations, which may cause inevitable estimation errors. Taking a step further beyond the above first objective, *the second objective of this paper* is to develop a generalized method for designing various IACG laws. This method aims to ensure that all derived guidance laws can achieve zero miss distance and meet the impact-angle constraint, while also providing accurate yet analytical time-to-go solutions even for highly nonlinear guidance geometry.

3) The purpose of time-to-go estimation is to achieve desired impact times on the target. Due to the errors in the time-to-go estimation, the performance of most existing ITCG or ITACG laws may significantly degrade when the heading error is large. Moreover, as previously mentioned, most ITCG and ITACG laws were developed based on the assumption of constant missile speed. However, this assumption may lead to mission failure in practice, as it

does not account for the effects of thrust and drag. Due to these reasons, *the third objective of this paper* is to extend the proposed guidance framework to obtain various ITCG and ITACG laws applicable to both constant-speed and varying-speed missiles.

III. Basic Time-to-Go Inversion Guidance

To realize the first guidance objective described in Sec. II.B, this section presents a generalized method for guidance law design, referred to as the *basic time-to-go inversion guidance*. In this new method, the time-to-go of the missile is first specified as an explicit function and then the corresponding guidance law is inversely derived. Two guidance laws are presented to illustrate the effectiveness of the proposed method. In this section, we assume that the missile speed remains constant for the simplicity of derivation.

A. Generalized Guidance Framework

Without considering the impact-angle constraint, the time-to-go of the missile is represented as

$$t_{go} = \frac{R}{V} f(\sigma) \quad (2)$$

where $f(\sigma)$ is a function of the lead angle defined on $[-\pi, \pi]$. Such a function $f(\sigma)$ needs to be further designed and its form is not unique. Before introducing the practical guidelines for designing $f(\sigma)$, we first derive a homing guidance law based on the time-to-go solution of the form (2). Taking the time derivative of Eq. (2) yields

$$\dot{t}_{go} = \frac{\dot{R}f(\sigma) + R\dot{f}(\sigma)}{V} \quad (3)$$

Defining $f'(\sigma) = df(\sigma)/d\sigma$, we rewrite $\dot{f}(\sigma)$ as

$$\dot{f}(\sigma) = \frac{df(\sigma)}{d\sigma} \frac{d\sigma}{dt} = f'(\sigma) \dot{\sigma} \quad (4)$$

Substituting Eqs. (1a) and (4) into Eq. (3), we obtain the time derivative of σ as

$$\dot{\sigma} = \frac{Vf(\sigma) \cos \sigma + V\dot{t}_{go}}{Rf'(\sigma)} \quad (5)$$

Finally, using $\dot{t}_{go} = -1$ and Eqs. (1d) and (5), we propose a basic guidance law in the following generalized form:

$$A^B = \frac{V^2}{R} \left[\frac{f(\sigma) \cos \sigma - 1}{f'(\sigma)} - \sin \sigma \right] \quad (6)$$

It is obvious that the basic guidance law (6) is inversely derived based on the time-to-go solution (2) without considering the impact-angle constraint. This is the reason why the proposed method is referred to as the *basic time-to-go inversion guidance*. Under the proposed guidance law (6), t_f can be exactly obtained at the initial launch moment by using Eq. (2). That is, $t_f = t_{go,0} = R_0 f(\sigma_0)/V$, where R_0 and σ_0 are the initial values of R and σ , respectively. In addition, the condition of $\dot{t}_{go} = -1$ is integrated into the guidance law (6). Therefore, during the entire guidance procedure, the time-to-go solution (2) will converge from $t_{go,0}$ to zero with a constant rate of -1 . This essentially ensures that t_{go} always matches the actual remaining flight time.

Note that the guidance law (6) is in a general form, with its specific expression being determined by $f(\sigma)$. To facilitate the design of $f(\sigma)$, we define an auxiliary function

$$F = -\frac{\dot{\sigma}}{\sigma} t_{go} \quad (7)$$

Substituting Eqs. (2) and (5) into Eq. (7), the auxiliary function becomes

$$F = \frac{f(\sigma) - f^2(\sigma) \cos \sigma}{f'(\sigma) \sigma} \quad (8)$$

which will be utilized for analyzing the properties of the lead angle and guidance command. At this point, we are ready to state the following new theorem:

Theorem 1. For the guidance law (6), if the function $f(\sigma)$ is appropriately designed such that

- $f(\sigma)$ and $f'(\sigma)$ are continuous on $[-\pi, \pi]$.
- $f(\sigma)$ is strictly increasing for $\sigma > 0$, with $f(0) = 1$. Besides, $f(\sigma) = f(-\sigma)$.
- $F > 0$ holds for all $\sigma \in [-\pi, \pi]$, and F is bounded and satisfies $F > 1$ when $\sigma = 0$.

then the missile can impact the target with zero terminal guidance command; furthermore, the accurate time-to-go can be analytically calculated via Eq. (2).

Proof. Zero miss distance: According to Eq. (2), we have

$$R = V t_{go} / f(\sigma) \quad (9)$$

As discussed above, the time-to-go in Eq. (2) will converge from its initial value $t_{go,0}$ to zero under the proposed guidance law (6). Also, the second condition in Theorem 1 ensures $f(\sigma) \geq 1$. With these conditions, we can deduce from Eq. (9) that R will converge to zero as $t_{go} \rightarrow 0$.

Zero terminal lead angle: Now, we utilize the auxiliary function F to analyze the property of lead angle. According to the definition in Eq. (8) and the third condition of Theorem 1, F is a continuous and nonsingular function with

respect to σ , satisfying $F > 0$ for all $\sigma \in [-\pi, \pi]$. By the properties of continuous functions, there must exist two constants $F_1^{\max} \geq F_1^{\min}$ such that

$$F_1^{\max} \geq F(\sigma) \geq F_1^{\min} > 0, \quad \forall \sigma \in [-\pi, \pi] \quad (10)$$

Then, substituting Eq. (10) into Eq. (7) yields

$$-\frac{F_1^{\max}}{t_{go}} \leq \frac{\dot{\sigma}}{\sigma} \leq -\frac{F_1^{\min}}{t_{go}} \quad (11)$$

which can be considered as the governing equation of the lead angle. For all $t \in [0, t_f]$, the closed-form solution of Eq. (11) can be found as

$$\sigma_0 \left(\frac{t_{go}}{t_{go,0}} \right)^{F_1^{\max}} \leq \sigma \leq \sigma_0 \left(\frac{t_{go}}{t_{go,0}} \right)^{F_1^{\min}} \quad (12)$$

As $t_{go} \rightarrow 0$, Eq. (12) becomes

$$\sigma_0 \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,0}} \right)^{F_1^{\max}} \leq \lim_{t_{go} \rightarrow 0} \sigma \leq \sigma_0 \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,0}} \right)^{F_1^{\min}} \quad (13)$$

which indicates that the lead angle will converge to zero at the final time.

Zero terminal command: When $\sigma \rightarrow 0$ and $t_{go} \rightarrow 0$, using the infinitesimal relation $\sin \sigma \sim \sigma$ and Eqs. (2) and (8), the guidance command (6) can be rewritten as

$$\lim_{t_{go} \rightarrow 0} A^B = \lim_{t_{go} \rightarrow 0} \frac{Vf(\sigma)}{t_{go}} \left[\frac{f(\sigma) \cos \sigma - 1}{f'(\sigma)} - \sigma \right] = - \lim_{t_{go} \rightarrow 0} V \left[F \frac{\sigma}{t_{go}} + f(\sigma) \frac{\sigma}{t_{go}} \right] \quad (14)$$

The third condition in Theorem 1 specifies that $F > 1$ when $\sigma = 0$. Since F is a continuous function of σ , there exists a small interval $[-\sigma_m, \sigma_m]$, where $\sigma_m > 0$ is a sufficiently small value, such that

$$F_2^{\max} \geq F(\sigma) \geq F_2^{\min} > 1, \quad \forall \sigma \in [-\sigma_m, \sigma_m] \quad (15)$$

where $F_2^{\max} \geq F_2^{\min} > 1$ are two constants. Because σ converges to zero at the final moment, it must necessarily enter the small interval $[-\sigma_m, \sigma_m]$ before $t = t_f$. The moment when the lead angle σ enters $[-\sigma_m, \sigma_m]$ is denoted as t_m . Then, resolving Eq. (11) on $[t_m, t_f]$, one has

$$\sigma_m \left(\frac{t_{go}}{t_{go,m}} \right)^{F_2^{\max}} \leq \sigma \leq \sigma_m \left(\frac{t_{go}}{t_{go,m}} \right)^{F_2^{\min}} \quad (16)$$

As $t_{go} \rightarrow 0$, one can deduce from Eq. (16) that

$$\frac{\sigma_m}{t_{go,m}} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,m}} \right)^{F_2^{\max} - 1} \leq \lim_{t_{go} \rightarrow 0} \frac{\sigma}{t_{go}} \leq \frac{\sigma_m}{t_{go,m}} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,m}} \right)^{F_2^{\min} - 1} \quad (17)$$

Due to $F_2^{\max} \geq F_2^{\min} > 1$, the above equation indicates that $\lim_{t_{go} \rightarrow 0} \sigma/t_{go} = 0$. Substituting this conclusion and $f(0) = 1$ into Eq. (14), it can be proved that the guidance command will converge to zero at the final time. *This completes the proof.*

Remark 1 (Physical meanings of Theorem 1). This remark discusses the physical meanings of the three conditions in Theorem 1. 1) The form of the basic guidance law (6) is determined by $f(\sigma)$ and $f'(\sigma)$. One may notice that if the first condition in Theorem 1 is satisfied, the guidance law (6) can generate a smooth acceleration command that can be easily tracked by the onboard autopilot. 2) If the second condition in Theorem 1 is met, the value of t_{go} increases as the heading error $|\sigma|$ grows. Therefore, the missile will follow a longer and more curved future trajectory when the heading error is large. In contrast, due to $f(0) = 1$, the time-to-go in Eq. (2) reaches its minimum value $t_{go} = R/V$ when the heading error is zero, resulting in a straight-line flight path towards the target. Furthermore, the symmetric condition $f(\sigma) = f(-\sigma)$ ensures that the basic guidance law (6) maintains the same properties regardless of the sign of σ . 3) Finally, according to the above analysis, the third condition in Theorem 1 guarantees a zero terminal guidance command, which could improve warhead lethality and reduce the possibility of command saturation at the end of engagement [27]. In summary, Theorem 1 rigorously ensures the success of the guidance mission while accommodating the typical flight behavior of actual missiles.

Remark 2 (Advantages of basic time-to-go inversion guidance). 1) The guidance law (6) is derived inversely without assuming a small heading error or involving other linearizing approximations. Consequently, the time-to-go in Eq. (2) is an accurate solution for all $\sigma \in [-\pi, \pi]$ as long as the conditions of Theorem 1 are satisfied. Even though there exist several ways to accurately calculate the time-to-go, we particularly introduce a purely analytical method that can further reduce the computational burden. 2) In practical applications, a large desired impact time is usually realized by directing the missile to follow a curved trajectory. This means that the heading error could be a large value during the guidance procedure. Owing to the precision of the time-to-go solution (2), the basic guidance law (6) offers flexibility to achieve precise impact-time control for a highly nonlinear geometry with large heading errors. 3) Since the permissible choice for the function $f(\sigma)$ is not unique, the proposed inversion method is a generalized tool for developing various new guidance laws.

B. Illustrative Examples

The three conditions in Theorem 1 are the practical guidelines for designing the function $f(\sigma)$. Two examples are presented herein to further discuss the proposed inversion method.

1. First Example

Let us formulate $f(\sigma)$ as a quadratic function

$$f_1(\sigma) = 1 + k_1\sigma^2 \quad (18)$$

where k_1 is a tuning parameter. Then, we have

$$f_1'(\sigma) = \frac{df_1(\sigma)}{d\sigma} = 2k_1\sigma \quad (19)$$

Substituting Eqs. (18) and (19) into Eq. (8), we find the auxiliary function as

$$F_1 = \frac{1 + k_1\sigma^2 - (k_1^2\sigma^4 + 2k_1\sigma^2 + 1) \cos \sigma}{2k_1\sigma^2} \quad (20)$$

When $\sigma = 0$, substituting $\cos \sigma \sim 1 - \sigma^2/2$ into Eq. (20), the auxiliary function F_1 is simplified as

$$F_1|_{\sigma=0} = \frac{1-2k_1}{4k_1} + \frac{1-2k_1}{2}\sigma^2 + \frac{k_1}{4}\sigma^4 = \frac{1-2k_1}{4k_1} \quad (21)$$

where the higher-order terms of σ have been ignored.

Now, the three conditions of Theorem 1 need to be satisfied by properly selecting the parameter k_1 . 1) As defined by Eqs. (18) and (19), $f_1(\sigma)$ and $f_1'(\sigma)$ are continuous over $[-\pi, \pi]$. The first condition in Theorem 1 always holds regardless of the value of k_1 . 2) According to Eq. (19), the second condition in Theorem 1 can be met by choosing $k_1 > 0$. 3) Using Eqs. (20) and (21), one may further obtain that the third condition in Theorem 1 can be fulfilled with $0 < k_1 < 1/6$. Summarizing the above analysis, $0 < k_1 < 1/6$ is the essential condition for satisfying all the three conditions required by Theorem 1.

