

A RECIPROCAL FUNCTION SOLUTION OF

LAPLACE'S EQUATION IN TWO DIMENSIONS :

WITH AN APPLICATION TO AN AEROFOIL SECTION.

By

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1. Introduction.

A general solution of Laplace's Equation, has been given §, by Professor Bairdow, as the solution of an integral equation. The form of this solution indicates that, for a particular boundary, successive integrations of the boundary values of the harmonic function ψ have to be performed. This requires, in a particular example, an initial knowledge of these boundary values, and the subsequent procedure has been illustrated analytically for the case of the torsion of an elliptic cylinder, § and graphically for boundaries of more arbitrary shape Δ .

The solution can be obtained however in another form, such that the knowledge of the boundary values of ψ is not required, until the final stage of the operations involved in solving a particular case. A function depending only on the shape of the boundary is developed, which can finally be combined with the boundary values, appropriate to any problem involving this boundary to give the corresponding solution.

The development of this "solving function" and the method of its application is dealt with in the present paper.

§ Bairdow and Berry, "Two dimensional solutions of Poisson's and Laplace's Equations". Proceedings of the Royal Society, A. Vol. 95. 1919 pp. 457 et seq.

Δ Bairdow and Pippard, "The Determination of Torsional Stresses in a shaft of any Cross Section". Proceedings of the Institution of Civil Engineers, Vol. CCXIV. Session 1921-1922 Part II.

Kirdany "Calculation of the Two dimensional motion of an inviscid fluid round an ordinary aerofoil on the assumption of the presence of circulation round the wing". Thesis for Ph.D degree in the University of London.

2. Nomenclature.

Successive or iterated integrals play a major part in the theory to be given, and as a guide to the order and manner of the integrations, a system of descriptive suffixes is adopted.

Consider a radius vector rotating counterclockwise at a point (Fig 1.) from an arbitrary known datum line. When it passes through a point, the angle it has turned through is described as θ , which is an angle measured from θ_0 to θ . For a radius vector centred at O , the angular position of P is defined by θ , which is an angle measured from θ_0 to θ .

If the data lines at θ_0 and θ are parallel and in the same direction, then when θ_0 and θ both have their smallest values

=

and in general

= where n is integral.

Hence

=

When θ_0 and θ are points on a closed curve, θ is a chord whose angle with the datum line at θ_0 gives θ . When θ is extremely close to θ_0 (Fig 2.), θ becomes a tangent to the curve at θ_0 , and the limiting value so obtained for the angle θ is described as θ_0 . If θ traverses the closed curve from this limiting position until again at θ_0 , the second limiting value of the angle is given by θ_0 .

Sometimes, when both θ_0 and θ (for example) are allowed to vary in position, it is desirable for greater clearness to write a differential

which expresses the differential at θ_0 of the angle θ .

The work which follows is mainly concerned with solutions of Laplace's equation for a region bounded by a single finite closed curve and extending to infinity in all directions. The closed curve is referred to hereafter as 'the boundary'.

A varying angle that is successively integrated times over this boundary is written. It is an angle that defines the position relative to a fixed point, of a point which traverses the whole boundary. If is integrated over the complete circuit with respect to an angle measured, as varies, from a fixed point, the result must depend only on the positions of and, or, more shortly, must be a function of and. If in turn, is considered as moving over the boundary, this function of, and becomes a variable which when integrated with respect to some angle, gives a function of, and only. Continuing the successive integration in this way, the () integration in this way, the () integration is performed with respect to an angle giving a function of, and, and finally, supposing, to vary, the integration is performed at giving a function of, and only, which varies as varies in position.

The process of iterated integration just described is expressed

and the suffixes, of the integrand and differentials exhibit the fact that the result is a function of, and only in that they all occur in pairs except, and.

When the stream-function at the boundary is integrated successively times it appears under the integral signs as. The result of the integrations, the last of which is at, is necessarily a function of, only.

6

and the shape of the boundary - a constant.

5



3. Solution of Laplace's Equation. §

Bairdow considers the equation

with the boundary values of either or given.
It can be solved in terms of simple sources on the boundary
if the motion is acyclic.

To find the distribution of the sources round the boundary,
the effect at (Fig.3.), a point either without or within
the closed boundary, of a number of elementary sources is
considered.

At , the elementary source , adds to the Stream
function at an amount

Hence

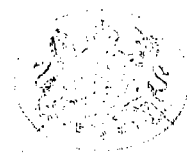
It often happens that the boundary value of is
unknown to the extent of a constant : equation (2) is therefore
written

This equation satisfies the necessary condition that
shall vanish when is at infinity, providing that
is continuous and acyclic.

Following Bairdow § and interchanging the variables,
it follows that

and also that for at , (Fig 4.) infinitely close
to the boundary, integrating contraclockwise from the
tangent at , (and remembering that)

§ See p.1. footnote (d)



the alternative positive and negative signs being respectively appropriate according as ϕ is the limiting position of a point in the field external or internal to the boundary.

The integral equation (4) is solved by Liouville's ^{*} method of successive substitutions, for

and similarly, considering ϕ as fixed, while ψ traverses the boundary

Likewise

Substituting

the solution develops as

(7) $\dagger$

For ϕ to be finite when ψ signs are taken, and single valued when ϕ signs are taken, it is necessary that

^{*} Cambridge Tracts No. 10 (see footnote (g) p. 14.) p. 14.

$\dagger$ This is the same as equations (9) and (10) on p. 460 of Roy. Soc. Proceedings given in footnote (g) p. 1.

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4. Reciprocal Form of Solution.

Consider the general term

of the series for (Art.3. eqn (7))

It implies that the boundary values () of the stream-function are integrated with respect to at all points in turn, and that the resulting continuous function is integrated with respect to , at all points in turn, and so on, until the successive integration is performed at , to give a value depending on the position of

This value may, however, be differently obtained.

Consider the contribution to the multiple integral (1) of an element. of the boundary located at ϕ (Fig 5.). The element is supposed sufficiently small for to be considered constant over it.

The contribution to the first integrations, performed at the points , is

which varies as , varies in position on the boundary.

The second integrations, at the points , show the contribution of the element to the second integral to be

The contribution to the third integral is

9
The replacing of by is a step to preserve the conventions set out in the last paragraph of Art.2.

8

Finally, it is seen that the contribution to the integral is

where the multiple integral is a function of , only.

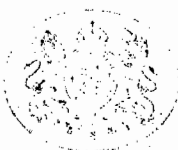
When the complete boundary is built up by considering all contributory elements such as simultaneously, the effect at , is

So when the elements are infinitely small

where

continuous in the range

Hence



This reciprocal relationship may be compared with
Volterra's Reciprocal Functions. ⁹

It is necessary that

(6) ⁷

⁹ Cambridge Tracts No. 10 (see footnote (5) p..) p.21.

⁷ Art.3. eqn.(8).

5. The Solving Function.

(a) From now onwards, the external field ^{only} will be considered.

$$\frac{1}{2} \mathcal{S}_{PQ} = \mathcal{O}_{PQ} + \frac{1}{\pi} \int_{\mathcal{O}_P}^{\pi + \mathcal{O}_P} \mathcal{O}_{Q,Q} \cdot d\mathcal{O}_{PQ} + \frac{1}{\pi} \int_{\mathcal{O}_P}^{\pi + \mathcal{O}_P} \mathcal{O}_{Q_2Q} \cdot d\mathcal{O}_{Q_2Q} \cdot d\mathcal{O}_{PQ} + \dots \quad (1)$$

is a solving function of the type desired, in that it depends only on the shape of the boundary.

In the original solution (equation (7) Art. 3), the increments $d\mathcal{O}$ are continuous during the integrations, so the angles $\mathcal{O}$ in (1) must be regarded as continuous in the ranges indicated by the limits. This means that the natural discontinuity which an angle $\mathcal{O}_{Q,Q}$ has at Q (See Fig. 6) must be made ineffective by an addition of π to the values of $\mathcal{O}_{Q,Q}$ occurring after Q in the course of the integrations. The result is that $\int \mathcal{O}_{Q,Q} d\mathcal{O}_{PQ}$, considered as a variable over the boundary with Q fixed, is a function discontinuous by π at Q . Continuing to preserve the continuity during integration, all the successive terms in (1) develop as functions discontinuous by π at Q .

The resulting series is divergent δ ; this and the cyclic nature of its terms make it unsuitable for graphical work. An equivalent convergent series can be deduced, however, from which this cyclicity is absent; its integral terms are continuous over the boundary, and so the preservation of continuity by additions of π during integrations, which is a hindrance from a graphical view point, is not required.

If functions of P_1 only are added to, or subtracted from the solving function as given by (1), the value of the right hand side of equation (5c) Art 4, is not affected, as these functions of P_1

δ of. later, example of Art. 7.



give zero when operated on by
replaced by a solving function

may be

and it is shown later that it is possible to find
values of such that the
terms, which form a double series, tend to two functions
of and . Calling these limiting values

(5)

By equation (6) Art. 4.

So in equation (5c), Art 4., may be replaced by

the terms of which converge absolutely to zero.

9

The fact that, for fixed, the integrals do not tend to
a constant for all points on the boundary, is worthy of
special notice.

The series in equation (5) is of no more use for graphical computation than that in equation (1), for the values of the and terms cannot be pre-determined; but it is possible to express as a fully determinate convergent series of iterated integrals whose basis is the angle considered 'discontinuously'. A 'discontinuous' integration is written

and is performed straightforwardly, allowing the discontinuity at to remain as such. The difference between the 'continuous' and discontinuous' methods of integration is illustrated in Fig. 7.

The integral (6) is a continuous acyclic function, and the result of successively integrating it will therefore be a continuous acyclic function.

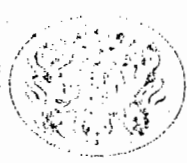
Consider the general term of the series for
(equation (1))

From Figs 6. & 7. it is clear that

Hence

and so on

Then the successive terms may be written



Putting

etc.

it follows that

If

then from equations (3), (8), and (9).

By equation (5) therefore,

all three of which series are absolutely convergent

(C) The most convenient of these for graphical work is (12b) but it is improved if the angles measured from points to points using constant direction datum lines, are replaced by angles measured to points from points, using datum lines which are tangents to the boundary curve at the points.

Referring to Fig 6.

So (12b) becomes

In practice, the integrals etc., are successively determined until graphical constancy representing (16) above is attained. This value is then subtracted from each of the evaluated integrals to give the series of equation (15)

is a constant for the boundary for fixed. footnote to p. Convergence to a function of P , and ϕ would have been a drawback, graphically.



(d) The Constants. C is given by equation (9) Art 5. by equation (3) Art 5a; by equations (10) and (11), Art 5b; by equations (15) and (16), Art 5c.

From equations (4) Art 5a, it follows that

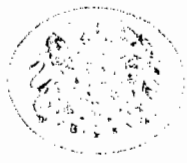
From equation (9) Art 3, and by the arguments of Art 4.

Hence, from (18) and (20)

Equations corresponding to (18) (2) and (21) follow similarly for

From equations (20) above, (18) (Art 5(b)), and (15) Art 5. (c).

and



6. It is proposed to work two examples illustrating the use of the theorem stated by equation 5, Art. 4.

The first (Art. 7.) consists of the mathematical development of the solving function for the field external to an Ellipse, and the combination of this function with the boundary values of (equation (5) Art. 4.) to give $\frac{1}{2} X$ over the boundary. It is then possible, if desired, to obtain for any position of P in the field by means of equation 3, ϕ Art. 3.

The second example (Art. 9) illustrates the procedure to be followed in cases where analytical methods are impossible or unsuitable. It consists of the graphical development of the solving function $\frac{1}{2} X$, in the form given by equation (16) Art. 5 c. for the field external to a boundary of Joukowski aerofoil shape; of the combination of $\frac{1}{2} X$ with the boundary values (equation (5) Art. 4) to obtain $\frac{1}{2} X$; and of the working of the right hand side of equation (4), Art. 3. as a check.

7. Solution For the Ellipse.

(a) The ellipse considered (Fig. 8) has its centre at O , and major and minor axes $2a$ and $2b$. The positions of the points P , are given by the eccentric angles θ , of the points Q by the eccentric angles ϕ , and of the points R by the eccentric angles ψ ; being measured from the positive direction of the x - axis.

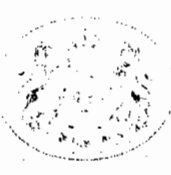
The ratio $\frac{a}{b}$ is called K_1 , and is the first of a series of terms, $\frac{a}{b} K_1, K_2, K_3, \dots$ which arise, in which

- - - - - (1)

It can easily be shown that

and that

ϕ Putting P for Q .



(b) Considering the original form (equation (1) Art. 5a.) obtained for the solving function, the first step is the evaluation of

where and are continuous in the range

From the geometry of Fig. 8,

------(2)

------(3)

and these angles are respectively continuous in the ranges and

Now the integration is performed over the latter range, so at it is necessary to introduce the term to neutralize the discontinuity which occurs in as Q passes through Q .

