



# Artefact Analysis of Multistatic Three-Dimensional SAR Imaging with Two Linear Arrays

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## Abstract

In recent years, radar technology has seen much improvement, making multistatic Synthetic Aperture Radar (SAR) sensing a realistic possibility, for example in satellite constellations or unmanned aircraft systems. With such systems, there then comes the requirement to investigate useful multistatic SAR geometries. We provide a microlocal analysis of a multistatic data acquisition geometry for three-dimensional SAR imaging, where the transmitters and receivers effectively constitute two linear arrays, which survey a region of interest (ROI). Using microlocal techniques, we show how the data acquisition geometry influences whether artefacts are likely to be present in the resultant image and how they can be avoided. Our main contribution is to provide a time-gating condition on the data which ensures that artefacts are not present in the ROI. As an independent verification, we provide several numerical simulations which follow a time-independent formulation.

**Keywords** SAR · Bistatic · Multistatic · Artefact · Singularity · Microlocal analysis

## 1 Introduction

In recent years, much research has been conducted on bistatic radar imaging [1–5]. The increased interest is partly due to improvements in technology which have made the proposition of multistatic radar sensing systems realisable. Microlocal analysis [6–9] has proved to be a powerful tool for understanding how backprojected images of scenes are formed and how image artefacts can appear. Understanding how and where these artefacts may form is vital so that one may have the opportunity to carefully design how the radar data are collected (the acquisition geometry) in order that

such artefacts may be avoided in certain parts of the scene, the so-called regions of interest (ROI).

We consider multistatic (multiple bistatic) radar imaging that employs independent radio wave sources and receivers distributed along a pair of horizontal straight line segments  $(\gamma_1, \gamma_2)$  which are at an angle  $\beta$  to each other. These line segments are parametrised by independent variables  $s$  and  $r$ , respectively. Note that this means that for each value of  $s$ , the echo is recorded at all receivers located along  $\gamma_2$  (parametrised by  $r$ ) and so our data have three degrees of freedom, namely  $r, s, t$ , where  $t$  is the fast time. Thus, for  $N$  transmitter positions and  $M$  receiver positions, we have  $MN$  bistatic radar pulse datasets. The SAR geometry is illustrated in Fig. 1. We consider various practical ways of generating the complete data set, as described in cases A–C below. In principle, we expect to be able to reconstruct three-dimensional objects, as we have three-dimensional data associated with  $r, s, t$ . Of course, we may reverse the roles of source and receiver here too if that is more suitable. Note that “multistatic SAR” in this paper is synonymous with linear arrays of transmitters and receivers.

The ability to scan a two-dimensional virtual aperture whilst constraining the transmitter and receiver (or transceiver) positions to lie only on two single lines could prove beneficial in practical scenarios where the region of space in between the lines may be inaccessible to sensors. It

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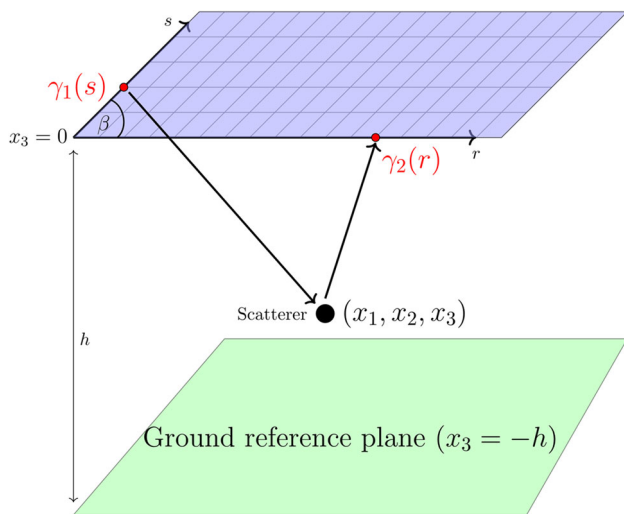


**Fig. 1** Illustration of the proposed acquisition geometry, showing generic positions of the transmitter  $\gamma_1(s)$ , receiver  $\gamma_2(r)$  and a scatterer at  $x$  with coordinates  $(x_1, x_2, x_3)$

may also be the case that accurately arranging or scanning a transmitter or receiver purely along two straight lines may prove more practical to set-up when compared to accurately scanning a two-dimensional range of transmitter or receiver positions.

It is noted that the work here, although directed to sensing with electromagnetic waves, is also relevant to synthetic aperture sensing with other wave modalities including sound waves, which has applications in underwater imaging, medical ultrasound and seismic waves to image underground structures, used for geological exploration, for example.

There are at least three different ways that the multistatic data may be collected, which we now describe.

**Case A:** One may have a set of  $M$  fixed transmitters distributed over  $\gamma_1$  and a set of  $N$  fixed receivers distributed over  $\gamma_2$ . The transmitters would then transmit one pulse, each in sequence, with the receivers providing  $N$  recordings for each of the  $M$  transmission pulses;

**Case B:** As with Case A, but the  $M$  transmitters being substituted by a single transmitter that flies along  $\gamma_1$ , transmitting  $M$  pulses sequentially along its aperture, whilst  $N$  receivers distributed on  $\gamma_2$  record the echo for each of the  $M$  transmissions;

**Case C:** As with Case B, however, the multiple receivers are substituted by a single receiver that flies back and forth along  $\gamma_2$  multiple times, at a speed which is much greater compared to the speed at which the transmitter is flying along  $\gamma_1$ . For each of the  $M$  transmitter locations along  $\gamma_1$ , the transmitter emits  $N$  separate pulses at times when the receiver is located at  $N$  determined positions along  $\gamma_2$ . The receiver rapidly completes a full sweep of  $\gamma_2$ , during which time, the transmitter has remained at approximately the same location on  $\gamma_1$ . The transmitter then moves on to the next transmission

site; the whole process is repeated with the receiver rapidly travelling in the opposite direction along  $\gamma_2$  compared to its previous sweep. For this case, only a single transmitter and a single receiver are required and this is physically realisable using two drones flying at different speeds, so in effect we have two slow time scales here. This set-up synthetically creates a two-dimensional aperture, and for this reason, we keep SAR in the title of the paper, although we have also shown that cases A and B provide other applications as well.

A special case of the acquisition geometry considered here was studied in [5], where the potential for extra artefacts to occur in the image was identified as a possibility. These artefacts may arise due to the breakdown of the local condition in the so-called Bolker condition. More specifically, the Bolker condition requires a certain projection map from the wavefront relation of the scattering operator into the cotangent space of the data manifold to be an embedding; see the discussion at the end of Sect. 2 for the terminology used here. The latter paper showed that the projection may fail to be an immersion (the local condition). This paper revisits the Bolker condition and shows that by designing the acquisition geometry appropriately (the extent and location of the source and receiver line segments), then it is possible to ensure that the Bolker condition is satisfied so that we can then be confident regarding the absence of the artefacts mentioned in [5]. In particular, in this paper, we provide a detailed calculation that shows how one may guarantee that the projection map is injective, putting wavefront set elements of the visible scatterers in a one-to-one correspondence with wavefront set elements of the data. We also consider in this paper the situation where the acquisition geometry has the sources and receivers arranged in two non-orthogonal straight line segments, thereby extending previous results, whilst at the same time showing that the reconstructions are stable with respect to the acquisition geometry.

We wish to analyse the image that could be obtained of scatterers, which in practice could include aircraft, buildings, ground terrain, etc., in the ROI (scene). This image is obtained by backprojecting the scattered wave field recorded at the receivers, which we call the data henceforth. We will presently explain what is meant by the process of backprojection but for now, suffice it to say that it is a concept that is omnipresent in almost all imaging methods. For example, in seismic imaging, we see the concept of “migrating” the seismograms to form an image; see [10] for example. In the radar community, the backprojection image formation is utilised and has been implemented on the simulated results described in Sect. 4 [11].

We are particularly interested in investigating whether the proposed SAR geometry would give rise to artefacts (false scatterers) in the image. When such artefacts occur, their contribution to the data is indistinguishable from that of a

scatterer which is truly present in the scene, and hence, it is imperative to determine their presence.

Our approach in this investigation is both theoretical through the use of microlocal analysis and also numerical to validate the theoretical predictions. Using microlocal analysis, we study the propagation of the non-smooth (singular) part of the radio waves (which concentrates at the wavefronts) and how the corresponding singular component of the scattered waves can be used to infer the non-smooth component of the wave speed, which appears as a scalar coefficient in the wave equation. The non-smooth component of the velocity field is a function of the refractive indices of the materials (such as air, building materials, terrain composition, etc.) encountered by the propagating radio waves. So ultimately, the image obtained is a representation of the estimated material composition of the scene.