Finally, substituting Eqs. (18) and (19) into Eq. (6), we find the basic time-to-go inversion guidance law of the first example as

$$A_1^B = \frac{V^2}{R} \left[\frac{\sigma \cos \sigma}{2} + \frac{\cos \sigma - 1}{2k_1\sigma} - \sin \sigma \right] \quad (22)$$

with a tuning parameter $0 < k_1 < 1/6$ and an accurate time-to-go solution

$$t_{go,1}^B = \frac{R}{V} [1 + k_1\sigma^2] \quad (23)$$

Note that the approximate time-to-go solution of the conventional PNG law is also a quadratic function of the lead angle, which is similar to Eq. (23). This solution was utilized in recent work [17] for inversely deriving a modified PNG law with innovative time-varying gains. Since the function $f(\sigma)$ can be freely designed subject to Theorem 1, the

time-to-go solution of the conventional PNG law and the advanced guidance law in [17] can be considered as a special case of our generalized method.

2. Second Example

Let us formulate $f(\sigma)$ as a trigonometric function

$$f_2(\sigma) = 1 + \frac{k_2 (\sigma - \sin \sigma)}{\sigma} \quad (24)$$

where k_2 is a tuning parameter. Then, we have

$$f_2'(\sigma) = \frac{df_2(\sigma)}{d\sigma} = \frac{k_2 (\sin \sigma - \sigma \cos \sigma)}{\sigma^2} \quad (25)$$

Substituting Eqs. (24) and (25) into Eq. (8), we obtain the auxiliary function as

$$F_2 = \left[1 + \frac{k_2 (\sigma - \sin \sigma)}{\sigma} \right] \frac{\sigma (1 - \cos \sigma) - k_2 \cos \sigma (\sigma - \sin \sigma)}{k_2 (\sin \sigma - \sigma \cos \sigma)} \quad (26)$$

Comparable to Eq. (21), when $\sigma = 0$, F_2 can be simplified as

$$F_2|_{\sigma=0} = \frac{3 - k_2}{2k_2} \quad (27)$$

Similar to the first example, it can be solved from Eqs. (24)~(27) that the parameter k_2 needs to be defined as $0 < k_2 < 1$ to satisfy all the three conditions required by Theorem 1.

Finally, substituting Eqs. (24) and (25) into Eq. (6), we obtain the basic time-to-go inversion guidance law of the second example as

$$A_2^B = \frac{V^2}{R} \frac{(k_2 + 1) \sigma^2 \cos \sigma - k_2 \sin^2 \sigma - \sigma^2}{k_2 (\sin \sigma - \sigma \cos \sigma)} \quad (28)$$

with a tuning parameter $0 < k_2 < 1$ and an accurate time-to-go solution

$$t_{go,2}^B = \frac{R}{V} \left[1 + \frac{k_2 (\sigma - \sin \sigma)}{\sigma} \right] \quad (29)$$

Because of the distinctive nature of the function $f_2(\sigma)$, the guidance law (28) and its time-to-go solution (29) have a unique form that has not been previously introduced in the existing literature.

Remark 3 (Implementation of basic time-to-go inversion guidance). The above two examples demonstrate that the proposed method allows for convenient implementation. By choosing different forms of the function $f(\sigma)$, one can readily develop a variety of new guidance laws with exact time-to-go solutions. Furthermore, when applying our

proposed method, we recommend the mission designer incorporate a tuning parameter into the function $f(\sigma)$, as in Eqs. (18) and (24). This parameter not only facilitates the satisfaction of Theorem 1 but also offers the flexibility to adjust the missile trajectory. For the two illustrative basic guidance laws (22) and (28), if their parameters k_1 and k_2 are selected as large values, the time-to-go in Eqs. (23) and (29) accordingly increases and thus the missile will detour along a longer flight trajectory. In contrast, selecting small values for k_1 and k_2 will cause the missile to rapidly fly toward the target in a shorter time.

C. Further Discussions on Basic Time-to-Go Inversion Guidance

A simple simulation case is presented herein to further discuss the significance of our proposed method. The following conventional PNG law is used as a benchmark comparison:

$$A^{\text{PNG}} = NV\dot{\lambda} = -\frac{NV^2}{R} \sin \sigma \quad (30)$$

where $N = 4$ is the navigation gain. The approximate time-to-go of the PNG law is $t_{go}^{\text{PNG}} \approx R/V [1 + \sigma^2/(4N - 2)]$, which is derived under the assumption of small heading errors.

In our simulations, the parameters of the two illustrative basic guidance laws (22) and (28) are chosen as $k_1 = 0.1$ and $k_2 = 0.8$. The initial LOS angle, target range, and the constant speed for the missile are selected as 0° , 10 km, and 250 m/s, respectively. Under various initial heading errors, the simulation results for the two basic guidance laws and the conventional PNG law are depicted in Fig. 2. Figure 3 compares the actual and estimated impact times of the three guidance laws for all $\sigma_0 \in [-\pi, \pi]$. The actual impact times are obtained after each simulation run. The estimated impact times of the three guidance laws are calculated at the initial moment using their time-to-go solutions, i.e., Eq. (23), Eq. (29), and $R_0/V [1 + \sigma_0^2/(4N - 2)]$. As demonstrated by these simulation results, the proposed method shows superior performance over the conventional PNG law as follows:

Convergence efficiency of heading error: As can be noticed from Fig. 2, all three guidance laws are capable of impacting the target with zero terminal lead angles and guidance commands. Under the conventional PNG law, the magnitude of the guidance command reaches the maximum value when $\sigma_0 = \pi/2$ and then decreases as the initial heading error approaches $\sigma_0 = \pi$. This feature is determined by the term of $\sin \sigma$ in the conventional PNG law (30). Thereby, when $\sigma_0 \in [\pi/2, \pi]$, the heading error of the PNG law converges slowly, leading to unnecessarily long flight trajectories and ultimately resulting in mission failure at $\sigma_0 = \pi$. In contrast, under the two basic guidance laws (22) and (28), the magnitude of the guidance command increases with the growth of σ_0 . Therefore, when $\sigma_0 \in [\pi/2, \pi]$, the heading error of our basic guidance laws converges more efficiently during the early phase, which enforces the missile's flight direction to turn to the target swiftly. In practice, this characteristic facilitates the rapid lock-on of the seeker on the target.

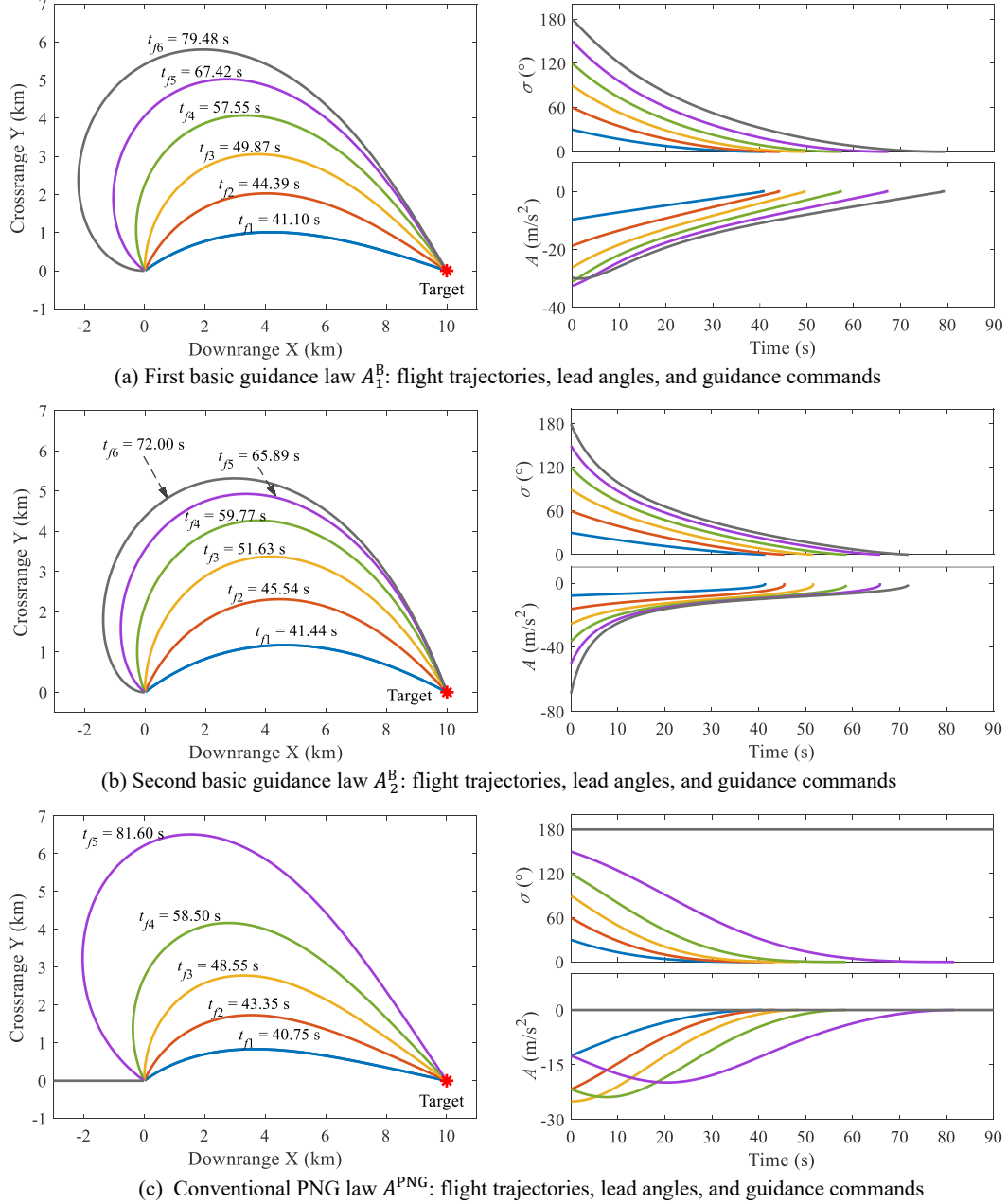


Fig. 2 Simulation results of the two basic guidance laws and the conventional PNG law under various initial heading errors ($|\sigma_0| = 30 : 30 : 180^\circ$).

Accuracy of time-to-go solution: As shown in Fig. 3, the actual impact time of the two basic guidance laws can be precisely estimated at the initial moment for all $\sigma_0 \in [-\pi, \pi]$. However, the estimation error for the time-to-go of the conventional PNG law becomes large when $|\sigma_0| > \pi/2$. Compared with the second basic guidance law (28), the impact time of the first basic guidance law (22) increases more rapidly with the growth of $|\sigma_0|$. This is because the functions $f_1(\sigma)$ and $f_2(\sigma)$ are selected in different forms. These simulation results fully validate the advantage of the proposed method in the context of accurate time-to-go estimation.

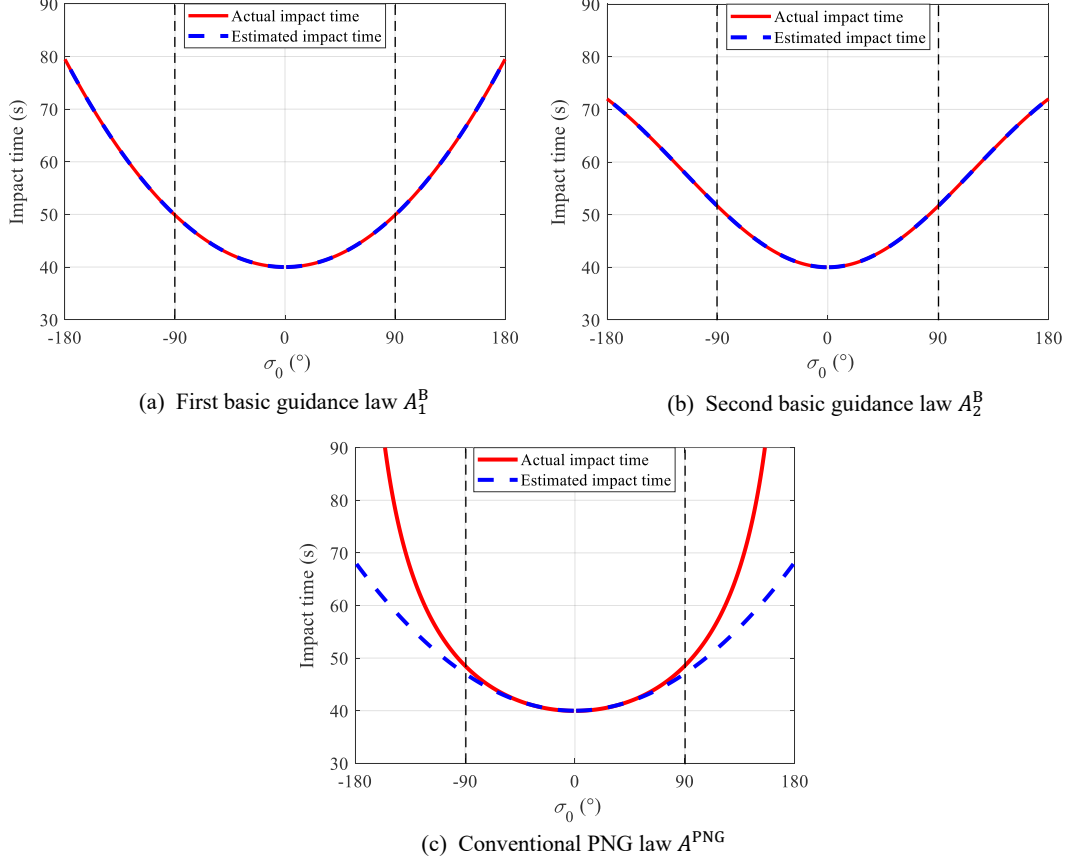


Fig. 3 Impact times of the two basic guidance laws and the conventional PNG law.