Also, to avoid any ambiguities in the course of the integration, it is advisable to write the angle as

------(4)

so that the inverse function($-X$) concerned may always represent an angle numerically less than , being then unique and continuous in the range

On these premises

-X- which is



From (3)

Referring to
Appendix I

----- (5)

where the inverse functions are cyclic

(c) The second integral of the series
has now to be evaluated.

It may be written

----- (6)

and

----- (7) ϕ

----- (7a)

Integrating this function as required by (6), the
preservation of the continuity at ϕ is again necessary;
the inverse term is however,

already continuous and single valued in the range

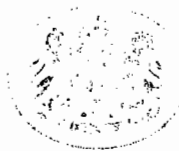
Hence, by (6) and (7a)

ϕ

is a function of P and ϕ ,

is a

function of ϕ and ϕ . The latter is obtained by
replacing P by ϕ in the former, that is, by replacing
by in (6)



Referring to
Appendix I

- - - - (8)

(d) Proceeding to the higher order integrals, the procedure is alternately that of sections (b) and (c) of this article, and

and so on.

All the terms not containing are functions of P only, so if Φ as given by equation (1), Art. 5. is written down, it can be immediately used as the solving function, the terms causing the divergence disappearing under the operator d_0 . It is more compact, however, to put

etc.



and

etc.

Then

etc.

So, (equation (2) Art. 5.)

- - - - - (9)

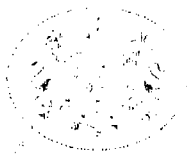
There is no advantage to be gained by subtracting
and from the terms in the manner

of equation (5) Art. 5. to give $\frac{1}{2}$; the form $\frac{1}{2}$
is the most suitable in the present example, and the
final integration $\oint$ may be immediately performed

- - - - - (10)

- - - - - (11)

$\oint$ of equation (5) Art. 4.



(11) is a series which converges to zero (see equations (4) Art 5a.)

(a) Now

--- (12)

The constant C may be obtained by equations (18) or (20) Art. 5 Using (19)

C = _____

----- (13)

So

----- (14) ρ

Equation (11) may be written, therefore,

----- (15)

By Appendix II

----- (16)

where

and hence

So, by equation 5c, Art. 4, putting

----- (17)

or

----- (18)

or

----- (19).X.X

ρ which of course satisfies equations (4) Art. 5a.
.X. Reference to p.473 of the Royal Society proceedings cited on p. 1 shows that this solution is immediately obtained from the summation of the external field. corresponding to
H of. Appendix III.

3. Mechanical Integration.

(a) The Apparatus.

The successive integrations are performed by means of the wheel-arm of a Coradi planimeter, assisted by the simple apparatus shown in Fig. 9 ^p and Fig. 10. This may be described as a frame, and consists of four pieces. Referring to Fig. 9, A is of transparent celluloid and carries a base line u, scored on the side next the paper; on this line is the needle-hole v, its position is, within limits, arbitrary, but most convenient in the position shown in the figure. B and C are celluloid strips between which the back-wheel of the planimeter is to run with a minimum of tolerance. To provide against variation in the width of the slot between the strips due to warping or slight inaccuracies in the making, they are secured to the base plate A by small nuts and bolts which pass through elongated holes (Fig. 9 (b)), allowing for minor adjustment when required. D is a millimetre scale on white celluloid, 24 cms long, and marked from -12 to +12 at centimetre intervals. The base line of the scale is a narrow groove which takes the tracing point of the planimeter, and at every millimetre it is provided with a minute cavity made with a needle, which is sufficient to hold the tracing point at the desired positions on the scale during movement of the system.

When in use, the planimeter arm rests with its back-wheel in the B C groove, its tracing point in the scale groove, and its recording wheel on the paper; the system is capable of rotating about a short needle .X. passing through the hole v.

-
- ^p The dimensions in Fig. 9, are those of the specimen actually used in the graphical work to be described. The instrument can be made to other dimensions, but those given are suitable for the usual size of planimeter (wheel arm 9" long.)
 - ^H The choice of unit is arbitrary, but the centimetre is the most convenient for the size of frame illustrated. The length of the scale is also a matter of choice; it should be as long as possible, and equal to the length of the B C slot.
 - ^X The needle must be short in order to clear the planimeter. About 1 cm. cut from an ordinary sewing needle is suitable.

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(b) The Method.

(1). The scale of the planimeter is adjusted so that when its point is at zero of the scale D, rotation causes no change in the planimeter reading.

(2). Before the apparatus is used for integration, the tracing point should be run the whole length of the scale with the frame stationary, and if, during the run, the reading of the planimeter changes, the strips B and C need adjustment. It is sometimes difficult, however, to obtain perfect and permanent constancy over this range, so the tracing point should always be moved to zero before the planimeter is read, to neutralize any error due to the two grooves not being truly parallel.

(3). To find the multiplier m with which to convert planimeter results to the values of the integrals they represent, the tracing point is placed on zero, a reading taken, and then moved to the point representing unity to the scale being used at the time. If the 10 cm. mark represents 1.0, the tracing point is moved to 10, and the system is rotated about v through . The tracing point is moved back to zero, and the reading taken. If the difference of the two readings is y , and if (as in work of the kind dealt with in this paper) $\frac{1}{2}$ of the integrals to be found will be required, m is obtained from

that is,

Both planimeter and celluloid scale are a little affected by climatic variations, so adjustments to preserve the zero and the parallelism of the grooves are sometimes necessary when the work extends over several days. When such adjustments are made, the value of the constant m needs revision in consequence.

(4). The procedure for integration over a closed boundary is as follows.

Points on the boundary are selected, if possible so that the variation of the function of their positions to be integrated may be considered linear ⁸ over the intervals between them, without affecting the order of accuracy possible. Tangents to the boundary curve are drawn at these points, which are recorded, with their positions on the boundary, and the corresponding boundary values of the function in the case, in the manner shown in Table I, Cols. A, B, I. The mean values of the function over the intervals are next written as in Col. IIa. of this table; they are means of adjacent values in Col. I.

This is convenient but not essential. The object is to be able to write the mean for the interval as the mean of the values at either end of it. If, for any reason, this degree of linearity is unattainable, the required mean for the interval must be otherwise obtained - by estimating from a curve for instance.



The preliminaries (1), (2) and (3) having been observed, the process of integration with respect to each of the points 0, 1, 2,15 ^p can be performed. As an example, consider the integration with respect to the point 6 (Col. A) in detail.

The frame is pivoted with v at the point 6 on the boundary, the base line u lying along the tangent at 6 (See Fig. 12.) It is intended always to rotate the frame and planimeter contra-clockwise ^X, so in this initial position, it is most convenient that the zero of the scale D be to the right of an observer supposed standing at 6, and looking normally into the bounded area. The planimeter is placed in position with its point at zero, and the reading noted. The point is then moved to the value .65, the mean for the interval 6-7, and, retaining it in this position, the system is rotated until the base line U, lies through the point 7. The tracing point is then moved to the value .75, and the system is rotated until u lies through 8; this procedure is continued until, with the point at .55, the base line once more lies on the tangent at 6, having turned through ^X. The point is brought to zero and the reading taken. The result of subtracting the second reading from the first is then written in Col. II b. This figure, multiplied by m is recorded in Col. II c, and is the integral multiplied by

If the column is completed by repeating the process with v at the points 0, 1,6,7,.....15, and if it is desired to treat Col. IIc, as Col. I has been treated, it is safe to assume that the approximately linear variation over the intervals 0-1, 1-2,.....which was true for Col. 1, is true for Col. IIc: for the change over any interval in Col. IIc, is always smaller than the corresponding change in Col. I.

^p or any other point on the boundary.

^X, which is consistent with the conventions of Art. 2 -7.



9. Solution for a Joukowski Aerofoil Section.

(a) The boundary chosen for exhibiting the graphical method of obtaining the solving function is the section of an aerofoil of the Joukowski type. At the time when the work was started, it was felt that the result for an aerofoil based on the Joukowski transformation

------(1)

was likely to be of more interest than the solution for an aerofoil of more arbitrary shape, but the choice was in one respect unfortunate, as the fine tail, ending in a cusp is unsatisfactory to deal with graphically in the same manner as the rest of the boundary. *H*

The actual section used was the one resulting from the transformation of a circle of 6.63 cms radius (P.19.11) the angle between the first and second axes being $5^{\circ} 43'$ and the origin of co-ordinates in the plane being 6 cms along the second axis from the point of intersection of the axes with the circumference.

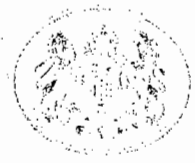
It is necessary in the course of the work to draw the aerofoil shape many times, so, rather than perform the usual geometrical construction *¶* on each occasion, a template was cut to the shape of the section in thin celluloid, with the second axis scribed on its under side as a line of reference.

The perimeter is 49.75 cms long, and a scale is supposed to traverse it contraclockwisely from zero at O (Fig. 11) round to 49.75 at O again. Positions of points on the boundary are given in terms of this scale.

(b). For one position of Q , the values of $\frac{1}{z}$ (equation 17, Art. 5c.) which are to be found at various points P_1 on the boundary from a continuous curve; positions for Q are chosen as far as possible in the way that best assists the drawing of curves to show the variation of $\frac{1}{z}$ for any one position of P_1 when Q varies in position. All the reasons leading to the selection of the various positions for Q in this example are not given here. In any particular example the desirability of certain choices makes itself apparent as the work progresses.

¶ See M. Glauert, Aeronautical Journal, July 1923 p, 348 et. seq. Geometrical construction p. 364.

H Professor Baird has suggested that such cusp-ended tails will be more conveniently dealt with by means of series evolved by considering doublets, (double sources) or even double-doublets distributed on the boundary, rather than by means of the simple source distribution which is the basis of the present work (Art. 2.)



In the Joukowski section, the tail point and a point on the nose are obvious choices. The next, however, is arbitrary. Let it be at the 27.22 cm. mark. The procedure is then as follows for q at this point.

At q , a tangent to the boundary curve is drawn. At suitable intervals (Fig. 12) angles are drawn, producing lines radiating from q and marking the positions of points 0, 1, 2.....15. Tangents are drawn at these points; their positions on the boundary, and their angular values expressed as multiples (h) of , are given by Cols. A, B, I, of table I.

The first step is to evaluate It may be written for the moment

It is convenient to express these integrals as multiples of , so the operation to be performed is

which has the dimensions of h. The next successive integral is given by and so on. With

this arrangement, the 10 cm mark on the scale D of the frame conveniently represents unity, so the constant m is given directly by the equation (1) Art. 8 b.

Col. IIA of table I, contains the main values of for the intervals between adjacent points, and, following the directions of Art. 8b, an integration over the boundary is performed with respect to each of the points 0, 1.....15 ϕ in turn. The planimeter results are in Col. IIB, and the values of are in

Col. IIC: these are multiples (h) of and are plotted in Fig. 13 on the perimeter ('S' from Col. B.)

Next, in Col. IIIa, means of values in the preceding column are written down, and these are integrated at each of the points P, - 0, 1, 2.....15, to produce Col. IIIc which contains the values of these

values (h) also are plotted against S in Fig. 13.

ϕ If the integration is with respect to ϕ , say, then ϕ is P, while 0, 1.....5, 7.....15 are positions of q . A note is given in appendix IV (a) pointing out the slightly different procedure when P, is at 0.

Proceeding similarly, the 3-fold, 4-fold, and subsequent integrals are evaluated and, being plotted (Fig. 13) show the convergence to a constant. The successive integration is continued until this constant line is reached: theoretically, an infinite number of integrations is required for this, but a line sufficiently closely approximating to the straight line is arrived at generally before twenty sets of successive integrations are completed. The first ten for q at 27.22 are shown in Fig. 13 and finally it is found that (Col. IIc - .316), (Col. IVc -.316), (Col. VIc - .316)..... are then written as in table 2, and at the points P, - 0, 1,15 the additions shown are made, giving a set of values of

which provide a smooth curve representing the variation of over the boundary. This curve is marked 27.22 in Fig. 17.

(a). The results of successive integrations for Q at 22.21 cms and 24.6 cms (nose) are given in Figs. 14 and 15, and show, with Fig. 13, how the sense of the convergence changes as q passes the nose of the section. The curve for 22.21 cms (Fig. 17) is roughly a reflection of that for 27.22 in consequence.

The positions chosen for Q are at 0, 2.3, 6.2, 17, 20, 22.21, 24.6, 27.22, 34, 39, 43.4 (49.75) cms. Curves of variation of over the boundary for some of these positions are shown in Fig. 16.

For each Q position, the procedure described in detail for Q at 27.22 cms is followed, and it is interesting to note that qualitative ideas of the rapidity of convergence to be expected for the curves corresponding to those of Fig. 13, are suggested by Fig. 16: for the more nearly the plotted values approximate to the straight lines AB, BC, CD, DE, BE in that figure, the more rapid is the convergence of the subsequent integrations. This shows that the convergence is most rapid around the nose, ϕ and slowest at 0.

