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METHODS FOR SEPARATING CLOSE NATURAL FREQUENCIES

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ABSTRACT

When close natural frequencies occur, they may cause complications when performing experiments and require special attention and treatment in modelling. It is shown that close and even identical natural frequencies can occur in linear structures. However, these are unlikely to occur if one parameter in the system is varied in an attempt to tune a system. Adding damping to a system with close natural frequencies may result in an adverse result where one mode has reduced damping and another increased damping. A method based on narrow band filtering is presented which enables close modes to be separated so that the natural frequency and damping of each mode can be identified. This method involves reversing the time history of vibration decay prior to filtering. Finally, the effect of nonlinearities is examined. A new algorithm based on barycentric interpolation is demonstrated. This algorithm can be used for model fitting linear or nonlinear systems and is shown to be near perfect. A case is demonstrated that uses the nonlinear model to separate close modes. However, a counter example shows that this may not be generalised.

Keywords: Structural dynamics, damping, nonlinearity, model fitting.

NOMENCLATURE

M	Mass matrix
C	Damping matrix
K	Stiffness matrix
F	Vector of forces
y	Displacement vector
ζ	Damping ratio
ω	Frequency

1. INTRODUCTION

Close natural frequencies can be a problem when analysing measurements or developing theories. Close natural frequences can occur naturally in structures or may be the results of fluid-structural coupling between components. Examples are acoustic structural coupling or heat exchanger tubes with similar natural frequencies. Symmetry is sometimes a cause, but it frequently "just happens" in experiments when a large number of modes are being examined.

A standard problem is the measurement of the natural frequency and damping of two modes with similar natural frequencies. Special signal processing methods are generally needed.

A full theory of coupled modes has yet to be developed particularly with respect to damping and nonlinearities. A simple question is what are the circumstances when two modes have the same natural frequency?

The objective of this paper is to investigate the circumstances that lead to close natural frequencies and to discuss methods for determining the properties of each mode. The general area of this research is known as blind separation of sources [1, 2, 3,4]. Examples are given of methods for linear and nonlinear systems.

2. THEORY

A general linear mechanical system can be modelled by a mass, damping and stiffness matrix with the equation of motion given by:

$$\mathbf{M}\,\ddot{\boldsymbol{y}} + \mathbf{C}\,\dot{\boldsymbol{y}} + \mathbf{K}\,\boldsymbol{y} = \mathbf{F} \tag{1}$$

where **M, C** and **K** are mass, damping and stiffness matrices, **y** is a vector of displacements and **F** a vector of forces. This system has eigen values and eigen vectors where the natural frequency and damping are the eigen values and the mode shapes are the eigen vectors.

A first point of investigation is to consider the circumstances under which close natural frequencies occur. The simplest case to study is a two degree-of-freedom system without damping. Such a system is shown in Fig 1.

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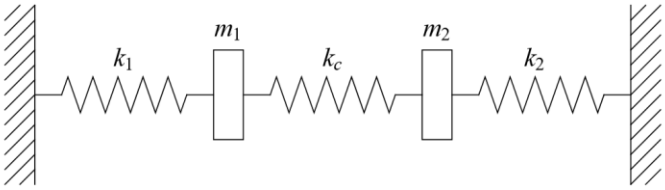


Figure 1: A TWO DEGREE OF FREEDOM SYSTEM WITH SPRINGS AND MASSES

Here there are two masses, m_1 and m_2 each connected to ground by springs k_1 and k_2. A third spring, k_c, couples the two masses. If the spring k_1 is varied the two natural frequencies of the system become close but do not become identical. An example of what happens is shown in Fig 2.

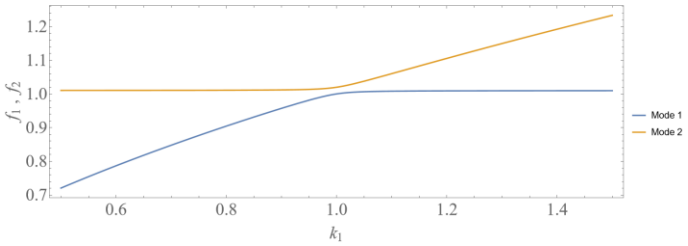


Figure 2: THE NATURAL FREQUENCIES AS A FUNCTION OF SPRING STIFFNESS.