IV. Angle-Constrained Time-to-Go Inversion Guidance

To address the more challenging second objective described in Sec. II.B, this section introduces a generalized method for designing IACG laws, referred to as the *angle-constrained time-to-go inversion guidance*. This section also discusses the proposed method for onboard implementation by introducing two examples. In this section, we still assume that the missile speed remains constant.

A. Generalized Guidance Framework

Ignoring the sideslip angle, the impact-angle constraint on a stationary target can be described by

$$\lambda \rightarrow \lambda_d \quad \text{as} \quad R \rightarrow 0 \quad (31)$$

where λ_d is the desired LOS angle at the final impact moment [27].

To proceed with our design, we define a new variable

$$\omega = \frac{1}{n+1}\sigma + \frac{n}{n+1}\lambda_e \quad (32)$$

where $n > 0$ is a tuning parameter and $\lambda_e = \lambda - \lambda_d$ is the impact-angle error. The parameter n is introduced in Eq. (32) to balance the weight between the heading error and the impact-angle error. Furthermore, due to $\sigma \in [-\pi, \pi]$ and $\lambda_e \in [-\pi, \pi]$, the variable ω always satisfies $\omega \in [-\pi, \pi]$.

Considering the impact-angle constraint, the time-to-go of the missile is represented as

$$t_{go} = \frac{R}{V} g(\omega) \quad (33)$$

where $g(\omega)$ is a function of the variable ω defined on $[-\pi, \pi]$. Similar to the results in Sec. III, prior to introducing the practical guidelines for designing $g(\omega)$, we first develop a generalized IACG law based on the time-to-go solution (33). Taking the time derivative of Eq. (33) leads to

$$\dot{t}_{go} = \frac{\dot{R}g(\omega) + R\dot{g}(\omega)}{V} \quad (34)$$

Defining $g'(\omega) = dg(\omega)/d\omega$, we rewrite $\dot{g}(\omega)$ as

$$\dot{g}(\omega) = \frac{dg(\omega)}{d\omega} \frac{d\omega}{dt} = g'(\omega) \dot{\omega} \quad (35)$$

Substituting Eqs. (1a) and (35) into Eq. (34), we find the time derivative of ω as

$$\dot{\omega} = \frac{Vg(\omega) \cos \sigma + Vi_{go}}{Rg'(\omega)} \quad (36)$$

One can readily obtain that $\dot{\lambda}_e = \dot{\lambda} = -V \sin \sigma / R$. Then, based on Eqs. (32) and (36), we have

$$\dot{\sigma} = \frac{V}{R} \left[(n+1) \frac{g(\omega) \cos \sigma + i_{go}}{g'(\omega)} + n \sin \sigma \right] \quad (37)$$

Finally, using $\dot{t}_{go} = -1$ and Eqs. (1d) and (37), we propose a generalized IACG law:

$$A^{IAC} = \frac{V^2}{R} \left[(n+1) \frac{g(\omega) \cos \sigma - 1}{g'(\omega)} + (n-1) \sin \sigma \right] \quad (38)$$

Because the impact-angle constraint is incorporated in our guidance design, the proposed method is named as the *angle-constrained time-to-go inversion guidance*. As previously mentioned, the parameter n can be used to balance the weight between σ and λ_e , which makes the IACG law (38) more general. If the initial magnitude of λ_e is large in practice, the mission designer may select a larger parameter n to allocate more control efforts for adjusting the impact-angle error. In contrast, if the initial magnitude of σ is large, a smaller value of n can be chosen to ensure the efficient convergence of the heading error.

Similar to the auxiliary function F , we introduce another auxiliary function to facilitate the design of $g(\omega)$ as

$$G = -\frac{\dot{\omega}}{\omega} t_{go} \quad (39)$$

Substituting Eqs. (33) and (36) into Eq. (39) results in

$$G = \frac{g(\omega) - g^2(\omega) \cos \sigma}{g'(\omega) \omega} \quad (40)$$

Now, the conditions for fulfilling the IACG law design are described by the following new theorem.

Theorem 2. For the guidance law (38), if $0 < \sigma_0/\lambda_{e0} \leq n$ is satisfied at the initial moment and the function $g(\omega)$ is appropriately designed such that

- $g(\omega)$ and $g'(\omega)$ are continuous on $\omega \in [-\pi, \pi]$.
- $g(\omega)$ is strictly increasing for $\omega > 0$, with $g(0) = 1$. Besides, $g(\omega) = g(-\omega)$
- $G\lambda_e > g(\omega) \sin \sigma$ when $\sigma = n\lambda_e$, and G is bounded and satisfies $G > 1$ when $\omega = 0$, $\sigma = 0$.

then the missile can impact the target with desired impact angle and zero terminal guidance command; furthermore, the accurate time-to-go can be analytically calculated via Eq. (33).

Proof. First, we analyze the behavior of the term σ/λ_e under the proposed guidance law. For the notational simplicity, let $\xi = \sigma/\lambda_e$. From Eq. (37), one has

$$\begin{aligned} \dot{\xi} &= \frac{\lambda_e \dot{\sigma} - \sigma \dot{\lambda}_e}{\lambda_e^2} \\ &= \frac{V}{R} \left[(n+1) \frac{g(\omega) \cos \sigma - 1}{g'(\omega) \lambda_e} + n \frac{\sin \sigma}{\lambda_e} + \frac{\sigma \sin \sigma}{\lambda_e^2} \right] \end{aligned} \quad (41)$$

The second condition in Theorem 2 gives $g(\omega) > 1$ when $\omega \neq 0$, while the third condition in Theorem 2 specifies $G\lambda_e > g(\omega) \sin \sigma$ when $\sigma = n\lambda_e$. Using these two conditions and Eqs. (32), (40), and (41), one can obtain

$$\dot{\xi} > 0 \quad \text{when} \quad \xi = 0; \quad \dot{\xi} < 0 \quad \text{when} \quad \xi = n$$

Consequently, $\mathcal{H} = \{\xi \in \mathbb{R} : 0 \leq \xi \leq n\}$ is an invariant set about ξ . That is, ξ is always in the set $\mathcal{H}$ if its initial value satisfies $\xi_0 \in \mathcal{H}$. Also, it can be deduced from Eq. (41) that there must exist a small interval $[0, \delta]$, where $0 < \delta < n$ is a constant, such that $\dot{\xi} > 0$ for all $\xi \in [0, \delta]$. Hence, if $\xi_0 > 0$ at $t = 0$, the condition $\xi > 0$ is essentially ensured for all $t \in [0, t_f]$. As stated in Theorem 2, $0 < \sigma_0/\lambda_{e0} \leq n$ needs to be satisfied at the initial launch moment. Then, according to the above analysis, $0 < \sigma/\lambda_e \leq n$ will hold for all $t \in [0, t_f]$. With this conclusion, Theorem 2 can be easily proved as follows.

Zero Miss Distance: One may obtain from Eq. (33) that

$$R = V t_{go} / g(\omega) \quad (42)$$

Similar to Eq. (9), it can be deduced from the above equation that $\lim_{t_{go} \rightarrow 0} R = 0$.

Convergence of impact-angle error: Substituting Eq. (33) into $\dot{\lambda}_e = -V \sin \sigma / R$ yields

$$\dot{\lambda}_e = -\frac{B}{t_{go}} \lambda_e \quad (43)$$

where $B = g(\omega) \sin \sigma / \lambda_e$. It is obvious that B is continuous over time. Due to $0 < \sigma / \lambda_e \leq n$, the term $\sin \sigma / \lambda_e$ is always bounded and satisfies $\sin \sigma / \lambda_e > 0$. Then, by the properties of continuous functions, there exist two positive constants $B^{\max} \geq B^{\min} > 0$ such that

$$B^{\max} \geq B \geq B^{\min} > 0, \quad \forall t \in [0, t_f] \quad (44)$$

Then, substituting Eq. (43) in Eq. (44) yields

$$-\frac{B^{\max}}{t_{go}} \leq \frac{\dot{\lambda}_e}{\lambda_e} \leq -\frac{B^{\min}}{t_{go}} \quad (45)$$

which can be considered as the governing equation of the impact-angle error. Because of $B^{\max} \geq B^{\min} > 0$, the closed-form solution of Eq. (45) can be found as

$$\lambda_{e0} \left(\frac{t_{go}}{t_{go,0}} \right)^{B^{\max}} \leq \lambda_e \leq \lambda_{e0} \left(\frac{t_{go}}{t_{go,0}} \right)^{B^{\min}} \quad (46)$$

When $t_{go} \rightarrow 0$, one obtains from Eq. (46) that

$$\lambda_{e0} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,0}} \right)^{B^{\max}} \leq \lim_{t_{go} \rightarrow 0} \lambda_e \leq \lambda_{e0} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,0}} \right)^{B^{\min}} \quad (47)$$

which indicates that the impact-angle error will converge to zero at the impact moment.

Zero terminal lead angle and command: Since $0 < \sigma / \lambda_e \leq n$ always holds and λ_e converges to zero at t_f , it can be inferred that both σ and ω will converge to zero as $t_{go} \rightarrow 0$. When σ and ω are sufficiently small, Eq. (38) becomes

$$\begin{aligned} \lim_{t_{go} \rightarrow 0} A^{\text{IAC}} &= \lim_{t_{go} \rightarrow 0} \frac{V g(\omega)}{t_{go}} \left[(n+1) \frac{g(\omega) \cos \sigma - 1}{g'(\omega)} + (n-1) \sigma \right] \\ &= - \lim_{t_{go} \rightarrow 0} V \left[(n+1) G \frac{\omega}{t_{go}} - (n-1) g(\omega) \frac{\sigma}{t_{go}} \right] \end{aligned} \quad (48)$$

The third condition in Theorem 2 gives that $G > 1$ when $\omega = 0$ and $\sigma = 0$. Because G is a continuous function, there exists small intervals $[-\omega_m, \omega_m]$ and $[-\sigma_m, \sigma_m]$, where $\omega_m > 0$ and $\sigma_m > 0$ are two small constants, such that

$$G^{\max} \geq G(\omega, \sigma) \geq G^{\min} > 1, \quad \forall \omega \in [-\omega_m, \omega_m] \text{ and } \sigma \in [-\sigma_m, \sigma_m] \quad (49)$$

where $G^{\max} \geq G^{\min} > 1$ are two constants. Then, substituting Eq. (49) into Eq. (39) yields

$$-\frac{G^{\max}}{t_{go}} \leq \frac{\dot{\omega}}{\omega} \leq -\frac{G^{\min}}{t_{go}} \quad (50)$$

Before the final impact, there must exist a time moment t_m such that, for all $t \in [t_m, t_f]$, ω remains within $[-\omega_m, \omega_m]$ and σ remains within $[-\sigma_m, \sigma_m]$. Then, comparable to the derivation of Eq. (17), one has

$$\frac{\omega_m}{t_{go,m}} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,m}} \right)^{G^{\max}-1} \leq \lim_{t_{go} \rightarrow 0} \frac{\omega}{t_{go}} \leq \frac{\omega_m}{t_{go,m}} \lim_{t_{go} \rightarrow 0} \left(\frac{t_{go}}{t_{go,m}} \right)^{G^{\min}-1} \quad (51)$$

Due to $G^{\max} \geq G^{\min} > 1$, the above equation indicates that $\lim_{t_{go} \rightarrow 0} \omega/t_{go} = 0$. Recalling the definition of ω and the condition of $0 < \sigma/\lambda_e \leq n$, we also have $\lim_{t_{go} \rightarrow 0} \sigma/t_{go} = 0$. Then, it rigorously ensures that the guidance command of Eq. (48) will converge to zero at the final time. *This completes the proof.*

Remark 4 (Physical meanings of Theorem 2). 1) Similar to our previous findings in Sec. III, the first condition of Theorem 2 also leads to a smooth guidance command that facilitates onboard implementation. 2) The second condition in Theorem 2 describes the relation between t_{go} and ω . According to the definition of ω in Eq. (32), the value of t_{go} is now determined by both the heading error and the impact-angle error. If the magnitude of ω is large, Eq. (33) will lead to a longer time-to-go, providing the missile with more guidance time to correct its heading error or impact-angle error. Besides, the above analysis has shown that $\omega = 0$ only occurs when $\sigma = \lambda_e = 0$. In this case, Eq. (33) will yield the minimum time-to-go (i.e., $t_{go} = R/V$) for guiding the missile to approach the target along a straight-line flight path. Furthermore, the symmetric condition $g(\omega) = g(-\omega)$ ensures that the guidance performance of the IACG law (38) will remain unchanged, even if the missile is launched in different directions with respect to the initial LOS. This aligns with the typical flight behavior of most actual missiles. 3) From the proof of Theorem 2, the first part of the third condition ensures the simultaneous convergence of σ and λ_e . In other words, one of σ or λ_e will not be zero if the other is nonzero, which could avoid singularities. The second part of the third condition in Theorem 2 guarantees a zero terminal guidance command.