The curve of variation of for each position chosen for Q is given in Fig. 17, forming a family of curves which vary regularly in shape and orientation as Q traverses the boundary.

ϕ compare Figs 13, 14, 15.



The curve for Q at O is shown incompletely and dotted, as the convergence of the integrals is extremely slow and the extrapolation found to be necessary implies an order of accuracy different from that of the other curves. A note on another aspect of the successive integration at O is given in Appendix IV (b).

(d). The variation of ϕ for Q varying, while P , remains fixed in any desired position, is now required by equation (5) Art. 4.

For this purpose, positions of P , are chosen at intervals of, in general, 1 cm. in the tail region and 2 cms elsewhere, and the curves of Fig. 17 are cross-plotted at these positions.

The result of this is shown in Figs. 18 and 19, which exhibit the solving function in a form suitable for the final step -

------(2)

As an example of this step, let the aerofoil section be supposed inclined so that the second axis is at 50° to a horizontal line through the tail point O : ϕ is represented by the ordinates of the points P , measured perpendicularly from this line. ϕ The positions chosen for the points P_1 (0, 1, 2, -----32) and their values are given in Cols. II and III of table 3.

The operation ϕ is now performed for P at each of the points 0, 1, 2,32. The constant C can be left undetermined for the moment.

For example, consider P_1 at 8, that is, at 12 cms from O . From the curve marked 12 in Fig. 18, values of ϕ are plotted against the S values which occur in Col. III of table 3, at the corresponding points on the boundary: thus for $S = 6$ cms, $\phi = -.38$ is plotted against $S = .89$; for $S = 20$ cms, $\phi = -.155$ is plotted against $S = 1.00$ etc. The result of this is the closed curve marked 12 in Fig. 21. The area of this curve represents ϕ to some scale, and is given as 189.8 cms² in Col. IV of table 3.

Precisely similar procedure is followed for all the positions, P_1 at 0, 1, 2,32, and the resulting curves, marked with the distances of P_1 from O , are shown in Figs. 20, 21, 22, 23: their areas complete Col. IV of Table 3.

ϕ of. Appendix V,

ϕ or

The determination of the scale for obtaining from Col. IV the actual values of ϕ is straightforward. In Figs. 20, 21, 22, 23,

10 cms horizontally represent 1 cm.

10 " vertically " 1 unit.

So 100 cms² represent 1 integral unit.

Hence, the required values of ϕ are given by Col. VI $\times \frac{1}{100}$; these values are in Col. V.

Col. VI contains

------(3)

obtained by adding Cols. III and V.

The constant C may be determined by equation (2) Art. 5 d., when the solution is complete. In the check now proposed, however, it is obtained without special integration.

The check consists of substitution in equation (4) Art. 3., which can be written

------(4)

Following the method of Art. 8b, the means of adjacent values in Col. VI are found, and integrations are performed for several points P, on the boundary. The results are in Col. VIII. The values of ϕ as given by (4) follow by subtracting Col. VIII, from Col. VI. They are given in Col. IX. Now this column should differ from Col. III, which contains the true values, by a constant C. By adding $C = 1.75$ to Col. IX, the check values of ϕ (Col. X) may be compared with the true values. The degree of agreement is shown in Fig. 24, where ϕ is plotted on the usual S base, with the check values indicated by ringed points. By subtracting $80 = 3.50$ from Col. VI, ϕ is obtained (Col. XI) and is plotted in Fig. 24.

The most noticeable discrepancies between true and check values of ϕ occur in the tail region, where it is difficult to deal accurately with the large values of ϕ occurring. Satisfactory agreement at the tail will only be obtained by working these on the boundary drawn to a larger scale, with a more powerful method of attack designed to deal adequately with the cusp ϕ . A boundary whose curvature was everywhere finite would have provided a more exact check on the graphical processes described.

TABLE I.

A	B (cmm.)	I		II		III		IV		V		VI					
		I (h)	a	b	a (h)	b	a (h)	b	a (h)	b	a (h)	b					
0	0	.133	.135	0912	.165	.15	0865	.164	.20	1102	.211	.21	1165	.223	.28	1333	.266
1	8.49	.14	.17	0743	.142	.15	1200	.230	.23	1101	.211	.25	1431	.279	.31	1365	.261
2	15.72	.20	.25	0845	.162	.20	1713	.328	.36	1471	.282	.30	1783	.342	.35	1602	.307
3	19.00	.30	.35	1212	.232	.27	2010	.384	.40	1705	.326	.34	1892	.362	.37	1706	.326
4	20.6	.40	.45	1646	.315	.36	2179	.417	.43	1842	.353	.36	1951	.374	.39	1762	.338
5	21.42	.50	.55	2066	.396	.44	2302	.441	.45	1935	.370	.38	1995	.386	.385	1802	.345
6	22.15	.60	.65	2513	.481	.52	2412	.461	.47	2020	.387	.395	2019	.387	.39	1833	.351
7	22.92	.70	.75	2929	.56	.59	2520	.483	.49	2107	.404	.41	2052	.392	.395	1860	.355
8	23.73	.80	.85	3273	.622	.63	2621	.502	.605	2123	.419	.425	2063	.396	.395	1890	.362
9	24.6	.90	.93	3343	.641	.62	2650	.509	.49	2260	.433	.43	2069	.396	.39	1917	.367
10	25.6	.96	.98	3092	.594	.55	2486	.476	.45	2242	.430	.42	2000	.383	.37	1936	.367
11	27.22	1.00 0	.02	2601	.500	.43	2171	.415	.36	2141	.410	.39	1900	.364	.35	1899	.364
12	30.50	.04	.06	1578	.360	.30	1576	.302	.26	1929	.369	.34	1710	.329	.30	1623	.349
13	35.48	.08	.09	1212	.232	.22	1090	.209	.20	1572	.302	.29	1446	.277	.27	1653	.317
14	39.36	.10	.11	1035	.198	.15	0976	.157	.15	1407	.269	.24	1339	.256	.24	1553	.298
15	44.2	.12	.125	0859	.164	.16	0835	.169	.165	1143	.219	.22	1193	.228	.225	1361	.261
0	49.75	.133		0812	.155		0855	.164		1102	.211		1165	.223		1333	.266

TABLE I. Contd.

A	VII			VIII			IX			X			
	VI.e-.25	a	b	e	b	a	e	b	a	b	e		
				(h)	(h)	(h)	(h)	(h)	(h)	(h)	(h)		
0	.006	.008	0754	.014	.264	.030	1712	.033	.283	.035	1926	.036	.288
1	.011	.034	2446	.047	.297	.064	1941	.037	.287	.066	2630	.054	.304
2	.057	.067	4246	.081	.331	.086	3196	.061	.311	.086	3799	.073	.323
3	.076	.082	4775	.091	.341	.093	3696	.071	.321	.075	4026	.075	.323
4	.088	.091	5022	.096	.346	.097	3959	.076	.326	.076	4209	.081	.331
5	.095	.098	5153	.099	.349	.100	4132	.079	.329	.080	4270	.082	.332
6	.101	.103	5255	.101	.351	.101	4266	.082	.332	.083	4330	.083	.333
7	.106	.110	5339	.102	.352	.102	4411	.085	.335	.086	4350	.083	.333
8	.112	.115	5389	.103	.353	.102	4563	.087	.337	.089	4386	.084	.334
9	.117	.117	5328	.102	.352	.100	4709	.090	.340	.090	4333	.083	.333
10	.117	.115	5081	.097	.347	.093	4745	.081	.341	.090	4198	.080	.330
11	.114	.106	4680	.090	.340	.081	4640	.089	.339	.086	3997	.077	.327
12	.099	.083	3967	.072	.322	.060	4321	.083	.333	.075	3636	.069	.319
13	.067	.058	2454	.047	.297	.041	3476	.067	.317	.061	2896	.055	.305
14	.048	.030	1838	.035	.285	.027	2912	.056	.306	.045	2374	.049	.299
15	.011	.003	0923	.018	.268	.016	1853	.035	.285	.034	2080	.040	.290
0	.006		0754	.014	.264		1712	.033	.283		1926	.036	.288

TABLE I.
CONT.

TABLE I. Contd.

A	a	XI b	a-.2b	c (h)	XII C	XIII C	XIV C	XV C	XVI C	XVII C	XVIII C
0	.048	2600	.050	.300	.306	.308	.311	.312	.313	.314	.315
1	.056	2096	.059	.309	.308	.312	.311	.314	.314	.315	.316
2	.065	3598	.069	.319	.314	.317	.315	.317	.315	.315	.316
3	.068	3728	.071	.321	.316	.318	.316	.317	.316	.315	.316
4	.070	3771	.073	.322	.317	.318	.316	.317	.316	.315	.316
5	.072	3804	.073	.323	.318	.318	.317	.317	.316	.315	.316
6	.073	3835	.073	.323	.319	.318	.317	.316	.316	.315	.316
7	.074	3863	.074	.324	.319	.320	.317	.318	.316	.317	.316
8	.076	3841	.074	.324	.319	.320	.317	.318	.316	.317	.316
9	.077	3824	.073	.323	.320	.319	.317	.317	.316	.316	.316
10	.077	3762	.072	.322	.320	.318	.318	.317	.316	.316	.316
11	.078	3646	.070	.320	.321	.317	.318	.317	.317	.316	.316
12	.070	3454	.066	.316	.319	.316	.317	.316	.317	.316	.316
13	.062	3092	.059	.309	.316	.311	.316	.316	.313	.316	.314
14	.055	2916	.056	.206	.313	.310	.313	.313	.316	.314	.315
15	.048	2641	.051	.301	.309	.319	.314	.314	.315	.315	.316
0		2600	.050	.300	.306	.308	.311	.312	.313	.314	.315

Tab. 2.

A	B (cms)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0
		0	3.49	15.72	19.00	20.5	21.42	22.19	22.92	23.73	24.6	25.6	27.22	30.50	35.48	39.36	44.2	49.75
IIIc-.316		-.161	-.174	-.184	-.024	-.001	.090	.105	.244	.312	.325	.273	.132	.044	-.084	-.118	-.152	-.161
IVc-.316		-.105	-.105	-.034	.010	.037	.054	.071	.088	.103	.117	.114	.094	.053	-.014	-.047	-.097	-.105
Vic-.316		-.060	-.055	-.009	.005	.023	.029	.035	.040	.046	.051	.051	.045	.033	.001	-.018	-.055	-.060
VIIIc-.316		-.033	-.029	-.025	.001	.010	.013	.016	.019	.021	.024	.025	.023	.017	.001	-.101	-.051	-.033
Xc-.316		-.019	-.016	-.004	.000	.004	.005	.007	.008	.009	.011	.011	.011	.008	-.001	-.006	-.017	-.019
XIIc-.316		-.010	-.008	-.002		.001	.002	.003	.003	.003	.004	.004	.005	.003	.000	-.003	-.007	-.010
XIVc-.316		-.005	-.005	-.001		.000	.001	.001	.001	.001	.001	.002	.002	.001		-.003	-.002	-.005
XVlc-.316		-.003	-.002	-.001			.000	.000	.000	.000	.000	.000	.001	.001		-.001	-.001	-.003
XVIIc-.316		-.001	-.001	000									.000	.000		-.001	-.000	-.001

$$\frac{1}{2\pi} \sum_{p,q}$$

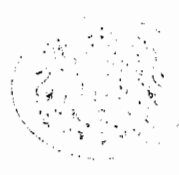
-.397	-.395	-.210	-.055	.073	.154	.238	.403	.495	.533	.435	.366	.180	-.097	-.207	-.362	-.397
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TABLE 2.

TABLE 3.