Note that the curves don't cross but change direction. This is known as "curve veering". In addition, it is notable that the two mode shapes do not change their phase so that the in-phase mode shape associated with the first natural frequence stays with that natural frequency throughout the changing stiffness values. Similarly, the out-of-phase mode shape stays with the second natural frequency as the stiffness is varied.

This behaviour askes the question; is possible to make the natural frequencies be identical? The answer is yes but more than one stiffness has to be changed. An example of this behaviour is shown in Fig 3

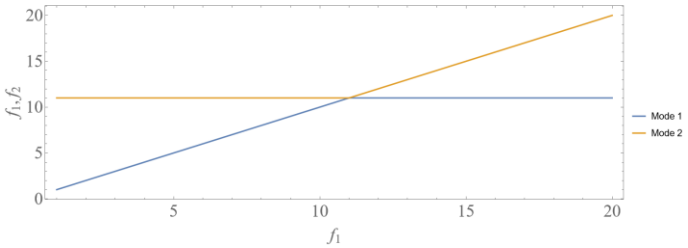


Figure 3: THE NATURAL FREQUENCIES WITH VARYING STIFFNESS MATRIX

Here the mass matrix has been kept constant, and the mode shapes have also been kept constant. The stiffnesses have been varied so that identical natural frequencies are achieved. By keeping the mass matrix and the mode shapes constant, there is enough information to freely choose the two natural frequencies.

For systems with more degrees of freedom identical frequencies may also be found.

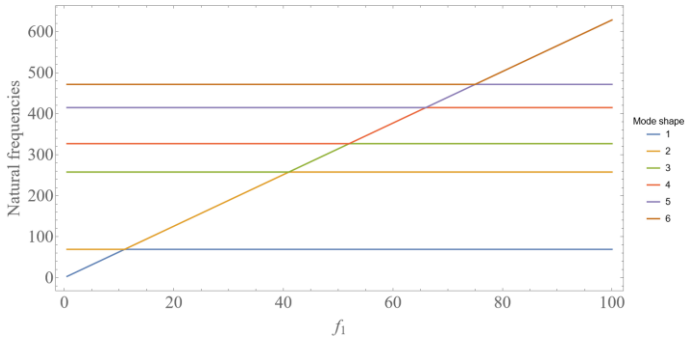


Figure 4: THE NATURAL FREQUENCIES OF A SYSTEM WITH SIX DEGREES-OF-FREEDOM WITH VARYING STIFFNESS MATRIX

Fig 4 gives an example with 6 degrees of freedom. The procedure was to define the mass matrix and the mode shapes and natural frequencies. The first natural frequency was then increased to create the figure. It is notable that although the first natural frequency is being increased the mode shape initially associated with the first natural frequency changes systematically when the changing natural frequency becomes identical with another natural frequency.

In conclusion it can be stated that identical natural frequencies in an un-damped structure can occur and veering is not always present.

3. DAMPED SYSTEMS

The investigation introduced in the previous section can now be developed by adding damping. Starting again with a two degree-of-freedom system damping can be added as shown in Fig 5.

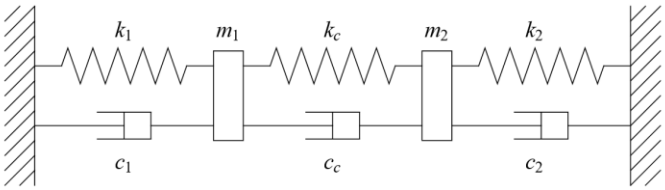


Figure 5: A TWO DEGREE OF FREEDOM SYSTEM WITH SPRINGS, DAMPERS AND MASSES. EXTERNAL FORCES MAY BE APPLIED TO THE MASSES TO GENERATE MOTION.