Remark 5 (Advantages of angle-constrained time-to-go inversion guidance). 1) For the first time, by utilizing the concept of inversion design, the proposed method addresses this challenging issue of time-to-go estimation for IACG laws. It offers a general tool for developing various IACG laws with globally accurate time-to-go solutions. 2) Theorem 2 also

gives the feasible initial condition for the guidance law (38), i.e., $0 < \sigma_0/\lambda_{e0} \leq n$. It is an analytical relation between the desired impact angle and the initial lead angle. Therefore, before the guidance implementation, the mission designer can properly select the initial launch conditions according to the final desired impact angle. 3) The guidance law (38) can be easily augmented by incorporating a bias command into it, which offers flexibility to achieve accurate impact time and angle control for large heading errors. 4) The conditions in Theorems 1 and 2 are similar, meaning that the design of the function $g(\omega)$ resembles that of $f(\sigma)$.

B. Illustrative Examples

As mentioned above, a design parameter n is integrated into the variable ω to further enhance the generality of the IACG law (38). In practical implementation, guidance designers can select this parameter according to the impact constraints and initial launch conditions of the missile. For simplicity, we directly select $n = 3$ in the following two examples to streamline our mathematical derivations.

1. First Example

Comparable to Eq. (18), $g_1(\omega)$ is formulated as

$$g_1(\omega) = 1 + h_1\omega^2 \quad (52)$$

where h_1 is a tuning parameter. In addition, one has

$$g'_1(\omega) = \frac{dg_1(\omega)}{d\omega} = 2h_1\omega \quad (53)$$

Substituting Eqs. (52) and (53) into Eq. (40), we find the auxiliary function as

$$G_1 = \frac{1 + h_1\omega^2 - (h_1^2\omega^4 + 2h_1\omega^2 + 1) \cos \sigma}{2h_1\omega^2} \quad (54)$$

When $\sigma = n\lambda_e$ and $n = 3$, one can obtain from Eq. (32) that $\omega = \sigma/2$. Using these relations, the inequality $G_1\lambda_e > g_1(\omega) \sin \sigma$, as required by the third condition in Theorem 2, can be rewritten as

$$\frac{2(1 - \cos \sigma)}{h_1\sigma} - \frac{\sigma \cos \sigma}{2} > 3 \sin \sigma \quad (55)$$

which is only related to σ . When $\omega = 0$ and $\sigma = 0$, G_1 can be simplified by substituting $\cos \sigma \sim 1 - \sigma^2/2$ into Eq. (54) as follows:

$$G_1|_{\omega, \sigma=0} = -\frac{1}{2} + \frac{\sigma^2}{4h_1\omega^2} + \frac{\sigma^2 + h_1^2\omega^2(\sigma^2 - 1)}{2} = -\frac{1}{2} + \frac{\sigma^2}{4h_1\omega^2} \Big|_{\omega, \sigma=0} \quad (56)$$

where the higher-order terms of σ and ω have been ignored.

Now, the three conditions of Theorem 2 need to be satisfied by properly selecting the parameter h_1 . 1) As shown in Eq. (52), the first condition of Theorem 2 always holds regardless of the value of h_1 . 2) According to Eq. (53), the second condition in Theorem 2 is satisfied if $h_1 > 0$ is selected. 3) The inequality $G_1 \lambda_e > g_1(\omega) \sin \sigma$ can be met by ensuring Eq. (55) holds for all $\sigma \in [-\pi, \pi]$, resulting in $h_1 < 2/7$. When $\omega = 0$ and $\sigma = 0$, the function G_1 in Eq. (56) is bounded and satisfies $G_1 > 1$ by selecting $h_1 > 1/6$. Note that the value of $\sigma^2/\omega^2|_{\omega, \sigma=0}$ in Eq. (56) needs to be solved using L'Hopital's rule, which is omitted here. In summary, $1/6 < h_1 < 2/7$ is the essential condition for satisfying all three conditions in Theorem 2.

Finally, substituting $n = 3$, $g_1(\omega)$, and $g'_1(\omega)$ into Eq. (38), we obtain the first illustrative IACG law as

$$A_1^{\text{IAC}} = \frac{2V^2}{R} \left[\frac{\cos \sigma - 1}{h_1 \omega} + \omega \cos \sigma + \sin \sigma \right] \quad (57)$$

with a tuning parameters $1/6 < h_1 < 2/7$ and an accurate time-to-go solution

$$t_{go,1}^{\text{IAC}} = \frac{R}{V} [1 + h_1 \omega^2] \quad (58)$$

2. Second Example

Comparable to Eq. (24), $g_2(\omega)$ is formulated as

$$g_2(\omega) = 1 + \frac{h_2 (\omega - \sin \omega)}{\omega} \quad (59)$$

where h_2 is a tuning parameter. Then, we have

$$g'_2(\omega) = \frac{dg_2(\omega)}{d\omega} = \frac{h_2 (\sin \omega - \omega \cos \omega)}{\omega^2} \quad (60)$$

Substituting Eqs. (52) and (53) into Eq. (40), we find the auxiliary function as

$$G_2 = \left[1 + \frac{h_2 (\omega - \sin \omega)}{\omega} \right] \frac{\omega (1 - \cos \sigma) - h_2 \cos \sigma (\omega - \sin \omega)}{h_2 (\sin \omega - \omega \cos \omega)} \quad (61)$$

Similar to Eq. (55), when $\sigma = n\lambda_e$ and $n = 3$, $G_2 \lambda_e > g_2(\omega) \sin \sigma$ is equivalent to

$$\left. \frac{G_2 \sigma}{g_2} \right|_{\omega=\sigma/2} > 3 \sin \sigma \quad (62)$$

Analogous to Eq. (56), when $\omega = 0$ and $\sigma = 0$, G_2 can be simplified as

$$G_2|_{\omega, \sigma=0} = -\frac{1}{2} + \frac{3\sigma^2}{2g_2\omega^2} \Big|_{\omega, \sigma=0} \quad (63)$$

Similar to the above first example, solving from Eqs. (59)~(63) reveals that the parameter h_2 should be defined as $1 < h_2 < 12/7$ to satisfy all three conditions as required by Theorem 2.

Finally, substituting $n = 3$, $g_2(\omega)$, and $g'_2(\omega)$ into Eq. (38), we determine the second illustrative IACG law as

$$A_2^{\text{IAC}} = \frac{4V^2}{R} \left[\frac{\omega^2(\cos \sigma - 1) + h_2(\omega^2 - \omega \sin \omega) \cos \sigma}{h_2(\sin \omega - \omega \cos \omega)} + \frac{\sin \sigma}{2} \right] \quad (64)$$

with a tuning parameter $1 < h_2 < 12/7$ and an accurate time-to-go solution

$$t_{g_2,2}^{\text{IAC}} = \frac{R}{V} \left[1 + \frac{h_2(\omega - \sin \omega)}{\omega} \right] \quad (65)$$

Remark 6 (Implementation of angle-constrained time-to-go inversion guidance). The IACG law (38) is a highly general form that includes a parameter n for balancing the weight between σ and λ_e . In practical applications, the guidance designer can first select this parameter based on the initial values of σ and λ_e , and then develop the corresponding guidance laws. For the purpose of simplicity, the two illustrative IACG laws (57) and (64) are obtained under $n = 3$. If n is selected as different values, the expressions of the two illustrative guidance laws and their parameter ranges will also change. It should be emphasized that the selection range of n as defined by Eq. (32) (i.e. $n > 0$) is a theoretical result without considering guidance performance. Extensive numerical simulations indicate that, in practice, n should be selected within $2 < n < 5$ to obtain proper acceleration commands and moderately curved flight trajectories. Furthermore, similar to the results in Sec. III, if h_1 and h_2 are chosen as large values for the two illustrative IACG laws, their time-to-go solutions in Eqs. (58) and (65) will increase and guide the missile to detour along a longer flight trajectory.

C. Further Discussions on Angle-Constrained Time-to-Go Inversion Guidance

In this subsection, a simple simulation case is presented to compare the performance of the above two IACG laws with that of the optimal IACG law in [26] of the form

$$A^{\text{OIAC}} = NV\dot{\lambda} + \frac{\alpha(N-1)V\dot{R}}{R}\epsilon \quad (66)$$

where $\epsilon = \gamma_d - \gamma_f$ is error between the desired flight-path angle γ_d and the final flight-path angle γ_f ; N and α are two guidance gains satisfying $(N-2)(\alpha-1) \geq 0$, $N+\alpha > 3$, and $N \neq \alpha+1$. Ignoring the sideslip angle, the desired

flight-path angle is equivalent to the desired LOS angle under the guidance geometry in Fig. 1, i.e., $\gamma_d = \lambda_d$. The final flight-path angle can be estimated as $\gamma_f = [N\lambda/(N-1) - \gamma/(N-1)]$ [26]. For such an optimal IACG law (54), its time-to-go estimation is

$$t_{go}^{OIAC} = \frac{R}{V} \left[1 + \frac{\sigma^2}{4N-2} - \frac{\alpha(N-1)\sigma\epsilon}{(2N-1)(N+\alpha)} + \frac{\alpha^2(N-1)^2\epsilon^2}{(2N-1)(\alpha+N)(2\alpha+1)} \right] \quad (67)$$

which was approximately derived in [31] based on the small-heading error assumption.

The parameters for the two illustrative IACG laws (57) and (64) are selected as $h_1 = 0.25$ and $h_2 = 1.8$, while the parameters for the optimal IACG law (66) are chosen as $N = 4$ and $\alpha = 2$ according to the simulation results in [31]. Other simulation settings remain unchanged from those in Sec. III.C. When applying our illustrative IACG laws, the missile launch direction and the desired impact direction should be on the same side of the initial LOS (i.e. $0 < \sigma_0/\lambda_{e0} \leq n$ as required by Theorem 2). This requirement is said to be very realistic in practice. For a wide range of initial heading errors and a selected desired impact angle of $\lambda_d = -60^\circ$ (i.e., $\lambda_{e0} = \lambda_0 - \lambda_d = 60^\circ$), the simulation results of the three guidance laws are compared in Fig. 4.

Because the impact-angle constraint is considered, the actual impact time of the missile is now determined by both the heading error and the impact-angle error. Thus, we compare the impact times of the three guidance laws under two cases: 1) λ_{e0} is fixed as 60° , while σ_0 varies from 0° to 180° ; and 2) σ_0 is fixed as 60° , while λ_{e0} varies from 30° to 180° . It is worth noting that, according to Theorem 2, the missile could fail to capture the target if the initial value of the impact-angle error is selected too small. Hence, for the second case mentioned above, our simulation starts with $\lambda_{e0} = 30^\circ$ rather than $\lambda_{e0} = 0^\circ$. The actual and estimated impact times of the three guidance laws are compared in Fig. 5. Similar to the results in Sec. III.C, the actual impact times are obtained after each simulation run. The estimated impact times of the three guidance laws are calculated at the initial moment using Eq. (58), Eq. (65), and Eq. (67), respectively. The main advantage of the proposed method manifests in the following two aspects.

Convergence efficiency of heading error: One may observe from Fig. 4 that the two illustrative IACG laws achieve the desired impact angle of -60° and result in zero terminal guidance command. For $\sigma_0 > \pi/2$, the flight trajectories of our illustrative IACG laws are more curved during the initial phase, in which the heading error converges efficiently and the flight direction of the missile turns to the target swiftly. As mentioned earlier, this property aids the seeker in locking onto the target, particularly when the missile is launched with a large initial heading error. However, under the optimal IACG law, the heading error converges slowly when $\sigma_0 > \pi/2$ and the missile fails to capture the target under the extreme condition of $\sigma_0 = \pi$.