35

I oint	II Dist from 0	III ψ	IV Area repi $(\psi d(\frac{1}{2}S))$	V $\frac{1}{\pi}(\psi d(\frac{1}{2}S))$	VI $\frac{1}{2}X+2C$ $=\frac{1}{2}X+Y$	VII Means from VI	VIII $\frac{1}{\pi}X.d0$	IX $[Y-C]$ $=VI-VIII$	X $[Y]$ $=XI+1.75$	XI $\frac{1}{2}X$ $=VI-3.5$
0	0	0	(-3090)	(-30.9)	(-30.9)	(-21.6)	(-30.9) 1.08	-1.08	.67	(-34.4)
1	1	.23	(-1250)	(-12.5)	(-12.3)	(-10.2)	11.075	-1.225	.525	(-15.8)
2	2	.45	-855	-8.55	-8.1	-6.05	6.92	-1.18	.87	-11.6
3	4	.70	-471	-4.71	-4.01	-3.245				-7.51
4	5	.80	-328	-3.28	-2.48	-1.84				-5.98
5	6	.89	-209	-2.09	-1.20	-.155	-.48	-.72	1.03	-4.70
6	6	1.02	-12.7	-.127	+.895	1.565				-2.61
7	10	1.09	+114.6	+1.146	2.235	2.605	+2.91	-675	1.075	-1.265
8	12	1.075	189.8	1.898	2.97	3.26				-.525
9	14	1.06	250.2	2.502	3.56	3.755	4.475	-.915	.835	+.06
10	16	1.005	294.1	2.941	3.945	4.075				.445
11	18	.98	322	3.22	4.20	4.295	5.15	-.95	.80	.70
12	20	1.00	338.8	3.388	4.39	4.420				.89
13	21	1.05	340	3.40	4.45	4.47				.95
14	22	1.16	333	3.33	4.49	4.645				.99
15	24	1.61	318.7	3.187	4.795	5.025				1.295
16	25	2.43	282	2.82	5.25	5.44				1.75
17	26	3.03	280	2.60	5.63	5.82	4.5	1.13	2.88	2.13
18	28	3.61	240	2.40	6.01	6.065				2.51
19	30	3.89	227	2.27	6.16	6.07	4.08	2.08	3.83	2.66
20	32	3.91	207.5	2.075	5.985	5.82	3.795	2.19	3.94	2.485
21	34	3.83	183	1.83	5.66	5.36				2.16
22	36	3.61	145	1.45	5.06	4.615	3.13	1.93	3.68	1.56
23	38	3.29	88.3	.883	4.175	3.57				.675
24	40	2.95	11.6	.116	2.965	2.23				-.535
25	42	2.36	-65.2	-.652	1.50	.97				-2.00
26	43	2.08	-164.3	-1.643	.435	-.25				-3.065
27	44	1.80	-273.7	-2.737	-.935	-2.775	-1.03	.095	1.945	-4.435
28	46	1.20	-581	-5.81	-4.61	-5.805	-4.06	-.55	1.20	-8.11
29	47.5	.74	-774	-7.74	-7.0	-7.805	-6.15	-.85	.90	-10.5
30	48	.59	-860	-8.60	-8.01	(-10)				-11.51
31	49				(-12)	(-21.5)				(-15.5)
32	49.75	0	(-3090)	(-30.9)	(-30.9)		(-30.9) 1.08	-1.08	.67	(-34.4)



APPENDIX I.

(a) It is required to evaluate the particular integral

-----(1)

Now

Hence as $=$

$=$

$=$

$=$

$=$

$=$

$=$

-----(2)

where

Replacing

and putting

----- (3)

Differentiating under the integral sign



Substituting

whence

$$\text{-----(4)}$$

where

$$\text{-----(5)}$$

It is convenient to express the integrant in partial fractions. Let

$$\text{-----(6)}$$

Then

$$\text{------(7)}$$

Solving these four equations simultaneously

$$\text{------(8)}$$



Substituting the expressions (8) in the identity (6) the expression (4) may be written,

----- (9)

Dealing with the integrand of the third integral term

---- (10)

So, subtracting (10) in (9), and evaluating the integrals,

---- (11)

----- (12)



Integrating

----- (13)

----- (14)

From (5)

B =

C =

D =

----- (15)



Also

----- (16)

Hence, by (12), (13) and (14)

---- (17)

(b) It is required to evaluate the particular integral

---- (18)

Proceeding as before, let

Then

----- (19)

By comparison with part (a), it is easily seen that the expression for $\frac{d}{dx}$ is obtained by changing the sign of the right hand side of (11), and changing the limits to $\tan^{-1} \frac{b}{a}$ and 0, the constants being given by putting $\frac{b}{a}$ for in (5). Proceeding, the expression (12) is repeated exactly, and so are (13) and (14). Substituting the values of B, C and D, it follows that

(c) So in general

4

APPENDIX II.

It is required to evaluate

Now

Hence

Putting
(1) becomes

The second integral in the brackets is zero.
Evaluating the first -

So

Now

Where u e

Similarly

i.e.

APPENDIX III.

SOLUTION FOR ELLIPSE - INTERNAL FIELD.

When the internal field is considered, it is easily seen by reference to equation (5b) Art 4. that the appropriate solving function is

and hence (Equation (5) Art 4. and Art 7c) that

which is the solution obtained by the direct method, on p.473 of the Roy. Soc. Proceedings given on p.1. footnote (p).

APPENDIX IV.

(a) When P is at O, it may be considered as the limiting position either of points on the lower surface or points on the upper surface, of the aerofoil.

In the former case, the frame is first of all lying on the tangent at O with the scale to the operator's right hand, if the section is placed as in Fig. 12. Referring to Table I, the planimeter having been read, the frame is turned from 0 to 1, with the tracing point at .135, and so on until with the point at .125, it is back in its initial position. The frame has, however, always to turn contraclockwisely through , so with the tracing point on .133 (not .135 or .125) the frame is turned through on to the tangent at O again, and the planimeter read. The difference in readings gives the desired integral.

Alternatively, of course, if P is supposed on the upper surface, the turn through with .133 on the scale is the first stage, and the traversing of the boundary the second. There is no difference in the result, and it is clear that the rotation through need not actually be performed, for the result of it is necessarily .133 (in this case), which can be added to the figure given by the rest of the boundary.

(b) When the point ϵ is at O, the singular nature of the cusp introduces a special consideration.

Considering the point as on the lower surface, the values of are those in Col. I of table 4. The point O has all values from 0 to 1.00. Cols. IIa, b...c, are only steps in the magnification of the values of Col. I to increase the accuracy of the graphical integration, which produces the values of given in Col. II f: here another point peculiar to this case becomes apparent. It is the two values assigned to O.

Wherever Q be on the boundary, it is clear (cf. Table 1) that the result of the first integral for P, at Q is necessarily .5. Remembering that O has originally all values of (H) between 0 and 1.00, this is clearly as true for ϵ at O as elsewhere on the boundary. But O is equally well the limiting position obtained as either 1 or 12 approaches infinitely closely to the tail, and the limit for from this viewpoint is 0.

The second integral for all points except O is unaffected by the .5 value in Col. I, but at O it is easily seen that two values are again available, namely (.5 - .005) and -.005 ~~and -.005~~. Similarly, the third integral at O gives (.495 - 0) and 0 and so on (see table 4.)

It follows that when a large number of successive integrations have been performed, and the constants subtracted, two ϵ values of are obtained for O;

if the (12b) form of equation 12, Art. 5b, is taken (12a) or (12c) would give an infinite number of values.

one is continuous with elsewhere on the boundary, but the other is a large positive singular value, [§] if it is assumed that the same applies to it.

Moreover, considering the point q at O to be on the upper surface, the (H) values are given by adding unity to Col. 1, and are therefore between .9 and 1.0. Their first integrals are of the same order, but the alternative value for O is again .5. After a large number of integrations the constant is found to be such that continuous values of are arrived at identical with those obtained by considering q to be on the lower surface at O , but the singular value at O is a large negative quantity.

These singularities at O are interesting, but do not appear to affect the finding of the solving function. In the graphical work, O is looked upon as the limitive position for a point approaching infinitely closely to it.

§ Not necessarily infinite.

TABLE 4.

A	B	I	II					III						
			a 1x10 ⁻¹	b	c	d	e (d)xm	f e-1.100 (.500)	a	b	c	d	e	f (.495)
0	0	1.00 To C	0	1.00	.80	5219	.100	0	0	1.00	.98	4940	.095	-.005
1	2.4	-.04	-.4	.60	.50	5011	.096	-.004	-.04	.96	.875	3371	.065	-.035
2	6.3	-.06	-.6	.40	.30	4630	.099	-.011	-.11	.89	.815	2067	.040	-.060
3	12.62	-.08	-.8	.20	.15	3837	.074	-.026	-.26	.74	.685	1480	.028	-.072
4	17.05	-.09	-.9	.10	.10	3286	.063	-.037	-.37	.63	.51	1341	.026	-.074
5	23.33	-.09	-.9	.10	.15	2016	.039	-.061	-.61	.39	.345	1500	.029	-.071
6	24.60	-.08	-.8	.20	.30	1545	.030	-.070	-.70	.30	.26	1793	.034	-.066
7	26.86	-.06	-.6	.40	.50	1160	.022	-.078	-.78	.22	.21	2324	.045	-.055
8	30.92	-.04	-.4	.60	.70	1098	.021	-.079	-.79	.21	.24	2765	.053	-.047
9	36.70	-.02	-.2	.80	.85	1423	.027	-.073	-.73	.27	.31	3469	.066	-.034
10	41.70	-.01	-.1	.90	.925	1849	.035	-.065	-.65	.35	.42	3915	.075	-.025
11	45.0	-.005	-.05	.95	.965	2581	.049	-.061	-.51	.49	.57	4207	.081	-.019
12	47.3	-.003	-.03	.98	.99	3341	.064	-.056	-.36	.64	.82	4517	.087	-.013
0	49.75	To 1.00	0	1.00			.100	0 (.500)	0	1.00		4940	.095	-.005

TABLE 4.

TABLE 4 Contd.

A	IV f	V f	VI f	VII f	VIII f	IX f	X f	XI f	XII f	XIII f
0	(.495) .00	(.491) -.004	(.490) -.001	(.487) -.003	(.486) -.001	(.483) -.003	(.482) -.001	(.480) -.002	(.479) -.001	(.477) -.002
1	-.014	-.034	-.018	-.031	-.020	-.029	-.021	-.027	-.021	-.025
2	.024	-.056	-.031	-.051	-.034	-.047	-.035	-.044	-.036	-.041
3	-.036	-.064	-.041	-.058	-.043	-.054	-.044	-.050	-.043	-.047
4	-.043	-.065	-.046	-.059	.047	-.055	-.046	-.051	-.045	-.048
5	-.056	-.062	-.054	-.057	-.052	-.054	-.050	-.051	-.046	-.048
6	-.061	-.059	-.056	-.055	-.053	-.052	-.051	-.050	-.049	-.048
7	-.066	-.053	-.060	-.052	-.055	-.050	-.052	-.049	-.049	-.047
8	-.067	-.050	-.061	-.049	-.056	-.048	-.053	-.047	-.049	-.045
9	-.065	-.040	-.060	-.043	-.055	-.044	-.052	-.043	-.048	-.042
10	-.061	-.033	-.056	-.037	-.052	-.039	-.048	-.039	-.045	-.039
11	-.048	-.025	-.046	-.029	-.042	-.030	-.039	-.031	-.037	-.031
12	-.035	-.018	-.033	-.021	-.030	-.022	-.029	-.024	-.027	-.022
0	.00	-.004	-.001	-.003	-.001	-.003	-.001	-.002	-.001	-.002

TABLE 4.
CONT.

APPENDIX V.

If θ is represented by ordinates measured
as shown in Fig. 25.

$$\dots\dots\dots(1)$$

where x and y are the co-ordinates of C with respect to
axes fixed relative to the boundary and θ is an angle
defining their orientation.

Thus if instead of the single angle of incidence chosen
in Art. 9 d., it is desired to consider a series of such
angles $\theta_1, \theta_2, \dots$, the procedure described in Art. 9 d.,
for the evaluation of (1) above need only be followed twice -
once to obtain C_x , once to obtain C_y .

being measured with respect to the "second axis" and a
perpendicular at O , as co-ordinate axes.

C_x and C_y are obtained by putting θ_1 and θ_2 in turn for
in equation (20) Art. 5 d.



In conclusion, the writer wishes to thank Professor Bairatow very sincerely for his help and advice while the material for this paper was being prepared.

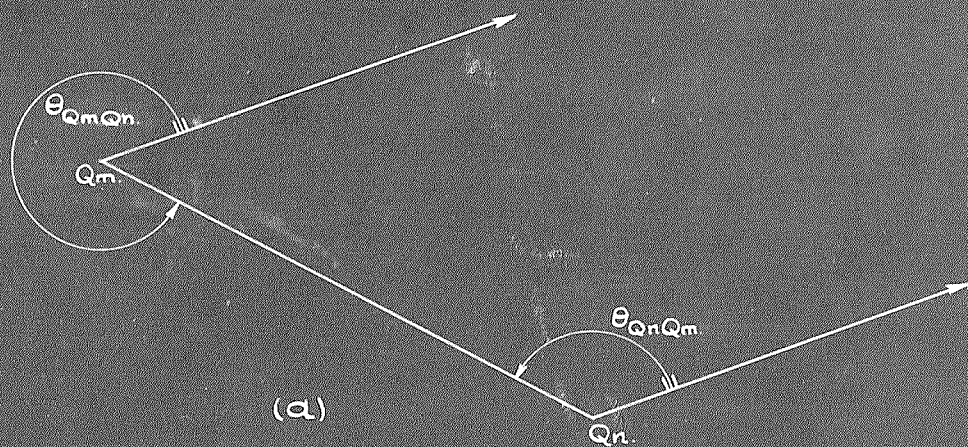


FIG. 1.