When making calculations for the natural frequencies of a damped system a formulation must be chosen so that an eigenvalue calculation may be made. The formulation shown is given in Equation 2:

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$$\begin{pmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \dot{Y} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{F} \end{pmatrix} \tag{2}$$

Here **M**, **C** and **K** are again the mass, damping and stiffness matrices. These have been assembled into two matrices which can be used in an eigen solver. Note that the over-dot above the $(Y, \dot{Y})^T$ vector applies to all entries in the vector, thus making it $(\dot{Y}, \ddot{Y})^T$. The result is that there are complex natural frequences whose real and imaginary parts reflect the damping and natural frequency. The eigenvalues may be expressed in the form:

$$-\zeta\,\omega_0 \;+\; i\,\sqrt{1-\zeta^2}\;\omega_0 \tag{3}$$

where $\boldsymbol{\omega_0}$ is the natural frequency in radians per second and $\boldsymbol{\zeta}$ is the damping ratio. The formulation being used here is the one used by control engineers when using Laplace transforms to work in the complex plane.

In order to investigate this system, the mass matrix and damping matrix are held constant and the stiffness values varied. The stiffness values were varied in the same manner as described in Section 2 and the changing value of the poles identified.
The natural frequencies follow the same pattern as the undamped system and identical natural frequencies can be found. The damping is more interesting.

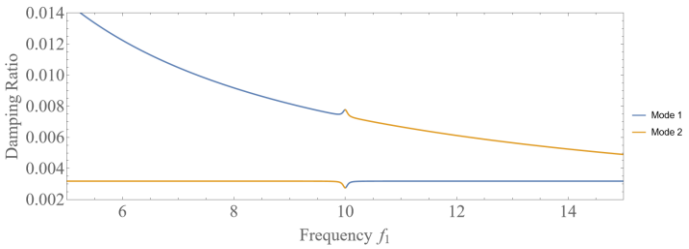


Figure 6: The DAMPING RATIO AS A FUNCTION OF NATURAL FREQUENCY OF A TWO DEGREE-OF-FREEDOM SYSTEM

Fig 6 shows the damping behaviour of the two modes. As the mode cross there is a sudden interaction between the damping values. One set of damping values increases and the other decreases. A sudden change in damping value particularly to a smaller value can be a concern in a mechanism relying on damping for stability.

4. SEPARATING MODES IN MEASURED DATA

Experimental data often contains close modes, and special methods are required to separate one mode from another. In this section separation of modes from linear systems is considered.

Fig 7 shows some spectra from a set of measurements obtained by using an impulse hammer. The Figure shows the response from 12 triaxial accelerometers to one strike by the impulse hammer. There are three modes in the interval displayed. As well as the two main modes at about 194 Hz and 202 Hz there is a third mode at 165 Hz. The mode at 165 Hz is overshadowed by the two main modes. This can be clearly seen in Fig 8 which expands the spectra around 165 Hz.

The first problem is how to separate the mode at 165Hz from the dominant modes. The method proposed uses a bandpass filter