Accurate time-to-go solution: By inspecting Fig. 4c, the time-to-go solution of the optimal IACG law (66) is said to be precise when the initial heading error and impact-angle error are sufficiently small. However, its estimated error increases with the growth of σ_0 or λ_{e0} . On the other hand, one can notice from Figs. 4a and 4b that the time-to-go

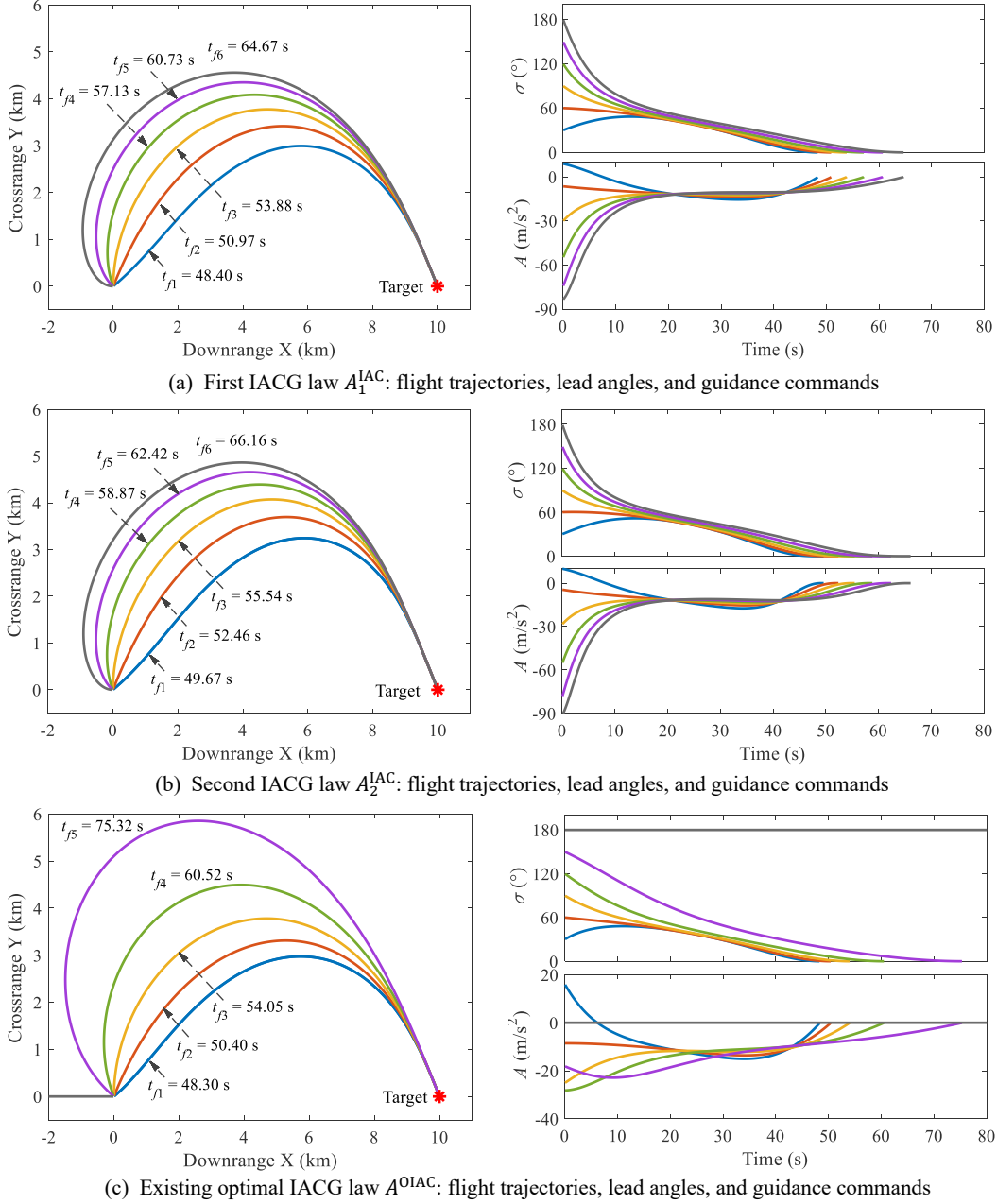


Fig. 4 Simulation results of the two proposed IACG laws and the optimal IACG law in [26] under various initial heading errors ($|\sigma_0| = 30 : 30 : 180^\circ$).

solutions of the two illustrative IACG laws are always accurate, even in the face of large initial heading errors and large impact-angle errors. Furthermore, the time-to-go of our illustrative IACG laws increases more rapidly with the increase of λ_{e0} . This is because $n = 3$ is selected for our guidance design, which gives λ_e a higher weight than σ in the time-to-go solution.

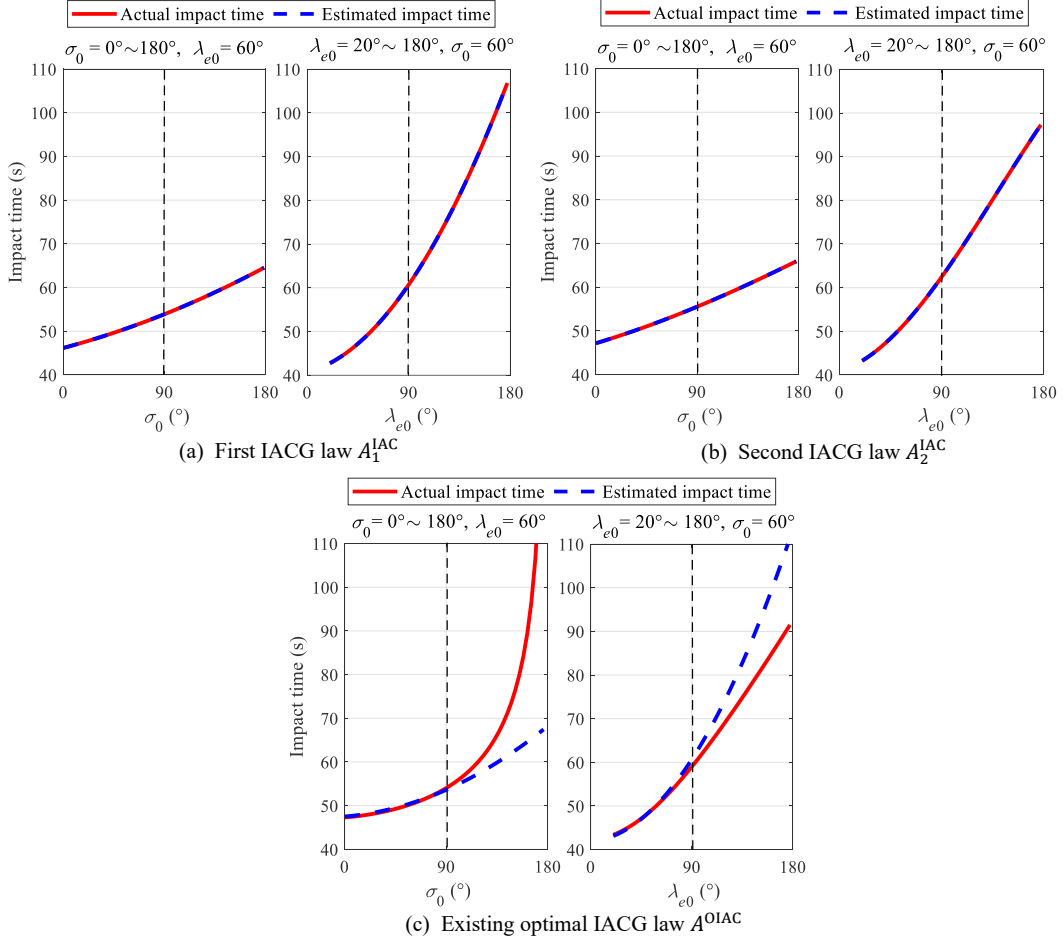


Fig. 5 Impact times of the two proposed IACG laws and the optimal IACG law in [26].

V. Augmented Time-to-Go Inversion Guidance for Impact-Time Control

To fulfill the third guidance objective described in Sec. II.B, this section augments the proposed basic and angle-constrained guidance methods with bias terms. These bias terms are specifically designed to guide the missile to meet the impact-time constraint under both constant and time-varying speeds.

A. ITCG and ITACG Laws for Constant-Speed Missiles

The desired value for the time-to-go is defined as $t_{go,d} = t_d - t$, where t_d is the desired final time selected by the guidance designer. Then, the error between the desired and actual time-to-go can be written as

$$e = t_{go,d} - t_{go} \quad (68)$$

where the value of t_{go} can be obtained using the accurate time-to-go solutions in Eqs. (2) and (33).

By augmenting the basic guidance law (6) and the IACG law (38), we propose a generalized ITCG law as follows:

$$A^{\text{ITC}} = A^{\text{B}} + \underbrace{\frac{V^2}{Rf'(\sigma)}K(e)}_{\text{Bias term}} \quad (69)$$

and a generalized ITACG law of the form

$$A^{\text{ITAC}} = A^{\text{IAC}} + \underbrace{\frac{V^2}{nRg'(\omega)}K(e)}_{\text{Bias term}} \quad (70)$$

where $K(e)$ is a function for ensuring the convergence of e . In this paper, $K(e)$ is designed as

$$K(e) = \frac{e_0^{1-p}}{\mu t_d(1-p)} \text{sgn}(e)|e|^p \quad (71)$$

where e_0 denotes the initial value of e , $0 < p < 1$ and $0 < \mu < 1$ are two design parameters.

Corollary 1. For missiles with constant speed, the proposed ITCG and ITACG laws (69) and (70), in which t_{go} is updated at each time step using the analytical time-to-go solutions (2) and (32), can achieve the desired impact constraints on the target.

Proof. Substituting the ITCG law (69) into Eq. (1d) gives

$$\dot{\sigma} = \frac{V}{R} \frac{f(\sigma) \cos \sigma - 1}{f'(\sigma)} + \frac{V}{Rf'(\sigma)}K(e) \quad (72)$$

Then, combining Eqs. (3), (4), and (72) yields

$$\dot{i}_{go} = -1 + K(e) = -1 + \frac{e_0^{1-p}}{\mu t_d(1-p)} \text{sgn}(e)|e|^p \quad (73)$$

Different from the basic guidance law (6), under which the time derivative of t_{go} is always equal to $\dot{i}_{go} = -1$, the bias term in the ITCG law (69) offers the flexibility to adjust $\dot{i}_{go}$. Taking the time derivative of e and substituting Eq. (73) into it, we find

$$\dot{e} = \dot{i}_{go,d} - \dot{i}_{go} = -\frac{e_0^{1-p}}{\mu t_d(1-p)} \text{sgn}(e)|e|^p \quad (74)$$

which can be rewritten as

$$\text{sgn}(e)|e|^{-p}de = -\frac{e_0^{1-p}}{\mu t_d(1-p)}dt \quad (75)$$

Integrating Eq. (75), the closed-form solution of e becomes

$$e(t) = \begin{cases} e_0[1 - t/(\mu t_d)]^{1/(1-p)} & \text{for } t < \mu t_d \\ 0 & \text{for } t \geq \mu t_d \end{cases} \quad (76)$$

where e_0 is the initial value of e . As can be seen in Eq. (76), the error e converges to zero at a specific time μt_d and remains zero thereafter. As e converges to zero, the bias term also goes to zero, which implies that the ITCG law (69) will revert to the original basic guidance law (6) for all $t \geq \mu t_d$. Thus, the proposed ITCG law (69) finally achieves zero miss distance, zero terminal command, as well as the desired impact time. Similarly, substituting the proposed ITACG law (70) into Eq. (1d) shows that the closed-form solution for the time-to-go error e is the same as that of Eq. (76). This means that $e = 0$ is also ensured for all $t \geq \mu t_d$ under the proposed ITACG law (70). As a result, the ITACG law (70) is able to realize zero miss distance, zero terminal command, and all desired impacts simultaneously. *This completes the proof.*

Remark 7 (Discussions of bias terms). Based on the above analysis, the parameters μ and p determine the convergence time and rate of the time-to-go error, respectively. Unlike the bias terms in [4, 5], which enforce e converge to zero exactly at $t = t_d$, the bias terms in this paper maintain $e = 0$ for all $t \geq \mu t_d$. There are two reasons for us to employ this design strategy. First, it offers flexibility to the guidance designer to freely determine the convergence time and rate of e . Moreover, it significantly simplifies our analysis process. The proposed ITCG and ITACG laws will revert to their original forms (i.e. Eqs. (6) and (38)) when $t \geq \mu t_d$. Therefore, the proposed ITCG and ITACG laws maintain the terminal performance of zero lead angle and zero terminal command, while also satisfying the desired impact constraints. In contrast, if e converges to zero exactly at $t = t_d$, the terminal performance of the guidance laws needs to be re-analyzed, and the parameters in the bias term should be selected very carefully. Note that our bias terms have the ability to resist external disturbances and measurement noises during the entire guidance procedure, even when $t \geq \mu t_d$. If e deviates from its equilibrium point when $t \geq \mu t_d$, the term $\text{sgn}(e)|e|^p$ in Eq. (71) will become nonzero and enforce e return to zero rapidly due to $p < 1$. Furthermore, in practice, we can choose a large value for μ (such as $\mu = 0.9$) to slow down the convergence of e and obtain proper acceleration commands.