Let us assume that before and after scattering, the radio waves are travelling in air with a constant speed of propagation  $c_0$ . When the waves enter other materials where the speed of propagation is  $c$ , then we are interested in estimating the non-smooth part of the deviation  $c_0 - c$  from the background speed, which then provides us with an image of the scene. It turns out to be more mathematically convenient to instead work with the difference  $c_0^{-2} - c^{-2}$ , which we call the reflectivity function,  $V$ , and the goal of the paper is to estimate the singular component of  $V$  from the scattered waves.

We will show how the scattered wavefield can be represented by a so-called Forward Operator,  $F$ , so that  $F : V \rightarrow d$  provides us with a model of the data  $d$ . We will show that, under standard assumptions, the operator  $F$  is a Fourier Integral Operator (FIO). In essence, the image of the scene is normally obtained by applying a weighted adjoint operator  $F^*$  of  $F$  to the data  $d$ . The application of  $F^*$  is essentially known as backprojection. A rough interpretation of backprojection is a weighted summation of the various parts of the data that contribute to a particular point in the scene, where we wish to construct  $V$ .

From previous experience (see [4, 12], [3], for example) we know the operator  $F$  is heavily influenced by the acquisition geometry, i.e. the location of the sources, receivers and recording interval used. The acquisition geometry we use in this study employs sources along a line segment and receivers are along another line segment at the same altitude as the sources, with the two-line segments intersecting at an angle  $\beta$  to each other. The reason for considering this set-up is that, intuitively, if the sources and receivers are collinear ( $\beta = 0$  or  $\beta = \pi$  in our set-up), a single scattering location would give rise to a whole ring of artefacts by symmetry considerations. Although this may not be immediately obvious, it will become clear after the analysis presented, with the reason being due to symmetries in the acquisition geometry. By inclining the line segments to each other, we break

this cylindrical symmetry and thereby avoid such circular artefacts. It will turn out that it is possible to do so with a carefully arranged acquisition geometry, combined with certain filters applied to the data  $d$  as a pre-processing step. However, constraining the sources and receivers to a plane will still give rise to artefacts which are mirror images of true scatters, reflected across the plane generated by the lines of sources and receivers.

Because the image we obtain is essentially  $F^*d = F^*FV$ , the normal operator  $F^*F$  becomes important to study. The sources and receivers lie at a fixed altitude and provided that we know that the scatterers recorded in the data are either below or above this height then our results may be summarised as follows. For a given aspect ratio between the line of sources and receivers, we identify a temporal window where no artefacts will be introduced into the scene. Conversely, for a given region of interest, we may choose an appropriate aspect ratio and temporal window in order to avoid such artefacts. If desirable, one may design a filter to pre-process the data so that it is unnecessary to place any further restrictions on the aspect ratio or temporal window wherein the data is recorded.

In Sect. 2, the forward model is described. Section 3 provides the analysis of non-perpendicular transmitter and receiver axes. Supporting numerical simulation results for four multistatic geometry scenarios are provided in Sect. 4. The simulations follow a time-independent formulation different to that of the previous sections, but closely aligned with common radar laboratory measurement setups [1, 13]. This also serves as independent verification of the method presented in Sect. 3. Conclusions are provided in Sect. 5.

## 2 The Forward Model

In this section, we recall a standard mathematical model, which uses geometrical optics to describe (i) the emission of radio waves from the transmitters, (ii) an approximation for the wave field that has scattered from inhomogeneities in the environment and (iii) the measurements of the resulting scattered waves at receivers. This whole process is captured by the so-called scattering operator, described below.

Let us initially suppose that the transmitter and receiver move along a pair of orthogonal lines at the same altitude. We parametrise these locations, using functions  $\gamma_1 : (s_{\min}, s_{\max}) \rightarrow \mathbb{R}^3$  and  $\gamma_2 : (r_{\min}, r_{\max}) \rightarrow \mathbb{R}^3$  with

$$\gamma_1(s) = (0, 2s, 0) \quad (2.1)$$

$$\gamma_2(r) = (2r, 0, 0). \quad (2.2)$$

We have set the altitude to be zero for mathematical convenience, meaning that the ground is located at  $z = -h$ , where

$h > 0$  represents the height at which the transmitters and receivers are located.

As each component of the electric field emitted by the transmitters satisfies a scalar wave equation, we will model the radio waves using a scalar wave equation. Such models are common in the literature; see for example [3], which we adapted here to obtain following approximate formula for  $u(s, r, t)$ , for waves emitted from the source location  $\gamma_1(s)$ , scattered and then measured at receiver location  $\gamma_2(r)$  at time  $t$ :

$$u(s, r, t) = \int e^{i\Phi(s, r, t, x, \omega)} a(s, r, t, x, \omega) V(x) dx d\omega \quad (2.3)$$

where the amplitude  $a(s, r, t, x, \omega)$  is a symbol that captures the geometrical spreading of the energy along the incident and scattered wavefronts and the phase function

$$\Phi(s, r, t, x, \omega) = \omega(t - |\gamma_1(s) - x|/c_0 - |x - \gamma_2(r)|/c_0) \quad (2.4)$$

comes from the geometrical optics model for high frequency ( $\omega \gg 1$ ) wave propagation. This amplitude also smoothly cuts off the data in order to avoid artefacts in the backprojected image later. The function (or distribution)

$$V(x) = \frac{1}{c_0^2} - \frac{1}{c^2(x)} \quad (2.5)$$

is associated with the deviation of the wave speed  $c$  from the background speed  $c_0$  in air. Such deviations are associated with changes in material properties, where one region of space is occupied by a (possibly inhomogeneous) material, other than air. It turns out that instead of tracking the difference  $c - c_0$ , it is more convenient to track the difference of their squared reciprocals. We observe that our sources are idealised as point sources (see [3]) in space and time. The approximation (2.4) is based on the Born approximation for single scattered waves. We also note that (2.3) is an *oscillatory integral* and as such may represent both a function as well as a distribution.

We consider measurements of  $u(s, r, t)$ , taken over the acquisition manifold

$$\mathcal{M} = \{(s, r, t) \mid s \in (s_{\min}, s_{\max}), r \in (r_{\min}, r_{\max}), t \in (0, T)\}, \quad (2.6)$$

where  $T > 0$  is the duration of the measurements. Since the amplitude  $a$  and phase  $\Phi$  in (2.3) satisfy the standard conditions [8] for a symbol and non-degenerate phase function, we have that the following forward model operator

$$F : V \mapsto u|_{(s, r, t) \in \mathcal{M}} \quad (2.7)$$

given by

$$FV(s, r, t) = \int e^{i\Phi(s, r, t, x, \omega)} a(s, r, t, x, \omega) V(x) dx d\omega$$

is a Fourier Integral Operator (FIO) [8]. We will exploit the well-known properties of FIOs to analyse the image obtained by backprojection methods later in this paper. The backprojection operator is the formal  $L^2$  adjoint of the operator  $F$ . More specifically, it is given by

$$F^*g(x) = \int e^{-i\Phi(s, r, t, x, \omega)} \overline{a(s, r, t, x, \omega)} g(s, r, t) dt dr ds d\omega.$$

The backprojection operator maps the singularities in the data back to the singularities in the reflectivity. This operator can potentially add new singularities in the form of artefacts, and our goal is to try to impose additional restrictions (see Remark 2.1) to avoid such artefacts in the reconstructed image.

We next study the canonical relation  $C$  associated with the FIO  $F$ . This is a useful object associated with an FIO, especially as it keeps track of the propagation of singularities. For the FIO  $F$ , the canonical relation  $C$  gives information on the propagation of singularities of the reflectivity function  $V$  to the data  $FV$ . We explain this in more detail after we have given the canonical relation associated with  $F$  and the dual operator  $F^*$ .

The canonical relation of  $F$  is given by

$$C = \left\{ (s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi; x, -\partial_x \Phi) : \partial_\omega \Phi = 0 \right\}$$

where  $\partial_s$  is the differential in the  $s$  variable, etc., [9]. It is a standard fact that if  $F$  is an FIO, so is  $F^*$ . The canonical relation  $C^t$  of  $F^*$  is obtained from that of  $F$  by switching the order of the variables. That is,

$$C^t = \left\{ (x, -\partial_x \Phi; s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi : \partial_\omega \Phi = 0 \right\}.$$

The canonical relation is a submanifold of the product cotangent bundle  $T^*\mathcal{M} \times T^*X$ . We recall that wavefront set of a distribution on an open set  $X$  is viewed as a subset of the cotangent bundle  $T^*X$ . Although it is possible to carry out all calculations using local coordinates on  $C$ , we note that the natural setting for such a relation is in the cotangent bundle and so we retain the notation, such as  $T^*X$ , etc., to reflect this.