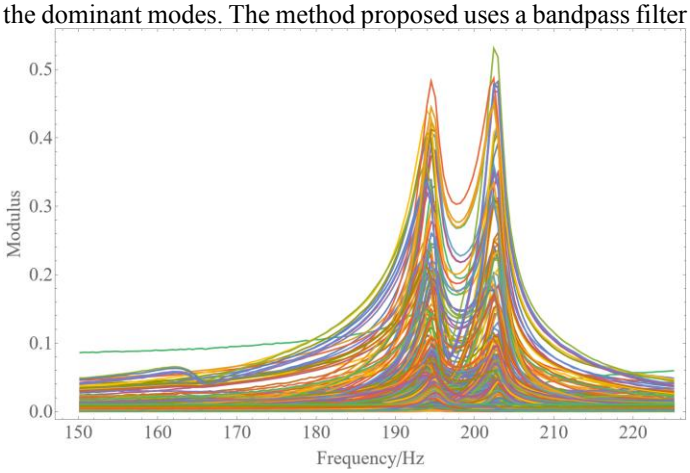


Figure 7: MEASURED SPECTRA FROM IMPULSE EXCITATION THERE ARE THREE MODES IN THE FIGURE

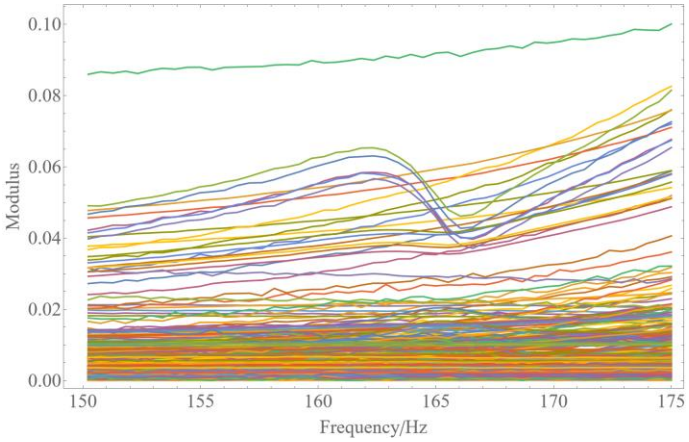


Figure 8: EXPANDED VERSION OF FIGURE 7 HIGHLIGHTING THE MODE AT APPROXIMATELY 165 HZ.

on the time histories. It is straightforward to make a digital filter and apply it to each of the time histories. The filter used was a digital bandpass Butterworth filter. However, if applied in the usual manner the time histories become convolved with the filter response. This is particularly the case with a narrow band filter which is used here. This convolution means that the properties of the data are lost [5]. This is due to the filter transfer function having poles which give rise to unwanted interactions.

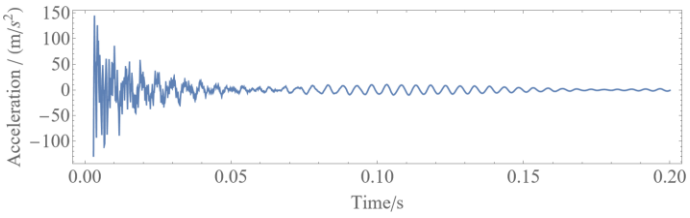


Figure 9: A TIME HISTORY FROM THE IMPULSE EXCITATION

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Fig 9 shows the unfiltered time history of an acceleration generated by an impulse excitation. The method proposed here starts by reversing the data and then passing it through the filter. The data now starts with negligible acceleration in the background noise and ends where the impulse was applied. With the data reversed the sudden excitation of the filter is prevented. A second part of the signal processing requires the filtering to stop precisely before the response to the impulse. The filtered data may then be reversed back to its original direction. Fig 10 shows the output of the filter for the input data in Fig 9.

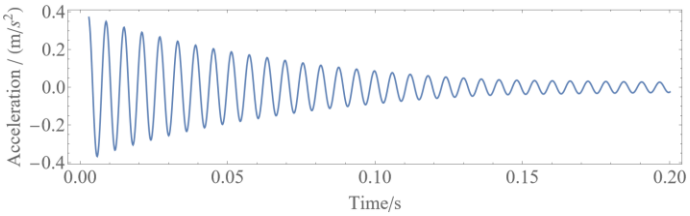


Figure 10: THE OUTPUT FROM THE FILTER CORRESPONDING TO THE INPUT OF FIGURE 9

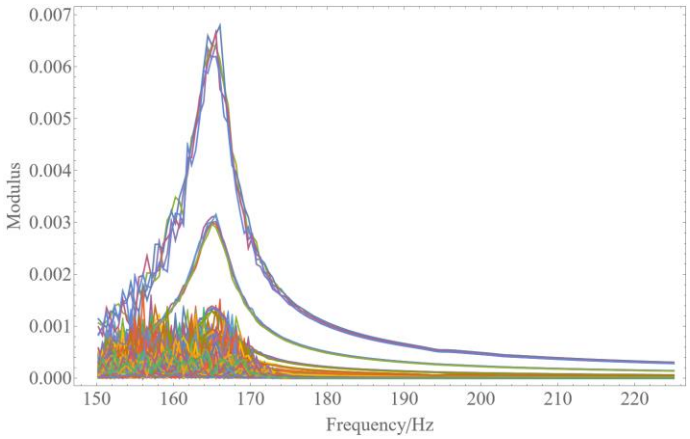


Figure 11: THE SPECTRUM OF THE FILTERED DATA

Following filtering of all the time histories the spectra were calculated and these are shown in Fig 11 This Figure may be compared to Fig 7. Note that the peaks of interest now stand out from the neighbouring modes. For these calculations the spectra have not been divided by the spectra of the force time history as is normal when calculating frequency response functions.