Finally, based upon the generalized formulations in Eqs. (69) and (70) and our previous results, we obtain the following two illustrative ITCG laws:

$$A_1^{\text{ITC}} = A_1^{\text{B}} + \frac{V^2}{2k_1 R \sigma} K(e) \quad (77)$$

$$A_2^{\text{ITC}} = A_2^{\text{B}} + \frac{V^2 \sigma^2}{k_2 R (\sin \sigma - \sigma \cos \sigma)} K(e) \quad (78)$$

and the following two illustrative ITACG laws:

$$A_1^{\text{ITAC}} = A_1^{\text{IAC}} + \frac{V^2}{6h_1 R \omega} K(e) \quad (79)$$

$$A_2^{\text{ITAC}} = A_2^{\text{IAC}} + \frac{V^2 \omega^2}{3h_2 R (\sin \omega - \omega \cos \omega)} K(e) \quad (80)$$

where A_1^{B} , A_2^{B} , A_1^{IAC} , and A_2^{IAC} are given in Eqs. (22), (28), (57), and (64), respectively. In addition, for these illustrative guidance laws, their actual time-to-go can be obtained via the accurate solutions in Eqs. (23), (29), (58), and (65), respectively.

In practical applications of the above ITCG and ITACG laws, the discontinuous $\text{sgn}(e)$ function in the bias term could cause command chattering when e is small. To address this issue, the $\text{sgn}(e)$ function in Eq. (71) can be approximated by

$$\text{sgmf}(e) = \frac{e}{|e| + a} \quad (81)$$

where $a > 0$ is a user-selected small constant. The function $\text{sgmf}(e)$ remains ± 1 for large magnitudes of e and only takes effects when e is near zero. By replacing $\text{sgn}(e)$ with $\text{sgmf}(e)$, the guidance command will be continuous and smooth when $e \rightarrow 0$.

B. ITCG and ITACG Laws for Varying-Speed Missiles

Although all the lateral guidance laws described in the preceding sections are developed under the assumption that the missile speed is constant, they can be readily extended to a realistic model with time-varying speed caused by drag and thrust. For a vertical guidance model, we need to include the lift and gravity in addition to the drag and thrust. However, for anti-ship missiles traveling in a horizontal plane with constant altitude, the effects of lift and gravity can be neglected. In this situation, the lateral acceleration A is generated by adjusting the sideslip angle. The dynamic model for the time-varying velocity V and the lateral acceleration are described as [4]

$$\dot{V} = \frac{F_T \cos \beta - F_D}{m} \quad (82a)$$

$$A = \frac{F_Y + F_T \sin \beta}{m} \quad (82b)$$

where F_T is the thrust, F_D is the drag force, F_Y is the lateral aerodynamic force, and β is the sideslip angle of the missile. In this paper, we consider a zero-thrust case without loss of generality because an active speed control is not required for our proposed guidance methods.

The aerodynamic models of F_Y and F_D are assumed as

$$F_Y = C_{Y_\beta} \beta q S \quad (83a)$$

$$F_D = \left[C_{D_0} + \kappa (C_{L_\alpha} \alpha_{cr})^2 + \kappa (C_{Y_\beta} \beta)^2 \right] q S \quad (83b)$$

where C_{D_0} is the zero-lift drag coefficient, C_{L_α} is the lift coefficient derivative, C_{Y_β} is the lateral-force coefficient derivative, α_{cr} is the trim angle-of-attack for sea-skimming without maneuver, S is the aerodynamic reference area, $q = 0.5\rho V^2$ is the dynamics pressure, and ρ is the air density.

The distance-to-go $D_{go} = D_f - D$, which describes the remaining flight distance of the missile, is introduced here to replace the time-to-go. Given that the missile velocity is tangential to the flight trajectory, the time derivative of D_{go} can be obtained as $\dot{D}_{go} = -V$. In fact, the guidance model in Eq. (1) is formulated in the time domain. According to the results in [22], this guidance model can also be expressed in the arc-length domain with respect to the flight distance D , as follows:

$$\frac{dR}{dD} = -\cos \sigma \quad (84a)$$

$$\frac{d\lambda}{dD} = -\frac{\sin \sigma}{R} \quad (84b)$$

$$\frac{d\gamma}{dD} = \frac{A}{V^2} \quad (84c)$$

$$\frac{d\sigma}{dD} = \frac{A}{V^2} + \frac{\sin \sigma}{R} \quad (84d)$$

Substituting the basic guidance law (6) into Eq. (84d), one has

$$\frac{d\sigma}{dD} = \frac{f(\sigma) \cos \sigma - 1}{R f'(\sigma)} \quad (85)$$

Since Eqs. (84a), (84b), and (85) do not depend on the missile speed V , the states R , λ , and σ with respect to D are independent of V . This implies that the trajectory shape of the missile is not affected by V [17, 32]. Thus, under the basic guidance law (6), the distance-to-go for constant-speed and varying-speed missiles is effectively the same, which can be calculated via Eq. (2), i.e., $D_{go}^B = R f(\sigma)$. By assuming a constant speed for the missile, we have shown the effectiveness of the basic guidance law (6) in the time domain using Eqs. (9)~(17). Actually, the basic guidance law (6) can also achieve zero miss distance and zero terminal command for varying-speed missiles. This conclusion can be easily demonstrated in the arc-length domain by following the analysis procedure in Eqs. (9)~(17). In this case, the primary task is to prove that R , σ , and A_M converge to zero as $D_{go} \rightarrow 0$.

Similarly, substituting the generalized IACG law (38) into Eq. (84d) yields

$$\frac{d\sigma}{dD} = (n+1) \frac{g(\omega) \cos \sigma - 1}{R g'(\omega)} + \frac{n \sin \sigma}{R} \quad (86)$$

which also indicates that the trajectory shape governed by the IACG law (38) is independent of V . Therefore, under the IACG law (38), the distance-to-go for varying-speed missiles is the same as that for constant speed missiles, i.e., $D_{go}^{IAC} = Rg(\omega)$. Then, by following the same analysis procedure in Eqs. (41)~(51), it can be proved that the IACG law (38) still ensures zero miss distance, desired impact angle, and zero terminal command under the missile speed variation.

In summary, Theorems 1 and 2 still hold for varying-speed missiles. The only difference is that the time-to-go is no longer an explicit function. Instead, the distance-to-go of the missile becomes explicit functions of σ or ω , as follows:

$$D_{go}^B = Rf(\sigma) \quad \text{or} \quad D_{go}^{IAC} = Rg(\omega) \quad (87)$$

In other words, the accurate length of the missile trajectory can be analytically obtained using Eq. (87). Due to $\dot{D}_{go} = -V$, the time-to-go can be calculated as

$$t_{go} = - \int_{D_{go}}^0 \frac{1}{V} dD_{go} \quad (88)$$

For constant-speed missiles, Eq. (88) reduces to $t_{go} = D_{go}/V$. However, for varying-speed missiles, solving Eq. (88) analytically is challenging because of the complicated dynamics of the missile speed. Thus, we need to use a numerical method to integrate the guidance model and estimate the time-to-go for varying-speed missiles.

Corollary 2. For missiles with time-varying speed, the proposed ITCG and ITACG laws (69) and (70), in which t_{go} is updated at each time step using a numerical integration method, can achieve the desired impact constraints on the target.

Proof. Similar to our previous analysis, substituting the generalized ITCG and ITACG laws (69) and (70) into the engagement model (84), we determine the derivative of D_{go} with respect to the flight distance D as

$$\frac{dD_{go}}{dD} = -1 + \frac{e_0^{1-p}}{\mu t_d(1-p)} \text{sgn}(e)|e|^p \quad (89)$$

According to the results in [22] and Eq. (88), we have $dD/dt = V$ and $dt_{go}/dD_{go} = 1/V$ and

$$\dot{t}_{go} = \frac{dt_{go}}{dD_{go}} \frac{dD_{go}}{dD} \frac{dD}{dt} = -1 + \frac{e_0^{1-p}}{\mu t_d(1-p)} \text{sgn}(e)|e|^p \quad (90)$$

which is identical to Eq. (73). Therefore, the convergence of the time-to-go error e is still the same as that in Eq. (76). This essentially ensures that our new ITCG and ITACG laws can achieve desired impact constraints and zero terminal commands, even when there is variation in missile speed. *This completes the proof.*

Remark 8 (Time-to-go estimation under time-varying speed). Because of the complicated nature of the missile speed variation, numerical iterative schemes are commonly used in recent studies for controlling the impact time [41]. These approaches may demand an optimization routine or an iterative process to determine the values of polynomial coefficients by trial and error, which increases the computational load. However, our time-to-go estimation method only requires a simple numerical integration over the distance-to-go D_{go} . Additionally, since D_{go} can be analytically calculated using Eq. (87), the proposed method provides accurate termination conditions for the integration, which is more tractable in practice. In this paper, we employ the Runge-Kutta method to integrate the guidance model and solve for t_{go} , with details presented in Sec. VI.C.

VI. Numerical Simulations

A few representative simulation cases are presented in this section to demonstrate the overall effectiveness of the proposed ITCG and ITACG laws. If not given specifically, the simulation conditions are the same as those in Sec. III.C. For all the cases, the control acceleration of the missile is limited by $|A|_{\max} = 100 \text{ m/s}^2$, and the simulation is terminated when $R < 1.0 \text{ m}$.

A. Numerical Validation of ITCG Laws (Constant-Speed Case)

The superiority of the two proposed ITCG laws (77) and (78) is validated herein, especially in comparison with two other similar guidance laws that were developed by using the command augmentation method. The parameters of the proposed ITCG laws are chosen as $k_1 = 0.1$, $k_2 = 0.8$, $p = 0.8$, $\mu = 0.9$, and $a = 0.01$. The missile speed is given as 250 m/s.

As mentioned earlier, one effective way to achieve the desired impact time is to incorporate a bias term with the conventional PNG law (30). Now, we compare the performance of the first ITCG law (77) with the well-known PN-based ITCG law. Note that the formulation of PN-based ITCG laws is not unique. For our comparison study, we utilize a guidance law recently introduced in [5], as follows:

$$A^{\text{PN-ITC}} = NV\dot{\lambda} + \frac{K(2N-1)V^2}{R\sigma t_{go,d}}e \quad (91)$$

where the parameters are selected as $N = 4$ and $K = 5$ for ensuring the zero terminal command. The approximate time-to-go solution for the conventional PNG law has been introduced in Sec. III.C. By choosing a desired impact time of 70 s and large initial heading errors, the simulation results of the first ITCG law (77) and the PN-based ITCG law in [5] are depicted in Fig. 6.

As we can see from Figs. 6a~6c, the proposed ITCG law (77) successfully attains the desired impact time for a wide range of initial heading errors, including the extreme case of $\sigma_0 = 180^\circ$. Its guidance command converges to zero at

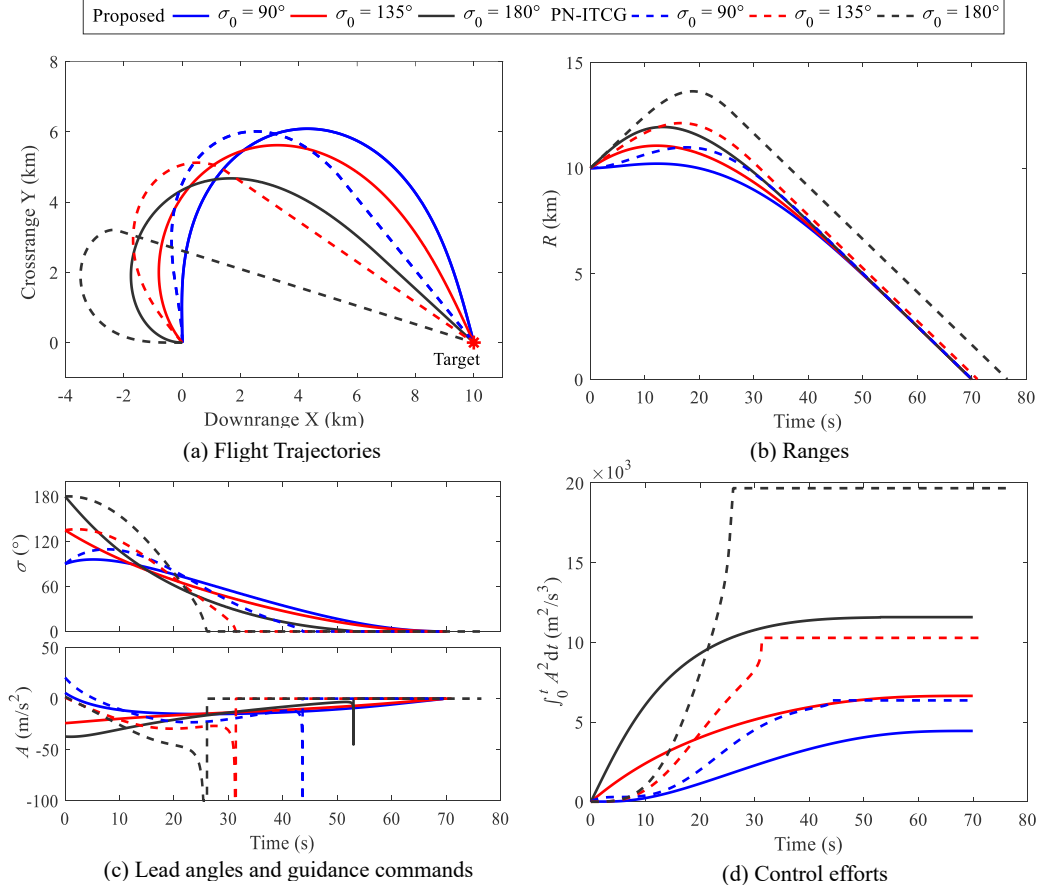


Fig. 6 Comparison of the first ITCG law (77) with the PN-based ITCG law in [5] under various σ_0 .

the final time, which aligns with the theoretical analysis in Sec. V.A. Furthermore, in Fig. 6d, relatively fewer control efforts can be observed for the proposed ITCG law (77). Nevertheless, the PN-based ITCG law leads to unnecessary detours in its flight trajectories and fails to satisfy the desired impact time when $\sigma_0 = 135^\circ$ and $\sigma_0 = 180^\circ$. In addition, as shown in Fig. 6c, the lead angle under the PN-based ITCG law converges to zero before the final collision, resulting in singularities in the guidance command. Note that under the condition of small initial heading errors, the PN-based ITCG law of [5] can achieve the desired impact time without any singularities. However, its performance is degraded when dealing with a large initial heading error or a long desired impact time. In such cases, the engagement geometry becomes highly nonlinear at the initial launch moment or during the guidance procedure, in which the time-to-go solution for the conventional PNG law is inaccurate.