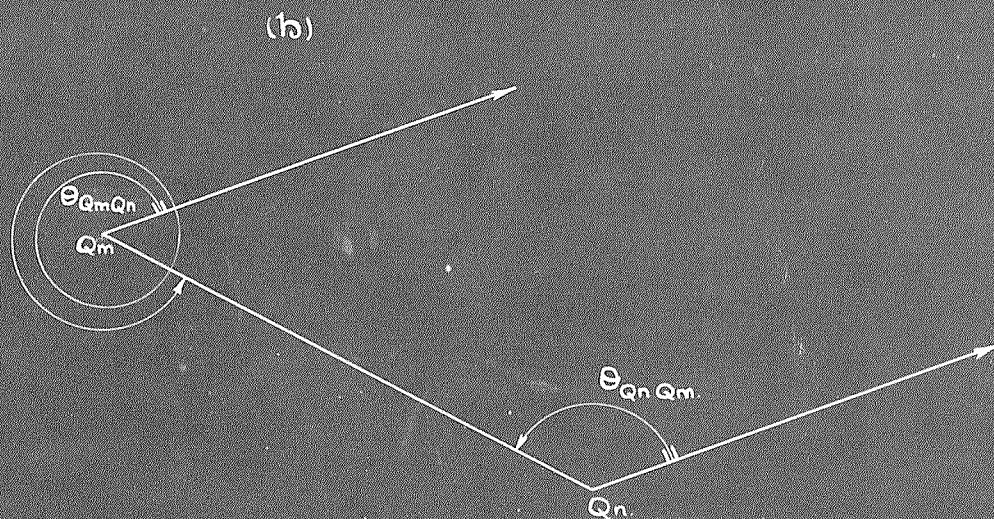
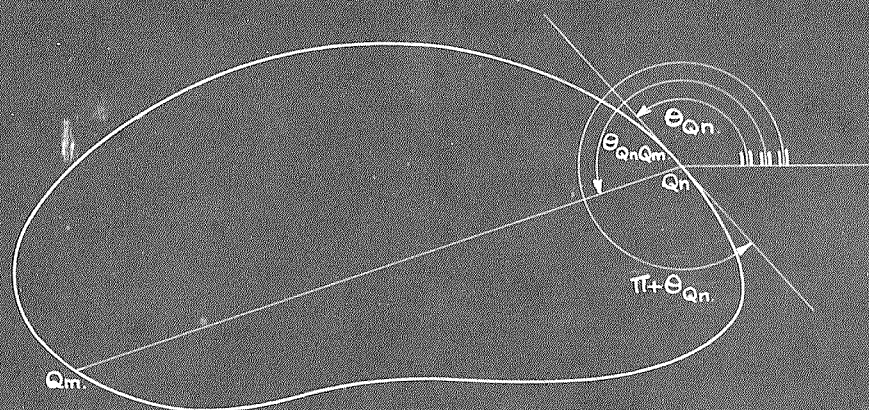


FIG. 2.

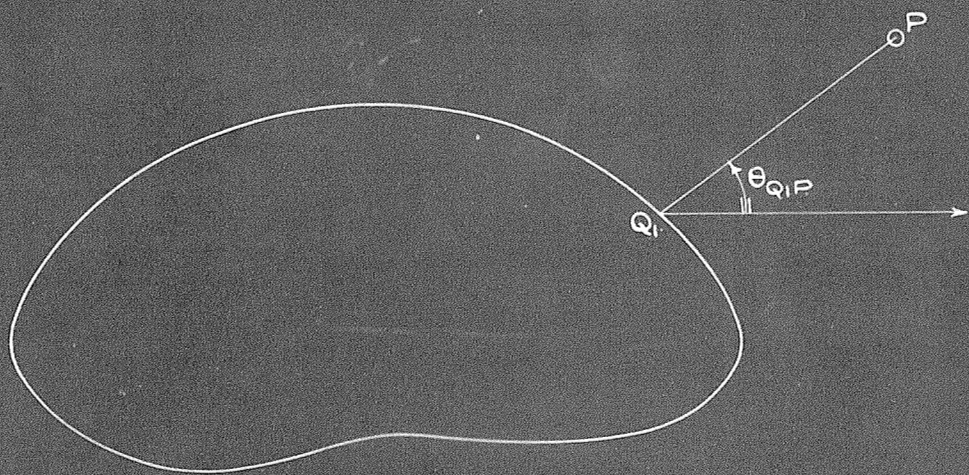


FIG. 3.

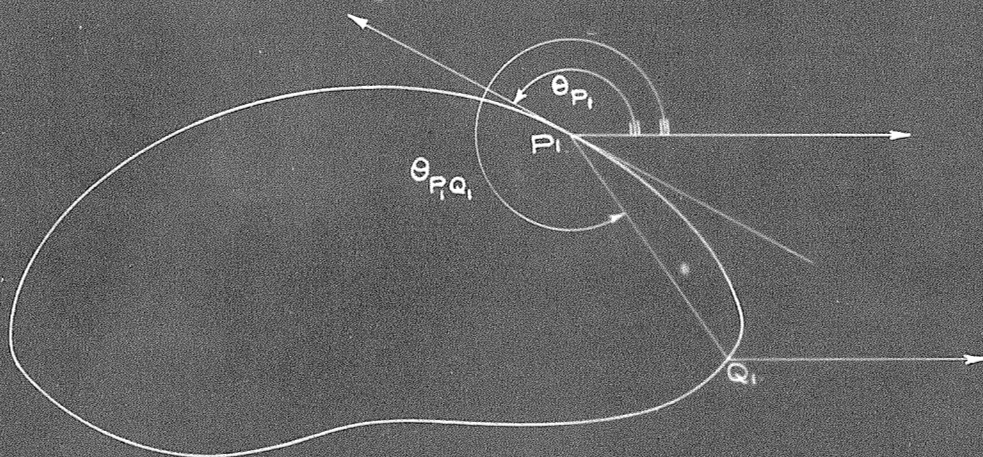


FIG. 4.

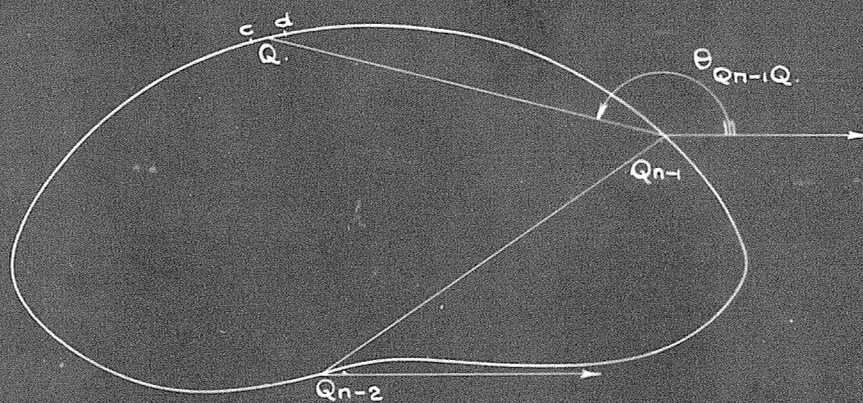


FIG. 5.

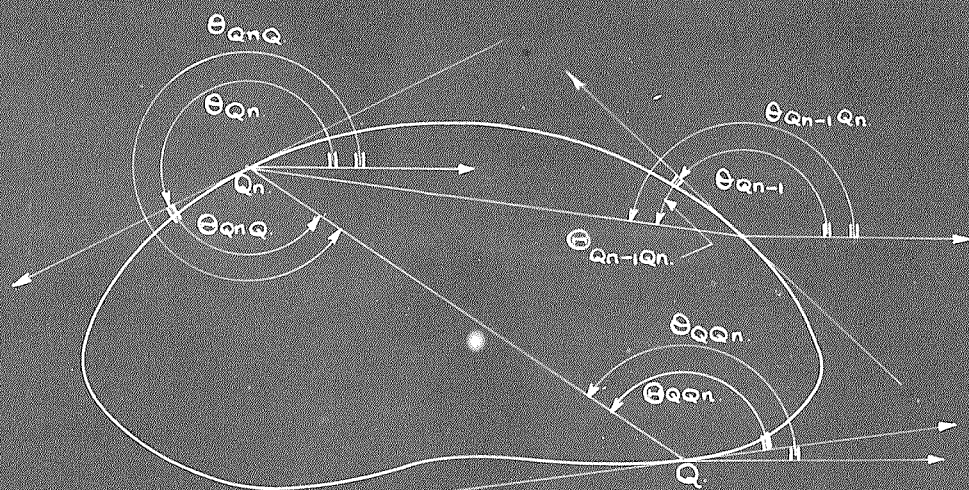


FIG. 6.

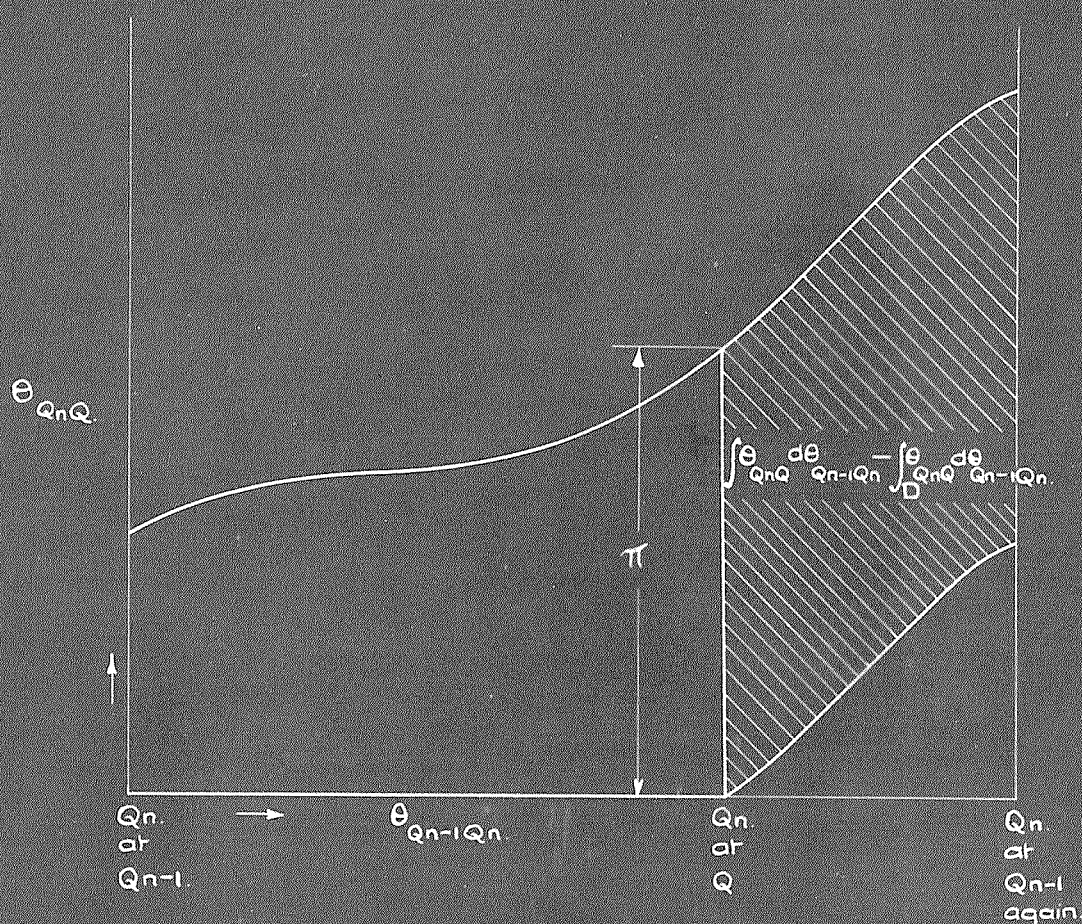
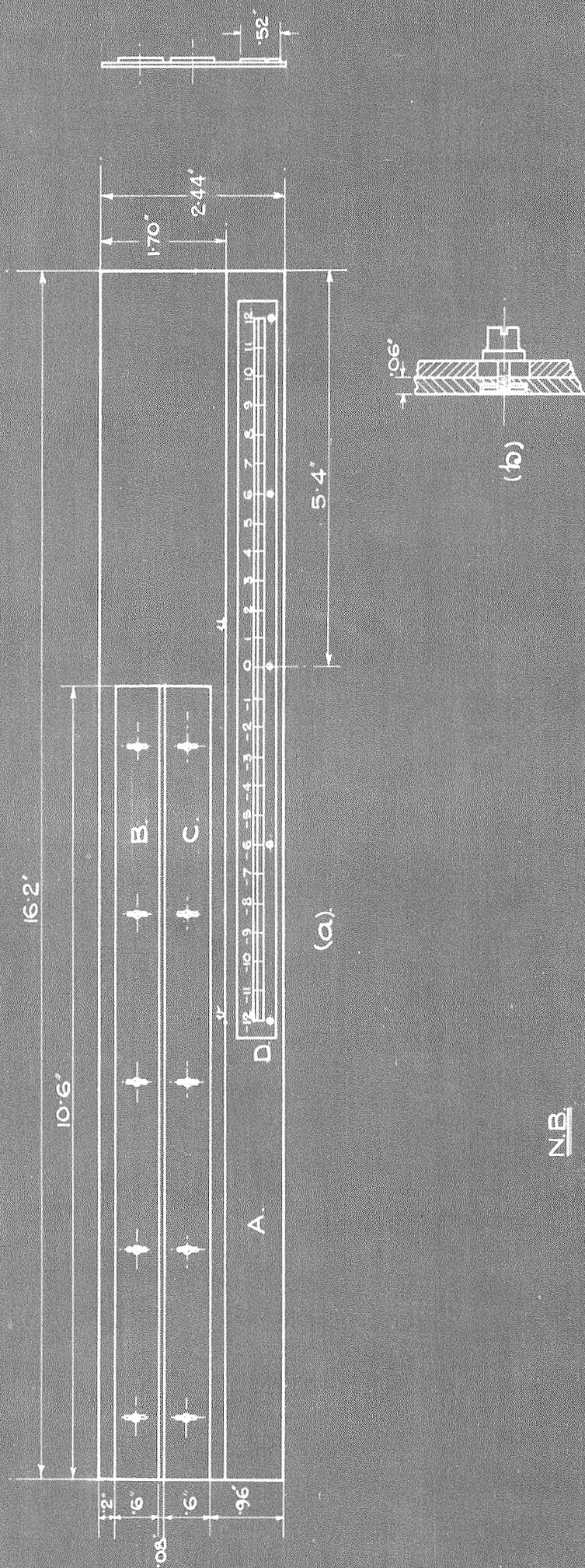


FIG. 7



FIG. 8.