More specifically, in our SAR context, recalling  $\Phi$  from (2.4), we have

$$C = \left\{ \left( s, r, \frac{1}{c_0} \left( |x - \gamma_1(s)| + |x - \gamma_2(r)| \right), \frac{\omega}{c_0} \frac{(x - \gamma_1(s)) \cdot \gamma_1'(s)}{|x - \gamma_1(s)|}, \right. \right. \\ \left. \left. \frac{\omega}{c_0} \frac{(x - \gamma_2(r)) \cdot \gamma_2'(r)}{|x - \gamma_2(r)|}, \omega; \right. \right. \\ \left. \left. x, \frac{\omega}{c_0} \left( \frac{x - \gamma_1(s)}{|x - \gamma_1(s)|} + \frac{x - \gamma_2(r)}{|x - \gamma_2(r)|} \right) \right\}.$$

As already mentioned, the canonical relation is a useful tool to keep track of propagation of singularities. Suppose the reflectivity has a wavefront direction at  $(x, -\partial_x \Phi)$ , then this wavefront direction is propagated to the wavefront directions  $\{(s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi)\}$  in the data provided the variables  $(s, r, t, x)$  satisfy  $\partial_\omega \Phi(s, r, t, x, \omega) = 0$ . The formal  $L^2$  adjoint operator,  $F^*$ , is also an FIO associated with the canonical relation  $C^t$  (which is  $C$  with  $T^*X$  and  $T^*Y$  coordinates flipped, if  $C = \{x, \xi; y, \eta\}$  then  $C^t = \{y, \eta; x, \xi\}$ ). In the context of what we have above,  $C^t$  is simply,

$$C^t = \left\{ \left( x, \frac{\omega}{c_0} \left( \frac{x - \gamma_1(s)}{|x - \gamma_1(s)|} + \frac{x - \gamma_2(r)}{|x - \gamma_2(r)|} \right); \right. \right. \\ \left. s, r, \frac{1}{c_0} (|x - \gamma_1(s)| + |x - \gamma_2(r)|), \right. \\ \left. \frac{\omega}{c_0} \frac{(x - \gamma_1(s)) \cdot \gamma_1'(s)}{|x - \gamma_1(s)|}, \right. \\ \left. \frac{\omega}{c_0} \frac{(x - \gamma_2(r)) \cdot \gamma_2'(r)}{|x - \gamma_2(r)|}, \omega \right\}.$$

As already mentioned, a standard method to form and analyse the image is provided by studying  $F^*$  and the normal operator  $F^*F$ , and in particular, studying what this normal operator does to the singularities of  $V$ . In order to perform this analysis, we study the composition of the canonical relations,  $C^t \circ C$ . This is given as follows:

$$C^t \circ C = \{(x, \xi, \tilde{x}, \tilde{\xi}) : y = \tilde{y}, \eta = \tilde{\eta}\},$$

with  $C^t = \{(x, \xi; \tilde{y}, \tilde{\eta})\}$  and  $C = \{(y, \eta; \tilde{x}, \tilde{\xi})\}$ .

This composition keeps track of how singularities of the reflectivity  $V$  are propagated to the data and subsequently how the singularities from the data are propagated back to the image (our reconstructed estimate of  $V$ ).

To be specific, in our context,  $C^t \circ C$  is as follows:

$$C^t \circ C = \left\{ \left( y, \frac{\omega}{c_0} \left( \frac{y - \gamma_1(s)}{|y - \gamma_1(s)|} + \frac{y - \gamma_2(r)}{|y - \gamma_2(r)|} \right); x, \right. \right. \\ \left. \frac{\omega}{c_0} \left( \frac{x - \gamma_1(s)}{|x - \gamma_1(s)|} + \frac{x - \gamma_2(r)}{|x - \gamma_2(r)|} \right) \right\} : \\ \omega \neq 0, |x - \gamma_1(s)| + |x - \gamma_2(r)| \\ = |y - \gamma_1(s)| + |y - \gamma_2(r)|, \\ \frac{(x - \gamma_1(s)) \cdot \gamma_1'(s)}{|x - \gamma_1(s)|} = \frac{(y - \gamma_1(s)) \cdot \gamma_1'(s)}{|y - \gamma_1(s)|}, \\ \frac{(x - \gamma_2(r)) \cdot \gamma_2'(r)}{|x - \gamma_2(r)|} = \frac{(y - \gamma_2(r)) \cdot \gamma_2'(r)}{|y - \gamma_2(r)|} \Big\}. \quad (2.8)$$

The composition of two FIOs is not always an FIO, unlike the situation for the case of pseudodifferential oper-

ators ( $\Psi$ DOs), which are special subclass of FIOs. In such instances, one is interested in analysing the left and right projection maps  $\pi_L$  and  $\pi_R$  from the canonical relation  $C$ , given by

$$\pi_L : \left\{ (s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi; x, -\partial_x \Phi) : \right. \\ \left. \partial_\omega \Phi = 0 \right\} \rightarrow \left\{ (s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi) \right\}$$

and

$$\pi_R : \left\{ (s, r, t, \partial_s \Phi, \partial_r \Phi, \partial_t \Phi; x, -\partial_x \Phi) : \right. \\ \left. \partial_\omega \Phi = 0 \right\} \rightarrow \left\{ (x, -\partial_x \Phi) \right\}$$

to understand whether the composition of the two FIOs  $F$  and  $F^*$  introduces artefacts in the reconstructed images. One instance where the composition becomes a  $\Psi$ DO is the case when the left projection map  $\pi_L$  is an embedding, that is, it is a proper injective immersion. This is the so-called Bolker condition. Roughly speaking, injectivity of  $\pi_L$  tells us that there are no artefacts in the reconstructed image, and injectivity of  $d\pi_L$  tells us that the normal operator (the composition of the forward operator with its  $L^2$  adjoint) has the additional structure of a pseudodifferential operator, a property that may be further exploited to provide a more quantitative estimate of  $V$ , although we do not pursue that in this work. In this special situation, we can be confident that no artefacts are introduced in the backprojected image. In the sections below, we have given sufficient conditions to ensure that there are no artefacts in the reconstructed image by showing that the Bolker condition is satisfied.

**Remark 2.1** [Remark on the Bolker Condition] In particular, we seek condition(s) which guarantee the absence artefacts in the reconstructed image, obtained from backprojection. Mathematically, this means we seek conditions under which the projection  $\pi_L$  is an injective immersion. For the special case of  $\beta = \pi/2$  (recall that  $\beta$  is the angle between  $\gamma_1$  and  $\gamma_2$ ; see Fig. 1), the results on the “walkaway geometry” in [5] show that if the traveltimes are sufficiently large, then  $\pi_L$  is an immersion. In this paper, we extend the work in [5] to an arbitrary value of  $0 < \beta \leq \pi/2$  and we also establish that  $\pi_L$  is injective for general  $0 < \beta \leq \pi/2$ , provided that the time is again sufficiently large, thereby establishing the Bolker condition to avoid artefacts.

### 3 Analysis for Non-perpendicular Transmitter and Receiver Arrays

In this section, we consider the composition of canonical relations  $C^t \circ C$  given in (2.8) for the case where  $\gamma_1$  and  $\gamma_2$  are two straight line segments, where a linear array of transmitters and receivers are, respectively, located and these lines are inclined at an angle  $\beta$  to each other.

In other words, with these data, we are interested in the composition of the forward scattering operator with that of its adjoint at the level of composition of their corresponding canonical relations.