Now, the performance of our second ITCG law (78) is compared with that of the varying-gain ITCG (VG-ITCG) law of [18]. Similar to our proposed method, in [18], the impact time control was also achieved for large heading errors by employing a bias term and an accurate time-to-go prediction. For a fixed initial heading error of 120° and different desired impact times, Fig. 7 presents the simulation results of these two guidance laws. As can be seen from Fig. 7, the flight trajectories, guidance commands, and control efforts of the two guidance laws are said to be similar when

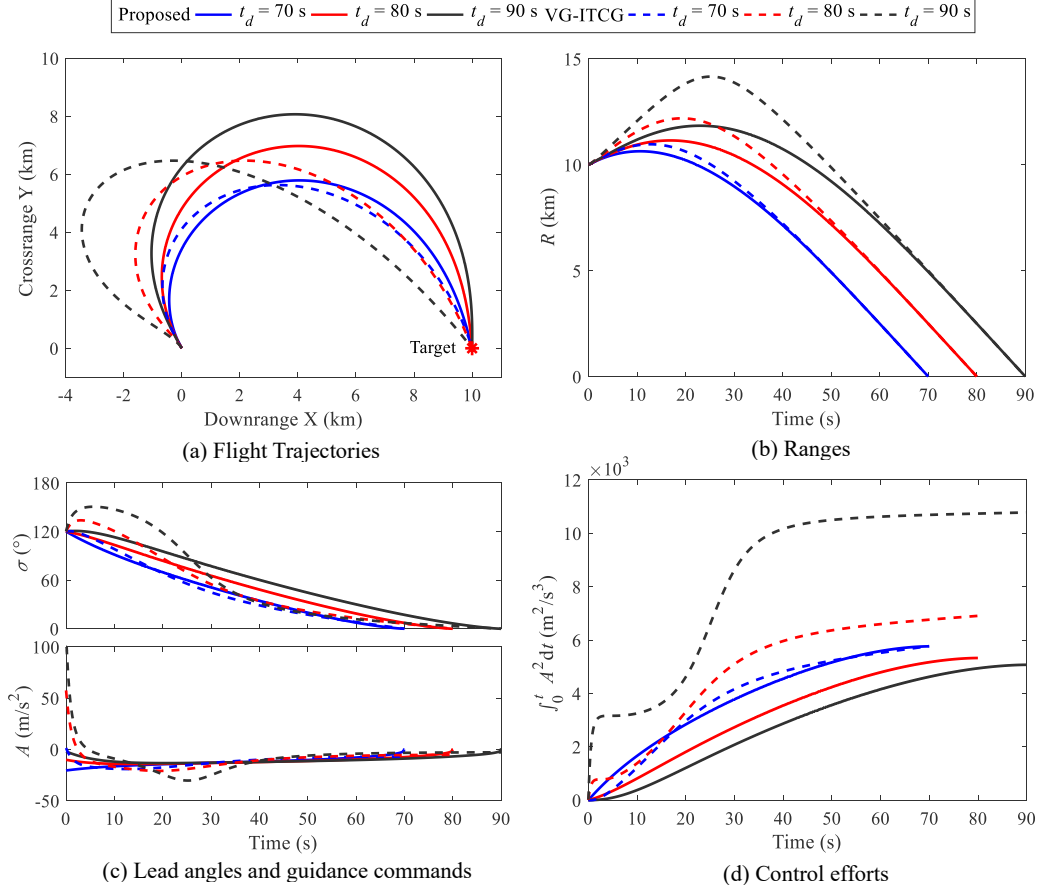


Fig. 7 Comparison of the second ITCG law (78) with the VG-ITCG law in [18] under various t_d .

$t_d = 70$ s. Nevertheless, when $t_d = 80$ s and 90 s, the VG-ITCG law produces a larger initial guidance command and forces the missile to follow a more curved trajectory, which results in more control efforts. Although the time-to-go can be precisely estimated under both of the two guidance laws, the proposed ITCG law (78) shows superior performance over the VG-ITCG law in [18] for large desired impact times.

B. Numerical Validation of ITACG Laws (Constant-Speed Case)

The effectiveness of the two proposed ITACG laws (79) and (80) is demonstrated herein in comparison with the ITACG law of [31] and the open-loop optimal solution for minimizing the control effort. The parameters of our new ITACG laws are set as $h_1 = 0.25$, $h_2 = 1.5$, $p = 0.8$, $\mu = 0.9$, and $a = 0.01$. The missile speed is still assumed as 250 m/s.

Until now, only a few studies have successfully achieved the desired impact time and angle using the command augmentation method. In [31], an impact-angle-constrained cooperative guidance law was developed by introducing a bias term into the optimal IACG law (66). This bias term, originally designed for cooperative guidance, can also be applied to a single missile to adjust the error between the desired and estimated time-to-go values. For the ITACG

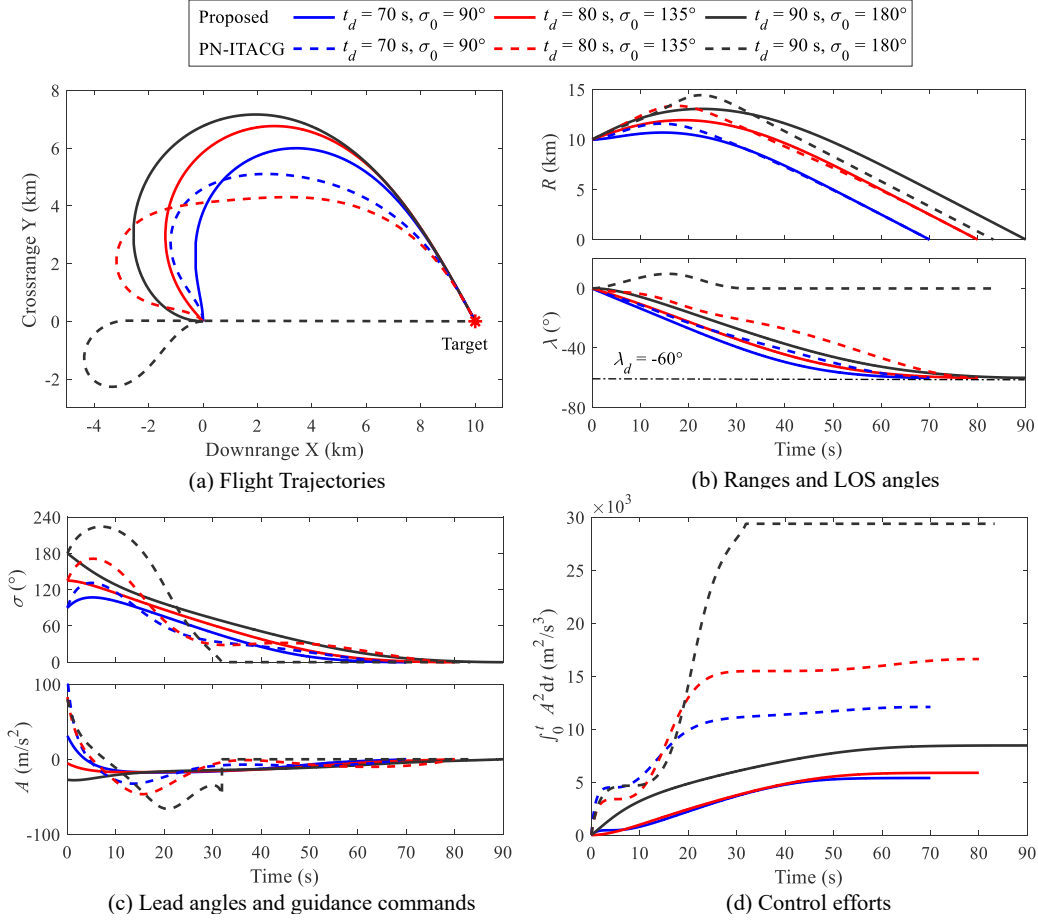


Fig. 8 Comparison of the first ITACG law (79) with the PN-based ITACG law in [31] under various σ_0 and t_d .

problem of a single missile, the advanced guidance law in [31] can be rewritten as

$$A^{\text{PN-ITAC}} = NV\dot{\lambda} + \frac{\alpha(N-1)V\dot{R}}{R}\epsilon + \frac{(2N-1)(N+\alpha)V^2}{NR\zeta t_{go}} [c_1|e|^{q_1}\text{sgn}(e) + c_2|e|^{q_2}\text{sgn}(e)] \quad (92)$$

where $\zeta = \sigma - \alpha(N-1)\epsilon/N(2K+1)$, and q_1, q_2, c_1 , and c_2 are guidance parameters. Note that the guidance law in [31] can also be considered as a PNG law with two bias terms for satisfying the desired impact time and impact angle, respectively. All the parameters for the PN-based ITACG law (92) are set as the same as those in [31] and the time-to-go estimation is calculated via Eq. (67), which is not an accurate solution.

In this simulation case, the desired impact angle is fixed to -60° , while the impact times and initial heading errors vary from -90° to -180° and 70 s to 90 s, respectively. Figure 8 compares the simulation results of the first ITACG law (79) with those of the PN-based ITACG law in [31]. One may observe from Figs. 8a~8c that the proposed ITACG law (79) achieves the desired impact constraints for all the cases, and its guidance command converges to zero at the impact moment. However, the PN-based ITACG law exhibits excessive regulating behaviors during the early phase and

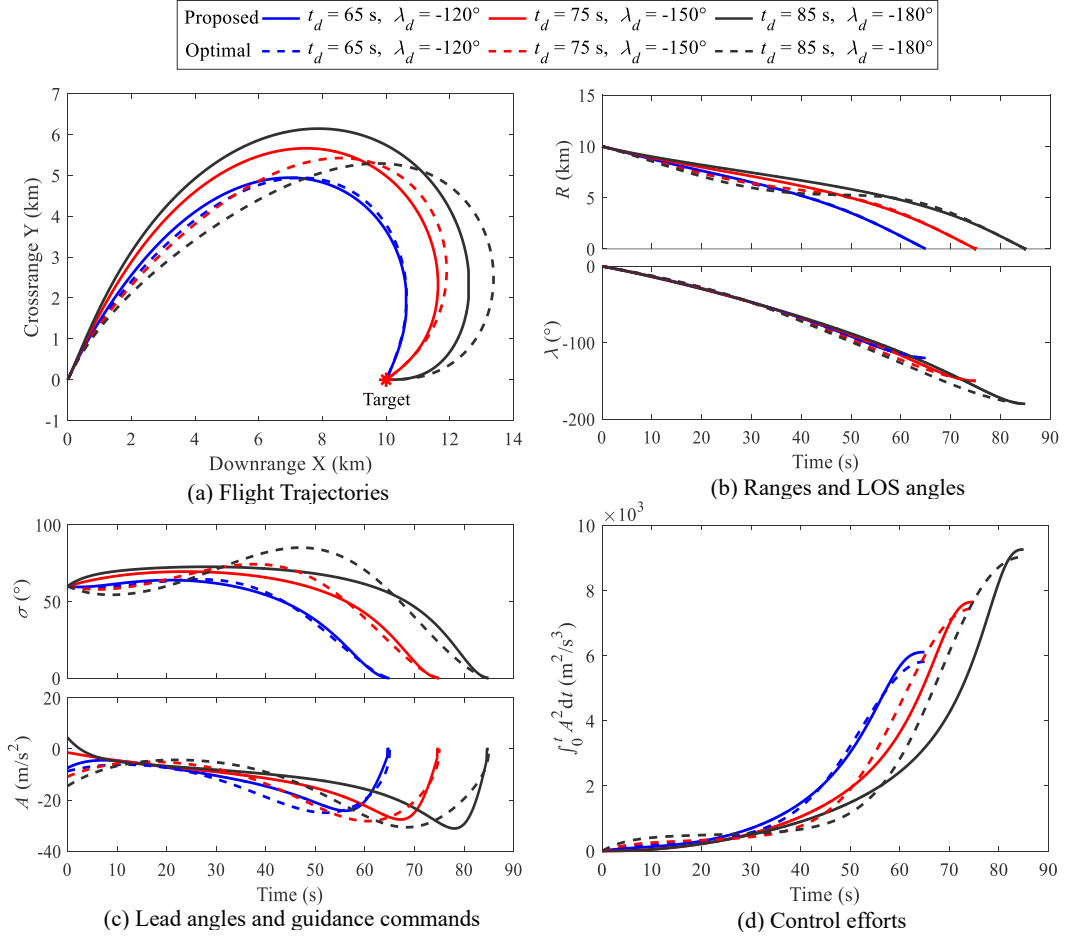


Fig. 9 Comparison of the second ITACG law (80) with the open-loop optimal solution under various λ_d and t_d .

results in mission failure under the extreme initial condition of $\sigma_0 = 180^\circ$. It should be emphasized that, when the initial heading error is small, the PN-based ITACG law can also meet the desired impact constraints with low control efforts [31]. Its performance degradation, as seen in Fig. 8, is caused by the inherent errors in its time-to-go estimation (67).