NB. Leading Dimensions only.

FIG. 9.

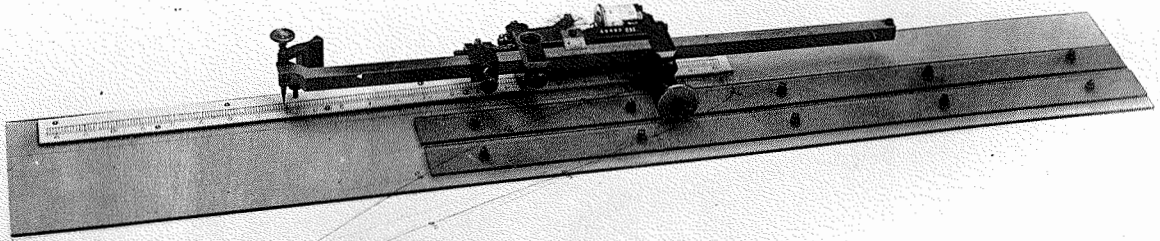


FIG. 10.

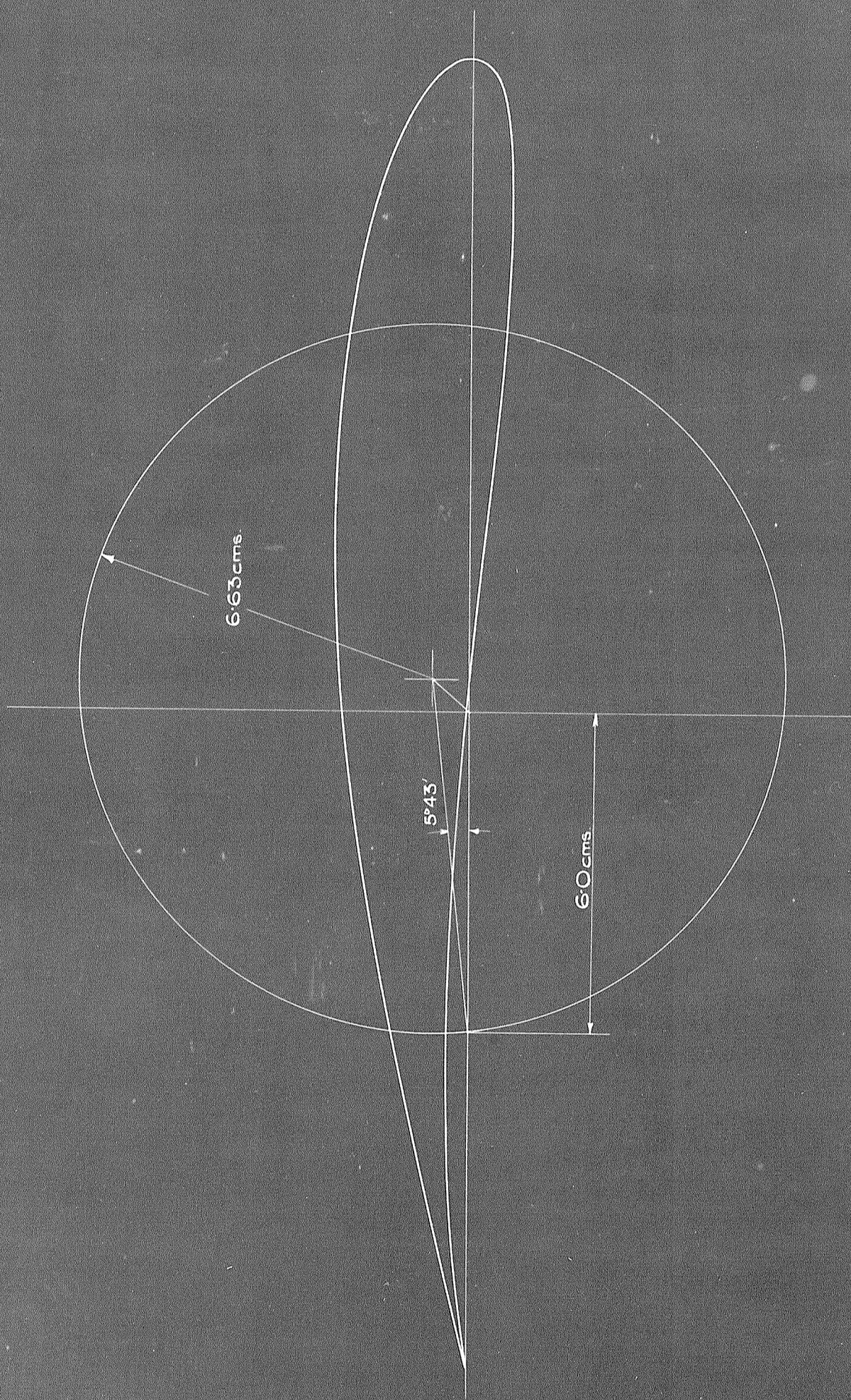


FIG. II.

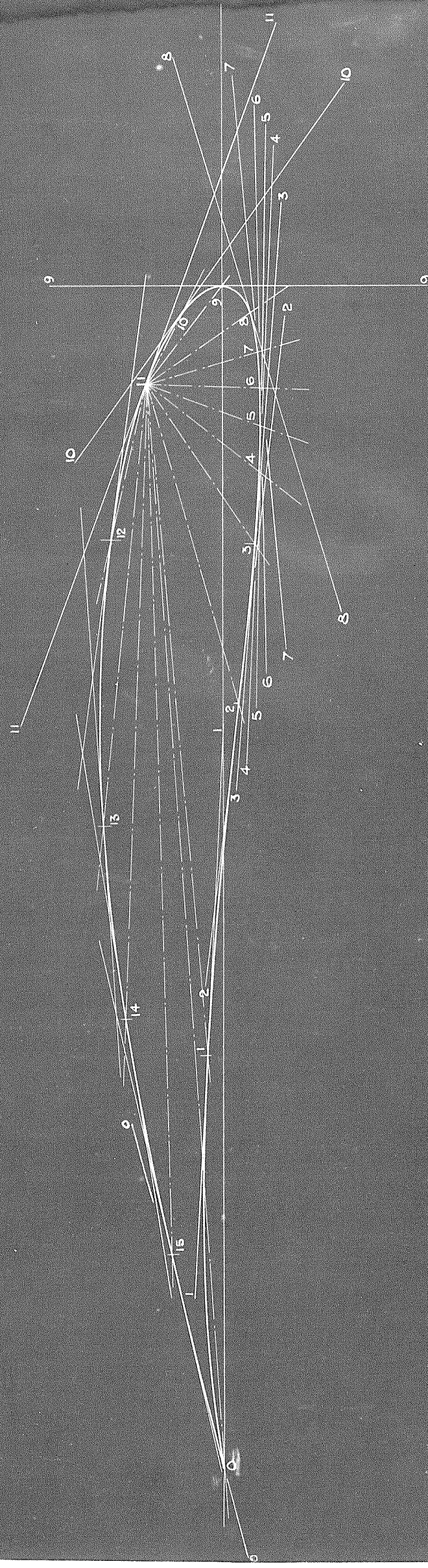


FIG. 12.

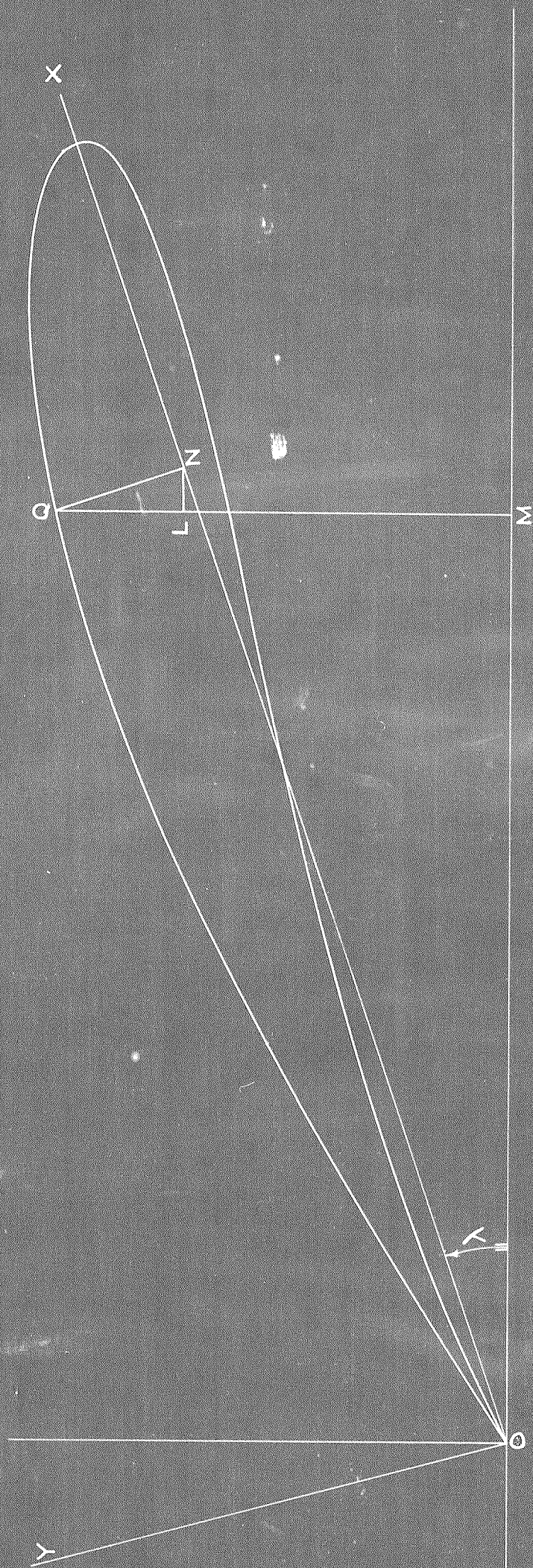
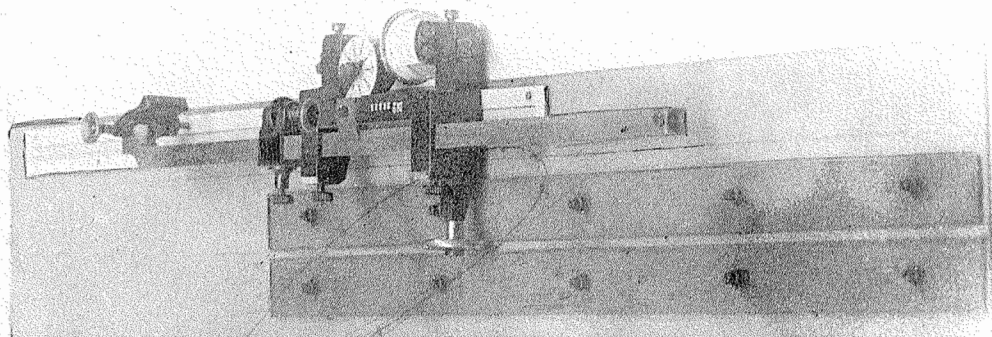


FIG. 25.