Recall that the forward operator for potentially non-perpendicular  $\gamma_1$  and  $\gamma_2$  is

$$FV(s, r, t) = \int e^{i\omega(t - \frac{1}{c_0}(|\gamma_1(s) - x| + |x - \gamma_2(r)|))} a(s, r, x, \omega) V(x) dx d\omega, \quad (3.1)$$

where  $\gamma_1(s) = (2s \cot \beta, 2s, 0)$  and  $\gamma_2(r) = (2r, 0, 0)$ . We note that when  $\cot \beta = 0$  (corresponding to when  $\beta = \pi/2$ ), the lines are perpendicular to each other. Recall that

$$C = \left\{ \left( s, r, \frac{1}{c_0} (|x - \gamma_1(s)| + |x - \gamma_2(r)|) \right), \frac{\omega(\gamma_1(s) - x) \cdot \gamma_1(s)}{c_0|\gamma_1(s) - x|}, \frac{\omega(\gamma_2(r) - x) \cdot \gamma_2(r)}{c_0|\gamma_2(r) - x|}, \omega; \right. \\ \left. x, \frac{\omega}{c_0} \left( \frac{\gamma_1(s) - x}{|\gamma_1(s) - x|} + \frac{\gamma_2(r) - x}{|\gamma_2(r) - x|} \right); \omega \neq 0 \right\}. \quad (3.2)$$

We note that  $(s, r, x, \omega)$  is a global parameterisation for  $C$ . We assume for simplicity that  $\cot \beta > 0$ .

Since we are interested in studying  $C^t \circ C$ , with  $\gamma_1$  and  $\gamma_2$  as above, our strategy is to analyse the following three equations; see (2.8):

$$\sqrt{(x_1 - 2s \cot \beta)^2 + (x_2 - 2s)^2 + x_3^2} \\ + \sqrt{(x_1 - 2r)^2 + x_2^2 + x_3^2} \\ = \sqrt{(y_1 - 2s \cot \beta)^2 + (y_2 - 2s)^2 + y_3^2} \\ + \sqrt{(y_1 - 2r)^2 + y_2^2 + y_3^2}, \quad (3.3)$$

$$\frac{\cot \beta (x_1 - 2s \cot \beta) + (x_2 - 2s)}{\sqrt{(x_1 - 2s \cot \beta)^2 + (x_2 - 2s)^2 + x_3^2}} \\ = \frac{\cot \beta (y_1 - 2s \cot \beta) + (y_2 - 2s)}{\sqrt{(y_1 - 2s \cot \beta)^2 + (y_2 - 2s)^2 + y_3^2}}, \quad (3.4)$$

$$\frac{x_1 - 2r}{\sqrt{(x_1 - 2r)^2 + x_2^2 + x_3^2}} = \frac{y_1 - 2r}{\sqrt{(y_1 - 2r)^2 + y_2^2 + y_3^2}}. \quad (3.5)$$

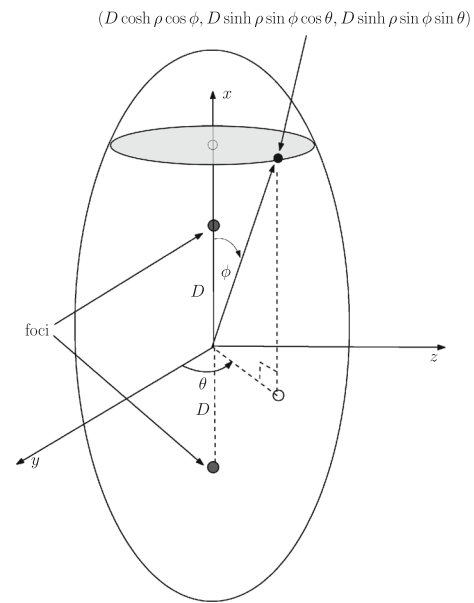


Fig. 2 Prolate spheroidal coordinates

Our goal is to show that the only intersection points of these three surfaces are the points

$$(x_1, x_2, x_3) = (y_1, y_2, y_3), \quad (x_1, x_2, x_3) = (y_1, y_2, -y_3). \quad (3.6)$$

Note that by straightforward verification, (3.6) satisfies the three Eqs. (3.3) - (3.5). Equation (3.6) tells us that the mirror artefact is always present in the reconstruction procedure involving the study of the composition operator  $F^*F$ . However, note that the mirror artefact appears in a region of non-interest in the sense that if we are only interested in imaging the region strictly below the data acquisition plane, then the mirror artefact would not be present. Imposing further restrictions so that the Bolker assumption is valid would show that no additional artefacts are present in the region of interest. This is what we do in the later calculations.

The three Eqs. (3.3)–(3.5) are easier to study in prolate spheroidal coordinates, as detailed below. Let us consider a generic ellipsoid of the form

$$\sqrt{(x_1 - 2s \cot \beta)^2 + (x_2 - 2s)^2 + x_3^2} \\ + \sqrt{(x_1 - 2r)^2 + x_2^2 + x_3^2} = t. \quad (3.7)$$

The foci of this ellipsoid are  $\gamma_1(s) = (2s \cot \beta, 2s, 0)$  and  $\gamma_2(r) = (2r, 0, 0)$ . The centre of this ellipsoid is  $(s \cos \beta + r, s, 0)$ . We apply a translation so that the ellipsoid is centred at the origin, which means the new foci are located at  $(s \cot \beta - r, s, 0)$  and  $(r - s \cot \beta, -s, 0)$ . Next we rotate the ellipsoid by an angle  $\alpha$  defined below, so that the ellipsoid

has an axis aligned with the  $x_1$ -axis. We define

$$\delta = \cot \beta, \quad \tan \alpha = \frac{s}{r - s\delta}. \quad (3.8)$$

The translation and rotation mentioned above are implemented via the following change of coordinates:

$$\tilde{x}_1 = \cos \alpha (x_1 - r - s \cot \beta) - \sin \alpha (x_2 - s) \quad (3.9)$$

$$\tilde{x}_2 = \sin \alpha (x_1 - r - s \cot \beta) + \cos \alpha (x_2 - s) \quad (3.10)$$

$$\tilde{x}_3 = x_3. \quad (3.11)$$

The ellipsoid defined by (3.7) is transformed in the  $\tilde{x}_1, \tilde{x}_2$  and  $\tilde{x}_3$  coordinates to

$$\sqrt{(\tilde{x}_1 + D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2} + \sqrt{(\tilde{x}_1 - D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2} = t \quad (3.12)$$

where

$$D = \sqrt{s^2 + (\delta s - r)^2}.$$

Hence, (3.3) transforms to

$$\begin{aligned} & \sqrt{(\tilde{x}_1 + D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2} + \sqrt{(\tilde{x}_1 - D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2} \\ &= \sqrt{(\tilde{y}_1 + D)^2 + \tilde{y}_2^2 + \tilde{y}_3^2} + \sqrt{(\tilde{y}_1 - D)^2 + \tilde{y}_2^2 + \tilde{y}_3^2}. \end{aligned} \quad (3.13)$$

Next, (3.4) transforms to

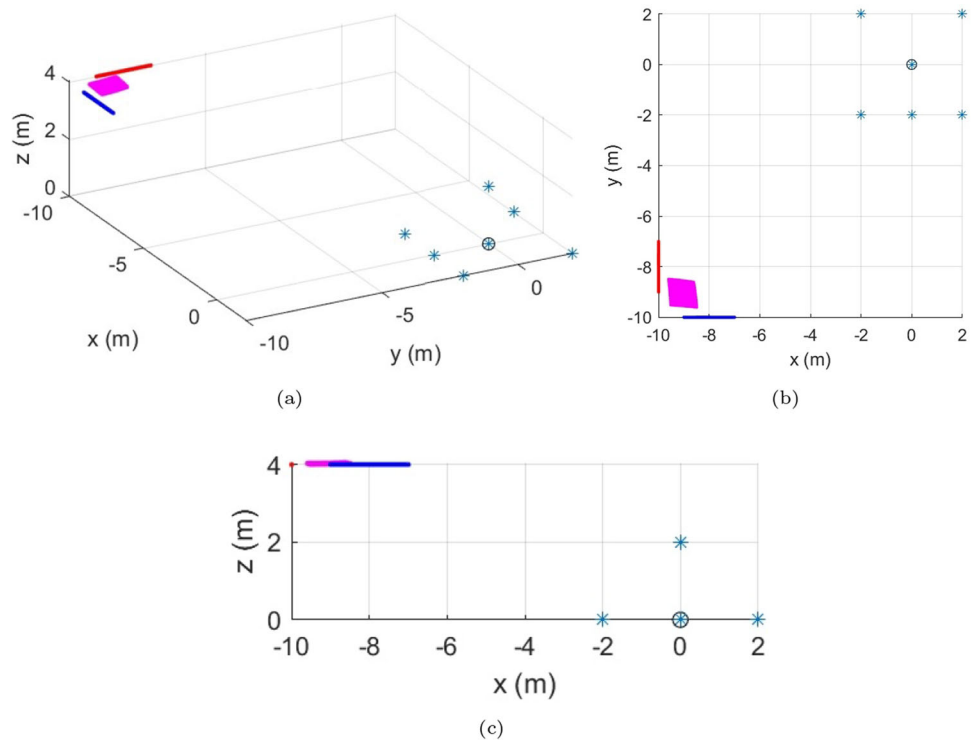
$$\begin{aligned} & \frac{\delta \left\{ \tilde{x}_1 \cos \alpha + \tilde{x}_2 \sin \alpha + D \cos \alpha \right\} + \left\{ -\tilde{x}_1 \sin \alpha + \tilde{x}_2 \cos \alpha - D \sin \alpha \right\}}{\sqrt{(\tilde{x}_1 + D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2}} \\ &= \frac{\delta \left\{ \tilde{y}_1 \cos \alpha + \tilde{y}_2 \sin \alpha + D \cos \alpha \right\} + \left\{ -\tilde{y}_1 \sin \alpha + \tilde{y}_2 \cos \alpha - D \sin \alpha \right\}}{\sqrt{(\tilde{y}_1 + D)^2 + \tilde{y}_2^2 + \tilde{y}_3^2}}. \end{aligned} \quad (3.14)$$