To further evaluate the applicability of the proposed method for large desired impact angles, the simulation results of the second ITACG law (80) are shown in Fig. 9, in which the initial heading error is fixed to 60° while the desired impact angles and impact times are selected from -120° to -180° and 65 s to 85 s, respectively. The open-loop optimal solutions, which are obtained via General Pseudospectral Optimal Control Software (known as GPOPS), are used as a benchmark in this group of simulations. These simulation results indicate that the proposed guidance law can be applied to a wide range of desired impact angles, including the extreme scenario with $\lambda_d = -180^\circ$. The control efforts of the proposed ITACG law are always close to that of the optimal solutions from GPOPS. Even though the optimization process is not integrated into our guidance design, the control efforts exerted by the proposed guidance laws remain advantageous in practical implementation. Furthermore, under various initial heading errors and desired impact times, the achievable impact-angle regions of the ITACG laws (79) and (80) are presented in Table 1. To the best

of our knowledge, the two proposed ITACG laws offer wider impact regions compared to most existing guidance laws, especially when the initial heading error is large.

Table 1 Achievable impact-angle regions of the two proposed ITACG laws (79) and (80) under various σ_0 and t_d .

$\sigma_0, ^\circ$	A_1^{ITAC} in Eq. (79)			A_2^{ITAC} in Eq. (80)		
	$t_d = 50$ s	$t_d = 70$ s	$t_d = 90$ s	$t_d = 50$ s	$t_d = 70$ s	$t_d = 90$ s
60	$[-27.1^\circ, -88.7^\circ]$	$[-29.4^\circ, -160.7^\circ]$	$[-31.4^\circ, -180.0^\circ]$	$[-24.8^\circ, -91.9^\circ]$	$[-26.7^\circ, -179.2^\circ]$	$[-33.3^\circ, -180.0^\circ]$
120	$[-23.2^\circ, -82.3^\circ]$	$[-26.8^\circ, -157.8^\circ]$	$[-30.1^\circ, -180.0^\circ]$	$[-21.9^\circ, -85.0^\circ]$	$[-24.5^\circ, -175.2^\circ]$	$[-31.8^\circ, -180.0^\circ]$
180	$[-16.3^\circ, -67.0^\circ]$	$[-22.4^\circ, -148.3^\circ]$	$[-28.0^\circ, -180.0^\circ]$	$[-15.8^\circ, -72.2^\circ]$	$[-18.9^\circ, -167.7^\circ]$	$[-30.3^\circ, -180.0^\circ]$

C. Cases with Missile Speed Variation

The proposed methods are now applied to a practical scenario where the missile speed varies subject to Eq. (82). The thrust F_T is maintained at 1.5 kN for the first 15 s, and is kept as zero thereafter. The initial mass is 400 kg with a constant fuel consumption rate 2.0 kg/s. The missile speed at the launch moment is set as 300 m/s. Without loss of generality, the dynamic model for the missile speed is parameterized with $S = 0.091 \text{ m}^2$, $\kappa = 0.06$, $C_{D0} = 0.1$, and $C_{L\beta} = C_{L\alpha} = 20$ [4]. To evaluate the robustness of our new guidance laws, the onboard autopilot is modeled as a first-order lag with a time constant of 0.3 s, and Gaussian noise with zero mean and standard deviation of 0.05 deg is added to the measurements of LOS and lead angles.

The time-to-go of the missile now needs to be estimated via numerical integration methods rather than relying on analytical solutions. In this paper, we present the application of the fixed-step Runge-Kutta method ("ode4" in MATLAB) to solve for t_{go} . The step size of integration is chosen as 0.05 s. The integration is terminated once the error between the distance-to-go in Eq. (87) and the actual flight distance is smaller than 1.0 m. The flight time recorded at the moment of integration stop is used as the estimated t_{go} . For different simulation cases in Figs. 10 and 11, this integration process is always stable during the whole guidance procedure, taking an average computational time of 5 ms (obtained via MATLAB R2022a on a PC with an Intel Core i7-8750H CPU @ 2.60 GHz and 16 GB RAM). By utilizing the fixed-step Runge-Kutta method, the time-to-go estimation error is always less than 0.06 s for all simulation runs shown in Figs. 10 and 11, which completely meets the requirements for practical applications. These simulation results highlight the effectiveness of our proposed guidance framework for varying-speed missiles, as it enables accurate estimation of t_{go} while significantly reducing the computational load for onboard implementation. Furthermore, one can further adjust the step size of the fixed-step Runge-Kutta method to balance the integration accuracy and computation efficiency. Other numerical methods, such as modified Euler method and variable-step Runge-Kutta method, can also be utilized to calculate t_{go} . The properties of different numerical integration methods are not discussed in this paper due to the space limitation.

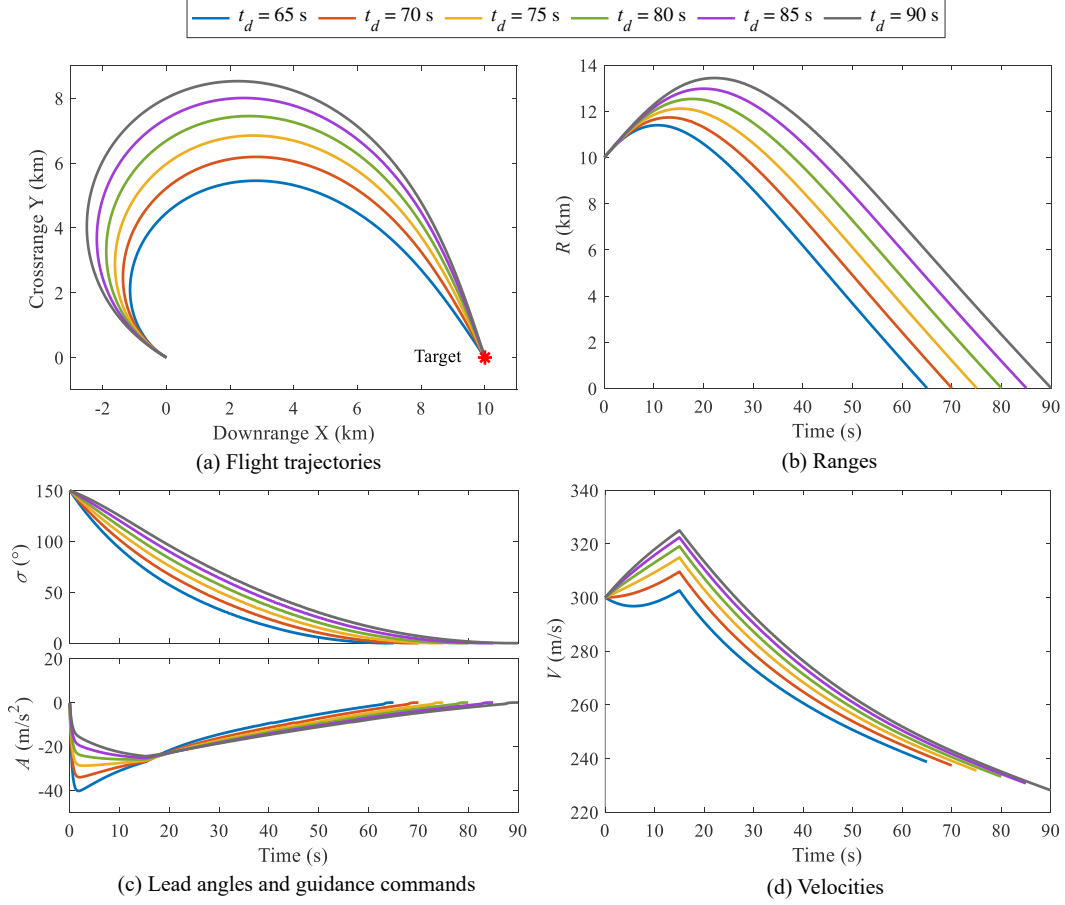


Fig. 10 Simulation results of the first ITCG law (77) under missile speed variation.

The simulation results of the ITCG law (77) and the ITACG law (79) for various desired impact constraints are presented in Figs. 9 and 10. As can be seen from these simulation results, the proposed ITCG and ITACG laws successfully meet the desired impact constraints under a practical scenario with autopilot lag, measurement noise, and variable missile speed. Besides, zero terminal lead angle and guidance command can be noticed for the proposed guidance laws, which is consistent with the theoretical analysis in Sec. V.B.

Conclusions

This paper has developed a generalized guidance framework based on an inversion design approach. Unlike all existing studies, our new framework allows for the derivation of various guidance laws and their globally precise time-to-go solutions without relying on linearizing approximations and numerical iterations. The challenging task of accurately estimating the time-to-go for homing missiles has been effectively addressed in this paper, even with the impact-angle constraint or large initial heading errors. Furthermore, the proposed method has been augmented to develop several ITCG and ITACG laws that are applicable to both constant-speed and varying-speed missiles. Numerical

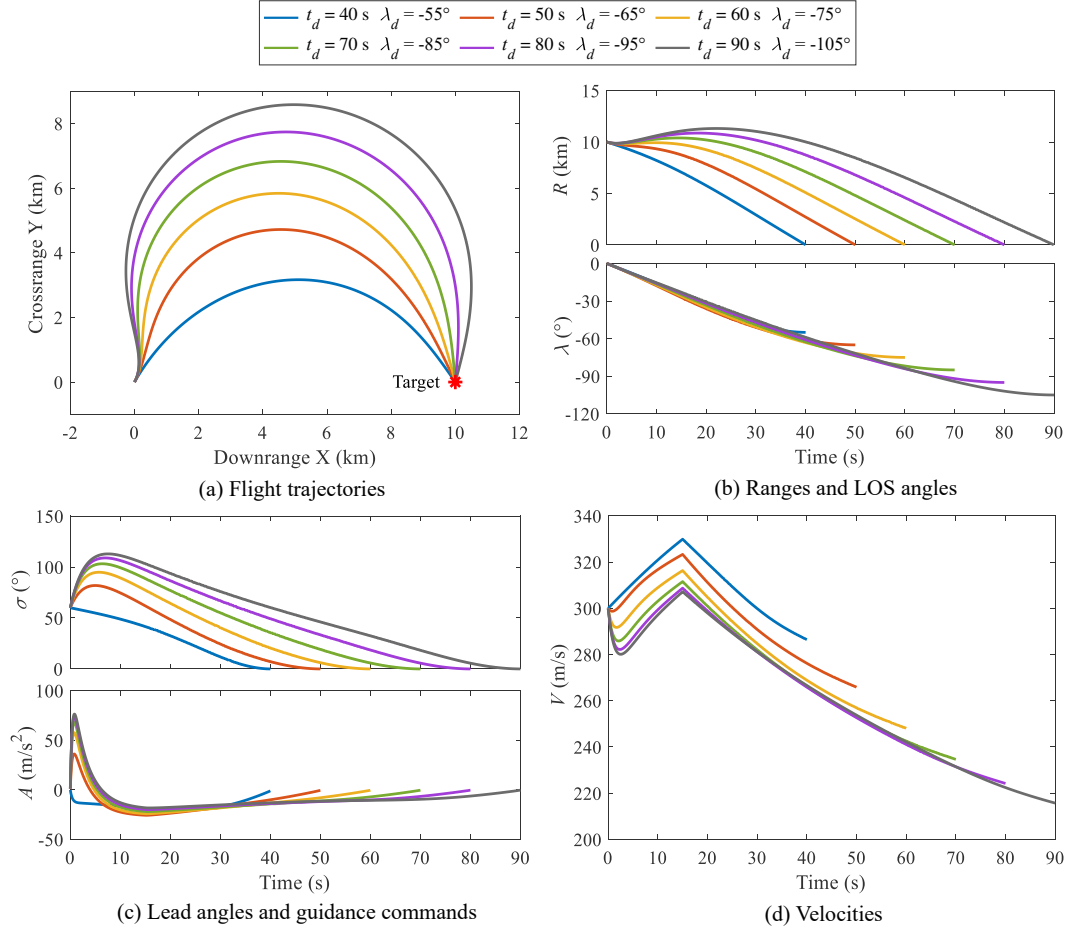


Fig. 11 Simulation results of the first ITACG law (79) under missile speed variation.

comparative study results have demonstrated the robustness, impact-time accuracy, and control effort efficiency of these new ITCG and ITACG laws.

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