The decay time history of Fig 10 is clearly a decaying oscillation, and this maybe modelled by fitting the measured and filtered data to obtain the frequency and damping. The model equation is:

$$e^{-\zeta\,\omega_0\,t}\left(A\cos\left(\sqrt{1-\zeta^2}\;\omega_0\,t\right)+\;B\sin\left(\sqrt{1-\zeta^2}\;\omega_0\,t\right)\right) \tag{4}$$

where A and B are coefficients to be determined in the model

This equation is a nonlinear function of the frequency and damping and good starting values are required. It has been found that an estimate of the frequency from the spectra is adequate and that an estimate of the damping order of magnitude is generally satisfactory. A commercial nonlinear data fitting method is used see Mathematica [6]. The fitted data overlays the measured time history and is not shown here. Fig 12 gives the residual of the fitting process. The axis scale has been expanded by a factor of 10 so that it can be seen. The background theory for this method is given in [5].

This fitting method can also be applied to the two peaks which are at 194 Hz and 202 Hz in the spectra. In this case a narrow band filter between 185 Hz and 210 Hz was used on the

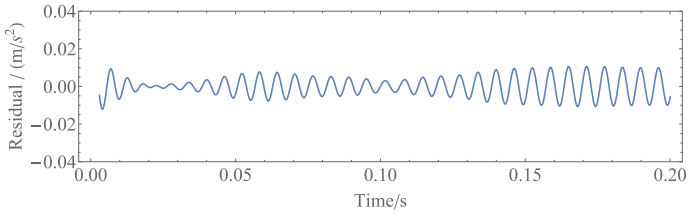


Figure 12: THE RESIDUAL OF THE FITTED DATA

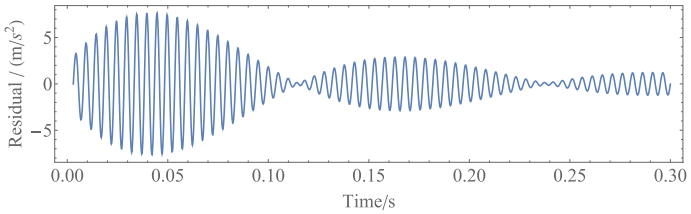


Figure 13: THE FILTERED DATA FROM THE TIME HISTORY OF FIGURE 9.

time histories with the data reversing procedure implemented as before. Figure 13 shows the resulting time history. The beating between the two modes is clearly visible. A time domain fitting procedure was again applied this time fitting two modes. The equation used was:

$$e^{-\zeta_1\,\omega_1\,t}\left(A_1\cos\left(\sqrt{1-\zeta_1{}^2}\;\omega_1\,t\right)+\;B_1\sin\left(\sqrt{1-\zeta_1^2}\;\omega_1\,t\right)+\right.$$
$$\left.e^{-\zeta_2\,\omega_2\,t}\left(A_2\cos\left(\sqrt{1-\zeta_2^2}\;\omega_2\,t\right)+\;B_2\sin\left(\sqrt{1-\zeta_2^2}\;\omega_2\,t\right)\right.\right. \tag{5}$$

where there are now two of each of the 4 parameters in each term.

The residual of the fit is shown in Fig 14.

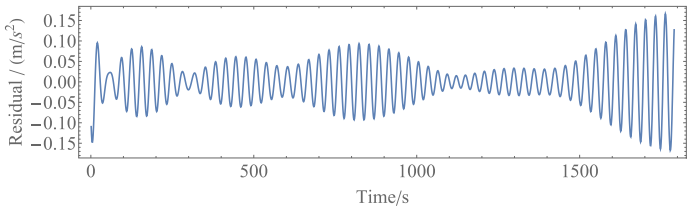


Figure 14: THE RESIDUAL FOLLOWING FITTING OF THE MODEL IN EQUATION 5 TO THE DATA IN FIGURE 13.

5. NONLINEAR DATA

The previous analysis has considered linear behaviour. If the data is nonlinear then there is no underlying theory that can be built upon. Nonlinear systems do experience behaviour similar to resonance and this can be found frequently in physical systems. The problem of close natural frequencies in nonlinear

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systems is thus a further issue to be resolved when addressing measurements and trying to develop models.