Similarly, (3.5) transforms to

$$\begin{aligned} & \frac{\tilde{x}_1 \cos \alpha + \tilde{x}_2 \sin \alpha - D \cos \alpha}{\sqrt{(\tilde{x}_1 - D)^2 + \tilde{x}_2^2 + \tilde{x}_3^2}} \\ &= \frac{\tilde{y}_1 \cos \alpha + \tilde{y}_2 \sin \alpha - D \cos \alpha}{\sqrt{(\tilde{y}_1 - D)^2 + \tilde{y}_2^2 + \tilde{y}_3^2}}. \end{aligned} \quad (3.15)$$

We now use prolate spheroidal coordinates to simplify these equations. We let

**Fig. 3** Different views of the simulation scene for the radar geometry with transmitter to receiver axis angle  $\beta = 90^\circ$ . The blue stars represent point scatterers; the black circle represents the scene centre; the blue and red lines represent the transmitter and receiver positions, respectively, and the magenta surface is constituted by the corresponding virtual equivalent monostatic antenna positions



$$\begin{aligned}\tilde{x}_1 &= D \cosh \rho \cos \phi, \\ \tilde{x}_2 &= D \sinh \rho \sin \phi \cos \theta, \\ \tilde{x}_3 &= D \sinh \rho \sin \phi \sin \theta.\end{aligned}\quad (3.16)$$

with  $D > 0$ ,  $\phi \in [0, \pi]$  and  $\theta \in [0, 2\pi]$ ; see Fig. 2.

We next substitute (3.16) into (3.13), (3.14), and (3.15). We start with (3.13). We get (denoting prime for the  $\tilde{y}$  variables)

$$\begin{aligned}D(\cosh \rho + \cos \phi) + D(\cosh \rho - \cos \phi) \\ = D(\cosh \rho' + \cos \phi') + D(\cosh \rho' - \cos \phi'),\end{aligned}$$

which gives  $\cosh \rho = \cosh \rho'$  and this in turn implies  $\rho = \rho'$ .

Next substituting (3.16) into (3.14), we get

Cross multiplying, we obtain

$$\begin{aligned}\cosh^2 \rho \cos \alpha \cos \phi + \cosh \rho \sinh \rho \sin \alpha \sin \phi \cos \theta \\ - \sinh \rho \sin \alpha \cos \phi' \sin \phi \cos \theta \\ - \cosh^2 \rho \cos \alpha \cos \phi' - \cosh \rho \sinh \rho \sin \alpha \sin \phi' \cos \theta' \\ + \sinh \rho \sin \alpha \cos \phi \sin \phi' \cos \theta' \\ + \cos \phi' \cos \alpha - \cos \phi \cos \alpha = 0.\end{aligned}\quad (3.20)$$

In order to simplify terms in (3.18) and (3.20), we let

$$\begin{aligned}A &= \cos \phi - \cos \phi' \\ B &= \sin \phi \cos \theta - \sin \phi' \cos \theta' \\ C &= \sin \phi' \cos \theta' \cos \phi - \sin \phi \cos \phi' \cos \theta.\end{aligned}\quad (3.21)$$

$$\begin{aligned}\frac{\delta \{ D \cos \alpha + D \cos \alpha \cosh \rho \cos \phi + D \sin \alpha \sinh \rho \sin \phi \cos \theta \}}{D(\cosh \rho + \cos \phi)} + \frac{\{ -D \sin \alpha \cosh \rho \cos \phi + D \cos \alpha \sinh \rho \sin \phi \cos \theta - D \sin \alpha \}}{D(\cosh \rho + \cos \phi)} \\ = \frac{\delta \{ D \cos \alpha + D \cos \alpha \cosh \rho \cos \phi' + D \sin \alpha \sinh \rho \sin \phi' \cos \theta' \}}{D(\cosh \rho + \cos \phi')} + \frac{\{ -D \sin \alpha \cosh \rho \cos \phi' + D \cos \alpha \sinh \rho \sin \phi' \cos \theta' - D \sin \alpha \}}{D(\cosh \rho + \cos \phi')}.\end{aligned}\quad (3.17)$$

Cross multiplying, we obtain

$$\begin{aligned}D\delta \cosh \rho \cos \alpha + D\delta \cosh^2 \rho \cos \alpha \cos \phi \\ + D\delta \sin \alpha \sinh \rho \cosh \rho \sin \phi \cos \theta \\ - s \cosh \rho - D \sin \alpha \cosh^2 \rho \cos \phi \\ + D \cos \alpha \sinh \rho \cosh \rho \sin \phi \cos \theta \\ + D\delta \cos \phi' \cos \alpha + D\delta \cos \alpha \cosh \rho \cos \phi \cos \phi' \\ + D\delta \sin \alpha \sinh \rho \sin \phi \cos \phi' \cos \theta \\ - s \cos \phi' - D \sin \alpha \cosh \rho \cos \phi \cos \phi' \\ + D \cos \alpha \sinh \rho \sin \phi \cos \phi' \cos \theta \\ = D\delta \cosh \rho \cos \alpha + D\delta \cosh^2 \rho \cos \alpha \cos \phi' \\ + D\delta \sin \alpha \sinh \rho \cosh \rho \sin \phi' \cos \theta' \\ - s \cosh \rho - D \sin \alpha \cosh^2 \rho \cos \phi' \\ + D \cos \alpha \sinh \rho \cosh \rho \sin \phi' \cos \theta' \\ + D\delta \cos \alpha \cos \phi + D\delta \cos \alpha \cosh \rho \cos \phi' \cos \phi \\ + D\delta \sin \alpha \sinh \rho \sin \phi' \cos \phi \cos \theta \\ - s \cos \phi - D \sin \alpha \cosh \rho \cos \phi \cos \phi' \\ + D \cos \alpha \sinh \rho \sin \phi' \cos \phi \cos \theta' = 0.\end{aligned}\quad (3.18)$$

Next we substitute (3.16) into (3.15) to get

$$\begin{aligned}\frac{\cosh \rho \cos \alpha \cos \phi + \sinh \rho \sin \alpha \sin \phi \cos \theta - \cos \alpha}{(\cosh \rho - \cos \phi)} \\ = \frac{\cosh \rho \cos \alpha \cos \phi' + \sinh \rho \sin \alpha \sin \phi' \cos \theta' - \cos \alpha}{(\cosh \rho - \cos \phi')}.\end{aligned}\quad (3.19)$$

Note that  $C$  can be written as

$$C = \sin \phi' \cos \theta' A - B \cos \phi'. \quad (3.22)$$

Simplifying (3.18) using (3.21), we get

$$\begin{aligned}A(\delta \cos \alpha - \sin \alpha) \sinh \rho + B(\cos \alpha + \delta \sin \alpha) \cosh \rho \\ - (\cos \alpha + \delta \sin \alpha) C = 0.\end{aligned}\quad (3.23)$$

Similarly, simplifying (3.20) using (3.21), we obtain

$$\begin{aligned}(\cosh^2 \rho - 1) \cos \alpha A + \cosh \rho \sinh \rho \sin \alpha B \\ + \sinh \rho \sin \alpha C = 0.\end{aligned}\quad (3.24)$$

Simplifying this further, we have

$$A \sinh \rho \cos \alpha + \cosh \rho \sin \alpha B + \sin \alpha C = 0. \quad (3.25)$$