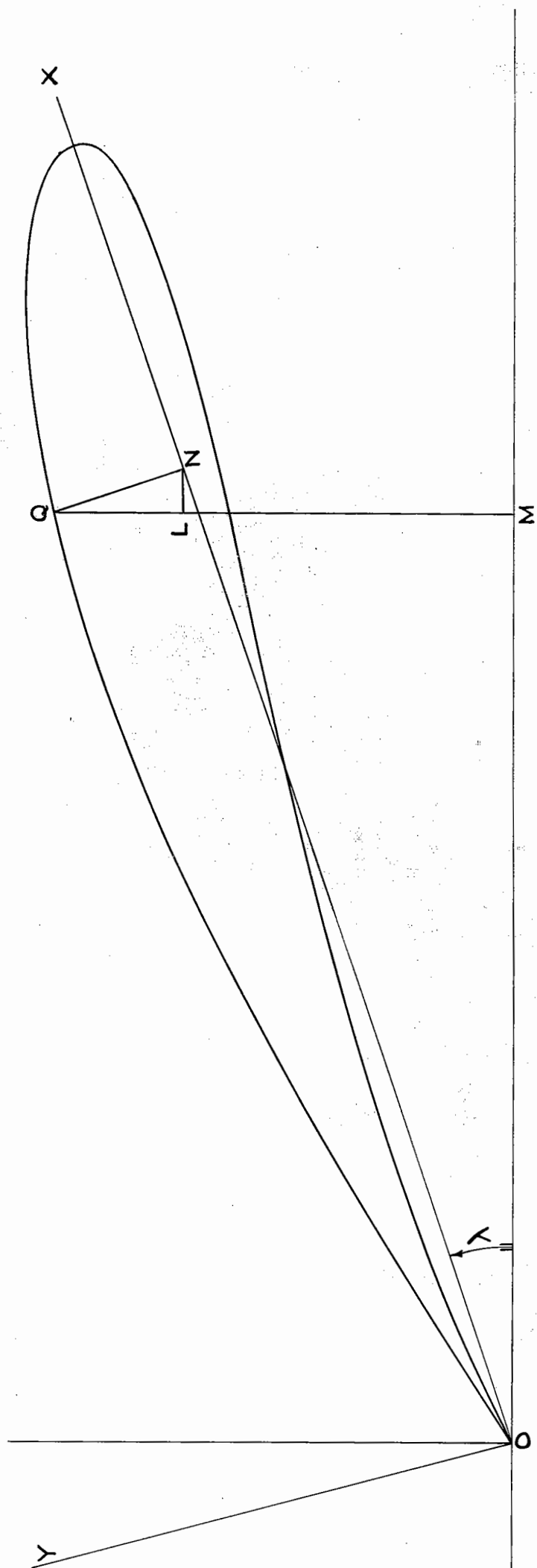


FIG. 25.

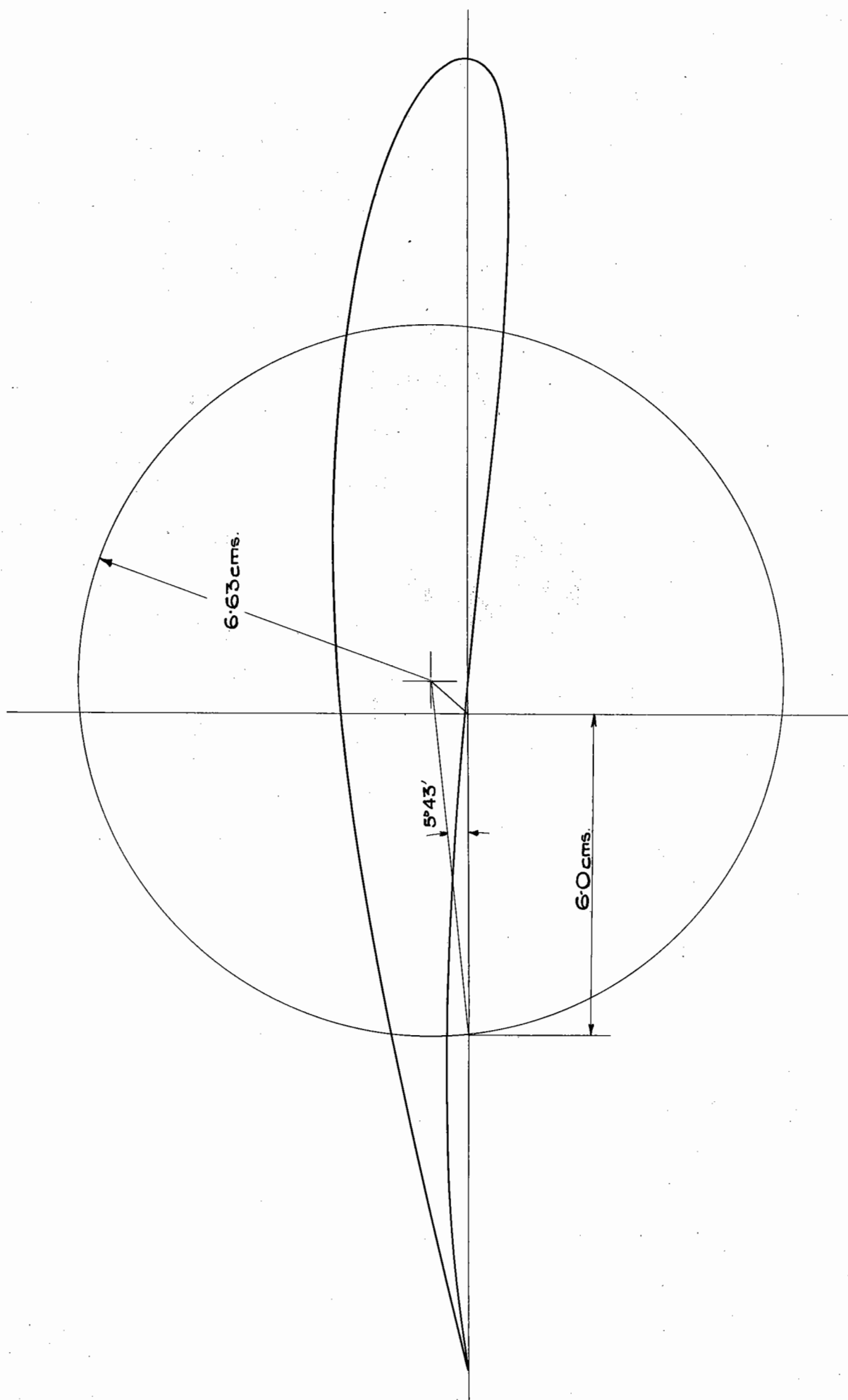
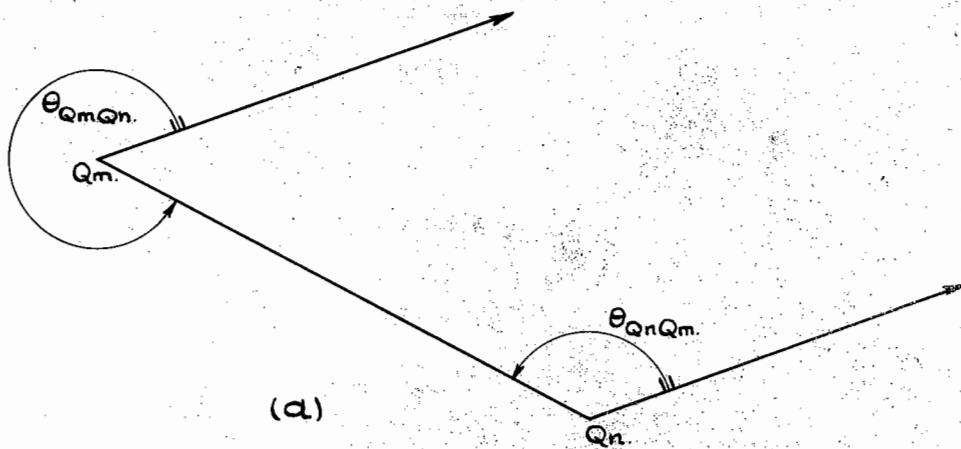
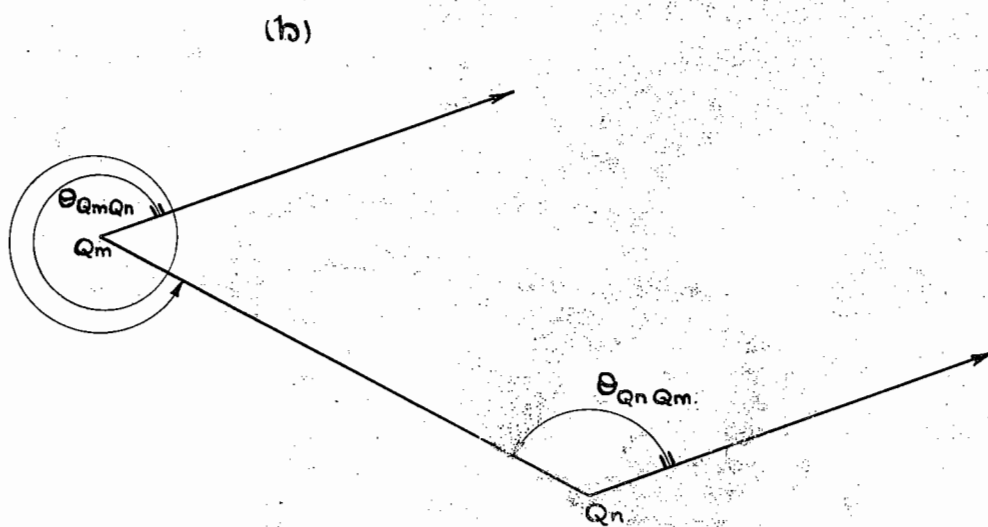


FIG. II.



(a)

FIG. 1.



(b)

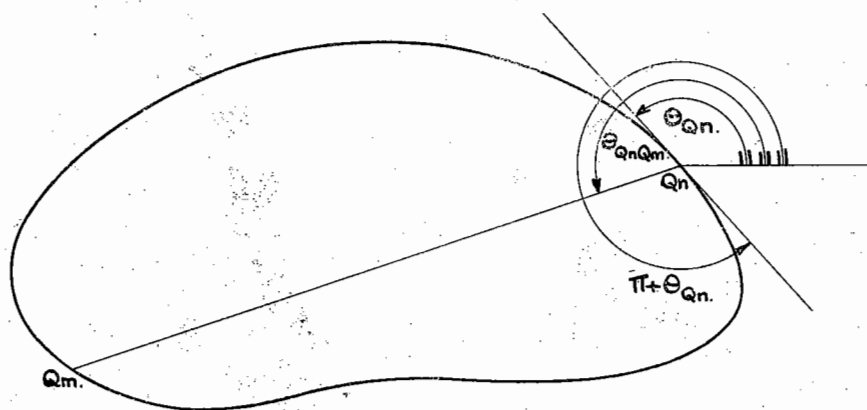


FIG. 2.

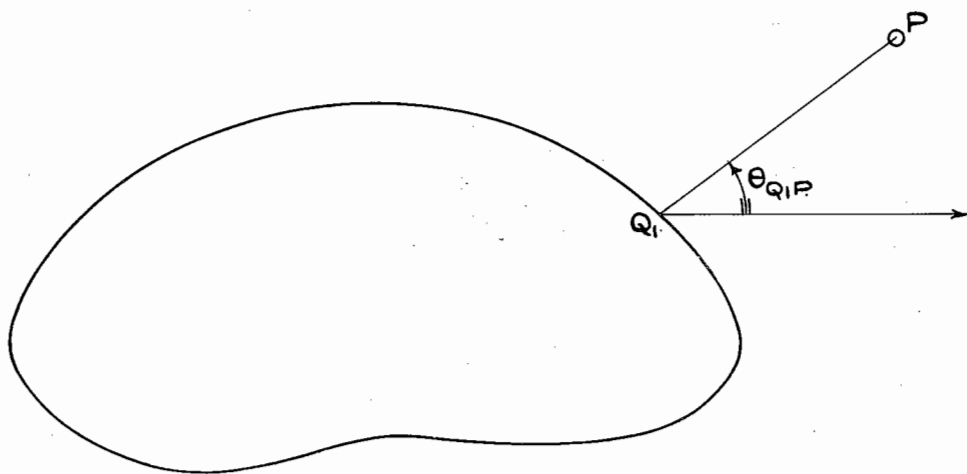


FIG. 3.

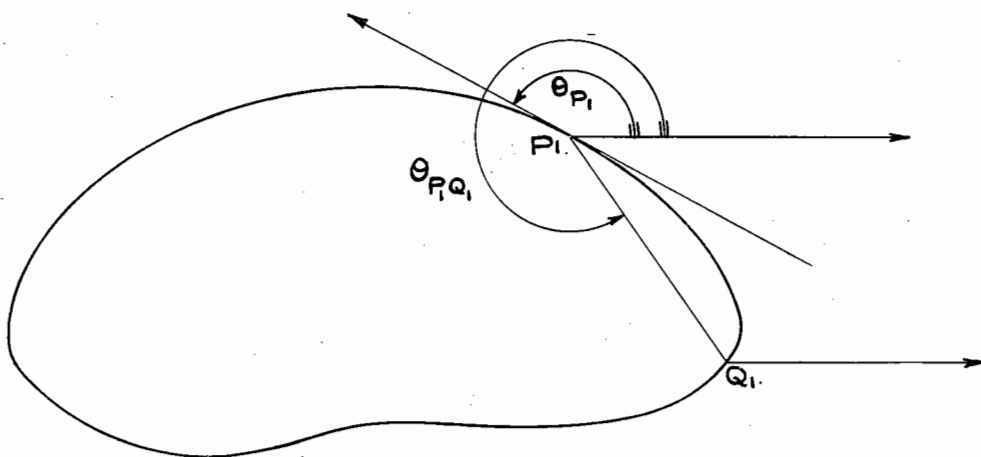


FIG. 4.