An attempt is now presented to use a new method that models close nonlinear resonances. The method is based on barycentric interpolation which can perfectly model linear or nonlinear spectra. When expressed in the time domain the model used for barycentric interpolation is written as:

$$\sum_{n=1}^{N} e^{-\zeta_n \omega_n t}\left(A_n \cos\left(\sqrt{1-\zeta_n^2}\ \omega_n t\right) + B_n \sin\left(\sqrt{1-\zeta_n^2}\ \omega_n t\right)\right) \tag{6}$$

The individual terms of this series can be recognised as the model for the decay of a linear single degree-of-freedom system. However, the series is best approached as a more general form of Fourier series where the frequencies are not restricted to be multiples of a fundamental and an additional decaying envelope is present. The series may be written in the frequency domain as:

$$\sum_{n=1}^{N} \frac{A_n s + A_n \zeta_n \omega_n + B_n \sqrt{1-\zeta_n^2}\ \omega_n}{s^2 + 2 s \zeta_n \omega_n + \omega_n^2} \tag{7}$$

This form has an excellent algorithm for fitting to spectra so that the various unknowns can be identified. More details can be found here [6]. The use of this algorithm for modelling close nonlinear modes is considered in this section.

Two problems can be identified when dealing with close nonlinear modes. The first is finding a model for the modes. The second is determining if modes can be separated as they can in linear system. It may be that nonlinear mode are coupled in a manner that prevents separation.

In order to illustrate and investigate the use of this model a nonlinear system was generated which contains nonlinear modes which can be separated. This was done by constructing two single degree-of-freedom systems and adding their decay time histories. Each system contains a cubic nonlinearity so that the stiffness was equal to $\omega_0^2\, y(t) + \beta\, y^3(t)$. The spectrum of the combined decays was then calculated, and the modulus spectrum is shown in Fig 15.

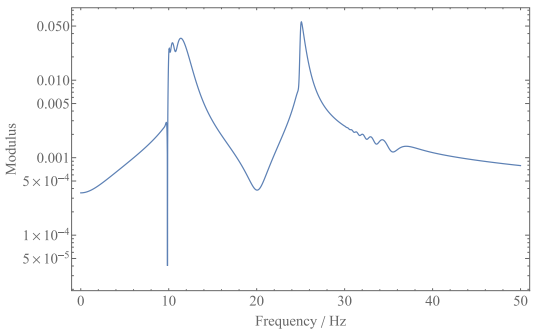


Figure 15: THE SPECTRUM OF THE TIME SERIES FOR A NONLINEAR SYSTEM

In the absence of the nonlinearity the two simulations would have natural frequencies and damping ratios of f_1 = 10 Hz, ζ_1 = 0.01, and of f_2 = 25 Hz, ζ_2 = 0.005 Examination of the spectrum shows that the peaks are distorted versions of resonance peaks and that there is spectral content at three times the fundamental frequency.

The spectrum in Fig 15 was modelled using the barycentric fitting algorithm [6]. The fit and the error are shown in Fig 16. The fit completely overlays the data. The horizontal pink lines are drawn to show the level of error in the fitting. It can be seen that the errors are at approximately 10^{-6} of the maximum value in

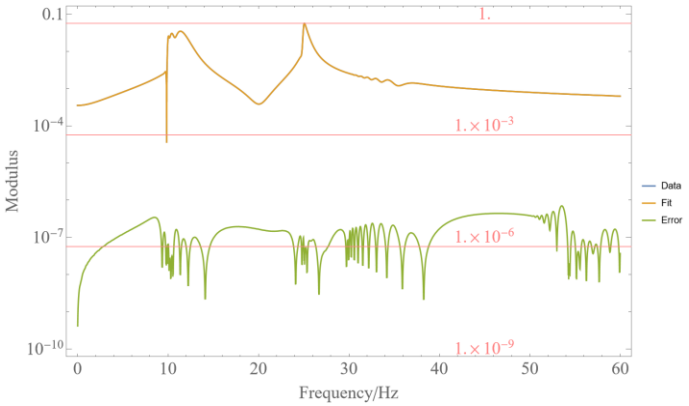


Figure 16: THE SPECTRUM, THE FITTED MODEL AND THE ERROR

the spectrum. It is clear that the algorithm provides an excellent model of the spectrum.

The second problem was then addressed to see if the modes could be separated. The fitted model was analysed to find the frequencies and damping ratios of the series in Equation 6. Fig 17 shows the natural frequencies and damping ratios.