We combine (3.22), (3.23), and (3.25) into a system:

$$\begin{aligned}A(\delta \cos \alpha - \sin \alpha) \sinh \rho + B(\cos \alpha + \delta \sin \alpha) \cosh \rho \\ - C(\cos \alpha + \delta \sin \alpha) = 0, \\ A \sinh \rho \cos \alpha + B \cosh \rho \sin \alpha + C \sin \alpha = 0 \\ A \sin \phi' \cos \theta' - B \cos \phi' - C = 0.\end{aligned}\quad (3.26)$$

Let us write this in matrix form:

$$\begin{pmatrix} \sinh \rho (\delta \cos \alpha - \sin \alpha) & \cosh \rho (\cos \alpha + \delta \sin \alpha) & -(\cos \alpha + \delta \sin \alpha) \\ \sinh \rho \cos \alpha & \cosh \rho \sin \alpha & \sin \alpha \\ \sin \phi' \cos \theta' & -\cos \phi' & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \quad (3.27)$$

We compute the determinant  $\Delta$  of the matrix on the left-hand side. We get

$$\Delta = \cosh \rho \sinh \rho + \sinh \rho \cos \phi' (\delta \sin 2\alpha + \cos 2\alpha) + \cosh \rho \sin \phi' \cos \theta' (\sin 2\alpha + 2\delta \sin^2 \alpha).$$

Next, let us impose conditions on  $\rho$  to make this determinant non-vanishing. We have

$$\begin{aligned} \Delta &\geq \cosh \rho \sinh \rho - (1 + |\delta|) \sinh \rho - (1 + 2|\delta|) \cosh \rho \\ &= \frac{e^{2\rho} - e^{-2\rho}}{4} - \frac{(1 + |\delta|)}{2} (e^\rho - e^{-\rho}) \\ &\quad - \frac{1 + 2|\delta|}{2} (e^\rho + e^{-\rho}) \\ &= \frac{e^{-2\rho}}{4} (e^{4\rho} - 1 - 2(1 + |\delta|)(e^{3\rho} - e^\rho) \\ &\quad - 2(1 + 2|\delta|)(e^{3\rho} + e^\rho)) \\ &= \frac{e^{-2\rho}}{4} (e^{4\rho} - 1 - (2 + 2|\delta| + 2 + 4|\delta|)e^{3\rho} \\ &\quad + (2 + 2|\delta| - 2 - 4|\delta|)e^\rho) \\ &= \frac{e^{-2\rho}}{4} (e^{4\rho} - 1 - (4 + 6|\delta|)e^{3\rho} - 2|\delta|e^\rho). \end{aligned} \quad (3.28)$$

Therefore, the only additional singularities arise exactly as follows:

$$((x_1, x_2, x_3) = (y_1, y_2, y_3), \quad ((x_1, x_2, x_3) = (y_1, y_2, -y_3)). \quad (3.29)$$

We additionally remark that the calculations in this section generalise the analysis made for the “walkaway geometry” in [5, §6.2] which only considered the case of  $\gamma_1$  and  $\gamma_2$  being perpendicular ( $\beta = 90^\circ$ ).

Next we compute the derivative of the left projection from the canonical relation. Using the parametrisation  $(s, r, x, \omega)$  for  $C$ , we have the left projection  $\pi_L$  to be

$$\pi_L(s, r, x, \omega) = (s, r, \frac{1}{c_0} (|x - \gamma_1(s)| + |x - \gamma_2(r)|), \frac{\omega(\gamma_1(s) - x) \cdot \gamma_1(s)}{c_0 |\gamma_1(s) - x|}, \frac{\omega(\gamma_2(r) - x) \cdot \gamma_2(r)}{c_0 |\gamma_2(r) - x|}, \omega),$$

with  $\gamma_1(s) = (2s \cot \beta, 2s, 0)$  and  $\gamma_2(r) = (2r, 0, 0)$ . For simplicity, we denote  $A = \sqrt{(x_1 - 2r)^2 + x_2^2 + x_3^2}$  and  $B = \sqrt{(x_1 - 2s \cot \beta)^2 + (x_2 - 2s)^2 + x_3^2}$ . To calculate the determinant of  $d\pi_L$ , it is enough to calculate the determinant of the following submatrix:

$$\begin{pmatrix} \frac{x_1 - 2r}{A} + \frac{x_1 - 2s \cot \beta}{B} & \frac{x_2}{A} + \frac{x_2 - 2s}{B} & \frac{x_3}{A} + \frac{x_3}{B} \\ \frac{x_2^2 + x_3^2}{A^3} & -\frac{(x_1 - 2r)x_2}{A^3} & -\frac{(x_1 - 2r)x_3}{A^3} \\ \frac{\cot \beta ((x_2 - 2s)^2 + x_3^2) - (x_1 - 2s \cot \beta)(x_2 - 2s)}{B^3} & \frac{(x_1 - 2s \cot \beta)^2 + x_3^2 - \cot \beta (x_2 - 2s)(x_1 - 2s \cot \beta)}{B^3} & -\frac{x_3 (\cot \beta (x_1 - 2s \cot \beta) + (x_2 - 2s))}{B^3} \end{pmatrix}.$$

We choose  $\rho \gg 1$ , so that  $1 < e^{3\rho}$  and  $e^\rho < e^{3\rho}$ . With this, we get

$$\begin{aligned} \Delta &> \frac{e^{-2\rho}}{4} (e^{4\rho} - (5 + 8|\delta|)e^{3\rho}) \\ &= \frac{e^\rho}{4} (e^\rho - (5 + 8|\delta|)). \end{aligned}$$

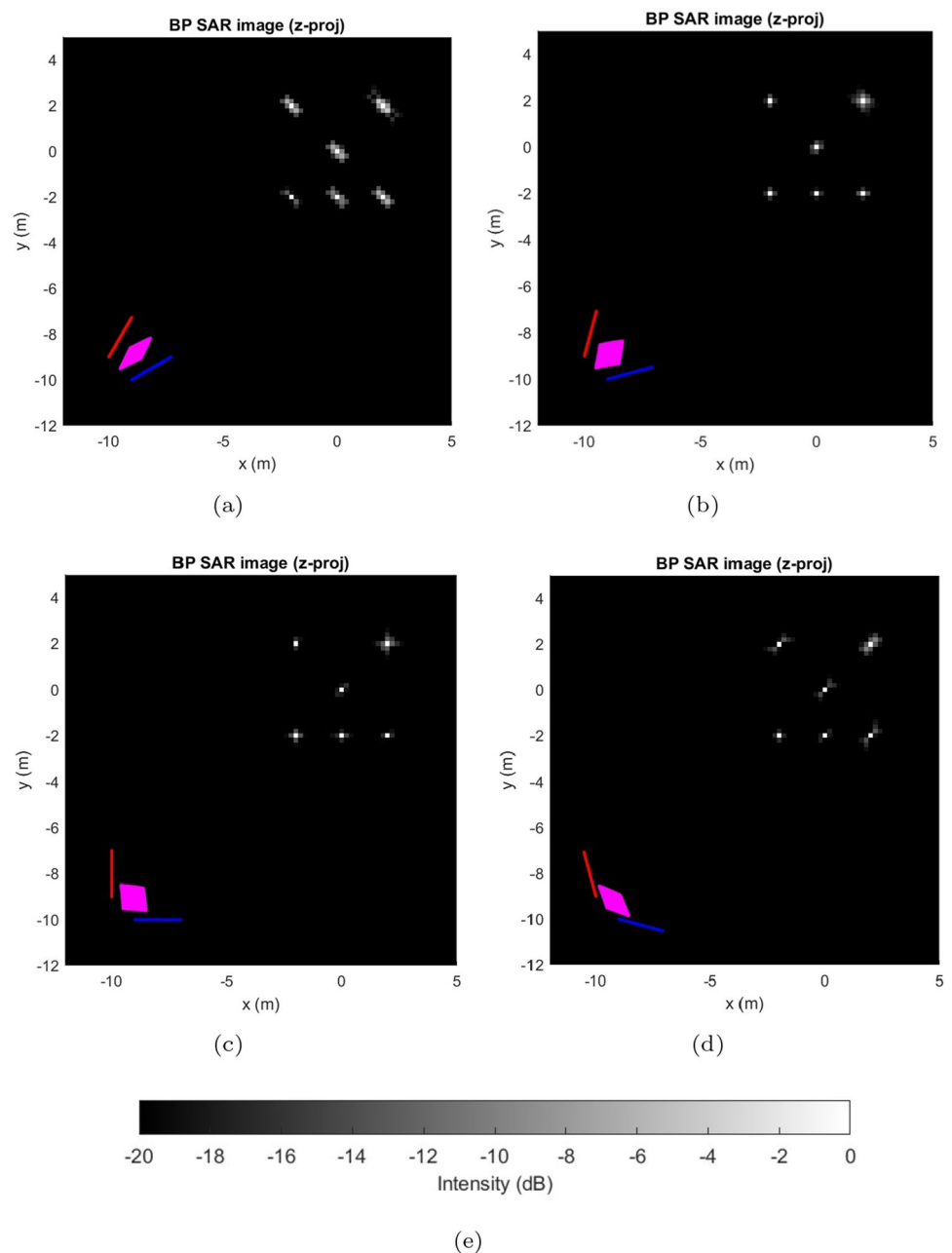
Given a fixed  $0 \leq |\delta| < \infty$ , we can choose  $\rho$  large enough so that  $\Delta > 0$ . This would mean  $A = B = C = 0$ . Beginning with  $A = 0$ , we get  $\cos \phi = \cos \phi'$ . Since  $0 \leq \phi, \phi' \leq \pi$ , we have that  $\phi' = \phi$ . Hence,  $\sin \phi = \sin \phi'$ , which then gives that  $\cos \theta = \cos \theta'$ . We then have  $\theta = \theta'$  or  $\theta' = 2\pi - \theta$ .