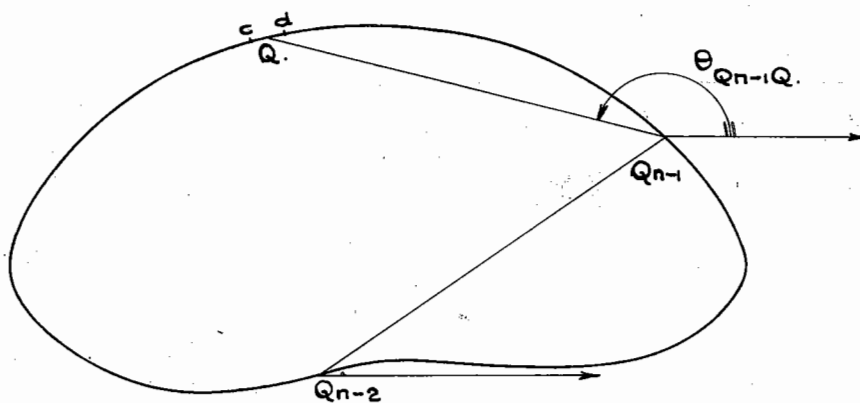


FIG. 5.

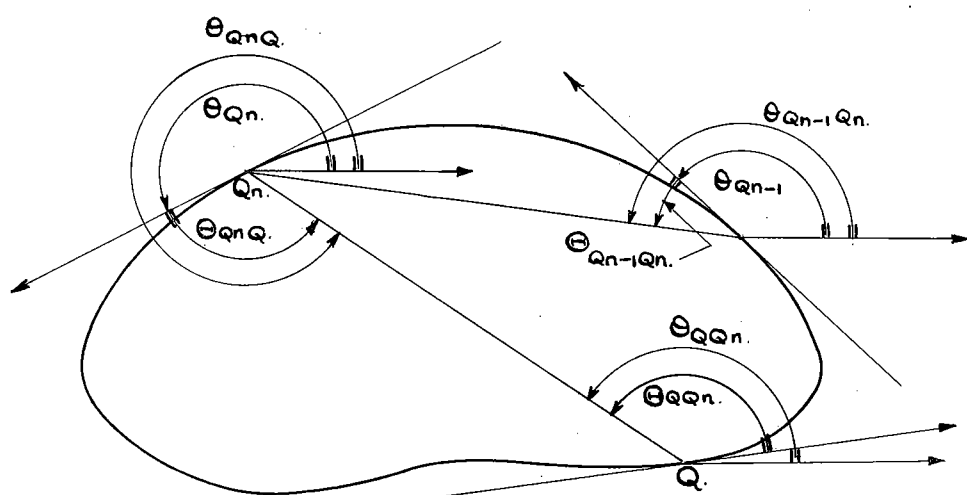


FIG. 6.

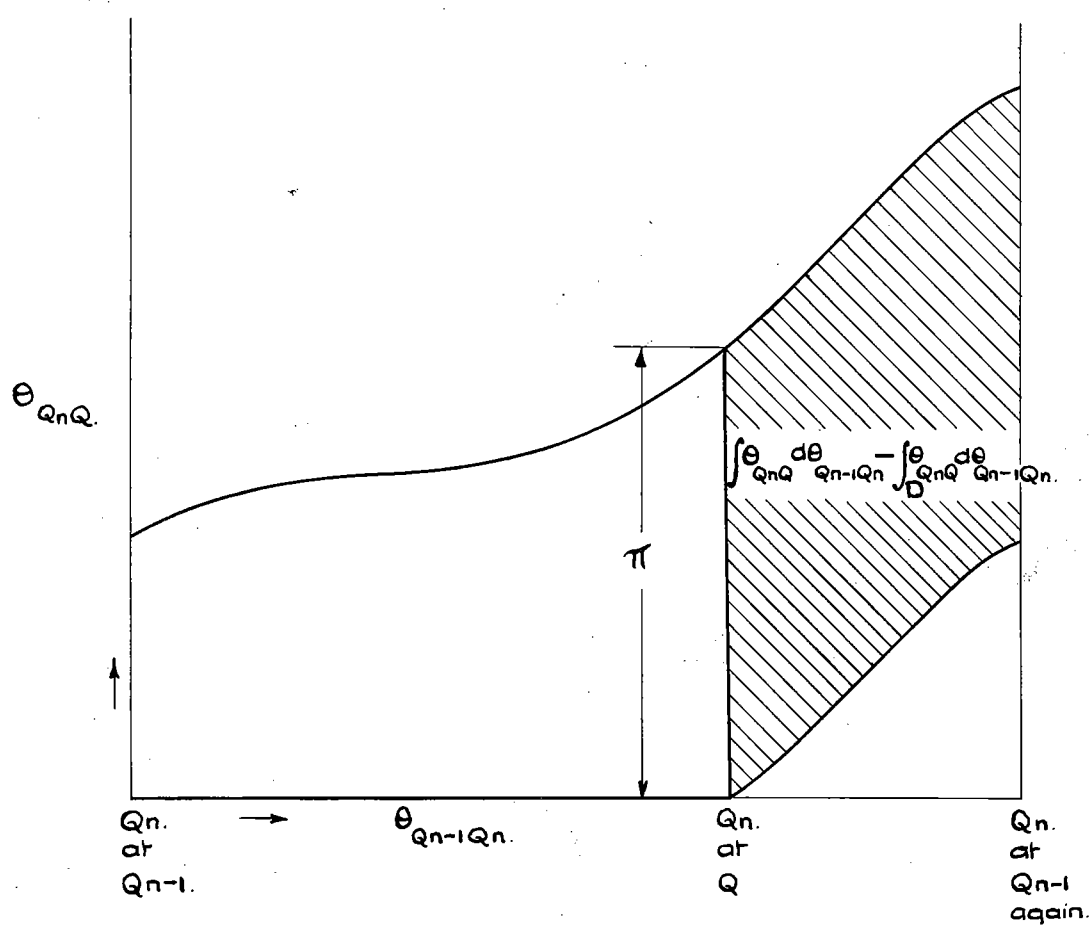


FIG. 7.

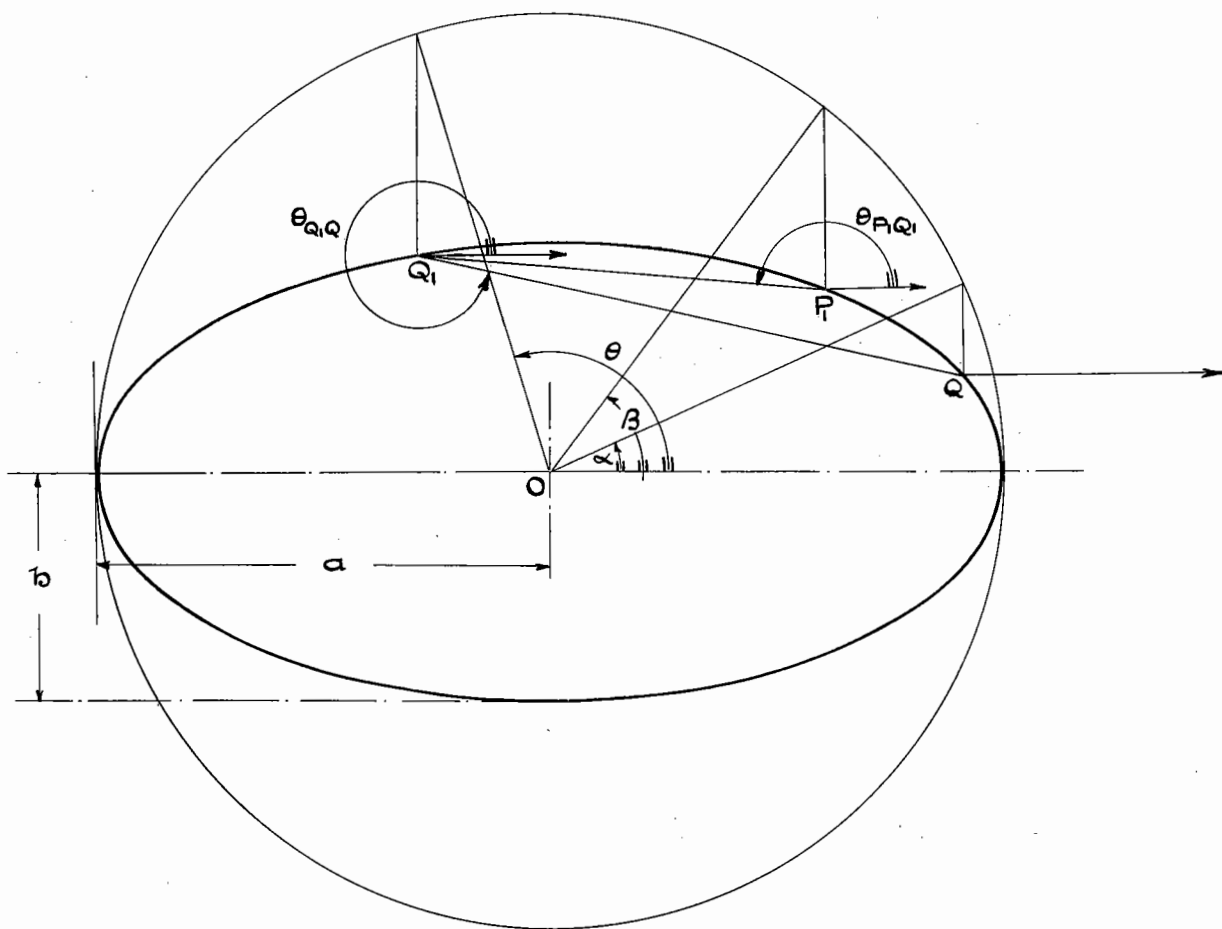
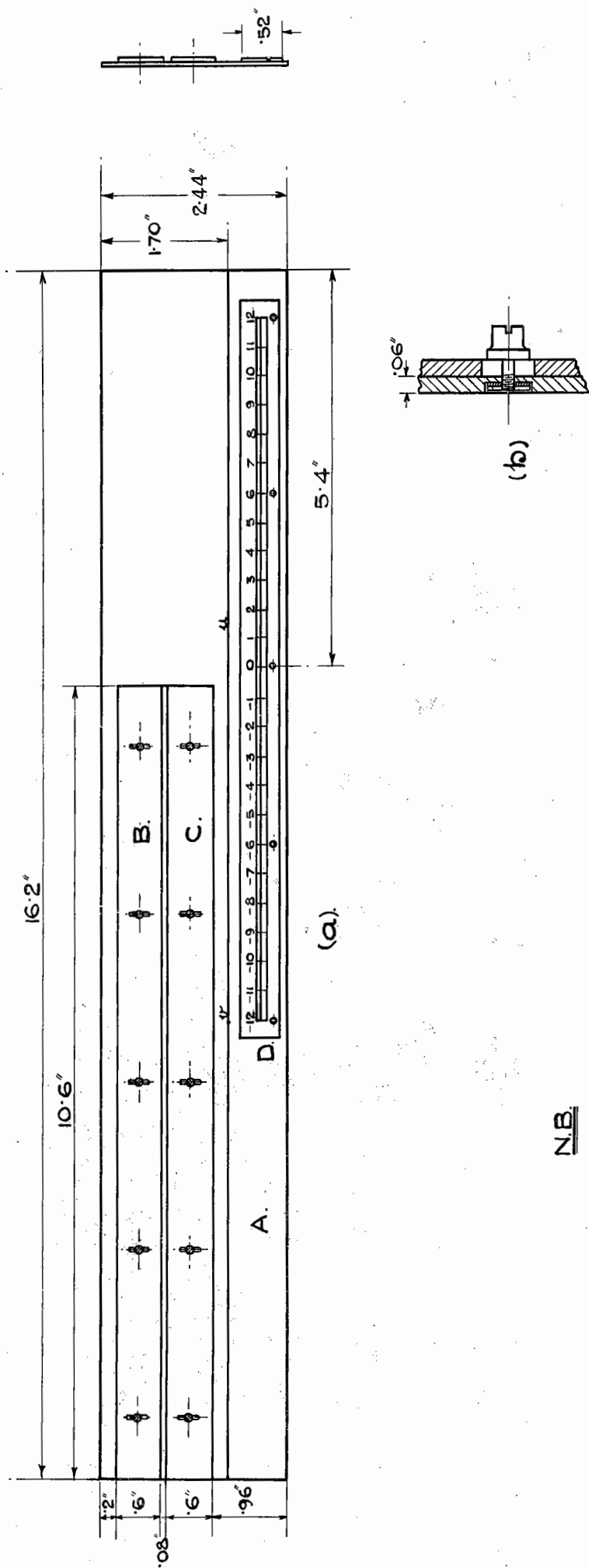


FIG. 8.



N.B. Leading Dimensions only.

FIG. 9.

0151-1