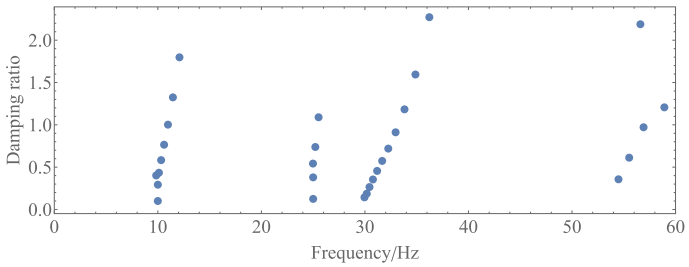


Figure 17: THE NATURAL FREQUENCIES AND DAMPING RATIOS OF THE MODEL FITTED TO THE SPECTRUM OF FIGURE 15.

If the frequencies are overlaid on the spectrum, Fig 18, then it can be seen which terms are modelling which parts of the spectrum. The frequencies that correspond to the first mode are shown in pink while the frequencies that model the second mode are shown in green. This selection could not be made automatically and the association of frequency with mode was made by hand. When this association was completed the model for each mode was constructed. In this case the simulation for each mode are available from the original simulation and a

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modelling error determined. The error for the first mode was 0.002 and the second 0.0017 This error is determined by calculating the absolute value of the error and dividing by the maximum value of the original simulation. This process has clearly been successful.

The next step was to see if a fully coupled nonlinear system can be decoupled in the same way. Simulated data was therefore generated for a two degrees-of-freedom system with cubic nonlinearity. For small motions the system is linear with natural frequencies and damping ratios identical to the previous system.

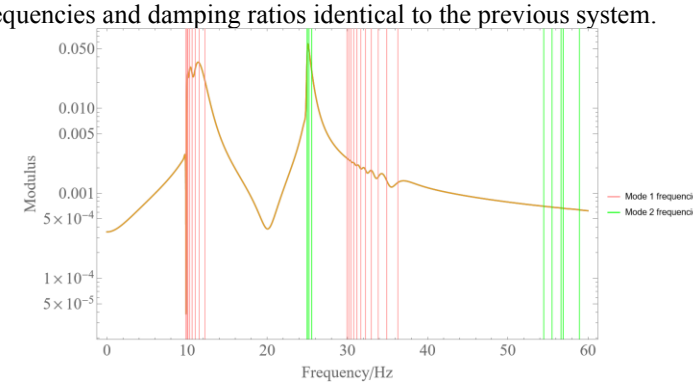


Figure 18: THE SPECTRUM WITH FITTED FREQUENCIES IDENTIFIED WITH MODES.

Time histories were simulated for the free vibration following an initial impulse. The spectra for the decay data are shown in Fig 19.

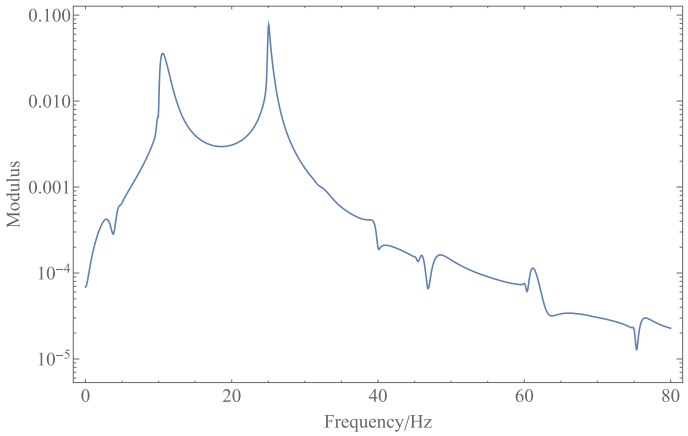


Figure 19: THE SPECTRUM OF A COUPLED NONLINEAR SYSTEM

This spectra is much more complicated than that of the first system, presumably due to the coupling between the two degrees of freedom.