The determinant of the matrix above is

$$\det(d\pi_L) = \frac{x_3}{A^2 B^2} \left\{ (x_1, x_2, x_3) \cdot \left\{ \frac{(x_1 - 2s \cot \beta, x_2 - 2s, x_3)}{B} + \frac{(x_1 - 2r, x_2, x_3)}{A} \right\} \right\}.$$

Obviously, this vanishes when  $x_3 = 0$ . Since  $x$  lies on the ellipsoid defined by  $|x - \gamma_1(s)| + |x - \gamma_r(s)| = t$  for  $t > 0$ , if the ellipsoids are large enough such that the origin is an interior point to such ellipsoids, then the second expression in  $\det(d\pi_L)$  is nonzero. Hence, if we were to restrict attention to the part of the domain where  $x_3 < 0$  (recall that we have let  $h = 0$  for simplicity), then  $d\pi_L$  is injective. Combining

**Fig. 4** Three-dimensional SAR image  $z$ -direction MIPs (top view) with SAR aperture overlaid, for radar geometries with transmitter to receiver axis angles  $\beta$  of **a**  $30^\circ$ , **b**  $60^\circ$ , **c**  $90^\circ$  and **d**  $120^\circ$ . The colour bar **e** represents normalised scattering intensity and pertains to all images



with the earlier calculations showing that for large enough ellipsoids, the left projection is injective as well, we have that the Bolker condition is satisfied. Therefore, the imaging operator  $F^*F$  restricted to domains where  $x_3 < 0$  is a  $\Psi$ DO, and thus, the image reconstruction procedure based on the composition of  $F$  with its  $L^2$  adjoint cannot create artefacts in reconstruction.

## 4 Numerical Simulations

To demonstrate the proposed multistatic SAR aperture approach, a free-space simulation was conducted for a specific scenario involving isotropic point scatterers, resulting in three-dimensional SAR imagery. The approach is a bistatic

**Fig. 5** Three-dimensional SAR image y-direction MIPs (side view) with SAR aperture overlaid, for radar geometries with transmitter to receiver axis angles  $\beta$  of **a**  $30^\circ$ , **b**  $60^\circ$ , **c**  $90^\circ$  and **d**  $120^\circ$ . The colour bar **e** represents normalised scattering intensity and pertains to all images

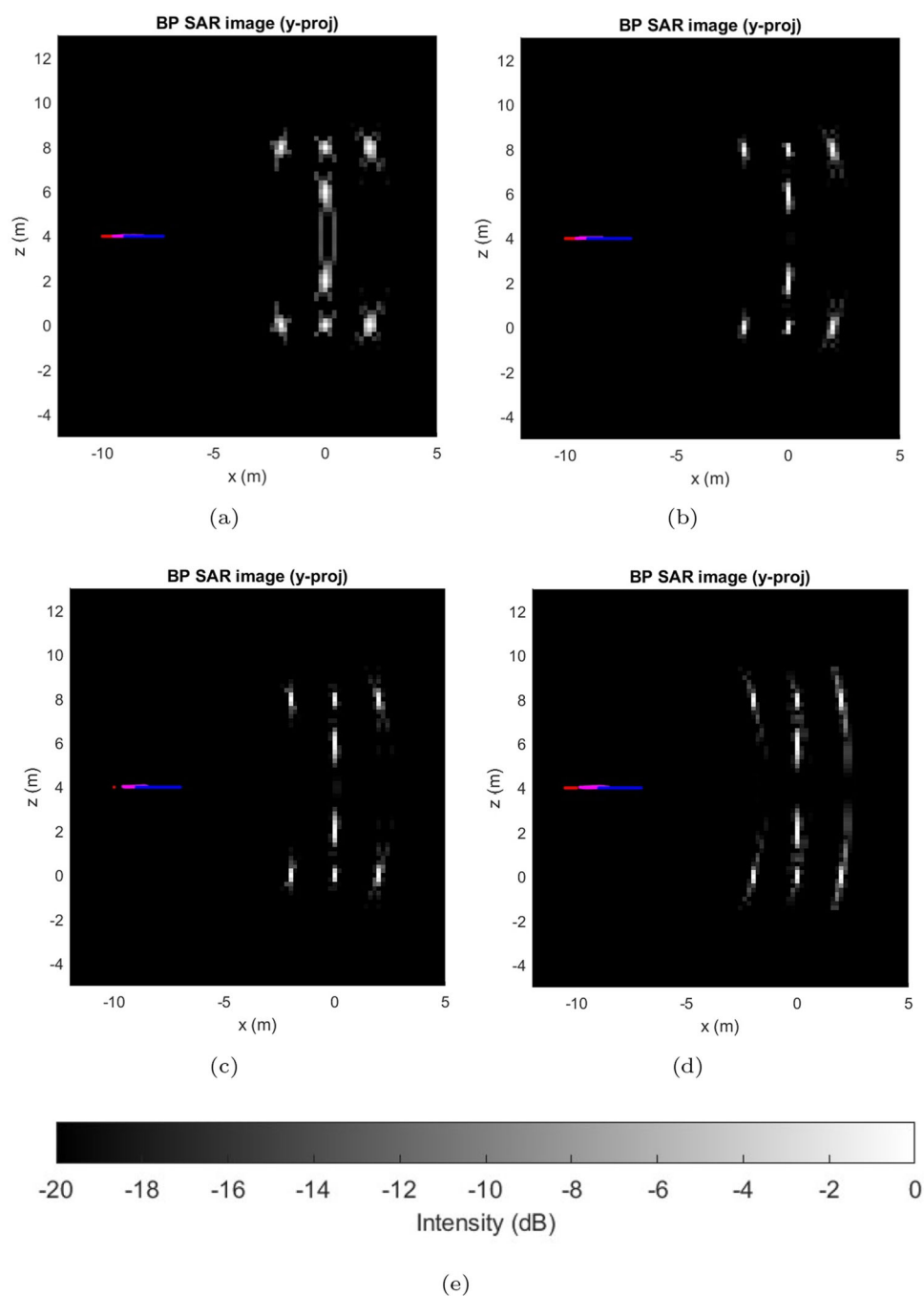


**Table 1** SAR geometry parameters

| Geometry            | $\eta$ (degrees) | $\psi_H$ (degrees) | $\psi_V$ (degrees) | $r_r$ (cm) | $r_{crH}$ (cm) | $r_{crV}$ (cm) |
|---------------------|------------------|--------------------|--------------------|------------|----------------|----------------|
| $\beta = 30^\circ$  | 8.6              | 2.7                | 2.6                | 15.0       | 31.6           | 33.2           |
| $\beta = 60^\circ$  | 10.6             | 4.8                | 2.3                | 15.1       | 18.1           | 37.7           |
| $\beta = 90^\circ$  | 12.2             | 6.3                | 1.8                | 15.1       | 13.6           | 47.3           |
| $\beta = 120^\circ$ | 13.3             | 7.4                | 1.3                | 15.1       | 11.7           | 69.0           |

generalisation of a monostatic time-independent version [1, 14] which aligns closely with the radar measurement procedure utilised at the Ground-Based SAR laboratory based at Cranfield University [1, 13]. The measurement provides phase history data, which is a function of frequency and antenna position. Hence, whilst these simulations provide a validation of the mathematical calculations in this paper, they also underpin the experimental measurements to be performed in future validation work. The simulation approach also serves as independent verification of the methods presented in Sect. 3.