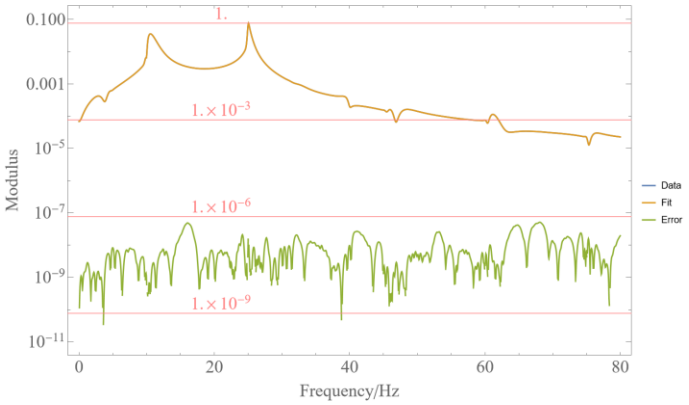


Figure 20: THE SPECTRUM, THE FIT AND THE ERROR.

The barycentric fitting algorithm was applied to the spectrum and produces a good model. A spectrum showing the data, the fit and the error is shown in Fig 20. The distribution of model frequencies and damping ratios are shown in Fig 21.

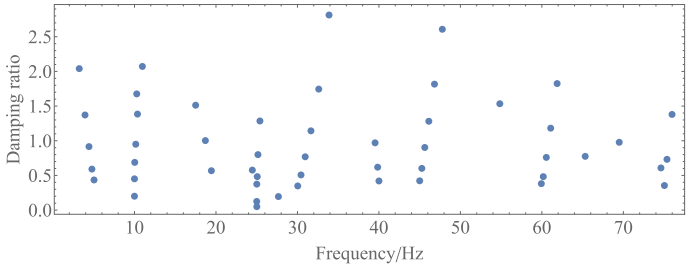


Figure 21: THE SPECTRUM, THE FIT AND THE ERROR

In this case it is not possible to sort the terms in the model into those associated with one mode and the other. The structure is clearly more complicated than in the first example. This may be because the modes cannot be separated or because no algorithm has yet been generated. Methods for separating modes is current work-in-progress.

6. DISCUSSION

The objective of this work was to investigate methods for modelling and separating close natural frequencies.

The modelling work described in Sections 2 and 3 shows that it is possible to have identical natural frequencies when there is no damping. However, these cannot be found by just changing one parameter; this leads to curve veering and does not result in identical natural frequencies. It is necessary to change several stiffness parameters to ensure identical natural frequencies. Such configurations of identical natural frequencies could occur by chance. An example of configurations in a six degree-of-freedom system illustrates how identical or close natural frequencies are possible.

The addition of damping to a two degree-of -freedoms system makes the natural frequences complex with the real part modelling the damping and the imaginary part the natural frequency. Identical natural frequencies are again possible, but the damping behaviour is unusual. When there are close natural

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frequencies then the damping of one system becomes large and the other one small. This is an interesting form of damping coupling.

A method for separating close modes was demonstrated using experimental data and a filtering procedure. The filtering was performed on the time histories of decay data with many modes. The key element was to reverse the data so that the filter characteristics were not excited. Using this technique very narrow band filters can be used which can suppress modes outside the filtering bandwidth. This has proved to be a very satisfactory process for separating modes.

Finally, the effect of nonlinearity was investigated. The use of the barycentric interpolation algorithm was demonstrated. This algorithm solved the problem of modelling a nonlinear system. Furthermore, the functions used in the modelling were very simple and familiar. The question of can nonlinear modes be separated was addressed. For a simple case this was shown to be possible, however, it was not possible for a more complicated case. This modelling approach shows potential and needs to be developed

7. CONCLUSIONS

The following conclusions may be drawn.

1. Close natural frequencies and identical natural frequencies can occur in linear structures if the mass and stiffness matrices are deliberately selected. Just changing one stiffness parameter does not typically tune a structure to have close natural frequencies.
2. The effect of damping on a linear structure does not prevent there being identical or close natural frequencies. The interaction between the damping of vibration modes becomes complex for close natural frequencies and may result in very small values of damping.
3. The modelling of nonlinear systems from vibration decay data has become possible using the new barycentric interpolation algorithm. This algorithm can provide a very general model for nonlinear system.
4. The barycentric interpolation algorithm may permit the separation of nonlinear modes and one case was demonstrated. However, it may not be possible to uncouple nonlinear modes or alternatively the algorithm for doing this has yet to be developed.

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Methods for separating close natural frequencies

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