The simulation calculates the phase history array contribution from each transmitter–receiver pair and from each scatterer. The phase history  $P$  is a three-dimensional array function of the transmitter and receiver positions and of the radar frequency  $\omega = 2\pi f$ , where  $f$  is the radar frequency in GHz,

$$P(f, s, r) = \sum_{n=1}^N a_n \exp \left[ -\frac{2\pi i f}{c_0} (|\gamma_1(s) - x_n| + |x_n - \gamma_2(r)|) \right] \quad (4.1)$$

where the summation is over  $N$  scatterers, each labelled by  $n$ , and  $a_n$  is a given scattering amplitude for each scatterer. Here,  $c_0 = 299792458$  m/s is the speed of light in vacuum.

Simulations were conducted for the cases with transmitter to receiver axis angle  $\beta$  of  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$ , and  $120^\circ$ . The  $90^\circ$  SAR geometry case is presented in Fig. 3, which shows different views of the scene. Note that in simulation Figs. 3, 4 and 5, the axis labels differ to those of the previous sections and are now  $x$ ,  $y$  and  $z$ , with the plane containing the receiver and transmitter positions, also referred to as the transceiver plane, at the height of  $H = 4$  m (at  $z = 4$  m).

The blue stars in Fig. 3 represent point scatterers, of which there are six at  $z = 0$  m and one at  $z = 2$  m. The black circle represents the chosen scene centre. The transmitter linear aperture is represented by the blue line and the receiver by the red line, each aperture with an extent of  $L_a = 2$  m, with the corresponding virtual equivalent monostatic (VEM) aperture displayed in magenta. The VEM aperture extent allows the estimation of the cross-range resolving capability of the SAR geometry at a point in the scene, which for Figs. 3, 4 and 5 is the designated scene centre. For any bistatic geometry scattering from a point in the scene, the VEM direction bisects the lines between the point to the transmitter and receiver positions. We set the distance of the VEM position from the point to be equal to the bistatic range to that point.

The mean range to the scene centre is about  $R_m = 14$  m, whereas the cross-range extent of the scatterer distribution is 4 m, so that the scatterer scene is in the SAR near-field (having significantly curved wavefronts over the scene).

The simulation implemented a stepped frequency radar waveform with a centre frequency  $f_c = 10$  GHz, a bandwidth  $B_w = 1$  GHz and sampled with the number of frequency points  $f_p = 512$ . This frequency sampling density gives an unambiguous range extent of  $L_r = c_0 f_p / 2B_w = 76.8$  m. This is the extent in range in which one will not expect Discrete Fourier transform aliasing to appear according to the Nyquist sampling criterion. For SAR imaging, a similar relation applies in cross-range, giving the required antenna position sampling density along the two antenna lines for a given desired unambiguous cross-range extent [15]. Considering the VEM antenna positions for geometries where  $L_a \ll R_m$ , the unambiguous cross-range extents in the horizontal ( $L_{crH}$ ) and vertical ( $L_{crV}$ ) directions for scatterers in the region of the scene centre can be shown to be approximately as follows:

$$\begin{aligned} L_{crH} &\approx \frac{c_0 b_p R_m}{f_c L_a \sin(\beta/2)} \sqrt{R_m^2 - H^2} \\ L_{crV} &\approx \frac{c_0 b_p R_m}{f_c L_a \cos(\beta/2)} \frac{\sqrt{R_m^2 - H^2}}{H} \end{aligned} \quad (4.2)$$

where  $b_p$  are the number of uniformly distributed antenna positions along each line and the other variables are defined earlier in this section. Hence, to achieve an image free of Discrete Fourier transform aliasing over the volume extent of a 30 m cube, for the different  $\beta$  cases, it was sufficient to use 128 transmitters arranged along the transmitter linear aperture and 128 receivers arranged along the receiver linear aperture. In total, the simulation implemented  $128^2 = 16384$  bistatic antenna measurements in this multistatic SAR scenario.

The image radar resolution in a given direction, at a point in a SAR image, is determined by the extent of the corresponding k-space image support in the spatial frequency domain, in the same direction [15, 16]. In the bistatic near-field case, this leads to approximations for resolution in range  $r_r$  and cross-range in the horizontal and vertical directions  $r_{crH}$  and  $r_{crV}$  as follows:

$$\begin{aligned} r_r &\approx \frac{c_0}{2B_w \cos(\eta/2)} \\ r_{crH} &\approx \frac{c_0}{2f_c \cos(\eta/2) \psi_H} \\ r_{crV} &\approx \frac{c_0}{2f_c \cos(\eta/2) \psi_V} \end{aligned} \quad (4.3)$$

where  $\eta$  is the mean bistatic angle for the scene point and  $\psi_H$  and  $\psi_V$  are the horizontal and vertical angular aperture extents subtended by the VEM directions about the scene point, measured in radians.

SAR image formation in a 30-m volumetric cube with 20 cm voxel spacing was performed on the phase history data,

using the bistatic backprojection image formation algorithm [11, 13, 15]. Maximum Intensity Projections (MIPs) of the resulting three-dimensional SAR images are presented in Fig. 4 for top views and Fig. 5 for side views. The intensity is normalised and scaled logarithmically in a 20 dB dynamic range. The SAR aperture positions are overlaid onto the SAR images to aid understanding. The top-view MIPs in Fig. 4 show well-resolved scatterers which correspond well to the simulated scene. The scatterer resolution is seen to vary from one position to another, and with  $\beta$ . The side-view MIPs show that the vertical radar resolution becomes coarse for scatterers located near to the transceiver plane. This is due to the reduced vertical angular extent of the BEM aperture as seen from these scatterer locations.

Table 1 shows the simulation parameters and estimates for resolution at the scene centre calculated with (4.3). The resolution values correspond well to those achieved in the simulations, and the resolution trends follow those one would expect from inspection of the virtual VEM apertures for the different  $\beta$  values.

The expected “up-down” mirrored ambiguity about the transceiver plane, described by (3.29), is evident in the MIPs; however, no other scatterer ambiguities were seen within the 30-m cube volumetric images.

## 5 Conclusion

This paper evaluates a two-line multistatic SAR aperture arrangement, with receivers along one line and transmitters along the other, designed to provide 3D imagery by forming a virtual 2D SAR aperture in between the lines. The paper provides a strategy to avoid image artefacts associated with multistatic SAR geometry.

We conclude that by time-gating in the fast and slow time variables, one may ensure that if  $\rho$  is large enough, the only additional singularities arise exactly as in (3.29), which describes the expected “up-down” mirrored ambiguity about the transceiver plane. We also note that we are explicitly ensuring that we avoid the potential artefacts that were identified in the “walkaway” acquisition geometry discussed in [5, §6.2]. Our calculation also gives additional insight as to how this extra artefact can arise.

Supporting numerical simulation results for four multistatic geometry scenarios are provided in Sect. 4. These follow a time-independent formulation different to that of the previous sections, but closely aligned with common radar laboratory measurement setups [1, 13]. Thus, the simulations also serve as independent verification of the methods presented in Sect. 3.

The analysis described in this paper, although directed to sensing with electromagnetic waves, is also relevant to synthetic aperture sensing with other wave modalities includ-

ing sound waves, which has applications in underwater and medical ultrasound imaging, and seismic waves to image underground structures, used for geological exploration for example.

Further work could include validation of the algebraic and numerical results by conducting real SAR measurements of the scenario at the Cranfield Ground-Based SAR laboratory, with different scene types, from canonical scatterers, to more realistic terrain with buried objects.

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**Data Availability** No datasets were generated or analysed during the current study.

## Declarations

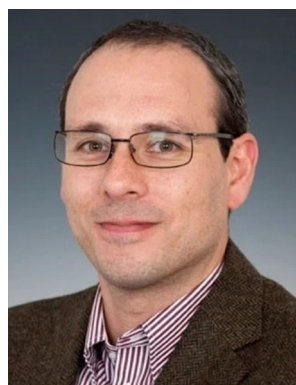
**Conflict of interest** The authors declare no conflict of interest.

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## **Artefact analysis of multistatic three-dimensional SAR imaging with two linear arrays**

**Andre, Daniel; Krishnan, Venkateswaran P.; Nolan, Clifford**

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